Sharif University of Technology — Finite Element Method and Applications (20-149)

Civil Engineering Department — Assignment Number 1

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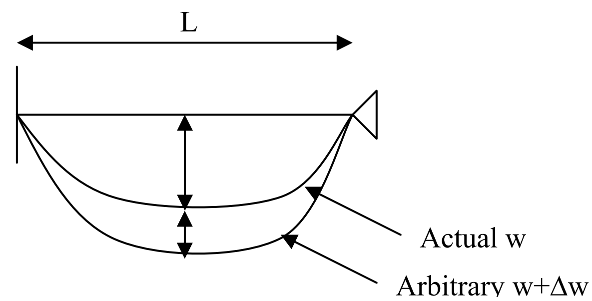
1) Potential Energy Theorem states that:
   Of all the w fields which satisfy compatibility and essential boundary conditions, the "actual one" which satisfies equilibrium and natural boundary conditions provides a minimum π (w).
   Prove the P.E.T. for the case of a beam

   Hint: Show $\pi(w + \Delta w) = \pi(w) + U(\Delta w)$

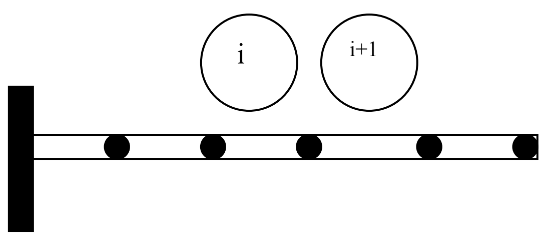
   When $U(\Delta w) = \frac{EI}{2}\int_0^l \Delta w_{xx}^{\ 2}\, dx$



2) Certain displacement continuity is a requirement for convergence of the finite element solution. This is historically the controversial point. You are asked to find continuity requirements for an element of beam such that the total potential energy of a structure would be equal to the sum of the P.E. of each element.

   Hint: Show $\pi_T = \sum_{i=1}^{no.\,of\ ele.} \pi_i$ $\qquad \pi_i = \text{potential energy of the } i^{th} \text{ element}$

   Calculate $= \pi_i\ (w + \Delta w) + \pi_{i+1}(w + \Delta w)$



3) The tapered beam shown in figure 3 is fixed at A, supported by a translational spring (K_1) at B and by a rotational spring (K_2) at c. The applied loads are shown.

   a) Construct the variational principle for this beam, and loads and boundary conditions as indicated.
   b) Perform the first variation and derive the boundary conditions that must be satisfied.
   c) How would such boundary conditions be satisfied in a two element solution?

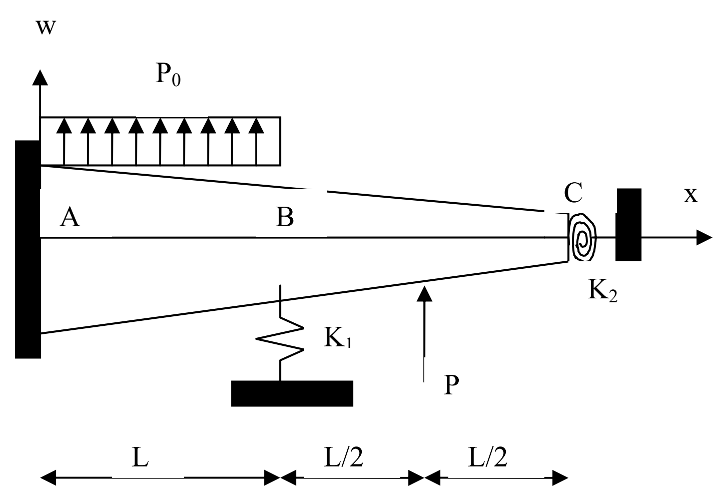


Figure 3

4) Consider the following variational principle:

$$J = \int_{\Omega} \frac{1}{2}\left\{ K_1\left(\nabla^2 w\right)^2 + K_2\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] + K_3 w^2 - qw \right\} dx\,dy$$
$$- \int_{SM} \overline{M}_n \frac{\partial w}{\partial n} ds + \int_{SQ} Q_n w ds$$

a) Perform first variation, i.e. δJ = 0, for extremum to derive the Eular equations (field equation and boundary conditions) and identify the essential and natural boundary conditions.
b) What is the number of rigid body modes?
c) Repeat part (b) if $K_3 = O$
d) Repeat part (b) if $K_2 = K_3 = O$

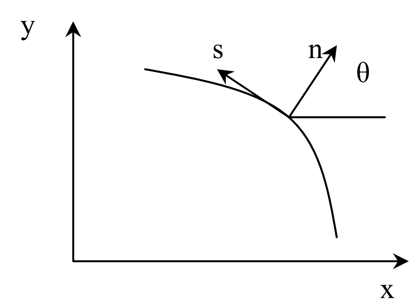
Use the following information:

$\nabla^2 w = w_{xx} + w_{yy}$ $\qquad\qquad$ $\nabla^4 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy}$

Coordinate transformation:

$$\begin{Bmatrix} n \\ s \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

$n = v_x \vec{i} + v_y \vec{j} \qquad \cos\theta = v_x \quad \sin\theta = v_y$



$$\frac{\partial w}{\partial x}=\frac{\partial w}{\partial n}\frac{\partial n}{\partial x}+\frac{\partial w}{\partial s}\frac{\partial s}{\partial x}=w_n\cos\theta-w_s\sin\theta$$

$$\frac{\partial w}{\partial y}=\frac{\partial w}{\partial n}\frac{\partial n}{\partial y}+\frac{\partial w}{\partial s}\frac{\partial s}{\partial y}=w_n\sin\theta+w_s\cos\theta$$

$$\frac{\partial^2 w}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)=\frac{\partial^2 w}{\partial n^2}\left(\frac{\partial n}{\partial x}\right)^2+2\frac{\partial^2 w}{\partial n\partial s}\left(\frac{\partial n}{\partial x}\right)\left(\frac{\partial s}{\partial x}\right)+\frac{\partial^2 w}{\partial s^2}\left(\frac{\partial s}{\partial x}\right)^2$$

$$w_{xx}=w_{nn}\cos^2\theta-2w_{ns}\sin\theta\cos\theta+w_{ss}\sin^2\theta$$

$$w_{yy}=w_{nn}\sin^2\theta+2w_{ns}\sin\theta\cos\theta+w_{ss}\cos^2\theta$$

$$w_{xy}=(w_{nn}-w_{ss})\sin\theta\cos\theta+2w_{ns}(\cos^2\theta-\sin^2\theta)$$

Also note that along the boundary $d_n=0$ *then* $d_x=-\sin\theta\, ds,\ dy=\cos\theta\, ds$
Integration by parts:

$$\int_\Omega \phi\frac{\partial\psi}{\partial x}\,dx\,dx=\oint\phi\psi\nu_x\,ds-\int_\Omega\psi\frac{\partial\phi}{\partial x}\,dx\,dy$$

$$\int_\Omega \phi\frac{\partial\psi}{\partial x}\,dx\,dx=\oint\phi\psi\nu_y\,ds-\int_\Omega\psi\frac{\partial\phi}{\partial y}\,dx\,dy$$

$$w_{xx}+w_{yy}=w_{nn}+w_{ss}$$

Caution: If you do not anticipate, ahead of time, many simplifications due to known results, this assignment can be unnecessarily long.
(please avoid this)

form of δJ

$$\delta J=\int_\Omega[......]\delta w\,dx\,dy\pm\int_{S-SQ}[......]\delta w\,ds\pm\int_{sq}[......]\delta w\,ds$$

$$\pm\int_{S-SM}[.......]\delta(\frac{\partial w}{\partial n})ds\pm\int_{SM}[......]\delta(\frac{\partial w}{\partial n})ds$$