تمرینهای فصل ۴

www.ASEC.blogfa.com

4-2)

a)

$\dot{x}_1 = u(t) - 3x_1 - kx_2$

$\dot{x}_2 = x_1 - 5x_2$

$y = x_2$

$A = \begin{bmatrix} -3 & -k \\ 1 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $D = 0$

b) $\det(A - \lambda I) = 0 \Rightarrow \det\begin{bmatrix} \lambda+3 & k \\ -1 & \lambda+5 \end{bmatrix} = 0 \Rightarrow (\lambda+3)(\lambda+5) + k = 0$

$\lambda^2 + 8\lambda + 15 + k = 0$

$\Rightarrow \lambda_{1,2} = \dfrac{-4 \pm \sqrt{16 - (15+k)}}{1} \Rightarrow \lambda = \begin{cases} -4 + \sqrt{1-k} \\ -4 - \sqrt{1-k} \end{cases}$

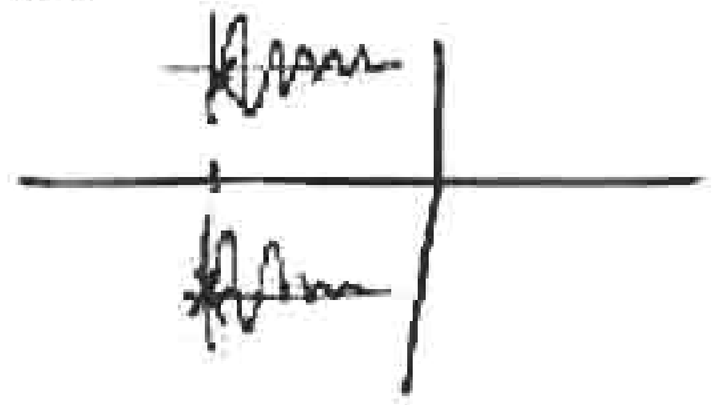
$1 - k < 0$
$\Rightarrow k > 1$
[illegible]

c) @ $k = 5$

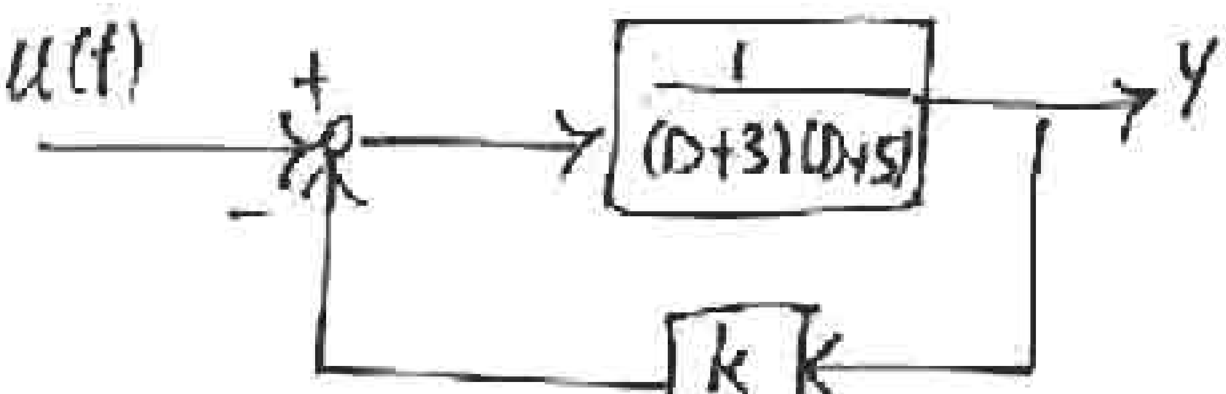
$\lambda = -4 \pm 2i$, Modes : $= e^{\lambda_1 t}, e^{\lambda_2 t}$

mode 1 : $e^{-4t}(i\sin 2t + \cos 2t)$,

mode 2 : $e^{-4t}(\cos 2t - i\sin 2t)$

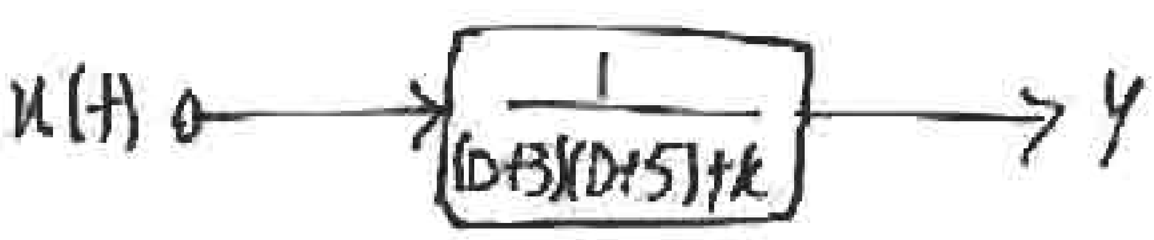


d)



[illegible]: $\dfrac{1}{(D+3)(D+5)+k}$

$\equiv$



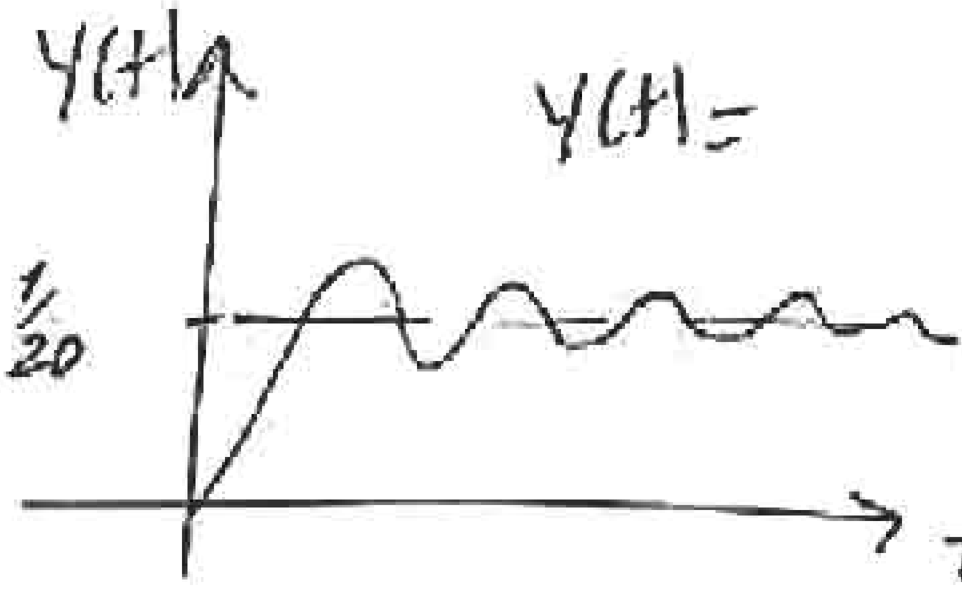
$(D+3)(D+5) + k = 0$

e) $(D+3)(D+5) + 5 = 0 \Rightarrow D_{1,2} = -4 \pm 2i \equiv \lambda_{1,2}$

f) $y(t) = \mathcal{L}^{-1}(G(s)\,1/s) = \int_0^t g(\tau)d\tau$, $g(\tau) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\left(\dfrac{1}{(s+4)^2+2^2}\right) = \frac{1}{2}e^{-4t}\sin 2t$

$\mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\left(\dfrac{1}{(s+3)(s+5)+5}\right) = \mathcal{L}^{-1}\left(\dfrac{1}{(s-(-4+2i))(s-(-4-2i))}\right) = \frac{1}{4i}e^{(-4+2i)t} - \frac{1}{4i}e^{(-4-2i)t}$

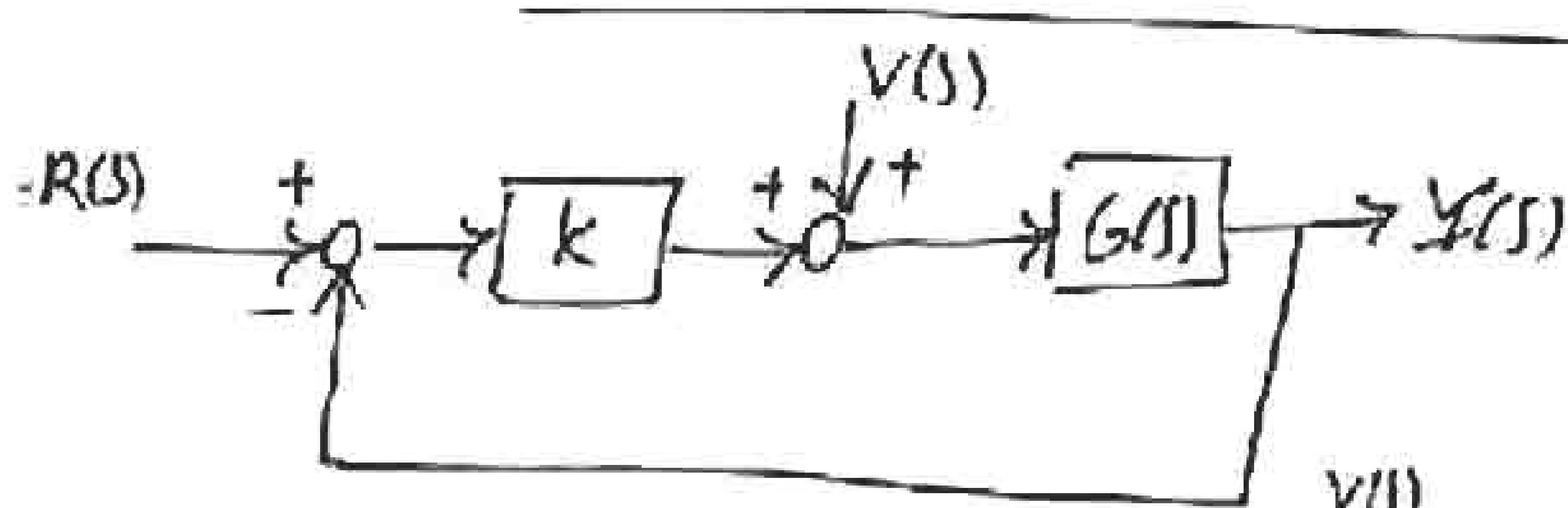
$= \frac{1}{4i}e^{-4t}(2i\sin 2t) = \frac{1}{2}e^{-4t}\sin 2t$ $\qquad y(t) = \int_0^t \frac{1}{2}e^{-4\tau}\sin 2\tau\, d\tau$



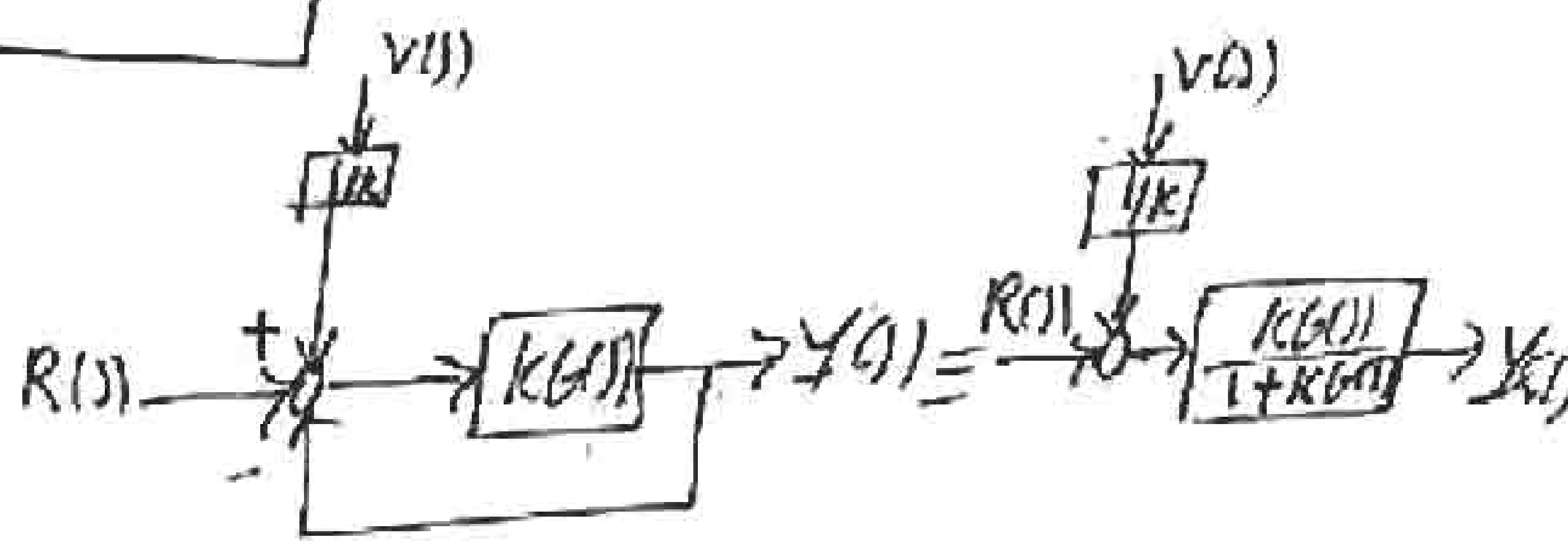
$Y(t) = \int_0^t \frac{1}{2} e^{-4\tau} \sin 2\tau \, d\tau = \frac{1}{20} - 0.1 \sin 2t \, e^{-4t} - 0.05 \, e^{-4t} \cos 2t$

f) $\lim_{t \to \infty} y(t) = \lim_{t \to \infty} = \frac{1}{20}$, $\lim_{s \to 0} s\left(G(s) \frac{1}{s}\right) = \lim_{s \to 0} G(s) = \frac{1}{20}$

4-3)



$G(s) = \dfrac{1}{(s+2)(s+3)}$



a)

$Y(s) = \dfrac{kG(s)}{1+kG(s)} R(s) + \dfrac{G(s)}{1+kG(s)} V(s)$

b) if K $\Uparrow \Rightarrow$ $\dfrac{G(s)}{k(\frac{1}{k} + G(s))} = \frac{1}{k} = 0 \Rightarrow Y(s) = \dfrac{kG(s)}{1+kG(s)} R(s) = R(s)$

c) $R(t) = u(t)$, $K = 0.25$ $\Big\}$ $y(t) = ?$, $\lim_{t \to \infty} y(t) = ?$

$Y(s) = \dfrac{0.25}{(s+2)(s+3)+0.25} R(s)$, $y(t) = \mathcal{L}^{-1}\left(\dfrac{0.25}{(s+2)(s+3)+0.25} \times \frac{1}{s}\right) =$

$\dfrac{1}{(s+2)(s+3)+0.25} = \dfrac{1}{(s+2.5)^2}$

$\mathcal{L}^{-1}\left(\dfrac{0.25}{s(s+2.5)^2}\right) = \mathcal{L}^{-1}\left(\dfrac{0.04}{s} - \dfrac{0.1}{(s+2.5)^2} + \dfrac{-0.04}{(s+2.5)}\right) = 0.04 - 0.1 t e^{-2.5t} - 0.04 e^{-2.5t}$

4-6) $A = \begin{bmatrix} -3 & 4 & 4 \\ 1 & -3 & -1 \\ -1 & 2 & 0 \end{bmatrix}$

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} -3-\lambda & 4 & 4 \\ 1 & -3-\lambda & -1 \\ -1 & 2 & -\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 0 & 4 \\ 1 & -2-\lambda & -1 \\ -1 & 2+\lambda & -\lambda \end{vmatrix} = \begin{vmatrix} -3-\lambda & 0 & 4 \\ 0 & 0 & -1-\lambda \\ -1 & 2+\lambda & -\lambda \end{vmatrix}$$

$$\Rightarrow -(\lambda+3)(\lambda+1)(\lambda+2) - 4(0) = 0 \Rightarrow (\lambda+1)(\lambda+2)(\lambda+3)=0 \Rightarrow \begin{cases} \lambda_1 = 1 \\ \lambda_2 = 2 \\ \lambda_3 = 3 \end{cases}$$

at $\lambda_1 = 1$

$$\begin{bmatrix} -4 & 4 & 4 \\ 1 & -4 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_2^1 \\ v_3^1 \end{bmatrix} = 0 \Rightarrow \begin{cases} -4v_1^1 + 4v_2^1 + 4v_3^1 = 0 \\ v_1^1 - 4v_2^1 - v_3^1 = 0 \\ -v_1^1 + 2v_2^1 - v_3^1 = 0 \end{cases} \qquad v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

at $\lambda_2 = 2$

$$\begin{bmatrix} -5 & 4 & 4 \\ 1 & -5 & -1 \\ -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} v_2^1 \\ v_2^2 \\ v_2^3 \end{bmatrix} = 0 \Rightarrow \begin{cases} -5v_1^2 + 4v_2^2 + 4v_3^2 = 0 \\ v_1^2 - 5v_2^2 - v_3^2 = 0 \\ -v_1^2 + 2v_2^2 - 2v_3^2 = 0 \end{cases} \quad -\tfrac{7}{8}v_1^2 = v_2^2 = -v_3^2 \qquad v_2 = \begin{bmatrix} -8/7 \\ 1 \\ -1 \end{bmatrix}$$

at $\lambda_3 = 3$

$$\begin{bmatrix} -6 & 4 & 4 \\ 1 & -6 & -1 \\ -1 & 2 & -3 \end{bmatrix} \begin{bmatrix} v^1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Rightarrow \begin{cases} -6v^1 + 4v_2 + 4v_3 = 0 \\ v_1 - 6v_2 - v_3 = 0 \\ -v_1 + 2v_2 - 3v_3 = 0 \end{cases} \Rightarrow v_3 = \begin{bmatrix} -10 \\ 1 \\ -1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & -8/7 & -10 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \qquad x = Tx^*, \quad T^{-1}AT = \Lambda$$

$$T^{-1} = \frac{\mathrm{adj}\,T}{\det T} =$$

4-9) $\dot{x}_1 = -2x_1 + u + x_2$

$\dot{x}_2 = u - x_1$

$y = -x_1 + x_2$

$$\Big\} \quad A = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = [-1, 1], \; D = 0$$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda+2) + 1 = 0 \Rightarrow \lambda^2 + 2\lambda + 1 = 0$$

$$\lambda_{1,2} = -1.$$

at $\lambda = -1$

$$\Rightarrow \begin{bmatrix} -1 & 1 \\ -1 & +1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow \begin{matrix} -v_1 + v_2 = 0 \\ v_1 + v_2 = 0 \end{matrix} \Rightarrow V = \begin{bmatrix} 1 \\ +1 \end{bmatrix}$$

repeated eigenvalue, we have: generalized eigen vectors

$(A - \lambda I) V_2 = V_1$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix} = \begin{bmatrix} 1 \\ +1 \end{bmatrix} \Rightarrow \begin{cases} -v'_1 + v'_2 = 1 \\ -v'_1 + v'_2 = 1 \end{cases} \Rightarrow v' = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\Lambda = T^{-1}AT = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Jordan Canonical form

$$A^* = T^{-1}AT = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix},$$

$$B^* = T^{-1}B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

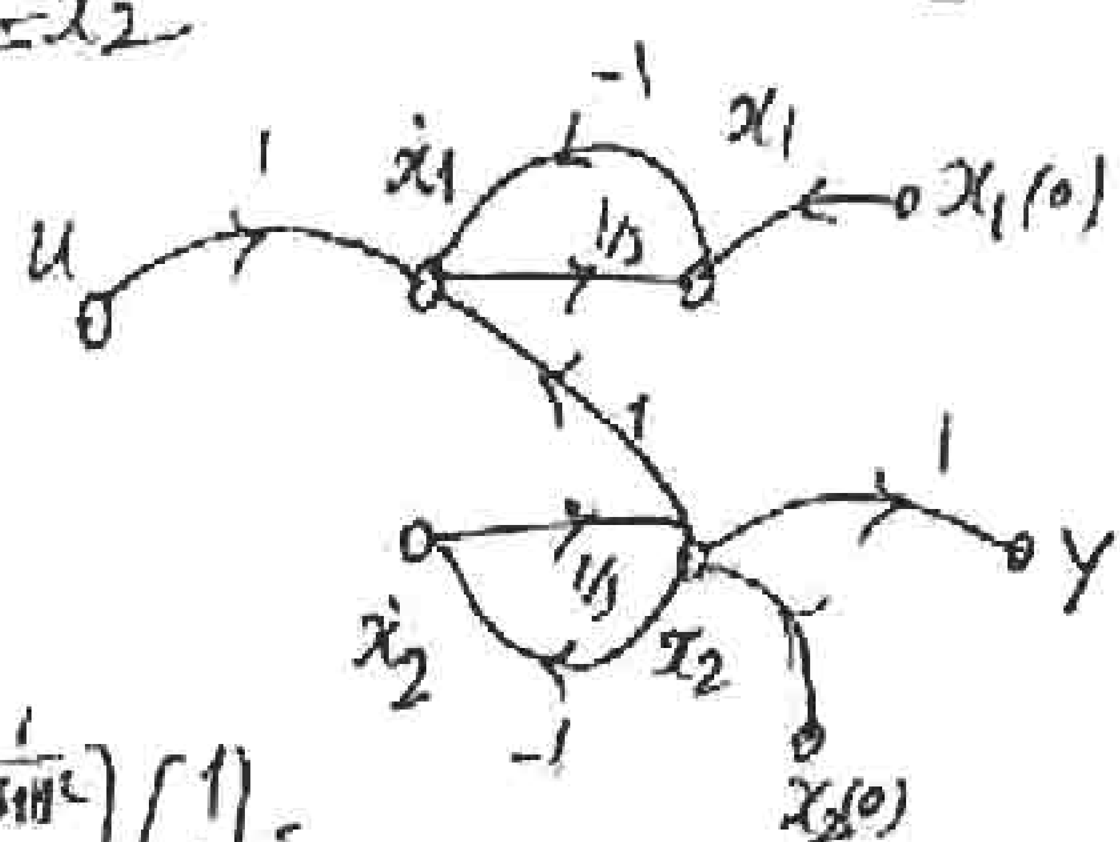
$$C^* = CT = [-1, 1] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = [0 \;\; 1]$$

$D^* = 0$

$\dot{x}_1 = -x_1 + x_2 + u$

$\dot{x}_2 = -x_2$

$y = x_2$



---

c) $G(s) = C(sI - A)^{-1}B$

$$= [-1 \;\; 1] \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [-1 \;\; 1] \begin{bmatrix} \frac{s}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s+2}{(s+1)^2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{1-s}{(s+1)^2} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{\frac{2}{(s+1)^2} = G(s)}$$

9-13)

$$G_1(s) = \frac{1}{s^2+\omega^2}$$

$\frac{1}{\omega}\sin\omega t$

$* u = \delta(t)$

$$Y(t) = \mathcal{L}^{-1}(G(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s-(\omega i))(s-(-\omega i))}\right) = \frac{1}{2\omega i}\mathcal{L}^{-1}\left(\frac{1}{s-\omega i} - \frac{1}{s+\omega i}\right)$$

$$= \frac{1}{2\omega i}\left(e^{\omega t i} - e^{-\omega t i}\right) = \frac{1}{2\omega i}(2i\sin\omega t) = \boxed{\frac{1}{\omega}\sin\omega t = y(t)}$$

$* u = u(t)$

$$y(t) = \int_0^t g(\tau)d\tau = \int_0^t \tfrac{1}{\omega}\sin\omega\tau\, d\tau = \tfrac{1}{\omega}\left(-\tfrac{1}{\omega}\right)\cos\omega\tau\Big]_0^t = \boxed{-\tfrac{1}{\omega^2}(\cos\omega t - 1) = y(t)}$$

$$G_2(s) = \frac{1}{(s+\delta)^2+\omega^2}$$

$\frac{1}{\omega}e^{-\delta t}\sin\omega t$

$u(t) = \delta(t)$

$$y(t) = \mathcal{L}^{-1}(G_2(s)) = \mathcal{L}^{-1}\left(\frac{1}{(s+\delta)^2+\omega^2}\right) = e^{-\delta t}\mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \boxed{e^{-\delta t}\left(\tfrac{1}{\omega}\sin\omega t\right) = y(t)}$$

$u(t) = u(t)$

$I_1 =$

$$y(t) = \int_0^t g(\tau)d\tau = \int_0^t e^{-\delta\tau}\left(\tfrac{1}{\omega}\sin\omega\tau\right)d\tau = \tfrac{1}{\omega}\left[-\tfrac{1}{\delta}e^{-\delta\tau}\sin\omega\tau\Big]_0^t + \tfrac{\omega}{\delta}\int_0^t e^{-\delta\tau}\cos\omega\tau\, d\tau\right]$$

$$= \tfrac{1}{\omega}\left(-\tfrac{1}{\delta}e^{-\delta t}\sin\omega t\right) + \tfrac{1}{\delta}\left(-\tfrac{1}{\delta}e^{-\delta\tau}\cos\omega\tau\Big]_0^t + \tfrac{-\omega}{\delta}\int_0^t e^{-\delta\tau}\sin\omega\tau\, d\tau\right)$$

$$\tfrac{1}{\delta}\left(-\tfrac{1}{\delta}e^{-\delta t}\cos\omega t + \tfrac{1}{\delta} - \tfrac{\omega}{\delta}I_1\right)$$

$$I_1 = -\frac{1}{\omega\delta}e^{-\delta t}\sin\omega t - \tfrac{1}{\delta^2}e^{-\delta t}\cos\omega t + \tfrac{1}{\delta^2} - \tfrac{\omega}{\delta^2}I_1$$

$$I_1 = \left(\frac{\delta^2}{\delta^2+\omega^2}\right)\left(1 - \frac{\delta}{\omega}e^{-\delta t}\sin\omega t - e^{-\delta t}\cos\omega t\right) = y(t)$$

4-14)

a) $\dot{x}_1 = ku - kx_2$

$\dot{x}_2 = x_1 - x_2$

$y = x_2$

$A = \begin{bmatrix} 0 & -k \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} k \\ 0 \end{bmatrix}$, $C = [0 \quad 1]$, $D = 0$

b) $\det(A - \lambda I) = \begin{vmatrix} -\lambda & -k \\ 1 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda+1) + k = 0 \Rightarrow \lambda^2 + \lambda + k = 0$

Eigenvalue $\lambda_{1,2} = \dfrac{-1 \pm \sqrt{1-4k}}{2}$

Eigen vectors:

c) $u(t) = c = cte$, the stable condition occured when $\dot{x} = 0$

$\Rightarrow \underbrace{\dot{x}}_{=0} = Ax + Bu = 0 \Rightarrow x_{st} = -A^{-1}Bu = -\begin{bmatrix} 0 & -k \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} k \\ 0 \end{bmatrix} u$

reversed reaction process

4-15) $G(s) = \dfrac{k}{Ts+1} - \dfrac{1}{s+1}$, $1 < k < T$

$u = \delta(t)$

$y(t) = \mathcal{L}^{-1}(G(s)) = \frac{k}{T} e^{-\frac{t}{T}} - e^{-t} = e^{-t}\left(\frac{k}{T} e^{-\frac{t}{T}} - 1\right)$

$u = u(t)$

$y(t) = \int_0^t g(\tau)\,d\tau = \int_0^t \mathcal{L}^{-1}(G(s))\,d\tau = \int_0^t \left(\frac{k}{T} e^{-\frac{\tau}{T}} - e^{-\tau}\right) d\tau = -ke^{-\frac{\tau}{T}} + e^{-\tau}\Big|_0^t = e^{-t} - ke^{-\frac{t}{T}} + k - 1$

$y(t) = e^{-t} - ke^{-\frac{t}{T}} + k - 1$

4-16)

$$\lim_{t\to\infty} y(t) = 0.8 = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\,(G(s)\cdot 1/s) = \lim_{s\to 0} G(s)$$

$$G(s) = \frac{\frac{k}{(s+1)(s+2)}}{1+\frac{k}{(s+1)(s+2)}} = \frac{k}{(s+1)(s+2)+k}$$

$$\lim_{s\to 0} G(s) = \frac{k}{2+k} = 0.8 \Rightarrow \frac{k}{2} = \frac{0.8}{\underbrace{1-0.8}_{0.2}} = 4 \Rightarrow \boxed{k=8}$$

4-18)

Euler equation:

$$\Sigma M_0 = I\alpha \Rightarrow mg\,l\sin\theta = -ml^2\ddot{\theta} \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

$$\begin{matrix} x_1=\theta \\ x_2=\dot{\theta}\end{matrix} \Bigg\} \Rightarrow \begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 \end{matrix} \Bigg\} \quad A = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} \begin{matrix} \to f_1 \\ \to f_2 \end{matrix}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{x=0} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}, \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

4-21)

$$G(s) = \frac{\frac{1}{s^2(s+2)}}{1+\frac{1}{s^2(s+2)}} = \frac{1}{s^2(s+2)+1}$$

$$Y(s) = \frac{1}{s^2}\cdot\frac{1}{s^2(s+2)+1}, \quad e(s) = R(s) - Y(s) = \frac{1}{s^2}\left(1 - \frac{1}{s^2(s+2)+1}\right) = \frac{1}{s^2}\cdot\frac{s^2(s+2)}{s^2(s+2)+1}$$

$$e(s) = \frac{s+2}{s^2(s+2)+1}$$

$$m(s) = \frac{1}{s}e(s) = \frac{s+2}{s\left[s^2(s+2)+1\right]},$$

$$\lim_{s\to 0} sY(s) = \infty, \quad \lim_{s\to 0} s\,m(s) = 2$$

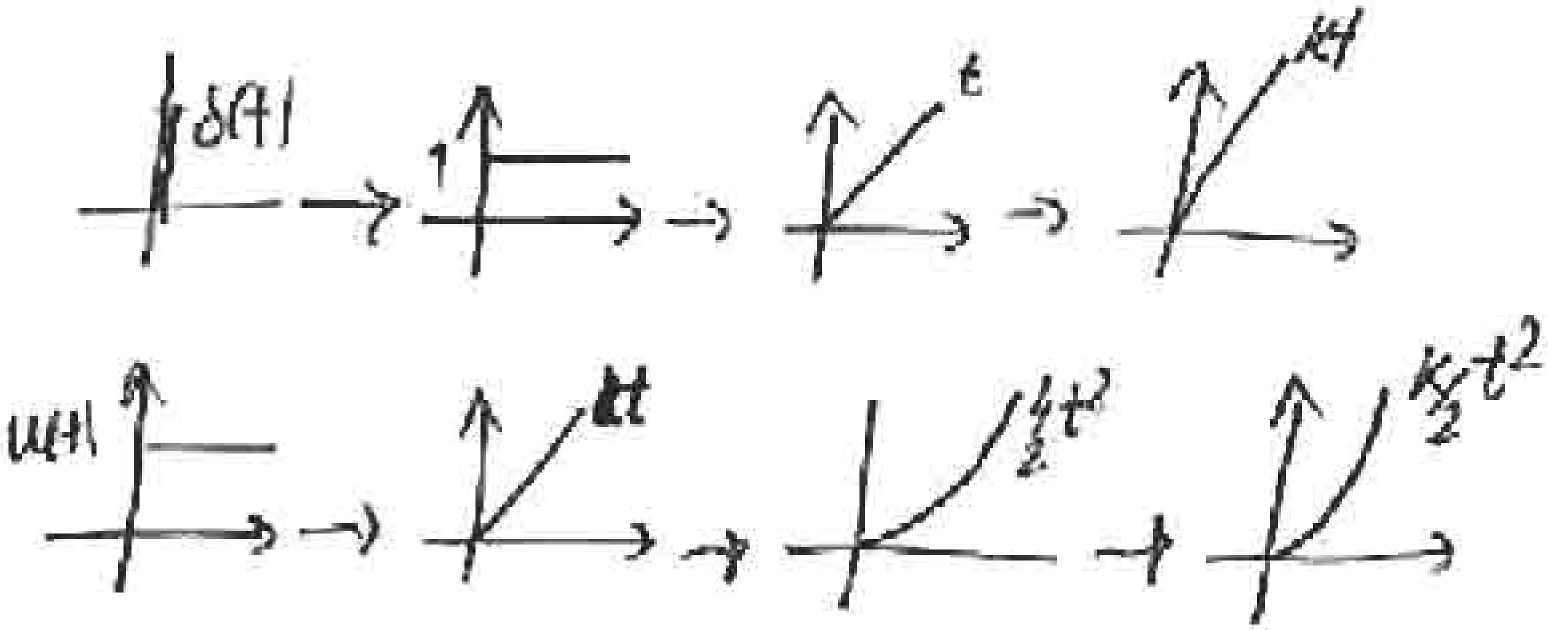
$$\lim_{s\to 0} s\,e(s) = \frac{0}{1} = 0$$

4-22)

$G(s) = k/s^2$

(a)

$$u(t) \rightarrow \boxed{\tfrac{1}{s}} \rightarrow \boxed{\tfrac{1}{s}} \rightarrow \boxed{k} \rightarrow y(t)$$



$y(t) = kt$

$y(t) = \frac{k}{2} t^2$

(b)

$$G_2(s) = \frac{k e^{-sT}}{(1+T_1 s)(1+T_2 s)}, \quad G_2' = \frac{k}{(1+T_1 s)(1+T_2 s)} \qquad T_1 < T_2$$

$u(t) = \delta(t)$ Impulse

$$y'(t) = \mathcal{L}^{-1}(G_2'(s)) = \frac{1}{T_1 - T_2} \mathcal{L}^{-1}\left(\frac{T_1}{1+T_1 s} - \frac{T_2}{1+T_2 s}\right) = \frac{1}{T_1 - T_2}\left(e^{-t/T_1} - e^{-t/T_2}\right) = \frac{1}{T_2 - T_1}\left(e^{-t/T_2} - e^{-t/T_1}\right)$$

$$y(t) = y'(t)\, u(t-T) = \frac{1}{T_2 - T_1}\left(\exp(-t/T_2) - \exp(-t/T_1)\right) u(t-T)$$

$u = u(t)$ step

$$y'(t) = \int_0^t g(\tau)\, d\tau = \int_0^t \left(\frac{1}{T_2 - T_1}\right)\left(e^{-\tau/T_2} - e^{-\tau/T_1}\right) d\tau$$

$$y(t) = y'(t)\, u(t-T)$$