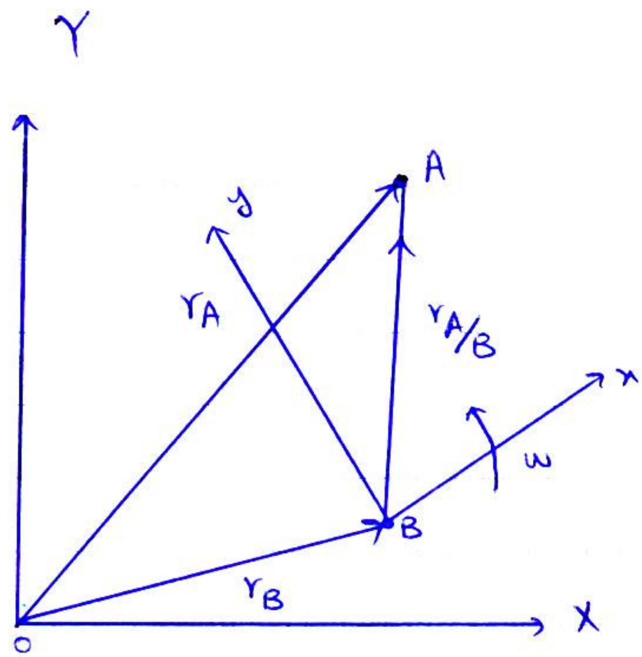
دینامیک

دکتر رضا انصاری

« درسنامه دینامیک»

سینماتیک و سنتیک اجسام صلب :



$$r_A = r_B + r_{\frac{A}{B}}$$

$$r_{\frac{A}{B}} = \vec{r} = x\vec{\imath} + y\vec{\jmath}$$

$$\dot{r} = (\dot{x}i + \dot{y}j) + (x\dot{i} + y\dot{j}) \longrightarrow \dot{r} = v_{rel} + \omega \times r$$

$$r_A = r_B + r_{\frac{A}{B}} \longrightarrow \dot{r}_A = \dot{r}_B + \dot{r}_{\frac{A}{B}}$$

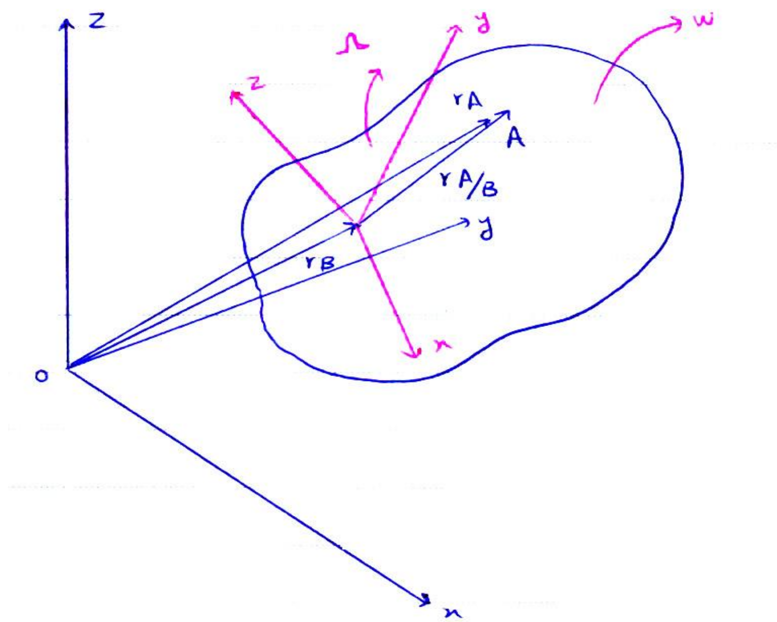
$$v_A = v_B + \omega \times r + v_{rel}$$

$$a_A = a_B + \dot{\omega} \times r + \omega \times (\omega \times r) + 2\omega \times v_{rel} + a_{rel}$$

$$V_{rel} = (\dot{x}i + \dot{y}j)$$

$$\dot{v}_{rel} = (\ddot{x}i + \ddot{y}j) + \left(\dot{x}\dot{i} + \dot{y}\dot{j}\right)$$

$$\dot{V}_{rel} = a_{rel} + \omega \times v_{rel}$$



$$\dot{\vec{i}} = \Omega \times \vec{i}$$

$$\dot{\vec{j}} = \Omega \times \vec{j}$$

$$\dot{k} = \Omega \times \vec{k}$$

$$r_A = r_B + r_{\frac{A}{B}}$$

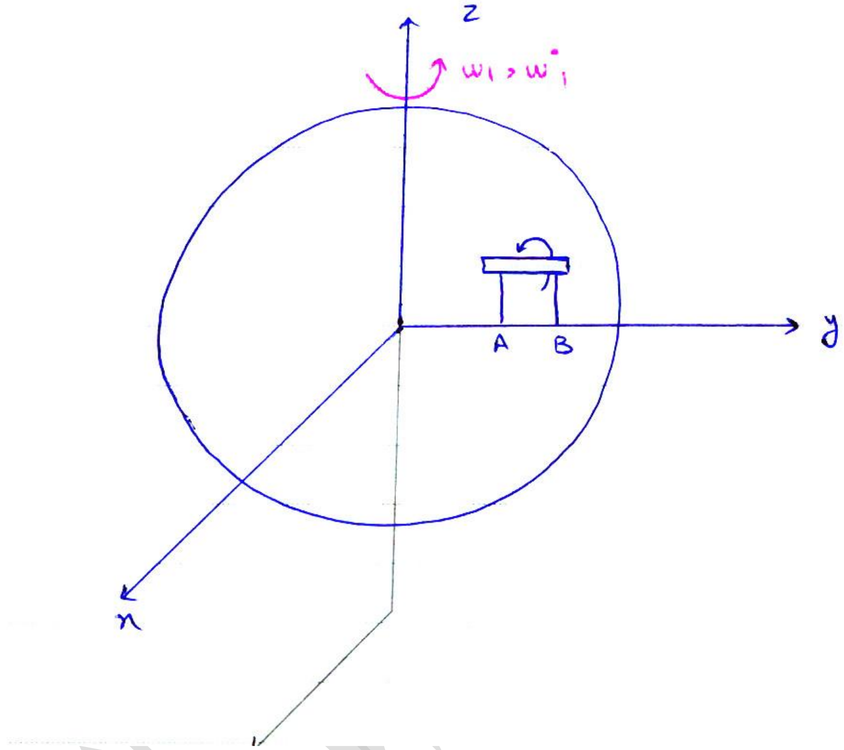
$$v_A = v_B + \Omega \times r_{\frac{A}{B}} + v_{rel}$$

$$\left(\frac{d}{dt}[\,]\right)_{XYZ} = \left(\frac{d}{dt}[\,]\right)_{XYZ} + \Omega \times [\,]$$

$$\left(\frac{d^2}{dt^2}[\,]\right)_{XYZ} = \left(\frac{d^2}{dt^2}[\,]\right)_{XYZ} + \dot{\Omega} \times [\,] + \Omega(\Omega \times [\,]) + 2\Omega \times \left(\frac{\mathrm{d}[\,]}{\mathrm{dt}}\right)_{\mathrm{XYZ}}$$

$$a_A = a_B + \dot{\Omega} \times \mathrm{r}_{\frac{A}{B}} + \Omega \times \left(\Omega \times \mathrm{r}_{\frac{A}{B}}\right) + 2\Omega \times \mathrm{v}_{\mathrm{rel}} + \mathrm{a}_{\mathrm{rel}}$$

مثال) مطلوبست تعیین سرعت و شتاب زاویه ای میله ی AB ؟



$$\vec{\Omega}_{AB} = \omega_2\vec{j} + \omega_1\vec{k}$$

$$\overrightarrow{\alpha_{AB}} = 0\times\vec{j} + \omega_1\vec{k}\times\omega_2\vec{j} + \dot{\omega}_1\vec{k} + 0 = -\omega_1\omega_2\vec{i} + \dot{\omega}_1\vec{k}$$

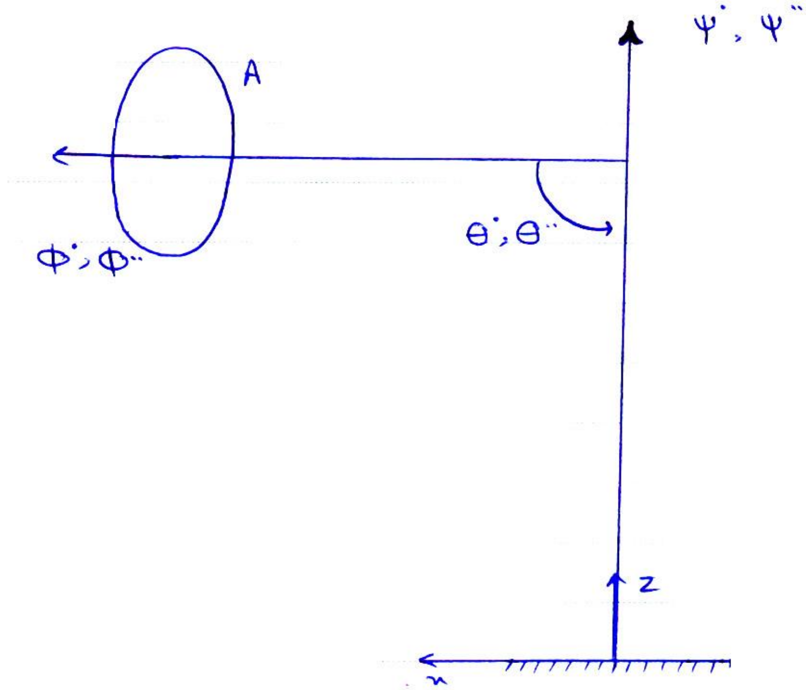
حال فرض می کنیم که ω وجود دارد و یک ω جدید به مجموعه سیتم در جهت i وارد می کنیم.

$$\overrightarrow{\Omega_{AB}} = \omega_2\vec{j} + \omega_1\vec{k} + \omega\vec{i}$$

$$\overrightarrow{\alpha_{AB}} = \dot{\omega}_2\vec{j} + \left(\omega_1\vec{k} + \omega\vec{i}\right)\times(\omega_2\vec{j}) + \dot{\omega}_1\vec{k} + \omega\vec{i}\times\left(\omega_1\vec{k}\right) + \dot{\omega}\vec{i} + 0$$

$$\overrightarrow{\alpha_{AB}} = (\dot{\omega} - \omega_1\omega_2)\vec{i} + (\dot{\omega}_2 - \omega_1\omega)\vec{j} + (\dot{\omega}_1 + \omega\omega_2)\vec{k}$$

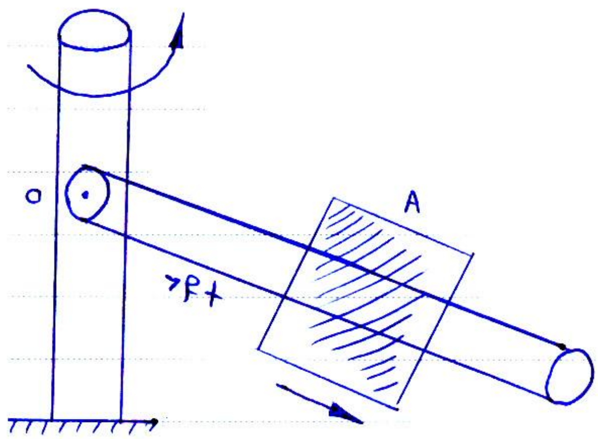
مثال) سرعت و شتاب زاویه ای دیسک A را محاسبه کنید ؟



$$\overrightarrow{\Omega_A} = \dot{\varphi}\vec{i} + \dot{\theta}\vec{j} + \dot{\psi}\vec{k}$$

$$\overrightarrow{\alpha_A} = \ddot{\varphi}\vec{i} + (\dot{\theta}\vec{j} + \dot{\psi}\vec{k}) \times \dot{\varphi}\vec{i} + \ddot{\theta}\vec{j} + (\dot{\psi}\vec{k}) \times \dot{\theta}\vec{j} + \ddot{\psi}\vec{k} + 0$$

مثال) سرعت و شتاب لغزنده ی A را محاسبه کنید ؟



$v_A = v_o + \omega \times oA + v_{rel}$

$v_o = 0$

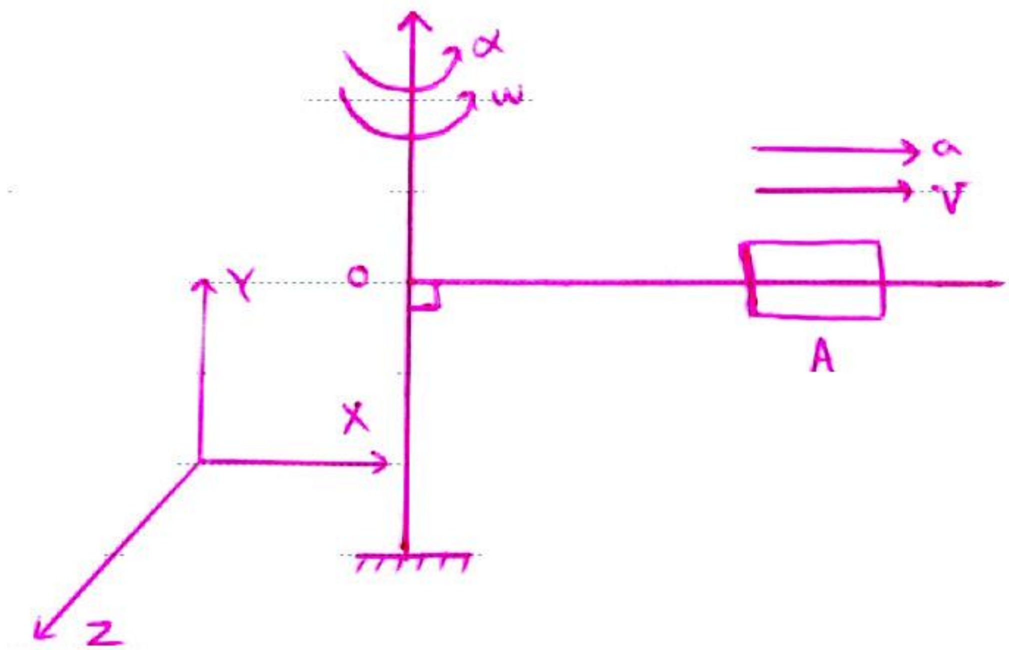
$\omega = 5j$

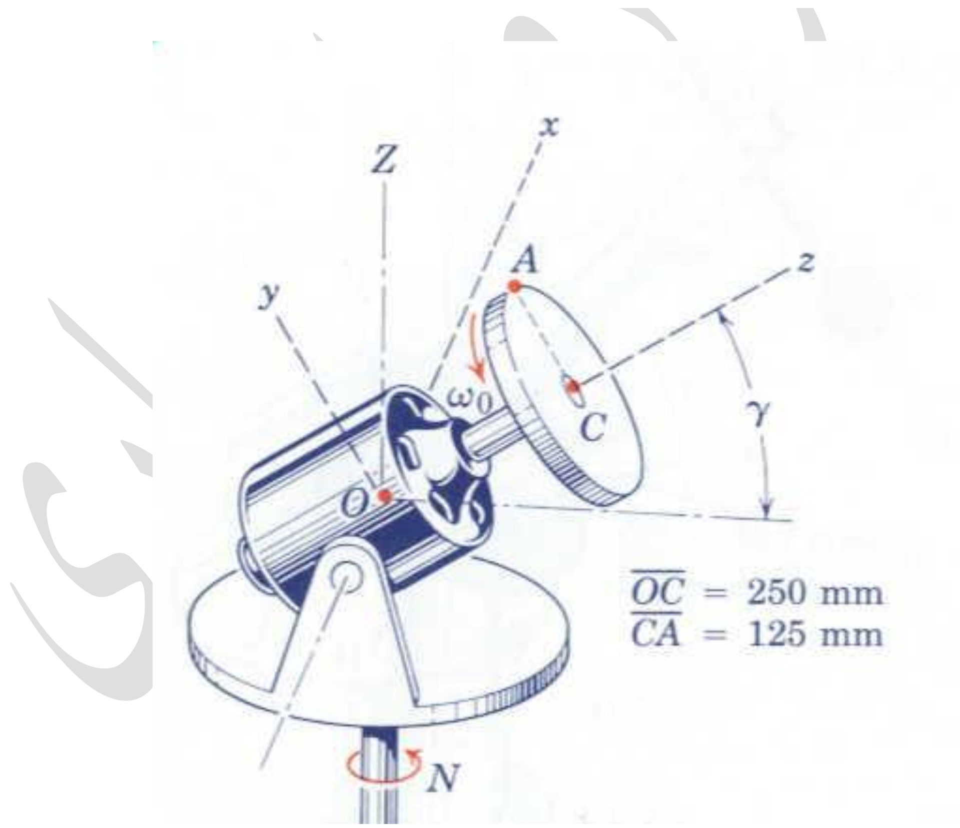
$oA = 7\vec{i}$

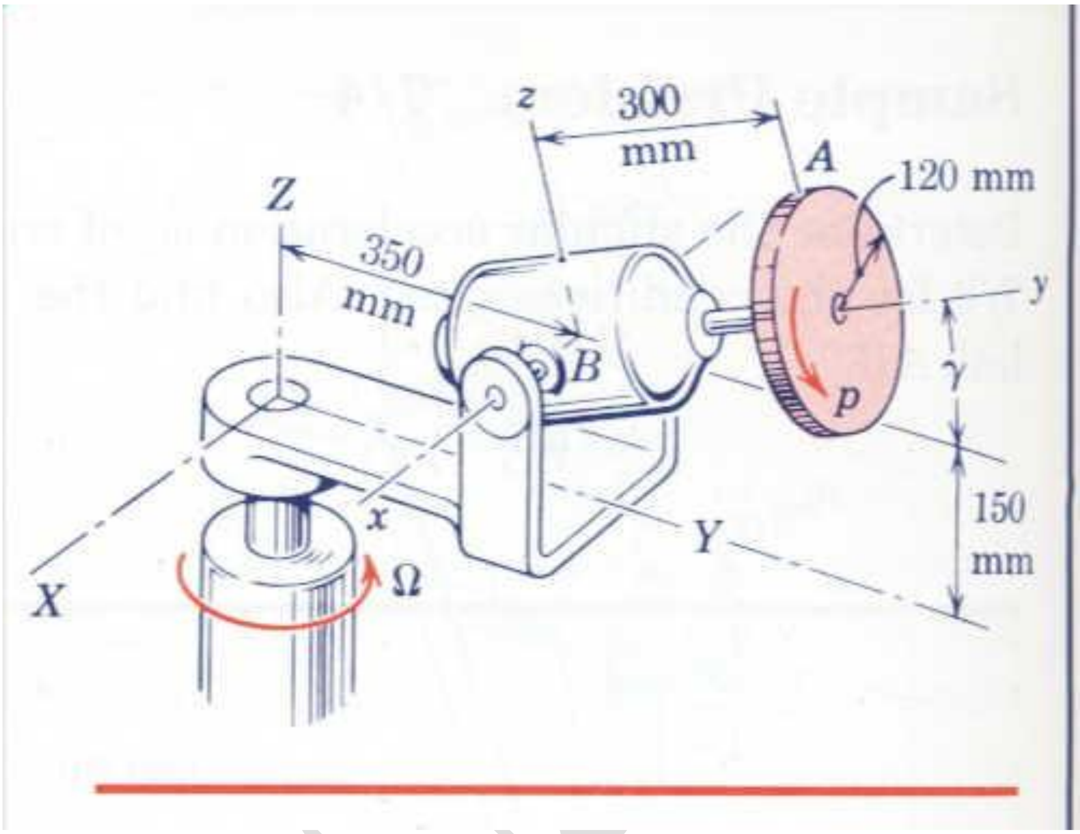
$v_{rel} = 10\vec{i}$

$a_A = a_O + \dot{\omega} \times oA + \omega \times (\omega \times oA) + 2\omega \times v_{rel} + a_{rel}$

$\vec{\dot{\omega}} = 5\vec{j} \quad , \overrightarrow{a_{rel}} = 4\vec{i}$


α
ω
a
V
Y
o
A
X
Z


x
Z
y
A
z
ω_0
C
γ
O
N
$\overline{OC}$ = 250 mm
$\overline{CA}$ = 125 mm



$$\overrightarrow{H_G} = \sum \overrightarrow{\rho_i} \times m_i \overrightarrow{\rho_i} = \int \vec{\rho} \times (\omega \times \rho) dm$$

$$H_G = \left(\int r^2 dm \right) \vec{\omega}$$

$$H_G = I\,\omega$$

$$I = \int \rho^2 dm$$

$$\sum M_G = \dot{H}_G = I\alpha$$

$$G = m\bar{v}$$

$$H_G = (H_G)_{abs} = (H_G)_{rel} = \sum \rho_i \times m_i \dot{\rho}_i$$

$$H_o = \sum r_i \times m_i v_i$$

$$(H_p)_{rel} = H_G + \bar{\rho} \times m\bar{v}_{rel}$$

$$\sum F = \dot{G} = m\bar{a}$$

$$\sum M_G = \dot{H}_G$$

$$\sum M_O = \dot{H}_O$$

$$\sum M_p = \dot{H}_G + \bar{\rho} \times m\bar{a} = (\dot{H}_p)_{rel} + \bar{\rho} \times ma_p = (\dot{H}_p)_{rel} - a_p \times m\bar{\rho}$$

$$H_G = \sum \rho_i \times m\dot{\rho}_i = \int \rho \times (\omega \times \rho)dm$$

$$H_O = \sum r_I \times m\dot{r}_i = \int r \times (\omega \times r)dm$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{\omega} = \omega_x\vec{i} + \omega_y\vec{j} + \omega_z\vec{k}$$

$$dH = \vec{i}[(y^2 + z^2)\omega_x - (xy)\omega_y - (xz)\omega_z]dm + \vec{j}[-(yx)\omega_x + (z^2 + x^2)\omega_y - (yz)\omega_z]dm + k[-(zx)\omega_x - (zy)\omega_y + (x^2 + y^2)\omega_z]dm$$

$$I_{xx} = \int (y^2 + z^2)dm\,,\ I_{yy} = \int (z^2 + x^2)dm\,,I_{zz} = \int (x^2 + y^2)\,dm$$

$$I_{xy} = \int xy\,dm \ , \qquad I_{xz} = \int xz\,dm\ ,\ I_{yz} = \int yz\,dm$$

$$H = (I_{xx}\omega_x - I_{xy}\omega_y - I_{xz}\omega_z)\vec{i} + (-I_{yx}\omega_x + I_{yy}\omega_y - I_{yz}\omega_z)\vec{j} + (-I_{zx}\omega_x - I_{zy}\omega_y + I_{zz}\omega_z)\vec{k}$$

$$\{H\} = [I]\{\omega\} \quad , \quad [I] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

$$\{H\} = \lambda\{\omega\}$$

$$[I]\{\omega\} = \lambda\{\omega\}$$

$$([I] - \lambda[I]_d)\{\omega\} = 0$$

$$|([I] - \lambda[I])| = 0$$

$$H_z = I_{zz}\omega_z$$

$$H = I\omega$$

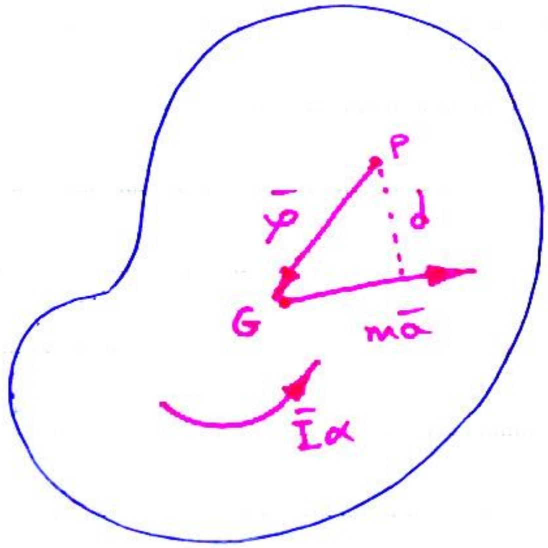
$$H_G = \bar{I}\omega$$

$$\dot{H}_G = \bar{I}\alpha$$

$$\begin{cases} \sum F = m\bar{a} \\ \sum M_G = \bar{I}\alpha \end{cases}$$

$$\sum M_O = I_O\alpha$$

$$\sum M_P = \bar{I}\alpha + m\bar{a}d$$



برای انتقال کامل (جابجایی محض) $\begin{cases}\sum F = m\bar{a} \\ \sum M_G = I\alpha\end{cases}$ و برای زمانی که چرخش حول یک نقطه داریم :

$$\sum M_O = \bar{I}\alpha + m(\bar{r}\alpha)\bar{r} = (\bar{I} + m\bar{r}^2)\alpha = I_O\alpha$$

$$\sum F = \dot{G}$$

$$\sum M = \dot{H}$$

$$\sum M = \left(\frac{dH}{dt}\right)_{xyz} = \left(\frac{dH}{dt}\right)_{xyz} + \Omega \times H = \left(\dot{H}_x i + \dot{H}_y j + \dot{H}_z k\right) + \Omega \times H$$

$$\Omega \times H = \begin{vmatrix} i & j & k \\ \Omega_x & \Omega_y & \Omega_z \\ H_x & H_y & H_z \end{vmatrix}$$

$$\begin{cases} \sum M_x = \dot{H}_x - H_y\omega_z + H_z\omega_y \\ \sum M_y = \dot{H}_y - H_z\omega_x + H_x\omega_z \\ \sum M_z = \dot{H}_z - H_x\omega_y + H_y\omega_x \end{cases}$$

$if \quad \omega = \Omega$

$$\sum M_x = \dot{\omega}_x I_{xx} + \omega_y\omega_z(I_{zz} - I_{yy}) + I_{xy}(\dot{\omega}_z\omega_x - \dot{\omega}_y) - I_{xz}(\dot{\omega}_z + \omega_y\omega_z) - I_{yz}({\omega_y}^2 - \omega^2{}_z)$$

$$\begin{cases} \sum M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \\ \sum M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \\ \sum M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \end{cases}$$

برای حرکت صفحه ای داریم :

$$\begin{cases} H_x = -I_{xz}\omega_z \\ H_y = -I_{yz}\omega_z \\ H_z = -I_{zz}\omega_z \end{cases}$$

$$\sum M_x = -I_{xz}\dot{\omega}_z + I_{yz}\omega^2{}_z$$

$$\sum M_y = -I_{yz}\dot{\omega}_z + I_{xz}\omega^2{}_z$$

$$\sum M_z = I_{zz}\dot{\omega}_z$$

هنگامی که در معادلات فوق بعد z نیز قابل اغماض باشد داریم :

$$H_x = 0$$

$$H_y = 0$$

$$H_z = -I_{zz}\omega_z$$

$$\sum M_x = 0$$

$$\sum M_y = 0$$

$$\sum M_z = I_{zz}\dot{\omega}_z$$

$$T = \frac{1}{2}m\overline{v^2} + \frac{1}{2}m_i\dot{\rho}_i.\dot{\rho}_i$$

$$= \frac{1}{2}\sum m_i\,(\vec{\omega}\times\overrightarrow{\rho_i}).(\vec{\omega}\times\vec{\rho}_i)$$

$$= \frac{1}{2}\sum m_i\,\vec{\omega}.\vec{\rho}\times(\vec{\omega}\times\vec{\rho}_i)$$

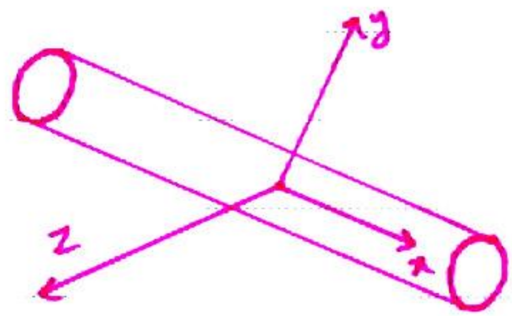
$$= \frac{1}{2}\vec{\omega}\sum m_i\,.\vec{\rho}_i\times(\vec{\omega}\times\vec{\rho}_i)$$

$$\frac{1}{2}\vec{\omega}\sum \rho\times(\omega\times\rho) = \frac{1}{2}\vec{\omega}\overrightarrow{H_G}$$

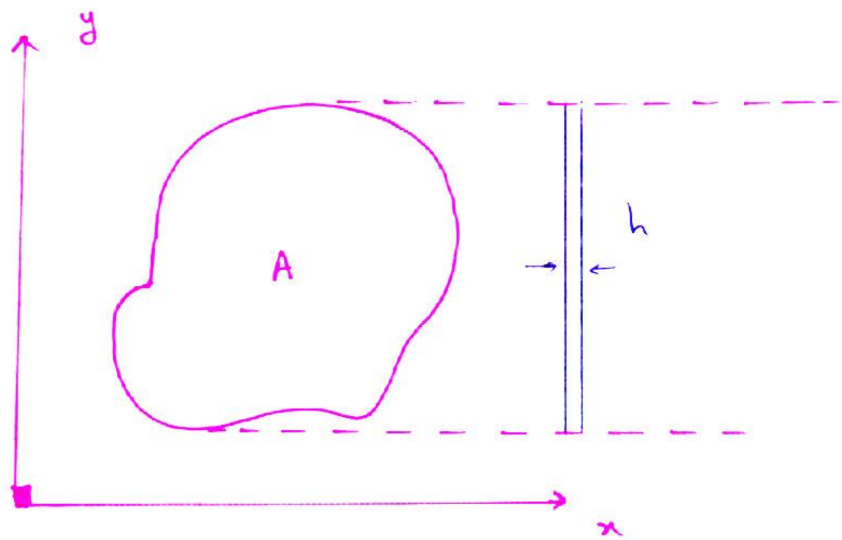
$$\longrightarrow T = \frac{1}{2}\bar{v}.G + \frac{1}{2}\omega H_G$$

$$T = \frac{1}{2}\vec{\omega}\overrightarrow{H_G}$$

دو بعدی $\longrightarrow \begin{cases} T = \frac{1}{2}m\overline{v^2} + \frac{1}{2}\bar{I}\omega^2 \\ T = \frac{1}{2}I_0\omega^2 \end{cases}$



$$I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}ml^2 & 0 \\ 0 & 0 & \frac{1}{12}ml^2 \end{bmatrix}$$



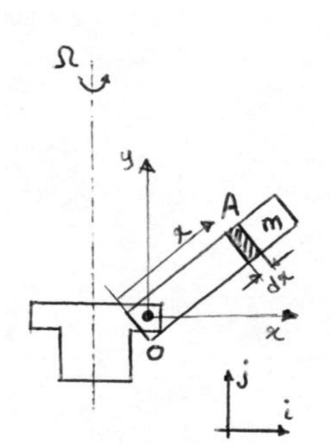
$$m = \rho A h$$

$$I_{xx} = \int (y^2 + z^2) dm = \rho h \int y^2 dA = \rho h I_x^A = \frac{m}{A} I_x^A$$

$$I = \frac{m}{A}\begin{vmatrix} I_x^A & -I_x^A & 0 \\ -I_x^A & I_x^A & 0 \\ 0 & 0 & J_O \end{vmatrix}$$

$$I_o = I_x + I_y$$

مثال) معادله حرکت پره هلیکوپتر را برای حرکت بالی آن تعیین نمائید. پره را بعنوان یک میله متناجس به جرم m در نظر گرفته واز وزن آن صرف نظر کنید.



$$a_A = a_o + \Omega \times \Omega + \mathrm{OA} + \dot{\Omega} \times \mathrm{OA} + 2 \times \Omega \times \mathrm{V_{rel}} + \mathrm{a_{rel}}$$

$$a_o = -R\Omega^2\mathrm{i} \quad , \Omega = \Omega\mathrm{j}$$

$$OA = xcos\theta i + xsin\theta j \ \ , \dot{\Omega} = 0$$

$$V_{rel} = -x\dot{\theta}sin\theta i + x\dot{\theta}cos\theta j$$

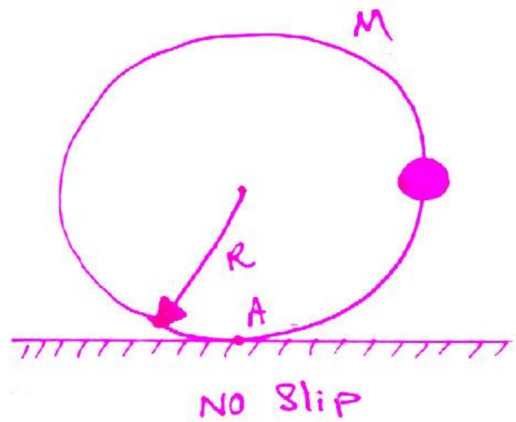
$$a_{rel} = -x\ddot{\theta}sin\theta i + x\ddot{\theta}cos\theta j - x\dot{\theta}^2cos\theta i - x\dot{\theta}^2sin\theta j$$

$$\Longrightarrow a_A = -\left(R\Omega^2 + \mathrm{x}\Omega^2\cos\theta + \mathrm{x}\ddot{\theta}\sin\theta + \mathrm{x}\dot{\theta}^2\cos\theta\right)\mathrm{i} + \left(\mathrm{x}\ddot{\theta}\cos\theta - \mathrm{x}\dot{\theta}^2\sin\theta\right)\mathrm{j} + 2\Omega\dot{\theta}\mathrm{x}\sin\theta\ \mathrm{k}$$

$$dF = \rho dx a_S \quad M_{oz} = 0$$

$$M_{oz} = \int OA \times d\,F_A \implies \frac{1}{3}mL^2\ddot{\theta} + \left(\frac{1}{2}mRL\Omega^2 + \frac{1}{3}mL^2\Omega^2cos\theta\right)sin\theta = 0$$

مثال) شتاب زاویه ی حلقه ای به جرم m در لحظه ی رها شدن نشان داده شده در شکل زیر را بدست آورید؟

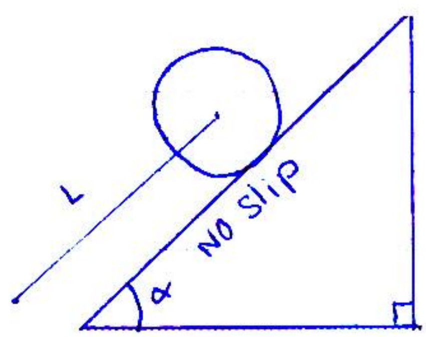


$$\sum M_A = I_A \alpha$$

$$I = \bar{I} + m_d^2$$

$$mgR = \left[(mR^2 + MR^2) + m\left(\sqrt{2}R\right)^2\right]\alpha \rightarrow \alpha = \frac{mg}{2(M+m)R}$$

مثال) دیسکی به جرم m که از بالای یک سطح شیب دار رها می شود سرعت آن را در انتهای آن با فرض غلط بدون لغزش بدست آورید ؟



راه حل اول :

$$mgsin\alpha - f = ma$$

$$f\ r = I\alpha = \left(\frac{1}{2}mr^2\right)\alpha \rightarrow f = \frac{1}{2}m\alpha$$

$$a = \frac{2}{3}gsin\alpha \ \rightarrow v = \sqrt{2aL} = 2\sqrt{\frac{gLsin\alpha}{3}}$$

راه حل دوم :

$$mgsin\alpha.r = \left(\frac{1}{2}mr^2 + mr^2\right)\alpha \rightarrow a = \frac{2}{3}gsin\alpha$$

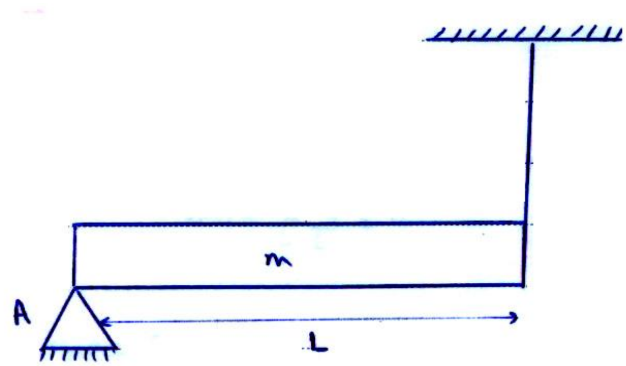
راه حل سوم :

$$v = r\omega$$

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}mr^2\right)\left(\frac{v}{r}\right)^2 = \frac{3}{4}mv^2$$

$$if\ \ T = U \ \ \rightarrow \ \ \frac{3}{4}mv^2 = mglsin\alpha \ \ \rightarrow \ \ v = 2\sqrt{\frac{gLsin\alpha}{3}}$$

مثال) در شکل زیر نیروی تکیه گاه A را در لحظه ی بریده شدن طناب بدست آورید ؟



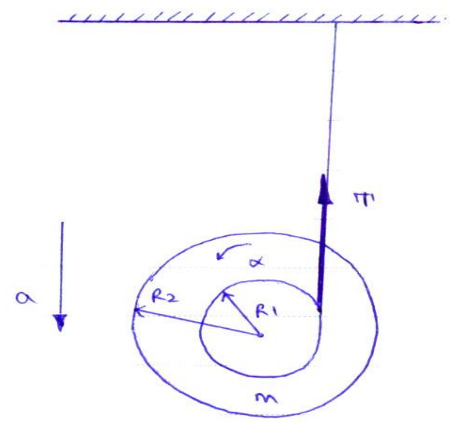
$$R_A = \frac{mg}{2}(static)$$

$$\sum M_A = I\alpha$$

$$\rightarrow \quad mg\frac{L}{2} = \left[\frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2\right]\alpha \quad \rightarrow \quad \alpha = \frac{3g}{2L}$$

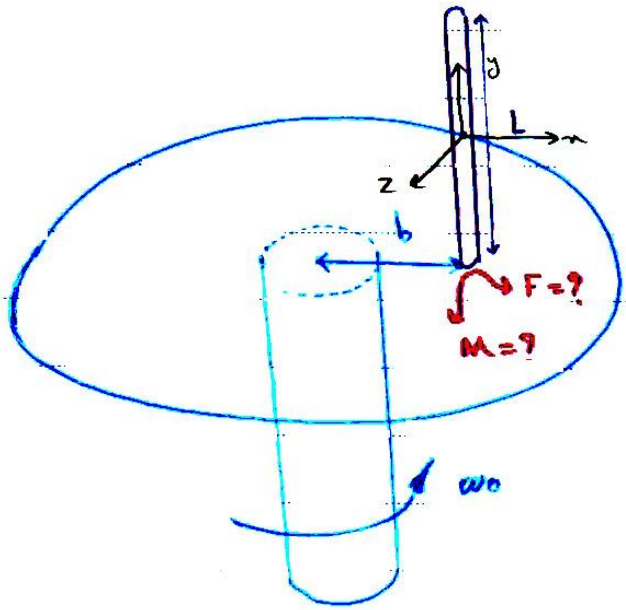
$$\begin{cases} mg - R_A = ma \\ a = \frac{L}{2}\alpha \end{cases} \rightarrow R_A = \frac{mg}{4}$$

مثال) مطلوبست تعیین شتاب زاویه ای در قرقره نشان داده ؟



$$\begin{cases} mg - T = ma \\ TR_1 = I\alpha \\ a = R_1\alpha \end{cases} \rightarrow \quad \alpha = \frac{g}{R_1}.\frac{1}{1 + \frac{I}{mR_1^2}}$$

مثال) نیرو و کوپل اعمالی از طرف دیسک به میله را در شکل زیر محاسبه نمایید ؟



$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ & I_{yy} & -I_{yz} \\ & & I_{zz} \end{bmatrix} \begin{Bmatrix} \dot{\omega_x} \\ \dot{\omega_y} \\ \dot{\omega_z} \end{Bmatrix} + \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix} [I] \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

$$I = \begin{bmatrix} \frac{1}{12} mL^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12} mL^2 \end{bmatrix}$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_z \end{Bmatrix} = 0 + \begin{bmatrix} 0 & 0 & \omega_0 \\ 0 & 0 & 0 \\ -\omega_0 & 0 & 0 \end{bmatrix} [I] \begin{Bmatrix} 0 \\ \omega_0 \\ 0 \end{Bmatrix}$$

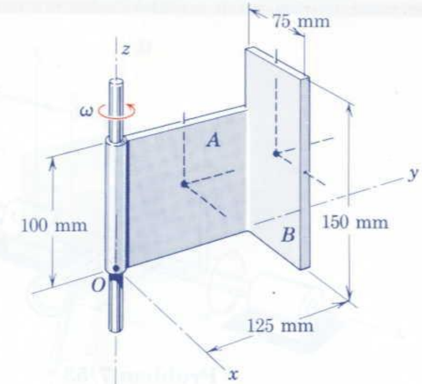
$$\begin{cases} M_x = 0 \\ M_y = 0 \\ M_z = 0 \end{cases} \rightarrow \qquad \begin{aligned} &F - mg\vec{j} = m(-b\omega_0^2)\vec{i} \\ &F = mg\vec{j} - m\omega_0^2\vec{i} \\ &M = M_0 + \left(-\frac{L}{2}\vec{j}\right) \times (mg\vec{j} - mb\omega^2{}_0\vec{i}) \end{aligned}$$

$$M = (M_o)_x\vec{i} + (M_o)_y\vec{j} + \left((M_o)_z - \frac{1}{2}mLb\omega_0^2\right)\vec{k}$$

$$\begin{cases} M_x = 0 & \rightarrow (M_o)_x = 0 \\ M_y = 0 & \rightarrow (M_o)_y = 0 \\ M_z = 0 & \rightarrow (M_o)_z = \dfrac{1}{2}mbL\omega_0^2 \end{cases}$$

Sample Problem 7/6

The bent plate has a mass of 70 kg per square meter of surface area and revolves about the z-axis at the rate $\omega = 30$ rad/s. Determine (a) the angular momentum **H** of the plate about point O and (b) the kinetic energy T of the plate. Neglect the mass of the hub and the thickness of the plate compared with its surface dimensions.



Solution. The moments and products of inertia are written with the aid of Eqs. B/3 and B/9 in Appendix B by transfer from the parallel ① centroidal axes for each part. First, the mass of each part is $m_A = (0.100)(0.125)(70) = 0.875$ kg, $m_B = (0.075)(0.150)(70) = 0.788$ kg.

Part A

$$[I_{xx} = \bar{I}_{xx} + md^2] \quad I_{xx} = \frac{0.875}{12}[(0.100)^2 + (0.125)^2] + 0.875[(0.050)^2 + (0.0625)^2] = 0.007\,47 \text{ kg}\cdot\text{m}^2$$

$$[I_{yy} = \tfrac{1}{3}ml^2] \quad I_{yy} = \frac{0.875}{3}(0.100)^2 = 0.002\,92 \text{ kg}\cdot\text{m}^2$$

$$[I_{zz} = \tfrac{1}{3}ml^2] \quad I_{zz} = \frac{0.875}{3}(0.125)^2 = 0.004\,56 \text{ kg}\cdot\text{m}^2$$

$$\left[I_{xy} = \int xy\,dm, \quad I_{xz} = \int xz\,dm\right] \quad I_{xy} = 0 \quad I_{xz} = 0$$

$$[I_{yz} = \bar{I}_{yz} + md_yd_z] \quad I_{yz} = 0 + 0.875(0.0625)(0.050) = 0.002\,73 \text{ kg}\cdot\text{m}^2$$

Part B

$$[I_{xx} = \bar{I}_{xx} + md^2] \quad I_{xx} = \frac{0.788}{12}(0.150)^2 + 0.788[(0.125)^2 + (0.075)^2] = 0.018\,21 \text{ kg}\cdot\text{m}^2$$

$$[I_{yy} = \bar{I}_{yy} + md^2] \quad I_{yy} = \frac{0.788}{12}[(0.075)^2 + (0.150)^2] + 0.788[(0.0375)^2 + (0.075)^2] = 0.007\,38 \text{ kg}\cdot\text{m}^2$$

$$[I_{zz} = \bar{I}_{zz} + md^2] \quad I_{zz} = \frac{0.788}{12}(0.075)^2 + 0.788[(0.125)^2 + (0.0375)^2] = 0.013\,78 \text{ kg}\cdot\text{m}^2$$

$$[I_{xy} = \bar{I}_{xy} + md_xd_y] \quad I_{xy} = 0 + 0.788(0.0375)(0.125) = 0.003\,69 \text{ kg}\cdot\text{m}^2$$

$$[I_{xz} = \bar{I}_{xz} + md_xd_z] \quad I_{xz} = 0 + 0.788(0.0375)(0.075) = 0.002\,21 \text{ kg}\cdot\text{m}^2$$

$$[I_{yz} = \bar{I}_{yz} + md_yd_z] \quad I_{yz} = 0 + 0.788(0.125)(0.075) = 0.007\,38 \text{ kg}\cdot\text{m}^2$$

The sum of the respective inertia terms gives for the two plates together

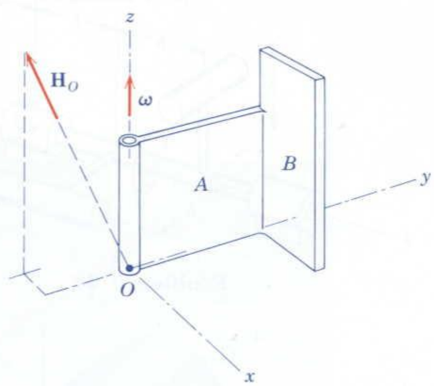
$$I_{xx} = 0.0257 \text{ kg}\cdot\text{m}^2 \qquad I_{xy} = 0.003\,69 \text{ kg}\cdot\text{m}^2$$
$$I_{yy} = 0.010\,30 \text{ kg}\cdot\text{m}^2 \qquad I_{xz} = 0.002\,21 \text{ kg}\cdot\text{m}^2$$
$$I_{zz} = 0.018\,34 \text{ kg}\cdot\text{m}^2 \qquad I_{yz} = 0.010\,12 \text{ kg}\cdot\text{m}^2$$

(a) The angular momentum of the body is given by Eq. 7/11, where $\omega_z = 30$ rad/s and ω_x and ω_y are zero. Thus,

② $$\mathbf{H}_O = 30(-0.002\,21\mathbf{i} - 0.010\,12\mathbf{j} + 0.018\,34\mathbf{k}) \text{ N}\cdot\text{m}\cdot\text{s} \qquad \textit{Ans.}$$

(b) The kinetic energy from Eq. 7/18 becomes

$$T = \tfrac{1}{2}\boldsymbol{\omega}\cdot\mathbf{H}_O = \tfrac{1}{2}(30\mathbf{k})\cdot 30(-0.002\,21\mathbf{i} - 0.010\,12\mathbf{j} + 0.018\,34\mathbf{k}) = 8.25 \text{ J}$$

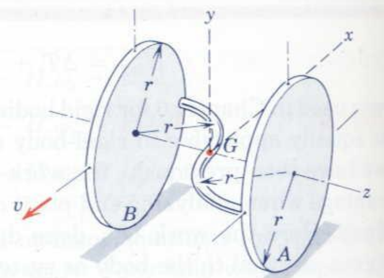


① The parallel-axis theorems for transferring moments and products of inertia from centroidal axes to parallel axes are explained in Appendix B and are most useful relations.

② Recall that the units of angular momentum may also be written in the base units as $\text{kg}\cdot\text{m}^2/\text{s}$.

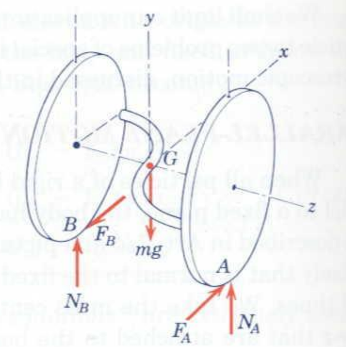
Sample Problem 7/7

The two circular disks, each of mass m_1, are connected by the curved bar bent into quarter-circular arcs and welded to the disks. The bar has a mass m_2. The total mass of the assembly is $m = 2m_1 + m_2$. If the disks roll without slipping on a horizontal plane with a constant velocity v of the disk centers, determine the value of the friction force under each disk at the instant represented when the plane of the curved bar is horizontal.



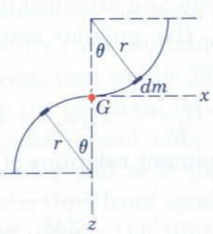
Solution. The motion is identified as parallel-plane motion since the planes of motion of all parts of the system are parallel. The free-body diagram shows the normal forces and friction forces at A and B and the total weight mg acting through the mass center G, which we take as the origin of coordinates that rotate with the body.

We now apply Eqs. 7/23, where $I_{yz} = 0$ and $\dot{\omega}_z = 0$. The moment equation about the y-axis requires determination of I_{xz}. From the diagram showing the geometry of the curved rod and with ρ standing for the mass of the rod per unit length, we have



① $\left[I_{xz} = \int xz\, dm\right] \qquad I_{xz} = \int_0^{\pi/2} (r \sin\theta)(-r + r\cos\theta)\rho r\, d\theta$

$$+ \int_0^{\pi/2} (-r\sin\theta)(r - r\cos\theta)\rho r\, d\theta$$



Evaluating the integrals gives

$$I_{xz} = -\rho r^3/2 - \rho r^3/2 = -\rho r^3 = -\frac{m_2 r^2}{\pi}$$

The second of Eqs. 7/23 with $\omega_z = v/r$ and $\dot{\omega}_z = 0$ gives

$[\Sigma M_y = -I_{xz}\omega_z^2] \qquad F_A r + F_B r = -\left(-\frac{m_2 r^2}{\pi}\right)\frac{v^2}{r^2}$

$$F_A + F_B = \frac{m_2 v^2}{\pi r}$$

But with $\bar{v} = v$ constant, $\bar{a}_x = 0$ so that

$[\Sigma F_x = 0] \qquad F_A - F_B = 0 \qquad F_A = F_B$

Thus,

$$F_A = F_B = \frac{m_2 v^2}{2\pi r} \qquad \textit{Ans.}$$

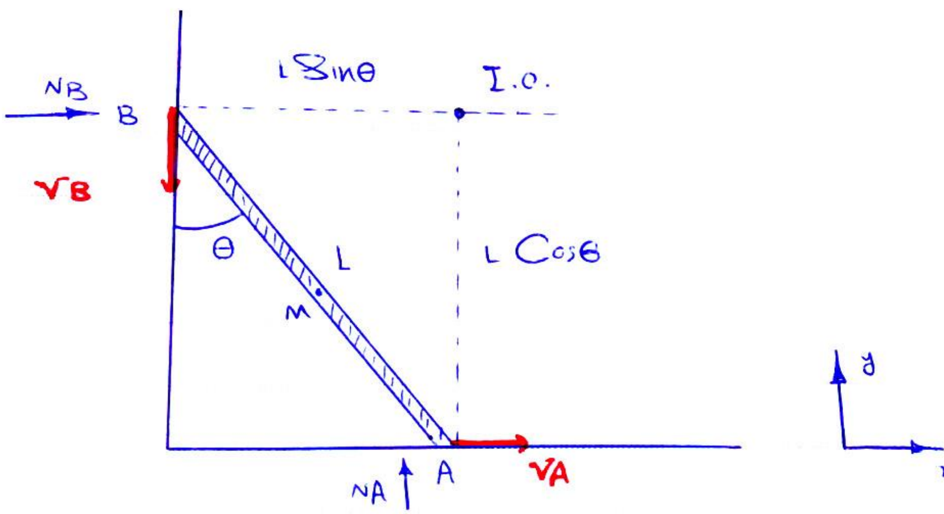
We also note for the given position that with $I_{yz} = 0$ and $\dot{\omega}_z = 0$, the moment equation about the x-axis gives

② $[\Sigma M_x = 0] \qquad -N_A r + N_B r = 0 \qquad N_A = N_B = mg/2$

① We must be very careful to observe the correct signs for each of the coordinates of the mass element dm that make up the product xz.

② When the plane of the curved bar is not horizontal, the normal forces under the disks are no longer equal.

مثال) میله ی AB به طول L بر روی کف و دیوار صافی می لغزد مطلوبست تعیین رابطه ی سرعت زاویه ای میله با سرعت دو انتهای آن ، شتاب زاویه ای میله و زاویه ای که میله دیوار را ترک می نماید و قتی میله از زاویه ی $\theta = \theta_0$ رها شده باشد ؟



$$\overrightarrow{v_B} = \overrightarrow{v_A} + \vec{\omega} \times \overrightarrow{r_{\frac{B}{A}}}$$

$$-v_B\vec{j} = v_A\vec{i} + \omega\vec{k} \times (-Lsin\theta\vec{i} + Lcos\theta\vec{j})$$

$$v_A = \omega Lcos\theta$$

$$v_B = \omega Lsin\theta$$

$$\overrightarrow{v_M} = \overrightarrow{v_A} + \vec{\omega} \times \overrightarrow{r_{AM}} = v_A\vec{i} + \omega\vec{k} \times \left(-\frac{L}{2}sin\theta\vec{i} + \frac{L}{2}cos\theta\vec{j}\right)$$

$$\rightarrow \ v_M = \frac{v_A}{2}(\vec{i} - tan\theta\vec{j})$$

راه حل دوم از طریق خود مرکز آن :

$$v_M = \frac{L\omega}{2}(cos\theta\vec{i} - sin\theta\vec{j}) = \frac{v_A}{2}(\vec{i} - tan\theta\vec{j})$$

$$\overrightarrow{a_M} = \overrightarrow{a_A} + \dot{\vec{\omega}} \times \overrightarrow{r_{AM}} + \vec{\omega} \times (\vec{\omega} \times \overrightarrow{r_{AB}})$$

$$a_{M_x}\vec{i} + a_{M_y}\vec{j} = a_A\vec{i} + \alpha\vec{k} \times \left(-\frac{L}{2}sin\theta\vec{i} + \frac{L}{2}cos\theta\vec{j}\right)$$

$$-\omega^2\left(-\frac{L}{2}sin\theta\vec{i} + \frac{L}{2}cos\theta\vec{j}\right)$$

$$a_{M_x} = \frac{L}{2}(\alpha sin\theta + \omega^2 cos\theta)$$

$$a_{M_y} = \frac{L}{2}(\alpha cos\theta - \omega^2 sin\theta)$$

$$N_B = m_{a_x}$$

$$mg - N_A = m_{a_y}$$

$$N_A\frac{L}{2}sin\theta - N_B\frac{L}{2}cos\theta = \left(\frac{1}{12}mL^2\right)\alpha$$

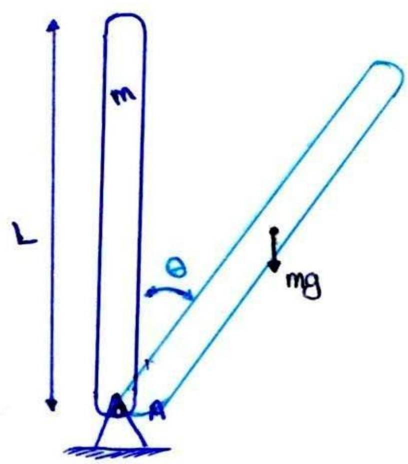
$$\alpha = \frac{3g}{2L}sin\theta$$

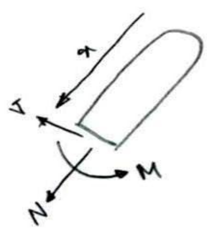
$$\int_0^\omega \frac{\omega d\omega}{d\theta} = \int_\theta^\theta \frac{3g}{2l}sin\theta$$

$$\omega = \frac{3g}{L}(cos\theta - cos\theta_0)$$

$$if\ \ N_B = 0 \quad \rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3}cos\theta_0\right)$$

مثال :





$$\sum M_A = I_A\alpha$$

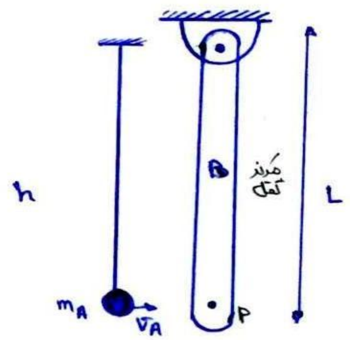
$$mg\sin\theta \times \frac{L}{2} = \frac{1}{3}mL^2\alpha$$

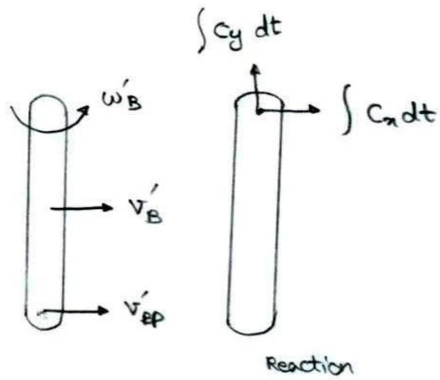
$$\alpha = \frac{3g}{2L}\sin\theta$$

$$M = \frac{mgL}{4}\sin\theta\left(\frac{x}{L}\right)^2\left(1-\frac{x}{L}\right)$$

$$\frac{dM}{dx} = 0 \ \rightarrow \ x = \frac{2}{3}L$$

مثال :





اینکه نیروی بعد از برخورد به چپ است یا راست معلوم نیست.

از علامت پریم برای بعد از برخورد استفاده می کنیم.

برای مجموعه می توان نوشت : $\sum M_{o} = 0$

$$e = \frac{v'_{BP} - v'_{A}}{v_{A} - 0}$$

قانون بقا را برای کل مجموعه می توان نوشت.

اندازه حرکت کل مجموعه ثابت است.

$$hm_A v_A = hm_A v_A' + \frac{1}{2} L m_B v_B{}' + \bar{I_B} w_B$$

$$v_B' = \frac{L}{2} w_B'$$

$$v_{BP}' = h w_B'$$

$$w_B' = \frac{(1+e) h m_A v_A}{h^2 m_A + \frac{1}{3} m_B L^2}$$

$$-h \int C_X \, dt = -\left(h - \frac{L}{2}\right) m_B v_B' + \bar{I_B} w'_B$$

$$\int C_x dt = \frac{\left(h - \frac{L}{2}\right) m_B v_B' - \bar{I_B} w'_B}{h} = C_{x(ave)} \, \Delta t$$

$$C_{X(ave)} = \frac{(1+e)\left(\frac{h}{2} - \frac{L}{3}\right) m_A m_B v}{\left(h^2 m_A + \frac{1}{3} m_B L^2\right) \Delta t}$$

C=0

$$h = \frac{2}{3} L$$

مرکز ضربه جایی است که در آن هیچ عکس ا لعملی نداریم.