Introduction to Hydrodynamics

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This is a script of a course we gave at the University Freiburg in the winter semester 2004/05. During 15 weeks, with 1.5 hours lecturing per week, it seemed impossible to us to cover all the subjects that belong to hydrodynamics. We would have needed way more time to do so. Therefor we picked subjects which we were most interested in, or which we really wanted to learn for our own interest...

As usual, at the end end of the semester time was flying, and with great regret we had to skip the chapters on instabilities and turbulence. We hope that at some point in the future we will give this lecture again, and with the gained experience we will cover these subjects, too.

If you find any statements and discussions which do seem erroneous to you, or in case you find any misprints (which might be plenty...), we would be really grateful to get your comments.

Freiburg, February 2005

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Introduction

Literature

D.J. Acheson: *Elementary Fluid Dynamics*, Clarendon Press, Oxford, 1990, ca. 40 Euro
D.J. Tritton: *Physical Fluid Dynamics*, Oxford Univ. Press, 1988, ca. 40 Eur
T.E. Faber: *Fluid Dynamics for Physicists*, Cambridge Univ. Press, 1995, ca. 55 Eur
M. van Dyke: *An Album of Fluid motions*, Parabolic Press, 1982

1.1 What is a fluid?

Generally we do not make a distinction between a liquid and a gas. In some sense water behaves very similar to air, except for the compressibility (even the kinematic viscosity of water is smaller and the one for air, because of the low density of the latter one; cf. Sect. 1.3).

The physical description of the fluid mechanics is largely based on the conservation of mass and Newton's law of motion as well as thermodynamics.

physical properties of fluids:

mass	[kg]	temperature	[K]	compressibility	
density	[kg/m^3]	pressure	[Pa]	viscosity (kinematic, ν)	[m^2/s]
velocity	[m/s]	heat conductivity		(dynamic, μ)	[kg/(m s)]

What are these quantities?!

In principle density, velocity and temperature are defined through statistical mechanics. They are properties of the microscopic distribution function of velocities of a large ensemble of molecules (including atoms). In equilibrium the velocity distribution (→ Brownian motions) is a Maxwellian. The (number) density is the integral of the distribution (0th moment), the velocity describes the bulk flow and the temperature the width of the velocity distribution. The pressure is the given e.g. through the ideal gas equation. Thus these properties are based on averaging over a large number of molecules. The viscosity is a transport property of the ensemble of molecules, as is the heat conductivity.

This shows that a description of fluid dynamics is valid only on scales larger than the molecular length scale L_{molecule}. Fluid dynamics is only valid on length scales $L_{\text{fluid}} \gg L_{\text{molecule}}$, and the volume L_{fluid}^3 contains a large number of molecules.

Different parcels of the fluid will interact on through e.g. molecular interactions such as collisions. Thus to "avoid" complications by such processes, the fluid length scale has to be much larger than the molecular mean free path λ, i.e. $L_{\text{fluid}} \gg \lambda$.

In a fluid mechanics one assumes that all molecular interactions can be approximated by transport processes such as viscosity, heat conduction, etc when considering length scales much larger than the molecular length scales L_{molecule} and λ.

Furthermore the time scales in fluid dynamics are much longer that the mean time between two collisions of the molecules.

1.2 Ideal Fluid

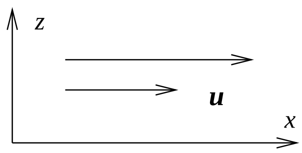
The main assumption is that an ideal fluid has zero viscosity. This is a fundamental difference to a real fluid with non-vanishing viscosity. It is not merely that the viscosity goes to zero, but also the differential equations describing a real and an ideal fluid have different character!

Even though widely used (especially in the past), the description of an ideal fluid often misses the crucial physics! In the 19th century one almost completely concentrated on ideal fluids, especially because of mathematical elegance.

One prominent example of a major shortcoming of an ideal fluid is it unability to explain why an airplane can fly (even though in many introductory physics textbooks one uses simple arguments of ideal fluids). It was not before Prandtl in the early 20th century that the important role of viscous effects was fully acknowledged. Among the other major revolutions in physics this one is often forgotten.

1.3 Viscosity, Reynolds number and turbulence

The viscous stress τ in a shear flow is defined to be proportional to the gradient of the velocity $\boldsymbol{u}$. Actually, τ can be considered as the force due to the shear per unit area. This unit area is perpendicular to the direction of the shear, e.g. if the shear is in z, the stress is a force per area in the x-y plane (see right).



Formally one can now Taylor expand τ in terms of the velocity. It is clear that the zeroth and first order terms are vanishing, as a constant flow has no shear. Thus the first non-vanishing term is

$$\text{viscous stress:} \qquad \tau = \mu \frac{\partial u}{\partial z} \qquad \left[\frac{\text{N}}{\text{m}^2} = \frac{\text{kg}}{\text{m s}^2}\right] \qquad \text{with dynamic viscosity:} \qquad \mu \quad \left[\frac{\text{kg}}{\text{m s}}\right] \tag{1.1}$$

This case with vanishing higher order terms is called a Newtonian fluid. Here the viscous stresses are approximated to be proportional to the gradients of the velocity. Implicitly this also assumes that the viscosity does not depend on the velocity, but it might well depend on temperature and/or pressure.

To estimate the viscous force per unit volume one has to compare the viscous stress across a small distance δz along the direction of the shear, i.e. at z and $z + \delta z$. The total viscous force F_μ then is the difference in stress $\tau_{z+\delta z} - \tau_z$ times the area $\delta x\, \delta y$. For infinitesimally small δz one can write

$$F_\mu = [\tau_{z+\delta z} - \tau_z]\, \delta x\, \delta y = \left[\mu \left(\frac{\partial u}{\partial z}\right)_{z+\delta z} - \mu \left(\frac{\partial u}{\partial z}\right)_z\right] \delta x\, \delta y = \frac{\partial}{\partial z}\left(\mu \frac{\partial u}{\partial z}\right) \delta x\, \delta y\, \delta z\ . \tag{1.2}$$

$$\text{viscous force per volume:} \qquad f_\mu = \mu \frac{\partial^2 u}{\partial z^2} \qquad \left[\frac{\text{N}}{\text{m}^3}\right] \tag{1.3}$$

In addition to the dynamic viscosity one often uses a kinematic viscosity,

$$\text{kinematic viscosity:} \qquad \nu = \frac{1}{\rho}\, \mu \qquad \left[\frac{\text{m}^2}{\text{s}}\right]. \tag{1.4}$$

A lot of problems can be characterized by dimensionless numbers. For example consider a flow of a fluid with (dynamic) viscosity μ, density ρ and velocity U through an infinitely long pipe with diameter L. These parameters μ, ρ, U and L basically define the whole problem.

One can now construct a dimensionless quantity from these parameters. At this point this choice seems to be arbitrary, but it is the simplest combination of these parameters giving a dimensionless number.

$$\text{Reynolds number:} \qquad \text{Re} = \frac{\rho\, U\, L}{\mu} = \frac{U\, L}{\nu}. \tag{1.5}$$

As we will see later this Reynolds number is found by comparing the inertial term in the momentum equation $\rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ with the viscous forces $\sim\rho\Delta\boldsymbol{u}$..

The Reynolds number is one of the most important quantities in hydrodynamics, as it characterizes the nature of a flow. For example for very large Reynolds numbers a flow changes is character from being laminar to turbulent. And still, turbulence is an active area of research.

1.4 Mach number

The Mach number compares the actual speed of the flow U to the sound speed c.

$$\text{Mach number:} \qquad \mathrm{Ma} = \frac{U}{c} \tag{1.6}$$

If the flow has a low Mach number the fluid will behave as being incompressible, i.e. at constant density. One can show that the fluctuations of the density roughly scale with the Mach number squared, i.e. $\Delta\rho/\rho \sim (\mathrm{Ma})^2$. Thus already at Ma$\approx$0.2 the density fluctuations are down to less than 5%. The high sound speed of a liquid (because of its high density) results in comparably small Mach numbers in liquid flows. This can be seen as the basic reason why a liquid is less compressible as a gas.

1.5 Motivation: some applications

Astrophysics: stellar convection; coronal flows; jets; supernovae...

Meteorology: hurricanes; jet steams; large scale convection ...

Geology: continental drift; mantle convection ...

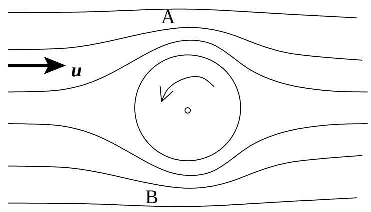
Cars & trains: optimal form of cars
"nice nose" is not most important feature of trains!
roof surface, body sides and underbelly cause almost 50% of drag (trains longer than cars)
trains entering and leaving tunnels

Aeronautics: How to fly; vortex generators at airplanes ...

Medicine: blood flow; transport of trace particles in blood; voice generation; cell diffusion ...

1.5.1 Medicine: keeping erythrocytes in the middle of the venes — Magnus effect

Following Bernoulli's law, $\frac{1}{2}\rho u^2 + p = constant$, the pressure at the sides of a ball in the wind at A and B is smaller than in the ambient wind. If the ball is rotating as indicated, then the (relative) speed will be increased at A and decreased at B, thus again because of Bernoulli's law the pressure at A will be lower than at B: the ball will be drifting in the direction of A!



This is a general property of a sphere or a cylinder, rotating around the axis (of symmetry) perpendicular of the flow. The sphere or cylinder will then feel a force perpendicular to the flow direction and the rotational axis. This effect is called the Magnus effect, for spheres sometimes Robins effect.

When particles such as erythrocytes are transported in the blood the velocity profile of the blood flow is maximum in the middle (see following section). Thus when the particle moves out of the central part of the vein the velocity gradient causes the particle to spin. The Magnus effect then forces the particle back to the central region.

1.5.2 Medicine: regulating blood fbw — Hagen-Poiseuille fbw

An illustrative example is the regulation of the blood flow in veins by slight changes of the diameter. For an ideal flow, i.e. with no viscosity, the mass flux Φ trough a pipe of radius a would change as $\Phi \propto a^2$, of course. Considering also the effects of viscosity (cf. Sect. 1.3) one finds a much stronger dependence of $\Phi \propto a^4$!

Discussing the viscous force in cylinder geometry rather than Cartesian geometry, instead of (1.3) one finds for the viscous force density in a pipe

$$f_{\mu,\mathrm{pipe}} = \mu \, \frac{1}{r} \, \frac{\partial}{\partial r} \left(r \, \frac{\partial u}{\partial r} \right) , \tag{1.7}$$

where r is the radial distance to the center of the cylinder. Let x be the axis along the cylinder.

The flow in the pipe is driven by a pressure gradient. Its component along the pipe is given in cylinder geometry by $\nabla_x p = (\partial p/\partial x)$.

Here we now assume that the pressure gradient is constant along the pipe, i.e. if p_1 and p_2 are the pressures at the end of the pile of length l, then the pressure gradient $\nabla_x p = G$, with $G = (p_1 - p_2)/l$.

The pressure gradient is balanced by the viscous forces, $f_{\mu,\mathrm{pipe}} + \nabla_x p = 0$, which can be easily integrated,

$$\mu \frac{\mathrm{d}}{\mathrm{d}r}\left(r\,\frac{\mathrm{d}u}{\mathrm{d}r}\right) = -G\,r \qquad \Rightarrow \qquad u \;=\; -\,\frac{G\,r^2}{4\,\mu} \;+\; A\ln r \;+\; B. \tag{1.8}$$

The constant A must vanish, $A = 0$ to keep u finite at $r{\to}0$, B can be calculated using the boundary condition that the velocity has to vanish at the (inner) surface of the pipe, i.e. u=0 at r=a, where a is the radius of the pipe. Thus one has

$$u = -\frac{G}{4\,\mu}\,(a^2 - r^2), \tag{1.9}$$

i.e. the velocity profile is a paraboloid. To obtain the total mass flux trough the pipe (assuming constant density ρ) one has to integrate the mass flux ρu over the cross section of the pipe,

$$\Phi \;=\; \int_0^a 2\pi r\,\rho u\,\mathrm{d}r \;=\; \frac{\pi}{8}\,\rho\,\frac{p_1 - p_2}{l}\,\frac{1}{\mu}\,a^4, \tag{1.10}$$

i.e. in a Hagen-Poiseuille flow trough a pipe with radius a, the mass flux pumped trough the pipe Φ varies very strongly with the radius, $\Phi \propto a^4$.

Therefore when regulating the blood flow the diameter of the veins has to be changed only very little, to get a large effect!

Chapter 2

Neglecting viscosity and compressibility: The Euler fluid

An Eulerian fluid is by definition incompressible and has no viscosity. Without viscosity the fluid cannot sustain any shear stress, and therefore the pressure is isotropic. Incompressibility does not mean uniform density, but rather that the density of a fluid element does not change when moving along with the fluid.

2.1 Conservation of mass: continuity equation

Considering a Volume $\mathcal{V}$ with the surface $\mathcal{A}$ the change of the mass M in this volume $\mathrm{d}M/\mathrm{d}t$ can only be caused by material flowing through a surface element $\mathrm{d}\boldsymbol{a}$ (normal to the surface) with density ρ and velocity $\boldsymbol{u}$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{V}}\rho\,\mathrm{d}V \;=\; -\int_{\mathcal{A}}\rho\,\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{a} \qquad\Longleftrightarrow\qquad \int_{\mathcal{V}}\left\{\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{u})\right\}\,\mathrm{d}V \;=\; 0 \tag{2.1}$$

Here on the left hand side the time derivative was pulled into the integrand and on the right hand side the integral theorem of Gauß was applied. As the above relation has to hold fo any volume, especially an infinitesimally small volume around any location in space, one finds the continuity equation

$$\text{continuity equation:}\qquad \frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\,\boldsymbol{u}) \;=\; 0 \tag{2.2}$$

2.2 The derivative in the co-moving frame and incompressibility

When describing a particle as a *point mass* its acceleration (in 1D) is given through

$$\text{particle dynamics:}\qquad \dot{u} \;=\; \frac{\mathrm{d}u}{\mathrm{d}t} \;=\; \frac{\mathrm{d}u}{\mathrm{d}x}\,\frac{\mathrm{d}x}{\mathrm{d}t} \;=\; \frac{\mathrm{d}u}{\mathrm{d}x}\,u \;=\; \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\,u^2\right) \tag{2.3}$$

Here all expressions contain a *full differentiation* rather than a *partial differentiation*! This is a feature of particle dynamics where u describes the velocity of the particle.

In contrast, in fluid dynamics u describes the velocity of the full flow field, i.e. it is a function of space *and* time. Thus we have to distinguish between the acceleration defined as the change of velocity at a fixed point in space, i.e. the partial derivative $\partial u/\partial t$, and the derivative in the co-moving frame.

A scalar quantity ϕ depending on space $\boldsymbol{x} = (x_1, x_2, x_3)$ and time t is given at $\boldsymbol{x}_0 = (x_{0,1}, x_{0,2}, x_{0,3})$ and t_0 in a frame of reference $S^{(0)}$. After an infinetesimal small time $\mathrm{d}t$, i.e. at $t = t_0 + \mathrm{d}t$, at the location $\boldsymbol{x} = \boldsymbol{x}_0 + \mathrm{d}\boldsymbol{x}$ with $\mathrm{d}\boldsymbol{x} = \boldsymbol{v}\mathrm{d}t$ its value $\phi_0 + \mathrm{d}\phi$ can be found through a Taylor expansion

$$\mathrm{d}\phi \;=\; \left.\frac{\partial\phi}{\partial t}\right|_{t_0}\mathrm{d}t \;+\; \left.\frac{\partial\phi}{\partial x_1}\right|_{x_{0,1}} v_1\mathrm{d}t \;+\; \left.\frac{\partial\phi}{\partial x_2}\right|_{x_{0,2}} v_2\mathrm{d}t \;+\; \left.\frac{\partial\phi}{\partial x_3}\right|_{x_{0,3}} v_3\mathrm{d}t \tag{2.4}$$

Thus $\mathrm{d}\phi$ is the change at $\boldsymbol{x}$ in the frame $S^{(0)}$. In another frame $S^{(1)}$ moving with $\boldsymbol{v}$ relative to $S^{(1)}$, $\boldsymbol{x}$ and $\boldsymbol{x}_0$ conincide. Therefore $\mathrm{d}\phi/\mathrm{d}t$ as given through (2.4) is the partial derivative (i.e. at a fixed point) in system $S^{(1)}$.

Therefore, in the case when $\boldsymbol{v} = \boldsymbol{u}$, i.e. the velocity of the fluid, $S^{(1)}$ becomes the frame moving with the flow. Therefore one can define the derivative

$$\frac{\mathrm{D}\phi}{\mathrm{D}t} \quad = \quad \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x_1}u_1 + \frac{\partial\phi}{\partial x_2}u_2 + \frac{\partial\phi}{\partial x_3}u_3 \quad = \quad \frac{\partial\phi}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\phi \tag{2.5}$$

as being the derivative in the co-moving frame, or the rate of change of ϕ in time when following the fluid. More generally one can define the operator

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\boldsymbol{u}\cdot\nabla) \tag{2.6}$$

often called the *convective derivative*. This operator can applied on any vector $\boldsymbol{A}$, but then one has to careful in evaluating $(\boldsymbol{u}\cdot\nabla)\boldsymbol{A}$ (cf. Sect. 2.4).

This discussion allows us to re-write the continuity equation (2.2) in the co-moving frame

$$\text{continuity equation:} \qquad \frac{\mathrm{D}\rho}{\mathrm{D}t} = -\,\rho\,\nabla\cdot\boldsymbol{u} \qquad \Longleftrightarrow \qquad \frac{\partial\rho}{\partial t} + (\boldsymbol{u}\cdot\nabla)\,\rho = -\,\rho\,\nabla\cdot\boldsymbol{u} \tag{2.7}$$

So far these considerations (i.e. in this and the preceeding section) where completely general.

If we now assume that a fluid element does not change density when moving along with the fluid, it is immediately clear that for a incompressible fluid we have

$$\text{incompressibile fluid:} \qquad \nabla\cdot\boldsymbol{u} = 0. \tag{2.8}$$

2.3 Conservation of momentum in the Euler fluid

For the Euler fluid we assume the force exerted across a surface element $\mathrm{d}\boldsymbol{a}$ normal to the surface $\mathcal{A}$ to be given by $p\ \mathrm{d}\boldsymbol{a}$, where p is the isotropic pressure. Now the net force on the volume $\mathcal{V}$ is

$$-\int_{\mathcal{A}} p\ \mathrm{d}\boldsymbol{a} = -\int_{\mathcal{V}} \nabla p\ \mathrm{d}V. \tag{2.9}$$

Here we used the Gauss theorem, i.e. $\int_{\mathcal{A}} \boldsymbol{x}\cdot\mathrm{d}\boldsymbol{a} = \int_{\mathcal{V}} \nabla\cdot\boldsymbol{x}\ \mathrm{d}V$, for a scalar ϕ, i.e. $\int_{\mathcal{A}} \phi\,\mathrm{d}\boldsymbol{a} = \int_{\mathcal{V}} \nabla\phi\,\mathrm{d}V$.

This force in (2.9) and the gravitational acceleration (integrated over the volume), i.e.,

$$\int_{\mathcal{V}} \rho\,\boldsymbol{g}\,\mathrm{d}V \tag{2.10}$$

cause the acceleration of the fluid in the co-moving frame of reference, i.e.

$$\int_{\mathcal{V}} \rho\,\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\,\mathrm{d}V \tag{2.11}$$

As the the forces in (2.9) – (2.11) have to balance for any volume, one finally arrives at the Euler equation:

$$\text{Euler equation:} \qquad \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\,\frac{1}{\rho}\,\nabla p + \boldsymbol{g}. \tag{2.12}$$

Together with (2.8), $\nabla\cdot\boldsymbol{u}$=0, this completely desribes the evolution of the pressure and the velocity for a given set of boundary and initial conditions.

2.4 Bernoulli's theorems

Using the definition of $\mathrm{D}/\mathrm{D}t$ in (2.6) and the vector identity $\nabla(\boldsymbol{u}^2) = 2(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - 2(\nabla\times\boldsymbol{u})\times\boldsymbol{u}$ one can re-write the Euler equation (2.12) in the case of *constant density* to yield

$$\frac{\partial\boldsymbol{u}}{\partial t} + (\nabla\times\boldsymbol{u})\times\boldsymbol{u} = -\,\nabla\left(\frac{p}{\rho} + \frac{1}{2}\,\boldsymbol{u}^2 + \phi\right), \tag{2.13}$$

where the gravitational potential ϕ was indroduced with $\boldsymbol{g}= -\,\nabla\phi$.

For a steady flow ($\partial/\partial t$=0) this reduces to

$$(\nabla \times \boldsymbol{u}) \times \boldsymbol{u} \;=\; -\nabla H \tag{2.14}$$

with the scalar

$$H \;=\; \frac{p}{\rho} + \frac{1}{2}\,\boldsymbol{u}^2 + \phi. \tag{2.15}$$

Taking the “dot product” of (2.14) the left hand side vanishes, as $(\nabla \times \boldsymbol{u}) \times \boldsymbol{u}$ is perpendicular to $\boldsymbol{u}$, and one finds

$$(\boldsymbol{u} \cdot \nabla) H = 0. \tag{2.16}$$

This implies that H is constant along a streamline. ¿From this it follows:

BERNOULLI I:

$$\text{In an ideal fluid flowing steadily, } \frac{p}{\rho} + \frac{1}{2}\,\boldsymbol{u}^2 + \phi \text{ is constant along a stream line.} \tag{2.17}$$

A flow field is defined irrotational, if

$$\text{irrotaional flow:} \qquad \nabla \times \boldsymbol{u} = 0. \tag{2.18}$$

In such a case, it follows immediately from (2.14) that ∇H=0, and thus:

BERNOULLI II:

$$\text{In an ideal fluid with a steady irrotational flow, } \frac{p}{\rho} + \frac{1}{2}\,\boldsymbol{u}^2 + \phi \text{ is constant troughout the flow field.} \tag{2.19}$$

2.5 What are curl and divergence?

2.5.1 Divergence and Gauss’ theorem

The interpretation of divergence is most easily demonstrated for the vector field of the mass flux.

Let $\boldsymbol{f} = \rho \boldsymbol{u}$ be the mass flux with the components $\boldsymbol{f} = (f_x, f_y, f_z)$. In the case of a box with sizes $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ in the respective directions the total mass per time flowing through the sides of the box at x and $x + \mathrm{d}x$ are

$$F(x) \;=\; f_x(x)\,\mathrm{d}y\,\mathrm{d}z \tag{2.20}$$

$$F(x+\mathrm{d}x) \;=\; f_x(x+\mathrm{d}x)\,\mathrm{d}y\,\mathrm{d}z \tag{2.21}$$

Using the definition of the (partial) derivative $\dfrac{\partial f}{\partial x} = \lim\limits_{\mathrm{d}x \to 0} \dfrac{f(x+\mathrm{d}x) - f(x)}{\mathrm{d}x}$ one can rewrite (2.21) for small $\mathrm{d}x$

$$F(x+\mathrm{d}x) = \left[f_x(x) + \frac{\partial f_x}{\partial x}\mathrm{d}x \right] \mathrm{d}y\,\mathrm{d}z \tag{2.22}$$

Now the mass loss or gain per time in the x-direction from x to $x + \mathrm{d}x$ is given by the difference of (2.22) and (2.20), i.e. $\dfrac{\partial f_x}{\partial x}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Similar arguments hold for the y- and z-direction, leading us to the definition of the divergence (in Cartesian coordinates) of a vector field $\boldsymbol{f}$

$$\operatorname{div} \boldsymbol{f} \;\equiv\; \nabla \cdot \boldsymbol{f} \;=\; \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}, \tag{2.23}$$

and $\operatorname{div}\boldsymbol{f}\,\mathrm{d}V$ with $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ describes the mass loss or gain per time of the volume $\mathrm{d}V$.

For a given volume $\mathcal{V}$ therefore $\int_{\mathcal{V}} \operatorname{div}\boldsymbol{f}\,\mathrm{d}V$ describes the mass loss or gain. As this loss or gain can only be through the surface $\mathcal{A}$ of the volume, it has to be equal to the surface integral of the component of the mass flux normal to the surface, i.e. we have

$$\text{Gauss' theorem:} \qquad \int_{\mathcal{V}} \operatorname{div}\boldsymbol{f}\,\mathrm{d}V = \int_{\mathcal{A}} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{a}, \tag{2.24}$$

where $\mathrm{d}\boldsymbol{a}$ is the surface element normal to the surface $\mathcal{A}$ pointing out of the enclosed volume $\mathcal{V}$.

Shrinking the volume $\mathcal{V}$ to zero leads us to a general definition of the divergence,

$$\operatorname{div}\boldsymbol{f} \;=\; \lim_{\mathcal{V} \to 0} \frac{1}{\mathcal{V}} \int_{\mathcal{A}} \boldsymbol{f} \cdot \mathrm{d}\boldsymbol{a}. \tag{2.25}$$

This nicely shows that a divergence free vector field $\nabla \cdot \boldsymbol{f} = 0$ means that the field $\boldsymbol{f}$ has no sources and sinks, which is well known e.g. for the magnetic field, $\nabla \cdot \boldsymbol{B} = 0$.

2.5.2 Curl, Stokes' theorem and vorticity

Think of a surface $\mathcal{A}$ that is enclosed by the curve $\mathcal{C}$. Then for a vector field $\boldsymbol{u}$ the integral over the surface and the closed line integral around the surface are connected by

$$\text{Stokes' theorem:} \qquad \int_{\mathcal{A}} (\nabla \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{a} \;=\; \int_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l}, \tag{2.26}$$

where $\mathrm{d}\boldsymbol{l}$ is the line element along $\mathcal{C}$. Thus similarly to above one can consider the curl as to be defined as the integral of the component of $\boldsymbol{u}$ along the curve when "going" around ("rotating" around) the surface $\mathcal{A} \to 0$. In this sense curl means the rotation locally, which must not be mixed up with the global rotation, which will be illustrated in the following.

In the case of the velocity $\boldsymbol{u}$, the curl is called

$$\text{vorticity:} \qquad \boldsymbol{\Omega} = \nabla \times \boldsymbol{u}. \tag{2.27}$$

For the interpretation let us assume a 2D flow field in the x-y-plane, i.e.

$$\boldsymbol{u} = u(x,y,t)\,\boldsymbol{e}_x \;+\; v(x,y,t)\,\boldsymbol{e}_y \qquad \longrightarrow \qquad \boldsymbol{\Omega} = \omega\,\boldsymbol{e}_z \quad ; \quad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{2.28}$$

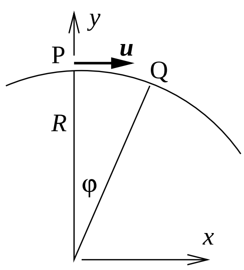
Now think of two perpendicular lines: AB in the x- and AC in the y-direction with (infinitesimal) lengths $\mathrm{d}x$ and $\mathrm{d}y$. The change of the y-component of the velocity along the x-direction, $\dfrac{v(x+\mathrm{d}x,y)-v(x,y)}{\mathrm{d}x} \equiv \dfrac{\partial v}{\partial x}$ corresponds to the angular velocity at B. Likewise $-\dfrac{\partial u}{\partial y}$ corresponds to the angular velocity at C. Thus the vorticity $\frac{1}{2}\omega$ represents the average angular velocity of two short line elements, which happen to be perpendicular (at one given instant). One has to emphasize that the vorticity is a measure of the *local* rotation of a fluid element, but it has nothing to do with the global rotation of the fluid.

For example a shear flow with $\boldsymbol{u} = (-\alpha\,y,\ 0,\ 0)$ has a constant vorticity $\Omega = (0,\ 0,\ \alpha)$, but is surely not rotating. One can also construct rotating flows which have zero vorticity, e.g. a cylinder flow in the azimuthal direction, $\boldsymbol{u} = \alpha/r\,\boldsymbol{e}_\varphi$.

2.6 Transverse pressure gradients —storms in a glass of water

If we stir the coffee in a cup, after removing the spoon we find the surface of the rotating coffee to be non-flat. What causes this effect and what shape does the surface have?

Let us assume the flow to be steady, i.e. $\partial\boldsymbol{u}/\partial t = 0$ or $\mathrm{D}\boldsymbol{u}/\mathrm{D}t = (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$. The flow is along a curved line in the horizontal plane. Consider the flow at a point P, where the (local) curvature radius is R and the flow is along the x-direction: at P we have $\boldsymbol{u} = (u_x, 0, 0)$, i.e. $u_y = u_z = 0$. Thus the longitudinal component of the acceleration at P is given by



$$\text{longitudinal acceleration} \qquad \left.\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right|_{\mathrm{P},x} = (\boldsymbol{u}\cdot\nabla)u_x = u_x \left.\frac{\partial u_x}{\partial x}\right|_{\mathrm{P}} = \frac{\partial\left(\frac{1}{2}u^2\right)}{\partial l},$$

where u is the magnitude of the flow (at P) and l the coordinate along a streamline.
Even though u_y is zero at P, in general the transverse component of the acceleration (i.e. along y) will not vanish,

$$\text{transverse acceleration} \qquad \left.\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right|_{\mathrm{P},y} = (\boldsymbol{u}\cdot\nabla)u_y = u_x \left.\frac{\partial u_y}{\partial x}\right|_{\mathrm{P}}. \tag{2.29}$$

We can now estimate the transverse acceleration by investigating the velocity at a location Q a bit downstream. ¿From the figure one readily finds

$$\left.u_y\right|_{\mathrm{Q}} = \left.\frac{\partial u_y}{\partial x}\right|_{\mathrm{P}} R\sin\varphi + \left.\frac{\partial u_y}{\partial y}\right|_{\mathrm{P}} R(1-\cos\varphi) \;\approx\; \left.\frac{\partial u_y}{\partial x}\right|_{\mathrm{P}} r\,\varphi, \tag{2.30}$$

for small angles φ, linear in φ. On the other hand $u_y\big|_Q$ is also simply given by $u_y\big|_Q = u\big|_Q \sin\varphi$. Assuming that the velocity does not change (much) we can replace $u\big|_Q = u$ and for small angles we finally find $u_y\big|_Q = u\,\varphi$. Equating this with (2.30) we find that

$$\left.\frac{\partial u_y}{\partial x}\right|_P = \left.\frac{u}{R}\right|_P .$$

As at P we have $u_x = u$, we can now write for (2.29)

$$\text{transverse acceleration} \qquad \left.\frac{D\boldsymbol{u}}{Dt}\right|_{P,y} = \left.\frac{u^2}{R}\right|_P . \tag{2.31}$$

This is corresponding to the centripetal acceleration of a particle moving in two dimensions along a trajectory with a radius of curvature R.

As the gravity does not play a role for the balance in the horizontal plane, we can write the transverse momentum balance following from the Euler equation (2.12) as

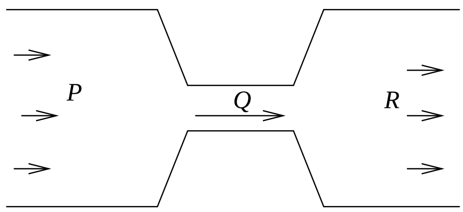
$$\frac{\partial p}{\partial r} = \frac{\rho\,u^2}{R} \tag{2.32}$$

In the case of the coffee in the rotating cup we can now estimate the transverse pressure gradient that is needed to support the surface to reach to larger heights when moving outwards. One finally finds the surface having the shape of a paraboloid.

2.7 A danger involved: Cavitation

Bernoulli's law as derived in Sect. 2.4 implies that when a fluid is forced through a narrow constriction, the pressure can become zero or even negative:

$$p_Q = p_P - \frac{1}{2}\,\rho\left(u_Q^2 - u_p^2\right) \tag{2.33}$$



Because of the conservation of the total mass flux, $A\rho u$, where A is the cross section of the tube, the flow speed in the constriction is given by $u_Q = u_P/C.$, with the contraction coefficient $C = A_Q/A_P$. Thus the pressure at Q is given by

$$p_Q = p_P - \frac{1}{2}\,\rho\,u_P^2\left(\frac{1}{C^2} - 1\right) \qquad \Rightarrow \qquad p_Q = 0 \ \text{ if:} \qquad C = \left(1 + \frac{p_p}{\frac{1}{2}\,\rho\,u_p^2}\right)^{-2} \tag{2.34}$$

As an example let us consider a 3/4 inch garden hose (diameter $d_P = 2cm$). Discharging water with a rate of about 9.4 liters per minute implies a velocity of the water of 0.5 m/s. When squeezing the hose with the fingers, the contraction coefficient has to be about $C = 1 : 28.5$ in order to have zero pressure in the constriction. This corresponds to a velocity $u_Q = 14$ m/s and a diameter of the constriction of about $d_Q = 4$ mm.

As we see typically $u_Q \gg u_P$ and therefore we can simplify (2.33) by neglecting u_P^2. Then we find

$$p_Q \approx p_P - \frac{1}{2}\,\rho\,u_Q^2 \tag{2.35}$$

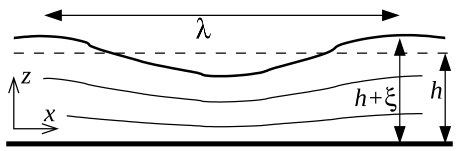
Thus for the "external" pressure given by the atmospheric pressure $p_P = p_A = 101\,300$ Pa, we find that velocities of about $u_Q = 10 \ldots 15$ m/s are sufficient in a flow of water to bring the local pressure p_Q down to zero!

For higher velocities, the pressure even becomes negative! What does this mean? A negative pressure would be absurd for a gas (but the above discussion is valid only for an incompressible fluid, so there is no problem here, as gases are not incompressible). For solids, however, a negative pressure simply means pulling at opposite ends of the solid — if the pressure becomes negative enough, the solid starts cracking and the piece is pulled apart. Sort of the same happens with an incompressible fluid when applying a negative pressure. At some point the fluid "cracks" and a bubble is formed filled with vapour — the liquid *cavitates*. These cavitation bubbles can grow either at the boundary to a solid surface of the flow or somewhere in the liquid. One example, where cavitation bubbles can occur are ship propellers and turbine blades.

What happens when the flow slows down again, i.e. after leaving the constriction? Of course the pressure rises again (and becomes positive) and the cavitation bubbles collapse. This collapse can reach considerable velocities and the bubbles vanish. This is the reason for a tap hissing when it is not quite closed, or when squeezing a hose.

In engineering cavitation is of quite some importance. When the bubbles collapse, the velocities involved can reach values well in excess of the flow speed. Thus when the bubble forms at the interface of the liquid to the solid, the collapse of the bubble can cause a localised impulse on the solid. This may lead to small cracks in the solids, which then might be sites where corrosion starts off, e.g. at the ship propellers or turbine blades. Therefore engineers try to avoid flow speeds of the water surrounding the turbine blades etc rising above 10 m/s. Then according to (2.35) the pressure will stay positive (see above) and no cavitation bubbles will form.

2.8 Shallow water waves



Let us a consider a wave with wavelength λ in water with an average depth h, where $h \ll \lambda$, i.e. the water is shallow. We do not consider any effects of boundary layers (i.e. h much larger than the boundary layer thickness) or surface tension. The latter assumption is valid if the wavelength is larger than several cm. The vertical displacement of the surface of the water due to the wave, ξ, is small in the sense that the amplitude, i.e. max($|\xi|$), is small compared to the water depth h.

The water is assumed to be "still", so only the waves are moving. We cange into a frame of reference moving along with the waves, so the waves appear steady, and the water flows as indicated by the flow lines in the figure, e.g. from left to right, generally along the x-direction.

While the flow is assumed to be constant in y, the wave manifests itself by the variation of the surface along the z-direction, the local height of the surface being $h + \xi$. In the frame of reference of the waves, the flow speed along the x direction will vary, as the cross section changes with the surface height. If $\overline{u}$ is the average speed and u_1 is the (small) deviation, the flow speed is given by $u = \overline{u} + u_1$. Mass conservation requires that

$$u\,(h + \xi) = (\overline{u} + u_1)\,(h + \xi) \;=\; \text{constant}$$

Assuming that the wave is weak, i.e. $\xi \ll h$ and $u_1 \ll \overline{u}$, we can linearise this equation and find

$$h\,u_1 + \;\overline{u}\,\xi = 0 \tag{2.36}$$

Applying Bernoulli's theorem (2.17) at the surface, where the pressure is the atmospheric pressure everywhere and the gravitational potential simply $\phi = g\,z$, we have

$$\frac{1}{2}\,u^2 + g\,(h + \xi) = \text{constant}$$

Again, linearising yields,

$$\overline{u}\,u_1 + g\,\xi = 0 \tag{2.37}$$

Combining (2.36) and (2.37) gives the average flow speed in the frame of reference of the waves

$$\overline{u} = \sqrt{g\,h}. \tag{2.38}$$

Now assuming a situation at the shore, where the water is more or less at rest, i.e. not drifting, we move from the frame of reference of the waves to the one of the fluid, and find that the speed of the waves is just given by the speed found in (2.38), but now in the other direction, of course

$$\text{speed of gravity shallow water waves} \qquad c = \sqrt{g\,h}. \tag{2.39}$$

These waves are called gravity waves, as the gravity is the force counteracting the growth of the waves. In water about 1 m deep, the wave speed is approximately 10 m/s.

When a wave from the ocean approaches the shore at an arbitrary angle, it becomes refracted, as the part of the wave front in deeper water is moving faster according to (2.39). Assuming that the depth of the water monotonically decreases towards the shore this means that the wave is refracted until the wave front is parallel to the shore. This is the reason why at the beach we see the waves mostly coming in parallel to the coast line.

Another consequence of (2.39) is that the speed of the gravity wave is independent of its wavelength. This implies that such waves are non-dispersive. This is true, however, only for very small amplitudes.

And yet another consequence is the braking of the waves. When the amplitude is no linger infinitesimal, the speed in the “upper” part of the wave is higher than in the “lower’ part. Thus the upper part overtakes the lower one and the wave brakes.

Chapter 3

Including viscosity: The Navier-Stokes equation

In this section we will include the effects of viscosity. This will lead to the Navier-Stokes equation for an incompressible fluid (3.9) as well as for the more general case of a compressible fluid (3.8), which is also valid e.g. for gases. The Navier-Stokes equation basically describes the momentum balance in a fluid and replaces the Euler equation (2.12). For the mass balance the investigations from Sect. 2.1 and 2.2 do still apply. It is only when including the effects of viscosity, a realistic description of a problem is possible.

3.1 Conservation of momentum

As a reminder we write the continuity equation (2.2) in index notation,

$$\partial_t \rho \;+\; \partial_k(\rho u_k) \;=\; 0. \tag{3.1}$$

Here ∂_t is short for the derivative with respect to time and ∂_k stands for the spatial derivative with respect to the k-component. In the following it will be more convenient to use this index notation. As usually, when using the index notation, one has to sum over repeated indices.

In analogy to the continuity equation (2.2), which relates the change of the mass density ρ to the divergence of the mass flux density $\rho\,\boldsymbol{u}$, one can write for the momentum $\rho\boldsymbol{u}$

$$\frac{\partial(\rho\boldsymbol{u})}{\partial t} \;+\; \nabla\cdot\mathsf{P} \;=\; 0 \qquad\Longleftrightarrow\qquad \frac{\partial(\rho u_i)}{\partial t} \;+\; \frac{\partial \mathit{\Pi}_{ik}}{\partial x_k} \;=\; 0. \tag{3.2}$$

Here P is the tensor of the momentum flux or stress tensor with its components $\mathit{\Pi}_{ik}$. These describe the flux of the i-momentum in the k-direction.

One can now split the momentum flux tensor $\mathit{\Pi}_{ik}$ into three parts: the advection of i-momentum in the k-direction through the mass flux, $\rho u_i\,u_k$; the isotropic contribution called pressure, $p\,\delta_{ik}$; the a non-isotropic part called viscous stresses, σ_{ik},

$$\mathit{\Pi}_{ik} \;=\; \rho u_i\,u_k \;+\; p\,\delta_{ik} \;+\; \sigma_{ik} \tag{3.3}$$

Substituting this into (3.2) we find

$$\rho\,\partial_t u_i \;+\; u_i\,\partial_t\rho \;+\; u_i\,\partial_k(\rho u_k) \;+\; \rho\,u_k\,\partial_k u_i \;+\; \partial_i p \;-\; \partial_k\sigma_{ik} \;=\; 0 \tag{3.4}$$

The sum of the second and third term vanishes because of the continuity equation (3.1) and we get

$$\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right)_i \;\equiv\; \partial_t u_i \;+\; u_k\,\partial_k u_i \;=\; -\,\frac{1}{\rho}\,\partial_i p \;+\; \frac{1}{\rho}\,\partial_k\sigma_{ik}, \tag{3.5}$$

where on the left hand side we have the time derivative in the co-moving frame of the velocity, i.e. the acceleration.

Now the final question concerns the form of the viscous stress tensor σ_{ik}. Following Newton's hypothesis, the viscous stresses are linear in the derivatives of the velocity ("shear"), as was already discussed in Sect. 1.3.

Furthermore one can assume that the action of the viscosity to be symmetric, i.e. $\sigma_{ik} = \sigma_{ki}$ This leads to the form of $\sigma_{ik} = \mu\,\widetilde{S}_{ik}$ with

$$\widetilde{S}_{ik} \;=\; \partial_i u_k + \partial_k u_i \tag{3.6}$$

Now the isotropic part of this tensor is the main diagonal, i.e. the average of the sum of the diagonal elements $\frac{1}{3}\mathrm{Trace}\big(\widetilde{\mathbf{S}}\big) = \frac{1}{3}\sum_i \widetilde{S}_{ii}$. One finds that $\frac{1}{3}\mathrm{Trace}\big(\widetilde{\mathbf{S}}\big) = \frac{2}{3}\nabla\cdot\boldsymbol{u}$. However, as we have defined the viscous stresses to have no isotropic part, as the isotropic part of the stress tensor is the pressure as introduced in (3.3), we have to correct $\widetilde{S}_{ik}$ defined in (3.6) with respect to the isotropic part, and subtract $\frac{2}{3}\nabla\cdot\boldsymbol{u}$.

Then the isotropic-free (i.e. traceless) and symmetric viscous stress tensor is given by

$$\sigma_{ik} \;=\; \mu\,\left(\frac{\partial u_k}{\partial x_i} \;+\; \frac{\partial u_i}{\partial x_k} \;-\; \delta_{ik}\,\frac{2}{3}\,\nabla\cdot\boldsymbol{u}\right). \tag{3.7}$$

Please note that in the case on an incompressible medium, i.e. $\nabla\cdot\boldsymbol{u} = 0$, already $\widetilde{S}_{ik}$ is traceless and the viscous stress tensor simplifies to $\sigma_{ik} = \mu\,(\partial_i u_k + \partial_k u_i)$.

As $(\nabla\cdot u) = \partial_l\,u_l$, we can write for the divergence of the viscous stress tensor when assuming a constant viscosity μ

$$\partial_k\sigma_{ik} = \mu\Big(\partial_i\,\partial_k\,u_k \;+\; \partial_k^2\,u_i \;-\; \frac{2}{3}\,\partial_i\,\partial_l\,u_l\Big) \;=\; \mu\Big(\varDelta\,u_i \;+\; \frac{1}{3}\,\partial_i\,(\nabla\cdot u)\Big).$$

Now finally we can write the equation of motion following (3.5) as

Navier-Stokes equation (compressible)
$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right) \;\equiv\; \rho\,\frac{\partial\boldsymbol{u}}{\partial t} + \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} \;=\; -\nabla p + \mu\Big(\varDelta\boldsymbol{u} + \frac{1}{3}\,\nabla\,(\nabla\cdot\boldsymbol{u})\Big) + \boldsymbol{f}_{\mathrm{ext}}, \tag{3.8}$$

where we now added external forces like gravity, e.g. $\boldsymbol{f}_{\mathrm{ext}} = \rho\,\boldsymbol{g}$. This equation is the *Navier-Stokes equation* for a compressible fluid.

In the case of an incompressible fluid, i.e. $\nabla\cdot\boldsymbol{u}$=0, using the vector identity $\varDelta\boldsymbol{u} = \nabla\,(\nabla\cdot\boldsymbol{u}) - \nabla\times(\nabla\times\boldsymbol{u})$ this equation simplifies to

Navier-Stokes equation (incompressible)
$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\right) \;\equiv\; \rho\,\frac{\partial\boldsymbol{u}}{\partial t} \;+\; \rho\,(\boldsymbol{u}\cdot\nabla)\,\boldsymbol{u} \;=\; -\,\nabla p \;-\; \mu\,\nabla\times(\nabla\times\boldsymbol{u}) \;+\; \boldsymbol{f}_{\mathrm{ext}}, \tag{3.9}$$

Thus in an incompressible fluid the effects of the viscous stresses are given by the curl of the vorticity $\boldsymbol{\Omega} = \nabla\times\boldsymbol{u}$ introduced in Sect. 2.5.2. This shows that the concept of vorticity is not only important when describing a non-viscous flow, but also when investigating the effects of viscosity.

Together with the continuity equation, either in its form of (2.2) or (2.7) the Navier-Stokes equation is describing the evolution of a fluid when taking viscosity into account. Of course, for the full description of a problem also a suitable set of initial and boundary conditions is needed.

It is obvious from (3.9) that the viscosity term cancels if the flow is irrotational, i.e., if vorticity of the flow vanishes: $\boldsymbol{\nabla}\times\boldsymbol{u} = 0$. Then the equation reduces to the Euler equation of motion.

3.2 No-slip boundary condition

The solution of the Navier-Stokes equation depends critically on the boundary condition. From experimental research one finds that the no-slip boundary conditions have to be applied for viscous fluids:

$$\boldsymbol{u}|_{\mathrm{boundary}} = \boldsymbol{u}_{\mathrm{boundary}}\;. \tag{3.10}$$

From most applications the problem is most conveniently solved putting the boundaries at rest, $\boldsymbol{u}_{\mathrm{boundary}} = 0$. Then the flow velocity has to vanish at the boundary.

This boundary conditions has important consequences on the flow. In particular, they imply that any flow has to decelerate towards the boundaries. The layer of deceleration is known as the boundary layer. In this boundary layer the viscosity term in the Navier-Stokes-equation $\nu\triangle\boldsymbol{u}$ can NOT be neglected even for high Reynolds numbers (low viscosity) since $\triangle\boldsymbol{u}$ is very large. In a way, it can be said that the existence of the boundary layer (which results from the no-slip boundary condition) manifests the essential difference between viscous and non-viscous fluids. In the limit of $\nu\longrightarrow 0$ the Navier-Stokes equation with no-slip boundaries does therefore not lead to the same result as the Euler equation with slippery boundaries.

In Sect. 4 we will see that the thickness, δ, of the boundary layer is related to the typical length scale, L, and the Reynolds number by $\delta^2/L^2 \sim \mathrm{Re}$.

3.3 Incompressible approximation

The assumption that the fluid is incompressible, i.e. that $\mathrm{D}\rho/\mathrm{D}t = 0$ or equivalently $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, facilitates the solution of a problem substantially. The obious question then is, when is this approximation valid? You may think the approximation to be valid for water, but do you expect it to be valid for air? Under what conditions? This section aims to give an answer on this[1].

Later in the lecture we will derive that the adiabatic speed of sound is given by

$$c_\mathrm{s} = \sqrt{\gamma \frac{p}{\rho}} \quad . \tag{3.11}$$

p and ρ denote the (average) gas pressure and density, respectively. γ is the ratio of specific heats, $\gamma = 1.4$ for air. Rewriting the upper equation, we find for the gas pressure, $p = \rho c_\mathrm{s}^2/\gamma \sim \rho c_\mathrm{s}^2$. By inspection of the equation of motion, however, it is evident (!) (we come back to this) that the pressure fluctuations $\triangle p$ within the gas associated with the fluid motion are of order ρU^2, where U is a typical flow velocity. Provided therefore that $U^2 \ll c_\mathrm{s}^2$ the fractional change in pressure,

$$\frac{\triangle p}{p} \sim \frac{\rho U^2}{\rho c_\mathrm{s}^2} \ ,$$

wrought by the fluid motion will be small, and will result in little expansion or compression of fluid elements. The latter implication from small pressure changes to small density changes can be drawn from assuming that the gas is isothermal. Then taking the total differential of the equation of state[2] $p = (\mathcal{R}/\mu)\rho\mathcal{T}$,

one obtains:

$$\frac{\triangle\rho}{\rho} = \frac{\triangle p}{p} \sim \frac{U^2}{c_\mathrm{s}^2} = \mathrm{Ma}^2 \ ,$$

i.e. if $\triangle p/p$ is small then $\triangle\rho/\rho$ is also small. Hence the answer to our question is: Low Mach number flows can savely be approximated as being incompressible. The condition for incompressibility is given by $\mathrm{Ma}^2 = U^2/c_\mathrm{s}^2 \ll 1$.

Now we come back to the exclamation mark. We consider a stationary flow: $\partial\boldsymbol{u}/\partial t = 0$. Outside of boundary layers, the viscosity term can be neglected, and from the advection and pressure gradient term we can estimate:

$$\triangle p \sim \rho U^2 \quad \text{q.e.d.}. \tag{3.12}$$

For very low Reynolds number, the boundary layer is very thick, i.e. it might be that there is no 'outside of boundary layers'. Then the treatment has to be done differently and one finds that the assumption of incompressibility is justified as long as[3]: $\mathrm{Ma}^2 = U^2/c_\mathrm{s}^2 \ll \mathrm{Re}$.

The sound velocity in air is $c_\mathrm{s} \sim 300\,\mathrm{m/s} \sim 1000\,\mathrm{km/h}$. Therefore, a very very strong wind at 30 m/s (you will have problems to stand upright in such a wind) has a Mach number of 0.1 and $\mathrm{Ma}^2 = 10^{-2} \ll 1$ and the corresponding air flow can be considered incompressible!

3.3.1 The kinetic energy of low Mach number flows

To gain some insight in low Mach number flow we want to compare its kinetic energy, E_kin to its internal (thermal) energy E_thermal[4]. The trick in the following consists in expressing the internal energy (which is given by $3/2p$) in terms of the sound speed, using (3.11).

$$E_\mathrm{thermal} = \frac{3}{2}p = \frac{3}{2\gamma}\rho c_\mathrm{s}^2 \sim \rho c_\mathrm{s}^2 \gg \rho U^2 \sim E_\mathrm{kin} \tag{3.13}$$

Hence, for a low Mach number flow the kinetic energy is small compared with the thermal energy. Even if the kinetic energy is fully dissipated the thermodynamic state does only change a tiny little bit. For example the water temperature in a water fall does not change, although kinetic energy is dissipated.

In particular it follows from the above that the flow velocity at low Mach number is small compared with the typical Brownian velocity of a molecule. In that respect, a low Mach number flow can be considered as a small fluctuation of the thermodynamic system at rest. Don't explain the latter remark to somebody who is standing in a wind of 30 m/s ($\mathrm{Ma} \approx 0.1$) having a hard time trying not to be blown away!

[1] The next paragraph is taken from Acheson, Sect. 3.1.

[2] μ denotes the mean melocular mass per mol. This form converts to $pV = NkT$ using: $N_\mathrm{A}k = \mathcal{R}$, $n = N/V$, $N = \nu N_\mathrm{A}$, with ν: number of mols, $\rho = \nu N_\mathrm{A} m/V$.

[3] cf. Tritton, 2nd ed., sect. 5.8.

[4] cf. Tritton: end of Sect. 5.8

3.4 Transformation to dimensionless variables: Reynolds number

In order to learn about the relative importance of the various terms in the equation of motion, it is of advantage to estimate a typical velocitiy, U, a typical length scale, L, and intermediate density and pressure values, ρ_0 and p_0, of the problem. This implies a typical time scale of $t_0 = L/U$. Then all variables in the equation motion are expressed in units of these typical values.

$$u = u_0\tilde{u}; \quad \rho = \rho_0\tilde{\rho}; \quad p = p_0\tilde{p}; \quad x = L\tilde{x}; \quad t = t_0\tilde{t}; \tag{3.14}$$

The variables with the tilde ($\tilde{}$) are dimensionles and should be of order unity. This is assuming that the problem allows to define one typical scale. Sometimes a typical scale does not exist and then the dimensionless variable is not of order unity everywhere in the flow.

Plugging the upper transformation into the equation of motion and omitting the tildes again, we obtain:

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\mathrm{Ma}^2}\boldsymbol{\nabla}p + \frac{1}{\mathrm{Re}}\triangle\boldsymbol{u}\,, \qquad \text{with } \mathrm{Ma} \sim \frac{p_0}{\rho_0 U^2} \quad \text{and} \quad \mathrm{Re} = \frac{UL}{\nu}. \tag{3.15}$$

The ratio between the advection and viscosity terms is expressed by the Reynolds number. Whether advection or viscosity term dominate the pressure gradient term or not, cannot be decided from the equation of motion alone, since the pressure and the pressure gradient are determined by equation of state and the equation.

3.4.1 High Reynolds number

For a high Reynolds number flow the viscosity term is negligible relative to the advection term and can therefore be neglected everywhere in the flow but not in the boundary layer. In the boundary layer $\triangle\boldsymbol{u}$ becomes large as a consequence of the no-slip boundary condition. The thickness of the boundary layer, δ, is related to the Reynolds number by $\delta/L \sim \sqrt{\mathrm{Re}}$ (cf. next section). Outside the boundary layer the viscosity term can be neglected. Complications occur for very high Reynolds numbers when stationary flows become instable and create turbulence. This will be dealt with in a later chapter.

Just to give some values: ν(air) = 1.5×10^{-5} m^2/s, and ν(water) = 1×10^{-6} m^2/s. That means that for both media the Reynolds numbers are very high if the typical length scales exceed some 10 cm, and the typical velocity exceeds 10 cm/s. Then Re $\sim 10^4$.

3.4.2 Low Reynolds number

Experiment: 2 Cylinders. Very viscous fluid inbetween, with a drop of ink somewhere in the middle. The outer one at rest, the inner one rotates slowly a few revolutions. The drop of ink is distributed. If the inner cylinder is rotated backwards the same amount of revolutions, the drop of ink is almost as concentrated as before, and at the same position. That means the flow is well ordered (no loss of entropy).

⇒ A flow at low Reynolds number is nearly reversible.

The reversibility of the “creeping motion” at low Reynolds number can be proven. The viscous fluid may extend over some region V, which is bounded by a closed surface S. Let the boundary condition on S be given by $\boldsymbol{u} = \boldsymbol{u}_\mathrm{B}(\boldsymbol{x})$. The slow flow equations are linear:

$$0 = -\boldsymbol{\nabla}p + \mu\triangle\boldsymbol{u} \quad \text{and} \quad \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \tag{3.16}$$

with ρ entering in $\mu = \nu/\rho$. It can be shown (cf., excercise) that there is at most one solution for a given boundary condition.

Assume a solution $\boldsymbol{u}(\boldsymbol{x})$ with the boundary condition $\boldsymbol{u}_\mathrm{B} = \boldsymbol{f}(\boldsymbol{x})$ is given ($\boldsymbol{x}$ on the closed surface S). Suppose that the boundary conditions are changed to $\boldsymbol{u}_\mathrm{B} = -\boldsymbol{f}(\boldsymbol{x})$. It is obvious from inspection of the slow flow equations that $-\boldsymbol{u}(\boldsymbol{x})$ constitutes *a* solution to the reversed problem. As we have mentioned above, this solution is the only solution, i.e., it is unique. We conclude that solutions to the slow flow equations obey reversibility, if the boundary conditions are reversed.

Examples: Creeping flow past a sphere; Corner eddies; Swimming at low Reynolds number; Swimming of a thin flexible sheet; Flow in a thin film; Flow in Hele-Shaw cell; Thin film flow down a slope;

3.4.3 Swimming at low Reynolds number

The trick: Do something that is not time-reversible, like swimming of a thin flexible sheet.

3.5 Some solutions

¿From the time independent Navier-Stokes equation of motion: $\rho(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla}p + \mu\triangle\boldsymbol{u}$, the following problem solution are addressed. The formulation in cartiesian coordinates is straight forward.

3.5.1 Poiseuille fbw

This problem was already dealt with in chapter 1 and the fist excercise sheet. There the governing equation were derived somewhat loosly. Now with the Navier-Stokes equation at hand, the equation of motions can readily be written down.

For the case of a flow between two parallel plates (without gravity), the Laplace operator is most conveniently formulated in cartesian coordinates. The flow streams in x-direction, and the plates extend indefinitly in z-direction. The y-axis is perpendicular to the plates. Assume that the flow is stationary $\partial\boldsymbol{u}/\partial t = 0$. The flow is unidirectional, $u_y = u_z = 0$. Since the distance between the plates is constant, $\partial u_x/\partial x = 0$.

The Navier Stokes equation in cartesian coordinates reduces to, denoting $u = u_x$:

$$\frac{\mathrm{d}p}{\mathrm{d}x} = \mu\triangle u \quad .$$

Expressing the Laplace operator, one obtains:

$$\mu\frac{\mathrm{d}^2u}{\mathrm{d}y^2} = G \qquad \text{with} \qquad G = \frac{\mathrm{d}p}{\mathrm{d}x}$$

$$\Rightarrow u = \frac{G}{2\mu}(a^2 - y^2)\ , \qquad \phi = \frac{2Ga^3}{3\mu}$$

with ϕ denoting the mass flow per unit length and time.

For the flow in the tube, cylindrical coordinates are appropriate to express the Laplace operator. Using $\boldsymbol{u}_\phi = \boldsymbol{u}_r = 0$, $\partial/\partial\phi\ldots = \partial/\partial z\ldots = 0$, and $u = u_z$, the Navier-Stokes equation reduces to its z-component, which itself simplifies to:

$$\mu\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = -r\frac{\mathrm{d}p}{\mathrm{d}z}$$

Now we define as $G = \mathrm{d}p/\mathrm{d}z$.

$$\Rightarrow u = \frac{G}{4\mu}(a^2 - r^2)\ , \qquad \Phi = \frac{\pi\rho Ga^4}{8\mu}$$

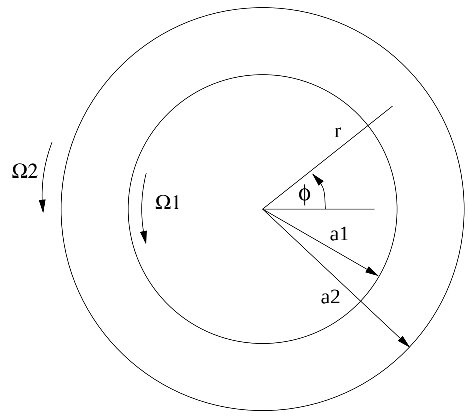
Entry length:

When the Reynolds number is less than about 30, the upper results always apply. At higher Reynolds number the results only apply after some distance down the pipe. This *entry length* x, is experimentally found to depend on the distance between the two plates (diameter of the tube), d, and the Reynolds number, Re:

$$\frac{x}{d} \sim \frac{\mathrm{Re}}{30}$$

3.5.2 Rotating Couette fbw or: How can the viscosity be measured?

We consider the rotating Couette flow at low Reynold numbers. Fluid is contained in the annulus between two long concentric cylinders of radii a_1 and a_2 rotating about their common axis with angular velocities Ω_1 and Ω_2. For now we assume that the velocity has only an azimuthal component: $u_r = u_z = 0$, and the azimuthal velocity only depends on the radial distance: $\partial/\partial z = \partial/\partial\phi = 0$. The latter will change for high Reynolds numbers when the flow becomes instable and turbulent. In the case we consider here, the continuity equation reduces to $\frac{\partial u_\phi}{\partial\phi} = 0$, which is already garanteed by the above assumptions. The azimuthal and radial component of the Navier-Stokes equation become:



$$0 = \mu\left(\frac{\mathrm{d}^2u_\phi}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u_\phi}{\mathrm{d}r} - \frac{u_\phi}{r^2}\right) \quad \text{and} \quad -\frac{\rho u_\phi^2}{r} = \frac{\mathrm{d}p}{\mathrm{d}r}$$

[5] with the boundary conditions:

$$u_\phi = \Omega_1 a_1 \quad \text{at} \quad r = a_1 \qquad \text{and} \qquad u_\phi = \Omega_2 a_2 \quad \text{at} \quad r = a_2 \;.$$

The first equation decouples from the second, i.e., u_ϕ is determined by visocous stresses only. The second equation determines the pressure distribution given by the balance between the pressure gradient and the centrifugal force $\rho u_\phi^2/r$ exerted by the circular motion.

Solution:

$$u_\phi = Ar + B/r \quad \text{with} \quad A = \frac{\Omega_2 a_1^2 - \Omega_1 a_1^2}{a_2^2 - a_1^2} \quad \text{and} \quad B = \frac{(\Omega_1 - \Omega_2)a_1^2 a_2^2}{a_2^2 - a_1^2}$$

The torque, Σ_1, acting on the inner cylinder (per unit length in the z-direction) is given by the viscous stress multiplied by the circumstance and the radius:

$$\Sigma_1 = \left[\mu r\frac{\partial(u_\phi/r)}{\partial r}\right]_{r=a_1} \cdot\ 2\pi a_1 \ \cdot\ a_1 = 4\pi\mu a_1^2 a_2^2\frac{\Omega_2 - \Omega_1}{a_2^2 - a_1^2}.$$

For the torque on the outer cylinder one obtains $\Sigma_2 = -\Sigma_1$!

[5]Deriving the Navier-Stokes equation in cylindrical coordinates, (r, ϕ, z), one has to consider that the nabla operators create extra terms as a consequence of

$$\frac{\partial\hat{\boldsymbol{e}}_r}{\partial\phi} = \hat{\boldsymbol{e}}_\phi, \qquad \frac{\partial\hat{\boldsymbol{e}}_\phi}{\partial\phi} = -\hat{\boldsymbol{e}}_r, \qquad \frac{\partial\hat{\boldsymbol{e}}_z}{\partial\phi} = 0\;,$$

which lead to the terms: u_ϕ/r^2 and u_ϕ^2/r.

Chapter 4

Boundary layer

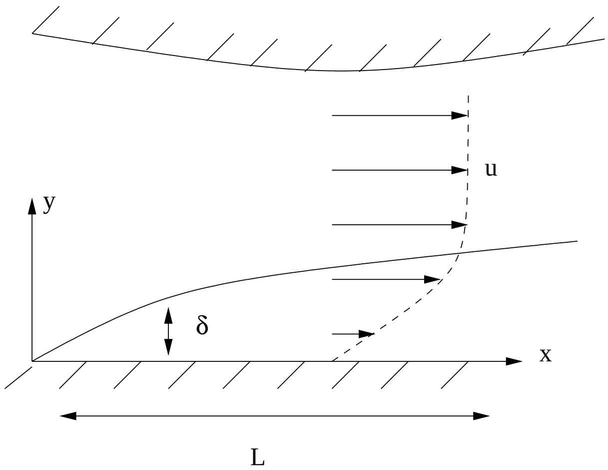
Outside the boundary layer: $\nabla \times \boldsymbol{u} = 0$. Vorticity is generated through viscosity in boundary layers! Therefore one might expect that the thickness, δ, depends on the Reynolds number of the flow. This will be addressed in Sect. 4.2. First we will derive the simplified equation of motion which governs the boundary layer.

(Wakes and jets as phenomena in a flow: Vortices that form in boundary layers are advected into the bulk of the flow. We will talk about this later in the lecture.)

4.1 Steady 2-D boundary layer equations

We derive the equations that govern a 2D steady boundary layer, adjacent to a rigid wall at $y = 0$. In boundary layer theory one assumes that u_x and u_y change much more rapidly with y than with x, meaning that

$$\left|\frac{\partial u_x}{\partial y}\right| \gg \left|\frac{\partial u_x}{\partial x}\right| \ .$$



Expressing the latter equation in length and velocity scales, it is found that $U/\delta \gg U/L$, and hence: $\delta \ll L$. Continuity yields:

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0 \tag{4.1}$$

The scale analysis of the continuity equation yields: $V \sim \delta U/L$, with U being the typical velocity in x-direction, and V being the typical velocity in y-direction, i.e., $V \ll U$. In the equation of motion the scale of the pressure difference in x-direction is denoted by Π and in y-direction by Λ. The x-component of the equation of motion reads, with the scale of the terms beneath:

$$\begin{array}{ccccc} u_x \dfrac{\partial u_x}{\partial x} & + u_y \dfrac{\partial u_x}{\partial y} = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & + \nu \dfrac{\partial^2 u_x}{\partial x^2} & + \nu \dfrac{\partial^2 u_x}{\partial y^2} \\ \sim \dfrac{U^2}{L} & \sim \dfrac{U^2}{L} & \sim \dfrac{\Pi}{\rho L} & \sim \nu \dfrac{U}{L^2} & \sim \nu \dfrac{U}{\delta^2} \end{array} \tag{4.2}$$

The y-component of the equation of motion read:

$$\begin{array}{ccccc} u_x \dfrac{\partial u_y}{\partial x} & + u_y \dfrac{\partial u_y}{\partial y} = & -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} & + \nu \dfrac{\partial^2 u_y}{\partial x^2} & + \nu \dfrac{\partial^2 u_y}{\partial y^2} \\ \sim \dfrac{U^2 \delta}{L^2} & \sim \dfrac{\delta U^2}{L^2} & \sim \dfrac{\Lambda}{\rho L} & \sim \nu \dfrac{U\delta}{L^3} & \sim \nu \dfrac{U}{L\delta} \end{array} \tag{4.3}$$

Viewing the last two equation as equations for the pressure gradients, it is seen that $L\Lambda \sim \delta\Pi$, i.e., $\Lambda \ll \Pi$! This justifies that the y-component of the equation of motion for boundary layers can be neglected and that $\partial p/\partial x$ can be written as $\mathrm{d}p/\mathrm{d}x$. Even more dramamtic, (4.2) can be further simplified: Since

$$\frac{\partial^2 u_x}{\partial x^2} \ll \frac{\partial^2 u_x}{\partial y^2} \ ,$$

the last term can be neglected, and the equation of motion for a steady 2-D boundary layer is given by

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \nu \frac{\partial^2 u_x}{\partial y^2} \tag{4.4}$$

4.2 How thick are boundary layers?

The key idea of boundary layer theory is that the rapid variations of u_x with y should be just sufficient to prevent the viscous term from being negligible, notwithstanding the small coefficient of viscosity, ν. This means that the advection terms are of the same order as the viscosity term, i.e., $U^2/L \sim \nu U/\delta^2$. This implies:

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{\mathrm{Re}}} \tag{4.5}$$

An alternative (of course in principle equivalent) way in estimating δ is by stating that outside the boundary, the viscosity term and $u_y \partial u_x/\partial y$ are negligible, i.e.,

$$u_x \frac{\partial u_x}{\partial x} \sim -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x}$$

The gas pressure gradient is also present in the boundary layer, and needs to be balanced there also by $u_x \partial u_x/\partial x$. Hence these two terms neutralize, and the other two terms also need to be of the same order:

$$u_y \frac{\partial u_x}{\partial y} \sim \nu \frac{\partial^2 u_x}{\partial y^2}$$

The latter estimate is equivalent to $\delta/L \sim 1/\sqrt{\mathrm{Re}}$. Hence, *a-posteriori* the initial assumption which has led to $\delta \ll L$ proves to be valid for sufficiently high Reynolds numbers.

4.3 Flow due to an impulsively moved plane boundary

Suppose 2D flow: $0 < y < \infty$, rigid boundary at $y = 0$. The rigid boundary is initially, $t = 0$, at rest and suddenly moved with velocity U in x direction. What velocity profile results? The no-slip boundary condition will effect the flow. The question you may want to ask is how and what rate does the moving boundary affect the fluid. One expects that the no-slip boundary will initially drive the fluid close to the boundary, and that the fluid starts moving further away from the boundary as time proceeds.

Since $u_y = 0$, we denote $u = u_x$, and note $\boldsymbol{\nabla} p = 0$, and $\partial u/\partial x = 0$. The Navier-Stokes equation for $u = u(y,t)$ becomes:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

The initial condition: $u(y, 0) = 0$. The boundary conditions: $u(0,t) = U$ and $u(\infty, t) = 0$[1].

The equation is unchanged under the transformation: $y \to \alpha y$ and $t \to \alpha^2 t$. This suggests that there are solutions which are functions of y and t simply through the single combination $y/\sqrt{\nu t}$ (ν is added to make it dimensionless). Thus we try

$$u = f(\eta), \quad \text{where} \quad \eta = \frac{y}{\sqrt{\nu t}} \ .$$

$$\left(\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -f'(\eta)\frac{y}{2\nu^{1/2}t^{3/2}} = -\frac{1}{2}f'(\eta)\frac{\eta}{t} \ , \qquad \frac{\partial u}{\partial y} = f'(\eta)\frac{\eta}{y} \ , \qquad \frac{\partial^2 u}{\partial y^2} = f''(\eta)\frac{1}{\nu t} \right)$$

[1]The problem is in fact identical with the problem of the spreading of heat through a thermally conducting solid when its boundary temperature is suddenly raised from zero to some constant.

For (4.3) we then obtain

$$f'' + \frac{1}{2}\eta f' = 0 \quad \Rightarrow \quad f''(\eta) = -\frac{1}{2}\eta f'(\eta) \quad \Rightarrow \quad f' = B\exp(-\eta^2/4)$$

Integrating once more, one obtains:

$$f(\eta) = A + B\int_0^{\eta} \exp(-s^2/4)\,\mathrm{d}s$$

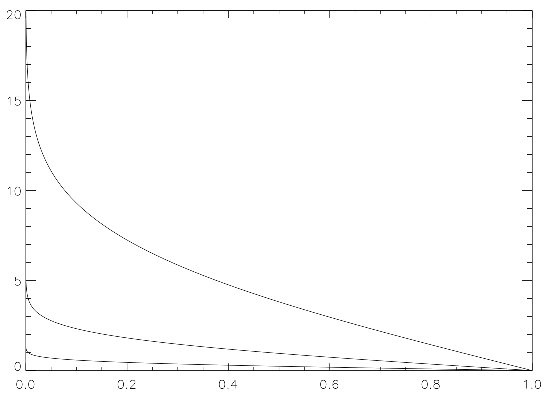
A and B are constants of intergration which are to be determined from the initial and boundary conditions, which reduce with $\eta = y/\sqrt{\nu t}$ to:

$$f(\infty) = 0\,, \quad f(0) = U\ .$$

Then, using that $\int_0^{\infty} \exp(x^2/a^2)\,\mathrm{d}x = a/2\sqrt{\pi}$, the solution to the problem (4.3) with $\eta = y/\sqrt{\nu t}$ is:

$$u = U\left[1 - \frac{1}{\sqrt{\pi}}\int_0^{\eta} \exp(-s^2/4)\,\mathrm{d}s\right] \quad .$$

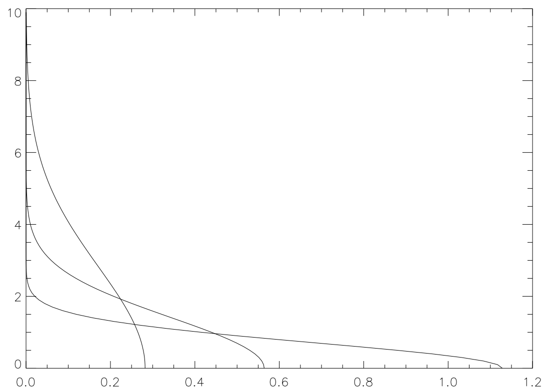
This solution is self-similar for y and t: At t_1, $u(\eta)$ is a function of $y/\sqrt{\nu t_1}$. At time t_2, $u(\eta)$ is the same (!) function of $y/\sqrt{\nu t_2}$.



Plotted: y versus u
Paramters: U = 1 m/s, ν = 1 cm^2/s
Time steps: $t = 1/4$ s $t = 1$ s and $t = 4$ s.

$u = 0.01U$ for $y \sim 4\sqrt{\nu t}$

4.3.1 Vorticity



$$\omega(y,t) = \boldsymbol{\nabla} \times \boldsymbol{u} = -\frac{\partial u}{\partial y} = \frac{U}{\sqrt{\pi\nu t}}\exp(-y^2/4\nu t)$$

Non-zero vorticity only in boundary layer. Vorticity spreads out in time, from the boundary layer into the fluid.

Plotted to the left: y versus ω.
Parameters: U = 1 m/s, ν = 1 cm^2/s
Time steps: $t = 1/4$ s $t = 1$ s and $t = 4$ s.

Some statements:

- Vorticity diffuses a distance of order $\sqrt{\nu t}$ in time t.
- If the typical length scale is given by L, the corresponding time scale is given by the diffusion time scale, τ_{diff}:

$$L \sim \sqrt{\nu\tau_{\mathrm{diff}}}\ , \quad \text{or} \quad \tau_{\mathrm{diff}} \sim \frac{L^2}{\nu}$$

4.3.2 Boundary layer and vorticity

Vorticity is generated in the boundary layer through the effects of viscosity! Consider the curl of the Navier-Stokes equation:

$$\boldsymbol{\nabla} \times \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \triangle \boldsymbol{u} \right] \quad ,$$

with the vector identity:

$$\boldsymbol{\nabla} \times (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{u} \cdot \boldsymbol{\nabla}(\boldsymbol{\nabla} \times \boldsymbol{u}) - [(\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \boldsymbol{\nabla}]\boldsymbol{u} + \boldsymbol{\nabla} \cdot \boldsymbol{u} \boldsymbol{\nabla} \times \boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u}$$

with $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$. Since $\boldsymbol{\nabla} \times \boldsymbol{\nabla} p = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, we obtain:

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \nu \triangle \boldsymbol{\omega}$$

This equation descrives the vorticity change of a fluid particle. The second term $\nu \triangle \boldsymbol{\omega}$ means that viscosity produces vorticity down a vorticity gradient. The first term describes the action of velocity variation on vorticity: vorticity twisting and stretching in 3D. In 2D the vorticity is perpendicular to the motion, $\boldsymbol{\omega} = (0, 0, \omega_z)$, and the first term reduces to $\omega_z \partial \boldsymbol{u}/\partial z$. Since $\boldsymbol{u}$ does not change with z, this terms vanishes in 2D. We conclude for inviscid fluids: *In the absence of the viscosity term, the vorticity of a 2D flow is constant,* $\mathrm{D}\boldsymbol{\omega}/\mathrm{D}t = 0$.

Stokes theorem:

$$\oint_C \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S.$$

$[\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \boldsymbol{u} \equiv 0; \boldsymbol{\nabla} \times \boldsymbol{\nabla} \phi \equiv 0;]$

4.4 Rotating fbws controlled by the boundary layers: Ekman layer

Consider a flow between $z = 0$ and $z = L$. The lower boundary rotates with Ω and the upper with $\Omega(1 + \epsilon)$, $\epsilon \ll 1$. The Reynolds number of a rotating flow is given by

$$\mathrm{Re} = \frac{\Omega L^2}{\nu} \; .$$

If Re is large we expect a thin viscous layer on both boundaries and an essentially inviscid 'interior' flow. How does the interior flow look like?

Since the fluid is rotating almost at uniform angular velocity Ω, it is approporiate to formulate the Navier-Stokes equation in a co-rotating frame of reference (with Ω):

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu \triangle \boldsymbol{u} \; , \qquad \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \; .$$

Here, $\boldsymbol{u}$ denotes the velocity relative to the rotating frame and the new terms come are the inertia forces: The third term is due to the Coriolis force and the fourth term is due to the centrifugal force. The centrifugal term, $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) = -\boldsymbol{\nabla}[\frac{1}{2}(\boldsymbol{\Omega} \times \boldsymbol{x})^2]$ can formally be cleared away to the pressure term:

$$p_R = p - \frac{1}{2}\rho(\boldsymbol{\Omega} \times \boldsymbol{x})^2 \; .$$

Now, we are interested in relative flow velocities which are much smaller than the rotation velocity:

$$U \ll \Omega L \quad \Rightarrow |(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}| \sim \frac{U^2}{L} \ll \Omega U \sim |2\boldsymbol{\Omega} \times \boldsymbol{u}|$$

Hence, in this approximation, known as *geostrophic approximation*, the advection term can be neglected in comparison with the Coriolis term. Many applications of this approximation are found in oceanography and meterology: The rotation velocity of the surface of the earth at the equator is $\Omega L = 40\,000\,\mathrm{km} / 24\,\mathrm{h} \sim 300\,\mathrm{m/s}$. Then a wind speed of $U = 10\,\mathrm{m/s}$ is still much smaller than the rotation of system.

Steady inviscid interior: For the steady inviscid flow, i.e. what we expect for the interior, we use cartesian coordinates (x, y, z), whith the rotation axis in z-direction: $\boldsymbol{\Omega} = (0, 0, \Omega)$. Then, with the velocity (u,v,w):

$$-2\Omega v = \frac{1}{\rho}\frac{\partial p_R}{\partial x}, \qquad 2\Omega u = -\frac{1}{\rho}\frac{\partial p_R}{\partial y}, \qquad 0 = -\frac{1}{\rho}\frac{\partial p_R}{\partial z}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

¿From these equations, assuming that ρ is constant, it follows that: (1) p_R is independent of z (third eq.). From the first two then it is obvious that (2) u and v are also independent of z. Moreover, substituting the first two into the last, $\partial w/\partial z = 0$! It follows that **$\boldsymbol{u}$ is independent of z!** The latter statement is the *far reaching* Taylor-Proudman theorem.

Einschub zum Taylor-Proudman Theorem: ρ sei nicht konstant! Es sei $p = f(\rho)$ (baroklin), $\boldsymbol{\Omega} = (0, 0, \omega)$ = constant, und $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$.

$$\boldsymbol{\nabla} \times \quad [\quad 2\boldsymbol{\Omega} \times \boldsymbol{u} + \frac{1}{\rho}\boldsymbol{\nabla} p = 0 \quad] \quad \Rightarrow \quad \boldsymbol{\nabla} \times (2\boldsymbol{\Omega} \times \boldsymbol{u}) + \frac{1}{\rho}\boldsymbol{\nabla} \times \boldsymbol{\nabla} p + \boldsymbol{\nabla}\left(\frac{1}{\rho}\right) \times \boldsymbol{\nabla} p = 0$$

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} p \equiv 0\ , \quad \boldsymbol{\nabla}\left(\frac{1}{\rho}\right) \times \boldsymbol{\nabla} p = -\frac{1}{\rho^2}\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} p = -\frac{1}{\rho^2}\frac{\mathrm{d}f}{\mathrm{d}\rho}\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla}\rho = 0 \text{ wegen } p = f(\rho)$$

$$\Rightarrow \quad \boldsymbol{\nabla} \times (\boldsymbol{\Omega} \times \boldsymbol{u}) = (\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} - \underbrace{(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\Omega} - \boldsymbol{u}\boldsymbol{\nabla} \cdot \boldsymbol{\Omega}}_{=0,\ \text{wegen } \boldsymbol{\Omega} = \text{konstant}} + \boldsymbol{\Omega}\boldsymbol{\nabla} \cdot \boldsymbol{u} = (\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \omega\frac{\partial \boldsymbol{u}}{\partial z} = 0!$$

Wesentliche Vorraussetzung, dass Taylor-Proudman Theorem gilt: $\boldsymbol{\nabla}\rho \times \boldsymbol{\nabla} p = 0$! (Das ist der Fall, wenn $p = f(\rho)$.)

Ekman boundary layer: We limit the following discussion for the case of the earth and consider the ocean surface as the boundary layer at $z = 0$. The rotation axis, Ω_E, is no longer parallel to the local vertical, $\hat{z}$. x and y axes lie in the surface. λ denotes the angle between equator plane and $\hat{z}$ in notherly direction. Assume as normal for the boundary layer that variations of $\boldsymbol{u}$ with z are much more rapid than those with x or y. Also we consider the case when the velocity of the interior is zero. The equations reduce to:

$$\nu\frac{\partial^2 u}{\partial z^2} + (2\Omega_E \sin\lambda)v = 0 \quad \text{and} \quad \nu\frac{\partial^2 v}{\partial z^2} - (2\Omega_E \sin\lambda)u = 0 \quad .$$

Eliminating v we obtain:

$$\frac{\partial^4 u}{\partial z^4} = -\left(\frac{2\Omega_E \sin\lambda}{\nu}\right)^2 u \quad .$$

There are four types of solutions: $u \propto \exp(\pm z/\epsilon)$, $\exp(\pm \mathrm{i} z/\epsilon)$, with $\epsilon = \sqrt{\nu/\Omega_E \sin|\lambda|}$. We are only interested in solutions that attenuate to zero for $z \to -\infty$, and which have the flow velocity U at the surface, which we choose to be in the x-direction: $\boldsymbol{u}(z = 0) = U\hat{x}$. Then an approriate solution is:

$$u = U\exp(z/\epsilon)\cos(z/\epsilon)\ , \quad \rightarrow \quad v = \frac{\sin|\lambda|}{\sin\lambda}U\exp(z/\epsilon)\sin(z/\epsilon)$$

Hence, $\boldsymbol{u}$ having a magnitude of U and pointing in x-direction at $z = 0$, attenuates over a depth ϵ by a factor 1/e, and as it attenuates it changes its direction. The angle θ of the flow with the x-axis is such that:

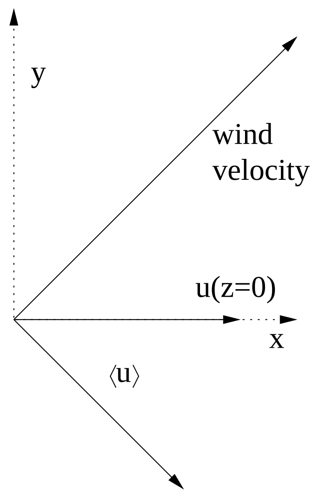
$$\tan\theta = \frac{u}{v} = \frac{\sin|\lambda|}{\sin\lambda}\tan(z/\epsilon)$$

Defining the mean velocity as $\langle\boldsymbol{u}\rangle = 1/\epsilon\int_{-\infty}^{0}\boldsymbol{u}\,\mathrm{d}z$, then, taking advangage of:

$$\int \exp(\alpha x)\sin\beta x\,\mathrm{d}x = \frac{\exp(\alpha x)}{\alpha^2 + \beta^2}(\alpha\sin\beta x - \beta\cos\beta x) \quad \text{and} \quad \int \exp(\alpha x)\cos\beta x\,\mathrm{d}x = \frac{\exp(\alpha x)}{\alpha^2 + \beta^2}(\alpha\cos\beta x + \beta\sin\beta x)$$

we obtain:

$$\langle u \rangle = -\frac{\sin|\lambda|}{\sin\lambda}\langle v \rangle = \frac{1}{2}U \quad .$$



The latter means that the mean boundary flow $\langle \boldsymbol{u} \rangle$ makes an angle of $\mp\pi/4$ with the x-axis, which coincides with the angle of the flow at $z = 0$. For the steady flow solution, there must be a stress that sustains such a flow. This stress can be supplied by wind. The stress components have to be:

$$s_{xz} = \nu\left(\frac{\partial u}{\partial z}\right)_{z=0} = \nu\epsilon U, \quad \text{and} \quad s_{yz} = \nu\left(\frac{\partial v}{\partial z}\right)_{z=0} = \nu\epsilon\frac{\sin|\lambda|}{\sin\lambda}U$$

The wind must exert this stress to the surface in order to garantee a steady solution of the upper type. Such a wind must make an angle the x-axis of $\pm\pi/4$.

Examples for Ekman layers: Northern wind along Californian coast drive surface towards west. Water at the coast is replaced by clean and cold water, i.e. nothern wind brings up fresh water at the shore.

4.5 Boundary layer separation

One consequence of the no-slip boundary condition for a flow that passes by a boundary. The flow velocity is reduced close to the boundary. From Bernoulli's law we know sum of the kinetic energy and gas pressure is constant along streamlines. Hence, due to the lower velocity, the kinetic energy is reduced and the gas pressure at that location close to the boundary needs to be increased. Such location of high gas pressure can drive small whirls which eventually may lead to boundary layer separation.

The theoretical description of boundary layer separation is complex and needs to be studied numerically. The description relies on the 'triple deck' structure.

Increasing the Reynolds number different types solutions are found in experiments:

1. For Reynolds numbers around 40, stationary eddys develop. For a cylinder, two eddies of opposite rotation form.

2. Increasing the Reynolds number further, the flow may enter a regime of oscillation: The sizes of the two eddies grow and shrink.

3. For Reynolds numbers around 100, eddies become separated, known as *Karman vortex street*. The eddies on the two sides of the cylinder are shed alternately, at a well defined frequency of approximately $0.1U/a$, which a being the radius of the cylinder.

4. For even higher Reynolds number the flow behind the cylinder (in the wake) becomes turbulent.

Chapter 5

Lift: Why can airplanes fly?

In this chapter we address the question of why airplanes fly. We will see that the simple answer is yielded in applying Bernoulli's equation, $p+\rho u^2/2 =$ constant, but deeper insights are necessary to understand what is going on! To this end we will touch the topics of circulation, and of the Kutta-Joukowski-Hypothesis and the Kutta-Joukowski-Theorem. We will demonstrate that the Bernoulli equation is not the full truth: Lift is only possible if the air is deflected by the aerofoil! This is related to the Newton's third law, *actio = reactio*, which also must be satisfied.

5.1 Circulation

The circulation is defined by the integral around a closed curve in the fluid:

$$\Gamma = \oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{u}) \cdot \mathrm{d}\boldsymbol{S}$$

For the second expression we have applied the Stokes theorem for any area S that is enclosed by the closed curve $\mathcal{C}$. For an Euler fluid (no viscosity, and constant density) it can be shown that the circulation does not change in time for a curve that is frozen into the fluid:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = 0$$

Proof: The curve $\mathcal{C}$ is parametrized by $\boldsymbol{l} = \boldsymbol{l}(s,t)$ with $0 \leq s \leq 1$, such that $\boldsymbol{l}(0,t) = \boldsymbol{l}(1,t)$, and $\mathrm{d}\boldsymbol{l}(s,t) = \partial \boldsymbol{l}/\partial s\ \mathrm{d}s$. The derivative is applied to each of the factors and the equation of motion is used to manipulate the first term of the sum:

$$\frac{\mathrm{D}}{\mathrm{D}t}\oint_{\mathcal{C}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{l} = \oint_{\mathcal{C}} \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{l} + \oint_{\mathcal{C}} \boldsymbol{u} \cdot \frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}\boldsymbol{l} = -\frac{1}{\rho}\oint_{\mathcal{C}} \boldsymbol{\nabla} p \cdot \mathrm{d}\boldsymbol{l} + \oint_0^1 \boldsymbol{u} \cdot \frac{\mathrm{D}}{\mathrm{D}t}\mathrm{d}\boldsymbol{l} = \underbrace{-\frac{1}{\rho}\oint_{\mathcal{C}} \mathrm{d}p}_{=0,\ \text{since } p(0)=p(1)} + \oint_0^1 \boldsymbol{u} \cdot \frac{\partial}{\partial s}\underbrace{\frac{\mathrm{D}\boldsymbol{l}}{\mathrm{D}t}}_{\boldsymbol{u}}\,\mathrm{d}s =$$

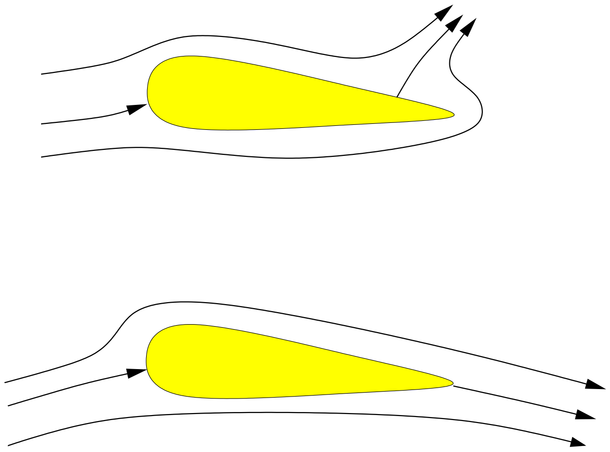
$$= \oint_0^1 \boldsymbol{u} \cdot \frac{\partial}{\partial s}\boldsymbol{u}\,\mathrm{d}s = \frac{1}{2}\oint_0^1 \frac{\partial \boldsymbol{u}^2}{\partial s}\,\mathrm{d}s = \frac{1}{2}[\boldsymbol{u}^2]_0^1 = 0 \quad .$$

5.2 Kutta-Joukowski-Hypothesis

An irrotational solution of the air flow streaming around the wing has no circulation. For such a solution the rear stagnation point is upstream of the trailing edge on the upper edge of the wing. A reverse pressure gradient is present that drives the flow from the trailing edge to the stagnation point. Such a solution has a singularity at the trailing edge, since the velocity is infinity there (upper sketch in the figure). It can be shown[1] that there is only one value for the circulation for which the flow speed is finite at the trailing edge (bottom sketch in the figure). This

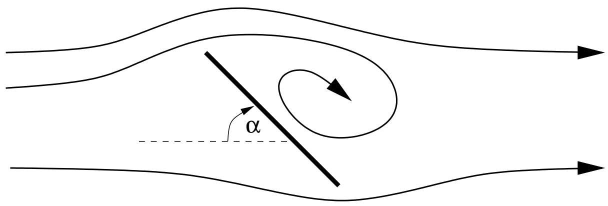
[1] This is quite complex, c.f. Acheson, Elementary fluid dynamics, chapter 4. The derivation involves the velocity potential, stream function, complex potential,, Milne-Thomson's circle theorem and conformal mapping.

is the solution in which the flow streams smoothly around the wing. The Kutta-Joukowski-Hypothesis states that the latter solution is the one that is observed.



Top: Irrotational flow with vanishing circulation. Stagnation point upstream of the trailing edge, such that the velocity at the trailing edge is infinite.
Bottom: Solution with finite circulation and finite velocity at the trailing edge. It is natural to hope (Kutta-Joukowski-Hypothesis) that this particular flow will correspond to the steady flow that is actually observed.

The Kutta-Joukowski Hypothesis implies that the flow around the aerofoil does not exhibit boundary layer separation, since the rear stagnation point is at the trailing edge of the aerofoil. Only then the aerofoil can experience lift. To produce such an airstream the geometric form of the aerofoil is crucial! There is a difference between a plate and an aerofoil.

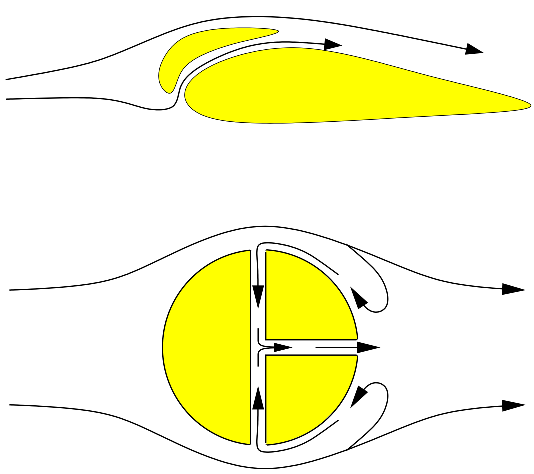


For a plate at a non-vanishing angle of attack, α, the airstreams stalls just after the leading edge, as depicted in the figure, and the plate will not experience lift (according to the Kutta-Joukowski hypothesis). The same happens to an aerofoil if the angle of attack becomes too large. If it exceeds some 8°, the air flow stalls and the aerofoil can not fly. This is of course understood in the context of Bernoulli's equation, if the air flow stalls above the wing, the velocity above is smaller than beneath the wing, leading to a gas pressure gradient that pulls the wing downward.

5.2.1 Difference between plate and aerofoil

For making a difference between a flat plate and an aerofoil, one has to suppress boundary layer separation of the flow. We want to mention three techniques.

The first is obvious and is called "stream lining". It consists in choosing an appropriate geometrical form, i.e. the leading edge is roundish and the trailing edge is sharp, and in between the aerofoil is as smooth as possible. The second technique aims at suppressing the reversed flow that tends to form from the trailing edge and is directed upstream, and may initiate boundary layer separation and thus stalls the air flow. This reversed flow is counter-acted by injecting fresh momentum into the boundary layer downstream of the leading edge (see figure). This technique is actually used in airplanes. The third technique does not aim at preventing the reverse flow, but it deviates the reverse flow such that it can not produce the boundary layer separation. The deviated flow is fed back into the overall flow where it does not harm, i.e. does not produce boundary layer separation.



5.2.2 Eddy (vortex) conservation

As mentioned above the only flow solution that has finite velocity at the trailing edge has a non-vanishing circulation: $\Gamma = \int_S \boldsymbol{\nabla} \times \boldsymbol{u}\, \mathrm{d}\boldsymbol{S} \neq 0$. This implies that the wing carries a bounded vortex with it. This eddy (vortex) is visible for the case of a plate, since the air flow stalls and an eddy forms in the wind shadow of the plate. This can nicely be experienced if you take a knife and a big pot of water. If you drag the knife with a finite angle of attack through the water you will see the bounded eddy. In case of an aerofoil (to be subject to the Kutta-Joukowski Hypothesis) the vortex is not readily seen, but is hidden in the fact that the air flow is deflected. But where is the counterpart of the vortex, which needs to be present in order to satisfy vortex (eddy) conservation? This counterpart forms when the movement starts. You will see the eddy at the trailing edge of the knife migrating sideways, shortly after you abruptly start the movement of the knife.

5.3 Kutta-Joukowski-Theorem

The Kutta-Joukowski-Theorem establishes the relation between lift and circulation. Again, an exact derivation is somewhat involved (c.f. Acheson, Sect. 4.11). The following derivation is not exact, but illustrative. Assume you have an aerofoil around which a flow streams. The flow velocity is denoted by U. The pressure on the upper edge of the aerofoil is denoted by p_t and at the bottom edge p_b. According to Bernoulli's equation the pressure difference is given by

$$p_b - p_t = \frac{1}{2}\rho(u_t^2 - u_b^2) = \frac{1}{2}\rho(u_t + u_b)(u_t - u_b) \sim \rho U(u_t - u_b)$$

The force acting on the aerofoil, L, called lift if the force acts upwards, is obtained by integrating along the wing, from the leading edge, $x = 0$, to the trailing edge, $x = c$:

$$L \sim \rho U \int_0^c (u_t - u_b)\, \mathrm{d}x$$

A close inspection of the integral reveals that it corresponds to the negative circulation aroung the wing. The circulation is given by:

$$\Gamma \sim \int_0^c u_b\, \mathrm{d}x + \int_c^0 u_t\, \mathrm{d}x = -\int_0^c (u_t - u_b)\, \mathrm{d}x$$

Hence, we can establish a relation between the lift and the circulation:

$$\boxed{L = -\rho U \Gamma}$$

Kutta-Joukowski theorem;
Lift theorem

This implies that if there is lift, the circulation is non-vanishing and negative, and the flow velocity above the wing is larger than beneath the wing.

5.4 Lift: the deflection of an air stream

It is important to realize that the conditions for non-vanishing lift can only be realized, if the air flow is deflected by the wing. Satisfying Newton's third law, *actio = reactio*, the lift must have a counter force. This counter force acts on the air and deflects it. In that respect both, Bernoulli's equation and the deflection of air (Newton's third law) are essential to produce lift!

Of course, Newton's third law is implicitly contained in the equation of motion[2]. This is most easily seen if one uses the Eulerian equation of motion (no viscosity) and integrates over a region, V. The surface of this Volume is denoted by ∂V:

$$\int_V \rho(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u}\, \mathrm{d}V = -\int \boldsymbol{\nabla} p\, \mathrm{d}V$$

[2] Note that the Bernoulli equation is also derived from the equation of motion.

Using the Einstein sum convention, the fact that $\partial u_i/\partial x_i = 0$, and Gauß's divergence theorem the left-hand side can be manipulated:

$$\int_V \rho u_j \frac{\partial u_i}{\partial x_j}\,\mathrm{d}V = \int_V \rho \frac{\partial}{\partial x_j} u_j u_i\,\mathrm{d}V = \int_{\partial V} \rho u_j u_i n_j\,\mathrm{d}S = \int_{\partial V} \rho u_i\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S$$

Applying Gauß's divergence theorem also on the right-hand side and putting the minus sign to the left-hand side, we obtain for the volume integral of the Euler equation:

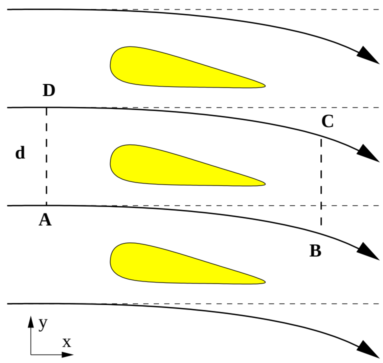
$$-\int_{\partial V} \rho\boldsymbol{u}\,\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{\partial V} p\boldsymbol{n}\,\mathrm{d}S \quad . \tag{5.1}$$

$\rho\boldsymbol{u}$ is the momentum per volume for a fluid element. $\boldsymbol{u}\cdot\boldsymbol{n}\,\delta S$ is the volume rate at which fluid leaves the volume V through the surface δS. Hence, the left-hand side gives the flux of momentum that leaves the volume V through its surface ∂V. The right-hand side is the integral of pressure acting on the surface, ∂V, which corresponds to the force that acts on the fluid volume, V.

5.4.1 Flow past a stack of aerofoils

In the case of a single aerofoil in an infinite expanse of fluid the deflection of air is somewhat hidden, since the deflection must tend to zero at infinity.

Therefore it is more illustrative to consider a stack of aerofoils which are separated by a distance d and are thought to repeat continuously to infinity in $+y$ and $-y$ direction. Then the flow that comes in horizontally in $+x$ direction, with $\boldsymbol{u}_1 = (U, 0)$, is deflected by the aerofoils. Since the flow is incompressible, the volume flux across AD must be equal to that across BC, meaning that the x-component of the velocity is constant. It follows that the flow velocity downstream of the aerofoils is given by $\boldsymbol{u}_2 = (U, v)$, with v denoting the y-component of the velocity downstream of the wings.



We now apply Eq. 5.1 to the fixed region ABCDA. The right-hand side is the force on the cross section and corresponds to the lift, L, if there is any. The left-hand side is the flux of momentum. Along the stream lines, i.e. for AB and DC, $\boldsymbol{u}\cdot\boldsymbol{n}$ vanishes, because $\boldsymbol{n}$ is perpendicular on $\boldsymbol{u}$. For AD, $\boldsymbol{u}$ is perpendicular to $\mathrm{d}\boldsymbol{l}$, and therefore flux of momentum through AD is also zero. The flux of momentum through BC is non-zero, because $\boldsymbol{u}\cdot\boldsymbol{n} = U$, and $\int_B^C \boldsymbol{u}\,\mathrm{d}\boldsymbol{l} = vd$, such that the left-hand side yields $-\rho U v d$. Thus, Eq. 5.1 yields

$$L \;=\; -\,\rho\,U\,v\,d$$

In this way, we see clearly how the lift is related to the deflection of the airstream: L is positive if v is negative. Moreover, the relation between deflection and circulation is also apparent: Using the same arguments as above the circulation, $\Gamma = \oint \boldsymbol{u}\,\mathrm{d}\boldsymbol{l}$ can be evaluated to be

$$\Gamma = v\,d \quad .$$

Hence again, the lift is given by $L \;=\; -\,\rho\,U\,\Gamma$. This demonstrates that the deflection of air is essential to obtain lift!

Chapter 6

Sound waves —linear analysis

In this chapter we will concentrate on perturbations in a gas. When a gas is disturbed, e.g. through the movement of a membrane of a loudspeaker, this perturbation is propagating away from the source. In the case of a gas the changes in velocity come along with changes in pressure (or density) — it is a compressible wave. As we are able to hear these waves (in a certain frequency range), we call these *sound waves*.

The governing equations are the continuity equation (2.2) and the Navier-Stokes equation (3.8) for the general case, i.e. including compressibility. We will limit our discussion here to the one-dimensional case in a gas with no bulk velocity, i.e. we consider a pressure perturbation propagating along the, e.g., z-direction, with the velocity perturbation having a component only in this direction. We include gravity as an external force $\boldsymbol{f}_{\rm ext} = \rho\,\boldsymbol{g}$ in the Navier-Stokes equation (3.8) and consider a wave propagating either in the case of absence of gravity (g=0; Sect. 6.2 & 6.3) or (anti) parallel to the gravitational acceleration (Sect. 6.4).

Instead of the dynamic viscosity μ we will use the kinematic viscosity $\nu = \mu/\rho$ (1.4), as in a gas the assumption of ν = const. is not too bad. In the following we will assume ν to be constant and for convenience define

$$\tilde{\nu} \;=\; \frac{4}{3}\,\nu \;=\; \frac{4}{3}\,\frac{\mu}{\rho} \;=\; \text{constant}$$

The one-dimensional equations for conservation of mass and momentum then read

$$\dot{\rho} + (\rho\,v)' \;=\; 0 \tag{6.1}$$

$$\rho\,\dot{v} + \rho\,v\,v' + p' \;=\; \rho\,\tilde{\nu}\,v'' - \rho\,g \tag{6.2}$$

These two equations have to be accompanied by an equation determining the pressure, normally the energy equation. Instead we will use an equation of state, here for an adiabatic process. This is a good assumption, as usually the gas element expanding or contracting when the wave passes by has no time to exchange energy with the surroundings. From thermodynamics one knows that in this case $p\,\rho^{-\gamma}$ is a constant for the respective gas parcel (not necessarily having the same value in the whole volume). Thus the time derivative in the co-moving frame has to vanish, cf. (2.6) in Sect. 2.2, i.e.

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(p\,\rho^{-\gamma}\right) = 0 \tag{6.3}$$

The adiabatic exponent γ is the ratio of specific heats and is about 1.4 for air at normal temperatures and pressures.

6.1 Small perturbations: linearizing the equations

Just as in perturbation theory we will now assume that an unperturbed state at rest exists, with density ρ_0, pressure p_0 and vanishing velocity $v_0 = 0$. The system is slightly perturbed, so that

$$v = v_1, \qquad \rho = \rho_0 + \rho_1, \qquad p = p_0 + p_1 \qquad ; \qquad \rho_1 \ll \rho_0, \qquad p_1 \ll p_0 \tag{6.4}$$

with the perturbations denoted by the index $_1$ being small compared to the unperturbed values. In the case of v_1 it is assumed that v_1 is smaller than the sound speed, see Sect. 6.1.3 and (6.13). The idea is then to neglect all terms quadratic or higher in the perturbations.

Let us first quickly investigate the non-perturbed state at rest. Then we have $v = 0$, $\rho = \rho_0$ and $p = p_0$, with $\dot{\rho}_0 = \dot{p}_0 = 0$. In this case the continuity equation (6.1) is trivially fulfilled and the momentum equation (6.2) reduces to the

$$\text{hydrostatic equilibrium} \qquad p_0' = -\rho_0\, g \tag{6.5}$$

Substituting the definitions (6.4), neglecting all terms of quadratic and higher order in v_1 and ρ_1, and remembering the hydrostatic equilibrium of the rest state (6.5), we are left just with

$$\dot{\rho}_1 + (\rho_0\, v_1)' = 0 \tag{6.6}$$

$$\rho_0\, \dot{v}_1 + p_1' = \rho_0\, \tilde{\nu}\, v_1'' - \rho_1\, g \tag{6.7}$$

Before applying these linearized equations to a problem, we have to establish a relation for the pressure perturbation p_1 based on the adiabatic assumption.

6.1.1 Adiabatic changes on a constant background

Let us first consider the simpler case with a background that is spatially constant, the spatial derivatives of the quantities at rest vanish, $\rho_0' = p_0' = 0$. In the adiabatic case described by (6.3) we know that p/ρ^γ is constant in each gas parcel. If it was p_0/ρ_0^γ in the unperturbed state it will remain so, i.e.

$$\frac{p}{\rho^\gamma} = \frac{p_0 + p_1}{(\rho_0 + \rho_1)^\gamma} = \frac{p_0}{\rho_0^\gamma} \qquad \Leftrightarrow \qquad \left(1 + \frac{p_1}{p_0}\right) \left(1 + \frac{\rho_1}{\rho_0}\right)^{-\gamma} = 1 \tag{6.8}$$

As $\rho_1 \ll \rho_0$ we can Taylor-expand $(1 + \rho_1/\rho_0)^{-\gamma} = (1 - \gamma\, \rho_1/\rho_0 + \ldots)$, where quadratic and higher orders in (ρ_1/ρ_0) denoted by "..." are neglected. Substituting this yields

$$1 + \frac{p_1}{p_0} - \gamma\, \frac{\rho_1}{\rho_0} - \gamma\, \frac{p_1\, \rho_1}{p_0\, \rho_0} + \ldots = 1, \tag{6.9}$$

We now first define a constant c, and will see later after (6.18) that it is the

$$\text{adiabatic sound speed} \qquad c = \left(\gamma\, \frac{p_0}{\rho_0}\right)^{1/2}. \tag{6.10}$$

Neglecting all terms quadratic or higher in the perturbed quantities (i.e. also $p_1\rho_1$), we can re-write (6.9) to give a relation between the pressure and density perturbations,

$$p_1 = c^2\, \rho_1. \tag{6.11}$$

Together with this relation, (6.6) and (6.7) are the linearized form of (6.1) and (6.2) for an adiabatic equation of state on a uniform background and fully describe the linear evolution of the perturbation, i.e. as long as the perturbations remain small.

6.1.2 Adiabatic changes in a stratified atmosphere

If we consider a problem where the gravity becomes important, we have to account for the stratification of the background atmosphere, i.e. for the hydrostatic equilibrium (6.5). Then the simple relation (6.11) no longer holds. In this case the adiabatic equation of state (6.3) has to be properly linearized. This is a bit tedious, but with some patience and the hydrostatic equilibrium (6.5) one arrives at

$$\dot{p}_1 = c^2 \dot{\rho}_1 + v_1 \left(c^2 \rho_0' + \rho_0\, g \right), \tag{6.12}$$

and for $\rho_0' = g = 0$ indeed (6.11) follows from this relation.

6.1.3 What means "small velocity perturbation"?

Above we assumed that v_1 should be small. However, as the non-perturbed state is assumed to be at rest, $v_0 = 0$, what does "v_1 is small" means?

This can be illustrated with the help of the continuity equation (6.6) An order of magnitude estimation gives $\rho_1/\tau \approx \rho_0\, v_1/L$ with the typical length scale L and time scale τ of the wave. Of course this time scales are related through the sound speed, $c \approx L/\tau$. Thus we can write for the velocity fluctuations

$$\frac{v_1}{c} \approx \frac{\rho_1}{\rho_0} \qquad \Rightarrow \qquad v_1 \ll c, \tag{6.13}$$

i.e. the velocity fluctuation with respect to the sound speed are of the same order than the density fluctuations and thus $v_1 \ll c$.

The same works also with the momentum equation (6.7) when comparing the two terms on the right hand side.

6.1.4 What means “small viscosity”?

One note has to be made concerning linearizing of the viscosity term. Of course this linearization does make sense only if the viscosity is not too large, i.e. if the effects of viscosity, represented by $\rho_0\, \tilde{\nu}\, v_1''$ in (6.7), are smaller than the force driving the sound wave, i.e. the pressure gradient, represented by $(c^2\, \rho_1)'$ in (6.7). For an order of magnitude estimation we introduce a length scale $L = 1/k$ through the wave number k of the propagating wave. Furthermore we assume that the perturbations of density and velocity are of the same order, i.e. $\rho_1/\rho_0 \approx v_1/c$. Now when comparing the terms $(c^2\, \rho_1)'$ and $\rho_0\, \tilde{\nu}\, v_1''$ in (6.7), *small viscosity* means

$$\frac{c}{\tilde{\nu}\, k} > 1 \tag{6.14}$$

Identifying k with $1/L$, the above right hand side is sort of a Reynolds number, cf. Sect. 1.3 and (1.5), i.e. the linearization is valid only for large Reynolds numbers.

This becomes important when discussing the sound wave in an viscous medium in Sect. 6.3 following (6.26).

6.2 Pure sound waves

If we now neglect any effects of viscosity ($\tilde{\nu}$=0)and gravity (g=0; ρ_0'=0), the linearized continuum and momentum equations (6.6) and (6.7) together with the adiabatic relation (6.11) simplify to

$$\dot{\rho}_1 + \rho_0\, v_1' \;=\; 0 \qquad ; \qquad \rho_0\, \dot{v}_1 + c^2\, \rho_1' \;=\; 0 \tag{6.15}$$

Taking the spatial derivative of the left hand equation and the time derivative of right equation,

$$\dot{\rho}_1' + \rho_0\, v_1'' \;=\; 0 \qquad ; \qquad \rho_0\, \ddot{v}_1 + c^2\, \dot{\rho}_1' \;=\; 0$$

we can combine these to give a differential equation for the velocity, (well, it is a wave equation)

$$\ddot{v}_1 - c^2\, v_1'' \;=\; 0 \tag{6.16}$$

Likewise we can get a wave equation for the density perturbation if we take the temporal derivative of the left and the spatial derivative of the right equation in (6.15).

As (6.16) has constant coefficients, we can make an exponential ansatz in time and space,

$$v_1 = \hat{v}\, \exp\left(ikz - i\omega t\right) \qquad \Rightarrow \qquad \begin{array}{ll} \ddot{v}_1 = -\omega^2\, v_1 & v_1' = i\, k\, v_1 \\ v_1'' = -k^2\, v_1 & \dot{v}_1'' = i\, \omega\, k^2\, v_1 \end{array} \tag{6.17}$$

If both ω and k are real, the real part of v_1, representing the observable quantity, is a sinusoidal wave. If then k>0 the wave propagates in the the +z-direction, if k<0 in the $-z$-direction.

Substituting the ansatz (6.17) in the equation governing the velocity perturbation (6.16) we find the dispersion relation

$$\omega^2 = c^2\, k^2 \tag{6.18}$$

As the group velocity is $\mathrm{d}\omega/\mathrm{d}k = c$, the sound speed is indeed c as mentioned above with equation (6.10). Here we have the special case that also the phase speed ω/k is given by the sound speed.

6.3 Sound waves being dissipated by viscosity

We will now include the effects of viscosity ($\tilde{\nu}\neq 0$) but still neglect gravity (g=0; ρ_0'=0). The linearized continuum and momentum equations (6.6) and (6.7) together with (6.11) then read

$$\dot{\rho}_1 + \rho_0\, v_1' \;=\; 0 \qquad ; \qquad \rho_0\, \dot{v}_1 + c^2\, \rho_1' \;=\; \rho_0\, \tilde{\nu}\, v_1'' \tag{6.19}$$

Combining them as following (6.15) yields

$$\ddot{v}_1 - c^2\, v_1'' \;=\; \tilde{\nu}\, \dot{v}_1'' \tag{6.20}$$

As this differential equation has constants coefficients, too, we can again use the exponential ansatz (6.17). Substituting this ansatz in (6.20) gives the dispersion relation

$$\omega^2 = c^2\, k^2 - i\, \tilde{\nu}\, \omega\, k^2. \tag{6.21}$$

For $\tilde{\nu}$= 0 this is the same as for a pure sound wave (6.18), of course.

Now the frequency ω will be complex in general, even if the (spatial) wavenumber k is real. In order to analyze this further we write

$$\omega = \omega_\mathrm{R} + i\omega_\mathrm{I} \tag{6.22}$$

where ω_R and ω_I are the real and complex part. Substituting this in (6.21) yields

$$\underbrace{\omega_\mathrm{R}^2 - \omega_\mathrm{I}^2}_{(2)} + \underbrace{2i\omega_\mathrm{R}\omega_\mathrm{I}}_{(1)} \;=\; \underbrace{c^2\, k^2}_{(2)} - \underbrace{i\, \tilde{\nu}\, \omega_\mathrm{R}\, k^2}_{(1)} + \underbrace{\tilde{\nu}\, \omega_\mathrm{I}\, k^2}_{(2)} \tag{6.23}$$

Here the imaginary parts (1) and the real parts (2) must independently fulfill this relation. Let us first compare the imaginary parts (1), which gives directly he imaginary part of the frequency,

$$\omega_\mathrm{I} = -\frac{1}{2}\, \tilde{\nu}\, k^2 < 0 \tag{6.24}$$

While k can be positive (propagation in $+z$) or negative (in $-z$), k^2 is always positive. As the viscosity $\tilde{\nu}$ is a positive quantity, too, the imaginary part of the frequency is always negative.

Thus the solution following the ansatz 6.17 is given through

$$v_1 = \hat{v}\; \exp\left(ikz - i\omega_\mathrm{R} t\right)\; \exp\left(\omega_\mathrm{I}\, t\right). \tag{6.25}$$

As ω_R and ω_I are real by definition, and as $\omega_\mathrm{I} < 0$, this solution is a wave which is exponentially decaying in time through the exp($\omega_\mathrm{I}\, t$) term.

This is exactly what is to be expected, namely that in the presence of viscosity a sound wave is damped. Furthermore we find that waves with large wave numbers k, i.e. small wavelengths are dissipated the quickest, because then ω_I is large. This also is sort of “common sense”.

Comparing the real parts (2) of (6.23) leads to the phase speed

$$v_\mathrm{phase}^2 \;=\; \left(\frac{\omega_\mathrm{R}}{k}\right)^2 \;=\; c^2 - \left(\tfrac{1}{2}\, \tilde{\nu}\, k\right)^2 \tag{6.26}$$

This shows that the phase speed is reduced compared to the case without viscosity, where it c. The stronger the viscosity, viz. damping, the smaller the phase speed. This result is also “common sense”, i.e. a sound wave damped in a viscous medium is propagating slower than in a non-viscous case.

As this linear analysis is limited to small viscosity, see Sect. 6.1.4 and (6.14), v_phase^2 will not become negative. But of course, this clearly shows the limitations of this linear analysis.

6.4 Sound waves in a stratified atmosphere

Turning to a sound wave propagating in an atmosphere subject to gravity we will again neglect viscosity.

As now the density of the non-perturbed state at rest is no longer spatially constant, we have to investigate the hydrostatic equilibrium (6.5) first. We assume that the gas is an perfect gas, so that the state at rest follows

$p_0 = \rho_0\, k_B T_0/m$, with a constant non-perturbed state temperature T_0 and mean molecular weight m. Using this, the hydrostatic equilibrium (6.5) can be re-written and easily integrated.

$$\frac{\rho_0'}{\rho_0} = -\frac{m\,g}{k_B T_0} = -\frac{1}{H} \qquad \Rightarrow \qquad \rho_0 = \rho_{00}\,\exp\left(-\frac{z}{H}\right) \quad \text{with scale height} \quad H = \frac{k_B T_0}{m\,g} \tag{6.27}$$

In the case of the earth's atmosphere $H \approx 8$ km. In the Sun's photosphere this is about 200 km (this is the reason why the solar limb appears so sharp, even with the best telescopes, which can currently resolve structures on the Sun of 100 km, at best).

It is important to note here, that the assumption of an isothermal atmosphere greatly simplifies the problem, especially the sound speed $c^2 = \gamma\, p_0/\rho_0 = \gamma\, H/\, g$ is a constant, as is $\rho_0'/\rho_0 = -1/H$. Nevertheless, the same could be done without this assumption of constant temperature, leading basically to the same result, but as not much more is learned from the general case (except some maths) we keep to the simplified version of an barometric atmosphere.

With gravity and neglecting viscous effects ($\tilde{\nu}$=0) the linearized continuum and momentum equations (6.6) and (6.7) and the linear adiabatic equation of state (6.12) read

$$\dot{\rho}_1 = \frac{\rho_0}{H}\, v_1 - \rho_0\, v_1' \,, \tag{6.28}$$

$$\dot{v}_1 = -\frac{p_1'}{\rho_0} - \frac{\rho_1}{\rho_0}\, g \,, \tag{6.29}$$

$$\dot{p}_1 = c^2\, \dot{\rho}_1 - v_1\, c^2\, \frac{\rho_0}{H} + v_1\, \rho_0\, g \,. \tag{6.30}$$

Here we used that the spatial derivative of the density of the rest state following (6.27) is $\rho_0' = -\rho_0/H$.

These three equations can now be combined to give one single differential equation for, e.g., the velocity perturbations v_1. For this one can take the spatial derivative of (6.30) and substitute $\dot{\rho}_1'$ using the spatial derivative of (6.28). The result then can be used to substitute $\dot{p}_1'$ in the temporal derivative of (6.29), and using (6.28) as well as $\rho_0' = -\rho_0/H$ from (6.27) one finally arrives at

$$\ddot{v}_1 - c^2\, v_1'' + \frac{c^2}{H}\, v_1' = 0 \,. \tag{6.31}$$

Just as before we use the exponential ansatz (6.17), which gives the dispersion relation

$$\omega^2 = c^2\, k^2 + i\,\frac{c^2}{H}\, k \tag{6.32}$$

As with the case including viscosity, here the frequency ω can be complex and we define $\omega = \omega_R + i\omega_I$ giving

$$\underbrace{\omega_R^2 - \omega_I^2}_{(2)} + \underbrace{2i\omega_R\omega_I}_{(1)} = \underbrace{c^2\, k^2}_{(2)} + \underbrace{i\,\frac{c^2}{H}\, k}_{(1)} \tag{6.33}$$

In principle the imaginary and the real part can be solved explicitly, but we will consider a somewhat simplified problem. For the imaginary parts (1) alone we find

$$\omega_I = \frac{c^2}{2\,H}\,\frac{k}{\omega_R} \tag{6.34}$$

We immediately see that the imaginary part of the frequency, ω_I is positive for an upward propagating wave (k>0) and negative for a downward propagating wave (k<0). Considering the solution (6.25) this implies that an upward propagating wave (propagating in denser material) will steepen, while a wave propagating downwards will be damped.

One can now assume that the atmosphere is only weakly stratified, i.e. that the phase speed of the wave propagating in the stratified atmosphere, $\omega_R/k \approx c$ is approximately the phase speed for a constant density, i.e.

the adiabatic sound speed, and thus $\omega_{\rm I} \approx c/(2H)$. Similar to the case with viscosity alone (6.25) we can now write for the solution of the velocity perturbation

$$v_1 \approx \hat{v}_1 \exp\left(ikz - i\omega_{\rm R}t\right) \exp\left(\frac{c}{2H}\,t\right) = \hat{v}_1 \exp\left(ikz - i\omega_{\rm R}t\right) \exp\left(\frac{z}{2H}\right) , \tag{6.35}$$

where we used that if the the phase speed is c, we can transform from time to space via $z = ct$. Comparing this with the solution of the hydrostatic equilibrium (6.27), we can write

$$v_1 \propto \frac{1}{\sqrt{\rho_0}} \exp\left(ikz - i\omega_{\rm R}t\right) , \tag{6.36}$$

i.e. in a weakly stratified atmosphere the perturbations are proportional to $1/\sqrt{\rho_0}$ (this also holds for the pressure perturbations).

Using the same approximation as above, i.e. $\omega_{\rm I}^2 \approx c^2/(4H^2)$, we can write for the real part of the dispersion relation (6.33)

$$\omega_{\rm R}^2 \approx c^2\,k^2 + \frac{c^2}{4\,H^2} = c^2\,k^2 + \omega_A^2. \tag{6.37}$$

Unlike for the case of a pure sound wave, here a minimum frequency exists for a wave propagating through a stratified atmosphere, ω_A. This frequency is called the *cut-off frequency*, as only waves with higher frequencies can propagate, e.g. in the solar atmosphere this frequency corresponds to about 3 minutes in the photosphere.

In a more general treatment, the cut-off frequency is given through the Brunt-Väisälä frequency

$$\text{cut-off frequency:} \quad \omega_A = \frac{\gamma}{2\sqrt{\gamma-1}}\,N \qquad \text{with} \qquad \text{Brunt-Väisälä frequency:} \quad N^2 = -g\,\frac{\rho_0'}{\rho_0} - \frac{g^2}{c^2}$$

It is easy to verify, that with our assumptions the cut-off frequency indeed is given by $\omega_A = c/(2H)$.

To summarize: in a barometrically stratified atmosphere, where the density is decreasing with height, a vertically propagating sound wave will steepen when propagating upwards and will be damped when propagating downwards. Furthermore only waves with frequencies larger than the cut-off frequency can propagate.

The physical relevance of these results is found e.g. in the solar atmosphere. There the convective overshoot at the surface of the Sun produces a whole range of sound waves. When propagating these sound waves steepen and their amplitude grows larger and larger. At some point the linear analysis presented in this chapter brakes down, of course. The waves steepen into shocks and their energy is dissipated. That this process operates on the Sun and might heat the solar outer atmosphere was first suggested independently in 1948 by Biermann and Schwarzschild. Today we know that the sound waves alone are not sufficient to heat the corona of the Sun to 10^6 K (while the visible surface is at $\sim$5800 K), but that magnetic fields play a crucial role. Nevertheless the heating processes due to sound waves still play an important role in the lower parts of a stellar atmosphere, namely the chromosphere of the Sun as well as of other stars.

The present discussion is limited to a linear analysis. A non-linear analysis for sound waves in a stratified atmosphere *including* viscous effects would lead to the following result. An upward propagating wave is steepening until it forms a shock and the amplitude of the velocity perturbations is comparable to the sound speed. Then the viscous dissipation is most efficient and the wave is quickly damped out. Thus the non-linear effects limit the wave amplitude to the sound speed. A detailed description of these processes is beyond the scope of this lecture. However, non-linear effects will be discussed in the next chapter.

Chapter 7

Surface waves and solitons

This chapter will deal with surface waves on an incompressible fluid, like waves on an ocean or in a channel. In the first part (Sect. 7.1) we discuss waves of small amplitude in deep water and find in Sect. 7.1.5 that these waves are dispersive, i.e. after some time a wave packet on deep water will become so broad with so small amplitude, that it practically vanishes. In Sect. 7.2 this will be expanded to water of finite depth. In contrast to this linear analysis, a non-linear treatment of surface waves on shallow water (Sect. 7.3) shows that these waves steepen, i.e. grow in amplitude. Finally one can find instances, where the dispersive effects exactly balance the non-linear effects of steepening. Such a perturbation, not changing its shape while propagating, is called soliton and is discussed at the end of this chapter in Sect. 7.4.

7.1 Surface waves on deep water —gravity waves

For the following we will restrict the discussion to two dimensions, with the wave propagating in the horizontal x-direction, and gravity pointing in negative z-direction. The *free* surface of the fluid is found at

$$\text{free surface:} \qquad z = \eta(x,t) \; . \tag{7.1}$$

Without the wave the surface would be at z=η=0. The respective velocity components are $(\boldsymbol{u})_x = u(x,z,t)$ and $(\boldsymbol{u})_z = v(x,z,t)$.

We assume the flow to be irrotational (cf. end of Sect. 3.1), i.e. $\nabla \times \boldsymbol{u} = 0$. This inplies that we do not have to consider viscous effects, as for an incompressible fluid in this case the viscous term drops out of the Navier-Sokes equation (3.9). Now the velocity vector is given through a (scalar) potential ϕ,

$$\boldsymbol{u} = \nabla\phi \qquad \rightarrow \qquad u = \frac{\partial\phi}{\partial x} \qquad ; \qquad v = \frac{\partial\phi}{\partial z} \; . \tag{7.2}$$

For an incompressible fluid we have $\nabla \cdot \boldsymbol{u} = 0$, thus the potential has to satisfy Laplace's equation, i.e.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \; . \tag{7.3}$$

7.1.1 The free surface

As effects like convection are not considered, any particle which is located at the surface will remain there. Defining $f = z - \eta$, this implies that $\mathrm{D}f/\mathrm{D}t = 0$ at the surface, as each surface elements remains there, i.e. $\partial f/\partial t + (\boldsymbol{u}\cdot\nabla)f = 0$. Using the definition of f and its partial derivatives, $\frac{\partial f}{\partial t}$=$-\frac{\partial\eta}{\partial t}$, $u\frac{\partial f}{\partial x}$=$-u\frac{\partial\eta}{\partial x}$, $v\frac{\partial f}{\partial z}$=$v$, one can write

$$\text{at the surface:} \qquad \frac{\partial\eta}{\partial t} + u\,\frac{\partial\eta}{\partial x} - v = 0 \; . \tag{7.4}$$

For a horizontal surface ($\partial\eta/\partial x$=0) one finds that v=$\partial\eta/\partial t$, which makes sense. For a stationary surface ($\partial\eta/\partial t$=0) one has v/u=$\partial\eta/\partial x$, i.e. streamlines are parallel to the surface, also making sense.

7.1.2 Using the Euler equation

Using the velocity potential (7.2) we can write the Euler equation (2.12) as

$$\frac{\partial}{\partial t}(\nabla\phi) = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}\boldsymbol{u}^2 + g\,z\right) . \tag{7.5}$$

One can now integrate this equation in z giving

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2\right) + g\,z \quad = \quad -\frac{p}{\rho} + \xi(t) . \tag{7.6}$$

Here $\xi(t)$ can be chosen arbitrarily, without any effect on the flow field, as $\boldsymbol{u}$ is given through *partial spatial* derivatives of the potential ϕ.

Assuming that at the surface $z = \eta$ the pressure being given by the atmospheric pressure p_0 constant in x and t, we can chose $\xi = p_0/\rho$ and find

$$\text{at the surface:} \qquad \frac{\partial\phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2\right) + g\,\eta \quad = \quad 0 . \tag{7.7}$$

This is often referred to as the *pressure condition*, as one assumes the pressure at the surface to be given through the atmosphere.

7.1.3 Wave solution for small amplitudes

The equations (7.4) and (7.7) describe the (non-linear) evolution at the surface.

As in Sect. 6.1 we are splitting the quantities in a time-independent "background value" and a perturbation to linearize the equations. Here we have the state of rest velocities to be zero and likewise the location of the surface should be around $z = 0$, and therefore we simply skip the index $_1$ for the perturbed quantities and assume that u, v, η and ϕ are small, and thus neglect terms quadratic and higher in these quantities.

Using this linearization we rewrite (7.4) using $v = \partial\phi/\partial z$, and (7.7) to give

$$\text{at the surface } z = \eta \ll 1 \ : \qquad \frac{\partial\phi}{\partial z} - \frac{\partial\eta}{\partial t} \quad = \quad 0 , \tag{7.8}$$

$$\frac{\partial\phi}{\partial t} + g\,\eta \quad = \quad 0 . \tag{7.9}$$

We will now make an ansatz for the surface displacement to be a sinusoidal wave,

$$\eta \quad = \quad \hat{\eta}\,\cos(kx - \omega t) . \tag{7.10}$$

This suggests that the solution of the potential has the form

$$\phi \quad = \quad \varphi(z)\,\sin(kx - \omega t) . \tag{7.11}$$

As the potential has to satisfy Laplace's equation (7.3), the amplitude $\varphi(z)$ must satisfy $\partial^2\varphi/\partial z^2 = k^2\varphi$, i.e.

$$\varphi = A\exp(kz) + B\exp(-kz) . \tag{7.12}$$

Assuming without loss of generality that $k{>}0$, we have to set $B = 0$ to ensure finite amplitudes "deep in the ocean", i.e. for $z \to -\infty$.

Using this solution and then substituting the ansatz for η and ϕ in the linearized wave equations (7.8) and (7.9), we find

$$A\,k - \hat{\eta}\,\omega \;=\; 0 \qquad ; \qquad -A\,\omega + g\,\hat{\eta} \;=\; 0 . \tag{7.13}$$

from which the dispersion relation follows,

$$\omega^2 \quad = \quad g\,k . \tag{7.14}$$

7.1.4 Going in circles

Before further discussing the dispersion relation, “particle paths” on the surface will be addressed quickly. Using (7.13) we have $A = \hat{\eta}\,\omega/k$, and thus

$$\phi = \hat{\eta}\frac{\omega}{k}\exp(kz)\sin(kx-\omega t) \qquad \Rightarrow \qquad u = \hat{\eta}\omega\exp(kz)\cos(kx-\omega t) \quad ; \quad v = \hat{\eta}\omega\exp(kz)\sin(kx-\omega t)$$

Assuming that the positions of particles in the fluid depart not much from an average position $(\bar{x},\bar{z})$, we might integrate the velocities $u = \mathrm{d}x_1/\mathrm{d}t$ and $v = \mathrm{d}z_1/\mathrm{d}t$ to give

$$x_1 = -\hat{\eta}\exp(k\bar{z})\sin(k\bar{x}-\omega t) \quad ; \quad z_1 = \hat{\eta}\exp(k\bar{z})\cos(k\bar{x}-\omega t)\ .$$

Thus “test particles” go in circles. At the surface the radius of these circles is just the amplitude of the wave, of course. It is important to note that even though each surface fluid element goes in circles, it remains on the surface all the time.

On the wave crests there is a forward motion, in the troughs a rearward motion, as is easily experienced when swimming in the ocean.

The water also goes in circles within the water, but the amplitude is rapidly decreasing. Thus almost all the energy of a surface wave is carried in a surface layer, its depth being comparable to the wavelength.

7.1.5 Dispersion of small amplitude surface waves in deep water

The most important result of this section is the dispersion relation (7.14) From this it quickly follows that the phase speed, i.e. the velocity of the individual crests is given by

$$c_{\text{phase}} = \frac{\omega}{k} = \sqrt{g/k}\ . \tag{7.15}$$

The group velocity is only half the phase speed

$$c_{\text{group}} = \frac{\partial\omega}{\partial k} = \frac{1}{2}\,c_{\text{phase}}\ . \tag{7.16}$$

Both speeds depend on the wave number, and thus the waves are dispersive. The energy of the wave is transported with the group velocity.

The most important result here is, that these waves are dispersive, i.e. the phase speed depends on the wave number, or in other words the phase speed differs from the group velocity. Thus an initially narrow packet of waves, consisting of a number of different frequencies will become broader in time. In the present case, the waves with large wavelengths (small k) propagate faster, while those with small wavelengths lag behind. This will ultimately lead to a broader wave packet with smaller amplitude. Thus, even in the absence of any viscous effects, the wave packet will finally dissappear, or more exactly: it will become so broad, that its amplitude is hardly recognizable.

7.2 Surface wave in water with finite depth

When considering small amplitude surface waves in water with finite depth h_0 all that changes is the boundary condition. Instead of the case of infinite depth, we now have account for that the fluid is bounded at $z = -h_0$, i.e. that its vertical velocity vanishes there, or in terms of the velocity potential

$$\left(\frac{\partial\phi}{\partial z}\right)_{z=-h_0} = 0\ .$$

With this boundary condition the constant B in (7.12) for the amplitude of the potential will no longer be zero. A more complicated but similar treatment the gives the dispersion relation

$$\omega^2 = g\,k\tanh(h_0 k) \qquad \Rightarrow \qquad c^2_{\text{phase}} = \frac{g}{k}\tanh(h_0 k)\ . \tag{7.17}$$

The consideration of finite depth just gives a factor of $\tanh(k\,h_0)$ to the solution of infinite depth (7.15). In deep water, i.e. when $h_0 \gg \lambda = k^{-1}$ we have $\tanh(h_0 k) \approx 1$, and thus the same dispersion relation as for infinitly deep water. In shallow water, where $h_0 \ll \lambda = k^{-1}$ we have $\tanh(h_0 k) \approx h_0 k$, and thus there the phase speed is $\sqrt{g\,h_0}$. This latter result was already found at the beginning in Sect. 2.8 by very basic arguments.

7.3 Finite amplitude waves in shallow water —nonlinear effects

We will now no longer assume that the perturbations are small. However, we will keep to the shallow water approximation, namely that the depth of the (undisturbed) water h_0 is small compared to any length scale of the wave L,

$$h_0 \ll L \; . \tag{7.18}$$

As before we consider a wave in the x-z plane. Then the components of the Euler equation (2.12) and the continuity equation in the case of incompressibility (2.8) read

$$\frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \; , \tag{7.19}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \; , \tag{7.20}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0 \; . \tag{7.21}$$

The *shallow water approximation* (7.18) now justifies to neglect the convective derivative $\mathrm{D}v/\mathrm{D}t$ in the vertical component of the Euler equation (7.20). This can be shown by a dimensional analysis of the above equations, e.g. with (7.18) it follows from (7.21) that $v/u \approx h_0/L \ll 1$.

Finally in the z-direction we are left with the hydrostatic equilibrium, and (7.20) can be written as

$$\frac{\partial p}{\partial z} = -\rho g \qquad \Rightarrow \qquad p = p_0 - \rho g \left(z - h\right) \; , \tag{7.22}$$

where we have directly integrated the hydrostatic equilibrium, assuming that the pressure at the surface at height h equals the constant atmospheric pressure p_0.

In contrast to the preceeding discussion of deep water waves we now choose $z = 0$ at the botton of the water and define the surface through $z = h(x,t)$.

Now the horizontal component of the Euler equation (7.19) simplifies to

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -g\,\frac{\partial h}{\partial x} \; ,$$

i.e. for any given fluid parcel the change of the horizontal velocity u does *not* depend on z and we can write instead of (7.19) the reduced form of the above equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\,\frac{\partial h}{\partial x} \; . \tag{7.23}$$

This is the first of the two shallow water equations.
To get the second one, we start with the continuity equation (7.21) and formally integrate with respect to z giving

$$v = -\frac{\partial u}{\partial x}\,z + f(x,t) \; , \tag{7.24}$$

where we have to add an arbitrary function $f(x,t)$ not depending on z. Using a no-slip boundary condition at the bottom of our problem, i.e. v=0 at z=0, we find $f(x,t)$=0.

With the help of the condition of the free surface (7.4), here taking the form

$$\text{at surface } z = h(x,t): \qquad \frac{\partial h}{\partial t} + u\,\frac{\partial h}{\partial t} - v = 0 \; .$$

we can re-write (7.24) to give the second shallow water equation

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} = -h\,\frac{\partial u}{\partial x} \; . \tag{7.25}$$

The coupled partial differential equations (7.23) and (7.25) fully describe the evolution of the surface and the horizontal velocities and thus fully describe the shallow water wave problem.

These equations can be written in a more symmetric form by introducing

$$c(x,t) = \sqrt{g\,h} \; , \tag{7.26}$$

which is similar to the phase speed in the linear analysis done for shallow water waves in Sect. 2.8, cf. equation (2.39). Substitution in (7.23) and (7.25) yields

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -2c\,\frac{\partial c}{\partial x} \qquad ; \qquad 2\frac{\partial c}{\partial t} + 2u\frac{\partial c}{\partial x} = -c\,\frac{\partial u}{\partial x}\,.$$

Adding and subtracting now finally gives the shallow water equations

$$\left\{\frac{\partial}{\partial t} + \left(u+c\right)\frac{\partial}{\partial x}\right\}\left(u+2c\right) = 0\,, \tag{7.27}$$

$$\left\{\frac{\partial}{\partial t} + \left(u-c\right)\frac{\partial}{\partial x}\right\}\left(u-2c\right) = 0\,. \tag{7.28}$$

Here we will not solve these equations, but only sketch one possible method, namely one based on characteristics. For this one defines a *characteristic curve* $x = x(s)$, $t = t(s)$ in the x-t plane. Here it is profitable when choosing $\mathrm{d}t/\mathrm{d}s = 1$ and $\mathrm{d}x/\mathrm{d}s = u \pm c$. One can then substitute this in (7.27) and (7.28), respectively depending on the sign, and will finally find that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(u \pm 2c\right) = 0\,.$$

Thus one arrives at the important statement that $u \pm 2c$ is constant along positive / negative characteristics.

7.3.1 A non-linear perturbation or wave

Now assume that a perturbation, e.g. a wave package or a Gaussian-shaped perturbation, is moving in the $+x$-direction into water at rest. As then in this problem each ("meaningful") characteristic of interest is connected to the water at rest, where the surface is at a constant height h_0, i.e. where $u = 0$, we have

$$u - 2c = -2c_0 = -\sqrt{g\,h_0}\,. \tag{7.29}$$

Then (7.28) is trivially satisfied, while (7.27) yields

$$\frac{\partial c}{\partial t} + \left(3c - 2c_0\right)\frac{\partial c}{\partial x} = 0\,. \tag{7.30}$$

This describes the non-linear evolution of the wave, here its surface height.

7.3.2 Non-linear evolution

To investigate this evolution, one might introduce a new variable ξ which, when substituted in (7.30), results in

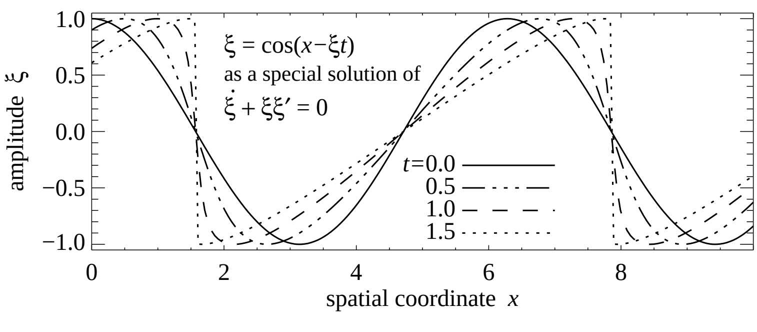
$$\xi = 3c - 3c_0 \qquad \Rightarrow \qquad \frac{\partial \xi}{\partial t} + \xi\,\frac{\partial \xi}{\partial x} = 0\,. \tag{7.31}$$

A general solution of this simple looking equation is

$$\xi = F(x - \xi t)\,, \tag{7.32}$$

as is easily proven. This solution implies that larger values of ξ propagate faster in the $+x$-direction than smaller ones. The figure below shows the evolution of $\xi = \cos(x - \xi t)$, which is one special function solving (7.31). Here one should recall that $c = \sqrt{g\,h}$ and $\xi = 3c - 3c_0$, i.e. that in the present case ξ basically describes the height of the surface of the water. Thus the parts of a perturbation of the surface which are at greater height ξ will propagate faster than those at lower height, so that ultimately the wave is steepening, as is demonstated in the figure below

This, of course, is well known from the breaking of waves on the ocean.



Temporal evolution of a special solution of (7.31). Very quickly the wave steepens and starts forming a shock front.

7.4 Solitons: the Korteweg-de Vries (KdV) equation

In Sect. 7.1 and especially in Sect. 7.1.5 we have seen, that a surface wave (on deep water) of small amplitude will vanish after some time because of dispersion (even in the absence of viscosity). However, when accounting for the non-linear character of the hydrodynamic equations, as done in Sect. 7.3, we see that a wave on shallow water will rapidly steepen. One can now think of a situation, where the steepening and dispersive effects just balance and a perturbation will propagate on the surface of the water without changing its shape. Such a phenomenon is called a *solitary wave* or *soliton* and is encountered not only in surface waves, but in many other branches of physics.

The first report of a soliton is from Scott Russel. In 1834 he observed a boat on a canal, which was drawn by horses, inducing a wave after suddenly being stopped. He followed this perturbation traveling with about 12 km/hour, first on foot then on horseback for more than a mile. The smooth perturbation was about 30 cm in height and about 9 m long and pretty much kept its shape. Later he studied such solitary waves in the laboratory.

In 1895 Korteweg and de Vries suggested a non-linear differential equation (7.34), which should describe the propagation of a surface perturbation in a canal. Actually one might use the propagation of an electro-magnetic wave between two parallel conducting planes as an inspiration. Here we will not discuss how this equation is derived, but just give the result, and then show that in two limiting cases one just finds the same results as presented above for the linear analysis with dispersion in Sect. 7.1 and for the non-linear steepening case in Sect. 7.3.

If the water has a depth of h_0, the amplitude of the perturbation is of the order of η_0, and its length scale is of order L, then for the case Korteweg and de Vries describe we have to assume

$$\frac{\eta_0}{h_0} \approx \frac{h_0^2}{L^2} \approx \text{“is small”} . \tag{7.33}$$

For the dispersive surface wave we assumed $\eta_0 \ll L \ll h_0$, for the non-linear surface wave in shallow water we used $\eta_0 \approx L \gg h_0$, now we assume that amplitude, wavelength and water depth are roughly comparable. Actually for the case Russel observed in 1834 (see above) we have η_0=0.3 m, L=10 m, and we might assume for the depth of the canal some h_0=3 m (which might be a bit too much...). Then both η_0/h_0 and h_0^2/L^2 in (7.33) are about 0.1.

With the phase speed of a (linear) shallow water wave $c_0 = \sqrt{gh_0}$, see Sect. 2.8, equation (2.38), the suggestion for the differential equation describing a soliton in a channel, i.e. the height of the surface perturbation η, is the

$$\text{Korteweg-de Vries equation:} \qquad \frac{\partial\eta}{\partial t} + c_0\frac{\partial\eta}{\partial x} + \frac{3}{2}\frac{c_0}{h_0}\,\eta\,\frac{\partial\eta}{\partial x} + \frac{1}{6}c_0h_0^2\frac{\partial^3\eta}{\partial x^3} = 0 . \tag{7.34}$$

When either the 3rd term (non-linear) or the fourth term (dispersive) are neglected, this equation is consistent with the special cases discussed earlier in Sect. 7.1 and 7.3. This will be shown below in Sect. 7.4.3 and 7.4.4. Before that we will show that there are indeed solutions to the Korteweg-de Vries equation preserving their shape.

7.4.1 A soliton solution for the Korteweg-de Vries equation

As a general analytical solution of the Korteweg-de Vries equation (7.34) is not possible, we shall simply seek if it indeed has a soliton solution. For this we make an ansatz that

$$\eta = f(x-vt) \tag{7.35}$$

is a special solution, with f being an arbitrary function; f and its derivatives f', f'' and f''' should vanish for $x-vt \to \pm\infty$. Here the constant v describes the propagation speed of the soliton. For any function f the above ansatz provides a structure that does not change its form while propagating. Substituting this ansatz into the Korteweg-de Vries equation (7.34) gives

$$\big(c_0-v\big)\,f' + \frac{3}{2}\frac{c_0}{h_0}\,ff' + \frac{1}{6}\,c_0h_0^2\,f''' = 0 . \tag{7.36}$$

Now this can be integrated to give

$$\big(c_0-v\big)\,f + \frac{3}{4}\frac{c_0}{h_0}\,f^2 + \frac{1}{6}\,c_0h_0^2\,f'' = 0 , \tag{7.37}$$

where we used the boundary condition that f and its derivatives vanish for $x - vt \to \pm\infty$. Multiplying by $(2h_0/c_0)f'$ one can integrate again and together with the boundary condition we find

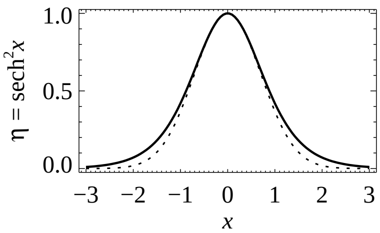
$$\frac{1}{3}\,h_0^3\,(f')^2 \;=\; (\hat{\eta} - f)\,f^2 \qquad \text{with} \qquad \hat{\eta} \;=\; 2\,h_0\left(\frac{v}{c_0} - 1\right) . \tag{7.38}$$

Finally we can take the square root, separate the variables and integrate using the substitution $f = \hat{\eta}\,\text{sech}^2\phi$ giving the solution

$$\eta \;=\; \hat{\eta}\,\text{sech}^2\left\{\left(\frac{3\,\hat{\eta}}{4\,h_0^3}\right)^{1/2}(x - vt)\right\} \qquad \text{with} \qquad v \;=\; c_0\left(1 + \frac{\hat{\eta}}{2\,h_0}\right) . \tag{7.39}$$

This is a *special* solution of the Korteweg-de Vries equation (7.34), but it should be kept in mind that it is not necessarily the only one! We only verified that the above ansatz (7.35) indeed is a solution.

$$\text{sech}^2 x \;=\; \frac{1}{\cosh^2 x} \;=\; \left(\frac{e^{-x} + e^{x}}{2}\right)^{-2}$$



The special solution represents a perturbation of the surface, which is roughly Gaussian like. The figure to the right shows the actual solution $\text{sech}^2 x$ (solid line) as compared to a Gaussian $\exp[-x^2]$ (dotted).

The solution (7.39) shows that a soliton is moving just a bit faster than a surface wave on shallow water with small amplitude. For Russel's observations mentioned at the beginning of the chapter we would find for the speed of the soliton about $v \approx 18$ km/hour, which roughly agrees with his estimation.

7.4.2 Propagation of the soliton

One of the most intriguing aspects of solitons is that they pass through each other (almost) without interaction. As a soliton with larger amplitude propagates faster, cf. (7.39), it will eventually overtake another soliton with smaller amplitude. After the overtaking procedure the two solitons remain unchanged! However there is a sign that an non-linear interaction took place, namely that the large amplitude soliton will arrive a bit early, the small amplitude soliton a bit late.

This feature of interaction-free passing through of solitons makes them quite special in physics. Since the discovery that the Korteweg-de Vries equation has a soliton solution, also other non-linear equations have been found allowing for solitons.

7.4.3 Neglecting the non-linear term of the KdV equation

The aim of this and the following subsection is to show that for two special cases the KdV equation gives the same results as found previously for dispersive and non-linear waves.

When we neglect the non-linear term in the KdV equation (7.34) it simplifies to

$$\frac{\partial\eta}{\partial t} \;+\; c_0\,\frac{\partial\eta}{\partial x} \;+\; \frac{1}{6}\,c_0 h_0^2\,\frac{\partial^3\eta}{\partial x^3} \;=\; 0 .$$

Substituting a wave ansatz $\eta = \hat{\eta}\cos(kx - \omega t)$ this gives the dispersion relation

$$\frac{\omega}{k} \;=\; c_0\left(1 + \frac{1}{6}\,h_0^2\,k^2\right) . \tag{7.40}$$

With the definition of $c_0 = \sqrt{g h_0}$ we can write the dispersion relation of surface waves on water with finite depth based on a linear analysis (7.17) as

$$\frac{\omega}{k} = c_0\left(\frac{\tanh(h_0 k)}{h_0 k}\right)^{1/2} . \tag{7.41}$$

When expanding this relation up to second order in $h_0 k$, one finds just the dispersion relation (7.40) for the KdV equation without the non-linear term. (The expansion is a bit lengthy, but doable...). In that way the KdV equation (for shallow water, i.e. $h_0 k \ll 1$) is consistent with the linear analysis of Sect. 7.1 and 7.2

7.4.4 Neglecting the dispersive term of the KdV equation

When neglecting the dispersive term $\frac{1}{6}c_0 h_0\, \partial^3\eta/\partial x^3$ in the KdV equation (7.34), only the non-linear term remains, and we should get a result similar to the discussion of non-linear waves in Sect. 7.3.

Therefore we start with the wave propagating into water at rest. Following (7.30) its evolution is described by

$$\frac{\partial c}{\partial t} + \big(3c - 2c_0\big)\frac{\partial c}{\partial x} = 0 ,$$

where through (7.26) the displacement η is hidden in $c = \sqrt{g\,h}$, with the height of the water h defining the displacement through $h = h_0 + \eta$, where h_0 is the height at rest. Furthermore we can use $c_0 = \sqrt{gh_0}$ and might write

$$c = c_0\left(1+\frac{\eta}{h_0}\right)^{\frac{1}{2}} \quad \Rightarrow \quad \frac{\partial c}{\partial t} = \frac{c_0}{2}\left(1+\frac{\eta}{h_0}\right)^{-\frac{1}{2}}\frac{\partial \eta}{\partial t} \quad ; \quad \frac{\partial c}{\partial x} = \frac{c_0}{2}\left(1+\frac{\eta}{h_0}\right)^{-\frac{1}{2}}\frac{\partial \eta}{\partial x} .$$

Substituting this in the equation above and multiplying it by $(2/c_0)\sqrt{1+\eta/h_0}$ gives

$$\frac{\partial \eta}{\partial t} + \left\{3\,c_0\left(1+\frac{\eta}{h_0}\right)^{\frac{1}{2}} - 2\,c_0\right\}\frac{\partial \eta}{\partial x} = 0 .$$

A Taylor expansion $\sqrt{1+\eta/h_0} \approx 1 + \frac{1}{2}\eta/h_0$ for small amplitudes $\eta \ll h_0$ gives

$$\frac{\partial \eta}{\partial t} + c_0\frac{\partial \eta}{\partial x} + \frac{3}{2}\frac{c_0}{h_0}\,\eta\,\frac{\partial \eta}{\partial x} = 0 .$$

And this is exactly the KdV equation (7.34) neglecting the dispersive term!

This discussion in the last two subsection shows that the KdV equation is correctly accounting for the dispersive and non-linear cases we discussed earlier in Sect. 7.1, 7.2 and 7.3.