ELASTICITY IN ENGINEERING MECHANICS

Elasticity in Engineering Mechanics, Third Edition Arthur P. Boresi, Ken P. Chong and James D. Lee

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PREFACE

The material presented is intended to serve as a basis for a critical study of the fundamentals of elasticity and several branches of solid mechanics, including advanced mechanics of materials, theories of plates and shells, composite materials, plasticity theory, finite element, and other numerical methods as well as nanomechanics and biomechanics. In the 21st century, the transcendent and translational technologies include nanotechnology, microelectronics, information technology, and biotechnology as well as the enabling and supporting mechanical and civil infrastructure systems and smart materials. These technologies are the primary drivers of the century and the new economy in a modern society.

Chapter 1 includes, for ready reference, new trends, research needs, and certain mathematic preliminaries. Depending on the background of the reader, this material may be used either as required reading or as reference material. The main content of the book begins with the theory of deformation in Chapter 2. Although the majority of the book is focused on stress–strain theory, the concept of deformation with large strains (Cauchy strain tensor and Green–Saint-Venant strain tensor) is included. The theory of stress is presented in Chapter 3. The relations among different stress measures, namely, Cauchy stress tensor, first- and second-order Piola–Kirchhoff stress tensors, are described. Molecular dynamics (MD) views a material body as a collection of a huge but finite number of different kinds of atoms. It is emphasized that MD is the heart of nanoscience and technology, and it deals with material properties and behavior at the atomistic scale. The differential equations of motion of MD are introduced. The readers may see the similarity and the difference between a continuum theory and an atomistic theory clearly. The theories of deformation and stress are treated separately to emphasize their independence of one another and also to emphasize their mathematical similarity. By so doing, one can clearly see that these theories depend only on approximations related to modeling of a continuous medium, and that they are independent of material behavior. The theories of deformation and stress are united in Chapter 4 by the introduction of three-dimensional stress–strain–temperature relations (constitutive relations). The constitutive relations in MD, through interatomic potentials, are introduced. The force–position relation between atoms is nonlinear and nonlocal, which is contrary to the situation in continuum theories. Contrary to continuum theories, temperature in MD is not an independent variable. Instead, it is derivable from the velocities of atoms. The treatment of temperature in molecular dynamics is incorporated in Chapter 4. Also the constitutive equations for soft biological tissues are included. The readers can see that not only soft biological tissue can undergo large strains but also exert an active stress, which is the fundamental difference between lifeless material and living biological tissue. The significance of active stress is demonstrated through an example in Chapter 6. The major portion of Chapter 4 is devoted to linearly elastic materials. However, discussions of nonlinear constitutive relations, micromorphic theory, and concurrent atomistic/continuum theory are presented in Appendices 4B, 4C, and 4D, respectively. Chapters 5 and 6 treat the plane theory of elasticity, in rectangular and polar coordinates, respectively. Chapter 7 presents the three-dimensional problem of prismatic bars subjected to end loads. Material on thermal stresses is incorporated in a logical manner in the topics of Chapters 4, 5, and 6.

General solutions of elasticity are presented in Chapter 8. Extensive use is made of appendixes for more advanced topics such as complex variables (Appendix 5B) and stress–couple theory (Appendixes 5A and 6A). In addition, in each chapter, examples and problems are given, along with explanatory notes, references, and a bibliography for further study.

As presented, the book is valuable as a text for students and as a reference for practicing engineers/scientists. The material presented here may be used for several different types of courses. For example, a semester course for senior engineering students may include topics from Chapter 2 (Sections 2-1 through 2-16), Chapter 3 (Sections 3-1 through 3-8), Chapter 4 (Sections 4-1 through 4-7 and Sections 4-9 through 4-12), Chapter 5 (Sections 5-1 through 5-7), as much as possible from Chapter 6 (from Sections 6-1 through Section 6-6), and considerable problem solving. A quarter course for seniors could cover similar material from Chapters 2 through 5, with less emphasis on the examples and problem solving. A course for first-year graduate students in civil and mechanical engineering and related engineering fields can include Chapters 1 through 6, with selected materials from the appendixes and/or Chapters 7 and 8. A follow-up graduate course can include most of the appendix material in Chapters 2 to 6, and the topics in Chapters 7 and 8, with specialized topics of interest for further study by individual students.

Special thanks are due to the publisher including Bob Argentieri, Dan Magers, and the production team for their interest, cooperation, and help in publishing this book in a timely fashion, to James Chen for the checking and proofreading of the manuscript, as well as to Mike Plesniak of George Washington University and Jon Martin of NIST for providing an environment and culture conductive for scholarly pursuit.

CHAPTER 1

INTRODUCTORY CONCEPTS AND MATHEMATICS

PART I INTRODUCTION

1-1 Trends and Scopes

In the 21st century, the transcendent and translational technologies include nanotechnology, microelectronics, information technology, and biotechnology as well as the enabling and supporting mechanical and civil infrastructure systems and smart materials. These technologies are the primary drivers of the century and the new economy in a modern society. Mechanics forms the backbone and basis of these transcendent and translational technologies (Chong, 2004, 2010). Papers on the applications of the theory of elasticity to engineering problems form a significant part of the technical literature in solid mechanics (e.g. Dvorak, 1999; Oden, 2006). Many of the solutions presented in current papers employ numerical methods and require the use of high-speed digital computers. This trend is expected to continue into the foreseeable future, particularly with the widespread use of microcomputers and minicomputers as well as the increased availability of supercomputers (Londer, 1985; Fosdick, 1996). For example, finite element methods have been applied to a wide range of problems such as plane problems, problems of plates and shells, and general three-dimensional problems, including linear and nonlinear behavior, and isotropic and anisotropic materials. Furthermore, through the use of computers, engineers have been able to consider the optimization of large engineering systems (Atrek et al., 1984; Zienkiewicz and Taylor, 2005; Kirsch, 1993; Tsompanakis et al., 2008) such as the space shuttle. In addition, computers have played a powerful role in the fields of computer-aided design (CAD) and computer-aided manufacturing (CAM) (Ellis and Semenkov, 1983; Lamit, 2007) as well as in virtual testing and simulation-based engineering science (Fosdick, 1996; Yang and Pan, 2004; Oden, 2000, 2006).

At the request of one of the authors (Chong), Moon et al. (2003) conducted an in-depth National Science Foundation (NSF) workshop on the research needs of solid mechanics. The following are the recommendations.

Unranked overall priorities in solid mechanics research (Moon et al., 2003)

1. Modeling multiscale problems:
   (i) Bridging the micro-nano-molecular scale
   (ii) Macroscale dynamics of complex machines and systems
2. New experimental methods:
   (i) Micro-nano-atomic scales
   (ii) Coupling between new physical phenomena and model simulations
3. Micro- and nanomechanics:
   (i) Constitutive models of failure initiation and evolution
   (ii) Biocell mechanics
   (iii) Force measurements in the nano- to femtonewton regime
4. Tribology, contact mechanics:
   (i) Search for a grand theory of friction and adhesion
   (ii) Molecular-atomic-based models
   (iii) Extension of microscale models to macroapplications
5. Smart, active, self-diagnosis and self-healing materials:
   (i) Microelectromechanical systems (MEMS)/Nanoelectromechanical systems (NEMS) and biomaterials
   (ii) Fundamental models
   (iii) Increased actuator capability
   (iv) Application to large-scale devices and systems
6. Nucleation of cracks and other defects:
   (i) Electronic materials
   (ii) Nanomaterials
7. Optimization methods in solid mechanics:
   (i) Synthesis of materials by design
   (ii) Electronic materials
   (iii) Optimum design of biomaterials
8. Nonclassical materials:
   (i) Foams, granular materials, nanocarbon tubes, smart materials
9. Energy-related solid mechanics:
   (i) High-temperature materials and coatings
   (ii) Fuel cells

10. Advanced material processing:
    (i) High-speed machining
    (ii) Electronic and nanodevices, biodevices, biomaterials
11. Education in mechanics:
    (i) Need for multidisciplinary education between solid mechanics, physics, chemistry, and biology
    (ii) New mathematical skills in statistical mechanics and optimization methodology
12. Problems related to Homeland Security (Postworkshop; added by the editor)
    (i) Ability of infrastructure to withstand destructive attacks
    (ii) New safety technology for civilian aircraft
    (iii) New sensors and robotics
    (iv) New coatings for fire-resistant structures
    (v) New biochemical filters

In addition to finite element methods, older techniques such as finite difference methods have also found applications in elasticity problems. More generally, the broad subject of approximation methods has received considerable attention in the field of elasticity. In particular, the boundary element method has been widely applied because of certain advantages it possesses in two- and three-dimensional problems and in infinite domain problems (Brebbia, 1988). In addition, other variations of the finite element method have been employed because of their efficiency. For example, finite strip, finite layer, and finite prism methods (Cheung and Tham, 1997) have been used for rectangular regions, and finite strip methods have been applied to nonrectangular regions by Yang and Chong (1984). This increased interest in approximate methods is due mainly to the enhanced capabilities of both mainframe and personal digital computers and their widespread use. Because this development will undoubtedly continue, the authors (Boresi, Chong, and Saigal) treat the topic of approximation methods in elasticity in a second book (Boresi et al., 2002), with particular emphasis on numerical stress analysis through the use of finite differences and finite elements, as well as boundary element and meshless methods.

However, in spite of the widespread use of approximate methods in elasticity (Boresi et al., 2002), the basic concepts of elasticity are fundamental and remain essential for the understanding and interpretation of numerical stress analysis. Accordingly, the present book devotes attention to the theories of deformation and of stress, the stress–strain relations (constitutive relations), nano- and biomechanics, and the fundamental boundary value problems of elasticity. Extensive use of index notation is made. However, general tensor notation is used sparingly, primarily in appendices.

In recent years, researchers from mechanics and other diverse disciplines have been drawn into vigorous efforts to develop smart or intelligent structures that can monitor their own condition, detect impending failure, control damage, and adapt to changing environments (Rogers and Rogers, 1992). The potential applications of such smart materials/systems are abundant: design of smart aircraft skin embedded with fiber-optic sensors (Udd, 1995) to detect structural flaws, bridges with sensoring/actuating elements to counter violent vibrations, flying microelectromechanical systems (Trimmer, 1990) with remote control for surveying and rescue missions, and stealth submarine vehicles with swimming muscles made of special polymers. Such a multidisciplinary infrastructural systems research front, represented by material scientists, physicists, chemists, biologists, and engineers of diverse fields—mechanical, electrical, civil, control, computer, aeronautical, and so on—has collectively created a new entity defined by the interface of these research elements. Smart structures/materials are generally created through synthesis by combining sensoring, processing, and actuating elements integrated with conventional structural materials such as steel, concrete, or composites. Some of these structures/materials currently being researched or in use are listed below (Chong et al., 1990, 1994; Chong and Davis, 2000):

- Piezoelectric composites, which convert electric current to (or from) mechanical forces
- Shape memory alloys, which can generate force through changing the temperature across a transition state
- Electrorheological (ER) and magnetorheological (MR) fluids, which can change from liquid to solid (or the reverse) in electric and magnetic fields, respectively, altering basic material properties dramatically
- Bio-inspired sensors and nanotechnologies, e.g., graphenes and nanotubes

The science and technology of nanometer-scale materials, nanostructure-based devices, and their applications in numerous areas, such as functionally graded materials, molecular-electronics, quantum computers, sensors, molecular machines, and drug delivery systems—to name just a few, form the realm of *nanotechnology* (Srivastava et al., 2007). At nanometer length scale, the material systems concerned may be downsized to reach the limit of tens to hundreds of atoms, where many new physical phenomena are being discovered. Modeling of nanomaterials involving phenomena with multiple length/time scales has attracted enormous attention from the scientific research community. This is evidenced in the works of Belytschko et al. (2002), Belytschko and Xiao (2003), Liu et al. (2004), Arroyo and Belytschko (2005), Srivastava et al. (2007), Wagner et al. (2008), Masud and Kannan (2009), and the host of references mentioned therein. As a matter of fact, the traditional material models based on continuum descriptions are inadequate at the nanoscale, even at the microscale. Therefore, simulation techniques based on descriptions at the atomic scale, such as molecular dynamics (MD), has become an increasingly important computational toolbox. However, MD simulations on even the largest supercomputers (Abraham et al., 2002), although enough for the study of some nanoscale phenomena, are still far too small to treat the micro-to-macroscale interactions that must be captured in the simulation of any real device (Wagner et al., 2008).

Bioscience and technology has contributed much to our understanding of human health since the birth of continuum biomechanics in the mid-1960s (Fung, 1967, 1983, 1990, 1993, 1995). Nevertheless, it has yet to reach its full potential as a consistent contributor to the improvement of health-care delivery. This is due to the fact that most biological materials are very complicated hierachical structures. In the most recent review paper, Meyers et al. (2008) describe the defining characteristics, namely, hierarchy, multifunctionality, self-healing, and self-organization of biological tissues in detail, and point out that the new frontiers of material and structure design reside in the synthesis of bioinspired materials, which involve nanoscale self-assembly of the components and the development of hierarchical structures. For example the amazing multiscale bones structure—from amino acids, tropocollagen, mineralized collagen fibrils, fibril arrays, fiber patterns, osteon and Haversian canal, and bone tissue to macroscopic bone—makes bones remarkably resistant to fracture (Ritchie et al., 2009). The multiscale bone structure of trabecular bone and cortical bone from nanoscale to macroscale is illustrated in Figure 1-1.1. (Courtesy of I. Jasiuk and E. Hamed, University of Illinois – Urbana). Although much significant progress has been made in the field of bioscience and technology, especially in biomechanics, there exist many open problems related to elasticity, including molecular and cell biomechanics, biomechanics of development, biomechanics of growth and remodeling, injury biomechanics and rehabilitation, functional tissue engineering, muscle mechanics and active stress, solid–fluid interactions, and thermal treatment (Humphrey, 2002).

Current research activities aim at understanding, synthesizing, and processing material systems that behave like biological systems. Smart structures/materials basically possess their own sensors (nervous system), processor (brain system), and actuators (muscular systems), thus mimicking biological systems (Rogers and Rogers, 1992). Sensors used in smart structures/materials include optical fibers, micro-cantilevers, corrosion sensors, and other environmental sensors and sensing particles. Examples of actuators include shape memory alloys that would return to their original shape when heated, hydraulic systems, and piezoelectric ceramic polymer composites. The processor or control aspects of smart structures/materials are based on microchip, computer software, and hardware systems.

Recently, Huang from Northwestern University and his collaborators developed the stretchable silicon based on the wrinkling of the thin films on a prestretched substrate. This is important to the development of stretchable electronics and sensors such as the three-dimensional eye-shaped sensors. One of their papers was published in *Science* in 2006 (Khang et al., 2006). The basic idea is to make straight silicon ribbons wavy. A prestretched polymer Polydimethylsiloxane (PDMS) is used to peel silicon ribbons away from the substrate, and releasing the prestretch leads to buckled, wavy silicon ribbons.

In the past, engineers and material scientists have been involved extensively with the characterization of given materials. With the availability of advanced computing, along with new developments in material sciences, researchers can now characterize processes, design, and manufacture materials with desirable performance and properties. Using nanotechnology (Reed and Kirk, 1989; Timp, 1999;

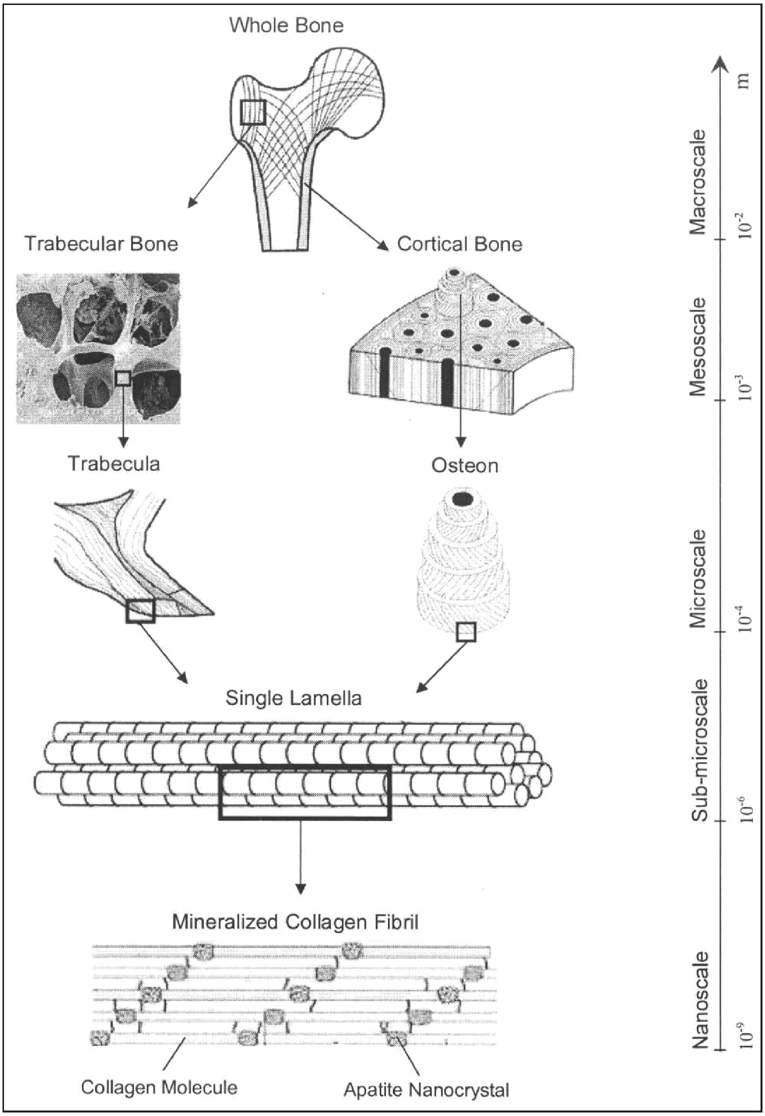


Figure 1-1.1

Chong, 2004), engineers and scientists can build designer materials molecule by molecule via self-assembly, etc. One of the challenges is to model short-term microscale material behavior through mesoscale and macroscale behavior into long-term structural systems performance (Fig. 1-1.2). Accelerated tests to simulate various environmental forces and impacts are needed. Supercomputers and/or workstations used in parallel are useful tools to (a) solve this multiscale and size-effect problem by taking into account the large number of variables and unknowns

MATERIALS		STRUCTURES		INFRASTRUCTURE
Nanolevel ∼	**microlevel ∼**	**mesolevel ∼**	**macro- ∼ level**	**systems integration**
Molecular Scale	***Microns***		***Meters***	***Up to km Scale***
nanomechanics	**micromechanics**	**mesomechanics**	**beams**	**bridge systems**
self-assembly	**microstructures**	**interfacial structures**	**columns**	**lifelines**
nanofabrication	**smart materials**	**composites**	**plates**	**airplanes**

Figure 1-1.2 Scales in materials and structures.

to project microbehavior into infrastructure systems performance and (b) to model or extrapolate short-term test results into long-term life-cycle behavior.

According to Eugene Wong, the former engineering director of the National Science Foundation, the transcendent technologies of our time are

- Microelectronics—Moore's law: doubling the capabilities every 2 years for the past 30 years; unlimited scalability
- Information technology: confluence of computing and communications
- Biotechnology: molecular secrets of life

These technologies and nanotechnology are mainly responsible for the tremendous economic developments. Engineering mechanics is related to all these technologies based on the experience of the authors. The first small step in many of these research activities and technologies involves the study of deformation and stress in materials, along with the associated stress–strain relations.

In this book following the example of modern continuum mechanics and the example of A. E. Love (Love, 2009), we treat the theories of deformation and of stress separately, in this manner clearly noting their mathematical similarities and their physical differences. Continuum mechanics concepts such as couple stress and body couple are introduced into the theory of stress in the appendices of Chapters 3, 5, and 6. These effects are introduced into the theory in a direct way and present no particular problem. The notations of stress and of strain are based on the concept of a continuum, that is, a continuous distribution of matter in the region (space) of interest. In the mathematical physics sense, this means that the volume or region under examination is sufficiently filled with matter (dense) that concepts such as mass density, momentum, stress, energy, and so forth are defined at all points in the region by appropriate mathematical limiting processes (see Chapter 3, Section 3-1).

1-2 Theory of Elasticity

The theory of elasticity, in contrast to the general theory of continuum mechanics (Eringen, 1980), is an ad hoc theory designed to treat explicity a special response of materials to applied forces—namely, the elastic response, in which the stress at every point P in a material body (continuum) depends at all times solely on the simultaneous deformation in the immediate neighborhood of the point P (see Chapter 4, Section 4-1). In general, the relation between stress and deformation is a nonlinear one, and the corresponding theory is called the *nonlinear theory of elasticity* (Green and Adkins, 1970). However, if the relationship of the stress and the deformation is linear, the material is said to be linearly elastic, and the corresponding theory is called the *linear theory of elasticity*.

The major part of this book treats the linear theory of elasticity. Although ad hoc in form, this theory of elasticity plays an important conceptual role in the study of nonelastic types of material responses. For example, often in problems involving plasticity or creep of materials, the method of successive elastic solutions is employed (Mendelson, 1983). Consequently, the theory of elasticity finds application in fields that treat inelastic response.

1-3 Numerical Stress Analysis

The solution of an elasticity problem generally requires the description of the response of a material body (computer chips, machine part, structural element, or mechanical system) to a given excitation (such as force). In an engineering sense, this description is usually required in numerical form, the objective being to assure the designer or engineer that the response of the system will not violate design requirements. These requirements may include the consideration of deterministic and probabilistic concepts (Thoft-Christensen and Baker, 1982; Wen, 1984; Yao, 1985). In a broad sense the numerical results are predictions as to whether the system will perform as desired. The solution to the elasticity problem may be obtained by a direct numerical process (numerical stress analysis) or in the form of a general solution (which ordinarily requires further numerical evaluation; see Section 1-4).

The usual methods of numerical stress analysis recast the mathematically posed elasticity problem into a direct numerical analysis. For example, in finite difference methods, derivatives are approximated by algebraic expressions; this transforms the differential boundary value problem of elasticity into an algebraic boundary value problem requiring the numerical solution of a set of simultaneous algebraic equations. In finite element methods, trial function approximations of displacement components, stress components, and so on are employed in conjunction with energy methods (Chapter 4, Section 4-21) and matrix methods (Section 1-28), again to transform the elasticity boundary value problem into a system of simultaneous algebraic equations. However, because finite element methods may be applied to individual pieces (elements) of the body, each element may be given distinct material properties, thus achieving very general descriptions of a body as a whole. This feature of the finite element method is very attractive to the practicing stress analyst. In addition, the application of finite elements leads to many interesting mathematical questions concerning accuracy of approximation, convergence of the results, attainment of bounds on the exact answer, and so on. Today, finite element methods are perhaps the principal method of numerical stress analysis employed to solve elasticity problems in engineering (Zienkiewicz and Taylor, 2005). By their nature, methods of numerical stress analysis (Boresi et al., 2002) yield approximate solutions to the exact elasticity solution.

1-4 General Solution of the Elasticity Problem

Plane Elasticity. Two classical plane problems have been studied extensively: plane strain and plane stress (see Chapter 5). If the state of plane isotropic elasticity is referred to the (x, y) plane, then plane elasticity is characterized by the conditions that the stress and strain are independent of coordinate z, and shear stress τ_{xz}, τ_{yz} (hence, shear strains γ_{xz}, γ_{yz}) are zero. In addition, for plane strain the extensional strain ϵ_z equals 0, and for plane stress we have $\sigma_z = 0$. For plane strain problems the equations represent exact solutions to physical problems, whereas for plane stress problems, the usual solutions are only approximations to physical problems. Mathematically, the problems of plane stress and plane strain are identical (see Chapter 5).

One general method of solution of the plane problem rests on the reduction of the elasticity equations to the solution of certain equations in the complex plane (Muskhelishvili, 1975).[1] Ordinarily, the method requires mapping of the given region into a suitable region in the complex plane. A second general method rests on the introduction of a single scalar biharmonic function, the Airy stress function, which must be chosen suitably to satisfy boundary conditions (see Chapter 5).

Three-Dimensional Elasticity. In contrast to the problem of plane elasticity, the construction of general solutions of the three-dimensional equations of elasticity has not as yet been completely achieved. Many so-called general solutions are really particular forms of solutions of the three-dimensional field equations of elasticity in terms of arbitrary, ad hoc functions. Particular examples of general solutions are employed in Chapter 8 and in Appendix 5B. In many of these examples, the functions and the form of solution are determined in part by the differential equations and in part by the physical features of the problem. A general solution of the elasticity equations may also be constructed in terms of biharmonic functions (see Appendix 5B). Because there is no apparent reason for one form of general solution to be readily obtainable from another, a number of investigators have attempted to extend the generality of solution form and show relations among known solutions (Sternberg, 1960; Naghdi and Hsu, 1961; Stippes, 1967).

1-5 Experimental Stress Analysis

Material properties that enter into the stress–strain relations (constitutive relations; see Section 4-4) must be obtained experimentally (Schreiber et al., 1973; Chong and Smith, 1984). In addition, other material properties, such as ultimate strength

[1]See also Appendix 5B.

and fracture toughness, as well as nonmaterial quantities such as residual stresses, have to be determined by physical tests.

For bodies that possess intricately shaped boundaries, general analytical (closed-form) solutions become extremely difficult to obtain. In such cases one must invariably resort to approximate methods, principally to numerical methods or to experimental methods. In the latter, several techniques such as photoelasticity, the Moiré method, strain gage methods, fracture gages, optical fibers, and so forth have been developed to a fine art (Dove and Adams, 1964; Dally and Riley, 2005; Rogers and Rogers, 1992; Ruud and Green, 1984). In addition, certain analogies based on a similarity between the equations of elasticity and the equations that describe readily studied physical systems are employed to obtain estimates of solutions or to gain insight into the nature of mathematical solutions (see Chapter 7, Section 7-9, for the membrane analogy in torsion). In this book we do not treat experimental methods but rather refer to the extensive modern literature available.[2]

1-6 Boundary Value Problems of Elasticity

The solution of the equations of elasticity involves the determination of a stress or strain state in the interior of a region R subject to a given state of stress or strain (or displacement) on the boundary B of R (see Chapter 4, Section 4-15). Subject to certain restrictions on the nature of the solution and of region R and the form of the boundary conditions, the solution of boundary value problems of elasticity may be shown to exist (see Chapter 4, Section 4-16). Under broader conditions, existence and uniqueness of the elasticity boundary value problem are not ensured. In general, the question of existence and uniqueness (Knops and Payne, 1971) rests on the theory of systems of partial differential equations of three independent variables.

In particular forms the boundary value problem of elasticity may be reduced to that of seeking a single scalar function f of three independent variables, say (x, y, z); that is, $f = f(x, y, z)$ such that the stress field of strain field derived from f satisfies the boundary conditions on B. In particular for the Laplace equation, three types of boundary value problems occur frequently in elasticity: the Dirichlet problem, the Neumann problem, and the mixed problem. Let $h(x, y)$ be a given function that is defined on B, the bounding surface of a simply connected region R. Then the Dirichlet problem for the Laplace equation is that of determining a function $f = f(x, y)$ that

1. is continuous on $R + B$,
2. is harmonic on R, and
3. is identical to $h(x, y)$ on B.

[2]*Experimental Mechanics* and *Experimental Techniques*, both journals of the Society for Experimental Mechanics (SEM), contain a wealth of information on experimental techniques. In addition, the American Society for Testing and Materials (ASTM) publishes the *Journal of Testing and Evaluation*, the *Geotechnical Testing Journal*, and other journals.

The Dirichlet problem has been shown to possess a unique solution (Greenspan, 1965). However, analytical determination of $f(x, y)$ is very much more difficult to achieve than is the establishment of its existence. Indeed, except for special forms of boundary B (such as the rectangle, the circle, or regions that can be mapped onto rectangular or circular regions), the problems of determining $f(x, y)$ do not surrender to existing analytical techniques.

The Neumann boundary value problem for the Laplace equation is that of determining a function $f(x, y)$ that

1. is defined and continuous on $R + B$,
2. is harmonic on R, and
3. has an outwardly directed normal derivative $\partial f/\partial n$ such that $\partial f/\partial n = g(x, y)$ on B, where $g(x, y)$ is defined and continuous on B.

Without an additional requirement [namely, that $f(x, y)$ has a prescribed value for at least one point of B], the solution of the Neumann problem is not well posed because otherwise the Neumann problem has a one-parameter infinity of solutions.

The mixed problem overcomes the difficulty of the Neumann problem. Again, let $g(x, y)$ be a continuous function on B' of R and let $h(x, y)$ be bounded and continuous on B'' of R, where $B = B' + B''$ denotes the boundary of region R. Then the mixed problem for the Laplace equation is that of determining a function $f(x, y)$ such that it

1. is defined and continuous on $R + B$,
2. is harmonic on R,
3. is identical with $g(x, y)$, on B', and
4. has outwardly directed normal derivative $\partial f/\partial n = h(x, y)$ on B''.

It has been shown that certain mixed problems have unique solutions[3] (Greenspan, 1965). Because, in general, the solutions of the Dirichlet and mixed problems cannot be given in closed form, methods of approximate solutions of these problems are presented in another book by the authors (Boresi et al., 2002). More generally, these approximate methods may be applied to most boundary value problems of elasticity.

PART II PRELIMINARY CONCEPTS

In Part II of this chapter we set down some concepts that are useful in following the developments in the text proper and in the appendices.

[3]These remarks are restricted to simply connected regions.

1-7 Brief Summary of Vector Algebra

In this text a boldface letter denotes a vector quantity unless an explicit statement to the contrary is given; thus, **A** denotes a vector. Frequently, we denote a vector by the set of its projections (A_x, A_y, A_z) on rectangular Cartesian axes (x, y, z). Thus,

$$\mathbf{A} = (A_x, A_y, A_z) \tag{1-7.1}$$

The magnitude of a vector **A** is denoted by

$$|\mathbf{A}| = A = (A_x^2 + A_y^2 + A_z^2)^{1/2} \tag{1-7.2}$$

We may also express a vector in terms of its components with respect to (x, y, z) axes. For example,

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z \tag{1-7.3}$$

where $\mathbf{i}A_x, \mathbf{j}A_y, \mathbf{k}A_z$ are components of **A** with respect to axes (x, y, z), and **i**, **j**, **k**, are unit vectors directed along positive (x, y, z) axes, respectively. In general, the symbols **i**, **j**, **k** denote unit vectors.

Vector quantities obey the *associative law of vector addition*:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + \mathbf{B} + \mathbf{C} \tag{1-7.4}$$

and the *commutative law of vector addition*:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \qquad \mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{B} + \mathbf{A} + \mathbf{C} = \mathbf{B} + \mathbf{C} + \mathbf{A} \tag{1-7.5}$$

Symbolically, we may represent a vector quantity by an arrow (Fig. 1-7.1) with the understanding that the addition of any two arrows (vectors) must obey the commutative law [Eq. (1-7.5)].

The *scalar product* of two vectors **A**, **B** is defined to be

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y + A_z B_z \tag{1-7.6}$$

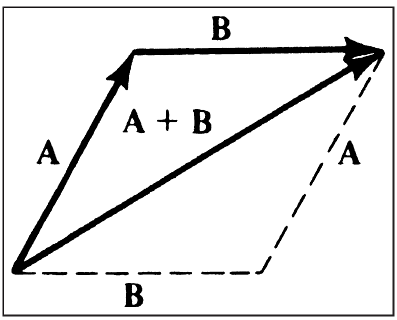


Figure 1-7.1

where the symbol · is a conventional notation for the scalar product. By the above definition, it follows that the scalar product of vectors is commutative; that is,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1-7.7}$$

A useful property of the scalar product of two vectors is

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta \tag{1-7.8}$$

where A and B denote the magnitudes of vectors **A** and **B**, respectively, and the angle θ denotes the angle formed by vectors **A** and **B** (Fig. 1-7.2).

If **B** is a unit vector in the x direction, Eqs. (1-7.3) and (1-7.8) yield $A_x = A\cos\alpha$, where α is the direction angle between the vector **A** and the positive x axis. Similarly, $A_y = A\cos\beta$, $A_z = A\cos\gamma$, where β, γ denote direction angles between the vector **A** and the y axis and the z axis, respectively. Substitution of these expressions into Eq. (1-7.2) yields the relation

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{1-7.9}$$

Thus, the direction cosines of vector **A** *are not independent*. They must satisfy Eq. (1-7.9).

The scalar product law of vectors has other properties in common with the product of numbers. For example,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \tag{1-7.10}$$

$$\begin{aligned}(\mathbf{A} + \mathbf{B}) \cdot (\mathbf{C} + \mathbf{D}) &= (\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} + (\mathbf{A} + \mathbf{B}) \cdot \mathbf{D} \\ &= \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C} + \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{D}\end{aligned} \tag{1-7.11}$$

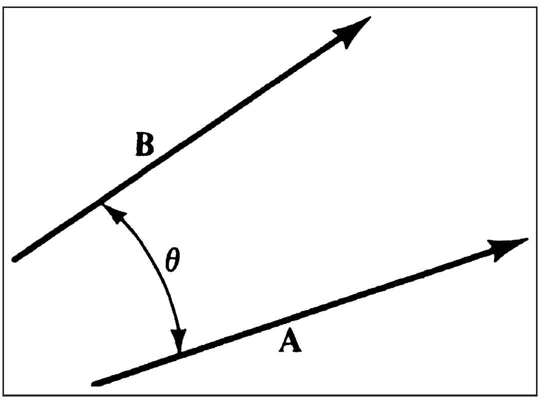


Figure 1-7.2

The *vector product* of two vectors **A** and **B** is defined to be a third vector **C** whose magnitude is given by the relation

$$C = AB \sin\theta \tag{1-7.12}$$

The direction of vector **C** is perpendicular to the plane formed by vectors **A** and **B**. The sense of **C** is such that the three vectors **A**, **B**, **C** form a right-handed or left-handed system according to whether the coordinate system (x, y, z) is right handed or left handed (see Fig. 1-7.3).

Symbolically, we denote the vector product of **A** and **B** in the form

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} \tag{1-7.13}$$

where **x** denotes vector product (or cross product). In determinant notation, Eq. (1-7.13) may be written as

$$\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{1-7.13a}$$

where (A_x, A_y, A_z), (B_x, B_y, B_z) denotes $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ projections of vectors $(\mathbf{A}, \mathbf{B})$, respectively.

The vector product of vectors has the following property:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \tag{1-7.14}$$

Accordingly, the vector product of vectors is not commutative.

The vector product also has the following properties:

$$\begin{aligned} \mathbf{R} \times (\mathbf{A} + \mathbf{B}) &= \mathbf{R} \times \mathbf{A} + \mathbf{R} \times \mathbf{B} \\ (\mathbf{A} + \mathbf{B}) \times \mathbf{R} &= \mathbf{A} \times \mathbf{R} + \mathbf{B} \times \mathbf{R} \end{aligned} \tag{1-7.15}$$

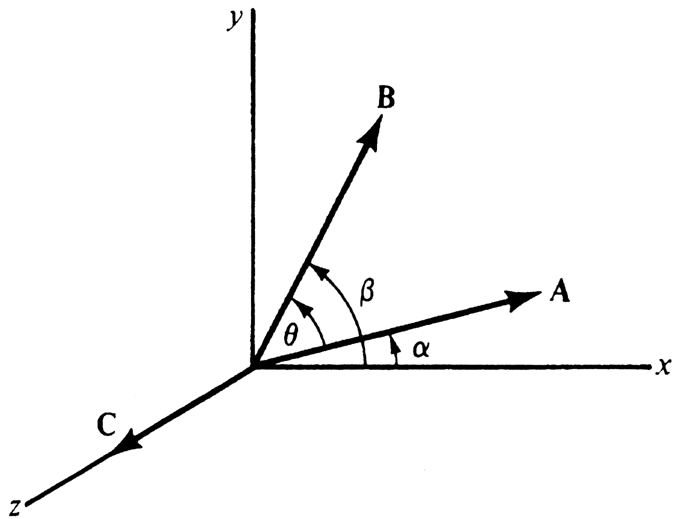
$$\begin{aligned} (\mathbf{A} + \mathbf{B}) \times (\mathbf{C} + \mathbf{D}) &= (\mathbf{A} + \mathbf{B}) \times \mathbf{C} + (\mathbf{A} + \mathbf{B}) \times \mathbf{D} \\ &= \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C} + \mathbf{A} \times \mathbf{D} + \mathbf{B} \times \mathbf{D} \end{aligned} \tag{1-7.16}$$

The scalar triple product of three vectors **A**, **B**, **C** is defined by the relation

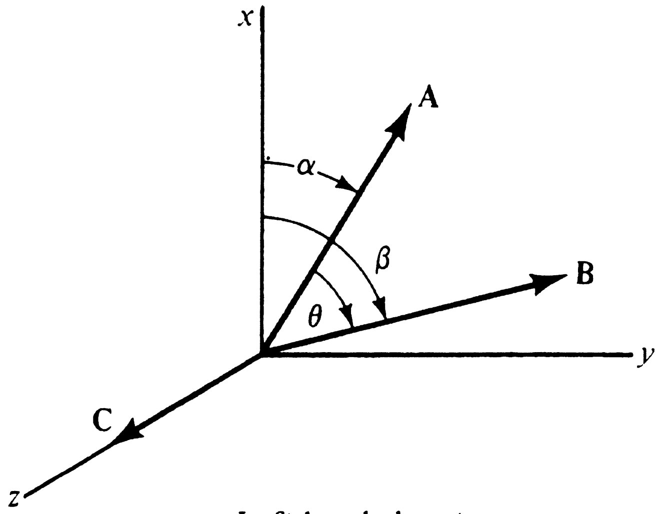
$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) \\ &\quad + A_z(B_x C_y - B_y C_x) \end{aligned} \tag{1-7.17}$$

In determinant notation, the scalar triple product is

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{1-7.18}$$



Right-handed system



Left-handed system

Figure 1-7.3

Because only the sign of a determinant changes when two rows are interchanged, two consecutive transpositions of rows leave a determinant unchanged. Consequently,

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) \tag{1-7.19}$$

Another useful property is the relation

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) \tag{1-7.20}$$

The vector triple product of three vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ is defined as

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}) \tag{1-7.21}$$

Furthermore,

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= \mathbf{A} \cdot \mathbf{B} \times (\mathbf{C} \times \mathbf{D}) = \mathbf{A} \cdot [\mathbf{C}(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})\mathbf{D}] \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})\end{aligned} \tag{1-7.22}$$

Equation (1-7.22) follows from Eqs. (1-7.20) and (1-7.21).

1-8 Scalar Point Functions

Any scalar function $f(x, y, z)$ that is defined at all points in a region of space is called a *scalar point function*. Conceivably, the function f may depend on time, but if it does, attention can be confined to conditions at a particular instant. The region of space in which f is defined is called a scalar field. It is assumed that f is differentiable in this *scalar field*. Physical examples of scalar point functions are the mass density of a compressible medium, the temperature in a body, the flux density in a nuclear reactor, and the potential in an electrostatic field.

Consider the rate of change of the function f in various directions at some point $P: (x, y, z)$ in the scalar field for which f is defined. Let (x, y, z) take increments (dx, dy, dz). Then the function f takes an increment:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{1-8.1}$$

Consider the infinitesimal vector $\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$, where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are unit vectors in the (x, y, z) directions, respectively. Its magnitude is $ds = (dx^2 + dy^2 + dz^2)^{1/2}$, and its direction cosines are

$$\cos\alpha = \frac{dx}{ds} \qquad \cos\beta = \frac{dy}{ds} \qquad \cos\gamma = \frac{dz}{ds}$$

The vector $\mathbf{i}\,(dx/ds) + \mathbf{j}\,(dy/ds) + \mathbf{k}\,(dz/ds)$ is a unit vector in the direction of $\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$, as division of a vector by a scalar alters only the magnitude of the vector. Dividing Eq. (1-8.1) by ds, we obtain

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

or

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma \tag{1-8.2}$$

From Eq. (1-8.2) it is apparent that df/ds depends on the direction of ds; that is, it depends on the direction (α, β, γ). For this reason df/ds is known as the *directional derivative* of f in the direction (α, β, γ). It represents the rate of change of f in the direction (α, β, γ). For example, if $\alpha = 0$, $\beta = \gamma = \pi/2$,

$$\frac{df}{ds} = \frac{\partial f}{\partial x}$$

This is the rate of change of f in the direction of the x axis.

Maximum Value of the Directional Derivative. Gradient. By definition of the scalar product of two vectors, Eq. (1-8.2) may be written in the form

$$\frac{df}{ds} = \mathbf{n} \cdot \text{grad } f \tag{1-8.3}$$

where $\mathbf{n} = \mathbf{i}\cos\alpha + \mathbf{j}\cos\beta + \mathbf{k}\cos\gamma$ is a unit vector in the direction (α, β, γ), and

$$\text{grad } f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} \tag{1-8.4}$$

is a vector point function (see Section 1-10) of (x, y, z) called the gradient of the scalar function f. Because $\mathbf{n}$ is a unit vector, Eq. (1-8.3) shows that $|\text{grad } f|$ is the maximum value of df/ds at the point P: (x, y, z) and that the direction of grad f is the direction in which $f(x, y, z)$ *increases* most rapidly. Equation (1-8.3) also shows that the directional derivative of f in any direction is the component of the vector grad f in that direction.

The equation $f(x, y, z) = C$ defines a family of surfaces, one surface for each value of the constant C. These are called level surfaces of the function f. If $\mathbf{n}$ is tangent to a level surface, the directional derivative of f in the direction of $\mathbf{n}$ is zero, as f is constant along a *level surface*. Consequently, by Eq. (1-8.3), the vector $\mathbf{n}$ must be perpendicular to the vector grad f when $\mathbf{n}$ is tangent to a level surface. Accordingly, the vector grad f at the point P: (x, y, z) is normal to the level surface of f through the point P: (x, y, z).

A symbolic vector operator, called *del* or *nabla*, is defined as follows:

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \tag{1-8.5}$$

By Eqs. (1-8.3), (1-8.4), and (1-8.5),

$$\text{grad } f = \nabla f$$

and

$$\frac{df}{ds} = \mathbf{n} \cdot \nabla f$$

By definition,

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1-8.6}$$

Consequently, the Laplace equation may be written symbolically as

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{1-8.7}$$

For this reason the symbolic operator ∇^2 is called the *Laplacian*.

1-9 Vector Fields

Assume that for each point P: (x, y, z) in a region there exists a vector point function $\mathbf{q}(x, y, z)$. This vector point function is called a *vector field.* It may be represented at each point in the region by a vector with length equal to the magnitude of $\mathbf{q}$ and drawn in the direction of $\mathbf{q}$. For example, for each point in a flowing fluid there corresponds a vector $\mathbf{q}$ that represents the velocity of the particle of fluid at that point. This vector point function is called the velocity field of the fluid. Another example of a vector field is the displacement vector function for the particles of a deformable body. Electric and magnetic field intensities are also vector fields. A vector field is often simply called a "vector."

In any continuous vector field there exists a system of curves such that the vectors along a curve are everywhere tangent to the curve; that is, the vector field consists exclusively of tangent vectors to the curves. These curves are called the *vector lines* (or *field lines*) of the field. The vector lines of a velocity field are called *stream lines.* The vector lines in an electrostatic or magnetostatic field are known as *lines of force.* In general, the vector function $\mathbf{q}$ may depend on (x, y, z) and t, where t denotes time. If $\mathbf{q}$ depends on time, the field is said to be *unsteady* or *nonstationary*; that is, the field varies with time. For a *steady field*, $\mathbf{q} = \mathbf{q}(x, y, z)$. For example, if a velocity field changes with time (i.e., if the flow is unsteady), the stream lines may change with time.

A vector field $\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ is defined by expressing the projections (u, v, w) as functions of (x, y, z). If (dx, dy, dz) is an infinitesimal vector in the direction of the vector $\mathbf{q}$, the direction cosines of this vector are $dx/ds = u/q, dy/ds = v/q$, and $dz/ds = w/q$. Consequently, the differential equations of the system of vector lines of the field are

$$\frac{ds}{q} = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \tag{1-9.1}$$

In Eq. (1-9.1) the components (u, v, w) are functions of (x, y, z). The finite equations of the system of vector lines are obtained by integrating Eq. (1-9.1). The theory of integration of differential equations of this type is explained in most books on differential equations (Morris and Brown, 1964; Ince, 2009).

If a given vector field $\mathbf{q}$ is the gradient of a scalar field f (i.e., if $\mathbf{q} = \text{grad } f$), the scalar function f is called a potential function for the vector field, and the vector field is called a potential field. Because grad f is perpendicular to the level surfaces of f, it follows that the vector lines of a potential field are everywhere normal to the level surfaces of the potential function.

1-10 Differentiation of Vectors

An infinitesimal increment $d\mathbf{R}$ of a vector $\mathbf{R}$ need not be collinear with the vector $\mathbf{R}$ (Fig. 1-10.1). Consequently, in general, the vector $\mathbf{R} + d\mathbf{R}$ differs from the vector $\mathbf{R}$ not only in magnitude but also in direction. It would be misleading to denote the magnitude of the vector $d\mathbf{R}$ by dR, as dR denotes the increment of the magnitude R. Accordingly, the magnitude of $d\mathbf{R}$ is denoted by $|d\mathbf{R}|$ or by another symbol, such as ds. The magnitude of the vector $\mathbf{R} + d\mathbf{R}$ is $R + dR$. Figure 1-10.1 shows that $|\mathbf{R} + d\mathbf{R}| \leq R + |d\mathbf{R}|$. Hence, $dR \leq |d\mathbf{R}|$.

If the vector $\mathbf{R}$ is a function of a scalar t (where t may or may not denote time), $d\mathbf{R}/dt$ is defined to be a vector in the direction of $d\mathbf{R}$, with magnitude ds/dt (where $ds = |d\mathbf{R}|$).

Vectors obey the same rules of differentiation as scalars. This fact may be demonstrated by the Δ method that is used for deriving differentiation formulas

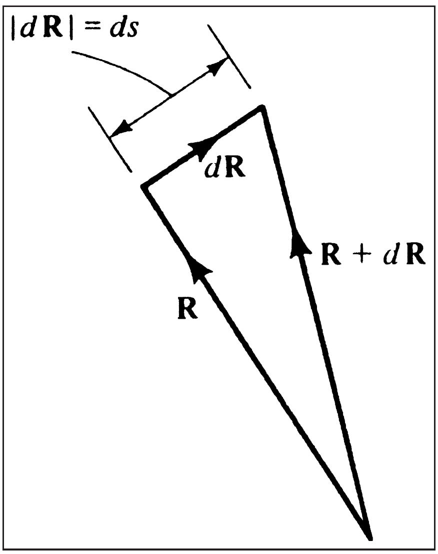


Figure 1-10.1

in scalar calculus. For example, consider the derivative of the vector function $\mathbf{Q} = u\mathbf{R}$, where u is a scalar function of t and $\mathbf{R}$ is a vector function of t. If t takes an increment Δt, $\mathbf{R}$ and u takes increments $\mathbf{\Delta R}$ and Δu. Hence,

$$\mathbf{Q} + \mathbf{\Delta Q} = (u + \Delta u)(\mathbf{R} + \mathbf{\Delta R})$$

Subtracting $\mathbf{Q} = u\mathbf{R}$ and dividing by Δt, we obtain

$$\frac{\mathbf{\Delta Q}}{\Delta t} = \mathbf{R}\frac{\Delta u}{\Delta t} + u\frac{\mathbf{\Delta R}}{\Delta t} + \Delta u\frac{\mathbf{\Delta R}}{\Delta t}$$

As $\Delta t \to 0, \Delta u \to 0$, $\mathbf{\Delta Q}/\Delta t \to d\mathbf{Q}/dt$, $\Delta u/\Delta t \to du/dt$, and $\Delta \mathbf{R}/\Delta t \to d\mathbf{R}/dt$. Hence,

$$\frac{d\mathbf{Q}}{dt} = \mathbf{R}\frac{du}{dt} + u\frac{d\mathbf{R}}{dt} \tag{1-10.1}$$

Equation (1-10.1) has the same form as the formula for the derivative of the product of two scalars.

Let $\mathbf{R} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ be a single vector (not a vector field) where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are unit vectors and (u, v, w) are the $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ projections of $\mathbf{R}$, respectively. Let (u, v, w) take increments (du, dv, dw). Then because $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are constants, $\mathbf{R}$ takes the increment $d\mathbf{R} = \mathbf{i}\,du + \mathbf{j}\,dv + \mathbf{k}\,dw$ where, in general, $d\mathbf{R}$ is not collinear with $\mathbf{R}$. If (u, v, w) are functions of the single variable t,

$$\frac{d\mathbf{R}}{dt} = \mathbf{i}\frac{du}{dt} + \mathbf{j}\frac{dv}{dt} + \mathbf{k}\frac{dw}{dt} \tag{1-10.2}$$

Hence, $d\mathbf{R}/dt$ is a vector in the direction of $d\mathbf{R}$, with magnitude $[(du/dt)^2 + (dv/dt)^2 + (dw/dt)^2]^{1/2}$.

If $\mathbf{R}$ is the position of a moving particle P measured from a fixed point O (Fig. 1-10.2), $d\mathbf{R}/dt$ is the velocity vector $\mathbf{q}$ of the particle. Likewise,

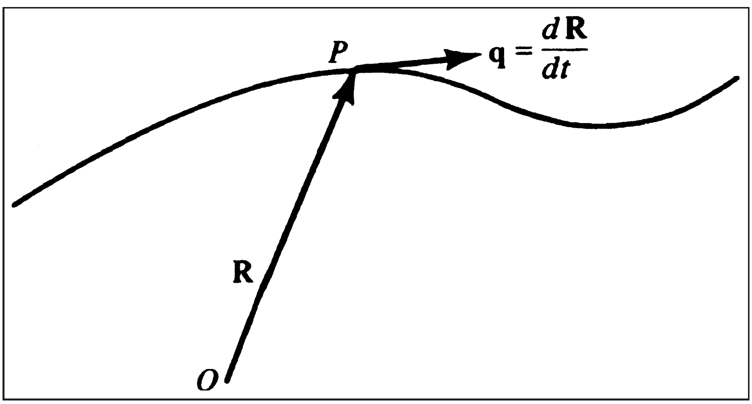


Figure 1-10.2

$d\mathbf{q}/dt = d^2\mathbf{R}/dt^2$ is the acceleration vector of the particle. Hence, the vector form of Newton's second law is

$$\mathbf{F} = m\frac{d^2\mathbf{R}}{dt^2} \tag{1-10.3}$$

1-11 Differentiation of a Scalar Field

Let $Q(x, y, z; t)$ be a scalar point function in a flowing fluid (such as temperature, density, a velocity projection, etc.). Then

$$dQ = \frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz + \frac{\partial Q}{\partial t}dt \tag{1-11.1}$$

Here (dx, dy, dz, dt) are arbitrary increments of coordinates (x, y, z) and time t. [In deformation theory, x, y, z are called *spatial* (*Eulerian*) *coordinates*; see Chapter 2.]

Let (dx, dy, dz) be the displacement that a particle of fluid experiences during a time interval dt. Then $dx/dt = u$, $dy/dt = v$, and $dz/dt = w$, where (u, v, w) is the velocity field. Hence, on dividing Eq. (1-11.1) by dt, we get

$$\frac{dQ}{dt} = u\frac{\partial Q}{\partial x} + v\frac{\partial Q}{\partial y} + w\frac{\partial Q}{\partial z} + \frac{\partial Q}{\partial t} \tag{1-11.2}$$

or, in vector notation,

$$\frac{dQ}{dt} = \mathbf{q} \cdot \text{grad } Q + \frac{\partial Q}{dt} \tag{1-11.3}$$

where $\mathbf{q}$ is the velocity field. Although Eq. (1-11.2) is derived for a scalar point function in a flowing fluid, it remains valid for any scalar point function $Q(x, y, z; t)$.

The distinction between $\partial Q/\partial t$ and dQ/dt is very important. The partial derivative $\partial Q/\partial t$ denotes the rate of change of Q at a fixed point of space as the fluid flows by. For steady flow, $\partial Q/\partial t = 0$. In contrast, dQ/dt denotes the rate of change of Q for a certain particle of fluid. For example, if Q is temperature, we determine $\partial Q/\partial t$ by holding the thermometer still. To determine dQ/dt, we must move the thermometer so that it coincides continuously with the same particle of fluid. This procedure, of course, is not feasible, but we do not need to make measurements with moving instruments because Eq. (1-11.2) gives the relation between the derivative dQ/dt and the derivative $\partial Q/\partial t$.

1-12 Differentiation of a Vector Field

If $\mathbf{Q}(x, y, z, t)$ is a vector field, Eq. (1-11.2) remains valid; that is,

$$\frac{d\mathbf{Q}}{dt} = u\frac{\partial \mathbf{Q}}{\partial x} + v\frac{\partial \mathbf{Q}}{\partial y} + w\frac{\partial \mathbf{Q}}{\partial z} + \frac{\partial \mathbf{Q}}{\partial t} \tag{1-12.1}$$

This follows from the fact that Eq. (1-11.2) is valid for each of the components of the vector **Q**. Equation (1-12.1) may be written in the form

$$\frac{d\mathbf{Q}}{dt} = (\mathbf{q} \cdot \nabla)\mathbf{Q} + \frac{\partial \mathbf{Q}}{t} \tag{1-12.2}$$

If $\mathbf{Q} = \mathbf{q}$, $d\mathbf{Q}/dt$ is the acceleration vector **a**. Consequently,

$$\mathbf{a} = \frac{d\mathbf{q}}{dt} = u\frac{\partial \mathbf{q}}{\partial x} + v\frac{\partial \mathbf{q}}{\partial y} + w\frac{\partial \mathbf{q}}{\partial z} + \frac{\partial \mathbf{q}}{\partial t} \tag{1-12.3}$$

or

$$\mathbf{a} = (\mathbf{q} \cdot \nabla)\mathbf{q} + \frac{\partial \mathbf{q}}{\partial t} \tag{1-12.4}$$

Thus, the acceleration field is derived from the velocity field.

1-13 Curl of a Vector Field

Let $\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ be a vector field. Then $\nabla \times \mathbf{q}$ is a vector field that is denoted by curl **q**. Hence, by Eq. (1-7.13),

$$\text{curl } \mathbf{q} = \nabla \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{1-13.1}$$

or

$$\text{curl } \mathbf{q} = \mathbf{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \mathbf{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \mathbf{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{1-13.2}$$

It can be shown that the vector field curl **q** is independent of the choice of coordinates. A physical significance is later attributed to curl **q** if **q** denotes the velocity of a fluid. Curl **q** may also be related to the rotation of a volume element of a deformable body (see Chapter 2).

1-14 Eulerian Continuity Equation for Fluids

Let $\mathbf{q} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$ be an unsteady *velocity* field of a compressible fluid. Let us consider the rate of mass flow out of a space cell $dx\,dy\,dz = dV$ fixed with respect to (x, y, z) axes (see Fig. 1-14.1). The mass that flows in through the face AB during a time interval dt is $\rho u\,dy\,dz\,dt$, where ρ is the mass density. The mass that flows out through the face CD during dt is $\{\rho u + [\partial(\rho u)/\partial x]\,dx\}\,dy\,dz\,dt$. Similar expressions are obtained for the mass flows out of the other pairs of faces. Accordingly, the net mass that passes out of the cell dV during dt is

$$\left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right] dV\,dt \tag{a}$$

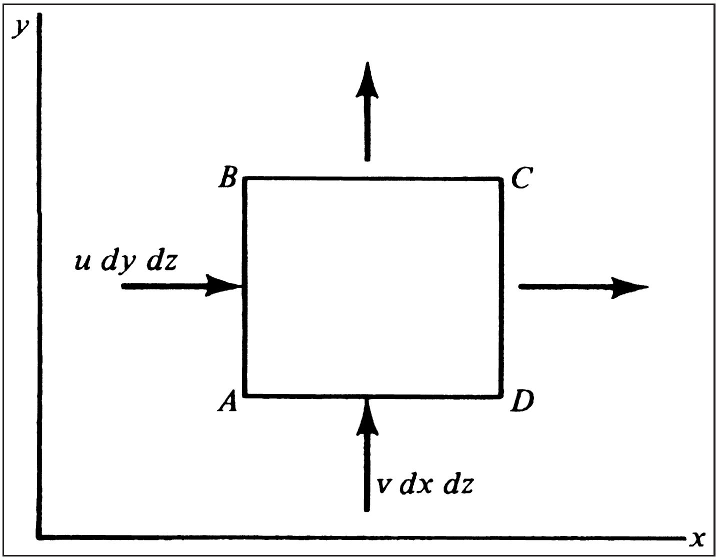


Figure 1-14.1

With the differential operator ∇ [see Eq. (1-8.5)] this may be written as

$$\nabla \cdot (\rho \mathbf{q})\, dV\, dt \tag{b}$$

The product $\rho\mathbf{q}$ is called *current density.*

If $\mathbf{a}(x, y, z; t)$ is any vector field, $\nabla \cdot \mathbf{a}$ is called the *divergence* of the field. Accordingly, the notation div $\mathbf{a}$ is sometimes used to denote $\nabla \cdot \mathbf{a}$. Note that div $\mathbf{a}$ is a scalar. Accordingly, by Eq. (b), the mass that flows out of the volume element dV during dt is

$$dV\, dt\, \mathrm{div}(\rho\mathbf{q}) \tag{c}$$

The name "divergence" originates in this physical idea.

Because mass is conserved in the velocity field of a fluid, the mass that passes into the fixed cell dV during time dt equals the increase of mass in the cell during dt. Now, the mass in the cell at the time t is $\rho\, dV$. Consequently, the increase of mass during dt is

$$\frac{\partial \rho}{\partial t} dV\, dt \tag{d}$$

Because Eq. (d) must be the negative of Eq. (c), we obtain

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho\mathbf{q}) = 0 \tag{1-14.1}$$

Equation (1-14.1) is known as the *Eulerian*[4] *continuity equation* for fluids. Any real velocity field must conform to this relation. For steady flow, the term $\partial\rho/\partial t$ disappears.

For an incompressible fluid, $\rho =$ constant. Consequently, the Eulerian form of the continuity equation for an incompressible fluid takes the simpler form:

$$\text{div}\,\mathbf{q} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1-14.2}$$

This is valid even for unsteady flow of an incompressible fluid. Liquids may usually be considered to be incompressible except in the study of compression waves.

The case in which the velocity $\mathbf{q}$ is the gradient of a scalar function has great theoretical importance, that is, the case where

$$\mathbf{q} = -\text{grad}\,\phi \tag{1-14.3}$$

where $\phi(x, y, z;\ t)$ is a scalar function. The flow is then said to be *irrotational* or *derivable from a potential function* ϕ. Then the velocity component in the direction of a unit vector $\mathbf{n}$ is

$$q_n = \mathbf{q}\cdot\mathbf{n} = -\mathbf{n}\cdot\text{grad}\,\phi \tag{1-14.4}$$

Hence, by Eq. (1-8.3),

$$q_n = -\frac{d\phi}{ds} \tag{1-14.5}$$

That is, q_n is equal to the negative of the directional derivative of ϕ in the direction $\mathbf{n}$.

Equation (1-14.3) may be written

$$u = -\frac{\partial\phi}{\partial x} \qquad v = -\frac{\partial\phi}{\partial y} \qquad w = -\frac{\partial\phi}{\partial z}$$

Accordingly, by Eq. (1-14.2) the continuity equation for irrotational flow of an incompressible fluid is

$$\nabla^2\phi = 0 \tag{1-14.6}$$

Thus, the continuity equation for irrotational flow of an incompressible fluid reduces to the Laplace equation (see Section 1-8). A general expression for the Laplace equation in orthogonal curvilinear coordinates in three-dimensional space is derived in Section 1-22.

[4]This form of the equation of continuity is referred to as the spatial form in modern continuum mechanics (see Chapter 2).

1-15 Divergence Theorem

Let $\mathbf{a}(x, y, z)$ be any continuous and differentiable vector field. We may regard $\mathbf{a}$ as current density in a hypothetical fluid. Then, by Eq. (c) of Section 1-14, div $\mathbf{a}\, dx\, dy\, dz$ is the net rate at which fluid flows out of the fixed space element $dx\, dy\, dz$. Hence, if R is a given fixed region of space that is bounded by a surface S, the net rate at which fluid passes out of R is

$$\iiint_R \operatorname{div} \mathbf{a}\, dx\, dy\, dz$$

This must also be the rate at which fluid passes through the surface S. If dS is an element of area of this surface with outward-directed unit normal $\mathbf{n}$, the rate of flow through dS is $\mathbf{a} \cdot \mathbf{n}\, dS$. Hence,

$$\iiint_R \operatorname{div} \mathbf{a}\, dx\, dy\, dz = \iint_S \mathbf{a} \cdot \mathbf{n}\, dS \tag{1-15.1}$$

Thus, a volume integral is transformed into a surface integral.

Equation (1-15.1) is known as the *divergence theorem* (also *Gauss's theorem*). It is purely mathematical; the reference to flow is simply an artifice to facilitate the derivation. Rigorous mathematical derivations of the theorem are given in books on advanced calculus (Goursat, 2005).

If (U, V, W) are the components of the vector $\mathbf{a}$, Eq. (1-15.1) may be expressed in scalar form:

$$\iiint_R \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right) dx\, dy\, dz = \iint_S (Un_1 + Vn_2 + Wn_3)\, dS = \iint_S a_n\, dS \tag{1-15.2}$$

where a_n denotes the projection of $\mathbf{a}$ in the direction of $\mathbf{n}$, and (n_1, n_2, n_3) are the direction cosines of the unit vector $\mathbf{n}$; the functions (U, V, W) are unrestricted, aside from the requirements of continuity and differentiability. The surface S may consist of a finite number of smooth parts that are joined together along edges. If the vector $\mathbf{n}$ is directed inward, the sign of the right side of Eq. (1-15.2) is reversed.

Many useful results can be obtained by giving special forms to the functions (U, V, W). For example, if $U = AB$, $V = W = 0$, we obtain

$$\iiint_R A\frac{\partial B}{\partial x}\, dx\, dy\, dz = -\iiint_R B\frac{\partial A}{\partial x} dx\, dy\, dz + \iint_S ABn_1\, dS \tag{1-15.3}$$

Corresponding results for y and z are obtained by setting $V = AB$, $U = W = 0$, and so on. These equations are similar in form to the formula for integration by parts of a single integral. Alternatively, if we take $V = W = 0$, Eq. (1-15.2) yields

$$\iiint_R \frac{\partial U}{\partial x} dx\, dy\, dz = \iint_R U n_1\, dS \tag{1-15.3a}$$

Similar results are obtained for $U = W = 0$ and $U = V = 0$. Equation (1.15.3a) is called *Gauss's theorem.* More generally, *Gauss's theorem* may be written in the form

$$\int_V \frac{\partial F_i}{\partial x_i}\, dV = \int_S F_i n_i\, dS \qquad i = 1, 2, 3 \tag{1-15.3b}$$

where $F_i = F_i(x_1,\ x_2,\ x_3)$, V denotes volume, S denotes surface of volume V with unit normal vector $\mathbf{n} : (n_1,\ n_2,\ n_3)$, and $x_1 \equiv x, x_2 \equiv y$, and $x_3 \equiv z$.

Another useful relation may be obtained as follows: Let $\mathbf{a}$ be the product of a scalar ϕ and a vector $\mathbf{A}$; that is,

$$\mathbf{a} = \phi \mathbf{A}$$

Then

$$\operatorname{div} \mathbf{a} = \phi \operatorname{div} \mathbf{A} + \frac{\partial \phi}{\partial x} A_x + \frac{\partial \phi}{\partial y} A_y + \frac{\partial \phi}{\partial z} A_z \tag{1-15.4}$$

or

$$\operatorname{div} \mathbf{a} = \phi \operatorname{div} \mathbf{A} + (\operatorname{grad} \phi) \cdot \mathbf{A}$$

Accordingly, Eq. (1-15.1) yields

$$\iint_S \phi A_n\, dS = \iiint_R [\phi \operatorname{div} \mathbf{A} + (\operatorname{grad} \phi) \cdot \mathbf{A}]\, dV \tag{1-15.5}$$

If, furthermore, the vector $\mathbf{A}$ is representable as the gradient of a scalar function $\psi(\mathbf{A} = \operatorname{grad} \psi)$, then by Eq. (1-14.5), $A_n = d\psi/dn$ and

$$\operatorname{div} \mathbf{A} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \nabla^2 \psi$$

Hence, for $\mathbf{A} = \operatorname{grad} \psi$, Eq. (1-15.5) becomes

$$\iint_S \phi \frac{\partial \psi}{\partial n}\, dS = \iiint_R \left[\phi \nabla^2 \psi + (\operatorname{grad} \phi) \cdot (\operatorname{grad} \psi)\right] dV \tag{1-15.6}$$

Equation (1-15.6) holds for any two functions ϕ and ψ that are finite, continuous, and twice differentiable in R.

If we subtract from Eq. (1-15.6) the equation obtained by interchanging ϕ and ψ, we obtain

$$\iint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS = \iiint_R \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) dV \tag{1-15.7}$$

Both Eqs. (1-15.6) and (1-15.7) are referred to as *Green's theorem.* They find extensive use in mathematical physics.

The above results are useful in transformations from volume to surface integrals and vice versa.

1-16 Divergence Theorem in Two Dimensions

The two-dimensional analog of Eq. (1-15.2) is

$$\iint_R \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) dx\, dy = \oint_C (U n_1 + V n_2)\, ds \tag{1-16.1}$$

where U and V are any continuous and differentiable functions of (x, y). Here R denotes a region of the (x, y) plane, and C is the curve that bounds the region R (Fig. 1-16.1). The unit normal vector (n_1, n_2) is directed outward. The element of arc length of the curve C is denoted by ds. The circle on the integral sign shows that the integration extends completely around the curve C, in the counterclockwise sense.

Referring to the figure, we have $n_1 = \cos\alpha$, and $n_2 = \sin\alpha$. Hence, $n_1\, ds = dy$, and $n_2\, ds = -dx$, where (dx, dy) is the displacement along the curve C, corresponding to the increment ds. Hence, by Eq. (1-16.1),

$$\iint_R \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) dx\, dy = \oint_C (U\, dy - V\, dx) \tag{1-16.2}$$

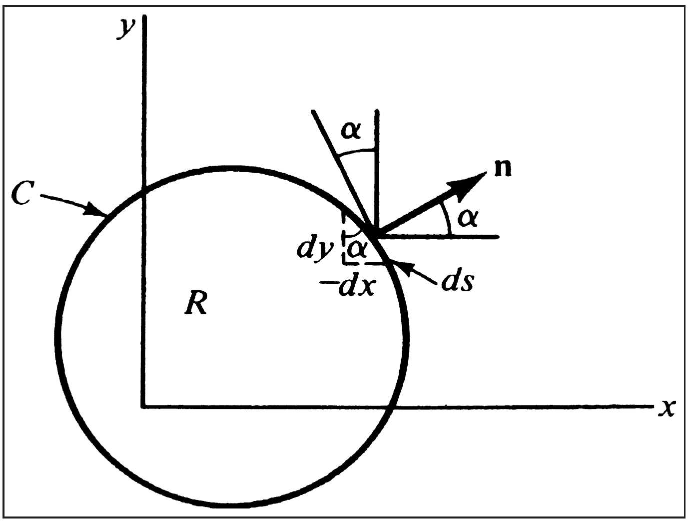


Figure 1-16.1

This relation is sometimes called *Green's theorem of the plane.* Another form of Green's theorem is obtained by the substitution $U = v$, $V = -u$. Then

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy = \oint_C (u\, dx + v\, dy) \tag{1-16.3}$$

With $U = AB$, $V = 0$, Eq. (1-16.2) yields

$$\iint_R A \frac{\partial B}{\partial x}\, dx\, dy = -\iint_R B \frac{\partial A}{\partial x}\, dx\, dy + \oint_C AB\, dy \tag{1-16.4}$$

Furthermore, analogous to the three-dimensional development of Eqs. (1-15.6) and (1-15.7), we have

$$\oint_C \phi \frac{\partial \psi}{\partial n}\, ds = \iint_R \left[\phi \nabla^2 \psi + (\text{grad}\, \phi) \cdot (\text{grad}\, \psi)\right] dx\, dy \tag{1-16.5}$$

$$\oint_C \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) ds = \iint_R \left(\phi \nabla^2 \psi - \psi \nabla^2 \phi\right) dx\, dy \tag{1-16.6}$$

where (ϕ, ψ) are functions of (x, y) only.

1-17 Line and Surface Integrals (Application of Scalar Product)

Line Integral. Consider a vector $\mathbf{F}$ defined at each point on a curve C (Fig. 1-17.1). The vector $\mathbf{F}$ forms an angle α with the tangent to the curve C at point P. In general, the vector $\mathbf{F}$ may vary in magnitude and direction along the curve. Let s be an arc length measured along the curve. The length of an infinitesimal element of the curve at point P is ds. The vector $\mathbf{ds}$ with magnitude ds is directed along the tangent line to the curve at point P (Fig. 1-17.1).

By Eq. (1-7.8), the projection of the vector $\mathbf{F}$ along the tangent to the curve is $\mathbf{F} \cdot \mathbf{ds} = F(\cos \alpha)\, ds$. The integral

$$\int_C \mathbf{F} \cdot \mathbf{ds} = \int_C F(\cos \alpha)\, ds \tag{1-17.1}$$

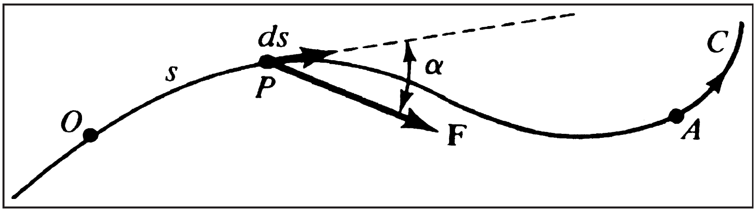


Figure 1-17.1

is called the line integral of the vector **F** along the curve C. The C in Eq. (1-17.1) denotes integration along the curve C. By Eq. (1-17.1) it is apparent that the line integral of a vector is the integral of the tangential component of the vector taken along a path.

The line integral Eq. (1-17.1) finds numerous applications in physical problems. For example, if **F** denotes a force that acts on a particle P that travels along curve C, the line integral of the tangential component of **F** from point O to point A represents the work performed by the force **F** as the particle travels from O to A. If **F** denotes the electric field intensity, that is, the force that acts on a unit charge in an electric field, the line integral between any two points represents the potential difference between the two points. If **F** denotes the velocity at any point in a fluid, the line integral taken around a closed path in the fluid represents the *circulation* of the fluid.

Surface Integral. In Section 1-15 it was shown that the volume of fluid that passes through a surface S in a unit time is

$$\iint_S \mathbf{q} \cdot \mathbf{n}\, dS = \iint_S q_n\, dS \tag{1-17.2}$$

where **q** is the velocity field and **n** is the unit normal to the surface. This integral is called the *surface integral* of the vector **q**. Accordingly, the expression *surface integral of a vector* denotes the integral of the normal component of the vector over a surface.

1-18 Stokes's Theorem

Equation (1-16.3) may be written as

$$\oint_C \mathbf{q} \cdot d\mathbf{r} = \iint_R \mathbf{n} \cdot \operatorname{curl} \mathbf{q}\, dS \tag{1-18.1}$$

where $d\mathbf{r} = (dx,\ dy)$, **q** denotes the vector $(u,\ v,\ w)$, and **n** now denotes the unit normal to the plane area R [directed in the positive z direction, if the coordinates $(x,\ y,\ z)$ are right handed]. Although Eq. (1-18.1) has been proven only if R is a region in the $(x,\ y)$ plane, it remains valid if R is any plane area in space with any orientation, for Eq. (1-18.1) is invariant under a coordinate transformation; that is, Eq. (1-18.1) does not depend on the choice of coordinates.

Our result may be generalized still further. The curve C need not be a plane curve; it may be any closed space curve, and R may be any surface S that caps this curve. Any capping surface of the curve C may be divided into infinitesimal cells. Each cell is a plane element of area. Consequently, Eq. (1-18.1) applies for any one of the cells. We may then sum Eq. (1-18.1) over all cells. Then the right side of the equation simply becomes the surface integral of curl **q** over the entire capping surface S of curve C. On the left side we have the sum of line integrals of

q about the boundaries of the cells. However, the line integrals over the boundaries of contiguous cells cancel, as any inner boundary of a cell is described twice, only in the positive sense and once in the negative sense. Consequently, only the line integral on the outer boundary C remains.

Accordingly, we have Stokes's theorem: *The line integral of a vector field about any closed curve equals the surface integral of the normal component of the curl of the vector over any capping surface.*

If **q** is a velocity field, then curl **q** is called the *vorticity vector.* Consequently, in the terminology of fluid mechanics Stokes's theorem is expressed as follows: *The circulation on any closed curve equals the flux of vorticity through the loop.*

1-19 Exact Differential

Let $M(x, y)$ and $N(x, y)$ be two functions of x and y such that $M, N, \partial M/\partial y$, and $\partial N/\partial x$ are continuous and single valued at every point of a simply connected[5] region. The differential expression $M\,dx + N\,dy$ is said to be *exact* if there exists a function $f(x, y)$ such that $df = M\,dx + N\,dy$. Now, by definition,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{1-19.1}$$

Consequently, if $M\,dx + N\,dy$ is exact, $M = \partial f/\partial x$, and $N = \partial f/\partial y$. Therefore,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{or} \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 \tag{1-19.2}$$

Accordingly, Eq. (1-19.2) is a *necessary condition* for $M\,dx + N\,dy$ to be an exact differential.

Equation (1-19.2) is also a *sufficient condition.* Assume that Eq. (1-19.2) is satisfied. Set

$$F(x, y) = \int M\,dx$$

where integration is performed with respect to x. Then $\partial F/\partial x = M$ and

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Therefore,

$$\frac{\partial}{\partial x}\left(N - \frac{\partial F}{\partial y}\right) = 0 \quad \text{or} \quad N = \frac{\partial F}{\partial y} + g(y)$$

Set $f(x, y) = F(x, y) + \int g(y)\,dy$. Then $N = \partial f/\partial y$ and $M = \partial F/\partial x = \partial f/\partial x$. Hence, $M\,dx + N\,dy = df$; that is, $M\,dx + N\,dy$ is an exact differential.

[5] A simply connected region has the property that any closed curve drawn on it can, by continuous deformation, be shrunk to a point without crossing the boundary of the region. For the significance of simple connectivity, see Courant (1992), Vol. II.

If $f = f(x, y, z)$, $df = P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dz$ where $P = \partial f/\partial x$, $Q = \partial f/\partial y$, and $R = \partial f/\partial z$, an argument analogous to the two-dimensional case leads to the necessary and sufficient conditions that *df* be an exact differential in the form

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \qquad \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0 \qquad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0 \tag{1-19.3}$$

1-20 Orthogonal Curvilinear Coordiantes in Three-Dimensional Space

Let three independent scalar functions (u, v, w) be defined in terms of three independent variables (x, y, z) as follows:

$$u = U(x, y, z) \qquad v = V(x, y, z) \qquad w = W(x, y, z) \tag{1-20.1}$$

By independent functions, we mean that Eqs. (1-20.1) yield unique solutions for (x, y, z):

$$x = X(u, v, w), \qquad y = Y(u, v, w), \qquad z = Z(u, v, w) \tag{1-20.2}$$

For example, if (x, y, z) represents rectangular Cartesian coordinates, and (u, v, w) represents cylindrical coordinates, Eq. (1-20.2) is of the form

$$x = u \cos v \qquad y = u \sin v \qquad z = w \tag{1-20.3}$$

If (u, v, w) represents spherical coordinates, Eq. (1-20.2) is of the form

$$x = u \sin v \cos w \qquad y = u \sin v \sin w \qquad z = u \cos v \tag{1-20.4}$$

If (u, v, w) are assigned constant values, Eq. (1-20.1) becomes

$$\begin{aligned} U_0(x, y, z) &= \text{const} = u_0 \\ V_0(x, y, z) &= \text{const} = v_0 \\ W_0(x, y, z) &= \text{const} = w_0 \end{aligned} \tag{1-20.5}$$

Equations (1-20.5) represent three surfaces in space, called *coordinate surfaces.* The intersection of any two of these surfaces (say, $U_0 = u_0$ and $V_0 = v_0$) determines a curve in space, the w *curvilinear coordinate line.* The u and v curvilinear coordinate lines are defined similarly. The three surface $U_0 = u_0$, $V_0 = v_0$, and $W_0 = w_0$ intersect at a point in space. Hence, a point in space is associated with each triplet (u_i, v_i, w_i).

If the three systems of surfaces defined by triplets (u_i, v_i, w_i) are mutually perpendicular (i.e., if the curvilinear coordinate lines through any point are mutually perpendicular), the curvilinear coordinate system is said to be *orthogonal.*

A very special case of an orthogonal curvilinear coordinate system is the rectangular Cartesian coordinate system. For rectangular coordinates,

$$x = u \qquad y = v \qquad z = w$$

Hence, three coordinate surfaces are the mutually perpendicular planes:

$$x = u_0 \qquad y = v_0 \qquad z = w_0$$

The intersection of any two of these planes is a *coordinate line*; for example, the intersection of planes $x = u_0$, $y = v_0$ determines a z coordinate line. Cylindrical coordinates [Eq. (1-20.3)] and spherical coordinate [Eq. (1-20.4)] are also examples of orthogonal curvilinear coordinate systems. Another example is elliptic coordinates.

1-21 Expression for Differential Length in Orthogonal Curvilinear Coordinates

Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be unit vectors along (x, y, z) axes, respectively. Let (u, v, w) be a system of orthogonal curvilinear coordinates. Let v and w be constant. Then at any point the tangent vector to the u coordinate line is

$$\mathbf{U} = x_u\mathbf{i} + y_u\mathbf{j} + z_u\mathbf{k} \tag{1-21.1}$$

where the u subscript denotes partial differentiation. Similarly, tangent vectors to the v and w coordinate lines are

$$\mathbf{V} = x_v\mathbf{i} + y_v\mathbf{j} + z_v\mathbf{k} \qquad \mathbf{W} = x_w\mathbf{i} + y_w\mathbf{j} + z_w\mathbf{k} \tag{1-21.2}$$

Vectors $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ are mutually perpendicular. Hence, by the scalar product definition of two vectors,

$$\mathbf{U} \cdot \mathbf{V} = \mathbf{V} \cdot \mathbf{W} = \mathbf{W} \cdot \mathbf{U} = 0 \tag{1-21.3}$$

Also, if (h_1, h_2, h_3) are the magnitudes of the lengths of vectors $(\mathbf{U}, \mathbf{V}, \mathbf{W})$, respectively, the scalar product definition yields

$$h_1^2 = \mathbf{U} \cdot \mathbf{U} \qquad h_2^2 = \mathbf{V} \cdot \mathbf{V} \qquad h_3^2 = \mathbf{W} \cdot \mathbf{W} \tag{1-21.4}$$

Hence, by Eqs. (1-20.2), (1-21.1), (1-21.2), and (1-21.4), $h_1 = h_1(u, v, w)$, $h_2 = h_2(u, v, w)$, and $h_3 = h_3(u, v, w)$.

Consider a line element PQ, where $P = P(x, y, z)$ and $Q = Q(x + dx, y + dy, z + dz)$. The differential length ds of the line element PQ is given by the relation

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{1-21.5}$$

By Eq. (1-20.2),

$$dx = x_u\,du + x_v\,dv + x_w\,dw$$
$$dy = y_u\,du + y_v\,dv + y_w\,dw \qquad (1\text{-}21.6)$$
$$dz = z_u\,du + z_v\,dv + z_w\,dw$$

Substituting Eqs. (1-21.6) into Eq. (1-21.5) and utilizing Eqs. (1-21.2), (1-21.3), and (1-21.4), we obtain

$$ds^2 = h_1^2\,du^2 + h_2^2\,dv^2 + h_3^2\,dw^2 \qquad (1\text{-}21.7)$$

Equation (1-21.7) expresses the differential length ds in terms of the orthogonal curvilinear coordinates (u, v, w). The coefficients (h_1, h_2, h_3) are called *Lamé coefficients.* The Lamé coefficients are equal in magnitude to the lengths of the vectors ($\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$) tangent to (u, v, w) coordinate lines, respectively. The quantities (h_1^2, h_2^2, h_3^2) are known as the *components of the metric tensor of space* (Synge and Schild, 1978).

1-22 Gradient and Laplacian in Orthogonal Curvilinear Coordinates

Consider the infinitesimal parallelepiped whose diagonal is the line element ds. The faces of the parallelepiped coincide with the planes $u =$ constant, $v =$ constant, $w =$ constant (Fig. 1-22.1).

The gradient u, (∇u) has the direction normal to the surface $u =$ constant; that is, the direction of $\mathbf{U}$ or the direction of the unit vector $\mathbf{U}/|\mathbf{U}| = \mathbf{U}/h_1$ [see

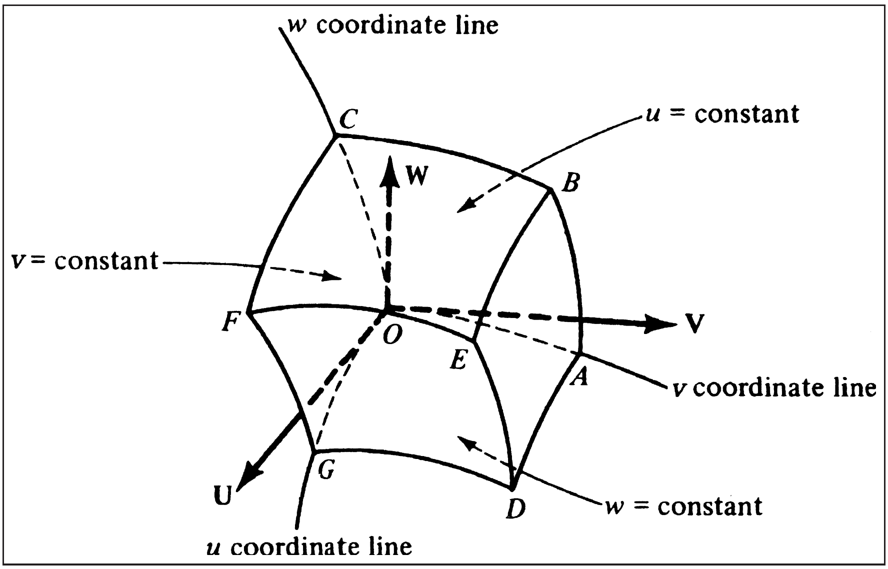


Figure 1-22.1

Eq. (1-21.4)]. The magnitude of ∇u is equal to the derivative of u in this direction. Hence, by Eq. (1-21.7), with v and w constant, the magnitude of ∇u is

$$\frac{du}{ds} = \frac{1}{h_1} \tag{1-22.1}$$

Hence, the gradient vector is

$$\nabla u = \frac{1}{h_1^2}\mathbf{U} \tag{1-22.2}$$

Similarly, the gradient of v and w are

$$\nabla v = \frac{1}{h_2^2}\mathbf{V} \qquad \nabla w = \frac{1}{h_3^2}\mathbf{W} \tag{1-22.3}$$

By the definition of ∇ and by the rule for partial differentiation, that is,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

if $f(u, v, w)$ is any scalar point function, then the gradient of f is

$$\nabla f = \frac{\partial f}{\partial u}\nabla u + \frac{\partial f}{\partial v}\nabla v + \frac{\partial f}{\partial w}\nabla w \tag{1-22.4}$$

Substituting Eqs. (1-22.2) and (1-22.3) into Eq. (1-22.4), we obtain

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u}\mathbf{u} + \frac{1}{h_2}\frac{\partial f}{\partial v}\mathbf{v} + \frac{1}{h_3}\frac{\partial f}{\partial w}\mathbf{w} \tag{1-22.5}$$

where $(\mathbf{u},\ \mathbf{v},\ \mathbf{w})$ are unit vectors in the directions of $(\mathbf{U},\ \mathbf{V},\ \mathbf{W})$, respectively; that is,

$$\mathbf{u} = \frac{\mathbf{U}}{h_1} \qquad \mathbf{v} = \frac{\mathbf{V}}{h_2} \qquad \mathbf{w} = \frac{\mathbf{W}}{h_3} \tag{1-22.6}$$

Equation (1-22.5) represents the gradient of a scalar in orthogonal curvilinear coordinates. Consequently, by Eq. (1-22.5), the expression for the operator ∇ in orthogonal curvilinear coordinates is

$$\nabla = \frac{1}{h_1}\mathbf{u}\frac{\partial}{\partial u} + \frac{1}{h_2}\mathbf{v}\frac{\partial}{\partial v} + \frac{1}{h_3}\mathbf{w}\frac{\partial}{\partial w} \tag{1-22.7}$$

To derive the expression for the Laplacian ∇^2, we first derive the expression for the divergence of a vector field, $\mathbf{Q} = (Q_1,\ Q_2,\ Q_3)$, that is, $\nabla \cdot \mathbf{Q}$, in orthogonal curvilinear coordinates.

Consider again the infinitesimal parallelepiped of Fig. 1-22.1. The lengths of its edges are $h_1\,du,\ h_2\,dv$, and $h_3\,dw$, and its volume is $h_1 h_2 h_3\,du\,dv\,dw$. To facilitate the calculation of the divergence of **Q**, we use Green's theorem for transforming volume integrals into surface integrals:

$$\underset{\substack{\text{through}\\\text{volume}}}{\iiint} (\nabla \cdot \mathbf{Q})\, dV = \underset{\substack{\text{over}\\\text{bounding}\\\text{surface}}}{\iint} \mathbf{Q} \cdot \mathbf{n}\, dS \tag{1-22.8}$$

The contribution of the surface *OABC*, Fig. (1-22.1), to the integral over the surface of the parallelepiped taken in the direction of the outward normal is $-Q_1 h_2\, dv\, h_3\, dw$. The contribution of the surface *DEFG* is

$$Q_1 h_2 h_3\, dv\, dw + \frac{\partial}{\partial u}(Q_1 h_2 h_3)\, du\, dv\, dw$$

Hence, the net contribution of the coordinate surfaces perpendicular to u coordinate lines is

$$\frac{\partial}{\partial u}(Q_1 h_2 h_3)\, du\, dv\, dw \tag{1-22.9}$$

Similarly, the contributions of the coordinate surfaces perpendicular to v and w coordinate lines, respectively, are

$$\frac{\partial}{\partial v}(Q_2 h_1 h_3)\, du\, dv\, dw \qquad \frac{\partial}{\partial w}(Q_3 h_1 h_2)\, du\, dv\, dw \tag{1-22.10}$$

Because the volume of the infinitesimal parallelepiped, Fig. 1-22.1, is infinitesimal,

$$\lim_{v \to 0} \iiint (\nabla \cdot \mathbf{Q})\, dV \to \nabla \cdot \mathbf{Q}\, h_1 h_2 h_3\, du\, dv\, dw \tag{1-22.11}$$

Consequently, by Eqs. (1-22.8) to (1-22.11),

$$\nabla \cdot \mathbf{Q} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(Q_1 h_2 h_3) + \frac{\partial}{\partial v}(Q_2 h_1 h_3) + \frac{\partial}{\partial w}(Q_3 h_1 h_2)\right] = \operatorname{div} \mathbf{Q} \tag{1-22.12}$$

Equation (1-22.12) represents the formula for the divergence of a vector field **Q** in terms of general three-dimensional orthogonal curvilinear coordinates.

Setting $\nabla f = \mathbf{Q}$ and noting by Eq. (1-22.5) that $Q_1 = (1/h_1)(\partial f/\partial u)$, $Q_2 = (1/h_2)(\partial f/\partial v)$, and $Q_3 = (1/h_3)(\partial f/\partial w)$, we obtain, by Eqs. (1-22.6) and (1-22.12),

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial w}\right)\right] \tag{1-22.13}$$

Equation (1-22.13) represents the Laplacian of a scalar function $f(u, v, w)$ in general three-dimensional orthogonal curvilinear coordinates. Hence, the Laplace equation $\nabla^2 f = 0$ in general three-dimensional orthogonal curvilinear coordinates is obtained by setting the right-hand side of Eq. (1-22.13) equal to zero.

For plane (two-dimensional) orthogonal curvilinear coordinates, $h_3 = 1$ and $\partial/\partial w = 0$.

PART III ELEMENTS OF TENSOR ALGEBRA

1-23 Index Notation: Summation Convention

Gibbs vector notation may be considered to replace and extend conventional scalar notation. For example, the scalar representation (F_x, F_y, F_z) of a force with respect to rectangular Cartesian axes is fully replaced by the vector notation **F**. Likewise, index notation may be considered to replace and extend Gibbs vector notation. Thus, the vector **F** may be represented by the symbol F_i, where the subscript (index) i is understood to take values $1, 2, 3$ (or the values x, y, z). Hence, the notation F_i is equivalent to (F_1, F_2, F_3) or to (F_x, F_y, F_z), where subscripts $(1, 2, 3)$ or subscripts (x, y, z) denote projections of the force along rectangular Cartesian coordinate axes $(1, 2, 3)$ or (x, y, z).

Restricting ourselves to rectangular Cartesian coordinates, we indicate coordinates by indices $(1, 2, 3)$ instead of letters (x, y, z). For example, the coordinate of a general point X in (x, y, z) space are denoted by $x_i = (x_1, x_2, x_3)$ or more briefly by x_i, with the understanding that i takes the values $(1, 2, 3)$. The coordinates of a specific point P are denoted by p_i, the letter p identifying the point and the index i, the separate coordinates (see Fig. 1-23.1). Similarly, axes (x, y, z) may be denoted by (x_1, x_2, x_3), or simply by x_i. Axes x_i may also be denoted by the notations $(01, 02, 03)$ or $(1, 2, 3)$.

The direction cosines of a line L with respect to axes x_i are denoted by $\alpha_1, \alpha_2, \alpha_3$ or briefly by α_i. Any other letter may replace α. For example, the direction cosines of line L may also be denoted by β_i, by m_i, by n_i, and so on.

The sum of two vectors q_i, r_i is $q_i + r_i$. The scalar product of two vectors u_α, v_α is [see Eq. (1-7.6)]

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 = \sum_{\alpha=1}^{3} u_\alpha v_\alpha \tag{1-23.1}$$

Equation (1-23.1) may be simplified by the use of conventional *summation* notation. For example, we may write Eq. (1-23.1) in the form

$$\mathbf{u} \cdot \mathbf{v} = u_\alpha v_\alpha \tag{1-23.2}$$

with the understanding that the *repeated Greek index* α implies summation over the values (1, 2, 3). Accordingly, if m_α and n_α denote the direction cosines of two

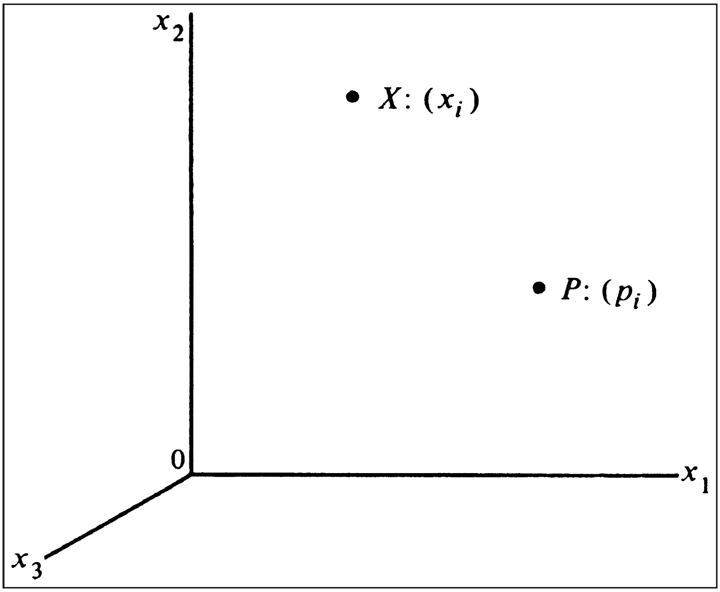


Figure 1-23.1

unit vectors directed along two lines M and N in (x, y, z) space, by the scalar product of vectors, the angle θ between lines M and N is given by the relation [see Eq. (1-7.8) and the discussion following it]

$$\cos\theta = m_\alpha n_\alpha \tag{1-23.3}$$

If lines M and N coincide, $\theta = 0$. Then Eq. (1-23.3) yields (with $m_\alpha = n_\alpha$)

$$m_1^2 + m_2^2 + m_3^3 = 1 \tag{1-23.4}$$

Accordingly, the sum of the squares of the direction cosines of a directed line in (x, y, z) space is equal to 1 [see Eq. (1-7.9)].

In general, a repeated index that is to be summed will be denoted by a Greek letter. We thus avoid the necessity of using some special notation for a repeated index that is not summed. Because the operation of summing is independent of the Greek index used to denote the summation process, the following representations of $\cos\theta$ are equivalent [see Eq. (1-23.3)]:

$$\cos\theta = m_\alpha n_\alpha = m_\beta n_\beta = m_\gamma n_\gamma = \cdots$$

as each of the representations denotes $m_1 n_1 + m_2 n_2 + m_3 n_3$. Accordingly, a repeated Greek index is called a *summing index* or a *dummy index.* An index that appears only once in a general term is called a *free index.* Thus, in the term

$A_{\alpha\beta\beta}$, the index β is a dummy index and the index α is a free index, the value of α being independent of the values of β. For example, if we assign the value 1 to α, the term $A_{\alpha\beta\beta}$ represents the sum $A_{111} + A_{122} + A_{133}$.

If a *repeated* index is *not* to be summed, we denote it by a *Latin letter* $(a, b, c, \ldots, z)$. Thus, $m_i n_i$ denotes any element of the set $(m_1 n_1,\ m_2 n_2,\ m_3 n_3)$, depending on the values assigned to i. For example, if $i = 2$, then $m_i n_i$ denotes the element $m_2 n_2$.

If several dummy indexes occur in a general term, summation is implied for each index separately. For example,

$$\begin{aligned} x_{i\alpha\beta} y_{\alpha\beta} &= x_{i1\beta} y_{1\beta} + x_{i2\beta} y_{2\beta} + x_{i3\beta} y_{3\beta} \\ &= x_{i11} y_{11} + x_{i12} y_{12} + x_{i13} y_{13} \\ &\quad + x_{i21} y_{21} + x_{i22} y_{22} + x_{i23} y_{23} \\ &\quad + x_{i31} y_{31} + x_{i32} y_{32} + x_{i33} y_{33} \end{aligned}$$

Thus, for every value of the free index i, there are nine terms in the sum $x_{i\alpha\beta} y_{\alpha\beta}$.

In modern algebra, the range of the index is often extended from $(1, 2, 3)$ to $(1, 2, 3, \ldots, n)$. Thus, we may write

$$A_{i\alpha} x_\alpha = A_{i1} x_1 + A_{i2} x_2 + \cdots + A_{in} x_n$$

where the summing index α takes values $(1,\ 2,\ 3, \ldots, n)$.

To avoid confusion, an index already appearing in a general term as a free index should not be used as a dummy index, as no meaning is given indexes that appear more than twice. Thus, notations such as $A_{\beta\beta} x_\beta$ should be avoided. For example, if $x = A_\alpha y_\alpha$ and $y_i = B_{i\alpha} z_\alpha$, the expression for x in terms of (z_1, z_2, z_3) is written

$$x = A_\alpha B_{\alpha\beta} z_\beta$$

not in the meaningless form

$$x = A_\alpha B_{\alpha\alpha} z_\alpha$$

Rectangular Arrays. A set of numbers arranged in the following form is called a *rectangular array*:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \tag{1-23.5}$$

where, in general, $m \neq n$.

More generally, such an array of numbers is called a *matrix.* In the study of matrix theory, extensive rules are laid down for the multiplication of matrices (Section 1-28). However, the role of products in matrix theory is to a large extent replaced by summation convention. A typical element of an array is denoted by a_{ij}, the index i referring to the ith row of the array and the index j to the jth column. For brevity, the entire array [Eq. (1-23.5)] is denoted by

$$[a_{ij}] \tag{1-23.6}$$

If $m = n$, the array is called a *square array*. In the theory of continuous media, we are concerned primarily with square arrays.

If the arrays $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$, ... all have the same number of rows and the same number of columns, a linear combination $[h_{ij}]$, of $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}]$, ... is defined by the elements

$$h_{ij} = Aa_{ij} + Bb_{ij} + Cc_{ij} + \cdots \tag{1-23.7}$$

where $A, B, C, \ldots$ are arbitrary constants independent of i and j. In particular, the *sum* $[a_{ij} + b_{ij} + c_{ij}]$ of the three arrays $[a_{ij}]$, $[b_{ij}]$, and $[c_{ij}]$ has the typical element $a_{ij} + b_{ij} + c_{ij}$.

A square array $[a_{ij}]$ is said to be *symmetric* if

$$a_{ij} = a_{ji} \tag{1-23.8}$$

for all pairs of values of i, j; a square array is said to be *skew symmetric* or *antisymmetric* if

$$a_{ij} = -a_{ji} \tag{1-23.9}$$

for all pairs of i, j. For an antisymmetric array, it follows, by Eq. (1-23.9), that $a_{ii} = a_{jj} = 0$.

An *arbitrary square array* (neither symmetric nor antisymmetric) may be represented as the sum of a symmetric array and an antisymmetric array. For example, any two numbers r and s can always be written in the form

$$r = \tfrac{1}{2}(x + y) \qquad s = \tfrac{1}{2}(x - y)$$

by letting

$$x = r + s \qquad y = r - s$$

Hence, we may express a typical element of the arbitrary square array $[a_{ij}]$ in the form

$$\begin{aligned} a_{ij} &= \tfrac{1}{2}(a_{ij} + a_{ij}) + \tfrac{1}{2}(a_{ji} - a_{ji}) \\ &= \tfrac{1}{2}(a_{ij} + a_{ji}) + \tfrac{1}{2}(a_{ij} - a_{ji}) \end{aligned}$$

or

$$a_{ij} = c_{ij} + d_{ij} \tag{1-23.10}$$

where

$$c_{ij} = \tfrac{1}{2}(a_{ij} + a_{ji}) = c_{ji}$$

denotes the elements of a symmetric square array, and

$$d_{ij} = \frac{1}{2}(a_{ij} - a_{ji}) = -d_{ji}$$

denotes the elements of an antisymmetric square array.

1-24 Transformation of Tensors under Rotation of Rectangular Cartesian Coordinate System

In this section we consider briefly some tensor transformations and properties that are important in the theory of deformable media. For simplicity, we restrict our discussion to rectangular Cartesian coordinates. Accordingly, the results presented here are special cases of more general tensor transformations (Synge and Schild, 1978; Spain, 2003).

Let (x, y, z) and (X, Y, Z) denote two right-handed rectangular Cartesian coordinate systems with common origin (Fig. 1-24.1). The cosines of the angles between the six coordinate axes may be represented in tabular form (Table 1-24.1). Each entry in Table 1-24.1 is the cosine of the angle between the two coordinate axes

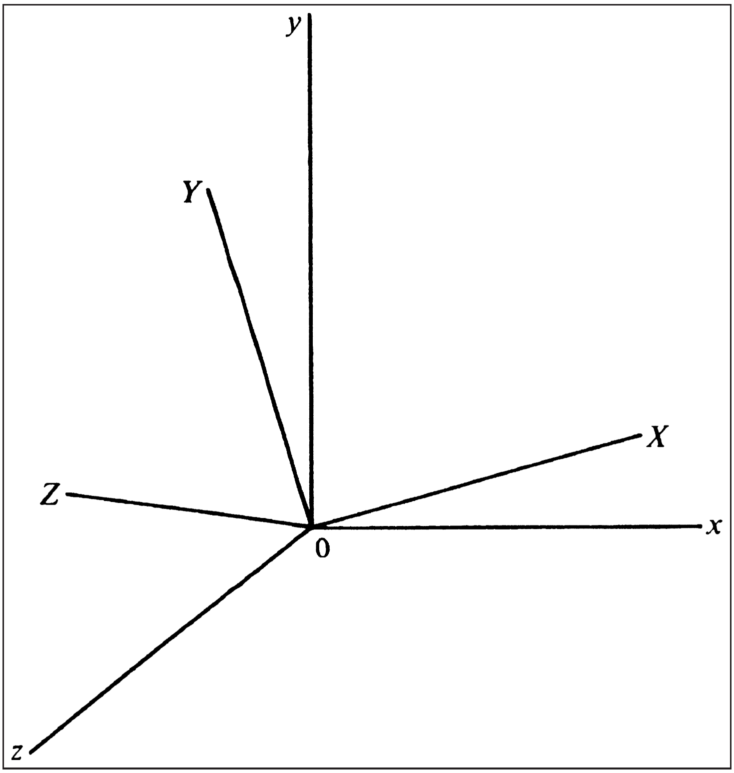


Figure 1-24.1

TABLE 1-24.1

	x	y	z
X	a_{11}	a_{12}	a_{13}
Y	a_{21}	a_{22}	a_{23}
Z	a_{31}	a_{32}	a_{33}

designated at the top of its column and left of its row. For example, a_{23} denotes the cosine of the angle between the Y axis and the z axis; that is, $a_{\alpha\beta}$ represents the direction cosines of the angle between the axes designated by the row α and the column β of Table 1-24.1. Because the elements of Table 1-24.1 are direction cosines, they satisfy the following relations (Eisenhart, 2005):

$$\begin{aligned} a_{1\beta}^2 + a_{2\beta}^2 + a_{3\beta}^2 &= 1 \qquad \beta = 1, 2, 3 \\ a_{\alpha 1}^2 + a_{\alpha 2}^2 + a_{\alpha 3}^2 &= 1 \qquad \alpha = 1, 2, 3 \end{aligned} \tag{1-24.1}$$

Equation (1-24.1) signifies that the sum of the squares of the elements of any row or column of Table 1-24.1 is 1. Furthermore, because the axes (X, Y, Z) are mutually perpendicular, we have

$$a_{\alpha 1}a_{\beta 1} + a_{\alpha 2}a_{\beta 2} + a_{\alpha 3}a_{\beta 3} = 0 \qquad \alpha, \beta = 1, 2, 3 \qquad \alpha \neq \beta \tag{1-24.2}$$

Similarly, because (x, y, z) are mutually perpendicular, we have further

$$a_{1\beta}a_{1\alpha} + a_{2\beta}a_{2\alpha} + a_{3\beta}a_{3\alpha} = 0 \qquad \alpha, \beta = 1, 2, 3 \qquad \alpha \neq \beta \tag{1-24.3}$$

Equations (1-24.2) and (1-24.3) signify that the sum of the products of corresponding elements in any two rows or any two columns in Table 1-24.1 is zero. In other words, they express the orthogonality of axes (X, Y, Z) and the orthogonality of axes (x, y, z). For this reason, they are called *orthogonality relations.*

Another important relation between the coefficients of Table 1-24.1 may be obtained as follows. Noting that the direction cosines of a unit vector with respect to (x, y, z) axes are identical to the projections of the unit vector on the coordinate axes, we regard the direction cosines (a_{11}, a_{12}, a_{13}) as the components on (x, y, z) axes of a unit vector in the X direction. Similarly, (a_{21}, a_{22}, a_{23}) and (a_{31}, a_{32}, a_{33}) represent unit vectors in the Y direction and the Z direction, respectively. Hence, by the vector product of vectors [see Eq. (1-7.13)], if the two coordinate systems (x, y, z) and (X, Y, Z) are both right handed (or both left handed), we obtain the vector relation

$$(a_{11}, a_{12}, a_{13}) = (a_{21}, a_{22}, a_{23}) \times (a_{31}, a_{32}, a_{33})$$

or, in scalar notation,

$$\begin{aligned} a_{11} &= a_{22}a_{33} - a_{23}a_{32} \\ a_{12} &= a_{31}a_{23} - a_{21}a_{33} \\ a_{13} &= a_{21}a_{32} - a_{22}a_{31} \end{aligned} \tag{1-24.4}$$

Similar relations hold for $(a_{21}, a_{22}, a_{23}), \ldots, (a_{13}, a_{23}, a_{33})$. In index notation, the entire set of relations may be written

$$a_{kr} = a_{ip}a_{jq} - a_{iq}a_{jp} \tag{1-24.5}$$

where (i, j, k), the first indexes of each direction cosine, may take any cyclic order of $1, 2, 3, 1, 2, \ldots$, and where (p, q, r), the second indexes of each direction cosine, take independently any cyclic order of $1, 2, 3, 1, 2, \ldots$. For example, let (i, j, k) be (2, 3, 1) and let (p, q, r) be (2, 3, 1). Then Eq. (1-24.5) yields

$$a_{11} = a_{22}a_{33} - a_{23}a_{32}$$

Similarly, $(i, j, k) = (1, 2, 3)$, $(p, q, r) = (3, 1, 2)$ yields

$$a_{32} = a_{13}a_{21} - a_{11}a_{23}$$

Equations (1-24.5) are also referred to as orthogonality relations, as they express the orthogonality of axes (x, y, z) and of axes (X, Y, Z).

In view of Eqs. (1-24.4), the second equation of Eqs. (1-24.1), with $\alpha = 1$, may be written

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{31}a_{23} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) = 1$$

Similar expressions hold for $\alpha = 2, 3$; $\beta = 1, 2, 3$.

In determinant notation, the above equation may be written in the form

$$\det a_{\alpha\beta} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 1 \tag{1-24.6}$$

where det denotes determinant. If the coordinate system is left handed, it may be shown that $\det a_{\alpha\beta} = -1$. Consequently, we have the following theorem:

Theorem 1-24.1. *Any one of the direction cosines of a set of right-handed (left-handed) rectangular Cartesian axes measured with respect to a second set of right-handed (left-handed) rectangular Cartesian axes is equal to its cofactor (the negative of its cofactor) in the determinant formed from the square array of direction cosines [see Eqs. (1-24.4) and (1-24.6)]. Furthermore, the numerical value of the determinant is 1(−1).*

In the following, we consider right-handed coordinate systems only.

Let the coordinates of a point P be (x, y, z) with respect to axes (x, y, z). Then, with respect to (X, Y, Z) axes, the coordinates of P may be expressed in terms of coordinates (x, y, z) by the equations

$$\begin{aligned} X &= a_{11}x + a_{12}y + a_{13}z \\ Y &= a_{21}x + a_{22}y + a_{23}z \\ Z &= a_{31}x + a_{32}y + a_{33}z \end{aligned} \tag{1-24.7}$$

For (X, Y, Z) axes with origin at (a_{10}, a_{20}, a_{30}), Eqs. (1-24.7) may be generalized by the substitution $X = X - a_{10}$, $Y = Y - a_{20}$, and $Z = Z - a_{30}$.

Conversely, with respect to (x, y, z) axes, the coordinates of P expressed in terms of (X, Y, Z) are given by the relations (because det $a_{\alpha\beta} = 1$)

$$\begin{aligned} x &= a_{11}X + a_{21}Y + a_{31}Z \\ y &= a_{12}X + a_{22}Y + a_{32}Z \\ z &= a_{13}X + a_{23}Y + a_{33}Z \end{aligned} \tag{1-24.8}$$

With the summation notation introduced in Section 1-23, Eq. (1-24.7) becomes

$$X_\alpha = a_{\alpha 1}x_1 + a_{\alpha 2}x_2 + a_{\alpha 3}x_3 \qquad \alpha = 1, 2, 3 \tag{1-24.9}$$

or

$$X_\alpha = a_{\alpha\beta}x_\beta \qquad \alpha, \beta = 1, 2, 3$$

Similarly, Eq. (1-24.8) may be written

$$x_\beta = a_{1\beta}X_1 + a_{2\beta}X_2 + a_{3\beta}X_3 \qquad \beta = 1, 2, 3$$

$$x_\beta = a_{\alpha\beta}X_\alpha \qquad \alpha, \beta = 1, 2, 3 \tag{1-24.10}$$

For given values of α and β, the value of $\alpha_{\alpha\beta}$ in Eq. (1-24.9) is identical to the value of $a_{\alpha\beta}$ in Eq. (1-24.10). This follows from the definition of the entries in Table 1-24.1.

With the understanding that α, β take values 1, 2, 3, Eqs. (1-24.9) and (1-24.10) are written

$$X_\alpha = a_{\alpha\beta}x_\beta \tag{1-24.11}$$

and

$$x_\beta = a_{\alpha\beta}X_\alpha \tag{1-24.12}$$

Because a repeated Greek index is always summed, it may be replaced by any convenient letter, as noted in Section 1-23. Accordingly, the following forms for Eq. (1-24.11) are all equivalent:

$$X_\alpha = a_{\alpha\beta}x_\beta = a_{\alpha\gamma}x_\gamma = a_{\alpha\zeta}x_\zeta$$

Scalars. Quantities such as temperature and density that may be represented by a single number—for example, 10°C or 30 g/cm^3—are called scalars. Under a transformation of coordinate axes, scalars remain unchanged; that is, scalars are invariant under coordinate transformations. For this reason, scalars are often called invariants. In tensor theory, scalars are called *tensors of zero* order.

Vectors. In summation notation a vector is represented by the symbol u_i (Section 1-23). Suppose the arrow OP representing the vector u_i is attached

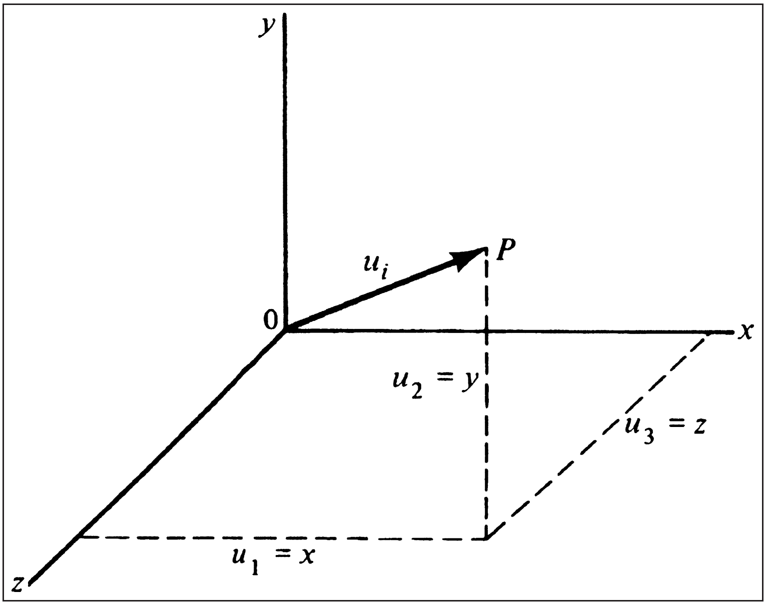


Figure 1-24.2

to a rectangular Cartesian coordinate system (x, y, z), as in Fig. 1-24.2. Then the coordinates of P correspond to the components (u_1, u_2, u_3) of vector u_i. Consequently, under a transformation from one rectangular Cartesian coordinate system to another, the components of a three-dimensional vector transform according to the relationship [see Eq. (1-24.11)]

$$U_\alpha = a_{\alpha\beta} u_\beta \tag{1-24.13}$$

The vector u_i remains fixed in space. Such sets of three components (i.e., vectors) are called *tensors of first order*. Tensors of first order require only one index for their representation. Multiplication of a first-order tensor by a zero-order tensor (i.e., multiplication of a vector by a scalar) yields another first-order tensor. For example, multiplications of u_i by a constant c yields cu_i. Hence, by Eq. (1-24.13), $a_{\alpha\beta}(cu_\beta) = c(a_{\alpha\beta}u_\beta) = cU_\alpha$. Thus, cu_i is a tensor of first order, as it obeys the rules of transformation of a tensor of first order. Furthermore, the addition of two tensors of first order (two vectors) yields a tensor of first order (a vector). For example, if u_p, v_P are two tensors of first order, by Eq. (1-24.13) we have

$$U_\alpha = a_{\alpha\beta} u_\beta \qquad V_\alpha = a_{\alpha\beta} v_\beta$$

Addition of these equations yields

$$U_\alpha + V_\alpha = a_{\alpha\beta} u_\beta + a_{\alpha\beta} v_\beta = \alpha_{\alpha\beta}(u_\beta + v_\beta)$$

Hence, $u_\beta + v_\beta$ is a tensor of first order, as it transforms according to Eq. (1-24.13).

Tensors of Higher Order. Multiplication of tensors of first order leads to quantities that are not tensors of zero or first order. For example, let u_ζ and v_η be two first-order tensors in the rectangular Cartesian coordinate system (x, y, z). Let U_α, V_β denote the corresponding tensors in the rectangular Cartesian coordinate system (X, Y, Z). Then, by Eq. (1-24.13),

$$U_\alpha V_\beta = (a_{\alpha\zeta} u_\zeta)(a_{\beta\eta} v_\eta) = a_{\alpha\zeta} a_{\beta\eta} u_\zeta v_\eta \tag{1-24.14}$$

or

$$W_{\alpha\beta} = a_{\alpha\zeta} a_{\beta\eta} w_{\zeta\eta}$$

where $W_{\alpha\beta} = U_\alpha V_\beta$ and $w_{\zeta\eta} = u_\zeta v_\eta$ represent the products of the vectors U_α, V_β in the (X, Y, Z) system and u_ζ, v_η in the (x, y, z) system, respectively.

Because both ζ and η are dummy indexes, for given values of α, β the right-hand side of Eq. (1-24.14) contains nine terms. Accordingly, Eq. (1-24.14) represents nine equations, each with nine terms. Quantities that transform according to Eq. (1-24.14) are called *tensors of second order*. In the symbolical representation of tensors of second order, two indexes are required. Many quantities other than the product of two vectors transform according to Eq. (1-24.14). For example, components of stress and of strain transform according to Eq. (1-24.14) under a change of rectangular coordinate systems (see Chapters 2 and 3). Accordingly, the components of stress and of strain form second-order tensors.

In a similar fashion, a *tensor of third order* is formed by multiplying together three first-order tensors, and so on. Thus, an nth-order tensor may be formed by multiplying together n first-order tensors. Essentially, this means that we have available means of specifying components of nth-order tensors with respect to any set of rectangular Cartesian axes and rules for transforming these components to any other set of rectangular Cartesian axes. Hence, the statement that a quantity is a tensor quantity may be proved by comparison with these known tensor transformations. For example, this technique was employed in the proof that the sum of two first-order tensors yields a first-order tensor.

In summary, a tensor of zero order (scalar) is a single quantity that depends on position in space but not on the coordinate system. A tensor of first order (vector) is a quantity whose components transform according to Eq. (1-24.13). Hence, with respect to a rectangular Cartesian coordinate system in three-dimensional space, a tensor of first order contains $3^1 = 3$ elements of components. A tensor of second order is a quantity that transforms according to Eq. (1-24.14). With respect to rectangular Cartesian coordinate systems in three-dimensional space, a second-order tensor has $3^2 = 9$ elements.

A tensor of nth order is a quantity whose components transform according to the rule[6]

$$T_{p_1 p_2 \cdots p_n} = a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n} t_{q_1 q_2 \cdots q_n} \tag{1-24.15}$$

[6]See Synge and Schild (1978). Here we let dummy indexes be denoted by $q_1, q_2, \ldots, q_n$.

With respect to rectangular Cartesian coordinate axes in three-dimensional space, an nth-order tensor has 3^n elements. Thus, a fourth-order tensor has 81 elements, a fifth-order tensor has 243 elements, and a tenth-order tensor has 59,049 elements.

In general developments of continuous-media mechanics, fourth-order tensors play a prominent role (Green and Zerna, 2002).

1-25 Symmetric and Antisymmetric Parts of a Tensor

If we interchange α and β in Eq. (1-24.14), we obtain

$$W_{\beta\alpha} = a_{\beta\zeta} a_{\alpha\eta} w_{\zeta\eta} \tag{1-25.1}$$

Because ζ and η are dummy indexes, we may interchange them. Thus, Eq. (1-25.1) may be written

$$W_{\beta\alpha} = a_{\beta\eta} a_{\alpha\zeta} w_{\eta\zeta} = a_{\alpha\zeta} a_{\beta\eta} w_{\eta\zeta} \tag{1-25.2}$$

Hence, comparing Eqs. (1-24.14) and (1-25.2), we see that $w_{\eta\zeta}$ transforms according to the same rule as $w_{\zeta\eta}$. The tensor $w_{\zeta\eta}$ is said to be *conjugate to* $w_{\eta\zeta}$. Thus, if $w_{\zeta n}$ is a tensor of second order, another tensor of second order is obtained by interchanging η and ζ. Consequently, $w_{\zeta\eta} + w_{\eta\zeta}$ and $w_{\zeta\eta} - w_{\eta\zeta}$ are tensors of second order. Symbolically, we may represent the tensors $w_{\zeta\eta}$ and $w_{\eta\zeta}$ as follows:

$$w_{\zeta\eta} = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix}$$

and

$$w_{\eta\zeta} = \begin{pmatrix} w_{11} & w_{21} & w_{31} \\ w_{12} & w_{22} & w_{32} \\ w_{13} & w_{23} & w_{33} \end{pmatrix}$$

Then

$$\begin{aligned} w_{\zeta\eta} + w_{\eta\zeta} &= \begin{pmatrix} 2\,w_{11} & w_{12} + w_{21} & w_{13} + w_{31} \\ w_{21} + w_{12} & 2\,w_{22} & w_{23} + w_{32} \\ w_{31} + w_{13} & w_{32} + w_{23} & 2\,w_{33} \end{pmatrix} \\ &= w_{\eta\zeta} + w_{\zeta\eta} \end{aligned} \tag{1-25.3}$$

and

$$\begin{aligned} w_{\zeta\eta} - w_{\eta\zeta} &= \begin{pmatrix} 0 & w_{12} - w_{21} & w_{13} - w_{31} \\ w_{21} - w_{12} & 0 & w_{23} - w_{32} \\ w_{31} - w_{13} & w_{32} - w_{23} & 0 \end{pmatrix} \\ &= -(w_{\eta\zeta} - w_{\zeta\eta}) \end{aligned} \tag{1-25.4}$$

Because $w_{\zeta\eta} + w_{\eta\zeta}$ is unaltered by interchanging ζ and η, it is called a symmetrical tensor of second order. However, when ζ and η are interchanged in $w_{\zeta\eta} - w_{\eta\zeta}$, each element changes in sign. Hence, $w_{\zeta\eta} - w_{\eta\zeta}$ is called an antisymmetrical tensor of second order. Also, by Eqs. (1-25.3) and (1-25.4),

$$w_{\zeta\eta} = \tfrac{1}{2}(w_{\zeta\eta} + w_{\eta\zeta}) + \tfrac{1}{2}(w_{\zeta\eta} - w_{\eta\zeta}) = S_{\zeta\eta} + A_{\zeta\eta} \tag{1-25.5}$$

where $S_{\zeta\eta}$ is a symmetric second-order tensor and $A_{\zeta\eta}$ is antisymmetric. Consequently, a second-order tensor may be resolved into symmetric and antisymmetric parts. Furthermore, because the antisymmetric part contains only three components, $w_{12} - w_{21}$, $w_{13} - w_{31}$, $w_{23} - w_{32}$, it may be associated with a vector u_i. Equation (1-25.5) is analogous to Eq. (1-23.10).

Problem. Let $w_{\zeta\eta} + w_{\eta\zeta} = 2S_{\zeta\eta} = 2S_{\eta\zeta}$ and $w_{\zeta\eta} - w_{\eta\zeta} = A_{\zeta\eta} = -(w_{\eta\zeta} - w_{\zeta\eta}) = -A_{\eta\zeta}$, where $w_{\zeta\eta}$ is a tensor of second order. Show that the product of the symmetric tensor $S_{\zeta\eta}$ and the antisymmetric tensor $A_{\zeta\eta}$ vanishes; that is, show that $S_{\zeta\eta}A_{\zeta\eta} = 0$.

1-26 Symbols δ_{ij} and ϵ_{ijk} (the Kronecker Delta and the Alternating Tensor)

The use of the following notation often simplifies the writing of equations:

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases} \tag{1-26.1}$$

The symbol δ_{ij} is called the *Kronecker delta.*

Using the notation δ_{ij} with respect to axes (x, y, z), we may write the second of Eqs. (1-24.1) and Eqs. (1-24.2) collectively as

$$a_{\alpha\gamma}a_{\beta\gamma} = \delta_{\alpha\beta} \tag{1-26.2}$$

Similarly, with respect to axes (X, Y, Z) we may express the first of Eqs. (1-24.1) and Eqs. (1-24.3) in the form

$$a_{\gamma\beta}a_{\gamma\alpha} = \delta_{\beta\alpha} \tag{1-26.3}$$

The Kronecker delta has the following important properties:

1. $\delta_{\lambda\lambda} = \delta_{11} + \delta_{22} + \delta_{33} = 3$
2. $\delta_{i\lambda}\delta_{j\lambda} = \delta_{ij}$
3. $p_{i\lambda}\delta_{j\lambda} = p_{ij}$

Property 3 is a generalization of 2. It is called the *rule of substitution of indexes*, as the multiplication of $\delta_{j\lambda}$ substitutes the index j for the index λ.

The set of quantities $\delta_{ij}, i, j = 1, 2, 3$ constitutes a *tensor of the second order.* To prove this, we must show that δ_{ij} transforms according to Eq. (1-24.14) under a transformation of rectangular Cartesian axes. The array δ_{ij} consists of the elements $\delta_{11} = 1$, $\delta_{22} = 1$, $\delta_{33} = 1$, $\delta_{12} = 0$, $\delta_{23} = 0$, and $\delta_{13} = 0$. Accordingly, if we set $\delta_{\sigma\gamma} = w_{\sigma\gamma}$ and substitute in Eq. (1-24.14), we get

$$W_{\alpha\beta} = \delta'_{\alpha\beta} = a_{\alpha\sigma} a_{\beta\gamma} \delta_{\sigma\gamma} = a_{\alpha 1} a_{\beta 1} + a_{\alpha 2} a_{\beta 2} + a_{\alpha 3} a_{\beta 3}$$

Hence, by Eqs. (1-26.1) and (1-26.2),

$$\delta'_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha = \beta \\ 0 & \text{for } \alpha \neq \beta \end{cases}$$

Thus, it follows that the array ($\delta_{11} = \delta_{22} = \delta_{33} = 1$, $\delta_{12} = \delta_{13} = \delta_{23} = 0$) is transformed into itself by the tensor transformation Eq. (1-24.14). This transformation is in accord with the definition of Eq. (1-26.1). Hence, $\delta_{\alpha\beta}$ is a second-order tensor. A tensor whose respective components (elements) are the same with respect to all sets of coordinate systems is called an *isotropic tensor*. In view of the fact that δ_{ij} is a tensor and in view of the substitution property 3 above, δ_{ij} is sometimes referred to as the *substitution tensor*.

Symbol ϵ_{ijk}. The symbol ϵ_{ijk} is defined as follows:

$$\epsilon_{ijk} \begin{cases} 1 & \text{if } i, j, k \text{ are in cyclic order } 1, 2, 3, 1, 2, \ldots \\ 0 & \text{if any two of } i, j, k \text{ are equal} \\ -1 & \text{if } i, j, k \text{ are in anticyclic order } 3, 2, 1, 2, 3, \ldots \end{cases} \tag{1-26.4}$$

For example,

$$\begin{aligned} \epsilon_{123} &= \epsilon_{312} = \epsilon_{231} = 1 \\ \epsilon_{112} &= \epsilon_{121} = \epsilon_{322} = \cdots = 0 \\ \epsilon_{321} &= \epsilon_{213} = \epsilon_{132} = -1 \end{aligned} \tag{1-26.5}$$

By definition of δ_{ij} and ϵ_{ijk}, it follows that

$$\epsilon_{ijk}\delta_{ij} = \epsilon_{iik} = 0 \qquad \text{no summation} \tag{1-26.6}$$

Furthermore, it follows by Eqs. (1-26.5) and (1-26.6) that

$$\epsilon_{\alpha\beta k}\delta_{\alpha\beta} = 0 \qquad \text{summed} \tag{1-26.7}$$

In terms of ϵ_{ijk}, the orthogonality relations [Eq. (1-24.5)] may be written

$$\epsilon_{ij\alpha} a_{\alpha n} = \epsilon_{\alpha\beta n} a_{i\alpha} a_{j\beta} \tag{1-26.8}$$

where i, j, n take independently any value 1, 2, 3. The proof of Eq. (1-26.8) is left for the problems.

The array ϵ_{ijk} transforms according to the rules of transformation of a third-order isotropic tensor. To show this, we note that a third-order tensor transforms according to the rule [see Eq. (1-24.15)]

$$T_{ijk} = a_{i\alpha}a_{j\beta}a_{k\gamma}t_{\alpha\beta\gamma} \tag{1-26.9}$$

Hence, we must show that $\epsilon_{\alpha\beta\gamma}$ transforms according to the rule

$$\epsilon_{ijk} = a_{i\alpha}a_{j\beta}a_{k\gamma}\epsilon_{\alpha\beta\gamma} \tag{1-26.10}$$

Substituting Eq. (1-26.8) into the right side of Eq. (1-26.10), we obtain

$$a_{k\gamma}a_{\alpha\gamma}\epsilon_{ij\alpha}$$

But $\alpha_{k\gamma}a_{\alpha\gamma} = \delta_{k\alpha}$, by Eq. (1-26.2). Hence,

$$a_{k\gamma}a_{\alpha\gamma}\epsilon_{ij\alpha} = \delta_{k\alpha}\epsilon_{ij\alpha} = \epsilon_{ijk}$$

Accordingly, Eq. (1-26.10) is verified. In view of the properties noted in Eqs. (1-26.4) and (1-26.10), the symbol ϵ_{ijk} is called the *alternating tensor.*

1-27 Homogeneous Quadratic Forms

The most general homogeneous quadratic form in the variables X_i, $i = 1, 2, 3$, may be written in index notation as

$$Q = a_{\alpha\beta}X_\alpha X_\beta \qquad \alpha,\ \beta = 1, 2, 3 \tag{1-27.1}$$

where a_{ij} denotes the following square array of real elements:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{1-27.2}$$

The quadratic form Q written in expanded form is

$$\begin{aligned} Q = a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 + (a_{12} + a_{21})X_1X_2 \\ + (a_{13} + a_{31})X_1X_3 + (a_{23} + a_{32})X_2X_3 \end{aligned} \tag{1-27.3}$$

The determinant det a_{ij} is called the *determinant* of the array [Eq. (1-27.2)]. The expression (1-27.1) [or Eq. (1-27.3)] is called the *quadratic form* associated with the array $[a_{ij}]$. Without loss of generality, the array may be assumed symmetrical; that is, we may set $a_{ij} = a_{ji}$. Then Eq. (1-27.3) becomes

$$Q = a_{11}X_1^2 + a_{22}X_2^2 + a_{33}X_3^2 + 2a_{12}X_1X_2 + 2a_{13}X_1X_3 + 2a_{23}X_2X_3 \tag{1-27.4}$$

where we have simply replaced the notation $(a_{12} + a_{21})$ in Eq. (1-27.3) by $2a_{12}$ in Eq. (1-27.4), and so on.

The equation

$$\begin{vmatrix} a_{11}-r & a_{12} & a_{13} \\ a_{21} & a_{22}-r & a_{23} \\ a_{31} & a_{32} & a_{33}-r \end{vmatrix} = 0$$

or, in the index notation,

$$|a_{ij} - r\delta_{ij}| = 0 \tag{1-27.5}$$

is called the *characteristic* equation of the array $[a_{ij}]$. The three roots (r_1, r_2, r_3) of Eq. (1-27.5) are called the *characteristic roots*, or *latent roots*, or *eigenvalues* of the array $[a_{ij}]$ (Eisenhart, 2005; Hildebrand, 1992). In general, the r_i are distinct. However, special cases may occur in which two or all of the r_i are equal.

A necessary and sufficient condition that a set of linear algebraic equations

$$c_{i\alpha}X_\alpha = 0 \qquad i = 1, 2, 3 \tag{1-27.6}$$

possess a solution other than the trivial solution $X_1 = X_2 = X_3 = 0$ is that the determinant of the coefficients $c_{i\alpha}$ of Eq. (1-27.6) vanishes (Pipes, 1959; Hildebrand, 1992). Accordingly,

$$|c_{i\alpha}| = 0 \tag{1-27.7}$$

represents a necessary and sufficient condition that Eq. (1-27.6) possess a solution $X_i (X_i \neq 0)$. Accordingly, by Eqs. (1-27.5), (1-27.6), and (1-27.7), it follows that for every r such that $|a_{ij} - r\delta_{ij}| = 0$, an array (X_i) exists such that

$$(a_{i\alpha} - r\delta_{i\alpha})X_\alpha = 0$$

Rewriting, we have

$$a_{i\alpha}X_\alpha = r\delta_{i\alpha}X_\alpha = rX_i \tag{1-27.8}$$

In other words, Eq. (1-27.5) expresses the necessary and sufficient condition that Eq. (1-27.8) possesses nontrivial solutions of X_i. The nontrivial solutions of Eq. (1-27.8) are called the *eigenvectors* of the array $[a_{ij}]$.

Let y_i denote any arbitrary array (y_1, y_2, y_3). Then, by Eq. (1-27.8), we obtain the *bilinear* form

$$a_{\alpha\beta}X_\beta y_\alpha = rX_\alpha y_\alpha \tag{1-27.9}$$

If $y_i = X_i$, we obtain the *quadratic* form (Hildebrand, 1992)

$$a_{\alpha\beta}X_\alpha X_\beta = rX_\alpha X_\alpha \tag{1-27.10}$$

Orthogonality of Eigenvectors. Consider the case where the array X_i corresponds to the array m_i of direction cosines [Eq. (1-23.4)]. Assume that there exist two nonequal characteristic roots $r^{(1)}$, $r^{(2)}$ of Eq. (1-27.5). Then the corresponding solutions (eigenvectors) of Eq. (1-27.8) may be denoted by ${m_i}^{(1)}$, ${m_i}^{(2)}$.

Accordingly, Eq. (1-27.8) becomes

$$\begin{aligned} a_{i\alpha}m_\alpha^{(1)} &= r^{(1)}m_i^{(1)} \qquad \text{for } r = r^{(1)} \\ a_{i\alpha}m_\alpha^{(2)} &= r^{(2)}m_i^{(2)} \qquad \text{for } r = r^{(2)} \end{aligned} \tag{1-27.11}$$

Multiplying the first of Eqs. (1-27.11) by $m_i^{(2)}$ and the second by $m_i^{(1)}$ and subtracting, we obtain (because $a_{ij} = a_{ji}$)

$$\left[r^{(2)} - r^{(1)}\right] m_\beta^{(1)} m_\beta^{(2)} = 0$$

However, because by hypothesis $r^{(2)} \neq r^{(1)}$, it follows that

$$m_\beta^{(1)} m_\beta^{(2)} = 0 \tag{1-27.12}$$

Accordingly, the directions (eigenvectors) $m_\beta^{(1)}$, $m_\beta^{(2)}$ that correspond to the characteristic roots $r^{(1)}$ and $r^{(2)}$ are orthogonal. Furthermore, if $r^{(1)}$ and $r^{(2)}$ are two distinct characteristic roots and $m_\beta^{(1)}$ and $m_\beta^{(2)}$ are the corresponding direction cosines, by Eq. (1-27.10) we have

$$a_{\alpha\beta}m_\alpha^{(1)}m_\beta^{(2)} = r^{(2)}m_\beta^{(1)}m_\beta^{(2)} = r^{(1)}m_\beta^{(1)}m_\beta^{(2)} = 0$$

Hence

$$a_{\alpha\beta}m_\alpha^{(1)}m_\beta^{(2)} = 0 \tag{1-27.13}$$

This result is equivalent to the vanishing of shearing stress (or strain) components relative to principal axes (see Chapters 2 and 3).

Finally, note that the characteristic roots $r^{(1)}$, $r^{(2)}$ are real. We prove this by contradiction as follows: Assume that $r^{(1)}$ is complex. Denote its complex conjugate by $\overline{r}^{(1)}$. Then, taking the complex conjugate of the first of Eqs. (1-27.11), we obtain

$$a_{\alpha\beta}\overline{m}_\beta^{(1)} = \overline{r}^{(1)}\overline{m}_\alpha^{(1)} \tag{1-27.14}$$

Multiplying (1-27.14) by $m_\alpha^{(1)}$, we get

$$a_{\alpha\beta}m_\alpha^{(1)}\overline{m}_\beta^{(1)} = \overline{r}^{(1)}m_\alpha^{(1)}\overline{m}_\alpha^{(1)} \tag{1-27.15}$$

Multiplying the first of Eqs. (1-27.11) by $\overline{m}_\alpha^{(1)}$, we get

$$a_{\alpha\beta}\overline{m}_\alpha^{(1)}m_\beta^{(1)} = r^{(1)}m_\alpha^{(1)}\overline{m}_\alpha^{(1)} \tag{1-27.16}$$

Comparison of Eqs. (1-27.15) and (1-27.16) yields

$$\left[\overline{r}^{(1)} - r^{(1)}\right] m_\alpha^{(1)}\overline{m}_\alpha^{(1)} = 0$$

Because $m_\alpha^{(1)}\overline{m}_\alpha^{(1)}$ is the sum of squares of real numbers, it cannot be zero unless $m_1 = m_2 = m_3 = 0$. However, this is not possible because, by Eq. (1-23.4),

$$m_1^2 + m_2^2 + m_3^2 = 1$$

Hence

$$r^{(1)} = \overline{r}^{(1)}$$

That is, $r^{(1)}$ is equal to its conjugate $\overline{r}^{(1)}$. Accordingly, $r^{(1)}$ must be real.

1-28 Elementary Matrix Algebra

The matrix algebra outlined in this section plays an important role in modern structural analysis and in numerical methods of continuum mechanics such as finite element methods.

In Section 1-23 we noted that the rectangular array of m rows and n columns of numbers a_{ij} is called a matrix (more explicitly, an m by n matrix or a matrix of order m by n). The elements a_{ij} may be real or complex numbers or, more generally, may be matrices themselves. However, unless we state otherwise, we take the numbers a_{ij} to be real. In Section 1-23 we denoted the array by $[a_{ij}]$ and considered several properties of the array in terms of the individual elements a_{mn}. However, it is frequently more economical to treat a matrix as a single entity, particularly in algebraic operations involving addition, subtraction, multiplication, and division of several arrays. Accordingly, we employ the notation

$$A = [a_{ij}] \qquad 1 \le i \le m \qquad 1 \le j \le n \tag{1-28.1}$$

where A denotes the m by n matrix [Eq. (1-23.5)] of the m by n elements a_{ij}.

If $m = 1$,

$$A = [a_{11}, a_{12}, \ldots, a_{1n}] = (a_{11}, a_{12}, \ldots, a_{1n}) \tag{1-28.2}$$

contains one row. Hence it is called a *row matrix*, where we use parentheses ( ) to denote a row matrix.

Alternatively, if $n = 1$,

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = \{a_{11}, a_{21}, \ldots, a_{m1}\} \tag{1-28.3}$$

contains one column. Hence it is called a *column matrix*, where, for economy of space, we use braces { } to denote a column matrix. Because the numbers $a_{11}, a_{12}, \ldots, a_{1n}$ (or the numbers $a_{11}, a_{21}, \ldots, a_{m1}$) may be taken as the components of a vector in n-dimensional space, it follows that a row matrix and a column matrix are sometimes called vectors of the first kind and of the second kind, respectively.

If all $a_{ij} = 0, 1 \le i \le m, 1 \le j \le n$, then the matrix $A = [a^{ij}] = [0]$ is called the *null matrix.*

The algebraic operations of addition, subtraction, multiplication, division, and so on of matrices are defined in terms of equivalent operations on the elements of the matrices. These operations are discussed next.

Matrix Addition. Let $A = [a_{ij}]$, $B = [b_{ij}]$, $1 \le i \le m, 1 \le j \le n$. Then the operation of addition, denoted by $A + B$, is defined by

$$A + B = [a_{ij} + b_{ij}] = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix} \tag{1-28.4}$$

Matrix Subtraction. The subtraction of matrices A, B, denoted by $A - B$, as in addition, requires that A, B be of the same order. By definition,

$$A - B = [a_{ij} - b_{ij}] \tag{1-28.5}$$

If $A = B$, that is, if $a_{ij} = b_{ij}$, $A - B = [0]$, the null matrix. In other words, two matrices A, B are said to be equal if they are of the same order and their difference is the null matrix.

Multiplication of a Matrix by a Scalar. Multiplication of a matrix A by a scalar s multiplies every element a_{ij} of A by s. Thus, $sA = s[a_{ij}] = [sa_{ij}]$. Analogously, division of a matrix A by a scalar s is defined by $(1/s)A = (1/s)[a_{ij}] = [(1/s)a_{ij}]$.

Multiplication of a Matrix by a Matrix. The operation of a matrix multiplication occurs in a number of situations. For example, in Section 1-24 we found that a rotation from one set of Cartesian axes x_α to another set X_α led to the result [Eq. (1-24.11)]

$$X_\alpha = a_{\alpha\beta} x_\beta \qquad \alpha, \beta = 1, 2, 3 \tag{1-28.6}$$

Similarly, a rotation from axes X_α to axes Y_α yields

$$Y_\alpha = b_{\alpha\beta} X_\beta \tag{1-28.7}$$

where $b_{\alpha\beta}$ are direction cosines between axes Y_α and X_β. Hence, substitution of Eqs. (1-28.6) into Eq. (1-28.7) yields a transformation from axes x_α directly to axes Y_α. Thus,

$$Y_\alpha = b_{\alpha\beta} a_{\beta\gamma} x_\gamma = c_{\alpha\gamma} x_\gamma \tag{1-28.8}$$

where

$$c_{\alpha\gamma} = b_{\alpha\beta} a_{\beta\gamma} \tag{1-28.9}$$

In matrix notation, we may write

$$X = Ax$$

$$Y = BX = BAx = Cx \tag{1-28.10}$$

where

$$A = [a_{\beta\gamma}] \qquad B = [b_{\alpha\beta}] \qquad C = [c_{\alpha\gamma}] \tag{1-28.11}$$

and where summation convention holds (Section 1-23).

Generalization of Eq. (1-28.10) leads to the following definition: Given $A = [a_{ij}]$, $1 \le i \le m, 1 \le j \le n$; $B = [b_{ij}]$, $1 \le i \le p, 1 \le j \le q$. The product AB is defined if and only if $p = n$. When $p = n$, matrices A and B are said to be *conformable* or to *conform.* The product of two conformable matrices A (of order m by n) and B (of order n by q) is a matrix C (of order m by q), with elements c_{ij} given by the rule

$$c_{ij} = b_{i\alpha}a_{\alpha j} = b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj} \tag{1-28.12}$$

This rule is summarized by the following statement: The matrix C, with elements c_{ij}, is obtained by multiplication of the elements of the ith row of matrix B into the elements of the jth column of matrix A.

In general, we note that the premultiplication BA of A by B is not equal to the postmultiplication AB of A by B. Thus, in general, $BA \neq AB$. In particular, BA and AB are both defined if and only if $m = q$ and $p = n$.

More generally, the product P of p matrices (extended product) $A_1, A_2, A_3, \ldots, A_p$ is defined by $P = A_1A_2A_3 \cdots A_p$, provided that in the order $A_1, A_2, A_3, \ldots, A_p$ two adjacent matrices conform. If $A_1 = A_2 = A_3 = \cdots = A_p = A$, we obtain $P = A^P$, the pth product of A.

Square Matrices. A matrix $A = [a_{ij}]$, $1 \le i \le m, 1 \le j \le n$ is said to be square if and only if $m = n$. A square matrix is said to be symmetric if and only if $a_{ij} = a_{ji}$. If $a_{ij} = 0$, for $i \neq j$, the matrix $A = [a_{ij}]$ is said to be a *diagonal matrix* and is denoted by $A = \text{diag}(a_{11}, a_{22}, \ldots, a_{nn})$. If A is a diagonal matrix and $a_{ii} = s$ for all i, A is called a *scalar matrix.* If, in addition, $s = 1$, the matrix A consists of diagonal elements all equal to 1. Then A is called the *unit matrix* and is denoted by the symbol I; that is, $A = I$. For any matrix B, we have $IB = BI = B$. Hence, I commutes with any matrix. Thus, the unit matrix operates on matrices in the same manner that the number 1 operates on real numbers.

Transpose of a Matrix. In operations with arrays $[a_{ij}]$, we must consider arrays $[a_{ji}]$. The matrix $[a_{ji}] = A^{\mathrm{T}}$ is called the *transpose* of the matrix $A = [a_{ij}]$, and the operation of forming the transpose A^{T} from matrix A is called *transposition.* In particular, the transpose P^{T} of a product $P = AB$ of matrices A, B is $P^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$, and, in like manner, the transpose of the extended product $P = A_1A_2 \cdots A_q$ is $P^{\mathrm{T}} = A_q^{\mathrm{T}}A_{q-1}^{\mathrm{T}}A_{q-2}^{\mathrm{T}} \cdots A_1^{\mathrm{T}}$.

Division of a Matrix by a Matrix. The operation of division is restricted to square matrices. Several preliminary notions are required: the concepts of the determinant $|a|$ of a square matrix $[a_{ij}]$, the cofactor A_{ij} of the elements a_{ij} in the determinant $|a|$, the adjoint matrix $\overline{A} = [A_{ji}]$ of a matrix, and the inverse of a matrix, denoted by A^{-1}.

We assume that the concept of determinant is familiar from elementary algebra. Then, for a square n by n matrix $A = [a_{ij}]$, $1 \leq i \leq n, 1 \leq j \leq n$, we have the associated determinant $|a|$ of the matrix $[a_{ij}]$, where the number $|a|$ is defined (Birkhoff and MacLane, 2008; Lancaster and Tismenetsky, 1985; Gilbert, 2008) by

$$a = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{vmatrix} \tag{1-28.13}$$

If $a \neq 0$, $A = [a_{ij}]$ is said to be nonsingular and possesses a *reciprocal* or *inverse matrix* A^{-1} such that

$$[a_{ij}]A^{-1} = AA^{-1} = I \tag{1-28.14}$$

where I is the unit matrix of the same order as A. Accordingly, the operation of matrix multiplication of a matrix by its inverse matrix is analogous to dividing a real number by itself. More generally, if B is any matrix conformable with the nonsingular inverse matrix A^{-1},

$$BA^{-1} = C \tag{1-28.15}$$

where C is a matrix. Equation (1-28.15) is sometimes referred to as the *division* of matrix B by matrix A. Accordingly, to divide matrix B by a conformable matrix A, we must first compute the inverse matrix A^{-1}. To compute A^{-1}, we first introduce the *adjoint matrix* $\overline{A}$ of A. The adjoint matrix $\overline{A}$ is defined by

$$\overline{A} = [A_{ji}] \tag{1-28.16}$$

where the element A_{ij} denotes the cofactor of the element a_{ij} in the determinant $|a|$ of the matrix $A = [a_{ij}]$, and $[A_{ij}]$ is the transpose of the matrix $[A_{ij}]$. The adjoint matrix A exists whether or not A is singular.

By definition of matrix multiplication and the theory of determinants, we have [Eq. (1-28.12)]

$$[a_{ij}][A_{ji}] = |a|I = S \tag{1-28.17}$$

where S is a diagonal (scalar) matrix, with $s_{ii} = |a|$, and $s_{ij} = 0$ for $i \neq j$. Dividing Eq. (1-28.17) by the determinant $|a|$ (assumed nonsingular), we obtain

$$I = \frac{[a_{ij}][A_{ji}]}{|a|} = \frac{A\overline{A}}{|a|} \tag{1-28.18}$$

Accordingly, comparison of Eqs. (1-28.14) and (1-28.18) yields

$$A^{-1} = \frac{[A_{ji}]}{|a|} = \frac{\overline{A}}{|a|} \tag{1-28.19}$$

The matrix A^{-1} is called the *inverse* or *reciprocal matrix* because of the property $AA^{-1} = I$. It plays the same role in matrix algebra as does division in ordinary algebra. Thus, if $AB = CD$ where A, B, C, D are appropriate matrices, premultiplication by A^{-1}, the inverse of A, yields $A^{-1}AB = IB = B = A^{-1}CD$.

1-29 Some Topics in the Calculus of Variations

Maxima, Minima, and Lagrange Multipliers. The problem of seeking maxima and minima of functions of several variables plays an important role in engineering. A generalization of the elementary theory of maxima and minima (or extrema) leads to the calculus of variations. For example, in the theory of extrema, consider the problem of determining for a given continuous function $f(x_1, x_2, \ldots, x_n)$ of the n variables $(x_1, x_2, \ldots, x_n)$ in a given region R, a point $(x_{p1}, x_{p2}, x_{p3}, \ldots, x_{pn})$ at which the function f attains *maximum* or *minimum values* (i.e., *extreme values* or simply *extrema*) with respect to all points of R in a neighborhood (vicinity) of the point $(x_{p1}, x_{p2}, \ldots, x_{pn})$. This problem always has a solution (point for which f is an extremum), because according to a theorem of Weierstrass, every function $f(x_1, x_2, \ldots, x_n)$ that is continuous in a closed bounded region R of the variables $(x_1, x_2, \ldots, x_n)$ possesses a maximum value and a minimum value in the interior of R or on the boundary of R. Analogous to the theory of a single variable, if the function $f(x_1, x_2, \ldots, x_n)$ is differentiable in R and if an extreme value is attained at an interior point $P : (x_{p1}, x_{p2}, \ldots, x_{pn})$, then the derivatives of f with respect to each of the x's vanish at P. The vanishing of the derivatives of f is a necessary condition for extrema. It is not sufficient, however, as an examination of the function $f(x) = x^3$ at the point $x = 0$ shows. More generally, we define a point at which all first-order derivatives of f vanish, hence at which $df = 0$, as a stationary point S. In turn, a *stationary point* S that furnishes a maximum value or a minimum value (an extreme value) in an allowable neighborhood of S is called an *extremum.*

In some problems the choice of points $(x_1, x_2, \ldots, x_n)$ is restricted to subregions of R by certain *equations of constraint* (or simply *constraints*). For example, consider the stationary values of the function $f(x_1, x_2, \ldots, x_n)$ of the n variables $(x_1, x_2, \ldots, x_n)$, continuous with continuous first partial derivatives, subject to the restrictions that the x's must satisfy m equations of constraint $(m < n)$

$$g_i(x_1, x_2, \ldots, x_n) = 0 \qquad i = 1, 2, \ldots, m \tag{1-29.1}$$

The direct approach to this problem is to eliminate m of the variables from f by means of Eq. (1-29.1). Then seek the stationary values of $f(y_1, y_2, \ldots, y_{n-m})$, where $y_1, y_2, \ldots, y_{n-m}$ denote the remaining $n - m$ variables. However, because the elimination of any m of the variables is arbitrary and the process of elimination from Eq. (1-29.1) may be nontrivial, an alternative approach attributed to Lagrange is often employed (the *Lagrange multiplier method*). This method has the advantage of retaining symmetry in the calculations (arbitrary elimination of m variables is avoided) and of routine elegance.

Lagrange's method of multipliers consists of forming a new function F such that

$$F(x_1, x_2, \ldots, x_n; \lambda_0, \lambda_1, \ldots, \lambda_m) = \lambda_0 f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i g_i(x_1, x_2, \ldots, x_n) \tag{1-29.2}$$

where the $\lambda_i, i = 0, 1, 2, \ldots, m$, are called the *Lagrange multipliers.* Then stationary values of F are sought over the unrestricted range of the variables $(x_1, x_2, \ldots, x_n)$ by the requirements

$$\begin{aligned} &\frac{\partial F}{\partial x_1} = 0 \qquad \frac{\partial F}{\partial x_2} = 0, \ldots, \frac{\partial F}{\partial x_n} = 0 \\ &\frac{\partial F}{\partial \lambda_1} = g_1 = 0 \qquad \frac{\partial F}{\partial \lambda_2} = g_2 = 0, \ldots, \frac{\partial F}{\partial \lambda_m} = g_m = 0 \end{aligned} \tag{1-29.3}$$

These equations suffice to determine the stationary points $(x_{p1}, x_{p2}, \ldots, x_{pn})$ and the Lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$. Because F is homogeneous in the λ's [Eq. (1-29.2)], we may take $\lambda_0 = 1$.

Equations (1-29.3) show that the stationary points for F are the same as the stationary values of f subject to the constraints of Eq. (1-29.1). The Lagrange multiplier method is useful in the theory of principal values of stress and strain (Chapters 2 and 3).

More generally, the above results may be summarized as follows[7]:

Given a function $f(x_1, x_2, \ldots, x_n)$ of n variables $(x_1, x_2, \ldots, x_n)$ subject to m constraints $g_i(x_1, x_2, \ldots, x_n) = 0, i = 1, 2, \ldots, m$. Let f and g_i possess continuous first partial derivatives in a region R of the x space. Furthermore, let the Jacobian J be nonzero; that is,

$$J = \frac{\partial(g_1, g_2, \ldots, g_m)}{\partial(a_1, a_2, \ldots, a_m)} \neq 0 \tag{1-29.4}$$

where the set of variables $(a_1, a_2, \ldots, a_m)$ is some selection of m variables from the extremum $(x_{p1}, x_{p2}, \ldots, x_{pn})$. Then the stationary values of f subjected to

[7]For an analytical proof of the Lagrange multiplier method, see Courant (1992), pp. 192–199 (footnote 5).

the constraints $g_i = 0, i = 1, 2, \ldots, m$ are identical to the stationary values of the function

$$F(x_1, x_2, \ldots, x_n; \lambda_1, \lambda_2, \ldots, \lambda_m) = f(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i g_i(x_1, x_2, \ldots, x_n) \quad (1\text{-}29.5)$$

In cases where the constraints $g_i = 0$ are algebraic relations, the Lagrange multipliers are constant parameters. However, more generally (Langhaar, 1989), when the equations of constraint require the x_i to be solutions of differential equations, the Lagrange multipliers may be functions of one or more of the variables x_i.

Variation of a Function. First Variation of an Integral. Stationary Value of an Integral. As in the theory of ordinary maxima and minima, the calculus of variations is concerned with the problem of extreme values (stationary values). However, in contrast to the ordinary extremum problem of a function of a finite number of independent variables, the calculus of variations deals with functions of functions, or simply functionals (Courant and Hilbert, 1989).

The simplest type of problem in the calculus of variations may be outlined as follows: Let $F(x, y, y')$ be a given function of the three arguments x, y, y' that is continuous and has continuous first and second derivatives in the region of the arguments. Because F is a function of x, an integral

$$I(y) = \int_{x_0}^{x_1} F(x, y, y')\,dx$$

becomes a definite number depending upon the behavior of the function $y = y(x)$, the *argument function*. That is, the integral $I(y)$ becomes a function of the argument function $y(x)$ or, in other words, a *functional*. The fundamental problem of the calculus of variations may be stated in this form: Among all functions $y = y(x)$ that are defined and continuous and possess continuous first and second derivatives in the interval $x_0 \le x \le x_1$ and for which boundary values $y_0 = y(x_0)$, $y_1 = y(x_1)$ are given, determine that function $y = u(x)$ for which the integral $I(y)$ has the smallest possible value (or the largest possible value). The conditions imposed upon the argument function $y(x)$ are called *conditions of admissibility*, and we speak of argument functions that satisfy the conditions of admissibility as *admissible functions*. The admissible functions $y(x)$ form a class C. In the above formulations we required that $y(x)$ be continuous with its first and second derivatives. Actually, the existence of $I(y)$ requires only that F, hence $y'(x)$, be sectionally continuous. The more restricted admissible conditions limit the class C in which functions $y(x)$ are sought. However, it may be shown that the function $y = f(x)$, which minimizes I when the broader class of admissible conditions is allowed, always lies in the more restricted class of admissible functions (Courant and Hilbert, 1989).

Accordingly, our objective is to determine *necessary conditions* that an admissible function $y = u(x)$ gives a maximum or minimum value (extreme value) to the integral $I(y)$. The method employed is analogous to that of the extreme problem of determining the extreme value of a function of a single variable. Thus, we assume that $y = u(x)$ is the solution, say, a minimum. [The problem of determining a maximum may be dispensed with, as the method of seeking a maximum is the same as that for seeking a minimum with F replaced by $-F$ in $I(y)$.] Then for any other admissible function the value of I must increase. Because we seek necessary conditions, it suffices to consider admissible functions that lie infinitesimally close to the solution $y = u(x)$. Hence, we consider the class of admissible functions

$$\overline{y} = y(x) + \varepsilon\eta(x) = y(x) + \delta y$$

where ε is a parameter and $\eta(x)$ is a function in the class of admissible functions (i.e., has continuous first and second derivatives in $x_0 \leq x \leq x_1$ and vanishes at $x = x_0$ and $x = x_1$). The quantity $\delta y = \varepsilon\eta(x)$ is called the variation of the function $y(x)$. Then if ε is sufficiently small, the admissible functions $\overline{y}$ lie in an arbitrarily small neighborhood of the extremum $y = u(x)$. Hence, the integral $J = I(y + \varepsilon\eta)$ may be regarded as a function of ε, which must attain a minimum at $\varepsilon = 0$ relative to all values of ε in a sufficiently small neighborhood of $\varepsilon = 0$. Consequently, $dJ/d\varepsilon|_{\varepsilon=0} = J'(0) = 0$ is a necessary condition that $I(y)$ attain a minimum for $y = u(x)$. More generally, without regard to maximum or minimum, we say that the integral I is *stationary* for $y = u(x)$. Thus, with $J(\varepsilon) = I(y + \varepsilon\eta) = \int_{x_0}^{x_1} F(x, y + \varepsilon\eta, y' + \varepsilon\eta')/dx$, differentiation yields the necessary condition

$$J'(0) = \int_{x_0}^{x_1} (F_y\eta + F_{y'}\eta')\, dx = 0$$

that $I(y)$ be stationary for all admissible $\eta(x)$. Integration by parts and use of the conditions $\eta(x_0) = \eta(x_1) = 0$ yield

$$J'(0) = \int_{x_0}^{x_1} \eta \left(F_y - \frac{d}{dx} F_{y'} \right) dx = 0$$

which must hold for arbitrary admissible functions η. Hence, by the fundamental theorem of the calculus of variations,[8]

$$F_y - \frac{dF_{y'}}{dx} = 0 \tag{1-29.6}$$

Equation (1-29.6) is the *Euler differential equation* for the intetgral $I(y)$. It is a necessary condition that $I(y)$ possess a *stationary value.*

Recalling the definition $\delta y = \varepsilon\eta(x)$, and noting that $\eta(x) = d\overline{y}/d\varepsilon$, we may interpret the symbol δ to denote the differential obtained when ε is regarded as the

[8]See Langhaar (1989). Langhaar gives an elegant approach to the derivation of the Euler equation in Section 3-2.

independent variable. Then the equation

$$\delta I = \eta \left. \frac{dJ}{d\varepsilon} \right|_{\varepsilon=0} = \int_{x_0}^{x_1} (\eta F_y + \eta' F_{y'})\, dx \tag{1-29.7}$$

is called the *first variation* of the integral I. Hence, the terminology *stationary character of an integral* means the same thing as *vanishing of the first variation* of the integral.

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CHAPTER 2

THEORY OF DEFORMATION

2-1 Deformable, Continuous Media

In classical mechanics a solid body is often assumed to be rigid; that is, distances between particles constituting the body are assumed to remain invariant under the action of applied forces. In many problems of dynamics of solids, the rigid-body approximation is sufficiently accurate for engineering purposes. However, the rigid-body approximation often leads to grossly incorrect results. For example, in describing the resistive behavior of a volume of fluid to applied forces, it is necessary to account for the change in form of the fluid.

The behavior of rigid bodies is treated in general mechanics. In the following pages we treat the behavior of deformable bodies. Actually, there are no rigid bodies in nature. All bodies are deformable to a greater or lesser degree; that is, the distance between the particles of real bodies always undergoes some change under the action of forces. The question of whether or not a body may be assumed rigid is a question of the range of validity of the rigid-body approximation.

Continuity. To describe in general the motion of a deformable medium, considered as a molecular structure, it would be necessary to write down the equations of motion of each molecule. However, this objective has been achieved only in the case of gases in states far removed from unstable states. In the case of solids, the present state of scientific development concentrates on nanomechanics and physics (Reed and Kirk, 1989; Timp, 1999). However, in this chapter, we are not generally interested in the motion of individual molecules of a medium. Rather, the problem is to describe the overall behavior of the medium—for example, to determine the velocity, acceleration, strain, stress, or temperature at any point in the medium. Consequently, we restrict our problem to the determination of mean or average values up to the microlevel (Fig. 1-1.2) in space and time, not to the determination of the motion of the molecules themselves. To accomplish this aim, we consider a medium to be contained in a volume (region) R. Each neighborhood of a material point P in R is *dense*. That is, each neighborhood in R contains sufficient material so that mean or average values in space and time exist. Thus, the concepts of density, temperature (or kinetic energy of the molecules), and so on are meaningful (Prandtl and Tietjens, 1957). Furthermore, we initially disregard motion; that is, for short periods of time each volume element contains the same molecules. In other words, we regard the molecular mean free paths to have negligible dimensions compared to the elements of volume considered. However, as evidenced in fluids by the phenomena of diffusion, internal friction (which results in a transfer of momentum between continuous boundaries of a medium), and heat conduction (which is due to transfer of kinetic energy at the molecular level), the constant interchange of molecules between volume elements, which in reality always exists, cannot be neglected for long periods of time.

In the following sections we treat deformation, that is, the changes in distance between material points of dense regions. The configuration or shape of the region is described by a continuous mathematical model whose geometrical points are associated in a one-to-one manner with the location of material particles of the region. When the configuration (shape) of a continuous model (region) changes under some physical action, we say that the region or material body is transformed (into a new shape or configuration). We assume that the transformation is continuous. Thus, neighborhoods of a material point P in the initial region R remain neighborhoods of the transformed material point $\mathscr{P}$ in the transformed region $\mathscr{R}$. Therefore, we regard tearing or fracture of the body as an extraordinary circumstance that requires special study. We dispense with such effects in this treatment. Before undertaking the treatment of deformation of deformable bodies, we will review a few basic concepts of rigid-body displacements, as they play a role in the general theory of deformable bodies.

2-2 Rigid-Body Displacements

Displacement of a Particle. By definition, the displacement of a particle is determined by its initial and final locations; the path of the particle between these points is irrelevant. A displacement of a particle is a vector quantity, as displacements of particles may be represented by arrows that combine by vector addition. For example, in Fig. 2-2.1, $\mathbf{q}_1$ denotes a displacement of a particle from point O to point P, and $\mathbf{q}_2$ denotes a displacement of the particle from P to Q. The resultant displacement from point O to point Q is represented by the vector sum $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$.

Translation. A mechanical system is said to undergo a translation if the displacement vectors for all its particles are equal. A translation is said to be a type

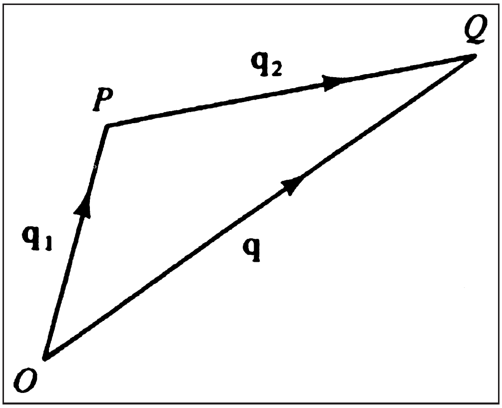

Figure 2-2.1

of “rigid-body displacement” because the distances between the particles of the systems are unchanged. A translation of a system may be represented by a single vector, which represents the displacement of any particle of the system.

A displacement of a rigid body is a translation if in the final position of the body all lines connecting particles of the body retain their original directions and senses (Boresi and Schmidt, 2000).

Rotation. A mechanical system is said to undergo a rotation through an angle θ about an axis x if all particles of the system describe circular arcs of angle θ with their centers on the x axis and with their planes perpendicular to the x axis. A rotation is a rigid-body displacement. In general, a rotation of a rigid body is not a vectorial quantity.

Plane Displacements. A rigid body is said to experience a plane displacement if the displacement vectors of all its particles are parallel to a plane. Translations and rotations are plane displacements. To study plane displacements of a rigid body, it is necessary to consider only those particles that lie in a cross-sectional plane parallel to the displacement vectors, because the displacement vectors do not vary along a normal to that plane. It may be shown that *any* plane displacement of a rigid body can be performed by a rotation (Boresi and Schmidt, 2000).

Other Theorems on Displacement of a Rigid Body. For general displacements of a rigid body, we have the following interesting theorems (Whittaker, 1999):

1. *Euler’s Theorem.* Any displacement of a rigid body that has one point fixed can be performed by a rotation about an axis through the fixed point.
2. *Chasles’s Theorem.* Any displacement of a rigid body can be performed by a screw motion, that is, by a rotation about an axis, combined with a translation parallel to the axis. Chasles’s theorem is a generalization of Euler’s theorem.

The general kinematical representation of a rigid-body displacement in terms of the components of the displacement vector is given in Section 2-15.

2-3 Deformation of a Continuous Region. Material Variables. Spatial Variables

Let R be a closed region that is occupied initially by a material body. In the theory of deformation we refer to region R as a medium or a deformable body, or simply as a body. Let the medium undergo a deformation (change in configuration) such that at some later time t it occupies the region $\mathscr{R}$, where we indicate the deformed or transformed state of the body by a script letter. Thus, the material body in region R is deformed into a region $\mathscr{R}$ by some physical action (Fig. 2-3.1). Under the deformation, material particles in the neighborhood of any point P in R remain neighborhoods of the transformed point $\mathscr{P}$ in $\mathscr{R}$.

To analyze the deformation of region R into region $\mathscr{R}$, some method of description of regions R and $\mathscr{R}$ is required. In the general theory of continuum mechanics, it is convenient to describe the region R by one system of curvilinear Euclidean coordinates (x_1, x_2, x_3) or (x, y, z) and region $\mathscr{R}$ by a second system (ξ_1, ξ_2, ξ_3) or (ξ, η, ζ). In turn, (x_1, x_2, x_3) and (ξ_1, ξ_2, ξ_3) may be considered to be coordinates associated with two separate reference frames,[1] A, B (Fig. 2-3.2). Hence, the geometrical point $P(x_1, x_2, x_3)$ in region R and the geometrical point $\mathscr{P}$ (ξ_1, ξ_2, ξ_3) into which $P(x_1, x_2, x_3)$ is transformed may be described in terms of coordinates (x_1, x_2, x_3), (ξ_1, ξ_2, ξ_3) measured in frames A and B, respectively.

Considering region R to be the initial (undeformed) configuration of the medium, the final (deformed) configuration $\mathscr{R}$ may be described by the equation

$$\xi_1 = \xi_1(x_1, x_2, x_3; t) \qquad \xi_2 = \xi_2(x_1, x_2, x_3; t) \qquad \xi_3 = \xi_3(x_1, x_2, x_3; t) \quad (2\text{-}3.1)$$

or, in index notation,

$$\xi_i = \xi_i(x_1, x_2, x_3; t) \qquad i = 1, 2, 3 \qquad (2\text{-}3.1\text{a})$$

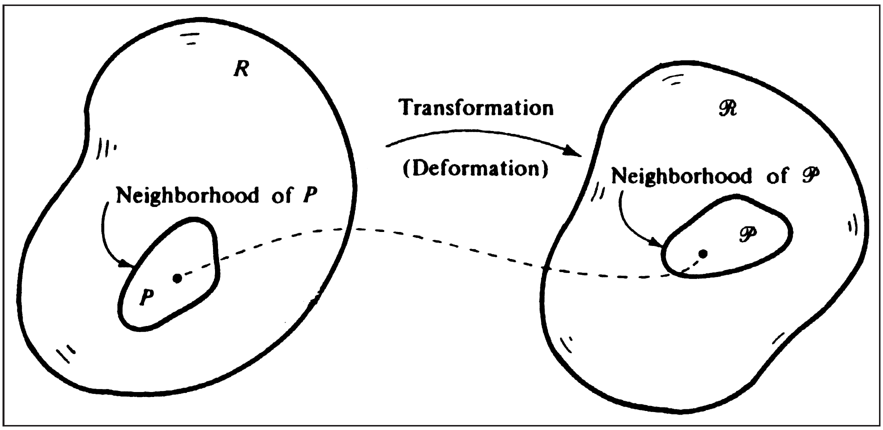

Figure 2-3.1

[1]See Boresi and Schnmidt (2000), Section 13.11, for a discussion of reference frames.

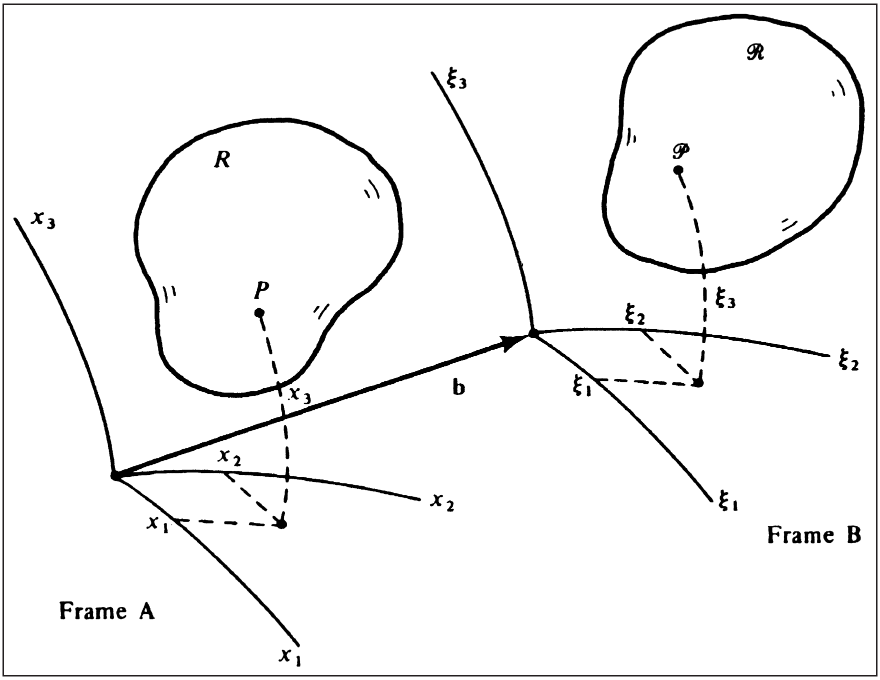

Figure 2-3.2 Curvilinear coordinates.

where (x_1, x_2, x_3) are restricted to region R, and (ξ_1, ξ_2, ξ_3) are restricted to region $\mathscr{R}$. Equations (2-3.1) define the location at time t of the particle P that initially lies at the point (x_1, x_2, x_3) in the underformed medium. We asume that (ξ_1, ξ_2, ξ_3) are continuous and differentiable in the variables $(x_1, x_2, x_3; t)$. Physically, this means that the medium does not rupture or separate into parts at any time during the deformation. Thus, to each point in region R there corresponds a point in region $\mathscr{R}$, and vice versa; mathematically speaking, Eqs. (2-3.1) denote a *one-to-one correspondence* between the points in regions R and $\mathscr{R}$. Consequently, Eqs. (2-3.1) possess single-valued solutions of the type

$$x_1 = x_1(\xi_1, \xi_2, \xi_3; t), \qquad x_2 = x_2(\xi_1, \xi_2, \xi_3; t), \qquad x_3 = x_3(\xi_1, \xi_2, \xi_3; t)$$

or

$$x_i = x_i(\xi_1, \xi_2, \xi_3; t), \qquad i = 1, 2, 3 \tag{2-3.2}$$

where the functions x_i are considered to be continuous and differentiable with respect to ξ_i. Equations (2-3.2) define the initial position x_i of a particle that is at point ξ_i in the deformed medium at time t.

Equations (2-3.1) and (2-3.2) allow a choice of independent variables, either (x_1, x_2, x_3) or (ξ_1, ξ_2, ξ_3). This choice gives fluid dynamics a dual nature. For example, when (x_1, x_2, x_3) are taken as independent variables, we fix our attention on a definite particle (x_1, x_2, x_3) or a definite portion of a medium (material), and we consider how it changes in space. This viewpoint is known as the *Lagrangian* (*material*) viewpoint, and $(x_1, x_2, x_3; t)$ are called *Lagrangian* (*material)* variables. When (ξ_1, ξ_2, ξ_3) are taken as independent variables, we fix in space a geometrical point (ξ_1, ξ_2, ξ_3), or a geometrical region, and we ask what particles pass through this point or region. This viewpoint is the *Eulerian* (*spatial*) viewpoint, and $(\xi_1, \xi_2, \xi_3; t)$ are called *Eulerian* (*spatial*) variables.

In describing a quantity Q, we may write it either in the *material form* $Q(x_1, x_2, x_3; t)$ or the *spatial form* $Q(\xi_1, \xi_2, \xi_3; t)$. For example, the spatial form is usually used in classical fluid mechanics to describe velocity, acceleration, and so forth. Also, many modern writers on large deformation theories of solids adopt $(\xi_1, \xi_2, \xi_3; t)$ as independent variables. However, although the spatial point of view simplifies the theory of stress, it introduces a natural difficulty in practical boundary value problems, as the deformed shape of the medium is not generally known in advance. Furthermore, when problems of deformable solids are formulated by means of energy principles, the initial coordinates (x_1, x_2, x_3) serve most simply and naturally as independent variables. The arbitrariness of selection of spatial or material variables does not arise in the classical small-deflection theories of elasticity and plasticity because there the points (x_1, x_2, x_3) and (ξ_1, ξ_2, ξ_3) are assumed to lie so close together that it is not necessary to distinguish between them; that is, the displacements are infinitesimally small. Kinematical consequences of this arbitrariness of choice of independent variables will be considered later.

By Eqs. (2-3.1) and (2-3.2), we note that for $t = 0$, $\xi_1 = x_0$, $\xi_2 = y_0$, and $\xi_3 = z_0$, where x_0, y_0, and z_0 denote initial values of ξ_i. Furthermore, we note that Eqs. (2-3.1) and (2-3.2) represent a triple infinity of curves, dependent on the initial point x_0, y_0, and z_0 chosen. Hence, for simplicity we consider a deformation at a given time $t =$ constant; that is, we consider the deformation between time $t = 0$ and time $t =$ constant. In this manner, we may omit t from Eqs. (2-3.1) and (2-3.2) entirely. Then

$$\xi_1 = \xi_1(x_1, x_2, x_3), \qquad \xi_2 = \xi_2(x_1, x_2, x_3), \qquad \xi_3 = \xi_3(x_1, x_2, x_3) \tag{2-3.3a}$$

and

$$x_1 = x_1(\xi_1, \xi_2, \xi_3), \qquad x_2 = x_2(\xi_1, \xi_2, \xi_3), \qquad x_3 = x_3(\xi_1, \xi_2, \xi_3) \tag{2-3.3b}$$

Although the general description of deformation described in terms of arbitrary curvilinear coordinates relative to two frames A and B (Fig. 2-3.2) has merit in a general study of deformation theory, the basic physical concepts may become more evident in terms of specific reference frames and coordinates. Accordingly, let us consider frames A, B to coincide with the rectangular Cartesian frame F. Hence, let x_i, ξ_i be rectangular Cartesian coordinates (Fig. 2-3.3). Then coordinates (x_1, x_2, x_3) denote rectangular Cartesian coordinates of point P in region R,

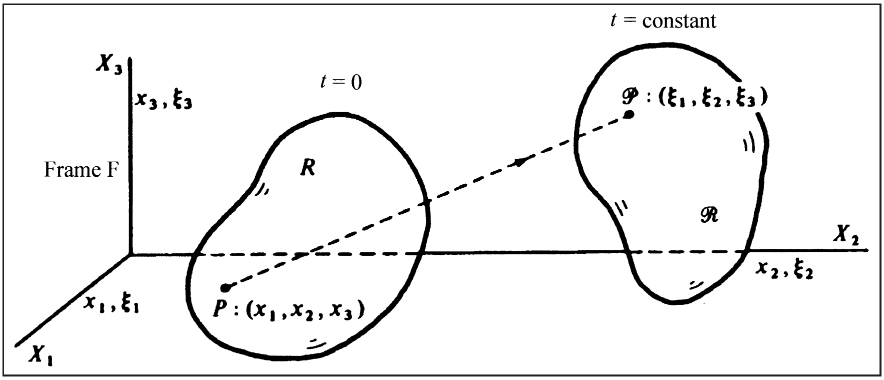

Figure 2-3.3 Frames A, B coincident with frame. $F(x_1, x_2, x_3)$

and (ξ_1, ξ_2, ξ_3) denote rectangular Cartesian coordinates of point $\mathscr{P}$ in region $\mathscr{R}$. Because Eqs. (2-3.3) define the final position (ξ_1, ξ_2, ξ_3) of a particle in terms of the initial position (x_1, x_2, x_3) of a particle, the quantities

$$u_1 = \xi_1 - x_1, \qquad u_2 = \xi_2 - x_2, \qquad u_3 = \xi_3 - x_3 \tag{2-3.4}$$

are called the *components of the displacement* of the particle P. The collection of elements (u_1, u_2, u_3) is called the *displacement of P*. The displacement is commonly represented by a displacement vector (see Section 2-2)

$$\mathbf{q} = \mathbf{i}u_1 + \mathbf{j}u_2 + \mathbf{k}u_3 \tag{2-3.5}$$

where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are unit vectors directed along positive (X_1, X_2, X_3) axes, respectively, and (u_1, u_2, u_3) are given by Eq. (2-3.4). By means of Eqs. (2-3.3), the displacement u_i may be expressed in terms of either the material coordinates x_i or the spatial coordinates ξ_i.

Remarks on Notations. In the above development we have employed indexing notation. Accordingly, we have denoted axes by the symbols x_i or (x_1, x_2, x_3), and associated quantities—for example, displacement components—by (u_1, u_2, u_3) with corresponding indexing. In many texts, conventional x, y, z notation is employed. For example, rectangular Cartesian axes are denoted by (x, y, z), and (x, y, z) displacement components are denoted by three corresponding symbols, say, (u, v, w). Then partial derivatives of (u, v, w) relative to (x, y, z) are denoted by $\partial u/\partial x, \partial u/\partial y, \partial u/\partial z, \ldots$, and so on, by $u_x, u_y, u_z, \ldots$, and so on. From time to time in the text we refer to x, y, z notation for convenience.

2-4 Restrictions on Continuous Deformation of a Deformable Medium

In the theory of functions, it is shown that Eqs. (2-3.1) possess a single-valued continuous solution of the type (2-3.2) if and only if the following determinant does not vanish in region R (Courant, 1992):

$$J = \begin{vmatrix} \dfrac{\partial \xi_1}{\partial x_1} & \dfrac{\partial \xi_1}{\partial x_2} & \dfrac{\partial \xi_1}{\partial x_3} \\ \dfrac{\partial \xi_2}{\partial x_1} & \dfrac{\partial \xi_2}{\partial x_2} & \dfrac{\partial \xi_2}{\partial x_3} \\ \dfrac{\partial \xi_3}{\partial x_1} & \dfrac{\partial \xi_3}{\partial x_2} & \dfrac{\partial \xi_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} \xi_{1,1} & \xi_{1,2} & \xi_{1,3} \\ \xi_{2,1} & \xi_{2,2} & \xi_{2,3} \\ \xi_{3,1} & \xi_{3,2} & \xi_{3,3} \end{vmatrix} = \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(x_1, x_2, x_3)} \neq 0 \tag{2-4.1}$$

where the subscripts $(\,, i)$ denote partial differentiations with respect to $x_i, i = 1, 2, 3$.

The determinant J of Eq. (2-4.1) is called the *functional determinant* or the *Jacobian* of the functions (ξ_1, ξ_2, ξ_3). The expression $\partial(\xi_1, \xi_2, \xi_3)/\partial(x_1, x_2, x_3)$ is a conventional notation for the Jacobian.

On the basis of the following physical argument, further restrictions may be placed on the Jacobian J of the deformation. If the particles of a body are not displaced at all, $\xi_1 = x_1$, $\xi_2 = x_2$, and $\xi_3 = x_3$. Then $J = 1$. Also, because the deformation is a continuous function of time (it does not occur instantaneously), the Jacobian is a continuous function of time. Hence, J is positive for small continuous deformation. Furthermore, J cannot become negative by a continuous deformation of the medium without passing through the excluded value zero [Eq. (2-4.1)]. It follows that J can never be negative. Therefore, we have the following theorem:

Theorem 2-4.1. *A necessary and sufficient condition for a continuous deformation to be physically possible is that the Jacobian J be greater than zero.*

Substituting Eqs. (2-3.4) into Eq. (2-4.1) we obtain

$$J = \begin{vmatrix} 1 + \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} \\ \dfrac{\partial u_2}{\partial x_1} & 1 + \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_3}{\partial x_1} & \dfrac{\partial u_3}{\partial x_2} & 1 + \dfrac{\partial u_3}{\partial x_3} \end{vmatrix} > 0 \tag{2-4.2}$$

In index notation (see Section 1-23), we may write

$$J = \det\left(\delta_{\alpha\beta} + \frac{\partial u_\alpha}{\partial x_\beta}\right); \qquad \alpha, \beta = 1, 2, 3$$

$$= \det(\delta_{\alpha\beta} + u_{\alpha,\beta}) > 0 \tag{2-4.3}$$

where

$$\delta_{\alpha,\beta} = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

is the Kronecker delta, $u_{\alpha,\beta} = \partial u_\alpha / \partial x_\beta$, and det stands for determinant.

When the displacement field (u_1, u_2, u_3) satisfies the condition $J > 0$, we say that the displacement field is *proper* and *admissible*, and the deformation is a *proper* and *admissible* deformation; or, for brevity, we may refer to the displacement and deformation simply as admissible.

Thus, for an admissible deformation of a medium, the displacement components (u_1, u_2, u_3) must satisfy Eq. (2-4.2). For example, we cannot subject a piece of rubber to displacement components $u_1 = -2x_1, u_2 = 0$, and $u_3 = 0$ because then $J = -1$. This type of displacement is called a *reflection about the* (x_2, x_3) *plane*, as the point (ξ_1, ξ_2, ξ_3) may be considered the image of the point (x_1, x_2, x_3) in a mirror that lies in the plane $x_1 = 0$ [see Eqs. (2-3.3) and (2-3.4)].

We may define deformation gradient tensor as

$$\xi_{i,\alpha} \triangleq F_{i\alpha} \equiv \frac{\partial \xi_i}{\partial x_\alpha} \tag{2-4.4}$$

Throughout this book $\triangleq$ is used as denoted by and "$\equiv$" is used as "defined as" or "indentical to".

Then, from Eq. (2-4.1), it is seen that the Jacobian is the determinant of the *deformation gradient tensor*:

$$J = \det(\xi_{i,\alpha}) \tag{2-4.5}$$

For *large strain theory*, which is useful in the study of biomechanics later, the *Cauchy strain tensor* is defined as

$$C_{\alpha\beta} \equiv \xi_{i,\alpha}\xi_{i,\beta} \tag{2-4.6}$$

We now define a set of three invariants of the Cauchy strain tensor as follows:

$$\begin{aligned} I_1(\mathbf{C}) &\triangleq I_1 \equiv \mathrm{tr}(\mathbf{C}) = C_{\alpha\alpha} \\ I_2(\mathbf{C}) &\triangleq I_2 \equiv \mathrm{tr}(\mathbf{C}^2) = C_{\alpha\beta}C_{\beta\alpha} \\ I_3(\mathbf{C}) &\triangleq I_3 \equiv \mathrm{tr}(\mathbf{C}^3) = C_{\alpha\beta}C_{\beta\gamma}C_{\gamma\alpha} \end{aligned} \tag{2-4.7}$$

Why do we call I_1, I_2, and I_3 invariants? Because I_1, I_2, and I_3 remain constants under the rotation of rectangular Cartesian system (coordinate transformation). Let's prove it. A tensor of nth order transforms according to Eq. (1-24.15), therefore one has

$$\overline{C}_{ij} = a_{i\alpha}a_{j\beta}C_{\alpha\beta} \tag{2-4.8}$$

where $\mathbf{a} = [a_{i\alpha}]$ is the transformation matrix between the two coordinate systems, (x, y, z) and $(\overline{x}, \overline{y}, \overline{z})$, that is,

$$\overline{x}_i = a_{i\alpha}x_\alpha \tag{2-4.9}$$

The transformation matrix has to satisfy [cf. Eqs. (1-26.2) and (1-26.3)]

$$a_{\alpha\gamma}a_{\beta\gamma} = a_{\gamma\alpha}a_{\gamma\beta} = \delta_{\alpha\beta} \tag{2-4.10}$$

Now one can readily verify that

$$I_1(\overline{\mathbf{C}}) = \mathrm{tr}(\overline{\mathbf{C}}) = \overline{C}_{ii} = a_{i\alpha}a_{i\beta}C_{\alpha\beta} = \delta_{\alpha\beta}C_{\alpha\beta} = C_{\alpha\alpha} = I_1(\mathbf{C}) \tag{2-4.11}$$

Readers may work out Problem 6 of Problem Set 2-4 to verify, similar to Eq. (2-4.11), that

$$I_2(\overline{\mathbf{C}}) = I_2(\mathbf{C}) \qquad I_3(\overline{\mathbf{C}}) = I_3(\mathbf{C}) \tag{2-4.12}$$

One may define another set of invariants as

$$\begin{aligned} \mathrm{I}_{\mathbf{C}} &\equiv I_1(\mathbf{C}) \\ \mathrm{II}_{\mathbf{C}} &\equiv \left\{I_1^2(\mathbf{C}) - I_2(\mathbf{C})\right\}/2 \\ \mathrm{III}_{\mathbf{C}} &\equiv \left\{2I_3(\mathbf{C}) - 3I_1(\mathbf{C})I_2(\mathbf{C}) + I_1^3(\mathbf{C})\right\}/6 \end{aligned} \tag{2-4.13}$$

It is obvious that $\mathrm{I}_{\overline{\mathbf{C}}} = \mathrm{I}_{\mathbf{C}}$, $\mathrm{II}_{\overline{\mathbf{C}}} = \mathrm{II}_{\mathbf{C}}$, and $\mathrm{III}_{\overline{\mathbf{C}}} = \mathrm{III}_{\mathbf{C}}$. What is the meaning of this set of invariants? The Cayley–Hamilton theorem (Hildebrand, 1992; Nair, 2009) says any second-order tensor **A** satisfies the following equation:

$$-\mathbf{A}^3 + \mathrm{I}_{\mathbf{A}}\mathbf{A}^2 - \mathrm{II}_{\mathbf{A}}\mathbf{A} + \mathrm{III}_{\mathbf{A}}\mathbf{I} = 0 \tag{2-4.14}$$

One may also verify that

$$-\lambda^3 + \mathrm{I}_{\mathbf{A}}\lambda^2 - \mathrm{II}_{\mathbf{A}}\lambda + \mathrm{III}_{\mathbf{A}} = 0 \tag{2-4.15}$$

where λ is the eigenvalue of **A**, that is,

$$\det(A_{\alpha\beta} - \lambda\delta_{\alpha\beta}) = 0 \tag{2-4.16}$$

and

$$\begin{aligned} \mathrm{I}_{\mathbf{A}} &= \mathrm{tr}(\mathbf{A}) = A_{11} + A_{22} + A_{33} \\ \mathrm{II}_{\mathbf{A}} &= \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} + \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \\ \mathrm{III}_{\mathbf{A}} &= \det(\mathbf{A}) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \end{aligned} \tag{2-4.17}$$

It is noticed from Eq. (2-4.6) that

$$\mathrm{III}_{\mathbf{C}} = \det(\xi_{i,\alpha}\xi_{i,\beta}) = \det(\xi_{i,\alpha})\det(\xi_{i,\beta}) = J^2 \tag{2-4.18}$$

Example 2-4.1. Proper and Admissible Displacement Field. Determine whether the displacement field

$$u_1 = x_1 - 2x_2, \qquad u_2 = 3x_1 + 2x_2, \qquad u_3 = 5x_3 \tag{a}$$

is proper and admissible.

By Eq. (2-4.2) the condition for a proper and admissible displacement field is $J > 0$. To examine this condition, we must compute the derivatives of u_i with respect to the coordiinates x_i. Thus,

$$\begin{aligned} \frac{\partial u_1}{\partial x_1} &= 1, & \frac{\partial u_2}{\partial x_1} &= 3, & \frac{\partial u_3}{\partial x_1} &= 0 \\ \frac{\partial u_1}{\partial x_2} &= -2, & \frac{\partial u_2}{\partial x_2} &= 2, & \frac{\partial u_3}{\partial x_2} &= 0 \\ \frac{\partial u_1}{\partial x_3} &= 0, & \frac{\partial u_2}{\partial x_3} &= 0, & \frac{\partial u_3}{\partial x_3} &= 5 \end{aligned} \tag{b}$$

Substitution of Eq. (b) into Eq. (2-4.2) yields

$$J = \begin{vmatrix} 2 & -2 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 6 \end{vmatrix} \tag{c}$$

Evaluation of the determinant in Eq. (c) yields $J = 72 > 0$. Hence, the displacement field, Eq. (a), is proper and admissible.

Problem Set 2-4

1. Determine whether $u_1 = k(x_2 - x_1)$, $u_2 = k(x_1 - x_2)$, and $u_3 = kx_1x_3$, where k is a constant, are continuously possible displacement components for a continuous medium. Consider (x_1, x_2, x_3) to be rectangular Cartesian coordinates.

2. Show that $u_1 = ax_2x_3$, $u_2 = bx_3x_1$, and $u_3 = cx_1x_2$, where (a, b, c) are constants, are the components of a continuously possible displacement vector.

3. A parallelepiped occupies the region $0 \le x \le L$, $-h \le y \le h$, and $-b \le z \le b$. It is deformed in such a manner that a material point $P(x, y, z)$ is displaced to $P^*(x^*, y^*, z^*)$, where $x^* = (C-y)\cos(x/C)$, $y^* = (C-y)\sin(x/C)$, and $z^* = z$, where C is a constant. Indicate the restrictions that must be imposed upon C in order that the displacement may be continuously possible. Here we use the notation $x \equiv x_1$, $y \equiv x_2$, and $z \equiv x_3$, and so on.

4. Determine whether

$$\begin{aligned} u_1 &= -k_1x_1x_2 \\ u_2 &= k_2(x_1^2 + \nu x_2^2 - \nu x_3^2) \\ u_3 &= k_3\nu x_2x_3 \end{aligned}$$

where (k_1, k_2, k_3) are constants and ν is Poisson's ratio, are possible continuous (x_1, x_2, x_3) displacement components for the deformation of a body of infinite dimensions. Consider (x_1, x_2, x_3) as Cartesian rectangular coordinates with origin in the body.

5. In the problem of twisting (torsion) of a cylindrical bar with an elliptic cross section, the (x_1, x_2, x_3) components of displacement are calculated to be

$$u_1 = -\theta x_2 x_3, \qquad u_2 = \theta x_1 x_3, \qquad u_3 = \frac{b^2 - a^2}{b^2 + a^2}\theta x_1 x_2$$

where x_3 is the coordinate along the axis of the bar, θ is the angle of twist in radians per unit length of the bar, and (a, b) are the major and minor axes of the elliptic cross section (see Chapter 7). Verify that this displacement field is admissible.

6. Verify that

$$I_2(\overline{\mathbf{C}}) = I_2(\mathbf{C})$$
$$I_3(\overline{\mathbf{C}}) = I_3(\mathbf{C})$$

7. Consider Eq. (2-4.17) as the definitions for $\mathrm{I}_{\mathbf{A}}, \mathrm{II}_{\mathbf{A}}$, and $\mathrm{III}_{\mathbf{A}}$, verify that

$$\mathrm{I}_{\mathbf{A}} \triangleq I_1(\mathbf{A})$$
$$\mathrm{II}_{\mathbf{A}} \triangleq \{I_1^2(\mathbf{A}) - I_2(\mathbf{A})\}/2$$
$$\mathrm{III}_{\mathbf{A}} \triangleq \{2I_3(\mathbf{A}) - 3I_1(\mathbf{A})I_2(\mathbf{A}) + I_1^3(\mathbf{A})\}/6$$

2-5 Gradient of the Displacement Vector. Tensor Quantity

The elements of the determinant of Eq. (2-4.2) play a fundamental role in the description of the behavior of deformable media. Physically, these components characterize the gradients of the displacement vector $\mathbf{q}$ with respect to the variable x_i.

With Eq. (2-3.5) and the definition of the vector operator ∇, we define the following operation:

$$\text{grad } \mathbf{q} = (\nabla, \mathbf{q}) = \mathbf{i}\frac{\partial \mathbf{q}}{\partial x_1} + \mathbf{j}\frac{\partial \mathbf{q}}{\partial x_2} + \mathbf{k}\frac{\partial \mathbf{q}}{\partial x_3} \tag{a}$$

where grad $\mathbf{q}$ and $(\nabla, \mathbf{q})$ are two conventional notations for the operation denoted on the right side of Eq. (a).

Expanding, by multiplying the right side of Eq. (a) term by term, we have

$$\begin{aligned} \text{grad}\,\mathbf{q} = {} & \mathbf{ii}\frac{\partial u_1}{\partial x_1} + \mathbf{ij}\frac{\partial u_2}{\partial x_1} + \mathbf{ik}\frac{\partial u_3}{\partial x_1} \\ & + \mathbf{ji}\frac{\partial u_1}{\partial x_2} + \mathbf{jj}\frac{\partial u_2}{\partial x_2} + \mathbf{jk}\frac{\partial u_3}{\partial x_2} \\ & + \mathbf{ki}\frac{\partial u_1}{\partial x_3} + \mathbf{kj}\frac{\partial u_2}{\partial x_3} + \mathbf{kk}\frac{\partial u_3}{\partial x_3} \end{aligned} \tag{2-5.1}$$

Thus, we may characterize the operation grad **q** by the following array of elements (matrix):

$$\begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_3}{\partial x_2} \\ \dfrac{\partial u_1}{\partial x_3} & \dfrac{\partial u_2}{\partial x_3} & \dfrac{\partial u_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} u_{1,1} & u_{2,1} & u_{3,1} \\ u_{1,2} & u_{2,2} & u_{3,2} \\ u_{1,3} & u_{2,3} & u_{3,3} \end{bmatrix} \tag{b}$$

where subscripts (, i) denote differentiation relative to x_i (Chapter 1, Section 1-23).

In contrast to the three-dimensional displacement vector **q**, which may be represented by the three components (u_1, u_2, u_3), grad **q** is a mathematical entity formed from three vector functions (the displacement or deformation gradients):

$$\frac{\partial \mathbf{q}}{\partial x_1}, \qquad \frac{\partial \mathbf{q}}{\partial x_2}, \qquad \frac{\partial \mathbf{q}}{\partial x_3} \tag{c}$$

or nine scalar functions:

$$u_{1,1}, \quad u_{1,2}, \quad u_{1,3}, \quad u_{2,1}, \quad u_{2,2}, \quad u_{2,3}, \quad u_{3,1}, \quad u_{3,2}, \quad u_{3,3} \tag{d}$$

Following Gibbs, such a quantity is called a *second-order tensor*; that is, grad **q** is a second-order tensor. In the theory of tensors, a vector is regarded as a first-order tensor, and a scalar, a zero-order tensor (see Chapter 1, Section 1-23).

Tensorial quantities have certain invariant properties under the transformation of coordinate axes. For example, the displacement vector **q** of a particle of a medium remains unchanged under a transformation from rectangular Cartesian coordinates (x_1, x_2, x_3) to cylindrical coordinates (r, θ, z). However, the components of **q** do not remain invariant. The property of invariance under coordinate transformation plays a fundamental role in mathematical physics (Morse and Feshbach, 1961).

Because the gradient of the displacement vector **q** is a second-order tensor, it may be represented as the sum of a symmetrical tensor of second order and an antisymmetrical tensor of second order (see Chapter 1, Section 1-24). Thus, we may write

$$\text{grad } \mathbf{q} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} + \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ \omega_{21} & 0 & \omega_{23} \\ \omega_{31} & \omega_{32} & 0 \end{bmatrix} \tag{2-5.2}$$

where

$$\begin{aligned} 2e_{\beta\alpha} &= 2e_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} \\ 2\omega_{\alpha\beta} &= u_{\beta,\alpha} - u_{\alpha,\beta} = -2\omega_{\beta\alpha} \end{aligned} \tag{2-5.3}$$

Thus, $e_{\alpha\beta}$ is symmetric in α, β, whereas $\omega_{\alpha\beta}$ is antisymmetric. Equation (2-5.2) signifies that if e_{ij} and ω_{ij} are known, grad $\mathbf{q}$ is determined. More briefly, we may write Eq. (2-5.2) in the form

$$\text{grad}\,\mathbf{q} = \mathbf{D} + \Omega \tag{2-5.4}$$

where

$$\mathbf{D} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} \tag{2-5.5}$$

denotes a symmetrical tensor of second order, and

$$\Omega = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix} \tag{2-5.6}$$

denotes an antisymmetrical tensor of second order. For small deformations we show later that $\mathbf{D}$ characterizes the strain components at a point in a medium, whereas $\boldsymbol{\Omega}$ characterizes the *mean* rotation of a volume element (i.e., ω_{12}, ω_{13}, and ω_{23}).

Problem. Show that any matrix may be expressed as the sum of a symmetric matrix ($a_{ij} = a_{ji}$) and an antisymmetrical matrix ($b_{ij} = -b_{ji}$). *Hint:* Show it for a 2×2 matrix. Is this transformation unique?

2-6 Extension of an Infinitesimal Line Element

In the theory of deformation of a continuous medium, the idea of the elongation of an infinitesimal line element ds is fundamental. In this section we derive a general expression for the magnification factor $d\mathscr{s}/ds$ of the line element of infinitesimal length ds, where $d\mathscr{s}$ denotes the extended length of the fiber. In the classical theories of elasticity and plasticity of massive bodies, approximations based on the concept of infinitesimally small displacements lead to a complete linearization of the theory of deformation. However, in a general treatment of bodies with one small dimension compared to the other dimensions (such as thin shells, bars, etc.) and in a general treatment of fluid mechanics, the nonlinear effects may be important. Hence, initially we develop the theory of deformation without using linearizing approximations. Later, we specialize the equations to obtain classical linear results.

Consider an infinitesimal fiber *PA* of length ds in a region R with direction defined by $\mathbf{N}$: (N_1, N_2, N_3) (Fig. 2-6.1). Under a deformation of the medium, the line element *PA* passes into the line element $\mathscr{P}\mathscr{A}$ of length $d\mathscr{s}$ and direction $\mathscr{N}$: $(\mathscr{N}_1, \mathscr{N}_2, \mathscr{N}_3)$; that is, under the deformation the particle at point $\mathscr{P}$: (x_1, x_2, x_3) moves to the point $\mathscr{P}$: (ξ_1, ξ_2, ξ_3), and the particle at A: $(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ moves to $\mathscr{A}$: $(\xi_1 + d\xi_1, \xi_2 + d\xi_2, \xi_3 + d\xi_3)$. In general, under the deformation

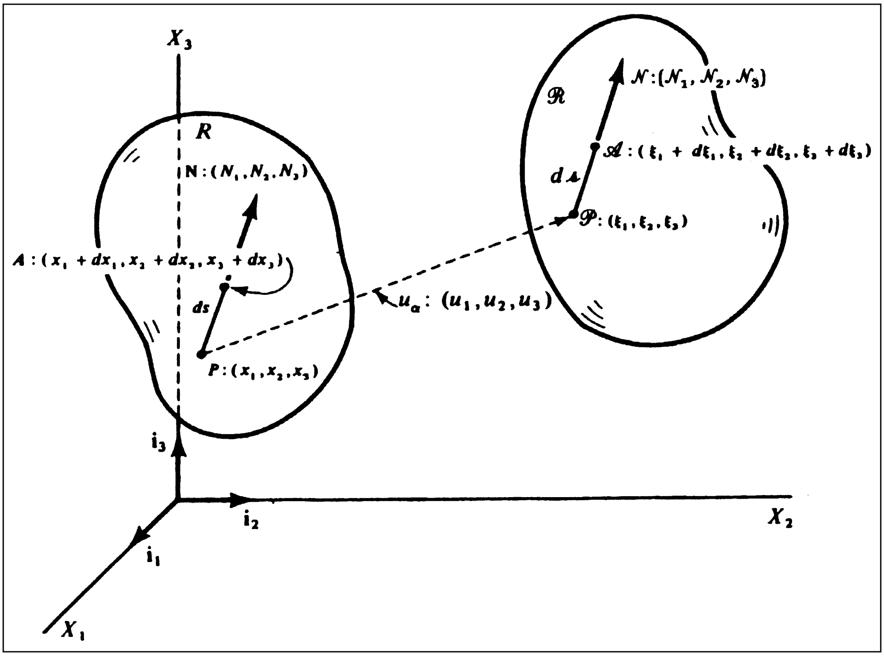

Figure 2-6.1 Extension of infinitesimal line element.

both the length and direction of *PA* are changed. Initially, we consider only the change in length of *PA*. Because the fiber remains continuous after the deformation, line element $\mathscr{PA}$ is an infinitesimal line element.

By geometry, we obtain the following expression for the square of the length of line element *PA*:

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{2-6.1}$$

or, in index notation (Chapter 1, Section 1-23 and 1-24),

$$ds^2 = \delta_{\alpha\beta}\, dx_\alpha\, dx_\beta = dx_\alpha\, dx_\alpha, \qquad \alpha = 1, 2, 3 \tag{2-6.2}$$

Similarly, for the deformed line element $\mathscr{PA}$, we have

$$\begin{aligned}(d\mathscr{s})^2 &= (d\xi_1)^2 + (d\xi_2)^2 + (d\xi_3)^2 \\ &= \delta_{\alpha\beta}\, d\xi_\alpha\, d\xi_\beta = d\xi_\beta\, d\xi_\beta \qquad \beta = 1, 2, 3\end{aligned} \tag{2-6.3}$$

To express the magnification factor $d\mathscr{s}/ds$ in terms of the displacement components $(u_1, u_2, u_3) = u_i = u_i(x_1, x_2, x_3)$ and $(dx_1, dx_2, dx_3) = dx_i$, we first express $(d\xi_1, d\xi_2, d\xi_3) = d\xi_i$ in terms of u_i and dx_i. Because $\xi_i = \xi_i(x_1, x_2, x_3) = \xi_i(\mathbf{x})$, where $\mathbf{x}$ stands collectively for (x_1, x_2, x_3), expressions for the total differentials

$d\xi_i$ in terms of dx_i are (Kaplan, 2002; Hildebrand, 1992)

$$\begin{aligned} d\xi_1 &= \xi_{1,1}\,dx_1 + \xi_{1,2}\,dx_2 + \xi_{1,3}\,dx_3 \\ d\xi_2 &= \xi_{2,1}\,dx_1 + \xi_{2,2}\,dx_2 + \xi_{2,3}\,dx_3 \\ d\xi_3 &= \xi_{3,1}\,dx_1 + \xi_{3,2}\,dx_2 + \xi_{3,3}\,dx_3 \end{aligned} \tag{2-6.4}$$

or

$$d\xi_\alpha = \frac{\partial \xi_\alpha}{\partial x_\beta} dx_\beta = \xi_{\alpha,\beta}\,dx_\beta, \qquad \alpha,\ \beta = 1,\ 2,\ 3 \tag{2-6.5}$$

where the subscripts (, β) denote partial derivatives with respect to x_β (Chapter 1, Section 1-23).

In symbol notation (see Chapter 1, Sections 1-12 and 1-22), Eq. (2-6.4) may be written in the form

$$d\mathscr{P} = (d\mathbf{P}\cdot\nabla)\mathscr{P} = d\mathbf{P}\cdot\nabla\mathscr{P} \tag{2-6.4a}$$

where

$$\begin{aligned} \mathbf{P} &= \mathbf{i}_1 x_1 + \mathbf{i}_2 x_2 + \mathbf{i}_3 x_3 \\ \mathscr{P} &= \mathbf{i}_1 \xi_1 + \mathbf{i}_2 \xi_2 + \mathbf{i}_3 \xi_3 \\ \nabla &= \mathbf{i}_1 \frac{\partial}{\partial x_1} + \mathbf{i}_2 \frac{\partial}{\partial x_2} + \mathbf{i}_3 \frac{\partial}{\partial x_3} \end{aligned}$$

The notation $(d\mathbf{P}\cdot\nabla)\mathscr{P}$ means that the scalar operation $d\mathbf{P}\cdot\nabla$ must be performed before the operation $\nabla\mathscr{P}$. The right side of Eq. (2-6.4a) is short-hand notation for this operation:

$$d\mathbf{P}\cdot\nabla = dx_1 \frac{\partial}{\partial x_1} + dx_2 \frac{\partial}{\partial x_2} + dx_3 \frac{\partial}{\partial x_3}$$

and

$$\begin{aligned} (d\mathbf{P}\cdot\nabla)\mathscr{P} = \mathbf{i}_1 &\left(\frac{\partial \xi_1}{\partial x_1} dx_1 + \frac{\partial \xi_1}{\partial x_2} dx_2 + \frac{\partial \xi_1}{\partial x_3} dx_3 \right) \\ &+ \mathbf{i}_2 \left(\frac{\partial \xi_2}{\partial x_1} dx_1 + \frac{\partial \xi_2}{\partial x_2} dx_2 + \frac{\partial \xi_2}{\partial x_3} dx_3 \right) \\ &+ \mathbf{i}_3 \left(\frac{\partial \xi_3}{\partial x_1} dx_1 + \frac{\partial \xi_3}{\partial x_2} dx_2 + \frac{\partial \xi_3}{\partial x_3} dx_3 \right) \end{aligned}$$

Hence, because $d\mathscr{P} = \mathbf{i}_1\,d\xi_1 + \mathbf{i}_2\,d\xi_2 + \mathbf{i}_3\,d\xi_3$, Eq. (2-6.4a) is equivalent to Eq. (2-6.4) or Eq. (2-6.5).

Alternatively, we may represent the increments dx_α in terms of $d\xi_\alpha$ as

$$d\mathbf{P} = (d\mathscr{P}\cdot\mathbf{\Delta})\mathbf{P} = d\mathscr{P}\cdot\mathbf{\Delta}\mathbf{P} \tag{2-6.4b}$$

where

$$\Delta = \mathbf{i}_1 \frac{\partial}{\partial \xi_1} + \mathbf{i}_2 \frac{\partial}{\partial \xi_2} + \mathbf{i}_3 \frac{\partial}{\partial \xi_3}$$

In addition, by the chain rule of differentiation, it can be shown that the displacement increments can be represented in the material description and the spatial description, respectively, as

$$d\mathbf{q} = (d\mathbf{P} \cdot \nabla)\mathbf{q} = d\mathbf{P} \cdot \nabla\mathbf{q} \tag{2-6.4c}$$

$$d\mathbf{q} = (d\mathscr{P} \cdot \mathbf{\Delta})\mathbf{q} = d\mathscr{P} \cdot \mathbf{\Delta}\mathbf{q} \tag{2-6.4d}$$

where [see Eq. (2-3.5)]

$$\mathbf{q} = \mathbf{i}_1 u_1 + \mathbf{i}_2 u_2 + \mathbf{i}_3 u_3$$

and the u_i are taken to be functions of x_i in Eq. (2-6.4c) and functions of ξ_i in Eq. (2-6.4d). See Example 2-6.1.

Substituting Eq. (2-3.4) into Eqs. (2-6.4) and (2-6.5), we obtain

$$\begin{aligned} d\xi_1 &= (1 + u_{1,1})\, dx_1 + u_{1,2}\, dx_2 + u_{1,3}\, dx_3 \\ d\xi_2 &= u_{2,1}\, dx_1 + (1 + u_{2,2})\, dx_2 + u_{2,3}\, dx_3 \\ d\xi_3 &= u_{3,1}\, dx_1 + u_{3,2}\, dx_2 + (1 + u_{3,3})\, dx_3 \end{aligned} \tag{2-6.6}$$

or

$$d\xi_\alpha = \left(\delta_{\alpha\beta} + \frac{\partial u_\alpha}{dx_\beta}\right) dx_\beta = (\delta_{\alpha\beta} + u_{\alpha,\beta})\ dx_\beta \tag{2-6.7}$$

where u_α is the displacement vector of point P under the deformation (Fig. 2-6.1). Hence, by Eqs. (2-6.2), (2-6.3), and (2-6.7), we obtain

$$\begin{aligned} \tfrac{1}{2}\left[d\mathscr{s}^2 - ds^2\right] &= \epsilon_{11}\ dx_1^2 + \epsilon_{22}\ dx_2^2 + \epsilon_{33}\ dx_3^2 + 2\epsilon_{12}\ dx_1\ dx_2 \\ &\quad + 2\epsilon_{13}\ dx_1\ dx_3 + 2\epsilon_{23}\ dx_2\ dx_3 = \epsilon_{\alpha\beta}\ dx_\alpha\ dx_\beta \end{aligned} \tag{2-6.8}$$

where the coefficients of $dx_1^2, dx_2^2, \ldots$ are given by the following expressions[2]:

$$\begin{aligned} \epsilon_{11} &= u_{1,1} + \tfrac{1}{2}\left(u_{1,1}^2 + u_{2,1}^2 + u_{3,1}^2\right) \\ \epsilon_{22} &= u_{2,2} + \tfrac{1}{2}\left(u_{1,2}^2 + u_{2,2}^2 + u_{3,2}^2\right) \\ \epsilon_{33} &= u_{3,3} + \tfrac{1}{2}\left(u_{1,3}^2 + u_{2,3}^2 + u_{3,3}^2\right) \\ 2\epsilon_{12} &= u_{2,1} + u_{1,2} + u_{1,1}u_{1,2} + u_{2,1}u_{2,2} + u_{3,1}u_{3,2} \\ 2\epsilon_{13} &= u_{3,1} + u_{1,3} + u_{1,1}u_{1,3} + u_{2,1}u_{2,3} + u_{3,1}u_{3,3} \\ 2\epsilon_{23} &= u_{3,2} + u_{2,3} + u_{1,2}u_{1,3} + u_{2,2}u_{2,3} + u_{3,2}u_{3,3} \end{aligned} \tag{2-6.9}$$

[2]Many authors use the notation γ_{12}, γ_{13}, γ_{23} where we have used $2\epsilon_{12}$, $2\epsilon_{13}$, $2\epsilon_{23}$. As we see later, the factors ϵ_{12}, ϵ_{13}, ϵ_{23} are components of a tensor of second order, whereas the factors γ_{12}, γ_{13}, γ_{23} are not.

or, in index notation,

$$2\epsilon_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha} + \frac{\partial u_\gamma}{\partial x_\alpha}\frac{\partial u_\gamma}{\partial x_\beta}$$
$$= u_{\alpha,\beta} + u_{\beta,\alpha} + u_{\gamma,\alpha}u_{\gamma,\beta} = 2\epsilon_{\beta\alpha} \qquad \alpha, \beta, \gamma = 1, 2, 3 \tag{2-6.10}$$

where $\epsilon_{\alpha\beta} = \epsilon_{\beta\alpha} = \epsilon_{\alpha\beta}(x_1, x_2, x_3)$.

The extension ratio e of the line element *PA* is defined by $e = (d\mathscr{s} - ds)/ds$. Hence, if we compute the ratio $d\mathscr{s}/ds$, we can compute the extension e. The ratio $d\mathscr{s}/ds$ may be expressed in terms of either x_i (material coordinates) or ξ_i (spatial coordinates), as x_i and ξ_i are related [see Eqs. (2-3.3) and (2-3.4)]. In the *material description*, we consider the displacement components u_α (hence ξ_α) to be functions of the material coordinates x_α. Then

$$\xi_\alpha(x_1, x_2, x_3) = x_\alpha + u_\alpha$$
$$u_\alpha = u_\alpha(x_1, x_2, x_3) \qquad \alpha = 1, 2, 3 \tag{a}$$

Consequently, Eqs. (2-6.10) represent the *material description* for the set of elements that determines the factor $\frac{1}{2}(d\mathscr{s}^2 - ds^2)$. In the *spatial description*, the displacement components are considered functions of the spatial coordinates ξ_i. Then

$$x_\alpha(\xi_1, \xi_2, \xi_3) = \xi_\alpha - u_\alpha$$
$$u_\alpha = u_\alpha(\xi_1, \xi_2, \xi_3) \qquad \alpha = 1, 2, 3 \tag{b}$$

Following a procedure analogous to that used to obtain Eqs. (2-6.10), we find for the spatial description the set of elements that determines the factor $\frac{1}{2}(d\mathscr{s}^2 - ds^2)$:

$$2\mathscr{E}_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} - u_{\gamma,\alpha}u_{\gamma,\beta} = 2\mathscr{E}_{\beta\alpha} \tag{2-6.11}$$

where we now consider $u_\alpha = u_\alpha$ (ξ_1, ξ_2, ξ_3) to be functions of the spatial coordinates ξ_α and the subscript (, α) denotes derivatives relative to ξ_α. Because the left side of Eq. (2-6.8) has the dimension $[L^2]$ and $dx_\alpha\, dx_\beta$ has the dimension $[L^2]$, $\epsilon_{11}, \epsilon_{12}, \ldots$ are dimensionless quantities. Also, by Eq. (2-6.10) $\epsilon_{\alpha\beta} = \epsilon_{\beta\alpha}$; that is, the terms $\epsilon_{\alpha\beta}$ form a symmetric array.

Equations (2-6.8) may be written in an alternative conventional form as follows. Let $(\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3)$ denote unit vectors directed along the positive (X_1, X_2, X_3) axes, respectively (Fig. 2-6.1). Then, because the unit vector **N** denotes the direction of *PA*, the direction cosine rule yields

$$\mathbf{N} = \mathbf{i}_1 N_1 + \mathbf{i}_2 N_2 + \mathbf{i}_3 N_3 \equiv N_i \tag{c}$$

where $(N_1, N_2, N_3) \equiv N_i$ are the direction cosines of **N** measured with respect to the (X_1, X_2, X_3) axes. Because the direction of **N** coincides with the direction of

(dx_1, dx_2, dx_3), we have

$$N_1 = \frac{dx_1}{ds} \qquad N_2 = \frac{dx_2}{ds} \qquad N_3 = \frac{dx_3}{ds} \tag{2-6.12}$$

Dividing Eq. (2-6.8) by ds^2 and utilizing Eq. (2-6.12), we obtain

$$\begin{aligned}\frac{1}{2}\left[\left(\frac{d\mathscr{s}}{ds}\right)^2 - 1\right] &= \epsilon_{11}N_1^2 + \epsilon_{22}N_2^2 + \epsilon_{33}N_3^2 \\ &\quad + 2\epsilon_{12}N_1N_2 + 2\epsilon_{13}N_1N_3 + 2\epsilon_{23}N_2N_3 \\ &= \epsilon_{\alpha\beta}N_\alpha N_\beta \end{aligned} \tag{2-6.13}$$

We may express Eq. (2-6.13) in the form

$$MF_A = \epsilon_{\alpha\beta}N_\alpha N_\beta \tag{2-6.14}$$

where we call

$$MF_A = \frac{1}{2}\left[\left(\frac{d\mathscr{s}}{ds}\right)^2 - 1\right] \tag{2-6.15}$$

the magnification factor of the extension of line element *PA*.

Equations (2-6.13) and (2-6.10) [or Eqs. (2-6.14) and (2-6.10)] define implicitly the magnification ratio $d\mathscr{s}/ds$ of an infinitesimal fiber ds that lies at point (x_1, x_2, x_3) (with direction cosines N), provided the displacement components u_α are known functions of (x_1, x_2, x_3). Because point (x_1, x_2, x_3) may be any point in the medium, and N_α may denote any direction, it follows that Eqs. (2-6.13) and (2-6.10) [or Eqs. (2-6.14) and (2-6.10)] describe the state of deformation of the entire medium.

It is important to note that the expressions for the elements $\epsilon_{\alpha\beta}$ [Eqs. (2-6.10)] are exact expressions; they are not a first-order approximation plus a second-order approximation. Furthermore, it may be shown that the set of elements $\epsilon_{\alpha\beta}$ possesses tensor character with respect to arbitrary coordinate transformations (Section 2-9). The elements $\mathscr{E}_{\alpha\beta}$ [Eqs. (2-6.11)] also possess tensor character. For these reasons, the arrays $\epsilon_{\alpha\beta}$, $\mathscr{E}_{\alpha\beta}$ are called *components of the strain tensors* relative to coordinates x_i, ξ_i, respectively. In particular, the array $\epsilon_{\alpha\beta}$ is referred to as the *Green–Saint-Venant strain tensor*, and the array $\mathscr{E}_{\alpha\beta}$ is called the *Almansi strain tensor* in honor of early investigators of deformable body mechanics.

In a rigid-body displacement of the medium, $d\mathscr{s} = ds$ for all infinitesimal line elements through any point in the medium. Then $\epsilon_{11} = \epsilon_{22} = \cdots = \epsilon_{23} = 0$. Conversely, if $\epsilon_{11} = \epsilon_{22} = \cdots = \epsilon_{23} = 0$, the displacement of the medium is a rigid-body displacement; that is, $d\mathscr{s} = ds$. Similar results hold for $\mathscr{E}_{\alpha\beta}$.

Although Eqs. (2-6.10) and (2-6.11) are derived for rectangular Cartesian coordinates, by means of tensor transformations they apply for arbitrary curvilinear Euclidean coordinates.

Derivations for arbitrary curvilinear Euclidean coordinates and for orthogonal curvilinear coordinates are given in the appendices of this chapter.

Example 2-6.1. Admissible Displacement Concepts in Material and Spatial Coordinates. Let the displacement vector be assumed to take the form $\mathbf{q} = \mathbf{q}(x_i, t) = (ax_1, bx_1, 0)$, where a and b are parameters independent of x_i. By Eqs. (2-3.4) and (2-6.4a), $\mathbf{q} = \mathscr{P} - \mathbf{P}$ or $\mathscr{P} = \mathbf{P} + \mathbf{q}$. In expanded form, we have, with Eq. (2-3.5),

$$\xi_1 = x_1 + u_1, \qquad \xi_2 = x_2 + u_2, \qquad \xi_3 = x_3 + u_3 \tag{a}$$

where $u_1 = ax_1$, $u_2 = bx_1$, and $u_3 = 0$. Alternatively, we may write Eq. (a) in the form

$$\xi_1 = (1 + a)x_1, \qquad \xi_2 = x_2 + bx_1, \qquad \xi_3 = x_3 \tag{b}$$

By Eq. (b) we may express (x_1, x_2, x_3) in terms of (ξ_1, ξ_2, ξ_3) as follows:

$$x_1 = \frac{\xi_1}{1 + a}, \qquad x_2 = \xi_2 - \frac{b\xi_1}{1 + a}, \qquad x_3 = \xi_3 \tag{c}$$

The admissibility of $\mathbf{q}$ is given by Eq. (2-4.2). By Eq. (2-4.1) we obtain

$$J = \frac{\partial(\xi_1, \xi_2, \xi_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} 1 + a & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 + a \tag{d}$$

Accordingly, if $a > -1$ and $J > 0$, then $\mathbf{q}$ is admissible.

By Eqs. (a), (b), and (c) we have

$$u_1 = ax_1, \qquad u_2 = bx_1, \qquad u_3 = 0 \tag{e}$$

where $u_i = u_i(x_1, x_2, x_3)$ and

$$u_1 = \frac{a}{1 + a}\xi_1, \qquad u_2 = \frac{b}{1 + a}\xi_1, \qquad u_3 = 0 \tag{f}$$

where $u_i = u_i(\xi_1, \xi_2, \xi_3)$. The partial derivatives of u_i with respect to x_i are, by Eq. (e),

$$\frac{\partial u_1}{\partial x_1} = a, \qquad \frac{\partial u_2}{\partial x_1} = b \tag{g}$$

All the other derivatives are zero. Similarly, the nonzero derivatives of u_i with respect to ξ_i are, by Eq. (f),

$$\frac{\partial u_1}{\partial \xi_1} = \frac{a}{1 + a}, \qquad \frac{\partial u_2}{\partial \xi_1} = \frac{b}{1 + a} \tag{h}$$

It is seen that $\partial u_1/\partial x_1 \neq \partial u_1/\partial \xi_1$ and $\partial u_2/\partial x_1 \neq \partial u_2/\partial \xi_1$. More generally, it may be shown that $\partial u_i/\partial x_j \neq \partial u_i/\partial \xi_j$. However, if the displacement components are very small (in this example, $a, b \ll 1$), $\partial u_i/\partial x_j \approx \partial u_i/\partial \xi_j$.

In particular, by Eqs. (g) and (h) for $a, b \ll 1$:

$$\frac{\partial u_1}{\partial x_1} = a \approx \frac{\partial u_1}{\partial \xi_1}, \qquad \frac{\partial u_2}{\partial x_1} = b \approx \frac{\partial u_2}{\partial \xi_1} \tag{i}$$

By Eq. (2-6.4c) we may express increments of the displacement components in terms of material coordinates x_i as

$$d\mathbf{q} = (du_1, du_2, du_3) = [dx_1, dx_2, dx_3] \begin{bmatrix} a & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

or

$$(du_1, du_2, du_3) = (a\ dx_1, b\ dx_1, 0) \tag{j}$$

Similarly, by Eq. (2-6.4d), in terms of spatial coordinates ξ_i,

$$(du_1, du_2, du_3) = \left(\frac{a}{1+a}\, d\xi_1, \frac{b}{1+a}\, d\xi_1, 0 \right) \tag{k}$$

Because, by Eq. (c), $x_1 = \xi_1/(1+a)$, it follows that $dx_1 = d\xi_1/(1+a)$. Hence, Eqs. (j) and (k) show that the increments of the displacement components are the same in material and spatial coordinates.

Problem Set 2-6

1. Measurements of a strained body yield the following data: $\epsilon_{11} = 0.002$, $\epsilon_{22} = 0.002$, and $\epsilon_{33} = -0.002$.

In the direction $(2/\sqrt{5}, 0, 1/\sqrt{5})$	$MF = 0.004$
In the direction $(3/\sqrt{10}, -1/\sqrt{10}, 0)$	$MF = 0.003$
In the direction $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$	$MF = 0.001$

Calculate $\epsilon_{12}, \epsilon_{13}, \epsilon_{23}$.

2. A straight bar of length L with end points 0 and 1 undergoes a displacement such that its length changes to L^*. Under this displacement the bar remains straight. Derive an expression for $(L^* - L)/L$, expressing the result in terms of the original length L and the displacement components (u_0, v_0), (u_1, v_1) of the end points of the bar. Let u be measured along the axial direction of the initial position of the bar and v be measured perpendicular to the bar in its initial position. Derive an approximate expression for

$(L^* - L)/L$ for the case where $u_1 - u_0 \ll L$, $v_1 - v_0 \ll L$. Derive an expression for the elongation of the bar in each case.

3. The deformation of a body is defined by the (x_1, x_2, x_3) displacement components:

$$u_1 = k(3x_1^2 + x_2), \qquad u_2 = k(2x_2^2 + x_3), \qquad u_3 = k(4x_3^2 + x_1)$$

where k is a positive constant. Compute the magnification of a line element ds that passes through the point (1, 1, 1) in the direction $n_1 = n_2 = n_3 = 1/\sqrt{3}$.

2-7 Physical Significance of ϵ_{ij}. Strain Definitions

Consider the deformation of a line element *PA*. Let ds be the original undeformed length of *PA*, and let ds^* be the deformed length of *PA*. Hence, under the deformation, the elongation of *PA* relative to its initial length ds is

$$e_A = \frac{ds^*}{ds} - 1 \tag{2-7.1}$$

If e_A is positive, the fiber *PA* elongates; if e_A is negative, *PA* contracts. Furthermore, because physically ds^* can never be zero under deformation, e_A can never attain the value -1; that is, -1 is a lower bound for e_A.

Substitution of Eq. (2-7.1) into Eq. (2-6.15) yields

$$MF_A = e_A + \tfrac{1}{2}e_A^2 = \boldsymbol{\epsilon}_{\alpha\beta} N_\alpha N_\beta \tag{2-7.2}$$

Equation (2-7.2) relates the relative elongation e_A of *PA* to the components $\epsilon_{\alpha\beta}$ of Eq. (2-6.10).

To examine the physical significance of the components $\epsilon_{\alpha\beta}$, we let $N_1 = 1$, $N_2 = N_3 = 0$; that is, we consider the deformation of a line element *PA* that originally is directed along the X_1 axis. Then, by Eq. (2-7.2),

$$e_1 + \tfrac{1}{2}e_1^2 = \epsilon_{11} \tag{2-7.3}$$

where the subscript 1 of e_1 now signifies that line *PA* is directed originally along the X_1 axis.

Solving Eq. (2-7.3) for e_1, we obtain

$$e_1 = \sqrt{1 + 2\epsilon_{11}} - 1 \tag{2-7.4}$$

Consequently, under a deformation of a medium, ϵ_{11} characterizes the *relative elongation* e_1 of an infinitesimal line element *PA* directed originally along the X_1 axis in the undeformed medium. Similarly, for line elements directed along the X_2

and X_3 axes, the relative elongations are given by the expressions

$$e_2 = \sqrt{1 + 2\epsilon_{22}} - 1, \qquad e_3 = \sqrt{1 + 2\epsilon_{33}} - 1$$

Hence, in general, we may write

$$e_i = \sqrt{1 + 2\epsilon_{ii}} - 1 = \epsilon_{ii} - \tfrac{1}{2}\epsilon_{ii}^2 + \cdots \tag{2-7.5}$$

Equations (2-7.5) represent the *relative elongation* of an infinitesimal line element PA that is directed initially in the X_i direction at a point P in the undeformed medium.

Large-Deformation Theory. In large-deformation theories many modern writers define MF_A [see Eq. (2-6.15)] *to be the strain of an infinitesimal line element PA at any point P in the medium*. Consequently, for large deformations the strain MF_A is given directly by Eq. (2-7.2), for an infinitesimal line element at point (x_1, x_2, x_3) (with direction cosines N_i with respect to X_i axes). Because N_i is relative to X_i, we also may write $MF_A \equiv MF_i$. Then, if the displacement components u_i are known functions of (x_1, x_2, x_3), Eqs. (2-7.2) and (2-6.10) determine the strain MF_i of any infinitesimal line element (with direction cosines N_i) in the medium.

Engineering Definition of Strain. In engineering practice it is common to *define the strain of an infinitesimal line element PA to be the elongation of the fiber PA relative to its initial length*. Consequently, the engineering definition of strain coincides with our definition of e_A [Eq. (2-7.1)].

If the relative elongation e_A is small compared to 1, the term $\frac{1}{2}e_A^2$ may be negligible in the expression $e_A + \frac{1}{2}e_A^2$. Then, in this case,

$$MF_i = e_i + \tfrac{1}{2}e_1^2 \approx e_i \tag{2-7.6}$$

Accordingly, the definition of strain of large-deformation theory [Eq. (2-6.15)] does not differ greatly from the engineering definition of strain [Eq. (2-7.1)], unless the relative elongations e_i are large.

Logarithmic Strain. Inherent in the preceding definitions is the condition that the strain be defined as a relation between the change in length of a line element and its initial length [see Eqs. (2-6.15) and (2-7.1)]. However, in creep and plasticity theories, particularly for large deformations, another definition of strain based on the instantaneous length of a line element is sometimes employed. For example, following Ludwig (1909), we define the increment $\Delta\epsilon_n$ of strain by the relation

$$\Delta\epsilon_n = \frac{\Delta l}{l} \tag{2-7.7}$$

where l denotes the instantaneous length of a finite line element. Accordingly, for an infinitesimal change dl, the infinitesimal increment $\Delta\epsilon_n$ of strain is defined by

$$d\epsilon_n = \frac{dl}{l} \tag{2-7.8}$$

Integration yields

$$\epsilon_n = \ln\frac{l}{l_0} = \log\left(1 + \frac{\Delta l}{l_0}\right)$$

or

$$\epsilon_n = \ln(1 + e_n) \tag{2-7.9}$$

where ln denotes natural logarithm, l_0 denotes the initial length of the line element, and, by Eq. (2-7.1), $e_n = \Delta l / l_0$ is the conventional engineering strain. The term ϵ_n is called the *natural* or *true* strain. Because of the relation expressed by Eq. (2-7.9), ϵ_n is sometimes called the *logarithmic* strain.

For $e_n^2 < 1$, Eq. (2-7.9) may be expanded in the following series form:

$$\epsilon_n = \log(1 + e_n) = e_n - \tfrac{1}{2}e_n^2 + \tfrac{1}{3}e_n^3 - \tfrac{1}{4}e_n^4 + \cdots, \qquad e_n^2 < 1 \tag{2-7.10}$$

Accordingly, for small strain e_n Eq. (2-7.10) yields the approximation

$$\epsilon_n = e_n, \qquad |e_n| \ll 1 \tag{2-7.11}$$

The definition embodied in Eq. (2-7.9) simplifies some of the equations of the mechanics of a deformable medium. However, in general, the use of natural strain tends to complicate the equations of deformable-body mechanics.

Other Strain Measures. Recalling Eqs. (2-6.2) and (2-6.3), we note that with Eq. (2-6.5) we may write relative to material coordinates

$$d\mathscr{s}^2 - ds^2 = 2\epsilon_{\alpha\beta}\, dx_\alpha\, dx_\beta \tag{2-7.12}$$

where

$$2\epsilon_{\alpha\beta} = C_{\alpha\beta} - \delta_{\alpha\beta} \tag{2-7.13}$$

and

$$C_{\alpha\beta} = \frac{\partial\xi_\theta}{\partial x_\alpha}\frac{\partial\xi_\theta}{\partial x_\beta} = \xi_{\theta,\alpha}\xi_{\theta,\beta} \tag{2-7.14}$$

is called the *Cauchy strain tensor* and in certain theories of continuum is used as a strain measure. Relative to spatial coordinates, Cauchy's strain tensor is given by

$$\mathscr{C}_{\alpha\beta} = \frac{\partial x_\theta}{\partial\xi_\alpha}\frac{\partial x_\theta}{\partial\xi_\beta} = x_{\theta,\alpha}x_{\theta,\beta} \tag{2-7.15}$$

For

$$\epsilon_{\alpha\beta} = 0$$

$$C_{\alpha\beta} = \mathscr{C}_{\alpha\beta} = \delta_{\alpha\beta}$$

Another measure of strain is the gradient of the displacement vector **q**, characterized by the nine deformation gradients [Eq. (d), Section 2-5]. For small-displacement theory, in particular, the symmetric tensor **D** is the matrix of the infinitesimal strain (see Section 2-15).

In view of the foregoing remarks, we observe that any definition of strain is arbitrary, as the deformation of a medium is described by Eq. (2-6.13) irrespective of the definition of strain. The theories of elasticity and plasticity do not require a definition of strain. However, in studies of strength of materials and in experimental work, it is customary to utilize the definition of strain expressed by Eq. (2-7.1). Accordingly, in this text we will usually employ the definition of strain given in Eq. (2-7.1). That is, we *define* strain of a line element to be the relative elongation of an infinitesimal line element.

2-8 Final Direction of Line Element. Definition of Shearing Strain. Physical Significance of $\epsilon_{ij}(i \neq j)$

In engineering theories of deformation, the angular change that occurs between any two originally mutually perpendicular line elements in a medium that undergoes deformation is *defined to be the shearing strain between the given line elements*. To examine the concept of shearing strain in terms of the components $\epsilon_{\alpha\beta}$ of Eq. (2-6.10), we first derive expressions for the directions cosines $\mathscr{N}_i$ of an infinitesimal line element in the deformed state in terms of its original direction cosines N_i relative to axis X_i and the displacement components u_i.

Final Direction of Line Element. Under deformation the line element dx_α deforms into the line element $d\xi_\alpha$, the lengths of these line elements being ds and $d\mathscr{s}$, respectively. By the definition of direction cosines, the direction cosines $\mathscr{N}_\alpha$ of $d\xi_\alpha$ are given by the equation

$$\mathscr{N}_\alpha = \frac{d\xi_\alpha}{d\mathscr{s}} \tag{a}$$

Analogously, the direction cosines N_α of dx_α are

$$N_\alpha = \frac{dx_\alpha}{ds} \tag{b}$$

Alternatively, we may write Eq. (a) in the form

$$\mathscr{N}_\alpha = \frac{d\xi_\alpha}{d\mathscr{s}}\frac{ds}{ds} = \frac{d\xi_\alpha}{ds}\frac{ds}{d\mathscr{s}} \tag{c}$$

where, by Eq. (2-3.3),

$$\frac{d\xi_\alpha}{ds} = \frac{\partial \xi_\alpha}{\partial x_\beta}\frac{dx_\beta}{ds}, \qquad \alpha,\ \beta = 1, 2, 3 \tag{d}$$

Furthermore, by Eqs. (2-3.4) or (2-6.7) we have

$$\frac{\partial \xi_\alpha}{\partial x_\beta} = \delta_{\alpha\beta} + \frac{\partial u_\alpha}{\partial x_\beta} = \delta_{\alpha\beta} + u_{\alpha,\beta} \tag{e}$$

where δ_{ij} is the Kronecker delta (Chapter 1, Section 1-25). Accordingly, by Eqs. (b), (d), and (e) we obtain

$$\frac{d\xi_\alpha}{ds} = \left(\delta_{\alpha\beta} + \frac{\partial u_\alpha}{\partial x_\beta}\right) N_\beta = (\delta_{\alpha\beta} + u_{\alpha,\beta})N_\beta \tag{f}$$

Now, Eq. (2-6.15) yields

$$\frac{ds}{d\mathscr{s}} = \frac{1}{\sqrt{1 + 2MF_i}} \tag{g}$$

Consequently, by Eqs. (c), (f) and (g) we obtain the following expression for the direction cosines $\mathscr{N}_\alpha$:

$$\mathscr{N}_\alpha \sqrt{1 + 2MF_i} = (\delta_{\alpha\beta} + u_{\alpha,\beta})N_\beta \tag{2-8.1}$$

Written out in detail, Eq. (2-8.1) becomes

$$\begin{aligned} \mathscr{N}_1 \sqrt{1 + 2MF_i} &= (1 + u_{1,1})N_1 + u_{1,2}N_2 + u_{1,3}N_3 \\ \mathscr{N}_2 \sqrt{1 + 2MF_i} &= u_{2,1}N_1 + (1 + u_{2,2})N_2 + u_{2,3}N_3 \\ \mathscr{N}_3 \sqrt{1 + 2MF_i} &= u_{3,1}N_1 + u_{3,2}N_2 + (1 + u_{3,3})N_3 \end{aligned} \tag{2-8.2}$$

Equation (2-8.1) expresses the final direction cosines $\mathscr{N}_\alpha$ of an infinitesimal line element with initial direction cosines N_α. The term MF_i in Eq. (2-8.1) is expressed in terms of the displacement u_α and the direction cosines N_α by Eqs. (2-6.10) and (2-6.14). In general, it may be shown that there is one direction N_α that remains invariant (unchanged) under the displacement u_α, that is, for which $N_1 = \mathscr{N}_1$, $N_2 = \mathscr{N}_2$, and $N_3 = \mathscr{N}_3$.

Definition of Shearing Strain. Next, let us consider *two* infinitesimal line elements *PA* and *PB* of lengths ds_1 and ds_2 emanating from point *P* in a medium and forming angle θ. Under a deformation the two line elements pass into the line element $\mathscr{PA}$ and $\mathscr{PB}$ with lengths $d\mathscr{s}_1$ and $d\mathscr{s}_2$ and with subtended angle ϑ (Fig. 2-8.1). In general, the plane *PAB* is nonparallel to the plane $\mathscr{PAB}$.

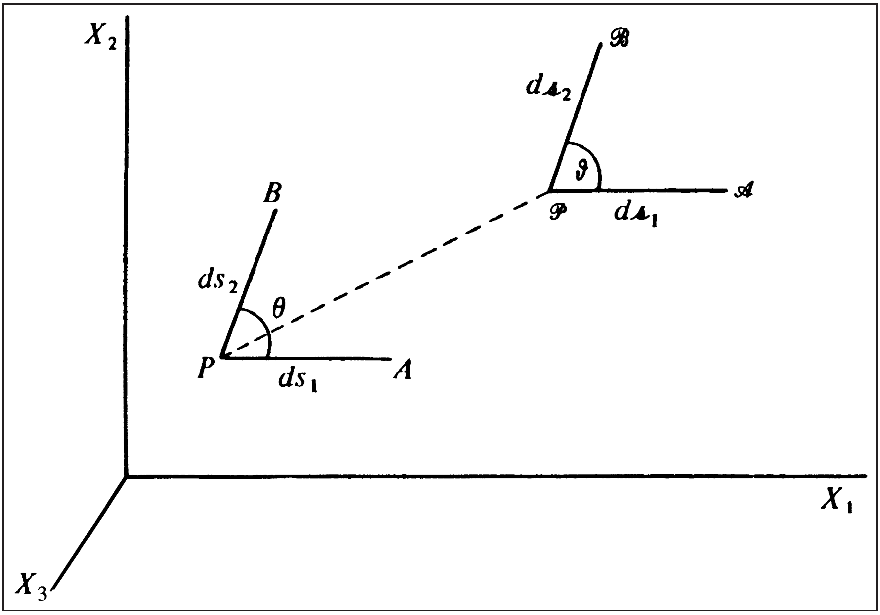

Figure 2-8.1

Let the direction cosines of lines *PA* and *PB* be M_α and N_α, respectively, with reference to (X_1, X_2, X_3) axes. Let $\mathscr{M}_\alpha$ and $\mathscr{N}_\alpha$ denote the corresponding direction cosines of $\mathscr{PA}$ and $\mathscr{PB}$. Hence, by the scalar product for vectors, the angle ϑ is defined by the equation

$$\cos\vartheta = \mathscr{M}_\alpha \mathscr{N}_\alpha \tag{h}$$

By Eq. (2-8.1), the directions $\mathscr{M}_\alpha$ and $\mathscr{N}_\alpha$ may be expressed in terms of M_α and N_α, respectively. Thus, substituting Eq. (2-8.1) into Eq. (h) and utilizing the condition $M_\alpha N_\alpha = \cos\theta$, we obtain, after employing the notation of Eq. (2-6.10),

$$\Gamma_{12} = \sqrt{(1 + 2MF_1)(1 + 2MF_2)}\ \cos\vartheta = \cos\theta + 2\epsilon_{\alpha\beta} M_\alpha N_\beta \tag{2-8.3}$$

where subscripts 1 and 2 on *MF* denote lines *PA* and *PB*, respectively. Accordingly, Eq. (2-8.3) defines the angle ϑ between the deformed line elements $\mathscr{PA}$, $\mathscr{PB}$ that initially subtend angle θ (Fig. 2-8.1). If the initial angle $\theta = 90^\circ$, $\cos\theta = 0$. Then

$$\Gamma_{12} = 2\epsilon_{\alpha\beta} M_\alpha N_\beta \tag{2-8.4}$$

Equation (2-8.4) forms the basis for definitions of shearing strain. For example, in theories of large deformation the quantity

$$\Gamma_{12} = \sqrt{(1 + 2MF_1)(1 + 2MF_2)}\cos\vartheta$$

is *defined to be the shearing strain* between the given line elements *PA* and *PB*. If the relative elongations of lines *PA* and *PB* are small compared to 1 [hence, if MF_1 and MF_2 are small compared to 1; see Eq. (2-7.2)], by Eq. (2-8.3) we see that $\Gamma_{12} \approx \cos\vartheta$. Furthermore, for small shearing strains, ϑ is approximately $\pi/2$. Hence, $\cos\vartheta = \sin[(\pi/2) - \vartheta] \approx (\pi/2) - \vartheta$. Accordingly, the definition of shearing strain used in large-deformation theory does not differ appreciably from the conventional definition, $\gamma_{12} = (\pi/2) - \vartheta$, unless shearing strains are so large that the above approximations are no longer valid. In this book we adhere to the conventional definition of shearing strain.

As in the case of the definition of strain of a line element (Section 2-7), the definition of shearing strain between line elements is of no fundamental importance in the theory. The intrinsic nature of the relative rotation of two originally mutually perpendicular line elements is described by Eq. (2-8.4), *independently of a definition of shearing strain.*

Physical Significance of $\epsilon_{ij}(i \neq j)$. To examine the physical significance of $\epsilon_{ij}(i \neq j)$, consider two line elements *PA* and *PB* initially directed along axes X_1 and X_2, respectively (Fig. 2-8.1, with $\theta = 90^\circ$). Let the direction cosines M_α and N_α of *PA* and *PB* be expressed with respect to the X_α axes. Then

$$\begin{aligned} (M_1, M_2, M_3) &= (1, 0, 0) \\ (N_1, N_2, N_3) &= (0, 1, 0) \end{aligned} \tag{i}$$

Substitution of Eqs. (i) into Eq. (2-8.3) yields

$$\Gamma_{12} = 2\epsilon_{12}$$

where subscripts $(1, 2)$ on Γ indicate that Γ is computed for lines *PA* and *PB* parallel to axes X_1 and X_2, respectively.

Similarly, for pairs of lines directed initially along axes (X_2, X_3) and axes (X_3, X_1),

$$\Gamma_{23} = 2\epsilon_{23}, \qquad \Gamma_{31} = 2\epsilon_{31}$$

Accordingly, for two lines directed initially along mutually perpendicular axes (i, j),

$$\Gamma_{ij} = 2\epsilon_{ij}, \qquad i \neq j \tag{2-8.5}$$

The sign of Γ_{ij} is determined by Eq. (2-8.3).

Equation (2-8.5) expresses the fact that the $\epsilon_{ij}(i \neq j)$ characterize the relative angular rotation between two initially mutually perpendicular lines (i, j) in the medium; that is, the $\epsilon_{ij}(i \neq j)$ are related to the shearing strain of the medium.

For small deformation we have, by Eq. (2-8.3),

$$\Gamma_{ij} = 2\epsilon_{ij} \approx \frac{\pi}{2} - \vartheta_{ij} = \gamma_{ij} \tag{2-8.6}$$

where γ_{ij} is the conventional definition of shearing strain. Accordingly, for small deformations $2\epsilon_{ij}$ is approximately equivalent to the conventional definition of shearing strain. A more detailed discussion of the nature of the approximations implied by Eqs. (2-7.6) and (2-8.6) is presented in Section 2-15.

Example 2-8.1. Computation of Shearing Strain. Let the displacement vector be given by $\mathbf{q} = (u_1, u_2, u_3) = (ax_2^2, 0, 0)$, where a is a constant coefficient. Compute the magnification factor, $\cos\vartheta$, and the shearing strain Γ_{12} between two line elements PA, PB in the X_1X_2 plane (Fig. 2-8.1), where P is located at the coordinate point $(x_1, x_2, x_3) = (0, 1, 0)$. Line PA is parallel to axis X_1, and PB is parallel to axis X_2. Hence, $\theta = \pi/2$, and $ds_1 = dx_1, ds_2 = dx_2$.

First, let us check to see if $\mathbf{q}$ is admissible. By Eq. (2-4.2) we have

$$J = \begin{vmatrix} 1 & 2ax_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0$$

Thus, $\mathbf{q}$ is admissible.

The angle ϑ between line elements $\mathscr{PA}$ and $\mathscr{PB}$ may be computed by vector analysis as follows. The direction cosines of lines PA and PB are $M_\alpha = (1, 0, 0)$ and $N_\alpha = (0, 1, 0)$, respectively. By Eq. (2-8.2) we may compute the direction cosines of lines $\mathscr{PA}$ and $\mathscr{PB}$. However, to do so we must compute the magnification factors MF_A and MF_B for lines PA and PB, respectively. The formula for the magnification factor is given by Eq. (2-6.14). By Eqs. (2-6.9) and with $(u_1, u_2, u_3) = (ax_2^2, 0, 0)$, we obtain the strain components as functions of a:

$$\begin{aligned} &\epsilon_{11} = \epsilon_{33} = \epsilon_{13} = \epsilon_{31} = \epsilon_{23} = \epsilon_{32} = 0 \\ &\epsilon_{12} = \epsilon_{21} = ax_2, \qquad \epsilon_{22} = 2a^2x_2^2 \end{aligned}$$

With these strain components and the direction cosines for PA and PB, we find from Eq. (2-6.15) the magnification factors MF_A and MF_B for line elements PA and PB, respectively, as $MF_A = 0, MF_B = 2a^2x_2^2$. For the point $(x_1, x_2, x_3) = (0, 1, 0)$, we have $MF_A = 0, MF_B = 2a^2$. Thus, for $x_2 = 1, x_1 = x_3 = 0$, we find by Eq. (2-8.2) for line elements $\mathscr{PA}$ and $\mathscr{PB}$ the direction cosines

$$\mathscr{M}_\alpha = (1, 0, 0) \quad \text{and} \quad \mathscr{N}_\alpha = (2a, 1, 0)/\sqrt{1 + 4a^2}$$

Now, by Eq. (h), Section 2-8, we obtain

$$\cos\vartheta = 2a/\sqrt{1 + 4a^2}$$

Letting γ_{12} be the decrease in angle between lines PA and PB, we have

$$\cos\vartheta = \cos\left(\frac{\pi}{2} - \gamma_{12}\right) = \sin\gamma_{12} = \frac{2a}{\sqrt{1 + 4a^2}}$$

Therefore, by Eq. (2-8.3), $\Gamma_{AB} = 2a$. Alternatively, in a very direct manner, Eq. (2-8.5) yields $\Gamma_{AB} = 2\epsilon_{12} = 2a$.

Problem Set 2-8

1. Show that

$$u_1 = a_0 + ax_2 - bx_3$$
$$u_2 = b_0 - ax_1 + cx_3$$
$$u_3 = c_0 - cx_2 + bx_1$$

where $a_0, b_0, c_0, \ldots$ are positive constants and u_1, u_2, u_3 are physically possible continuous displacement components. Assume $a_0, b_0, \ldots$ are very small compared to 1. Derive expressions in terms of the constants $a, b, \ldots$ for the direction cosines (n_1, n_2, n_3) of the line element that maintains a fixed direction under the displacement. Evaluate the expressions for $a = b = c$.

2. The displacement components for a body are

$$u_1 = 2x_1 + x_2, \qquad u_2 = x_3, \qquad u_3 = x_3 - x_2$$

(a) Verify that this displacement vector is physically possible for a continuously deformed body.
(b) Determine the strain in the direction $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$.
(c) Determine the direction cosines of the element in the undeformed medium that ends up in the x_3 direction in the deformed medium.
(d) Determine the change in angle between the lines whose directions in the undeformed medium were 1, 0, 0 and $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$.

3. The displacement components for a body are

$$u_1 = 2x_1, \qquad u_2 = 3x_2 + x_3, \qquad u_3 = x_3 - x_2$$

(a) Verify that this is a physically possible set of displacements for a continuously deformed body.
(b) Determine the strain in the direction $1/\sqrt{3}$, $1/\sqrt{3}$, $1/\sqrt{3}$ and the shear strain between this direction and the direction $-1\sqrt{2}$, $1/\sqrt{2}$, 0.
(c) Determine the direction cosines of the element in the undeformed medium that ends up in the x_2 direction in the deformed medium.

2-9 Tensor Character of $\epsilon_{\alpha\beta}$. Strain Tensor

In this section we consider the laws of transformation of Eq. (2.-6.10) under a rotation from rectangular Cartesian coordinates x_α to rectangular Cartesian coordinates y_α.

Let a medium undergo a deformation. Then the extension of an infinitesimal line element ds in the medium is characterized by the relation [see Eq. (2.-6.8)]

$$(ds^*)^2 - (ds)^2 = 2\epsilon_{\alpha\beta}\, dx_\alpha\, dx_\beta \tag{a}$$

where the components $\epsilon_{\alpha\beta}$ are functions of x_α.

Because $(ds^*)^2 - (ds)^2$ represents the physical extension of line ds, it remains invariant under a transformation of coordinates. Hence, we may also write

$$(ds^*)^2 - (ds)^2 = 2E_{\gamma\delta} dy_\gamma dy_\delta \tag{b}$$

where the components $E_{\gamma\delta}$ are functions of y_α. The components $E_{\gamma\delta}$ are symmetrical components that determine the quantity $(ds^*)^2 - (ds)^2$ in terms of y_α coordinates.

Now, under a transformation of coordinates from x_α to y_α, dx_α transforms according to the rule [see Eq. (1-24.12)]

$$dx_\alpha = a_{\gamma\alpha}\, dy_\gamma \tag{c}$$

where $a_{\gamma\alpha}$ denote the direction cosines between axes y_γ and x_α.

Substituting Eq. (c) into Eq. (a) and equating the resulting expression to Eq. (b), we obtain

$$E_{\gamma\delta}\, dy_\gamma\, dy_\delta = \epsilon_{\alpha\beta}\, a_{\gamma\alpha}\, a_{\alpha\beta}\, dy_\gamma\, dy_\delta$$

or

$$(E_{\gamma\delta} - \epsilon_{\alpha\beta} a_{\gamma\alpha} a_{\delta\beta})\, dy_\gamma\, dy_\delta = 0 \tag{d}$$

Because the y_γ are independent and both $E_{\gamma\delta}$ and $\epsilon_{\alpha\beta}$ are symmetric, Eq. (d) is satisfied identically (Synge and Schild, 1978) if

$$E_{\gamma\delta} = \epsilon_{\alpha\beta} a_{\gamma\alpha} a_{\delta\beta} \tag{2-9.1}$$

Thus, $\epsilon_{\alpha\beta}$ transforms according to the rule of transformation of a second-order tensor [see Eq. (1-24.14)]. For this reason, the following matrix (i.e., the array of elements $\epsilon_{\alpha\beta}$; $\epsilon_{\alpha\beta} = \epsilon_{\beta\alpha}$) is called the *strain tensor*:

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix} \tag{2-9.2}$$

Individual elements of this matrix are called *components of the strain tensor*. If the six components of the strain tensor are known, the deformation of the medium is defined by Eq. (2-6.14) [or Eq. (2-7.2)]. Because the right side of Eq. (2-6.14) [or Eq. (2-7.2)] is a quadratic form in the variables N_α, Eq. (2-9.2) is the matrix of coefficients of the quadratic form (see Chapter 1, Section 1-27), which defines the deformation of the medium. The matrix is symmetric with respect to its principal diagonal (the diagonal containing $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}$); thus, by the theory of quadratic forms (Birkhoff and MacLane, 2008), it follows that MF_i defined by Eq. (2-6.14) possesses three stationary values.[3] In the theory of quadratic forms, these stationary real values are called *principal values*; in the theory of strain, they are called *principal strains*. The following section presents a geometrical interpretation of principal strains.

Alternatively, a rotation from axes y_α to axes x_α yields the relation

$$\epsilon_{\alpha\beta} = a_{\gamma\alpha} a_{\delta\beta} E_{\gamma\delta} \tag{2-9.3}$$

Equations (2-9.1) and (2-9.3) represent the law of transformation of strain components from one system of rectangular Cartesian coordinate axes to another (see Chapter 3, Sections 1-23 and 1-24).

2-10 Reciprocal Ellipsoid. Principal Strains. Strain Invariants

The results obtained in this section are purely geometric in nature. Hence, they apply to any symmetrical tensor of second order; for example, the results pertain to the theories of strain, stress, and moments of inertia. However, for continuity in treatment we employ the notations of the theory of strain developed in the previous sections.

Let a small volume V, enclosed by the surface S, contain point P of a medium. Under a deformation of the medium, the strain of all infinitesimal elements emanating from P may be computed by Eq. (2-6.14). Because the surface S is initially closed, it is closed after the deformation (see Corollary 2-17.2).

To represent geometrically the behavior of this surface under deformation, consider a particular infinitesimal line element *PA* emanating from point P with direction cosines N_α. In the direction of line *PA* extended, mark off a length[4]

$$PQ = \frac{ds}{d\mathscr{s}} = \frac{1}{1 + e_i} = \frac{1}{\sqrt{1 + 2MF_i}} \tag{a}$$

where MF_i, defined by Eq. (2-6.14), and e_i, defined by Eq. (2-7.1), are related by Eq. (2-7.2). We seek the locus of point Q for all possible directions of line *PA*. Accordingly, we let point P be the origin of coordinate axis y_α parallel, respectively, to coordinate axes X_α. Then, in the y_α coordinate system, the coordinates of point Q are

$$y_\alpha = \frac{ds}{d\mathscr{s}} N_\alpha \tag{b}$$

[3]A geometrical interpretation is given in Section 2-10. See also Chapter 3, Section 1-27.
[4]This particular choice of length simplifies the development.

Hence, Eqs. (a) and (b) yield

$$N_\alpha = (1 + e_i)y_\alpha = \sqrt{1 + 2MF_i}\, y_\alpha \tag{c}$$

Substitution of Eq. (c) into Eq. (2-6.14) yields

$$\begin{aligned}(1 + 2\epsilon_{11})y_1^2 + (1 + 2\epsilon_{22})y_2^2 + (1 + 2\epsilon_{33})y_3^2 + 4\epsilon_{12}y_1y_2 \\ + 4\epsilon_{13}y_1y_3 + 4\epsilon_{23}y_2y_3 = 1\end{aligned} \tag{d}$$

or, in index notation

$$(\delta_{\alpha\beta} + 2\epsilon_{\alpha\beta})y_\alpha Y_\beta = 1 \tag{e}$$

Letting $y_1 = X,\ y_2 = Y$, and $y_3 = Z$, we may write Eq. (d) in the form

$$\begin{aligned}F(X, Y, Z) = (1 + 2\epsilon_{11})X^2 + (1 + 2\epsilon_{22})Y^2 + (1 + 2\epsilon_{33})Z^2 \\ + 4\epsilon_{12}XY + 4\epsilon_{13}XZ + 4\epsilon_{23}YZ = 1\end{aligned} \tag{2-10.1}$$

Accordingly, the function $F(X, Y, Z)$ is a second-degree algebraic equation[5] in the variables (X, Y, Z), the coordinates of point Q. Because point Q is real and finite for nonzero values of e_i, the equation $F(X, Y, Z) = 1$ represents an ellipsoid with center P. This ellipsoid is called the *reciprocal strain ellipsoid*.[6] There is one reciprocal strain ellipsoid associated with each point P of the medium.

Once the reciprocal strain ellipsoid has been determined, the relative elongation (or dilatation) e_i of any infinitesimal line element *PA* emanating from point P may be interpreted as follows. Consider the extension of line *PA*. This extension pierces the reciprocal strain ellipsoid at point Q. Hence, by Eq. (a), the dilatation e_i of line *PA* is

$$e_i = \frac{1}{PQ} - 1 \tag{2-10.2}$$

The dilatation may be positive, negative, or zero. To examine the various possibilities, consider a sphere with origin at P and with radius equal to 1. This sphere cuts the reciprocal strain ellipsoid along a curve C, which divides the ellipsoid into two regions: region R_e, exterior to the sphere, and region R_i, interior to the sphere. If the line extended along the direction *PA* pierces the ellipsoid in region R_e, then we have $PQ > 1$, and the dilatation of line *PA* is negative; that is, line *PA* contracts. If the extended line pierces the ellipsoid in region R_i, then we have $PQ < 1$, and the dilatation of line *PA* is positive; that is, line *PA* elongates. Finally, if the extended line pierces the sphere on the curve C, then we have $PQ = 1$, and the dilatation of *PA* is zero. Accordingly, it follows that if point Q is inside the

[5]More generally, $F(X, Y, Z)$ is said to be a quadratic form in the variables (X, Y, Z). See Birkhoff and MacLane (2008).

[6]This ellipsoid is referred to as the reciprocal strain ellipsoid by Love because the strain is inversely proportional to line PQ [see Eq. (2-10.2)]. See Love (2002), pp. 36–37.

sphere, e_i is positive; if Q is outside the sphere, e_i is negative; and if Q is on curve C, $e_i = 0$.

In contrast, if the sphere with origin at P and with unit radius does not cut the ellipsoid, either all elements emanating from point P elongate (the ellipsoid is interior to the sphere, as $PQ < 1$ for all points on the ellipsoid) or all elements contract (the ellipsoid is exterior to the sphere, as $PQ > 1$ for all points on the ellipsoid).

Principal Strains. Relative Elongations of the Axes of the Reciprocal Strain Ellipsoid. Consider three line elements that emanate from point P in an undeformed medium and that are directed along the axes of the reciprocal strain ellipsoid at P. The relative elongations (e_1, e_2, e_3) of these line elements are called the *principal strains of the ellipsoid*. The *initial* directions of the line elements that experience the principal strains are called *principal directions in the undeformed medium*. Their *final* directions are called *principal directions in the deformed medium*.

Let A, B, C denote the end points of the three semi-axes of the reciprocal strain ellipsoid. Then the relative elongation of the line element PA is

$$e_1 = \frac{1}{PA} - 1$$

Similarly, the relative elongations of line elements PB and PC are

$$e_2 = \frac{1}{PB} - 1 \qquad e_3 = \frac{1}{PC} - 1$$

Accordingly, in order to calculate (e_1, e_2, e_3), we must determine the axes (PA, PB, PC) of the reciprocal strain ellipsoid [Eq. (2-10.1)].

By the theory of geometry (Eisenhart, 1939),[7] we have [with Eq. (2-10.1)]

$$\begin{vmatrix} \epsilon_{11} - \phi & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} - \phi & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} - \phi \end{vmatrix} = 0 \tag{2-10.3}$$

where $\phi = (r - 1)/2$. Corresponding to the three roots (ϕ_1, ϕ_2, ϕ_3) of Eq. (2-10.3), there exist the three quantities (r_1, r_2, r_3), the squares of the inverses of the lengths of the axes $(1/PA)^2$, $(1/PB)^2$, and $(1/PC)^2$, respectively. Consequently, the relative elongations (e_1, e_2, e_3) are given by the relations

$$e_1 = \sqrt{r_1} - 1 \qquad e_2 = \sqrt{r_2} - 1 \qquad e_3 = \sqrt{r_3} - 1 \tag{2-10.4}$$

[7] Also see Section 2-11, where we treat the problem of principal strains using a somewhat different method.

We may also show that the shearing strains vanish between line elements in the principal directions. For example, consider two infinitesimal line elements PQ_1 and PQ_2 initially emanating from point P with direction cosines M and N. (Lines PQ_1 and PQ_2 are assumed perpendicular.) Then, analogous to the develoment of Section 2-7, we obtain the following expression for the angle ϑ subtended by the line elements after the deformation:

$$\Gamma_{12} = \sqrt{(1+2\lambda_1)(1+2\lambda_2)}\cos\ \vartheta = (\delta_{\alpha\beta} + 2\epsilon_{\alpha\beta})M_\alpha N_\beta \tag{f}$$

where, for simplicity, we have set $\lambda_i = MF_i$.

Substitution of $M_\alpha = X_\alpha\sqrt{1+2\lambda_1}$ and $N_\beta = Y_\beta\sqrt{1+2\lambda_2}$ [see Eq. (c)] into Eq. (f) yields

$$\begin{aligned}\cos\ \vartheta &= (\delta_{\alpha\beta} + 2\epsilon_{\alpha\beta})X_\alpha Y_\beta \\ &= (1+2\epsilon_{11})X_1Y_1 + (1+2\epsilon_{22})X_2Y_2 \\ &\quad + (1+2\epsilon_{33})X_3Y_3 + 2\epsilon_{12}(X_1Y_2 + X_2Y_1) \\ &\quad + 2\epsilon_{13}(X_1Y_3 + X_3Y_1) + 2\epsilon_{23}(X_2Y_3 + X_3Y_2)\end{aligned} \tag{g}$$

Utilizing Eq. (2-10.1) (with the notation $X = X_1, Y = X_2, Z = X_3$), we may write Eq. (g) in the form

$$\cos\vartheta = \frac{1}{2}\left[Y_1\frac{\partial F}{\partial X_1} + Y_2\frac{\partial F}{\partial X_2} + Y_3\frac{\partial F}{\partial X_3}\right]$$

Hence, a necessary and sufficient condition that $\vartheta = \pi/2$ is that

$$Y_1\frac{\partial F}{\partial X_1} + Y_2\frac{\partial F}{\partial X_2} + Y_3\frac{\partial F}{\partial X_3} = 0 \tag{h}$$

Accordingly, Eq. (h) is the condition for the shearing strain between lines PQ_1 and PQ_2 to vanish; that is, lines PQ_1 and PQ_2 remain orthogonal under the deformation. However, Eq. (h) is also a necessary and sufficient condition that PQ_1 and PQ_2 are conjugate[8] diameters of the reciprocal strain ellipsoid [Eq. (2-10.1)]. Hence, we may conclude that there exist three line elements emanating from point P that form a rectangular triad before and after the deformation; they are the infinitesimal line elements directed along the axes of the reciprocal strain ellipsoid at P (see also Section 2-11).

[8]A diameter is said to be conjugate to the ellipsoid if it is parallel to the tangents to the ellipsoid drawn away from the end points of any other diameter. Because PQ_1 and PQ_2 are orthogonal, they must lie along the axes of the ellipsoid.

Invariants of Reciprocal Strain Ellipsoid. Under a transformation of rectangular Cartesian axes, the coefficients $\delta_{\alpha\beta} + 2\epsilon_{\alpha\beta}$ of the strain ellipsoid are changed. However, because the principal strains are physical quantities independent of any coordinate system, they remain invariant under coordinate transformations.

It may also be shown (see Section 2-11) that the following quantities are invariants under coordinate transformations; that is, they depend only on the physical nature of the deformation:

$$\epsilon_{11} + \epsilon_{22} + \epsilon_{33}$$

$$\epsilon_{12}^2 + \epsilon_{13}^2 + \epsilon_{23}^2 - (\epsilon_{22}\epsilon_{33} + \epsilon_{33}\epsilon_{11} + \epsilon_{11}\epsilon_{22})$$

$$\epsilon_{11}\epsilon_{22}\epsilon_{33} + 2\epsilon_{12}\epsilon_{13}\epsilon_{23} - \epsilon_{11}\epsilon_{23}^2 - \epsilon_{22}\epsilon_{13}^2 - \epsilon_{33}\epsilon_{12}^2$$

These quantities are called invariants of the reciprocal strain ellipsoid, or simply strain invariants.

2-11 Determination of Principal Strains. Principal Axes

The geometrical treatment of Section 2-10 offers some physical insight into the concepts of principal strains and principal directions. However, a more direct symmetrical approach may be employed. Essentially, the determination of principal strains reduces to the problem of computing the directions for which the relative elongation e_i assumes extremal values.

Consequently, let us consider the law of transformation of the strain tensor $\epsilon_{\alpha\beta}$. By Eq. (2-9.1), under a transformation from rectangular Cartesian coordinates x_α to rectangular Cartesian coordinates y_α, the components $E_{\gamma\delta}$ of the strain tensor $\epsilon_{\alpha\beta}$ in the coordinate system y_α are

$$E_{\gamma\delta} = \epsilon_{\alpha\beta} a_{\gamma\alpha} a_{\delta\beta} \tag{a}$$

where $a_{\alpha\beta}$ are the direction cosines between axes y_α and x_β. Expanding Eq. (a), we obtain

$$\begin{aligned}
E_{11} &= \epsilon_{11}a_{11}^2 + \epsilon_{22}a_{12}^2 + \epsilon_{33}a_{13}^2 + 2\epsilon_{12}a_{11}a_{12} + 2\epsilon_{13}a_{11}a_{13} + 2\epsilon_{23}a_{12}a_{13} \\
E_{22} &= \epsilon_{11}a_{21}^2 + \epsilon_{22}a_{22}^2 + \epsilon_{33}a_{23}^2 + 2\epsilon_{12}a_{21}a_{22} + 2\epsilon_{13}a_{21}a_{23} + 2\epsilon_{23}a_{22}a_{23} \\
E_{33} &= \epsilon_{11}a_{31}^2 + \epsilon_{22}a_{32}^2 + \epsilon_{33}a_{33}^2 + 2\epsilon_{12}a_{31}a_{32} + 2\epsilon_{13}a_{31}a_{33} + 2\epsilon_{23}a_{32}a_{33} \\
E_{12} &= \epsilon_{11}a_{11}a_{21} + \epsilon_{22}a_{12}a_{22} + \epsilon_{33}a_{13}a_{23} + \epsilon_{12}(a_{11}a_{22} + a_{12}a_{21}) \\
&\quad + \epsilon_{13}(a_{11}a_{23} + a_{13}a_{21}) + \epsilon_{23}(a_{12}a_{23} + a_{13}a_{22}) \\
E_{13} &= \epsilon_{11}a_{11}a_{31} + \epsilon_{22}a_{12}a_{32} + \epsilon_{33}a_{13}a_{33} + \epsilon_{12}(a_{11}a_{32} + a_{12}a_{31}) \\
&\quad + \epsilon_{13}(a_{11}a_{33} + a_{13}a_{31}) + \epsilon_{23}(a_{12}a_{33} + a_{13}a_{32}) \\
E_{23} &= \epsilon_{11}a_{21}a_{31} + \epsilon_{22}a_{22}a_{32} + \epsilon_{33}a_{23}a_{33} + \epsilon_{12}(a_{21}a_{32} + a_{22}a_{31}) \\
&\quad + \epsilon_{13}(a_{21}a_{33} + a_{23}a_{31}) + \epsilon_{23}(a_{22}a_{33} + a_{23}a_{32})
\end{aligned} \tag{2-11.1}$$

Now let the y_1 axis be parallel to the direction for which the relative elongation e_1 takes an extremal value. By Eq. (2-7.1), Eq. (b) of Section 2-9, and Eq. (2-11.1), we obtain

$$e_{y_1} + \tfrac{1}{2}(e_{y_1})^2 = E_{11}$$

or

$$e_{y_1} = \sqrt{1 + 2E_{11}} - 1 \quad \text{because} \quad e_{y_1} > -1 \tag{b}$$

Consequently, we see that the problem of determining an extremal value for the relative elongation e_{y_1} is equivalent to the computation of an extremal value of the strain component E_{11}. In turn, the problem of computing an extremal value of the component E_{11} reduces to the determination of the initial direction (a_{11}, a_{12}, a_{13}) of the infinitesimal line element for which E_{11} attains an extremal value under a deformation. Thus, we seek stationary values of E_{11} (i.e., values for which $\partial E_{11}/\partial a_{1\alpha} = 0$) under the restriction that $a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$.

The extremal values of the relative elongation e_i are called the *principal values of the deformation*, or simply the *principal strains*. (Alternatively, we may refer to extreme value of E_{ii} as the principal strains.) Again, the initial direction $(a_{\alpha\beta})$ along which e_i attains stationary values are called the *principal directions* (or *axes*) of strain.

We show that there are three initially mutually orthogonal principal directions for which e_i takes extremal values. Furthermore, it will be shown that in the deformed position the shearing strain between principal axes vanishes. Hence, under the deformation, principal axes remain mutually orthogonal. Accordingly, at each point in a body there exists a set of three mutually orthogonal principal axes that remain mutually orthogonal under a deformation.

As noted above, the mathematical problem of determining the extremal of E_{11} essentially consists of determining the directions (a_{11}, a_{12}, a_{13}) for which

$$\frac{\partial E_{11}}{\partial a_{11}} = \frac{\partial E_{11}}{\partial a_{12}} = \frac{\partial E_{11}}{\partial a_{13}} = 0 \tag{c}$$

where

$$a_{11}^2 + a_{12}^2 + a_{13}^2 - 1 = 0 \tag{d}$$

We may solve this problem straightaway by eliminating one of the $a_{\alpha\beta}$ between Eqs. (c) and (d). Thus, the problem may be reduced to seeking extremal values of E_{11} as a function of two variables (say, a_{11} and a_{12}). However, this procedure of elimination is rather complicated algebraically, as Eq. (d) is of second degree. Consequently, rather than proceed directly into these difficulties, we seek extremals of E_{11} by a more elegant symmetrical technique called the *Lagrange multiplier method*.[9]

[9]See Chapter 3, Section 1-29. See also Courant (1992).

Accordingly, we consider the function

$$H = E_{11} - L(a_{11}^2 + a_{12}^2 + a_{13}^2 - 1) \tag{e}$$

where L is an underdetermined constant called the Lagrange multiplier. In the manner of Lagrange, we ignore initially the condition of Eq. (d). Thus, we seek the direction for which H attains an extremal value. This direction is a function of L. Because it provides a stationary value of H, it also provides a stationary value for E_{11} in the region restricted by the condition $a_{11}^2 + a_{12}^2 + a_{13}^2 - 1 = 0$. This follows from the fact that extremal values of H and E_{11} coincide in the region $a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$ [see Eq. (e)].

Substituting the expression E_{11} [Eq. (2-11.1)] into Eq. (e), and setting partial derivatives of H with respect to (a_{11}, a_{12}, a_{13}) equal to zero, we obtain

$$\begin{aligned} (\epsilon_{11} - L)a_{11} + \epsilon_{12}a_{12} + \epsilon_{13}a_{13} &= 0 \\ \epsilon_{12}a_{11} + (\epsilon_{22} - L)a_{12} + \epsilon_{23}a_{13} &= 0 \\ \epsilon_{13}a_{11} + \epsilon_{23}a_{12} + (\epsilon_{33} - L)a_{13} &= 0 \end{aligned} \tag{2-11.2}$$

Equations (2-11.2) are a set of three homogeneous linear algebraic equations in (a_{11}, a_{12}, a_{13}). Because $a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$, the trivial solution $a_{11} = a_{12} = a_{13} = 0$ is excluded. Hence, by the theory of linear algebraic equations (Hildebrand, 1992), Eqs. (2-11.2) possess a solution if and only if the determinant of the coefficient of (a_{11}, a_{12}, a_{13}) vanishes identically.

Thus, we obtain the result

$$F(L) = \begin{vmatrix} \epsilon_{11} - L & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} - L & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} - L \end{vmatrix} = 0 \tag{2-11.3}$$

Equation (2-11.3) is a third-degree algebraic equation in the Lagrange multiplier L. Inspection of Eq. (2-11.3) shows that the highest degree term in L is $-L^3$. Hence, for large positive values of L, $F(L)$ is negative. For large negative values of L, $F(L)$ is positive. Accordingly, because $F(L)$ is a continuous cubic function of L, it must pass through the value zero at least once for real values of L. Consequently, Eq. (2-11.3) possesses at least one real root, say, L_1.

Substitution of L_1 into Eqs. (2-11.3) yields the following relations for the principal direction ξ_i corresponding to the root L_1:

$$\begin{aligned} (\epsilon_{11} - L_1)\xi_1 + \epsilon_{12}\xi_2 + \epsilon_{13}\xi_3 &= 0 \\ \epsilon_{12}\xi_1 + (\epsilon_{22} - L_1)\xi_2 + \epsilon_{23}\xi_3 &= 0 \\ \epsilon_{13}\xi_1 + \epsilon_{23}\xi_2 + (\epsilon_{33} - L_1)\xi_3 &= 0 \end{aligned} \tag{f}$$

where

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1 \tag{g}$$

Multiplying the first, second, and third of Eqs. (f) by ξ_1, ξ_2, and ξ_3, respectively, adding, and utilizing Eq. (g), we obtain

$$L_1 = \epsilon_{\alpha\beta}\xi_\alpha\xi_\beta \tag{h}$$

Comparison of Eq. (h) and the first of Eqs. (2-11.1) shows that

$$L_1 = \text{extremal } E_{11} \tag{2-11.4}$$

Consequently, the value L_1 of the Lagrange multiplier L also corresponds to an extremal value of e_i [see Eq. (b)].

Furthermore, it follows from the fourth and fifth parts of Eqs. (2-11.1), Eqs. (f), and the orthogonality conditions [see Eqs. (1-24.2) and (1.24.3)] for coordinate axes y_α that

$$E_{12} = E_{13} = 0 \tag{i}$$

Equation (i) signifies that the shearing strains between line elements directed along y_1, y_2 axes and along y_1, y_3 axes vanish identically. However, by the last of Eqs. (2-11.1) and by Eqs. (f), we note that $E_{23} \neq 0$. Furthermore, the condition that E_{11} attains an extremal value in the direction of axis y_1 does not ensure that the relative elongation e_i takes on extremal values in the directions of axes y_2, y_3.

Hence, having established the existence of one real root ($L_1 =$ extremal E_{11}) for Eq. (2-11.3), we now proceed to examine the two remaining roots. Accordingly, we consider a second transformation to coordinate axes Y_α, which leaves E_{11} invariant; that is, we consider a rotation of axes with respect to axes y_1 (see Section 2-10). Thus, we let $Y_1 = y_1$ under a rotation through an angle θ with respect to axis y_1 (Fig. 2-11.1). The direction cosines of this transformation are given in the following table:

	y_1	y_2	y_3
Y_1	1	0	0
Y_2	0	$\cos\theta$	$\sin\theta$
Y_3	0	$-\sin\theta$	$\cos\theta$

Let the strain components in the coordinate system Y_α be distinguished by a prime from those in the y_α system. Then, by Eqs. (2-11.1), we have

$$\begin{aligned}
E'_{11} &= E_{11} \\
E'_{22} &= E_{22}\cos^2\theta + E_{33}\sin^2\theta + E_{23}\sin 2\theta \\
E'_{33} &= E_{22}\sin^2\theta + E_{33}\cos^2\theta - E_{23}\sin 2\theta \\
E'_{12} &= E'_{13} = 0 \\
E'_{23} &= (E_{33} - E_{22})\sin\theta\,\cos\theta + E_{23}\cos 2\theta
\end{aligned} \tag{2-11.5}$$

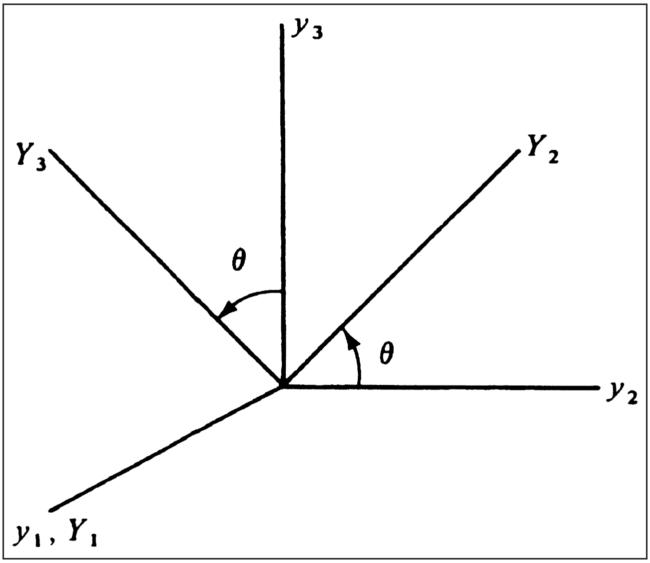

Figure 2-11.1

By Eqs. (2-11.5) the strain components $E'_{\alpha\beta}$ are expressed as functions of the rotation θ. Hence, we seek the values of θ for which E'_{22} and E'_{33} are stationary. Differentiation of E'_{22} with respect to θ yields the condition

$$\tan\ 2\theta = \frac{2E_{23}}{E_{22} - E_{33}} \tag{2-11.6}$$

Accordingly, E'_{22} is a real extremal value for θ determined by Eq. (2-11.6). Similarly, the value of θ given by Eq. (2-11.6) yields a stationary real value of E'_{33}. Consequently, we have shown that the relative elongations e_i attain stationary real values in three mutually orthogonal directions; that is, Eq. (2-11.3) possesses three real roots $L_1,\ L_2,\ L_3$, these roots being equal to the extremal values $E'_{11},\ E'_{22},\ E'_{33}$. Furthermore, substitution of Eq. (2-11.6) into the last of Eqs. (2-11.5) shows that $E'_{23} = 0$. Consequently, all the shearing strains between line elements in the principal directions (axes) vanish. Moreover, the extremal values of the relative elongations in the direction of the principal axes are determined by substitution of the extremal values $E'_{11},\ E'_{22},\ E'_{33}$ into Eq. (2-7.5).

The three sets of direction cosines, say, $\xi_{\alpha'},\ \eta_\alpha,\ \zeta_\alpha$, of the three principal axes in the undeformed medium may be determined by solving Eqs. (2-11.2) with L equal to $L_1,\ L_2,\ L_3$, respectively, subject to the restriction that $\xi_\alpha\xi_\alpha + \eta_\alpha\eta_\alpha + \zeta_\alpha\zeta_\alpha = 1$. With $\xi_\alpha,\ \eta_\alpha,\ \zeta_\alpha$ known, the principal directions, say, $\xi^*_\alpha,\ \eta^*_\alpha,\ \zeta^*_\alpha$, in the deformed medium may be computed by means of Eqs. (2-7.5), provided (u_1, u_2, u_3) are known.

Because the shearing strains between line elements directed along principal axes vanish, we have shown that at each point in a medium that undergoes deformation, there exists a set of three directions that are mutually orthogonal before and after the deformation.

Special Cases. Special cases arise in principal axes theory when two or more of the principal strains are equal. For example, let X_1, X_2, X_3 denote principal axes at a point O in a medium. Let the associated principal strains be L_1, L_2, L_3, respectively. If $L_2 = L_3$, then principal axes X_2, X_3 may be chosen to be any two mutually perpendicular axes that lie in the $X_2 - X_3$ plane. If $L_1 = L_2 = L_3$, then principal axes X_1, X_2, X_3 may be chosen to be any three mutually perpendicular axes at point O.

Example 2-11.1. Computation of Principal Strains and Principal Strain Directions. Given a strain tensor

$$\epsilon = \begin{bmatrix} 0.01 & 0.003 & 0 \\ 0.003 & 0.002 & 0 \\ 0 & 0 & 0.006 \end{bmatrix} \tag{a}$$

We wish to compute the principal strains and their directions. By Eq. (2-11.3) the principal strains are the roots L_i of the determinantal equation

$$F(L) = \begin{vmatrix} 0.01 - L & 0.003 & 0 \\ 0.003 & 0.002 - L & 0 \\ 0 & 0 & 0.006 - L \end{vmatrix} = 0 \tag{b}$$

Expanding Eq. (b) we find

$$(0.006 - L)(L - 0.011)(L - 0.001) = 0 \tag{c}$$

Thus, the roots (principal strains or eigenvalues) are

$$L_1 = 0.011 \qquad L_2 = 0.006 \qquad L_3 = 0.001 \tag{E2-11.1}$$

where we have taken the order $L_1 > L_2 > L_3$.

With each principal strain there is an associated principal strain direction (or eigenvector). Thus, let $\xi = (\xi_1, \xi_2, \xi_3)$ be the principal direction associated with the principal strain L_1. By Eqs. (f) and (g) of Section 2-11, we have

$$\begin{aligned} (0.01 - 0.011)\xi_1 + 0.003\xi_2 + 0.0\xi_3 &= 0 \\ 0.003\xi_1 + (0.002 - 0.011)\xi_2 + 0.0\xi_3 &= 0 \\ 0.0\xi_1 + 0.0\xi_2 + (0.006 - 0.011)\xi_3 &= 0 \\ \xi_1^2 + \xi_2^2 + \xi_3^2 &= 1 \end{aligned} \tag{d}$$

By the first of Eqs. (d), we find $\xi_1 = 3\xi_2$. By the third of Eqs. (d), we find $\xi_3 = 0$, and substituting into the last of Eqs. (d), we get $9\xi_2^2 + \xi_2^2 = 1$. Thus, the direction cosines (ξ_1, ξ_2, ξ_3) of the principal direction ξ are

$$\xi_1 = \pm\frac{3}{\sqrt{10}} \qquad \xi_2 = \pm\frac{1}{\sqrt{10}} \qquad \xi_3 = 0 \tag{e}$$

where either the top (+) signs or the bottom (−) signs must be used.

Similarly, the other principal directions (corresponding to the principal strains L_2 and L_3) may be computed. These computations are left for the reader as a learning exercise. As noted above, more generally the principal strains are eigenvalues and the principal directions are eigenvectors of the mathematical eigenproblem (Morse and Feshbach, 1961; Bathe, 1995, 2003).

Computer subroutines (Smith and Griffiths, 2004) and commercial computer programs are available for digital computers for the calculation of eigenvalues and eigenvectors. Such programs are also available for some handheld calculators and palmtops.

Equivalent computations hold for the stress tensor (principal stresses and principal stress directions) (see Chapter 3, Section 3-5).

Problem Set 2-11

1. Determine the principal strains for Problem 2-6.1.

2. Noting the condition $l^2 + m^2 + n^2 = 1$, where l, m, n are direction cosines, use the Lagrange multiplier method to seek extreme values of MF_A [Eq. (2-7.2) and Chapter 1, Section 1-29].

3. The displacement components (u_1, u_2, u_3) are given by the relations $u_1 = x_1 - 2x_2$, $u_2 = 3x_1 + 2x_2$, $u_3 = 5x_3$. Verify that this displacement vector is continuously possible for a continuously deformed body. Determine the principal strains. Determine the principal axes of strain in the undeformed medium and in the deformed medium.

4. A body is strained so that $\epsilon_{11} = 0.002$, $\epsilon_{22} = -0.002$, $\epsilon_{33} = 0$, $2\epsilon_{13} = 0.004$, $\epsilon_{12} = 0$, and $\epsilon_{23} = 0$. Derive equations that determine the directions of the sides of a cubic element in the body whose angles are preserved under the strain.

5. A set of displacements for a deformable body are given as

$$u_1 = 2x_1 - x_2 \qquad u_2 = x_2 - 2x_1 \qquad u_3 = x_3$$

(a) Determine if this is a possible set of displacements for a continuously deformed body.

(b) Determine the principal strains.

(c) Determine the direction of the maximum principal strain in the undeformed medium.

6. Let axes (x, y, z) be principal axes of strain. Let principal strains be $(\epsilon_1, \epsilon_2, \epsilon_3)$. Two perpendicular line elements (element 1 and element 2) lie in the octahedral plane (see Section 2-12 for definition of octahedral plane) in the first octant of the coordinate system. Element 1 is parallel to the (x, y) plane. Determine the direction cosines of elements 1 and 2. Determine Γ between elements 1 and 2 in terms of $(\epsilon_1, \epsilon_2, \epsilon_3)$. Determine the magnification of element 1 in terms of $(\epsilon_1, \epsilon_2, \epsilon_3)$.

7. Consider the motion of a continuum from initial position $\mathbf{X} = (X_1, X_2, X_3)$ to the final position $\mathbf{x} = (x_1, x_2, x_3)$. The relation $\mathbf{X}$ and $\mathbf{x}$ is given by

$$x_k = (\delta_{kK} + B_{kK})X_K \qquad k, K = 1, 2, 3$$

where δ_{kK} is the Kronecker delta and where we let K be a summing index. Hence,

$$\delta_{kK} = 1 \qquad k = K \qquad \delta_{kK} = 0 \qquad k \neq K$$

The elements B_{kK} are constants.

(a) Determine the displacement components U_K.

(b) Determine the material strain tensor symmetric components e_{KL} and the antisymmetric components ω_{KL} [Eqs. (2-5.2) and (2-2.3)].

(c) Determine the requirements for the approximations $\epsilon_{KL} \approx e_{KL}$, where ϵ_{KL} is the material strain tensor [Eq. (2-6.10)].

(d) Let $B_{kK} = 1$ for all k, K. Determine the principal strains and principal directions for the point $X_1 = X_2 = X_3 = 1$.

8. A body is strained so that $\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = e$, where e is a constant. Determine the principal strains. Then, write down the three systems of equations that determine the principal directions. Here we employ the notation $x \equiv x_1$, $y \equiv x_2$, $z \equiv x_3$, $\epsilon_x \equiv \epsilon_{11}$, $\epsilon_y \equiv \epsilon_{22}$, $\epsilon_z \equiv \epsilon_{33}$, $\gamma_{xy} \equiv 2\epsilon_{12}$, $\gamma_{xz} \equiv 2\epsilon_{13}$, $\gamma_{yz} \equiv 2\epsilon_{23}$.

9. Figure P2-11.9 represents the centerline of the cross section of a cylindrical shell. When the shell buckles, the particle that lies at point (x, y, z) of the middle surface is displaced to the point (x^*, y^*, z^*). (The x axis is perpendicular to the paper, outward.) In terms of the axial, circumferential, and radial displacement components (u, v, w), the angular coordinate θ, and the initial coordinate x, derive expressions for the coordinates (x^*, y^*, z^*).

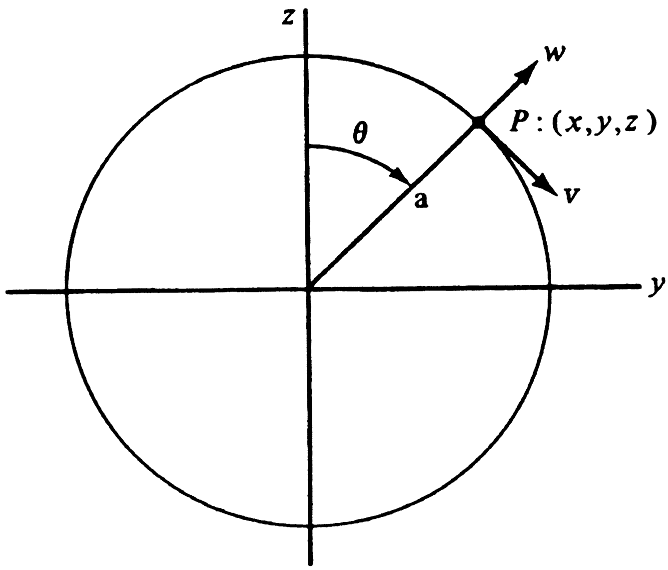

Figure P2-11.9

2-12 Determination of Strain Invariants. Volumetric Strain

In Eq. (2-11.8), denote L by λ_i. Then, expanding, we obtain

$$\lambda_i^3 - J_1\lambda_i^2 + J_2\lambda_i - J_3 = 0 \tag{2-12.1}$$

where

$$J_1 = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{\alpha\alpha} = \delta_{\alpha\beta}\epsilon_{\alpha\beta}$$

$$J_2 = \epsilon_{11}\epsilon_{22} + \epsilon_{11}\epsilon_{33} + \epsilon_{22}\epsilon_{33} - \epsilon_{12}^2 - \epsilon_{13}^2 - \epsilon_{23}^2$$

$$= \begin{vmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{12} & \epsilon_{22} \end{vmatrix} + \begin{vmatrix} \epsilon_{11} & \epsilon_{13} \\ \epsilon_{13} & \epsilon_{23} \end{vmatrix} + \begin{vmatrix} \epsilon_{22} & \epsilon_{23} \\ \epsilon_{23} & \epsilon_{33} \end{vmatrix} = \delta_{\alpha\beta} \text{ cof } \epsilon_{\alpha\beta} \tag{2-12.2}$$

$$J_3 = \begin{vmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{vmatrix} = \det(\epsilon_{\alpha\beta})$$

It is shown in the theory of algebraic equations that the coefficients (J_1, J_2, J_3) are related to the three roots ($\lambda_1, \lambda_2, \lambda_3$) of Eq. (2-12.1) by the following equation:

$$\begin{aligned} J_1 &= \lambda_1 + \lambda_2 + \lambda_3 \\ J_2 &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ J_3 &= \lambda_1\lambda_2\lambda_3 \end{aligned} \tag{2-12.3}$$

Comparing Eqs. (2-12.2) and (2-12.3), we see that Eqs. (2-12.3) are a special case of Eqs. (2-12.2), that is, the case when axes (X_1, X_2, X_3) are the principal axes and $\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$. Accordingly, because the principal deformations[10] ($\lambda_1, \lambda_2, \lambda_3$) are physical quantities independent of the coordinate system, Eqs. (2-12.3) state that the quantities (J_1, J_2, J_3) are independent of the coordinate system; that is, (J_1, J_2, J_3) are invariant (unchanged) under coordinate transformations. Consequently, the coefficients (J_1, J_2, J_3) of Eq. (2-12.1) are called the *invariants of the strain tensor* $\epsilon_{\alpha\beta}$, or, briefly, the *strain invariants*.

Alternatively, the invariance of J_1, J_2, J_3 may be shown by direct calculations. For example, by Eq. (2-9.3) the strain components $\epsilon_{\alpha\beta}$, $E_{\alpha\beta}$ relative to the two sets of axes y_α and x_β are related by the equation

$$\epsilon_{\alpha\beta} = a_{\gamma\alpha}a_{\delta\beta}E_{\gamma\delta} \tag{a}$$

where $a_{\alpha\beta}$ are the direction cosines between the axes y_α and x_β, respectively (see Section 1-24). Substituting Eq. (a) into Eq. (2-12.2), we have [with $\alpha = \beta$ in Eq. (a)], using Eq. (1-26.2),

$$J_1 = \epsilon_{\alpha\alpha} = a_{\gamma\alpha}a_{\delta\alpha}E_{\gamma\delta} = \delta_{\gamma\delta}E_{\gamma\delta} = E_{\gamma\gamma} = E_{\alpha\alpha}$$

[10]Because $\lambda_i = e_i + \frac{1}{2}e_i^2$, λ_i is called a *principal strain*.

Thus, J_1 is invariant under a rotation of axes x_α into axes y_α; that is, it is computed in the same manner relative to axes x_α or axes y_α, and it has the same numerical value relative to either axis.

Similarly, for J_2 [Eq. (2-12.2)] we have

$$J_2 = \delta_{\alpha\beta} \text{ cof } \epsilon_{\alpha\beta} = \tfrac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta}\epsilon_{\iota\alpha\gamma}\epsilon_{\kappa\beta\delta} \tag{b}$$

where ϵ_{ijk} is the alternating tensor [Eq. (1-26.4)]. Substitution of Eq. (a) into Eq. (b) yields

$$J_2 = \tfrac{1}{2}(a_{\zeta\alpha}a_{\eta\beta}E_{\zeta\eta})(a_{\mu\gamma}a_{\nu\delta}E_{\mu\nu})\epsilon_{\iota\alpha\gamma}\epsilon_{\kappa\beta\delta} \tag{c}$$

or, regrouping, we obtain

$$J_2 = \tfrac{1}{2}E_{\zeta\eta}E_{\mu\nu}(\epsilon_{\iota\alpha\gamma}a_{\zeta\alpha}a_{\mu\gamma})(\epsilon_{\kappa\beta\delta}a_{\eta\beta}a_{\nu\delta})$$

which [with Eq. (1-26.10) and the result following Eq. (1-26.10)] reduces to

$$J_2 = \tfrac{1}{2}E_{\zeta\eta}E_{\mu\nu}\epsilon_{\iota\zeta\mu}\epsilon_{\kappa\eta\nu} = \tfrac{1}{2}E_{\alpha\beta}E_{\gamma\delta}\epsilon_{\iota\alpha\gamma}\epsilon_{\kappa\beta\delta} \tag{d}$$

where the right side of Eq. (c) is obtained by changing summing indices (Chapter 1, Section 1-23). Thus, the computation of J_2 relative to axes x_α is the same as it is relative to axes y_α, Eqs. (b) and (c), respectively; and it has the same numerical value whether expressed in terms of $\epsilon_{\alpha\beta}$ or $E_{\alpha\beta}$. Consequently, J_2 is invariant under a rotation of axes.

Finally, it can be shown that

$$\begin{aligned} J_3 &= \det(\epsilon_{\alpha\beta}) = \tfrac{1}{6}\epsilon_{\alpha\beta}\epsilon_{\delta\gamma}\epsilon_{\zeta\eta}\epsilon_{\zeta\alpha\delta}\epsilon_{\eta\beta\gamma} \\ &= \tfrac{1}{6}E_{\alpha\beta}E_{\delta\gamma}E_{\zeta\eta}\epsilon_{\zeta\alpha\delta}\epsilon_{\eta\beta\gamma} \end{aligned} \tag{d}$$

hence that J_3 is invariant under a rotation of axes.

Volumetric Strain. Under a deformation, an initial volume element dV is deformed into a volume element $d\mathscr{V}$. In accordance with Eq. (2-6.15), we define the volumetric strain e by the relation

$$e = \frac{1}{2}\left[\left(\frac{d\mathscr{V}}{dV}\right)^2 - 1\right] \tag{2-12.4}$$

The volumetric strain e may be expressed in terms of the components $\epsilon_{\alpha\beta}$ directly if $d\mathscr{V}/dV$ is evaluated for an arbitrary volume element dV. However, considerable algebraic labor is avoided through the use of principal strains. Accordingly, we consider an infinitesimal rectangular parallelepiped with edges along principal axes. Because line elements in the principal directions remain mutually perpendicular under a deformation, the parallelepiped remains rectangular in the deformed state.

Consequently, Eq. (2-6.15) yields the result

$$\left(\frac{d\mathscr{s}_1}{ds_1}\right)^2 = 1 + 2\lambda_1 \tag{a}$$

where λ_1 is the principal strain of one edge of the rectangular parallelepiped whose edges are (ds_1, ds_2, ds_3). Similar expressions pertain for ds_2 and ds_3. Hence, because $dV = ds_1\, ds_2\, ds_3$, by Eq. (a) we have

$$\left(\frac{d\mathscr{V}}{dV}\right)^2 = (1 + 2\lambda_1)(1 + 2\lambda_2)(1 + 2\lambda_3) \tag{b}$$

where $(\lambda_1, \lambda_2, \lambda_3)$ are principal strains.

By Eqs. (2-12.3), (2-12.4), and (b), the volumetric strain (also called cubical dilatation, or simply dilatation) e may be expressed in terms of the strain invariants (J_1, J_2, J_3) as

$$e = J_1 + 2J_2 + 4J_3 \tag{2-12.5}$$

Accordingly, the volumetric strain e is also an invariant quantity. If (J_1, J_2, J_3) are expressed in terms of the components $\epsilon_{\alpha\beta}$ of the strain tensor [Eqs. (2-12.2)], Eq. (2-12.5) defines the volumetric strain e in terms of $\epsilon_{\alpha\beta}$. Furthermore, because J_2 and J_3 are, respectively, second- and third-order terms in the principal strains λ_i [see Eqs. (2-12.3)], to first-order terms in λ_i the volumetric strain is

$$e = J_1 = \epsilon_{11} + \epsilon_{22} + \epsilon_{33} = \epsilon_{\alpha\alpha} \tag{2-12.6}$$

Consequently, if quadratic terms in u_i and their derivatives are small compared to corresponding linear terms in u_i, Eqs. (2-12.6) and (2-6.9) yield

$$e = u_{1,1} + u_{2,2} + u_{3,3} \tag{2-12.7}$$

Hence, Eq. (2-12.7) represents an approximate expression for the volumetric strain in terms of the (x_1, x_2, x_3) derivatives of (u_1, u_2, u_3), respectively. For an incompressible medium, $e = 0$. For example, for infinitesimal small displacements of an incompressible fluid,

$$e = J_1 = u_{1,1} + u_{2,2} + u_{3,3} = 0 \tag{2-12.8}$$

Mean and Deviator Strain Tensor. Experiments indicate that yielding and plastic deformation of many metals are essentially independent of mean strain ϵ_m, where by definition

$$\epsilon_m = \frac{\epsilon_x + \epsilon_y + \epsilon_z}{3} = \frac{\epsilon_{11} + \epsilon_{22} + \epsilon_{33}}{3} = \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} = \frac{1}{3}J_1 \tag{2-12.9}$$

Hence, plasticity theories often postulate that plastic behavior of materials is related primarily to that part of the strain tensor that is independent of ϵ_m. Accordingly, we write the strain tensor [Eq. (2-9.2)] in the form

$$D = D_m + D_d \tag{2-12.10}$$

Here, D symbolically represents the strain (deformation) tensor and

$$D_m = \begin{pmatrix} \epsilon_m & 0 & 0 \\ 0 & \epsilon_m & 0 \\ 0 & 0 & \epsilon_m \end{pmatrix} = \epsilon_m D_1 \tag{2-12.11}$$

where

$$D_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2-12.12}$$

represents the *unit tensor* and

$$D_d = \begin{pmatrix} \overline{\epsilon}_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \overline{\epsilon}_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \overline{\epsilon}_{33} \end{pmatrix} \tag{2-12.13}$$

where

$$\begin{aligned} \overline{\epsilon}_{11} &= \epsilon_x - \epsilon_m = \epsilon_{11} - \epsilon_m \\ \overline{\epsilon}_{22} &= \epsilon_y - \epsilon_m = \epsilon_{22} - \epsilon_m \\ \overline{\epsilon}_{33} &= \epsilon_z - \epsilon_m = \epsilon_{33} - \epsilon_m \end{aligned} \tag{2-12.14}$$

The validity of Eq. (2-12.10) follows from the definition of a tensor (Synge and Schild, 1978).

The tensor D_m is called the *mean* strain tensor. The tensor D_d is called the *deviator* strain tensor. Accordingly, the components

$$\overline{\epsilon}_{11}, \overline{\epsilon}_{22}, \overline{\epsilon}_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23}$$

are called the components of the deviator strain tensor.

If (x_1, x_2, x_3) are principal axes of strain,

$$\begin{aligned} \epsilon_x = \epsilon_{11} = \lambda_1 \qquad & \epsilon_y = \epsilon_{22} = \lambda_2 \\ \epsilon_z = \epsilon_{33} = \lambda_3 \qquad & \epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0 \end{aligned} \tag{2-12.15}$$

and the above equations are simplified accordingly, then

For D_m :

$$\begin{aligned} J_{1m} &= J_1 = 3\epsilon_m \\ J_{2m} &= \tfrac{1}{3} J_1^2 = 3\epsilon_m^2 \\ J_{3m} &= \tfrac{1}{27} J_1^3 = \epsilon_m^3 \end{aligned} \tag{2-12.16}$$

For $D_d : J_{1d} = 0$

$$
\begin{aligned}
J_{2d} &= J_2 - \tfrac{1}{3}J_1^2 = -\tfrac{1}{6}[(\lambda_1 - \lambda_2)^2 + (\lambda_2 - \lambda_3)^2 + (\lambda_3 - \lambda_1)^2] \\
&= \overline{\epsilon}_{11}\overline{\epsilon}_{22} + \overline{\epsilon}_{11}\overline{\epsilon}_{33} + \overline{\epsilon}_{22}\overline{\epsilon}_{33} \qquad (2\text{-}12.17) \\
J_{3d} &= \tfrac{1}{27}(2\lambda_1 - \lambda_2 - \lambda_3)(2\lambda_2 - \lambda_3 - \lambda_1) \\
&\quad \times (2\lambda_3 - \lambda_1 - \lambda_2) \\
&= (\lambda_1 - \epsilon_m)(\lambda_2 - \epsilon_m)(\lambda_3 - \epsilon_m) \\
&= \overline{\overline{\epsilon}}_{11}\,\overline{\overline{\epsilon}}_{22}\,\overline{\overline{\epsilon}}_{33}
\end{aligned}
$$

where $\overline{\overline{\epsilon}}_{11} = \lambda_1 - \epsilon_m, \overline{\overline{\epsilon}}_{22} = \lambda_2 - \epsilon_m, \overline{\overline{\epsilon}}_{33} = \lambda_3 - \epsilon_m$ denote the principal values of $\overline{\epsilon}_{11}, \overline{\epsilon}_{22}, \overline{\epsilon}_{33}$.

Octahedral Strains. Consider the octahedral planes defined as planes whose normals satisfy the relations $N_1^2 = N_2^2 = N_3^2 = \frac{1}{3}$ with respect to principal strain axes.

Then

$$\lambda_{\text{oct}} = \epsilon_{\text{oct}} = \frac{\epsilon_{11} + \epsilon_{22} + \epsilon_{33}}{3} = \epsilon_m = \frac{\lambda_1 + \lambda_2 + \lambda_3}{3} \qquad (2\text{-}12.18)$$

The strain ϵ_{oct} is called the *octahedral strain*. Similarly, the maximum shearing strain between an octahedral plane and its normal,

$$\Gamma = \gamma_{\text{oct}} = \tfrac{2}{3}\sqrt{(\lambda_1 - \lambda_2)^2 + (\lambda_1 - \lambda_3)^2 + (\lambda_2 - \lambda_3)^2} \qquad (2\text{-}12.19)$$

is called the octahedral *shearing strain*. It plays a significant role in certain theories of plasticity.

Plane Strain. A special case of strain that plays an important role in the plane theory of deformation is that of *plane strain*. For example, if two of the displacement components (say u_1, u_2) are functions of two coordinates only (say x_1, x_2) and if the third component $u_3 =$ constant, the state of the deformation is called a *plane strain* relative to the (x_1, x_2) plane. It follows that

$$\epsilon_{33} = \epsilon_{13} = \epsilon_{23} = 0 \qquad (2\text{-}12.20)$$

and ϵ_{11}, ϵ_{22}, ϵ_{12} are functions of (x_1, x_2) only. Such a state of deformation exists in a cylinder constrained at its end faces such that the axial displacement of all points in the cylinder is prevented ($u_3 = 0$). In the case of plane strain relative to the (x_1, x_2) plane, the three-dimensional deformation problem in (x_1, x_2, x_3) is reduced to a two-dimensional problem in (x_1, x_2) (see Chapter 5).

2-13 Rotation of a Volume Element. Relation to Displacement Gradients

Consider an infinitesimal volume element surrounding any particle P of a continuous media. Under a deformation this infinitesimal volume is altered not only in position but also in dimensions and shape. We define the *rotation of the volume element* to be the mean value of the rotation experienced by the set of infinitesimal line elements emanating from point P. Accordingly, in our present discussion we are concerned only with rotations of line elements. Hence, we let the position $\mathscr{P}$ of the particle in the deformed medium coincide with its position P in the undeformed medium.

Initially, we consider the rotation of a single infinitesimal line element PA_1 that lies in the (x_1, x_2) plane. After the deformation, the line PA_1 coincides with the direction $\mathscr{P}\mathscr{A}$ (or $P\mathscr{A}$) (see Fig. 2-13.1).

The orthogonal projection of the vector $\mathscr{P}\mathscr{A}$ on the (X_1, X_2) plane is $\mathscr{P}\mathscr{A}_1$. The angle $\phi_{X_3} = \vartheta - \theta$ between lines PA_1 and $\mathscr{P}\mathscr{A}_1$ is understood to be the rotation of line element PA_1 about the X_3 axis. Accordingly, by Fig. 2-13.1 we have the relations

$$\tan\theta = \frac{dx_2}{dx_1} \qquad \tan\vartheta = \frac{d\xi_2}{d\xi_1} \tag{a}$$

Furthermore, the deformation is defined by the equations $\xi_i = \xi_i(x_1, x_2, x_3)$, $i = 1, 2, 3$, with the understanding that (xi_1, xi_2, xi_3) vanish at $x_1 = x_2 = x_3 = 0$. Because $\mathscr{P}\mathscr{A}_1$ lies in the (x_1, x_2) plane, $\mathscr{P}\mathscr{A}_1 = (d\xi_1, d\xi_2, 0)$. Hence, for $\mathscr{P}\mathscr{A}_1$, by Eqs. (2-5.3), (2-6.4) and (2-6.6), we obtain the relations

$$\begin{aligned} d\xi_1 &= (1 + e_{11})dx_1 + (e_{21} + \omega_{21})\,dx_2 \\ d\xi_2 &= (e_{12} + \omega_{12})dx_1 + (1 + e_{22})\,dx_2 \end{aligned} \tag{b}$$

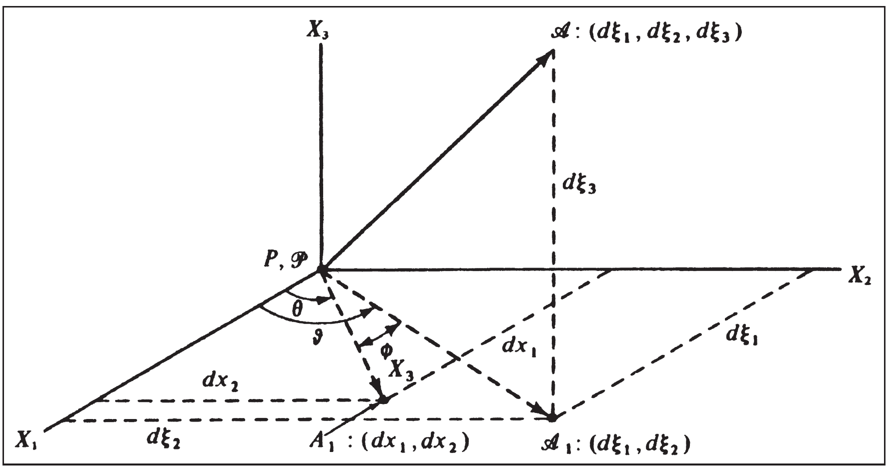

Figure 2-13.1

Hence, Eqs. (a) and (b) yield

$$\tan\vartheta = \frac{e_{12} + \omega_{12} + (1 + e_{22})\tan\theta}{1 + e_{11} + (e_{21} + \omega_{21})\tan\theta} \tag{c}$$

The angle of rotation of $\mathbf{PA}_1$ about the X_3 axis is $\phi_{X_3} = \vartheta - \theta$. Moreover, the tangent of the difference of two angles may be written in the form

$$\tan\phi_{X_3} = \frac{\tan\vartheta - \tan\theta}{1 + \tan\vartheta\tan\theta} \tag{d}$$

Accordingly, substitution of Eq. (c) into Eq. (d) yields, after simplification,

$$\tan\phi_{X_3} = \frac{\omega_{12} + \frac{1}{2}(e_{22} - e_{11})\sin 2\theta + e_{12}\cos 2\theta}{1 + e_{11}\cos^2\theta + e_{22}\sin^2\theta + e_{21}\sin 2\theta} \tag{2-13.1}$$

Equation (2-13.1) expresses the rotation ϕ_{X_3} of line element $\mathbf{PA}_1$ about the X_3 axis in terms of $\tan\phi_{X_3}$. It is desirable to obtain an expression for ϕ_{X_3} in explicit form. With ϕ_{X_3} determined, we have, by definition of mean rotations $\overline{\phi}$, for the mean rotation of the volume element about the X_3 axis,

$$\overline{\phi}_{X_3} = \frac{1}{2\pi}\int_0^{2\pi} \phi_{X_3} d\theta \tag{e}$$

One method of approach might be by numerical integration after substitution of Eq. (2-13.1) into Eq. (e). In general, however, $\overline{\phi}_{X_3}$ cannot be obtained in closed form, and, consequently, the effect of $\omega_{\alpha\beta}$ and $e_{\alpha\beta}$ remain coupled in the mean rotations.[11] Hence, the assertion made in Section 2-5 to the effect that the $\omega_{\alpha\beta}$ may be related to mean rotations of volume elements is not generally true. Accordingly, we consider cases for which the assertion is valid. For this purpose we restrict the discussion to the case where

$$|e_{11}| \ll 1 \qquad |e_{12}| = |e_{21}| \ll 1 \qquad |e_{22}| \ll 1$$

Then Eq. (2-13.1) may be written

$$\tan\phi_{X_3} \approx \omega_{12} + \tfrac{1}{2}(e_{22} - e_{11})\sin 2\theta + e_{12}\cos 2\theta = \omega_{12} + F(\theta)$$

where

$$|F(\theta)| = \left|\tfrac{1}{2}(e_{22} - e_{11})\sin 2\theta + e_{12}\cos 2\theta\right| \ll 1$$

[11]In an extensive study of the mean rotation of an elastic solid around a coordinate axis, Elder et al., (1984) have developed a Fourier series expansion for the mean angle of rotation in terms of certain parameters. The expansion is valid for finite strains not exceeding $\frac{1}{2}$.

The determination of angle ϕ_{X_3} now reduces to two cases: (a) $|\omega_{12}| \ll 1$ and (b) $|\omega_{12}|$ sufficiently small so that a Taylor series expansion of ϕ_{X_3} about ω_{12}, retaining at most two terms, is permissible. We now consider these cases in order.

Case a. Because $|e_{\alpha\beta}| \ll 1$ is assumed, with $|\omega_{12}| \ll 1$, Eq. (2-13.1) yields the result

$$\phi_{X_3} = \omega_{12} + e_{12}\cos 2\theta + \tfrac{1}{2}(e_{22} - e_{11})\sin 2\theta \tag{f}$$

Moreover, by the definition of the mean rotation $\overline{\phi}$ of a volume element, we have, for the mean rotation of the volume element about the X_3 axis,

$$\overline{\phi}_{X_3} = \frac{1}{2\pi}\int_0^{2\pi} \phi_{X_3} d\theta \tag{g}$$

Substitution of Eq. (f) into Eq. (g) yields, after integration,

$$\overline{\phi}_{X_3} = \omega_{12} \tag{g}$$

Similar expressions hold for the rotations of the volume element about X_1 and X_2 axes. Hence, we have for the mean rotation of the volume element about (X_1, X_2, X_3) axes, respectively,

$$\overline{\phi}_{X_1} = \omega_{23} \qquad \overline{\phi}_{X_2} = \omega_{31} \qquad \overline{\phi}_{X_3} = \omega_{12} \tag{2-13.2}$$

Consequently, the quantities $(\omega_{12}, \omega_{13}, \omega_{23})$ may be interpreted as the rotation vector

$$\begin{aligned} \boldsymbol{\omega} &= \mathbf{i}\omega_{23} + \mathbf{j}\omega_{31} + \mathbf{k}\omega_{12} \\ &= \tfrac{1}{2}\text{curl } \mathbf{q} \end{aligned} \tag{2-13.3}$$

for a volume element, provided $e_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are very small compared to 1. Hence, we have demonstrated the assertion made in Section 2-5 that $\mathbf{\Omega}$ characterizes a mean rotation of a volume element when the strains are small. In other words, for $|e_{\alpha\beta}| \ll 1$, $|\omega_{\alpha\beta}| \ll 1$, the mean rotation of a volume element may be characterized by the components of the antisymmetric part of the gradient of the displacement.

Case b. We assume that $|\omega_{12}|$ is sufficiently small to allow a Taylor series expansion of $\phi_{X_3} = \arctan\,[\omega_{12} + F(\theta)]$, retaining, at most, two terms. Hence, by the Taylor series expansion, we have, retaining two terms,

$$\phi_{X_3} \approx \phi_{X_3}(0) + \left.\frac{\partial \phi_{X_3}}{\partial \omega_{12}}\right|_0 F(\theta) = \arctan \omega_{12} + \frac{F(\theta)}{1 + \omega_{12}^2}$$

Accordingly, by definition of mean rotation, we find

$$\overline{\phi}_{X_3} = \arctan \omega_{12} \tag{2-13.4}$$

because $\int_0^{2\pi} F(\theta)\, d\theta = 0$. Hence,

$$\omega_{12} = \tan \overline{\phi}_{X_3}$$

That is, ω_{12} is equal to the tangent of the mean rotation $\overline{\phi}_{X_3}$ of the volume element about axis X_3. Similar results hold for ω_{23}, ω_{31} in this case. Because the arctangent is multivalued, in lieu of other information, Eq. (2-13.3) must be restricted to one branch of the function, say, $-\pi/2 \le \phi_{X_3} \le \pi/2$.

We note that the requirement $|e_{\alpha\beta}| \ll 1$ is *sufficient* for interpreting $\omega_{\alpha\beta}$ in terms of mean volume rotation, as may be seen by considering the displacement field $u_1 = -(1 - \cos\theta)X_1 - (\sin\theta)\, X_2, u_2 = (\sin\theta)X_1 - (1 - \cos)X_2$, as then $e_{11} = e_{22} = \cos\theta - 1, e_{12} = 0$, and $\omega_{12} = \sin\theta$. Hence, Eq. (2-13.1) reduces to an identity. Then, by definition of mean value of rotation, we obtain $\omega_{12} = \sin\phi_{X_3}$; that is, ω_{12} is related to volume rotation.

Problem. Let $u_1 = a + cx_2, u_2 = b - cx_1$. Hence, compute the strain components of $\epsilon_{\alpha\beta}$. Discuss the possibility of interpretation of ω_{12} in terms of the volume rotation.

Example 2-13.1. Volumetric Rotation. Let the displacement components be given as

$$u_1 = C(10x_1 - 3x_2) \qquad u_2 = C(3x_1 + 2x_2) \qquad u_3 = 6Cx_3 \tag{a}$$

where C is a small positive constant. By the theory of Section 2-13, the mean rotation of a volume element is characterized by $\omega_{\alpha\beta}$, where the $\omega_{\alpha\beta}$ are related to u_α by Eq. (2-5.3). Thus, the components of mean rotation for $|e_{\alpha\beta}| \ll 1$ [see Eq. (2-5.3)] are

$$\begin{aligned} \omega_{12} &= \tfrac{1}{2}(u_{2,1} - u_{1,2}) = 3C \\ \omega_{23} &= \tfrac{1}{2}(u_{3,2} - u_{2,3}) = 0 \\ \omega_{31} &= \tfrac{1}{2}(u_{1,3} - u_{3,1}) = 0 \end{aligned} \tag{b}$$

Equation (b) indicates that the mean rotation of the volume element is constant throughout the region and is directed around axis x_3 [see Eq. (2-13.3)].

Problem Set 2-13

1. The classical small-displacement theory of elasticity yields the following (x, y, z) displacement components for a beam subjected to pure bending: $u = -k_1xy, v = k_2(x^2 + vy^2 - vw^2), w = k_3vyz$. Compute the rotations of a volume element in the beam with respect to (x, y, z) axes, respectively.

2. For a bar stretched by its own weight, the classical small-displacement theory of elasticity yields the displacement components:

$$u = -C_1zx \qquad v = -C_1zy \qquad w = \tfrac{1}{2}C_1(x^2 + y^2) + C_2z^2 + C_3$$

where C_1, C_2, C_3 are constants. Compute the rotations of a volume element in the body with respect to (x, y, z) axes, respectively.

3. The rectangular Cartesian displacement components of an arbitrary point in a body are given by the relations

$$u = a_1 x + a_2 y + a_3 z$$
$$v = b_1 x + b_2 y + b_3 z$$
$$w = c_1 x + c_2 y + c_3 z$$

Show that a spherical surface with center at the origin of coordinate system (x, y, z) is transformed into a quadratic surface. Show that if $\epsilon_{11} = \epsilon_{22} = \epsilon_{33}$ are principal strains, the spherical volume element enclosed in the spherical surface remains spherical under the deformation.

4. A displacement field (u_1, u_2, u_3) is defined for all (x_1, x_2) and for $x_3 \geq 0$ by the relations

$$u_1 = -\epsilon x_3 x_1 \qquad u_2 = -\epsilon x_3 x_2 \qquad u_3 = \tfrac{1}{2}\epsilon(x_1^2 + x_2^2) + \tfrac{1}{2}\epsilon A x_3^2$$

where $\epsilon > 0$ and A are constants.

(a) Determine the value of A that ensures that this field is a physically possible continuous field (an admissible field).

(b) With the value of A determined in part (a), compute the three strain invariants.

(c) With the value of A determined in part (a), compute the rotations of a volume element with respect to (x_1, x_2, x_3) axes, respectively.

5. Outline a method of computing the octahedral shearing strain γ_{oct} defined as the maximum value of Γ measured between the normal to an octahedral plane and any line in the octahedral plane. Hence, verify Eq. (2-12.19).

6. The following set of displacement components is given for a deformable body in the region:

$$x^2 + y^2 \leq a \qquad |z| \leq L$$

where a and L are constants:

$$u = x(\cos kz - 1) - y(\sin kz)$$
$$v = x(\sin kz) + y(\cos kz - 1) \qquad k = \text{const} > 0$$
$$w = 0$$

(a) Determine whether these displacement components may represent a continuous deformation.

(b) Determine the strain components.

(c) Determine the volumetric strain.

(d) Determine the volumetric rotation for small angles of rotation and small $e_{\alpha\beta}$.

7. For the displacement field of Example 2-13.1, determine the tangent of the angle of rotation about the x_2 axis of a line element initially parallel to the x_3 axis, and about the x_1 axis of a line element initially parallel to the x_2 axis.

2-14 Homogeneous Deformation

In the preceding sections we considered general deformation of a continuous medium. For this case we discovered that the deformation of the medium is characterized by the six strain components $\epsilon_{\alpha\beta}$. These six components are expressed in Lagrange coordinates by Eq. (2-6.9). Furthermore, in Section 2-10 we noted that the deformation may be described by a family of strain ellipsoids, one ellipsoid for each particle of the medium.

In this section we consider a special kind of deformation; that is, we let the final position (ξ_1, ξ_2, ξ_3) of a particle of the medium be a linear function of its initial position (x_1, x_2, x_3). Thus, we take

$$\begin{aligned} \xi_1 &= c_{10} + (1 + c_{11})x_1 + c_{12}x_2 + c_{13}x_3 \\ \xi_2 &= c_{20} + c_{21}x_1 + (1 + c_{22})x_2 + c_{23}x_3 \\ \xi_3 &= c_{30} + c_{31}x_1 + c_{32}x_2 + (1 + c_{33})x_3 \end{aligned} \tag{2-14.1}$$

where c_{ij}, $i = 1, 2, 3$; $j = 0, 1, 2, 3$, are arbitrary constants.

By Eqs. (2-3.4) and (2-14.1) we obtain

$$\begin{aligned} u_1 &= c_{10} + c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ u_2 &= c_{20} + c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \\ u_3 &= c_{30} + c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \end{aligned} \tag{2-14.2}$$

For $x_1 = x_2 = x_3 = 0$, Eqs. (2-14.2) yield $u_1 = c_{10}, u_2 = c_{20}, u_3 = c_{30}$. Accordingly, (c_{10}, c_{20}, c_{30}) represent a translation of the origin of the (x_1, x_2, x_3) coordinate system. Furthermore, by Eq. (2-6.9), we see that (c_{10}, c_{20}, c_{30}) produce no strain in the medium. Consequently, they represent a rigid-body translation of the medium (see Section 2-2). Hence, we discard (c_{10}, c_{20}, c_{30}) from Eqs. (2-14.2):

$$\begin{aligned} u_1 &= c_{11}x_1 + c_{12}x_2 + c_{13}x_3 \\ u_2 &= c_{21}x_1 + c_{22}x_2 + c_{23}x_3 \\ u_3 &= c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \end{aligned} \tag{2-14.3}$$

In index notation, Eqs. (2-14.3) is

$$u_\alpha = c_{\alpha\beta} x_\beta \tag{2-14.4}$$

Substitution of Eqs. (2-14.3) into Eq. (2-6.14) yields

$$\epsilon_{\alpha\beta} = \tfrac{1}{2}(c_{\alpha\beta} + c_{\beta\alpha} + c_{\theta\alpha} c_{\theta\beta}) \tag{2-14.5}$$

or

$$
\begin{aligned}
\epsilon_{11} &= c_{11} + \tfrac{1}{2}(c_{11}^2 + c_{21}^2 + c_{31}^2) \\
&\vdots \\
2\epsilon_{12} &= c_{21} + c_{12} + c_{11}c_{12} + c_{21}c_{22} + c_{31}c_{32} \\
&\vdots
\end{aligned}
\qquad (2\text{-}14.6)
$$

Hence, the six strain components $\epsilon_{\alpha\beta}$ are constant throughout the medium if (ξ_1, ξ_2, ξ_3) are linear functions of (x_1, x_2, x_3). A deformation for which the strain components are constant is called a *homogeneous deformation* or a *homogeneous state of strain*. It follows that the ellipsoids of strain for each point are equal, and they are identically oriented. Consequently, the relative elongation of an infinitesimal line element depends only on its direction; it does not depend on its location in the body. Likewise, the change in angle between two infinitesimal line elements does not depend on their location but only on their initial directions [see Eq. (2-8.3)]. In particular, it follows that two initially parallel infinitesimal lines remain parallel under the deformation.

Geometric Properties. Under a homogeneous deformation, certain finite geometric entities also remain unchanged. For example, consider a straight line under a homogeneous strain. The general equation of a straight line may be written parametrically as

$$x_1 = a_1 + b_1 t \qquad x_2 = a_2 + b_2 t \qquad x_3 = a_3 + b_3 t \qquad (2\text{-}14.7)$$

where $a_1, a_2, a_3, b_1, b_2, b_3$ are constants and t is a parameter. Under the deformation, (x_1, x_2, x_3) is transformed into (ξ_1, ξ_2, ξ_3), or

$$\xi_1 = x_1 + u_1 \qquad \xi_2 = x_2 + u_2 \qquad \xi_3 = x_3 + u_3 \qquad (2\text{-}14.8)$$

Substitution of Eqs. (2-14.3) and (2-14.7) into (2-14.8) yields

$$
\begin{aligned}
\xi_1 &= (1 + c_{11})a_1 + c_{12}a_2 + c_{13}a_3 + [(1 + c_{11})b_1 + c_{12}b_2 + c_{13}b_3]t \\
\xi_2 &= \cdots \\
\xi_3 &= \cdots
\end{aligned}
\qquad (2\text{-}14.9)
$$

Equation (2-14.9) is again a linear equation in t; accordingly, it is the equation of a straight line. Thus, we have proved that finite straight lines remain straight under a homogeneous deformation. Furthermore, because a plane is generated by all straight lines through two given nonskew straight lines, planes also remain plane under the deformation.

Similarly, it may be proved that under a homogeneous deformation, parallel lines remain parallel; hence, parallel planes remain parallel, and any parallelepiped remains a parallelepied because it is constructed from pairs of parallel planes.

Moreover, under a homogeneous deformation, a spherical surface in the medium is transformed in general into an ellipsoid, provided the displacement remains finite.

Pure Strain (Dilatation). A homogeneous deformation is called a *pure strain* or *dilatation* when there exist in the undeformed medium three axes that remain unaltered under the deformation. In general, there exists in a homogeneous deformation three rectangular Cartesian axes (x_1, x_2, x_3), which are transformed into three other rectangular Cartesian axes (X_1, X_2, X_3). The deformation (strain) is *pure* when (x_1, x_2, x_3) and (X_1, X_2, X_3) coincide. Then the axes of (x_1, x_2, x_3) and (X_1, X_2, X_3) constitute the principal axes of the deformation (see Sections 2-10 and 2-11) in the undeformed medium and the deformed medium, respectively.

Accordingly, let us take the axes of the rectangular Cartesian system (X_1, X_2, X_3) as principal axes. Let (x_1, x_2, x_3) be the coordinates of point P in the undeformed medium; let (ξ_1, ξ_2, ξ_3) be coordinates of $\mathscr{P}$ after a homogeneous deformation. Then, by Eqs. (2-14.1),

$$\begin{aligned} \xi_1 &= (1 + c_{11})x_1 + c_{12}x_2 + c_{13}x_3 \\ \xi_2 &= c_{21}x_1 + (1 + c_{22})x_2 + c_{23}x_3 \\ \xi_3 &= c_{31}x_1 + c_{32}x_2 + (1 + c_{33})x_3 \end{aligned} \tag{2-14.10}$$

where we have discarded the rigid-body translation (c_{10}, c_{20}, c_{30}). By the definition of *pure strain*, if point P is on the X_1 axis, the corresponding point $\mathscr{P}$ must also be on the X_1 axis. Then, because (x_2, x_3) are zero, (ξ_2, ξ_3) must also be zero. Hence, because x_1 is not zero, it follows by Eqs. (2-14.10) that $c_{21} = c_{31} = 0$. Similarly, by considering points on the axes X_2 and X_3, $c_{12} = c_{13} = c_{32} = c_{23} = 0$. Consequently, referred to principal axes, a pure strain (dilatation) is defined by the equations

$$\xi_1 = (1 + c_{11})x_1 \qquad \xi_2 = (1 + c_{22})x_2 \qquad \xi_3 = (1 + c_{33})x_3 \tag{2-14.11}$$

Accordingly, the constants (c_{11}, c_{22}, c_{33}) in Eqs. (2-14.3) are the relative elongations of line elements that coincide with the principal axes. They are called the *coefficients of the principal dilatations*.

By Eqs. (2-14.11), we see that a dilatation is characterized by three extensions (for positive c_{11}, c_{22}, c_{33}) or contractions (for negative c_{11}, c_{22}, c_{33}) parallel to three rectangular axes. A dilatation is said to be simple if $c_{11} = c_{22} = c_{33}$. For example, a simple dilatation occurs when an isotropic medium is subjected to uniform external pressure. In spherical coordinates, a simple dilatation is given by the relations

$$u = Cr \qquad v = w = 0 \qquad c_{11} = c_{22} = c_{33} = C \tag{2-14.12}$$

where (u, v, w) are the displacement components in the (r, ϕ, θ) directions, respectively.

The type of deformation that occurs in a cylindrical or prismatic bar subjected to uniform tension or compression is given in cylindrical coordinates by the equations

$$u = Cr \qquad v = 0 \qquad w = Kz \qquad c_{11} = c_{22} = C \qquad c_{33} = K \tag{2-14.13}$$

where (u, v, w) are in the (r, ϕ, z) directions, respectively. In this case, K and C have opposite signs. The ratio $-C/K$ is a dimensionless characteristic constant of the material, called Poisson's ratio. For most metals, Poisson's ratio is a number that lies in the range $\frac{1}{4}$ to $\frac{1}{3}$.

In general, for pure strain, we have the following theorem[12]:

Theorem 2-14.1. *A necessary and sufficient condition that a homogeneous deformation to be a pure strain (dilatation) is that the displacement of components* (u_1, u_2, u_3) *be the* (x_1, x_2, x_3) *derivatives, respectively, of a function* ϕ *of second degree in* (x_1, x_2, x_3).

Furthermore, it may be shown that any homogeneous strain may be produced in a body by a suitable pure strain followed by a properly chosen rotation.[13]

2-15 Theory of Small Strains and Small Angles of Rotation

In the previous sections we examined the general deformation of a continuous medium. In certain cases we simplified the general results for small values of strains and small angles of rotation. In this section we discuss further the simplifications obtained when the strain is small compared to 1 and when the angles of rotation are either of the same order or smaller.

Using the notation of Eq. (2-5.7), we may write Eq. (2-6.13) in the form

$$
\begin{aligned}
\epsilon_{11} &= e_{11} + \tfrac{1}{2}[e_{11}^2 + (e_{12} + \omega_{12})^2 + (e_{13} + \omega_{13})^2] \\
\epsilon_{22} &= e_{22} + \tfrac{1}{2}[e_{22}^2 + (e_{21} + \omega_{21})^2 + (e_{23} + \omega_{23})^2] \\
e_{33} &= e_{33} + \tfrac{1}{2}[e_{33}^2 + (e_{31} + \omega_{31})^2 + (e_{32} + \omega_{32})^2] \\
2\epsilon_{12} &= 2e_{12} + e_{11}(e_{21} + \omega_{21}) + (e_{12} + \omega_{12})e_{22} \\
&\quad + (e_{13} + \omega_{13})(e_{23} + \omega_{23}) \qquad (2\text{-}15.1) \\
2\epsilon_{13} &= 2e_{13} + e_{11}(e_{31} + \omega_{31}) + (e_{12} + \omega_{12})(e_{32} + \omega_{32}) \\
&\quad + (e_{13} + \omega_{13})e_{33} \\
2\epsilon_{23} &= 2e_{23} + (e_{21} + \omega_{21})(e_{31} + \omega_{31}) + e_{22}(e_{32} + \omega_{32}) \\
&\quad + (e_{23} + \omega_{23})e_{33}
\end{aligned}
$$

or, in index notation,

$$2\epsilon_{\alpha\beta} = 2e_{\alpha\beta} + (e_{\alpha\theta} + \omega_{\alpha\theta})(e_{\beta\theta} + \omega_{\beta\theta}) \qquad (2\text{-}15.1a)$$

[12]See Love (2002), p. 38.
[13]See Love (2002), p. 69; see also Section 2-15.

To examine the reduction of Eqs. 2-15.1 to the classical approximations of small strains and small rotations, we rewrite the first of Eqs. 2-15.1 in the form

$$(1 + 2\epsilon_{11}) = (1 + e_{11})^2 + (e_{12} + \omega_{12})^2 + (e_{13} + \omega_{13})^2 \tag{2-15.2}$$

Next, we note that Eq. (2-15.2) is satisfied identically by the relations

$$\begin{aligned} \frac{1 + e_{11}}{\sqrt{1 + 2\epsilon_{11}}} &= \cos\alpha_1 \\ \frac{e_{12} + \omega_{12}}{\sqrt{1 + 2\epsilon_{11}}} &= \sin\alpha_1 \cos\beta_1 \\ \frac{e_{13} + \omega_{13}}{\sqrt{1 + 2\epsilon_{11}}} &= \sin\alpha_1 \sin\beta_1 \end{aligned} \tag{2-15.3}$$

where $\cos\alpha_1 = \mathscr{N}_1$ [see Eq. (2-8.2)] is the direction cosine with respect to the x_1 axis of a line element with initial direction cosines $N_1 = 1$, $N_2 = N_3 = 0$ in the undeformed state [see Eq. (2-6.12)]. The angle β_1 is merely a variable of transformation. It does not enter into the principal results of the argument to follow.

Now if the rotation α_1 is small, $\mathscr{N}_1 = \cos\alpha_1 \approx 1 - (\alpha_1^2/2)$. Accordingly, for small rotations, by the first of Eqs. (2-15.3), we obtain

$$\frac{1 + e_{11}}{\sqrt{1 + 2\epsilon_{11}}} \approx 1 - \frac{\alpha_1^2}{2}$$

In a similar manner, the second and third equations of Eqs. 2-15.1 yield equivalent results. Thus,

$$\begin{aligned} &\frac{1 + e_{11}}{\sqrt{1 + 2\epsilon_{11}}} \approx 1 - \frac{\alpha_1^2}{2} \qquad \frac{1 + e_{22}}{\sqrt{1 + 2\epsilon_{22}}} \approx 1 - \frac{\alpha_2^2}{2} \\ &\frac{1 + e_{33}}{\sqrt{1 + 2\epsilon_{33}}} \approx 1 - \frac{\alpha_3^2}{2} \end{aligned} \tag{2-15.4}$$

where (α_1, α_2, α_3) denote the angles of rotation of lines initially directed along (x, y, z) axes, respectively.

For strains (ϵ_{11}, ϵ_{22}, ϵ_{33}) small compared to 1, we may write

$$(1 + 2\epsilon_{ii})^{1/2} = 1 + \epsilon_{ii} - \tfrac{1}{2}\epsilon_{ii}^2 + \cdots \tag{2-15.5}$$

where $i = 1, 2, 3$. Substitution of Eqs. (2-15.5) into Eqs. (2-15.4) yields, to second-degree terms in the rotation α_i,

$$\begin{aligned} \epsilon_{11} - e_{11} &= \frac{\alpha_1^2}{2} \\ \epsilon_{22} - e_{22} &= \frac{\alpha_2^2}{2} \\ \epsilon_{33} - e_{33} &= \frac{\alpha_3^2}{2} \end{aligned} \tag{2-15.6}$$

or

$$\epsilon_{ii} - e_{ii} = \frac{\alpha_i^2}{2} \qquad i = 1, 2, 3 \tag{2-15.7}$$

Equations (2-15.7) indicate that if the strains are small [Eq. (2-15.5)] and if the rotations are sufficiently small [Eqs. (2-15.4)], the difference between ϵ_{ii} and e_{ii} is of the order of the square of the angle α_i of rotation.

To examine the reduction of the last three equations of Eqs. 2-15.1, we note that in a manner analogous to the derivation of Eqs. (2-15.3), the second of Eqs. 2-15.1 yields the relations

$$\begin{aligned} \frac{e_{12} - \omega_{12}}{\sqrt{1 + 2\epsilon_{22}}} &= \sin\alpha_2 \sin\beta_2 \\ \frac{1 + e_{22}}{\sqrt{1 + 2\epsilon_{22}}} &= \cos\alpha_2 \\ \frac{e_{23} + \omega_{23}}{\sqrt{1 + 2\epsilon_{22}}} &= \sin\alpha_2 \cos\beta_2 \end{aligned} \tag{2-15.8}$$

Then by Eqs. (2-15.3), (2-15.8), and the fourth of Eqs. 2-15.1, we find

$$\begin{aligned} \frac{2\epsilon_{12}}{\sqrt{1 + 2\epsilon_{11}}\sqrt{1 + 2\epsilon_{22}}} &= \cos\alpha_1 \sin\alpha_2 \sin\beta_2 + \cos\alpha_2 \sin\alpha_1 \cos\beta_1 \\ &\quad + \sin\alpha_1 \sin\alpha_2 \sin\beta_1 \cos\beta_2 \end{aligned} \tag{2-15.9}$$

If the strains (ϵ_{11}, ϵ_{22}) are small compared to 1, and if the angles (α_1, α_2) of rotation are small, Eq. (2-15.9) yields the approximation

$$2\epsilon_{12} \approx \alpha_2 \sin\beta_2 + \alpha_1 \cos\beta_1 + \alpha_1\alpha_2 \sin\beta_1 \cos\beta_2 \tag{2-15.10}$$

Also, Eqs. (2-15.3) and (2-15.8) yield

$$\begin{aligned} e_{12} + \omega_{12} &\approx \alpha_1 \cos\beta_1 \\ e_{12} - \omega_{12} &\approx \alpha_2 \sin\beta_2 \end{aligned} \tag{2-15.11}$$

Finally, Eqs. (2-15.10) and (2-15.11) yield, to second-degree terms in (α_1, α_2),

$$2(\epsilon_{12} - e_{12}) = \alpha_1\alpha_2 \sin\beta_1 \cos\beta_2$$

By an entirely similar argument, similar expressions for $2(\epsilon_{13} - e_{13})$, $2(\epsilon_{23} - e_{23})$ are obtained from the last two of Eqs. 2-15.1. Hence, we obtain, to second-degree terms in (α_1, α_2, α_3),

$$\begin{aligned} 2(\epsilon_{12} - e_{12}) &= \alpha_1\alpha_2 \sin\beta_1 \cos\beta_2 \\ 2(\epsilon_{13} - e_{13}) &= \alpha_1\alpha_3 \cos\beta_1 \sin\beta_3 \\ 2(\epsilon_{23} - e_{23}) &= \alpha_2\alpha_3 \sin\beta_2 \cos\beta_3 \end{aligned} \tag{2-15.12}$$

Equations (2-15.12) show that for small strains and small rotations (ϵ_{12}, ϵ_{13}, ϵ_{23}) differ from (e_{12}, e_{13}, e_{23}), respectively, by second-degree terms in the angles of rotation. Consequently, for sufficiently small strains and small rotations, the second-degree terms in (e_{11}, e_{12}, e_{13}) may be discarded in the expression for ϵ_{11} [Eqs. (2-15.1)], as these terms yield second-degree terms in ϵ_{11} and fourth-degree terms in (α_1, α_3). Hence,

$$\epsilon_{11} \approx e_{11} + e_{12}\omega_{12} + e_{13}\omega_{13} + \tfrac{1}{2}(\omega_{12}^2 + \omega_{13}^2)$$

Furthermore, substitution into this relation the expressions for (e_{12}, e_{13}) [see Eqs. (2-15.11) and (2-15.12)] yields square terms in the strains and cubic terms in the rotations. Hence, to second-degree terms, we obtain

$$\epsilon_{11} = e_{11} + \tfrac{1}{2}(\omega_{12}^2 + \omega_{13}^2)$$

By an analogous argument, similar expressions for (ϵ_{22}, ϵ_{33}, ϵ_{12}, ϵ_{13}, ϵ_{23}) are obtained. Thus, to this degree of approximation we find

$$\begin{aligned}
\epsilon_{11} &= e_{11} + \tfrac{1}{2}(\omega_{12}^2 + \omega_{13}^2) \\
\epsilon_{22} &= e_{22} + \tfrac{1}{2}(\omega_{21}^2 + \omega_{23}^2) \\
\epsilon_{33} &= e_{33} + \tfrac{1}{2}(\omega_{31}^2 + \omega_{32}^2) \\
2\epsilon_{12} &= 2e_{12} + \omega_{13}\omega_{23} \\
2\epsilon_{13} &= 2e_{13} + \omega_{12}\omega_{32} \\
2\epsilon_{23} &= 2e_{23} + \omega_{21}\omega_{31}
\end{aligned} \tag{2-15.13}$$

or

$$2\epsilon_{\alpha\beta} = 2e_{\alpha\beta} + \omega_{\alpha\theta}\omega_{\beta\theta} \qquad \alpha, \beta, \theta = 1, 2, 3 \tag{2-15.13a}$$

Finally, if the squares and the products of ω_{ij} may be neglected compared to the strains, we may discard the ω terms from Eqs. (2-15.13). Accordingly, if the strains and the angles of rotation are sufficiently small compared to 1, and if the rotations are sufficiently small compared to the strains, we may neglect all quadratic terms in Eqs. 2-15.1. This approximation is equivalent to discarding the quadratic terms in Eqs. (2-6.9) [or in Eqs. (2-6.10)]. Then we obtain the approximation

$$\begin{aligned}
\epsilon_{11} &= e_{11} = u_x = \frac{\partial u}{\partial x} = u_{1,1} \\
\epsilon_{22} &= e_{22} = v_y = \frac{\partial v}{\partial y} = u_{2,2} \\
\epsilon_{33} &= e_{33} = w_z = \frac{\partial w}{\partial z} = u_{3,3}
\end{aligned}$$

$$2\epsilon_{12} = 2e_{12} = v_x + u_y = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = u_{2,1} + u_{1,2} \tag{2-15.14}$$

$$2\epsilon_{13} = 2e_{13} = w_x + u_z = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = u_{3,1} + u_{1,3}$$

$$2\epsilon_{23} = 2e_{23} = w_y + v_z = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = u_{3,2} + u_{2,3}$$

or, in index notation,

$$2\epsilon_{\alpha\beta} = \left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) = u_{\alpha,\beta} + u_{\beta,\alpha} \tag{2-15.15}$$

Accordingly, Eqs. (2-15.14) or (2-15.15), which form the basis of classical small-displacement theory, imply that the strains and the angles of rotation are both small compared to 1. Moreover, they imply that quadratic terms in ω_{ij} may be neglected in comparison to linear terms in e_{ii} [see Eqs. 2-15.1 and (2-6.9)]. The latter condition is not satisfied for general displacements of flexible bodies, such as slender rods and thin plates. For example, if we roll up a large thin sheet into the shape of a cylinder, the strains are quite small, even though the displacements are very large. However, the angles of rotation of line elements in the sheet are very large. Hence, the rotational terms ω_{ij} cannot be discarded from Eqs. 2-15.1, and the approximations entailed in Eqs. (2-15.15) are no longer valid. In any case, however, Eqs. (2-15.15) are valid approximations, provided the displacement components are infinitesimally small.

Consequently, although the classical theory of small displacements is applicable to a large class of problems, it must be used with caution. It is applicable to massive bodies (thick bars and thick plates) but it may yield results grossly in error when applied to thin flexible bodies[14] (e.g., thin shells).

Aside from its importance in the classical small-displacement theory of solids, the theory of infinitesimally small deformation (displacement) finds applications in the theory of liquids and gases. For example, in considering the motion of a particle of fluid from the time t to the time $t + dt$, where dt is an infinitesimally small period of time, the particle undergoes an infinitesimally small displacement. Hence, a particle at point P at time t is at point $\mathscr{P}$ at time $t + dt$. The displacement vector $\mathbf{P}\mathscr{P}$ has the (x_1, x_2, x_3) projections

$$\dot{u}_1\, dt \qquad \dot{u}_2\, dt \qquad \dot{u}_3\, dt$$

where here $(\dot{u}_1, \dot{u}_2, \dot{u}_3)$ designate the (x_1, x_2, x_3) projections of the velocity vector of the particle. Hence, to apply the theory of small displacements to fluids, we may in many cases simply let the displacement components be given by the relations

$$u_1 = \dot{u}_1\, dt \qquad u_2 = \dot{u}_2\, dt \qquad u_3 = \dot{u}_3\, dt \tag{2-15.16}$$

[14]See *Thin-Walled Structures Journal*, ed. by J. Loughlan and K. P. Chong, Oxford, UK: Elsevier Science.

Simple Shear. A simple shear parallel to the (x_1, x_2) plane and along the x_1 axis is said to exist in a medium when the displacement function is of the form

$$u_1 = kx_2 \qquad u_2 = 0 \qquad u_3 = 0 \tag{a}$$

By Eqs. (2-15.15) and (a), we obtain

$$2\epsilon_{12} = 2e_{12} = k \tag{b}$$

Thus, comparing Eqs. (2-8.5) and (b), we see that k is the shearing strain between two line elements originally parallel to the x_1 and x_2 axes, respectively. It is the tangent of the angle ϕ through which a line parallel to the x_2 axis is turned by the deformation (see Fig. 2-15.1). More explicitly, for small strains and small rotations [Eq. (2-15.16)], k is equal to ϕ.

Dilatation. By Eq. (2-7.5) the dilatation at a point P in the direction of axis i is

$$e_i = \sqrt{1 + 2\epsilon_{ii}} - 1$$

Because ϵ_{ii} is small compared to 1, a binomial expansion of the radical yields, to first-degree terms,

$$e_i = \epsilon_{ii}$$

Accordingly, for the small-displacement theory the coefficients $(\epsilon_{11}, \epsilon_{22}, \epsilon_{33})$ are the coefficients of dilatations of elements parallel to coordinate axes (1, 2, 3).

Irrotational Strain. By Eqs. (2-14.10) and (2-14.11) we note that the terms $(1 + c_{11})x_1$, $(1 + c_{22})x_2$, $(1 + c_{33})x_3$ determine the part of a homogeneous deformation that is called a *pure strain*. Furthermore, by Eqs. (2-3.4), (2-5.3), and (2-14.10),

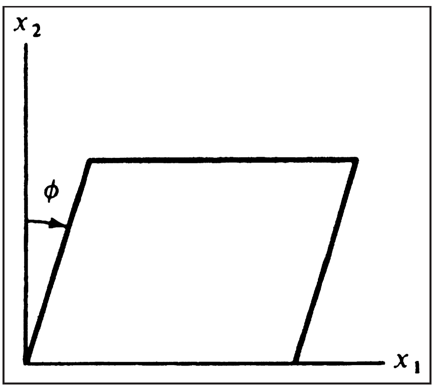

Figure 2-15.1

we have

$$\begin{aligned} 2\omega_{12} &= c_{21} - c_{12} \\ 2\omega_{31} &= c_{13} - c_{31} \\ 2\omega_{23} &= c_{32} - c_{23} \end{aligned} \tag{2-15.17}$$

It follows from the definition of the rotation of a volume element that for small strains ($e_{ij} \ll 1$) and small angles of rotation (ω_{ij} of the same order as e_{ij}), the rotation of a volume element is determined by the constants $c_{12}, c_{21}, \ldots$. Thus, a homogeneous deformation is characterized by a translation [c_{10}, c_{20}, c_{30}; see Eqs. (2-14.1)], a pure strain [Eq. (2-14.11)], and a rotation [Eq. (2-15.17)]. In the absence of translation, it follows that any homogeneous deformation such that $c_{21} = c_{12} = c_{13} = c_{31} = c_{32} = c_{23} = 0$, hence such that $\omega_{12} = 0$, $\omega_{13} = 0$, $\omega_{23} = 0$, is a *pure* strain. Because ω_{12}, ω_{31}, ω_{23} characterize a rotation, a pure strain is said to be irrotational.

Rigid-Body Displacements. The simplest type of displacement vector is $\mathbf{q} =$ constant. This type of motion defines a translation of a medium. A second group of rigid motions is that of rotations. These two groups (translations and rotations) combine to form the group of all rigid-body displacements, as any rigid displacement may be composed of a rotation about an axis in a fixed direction, plus a translation that depends on the choice of this axis (see Chasles's theorem, Section 2-2).

We now proceed to show that a rigid displacement is represented by a linear function of the coordinates. However, we shall see that it is not the most general linear functions, as certain coefficients must be restricted appropriately. To derive the displacement vector, we note that for infinitesimally small displacements a rigid-body displacement is characterized by the vanishing of the six strain components e_{ij} [see Eq. (2-15.14)].

Hence, in x, y, z notation,

$$\begin{gathered} \frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial w}{\partial z} = 0 \\ \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \qquad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \end{gathered} \tag{2-15.18}$$

Differentiating the last three of Eqs. (2-15.18) with respect to (x, y, z), respectively, we obtain

$$\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial z} = 0 \qquad \frac{\partial^2 u}{\partial z \partial y} + \frac{\partial^2 w}{\partial x \partial y} = 0 \qquad \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial z} = 0$$

Solving, we obtain

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial^2 v}{\partial z \partial x} = \frac{\partial^2 w}{\partial x \partial y} = 0$$

Hence, the equations

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial y \partial z} = 0$$

show that u is a linear sum of a function of y and a function of z. Accordingly,

$$u = f_1(y) + f_2(z)$$

Likewise,

$$v = f_3(x) + f_4(z)$$
$$w = f_5(x) + f_6(y)$$

By the equation $(\partial w/\partial y) + (\partial v/\partial z) = 0$, we have

$$f_6' + f_4' = 0$$

where primes denote derivatives. However, because f_6 is a function of y and f_4 is a function of z, the most general possibility is

$$f_6'(y) = -f_4'(z) = c_1 \tag{c}$$

where c_1 is a constant. Similarly we have

$$f_2'(z) = -f_5'(x) = c_2 \tag{d}$$

$$f_3'(x) = -f_1'(y) = c_3 \tag{e}$$

Integrating Eqs. (c), (d), and (e), and substituting the results into the expressions for (u, v, w), we obtain

$$\begin{aligned} u &= a_0 + c_2 z - c_3 y \\ v &= b_0 + c_3 x - c_1 z \\ w &= c_0 + c_1 y - c_2 x \end{aligned} \tag{2-15.19}$$

where (a_0, b_0, c_0) are constants of integration. Equations (2-15.19) define the *infinitesimal displacement components of a rigid-body motion*, the translation being (a_0, b_0, c_0) and the rotation (c_1, c_2, c_3).

Example 2-15.1. Comprehensive Analysis of Deformation. The displacement vector of a continuous medium is given by the components

$$u_1 = C(10x_1 + 3x_2) \qquad u_2 = C(3x_1 + 2x_2) \qquad u_3 = 6Cx_3$$

relative to rectangular Cartesian axes (x_1, x_2, x_3), where C is a constant. We wish to compute characteristic deformation quantities associated with this displacement.

First consider the admissibility of the displacement field. By Eq. (2-4.2) we find, with Eq. (a), $J = (1 + 6C)(1 + 11C)(1 + C) > 0$ for admissibility of the field. Examination of J as a function of C reveals that $J > 0$ for $-1 < C < -\frac{1}{6}$ and for $C > -\frac{1}{11}$; hence, the displacement field is proper and admissible for these values of C.

By Eq. (2-6.9) the associated strain components are

$$\begin{aligned} \epsilon_{11} &= 10C + 54.5C^2 \qquad \epsilon_{22} = 2C + 6.5C^2 \qquad \epsilon_{33} = 6C + 18C^2 \\ 2\epsilon_{12} &= 6C + 36C^2 \qquad 2\epsilon_{13} = 2\epsilon_{23} = 0 \end{aligned} \tag{b}$$

Equations (b) include the nonlinear effects related to C^2. If $C \ll 1$, the $\epsilon_{\alpha\beta}$ may be approximated by $e_{\alpha\beta}$ [see Eqs. (2-15.14)]. Then

$$\begin{aligned} \epsilon_{11} \approx e_{11} = 10C \qquad \epsilon_{22} \approx e_{22} = 2C \qquad \epsilon_{33} \approx e_{33} = 6C \\ 2\epsilon_{12} \approx 6C \qquad 2\epsilon_{13} \approx e_{13} = 0 \qquad 2\epsilon_{23} \approx e_{23} = 0 \end{aligned} \tag{c}$$

Also for $C \ll 1$, the volumetric strain e is approximated as follows [see Eqs. (2-12.2) and (2-12.5)]:

$$e = J_1 + 2J_2 + 4J_3 \approx J_1 \approx e_{11} + e_{22} + e_{33} = 18C \tag{d}$$

The principal values (strains) of the deformation are determined as the roots L_i of the determinant [Eq. (2-11.3)]

$$F(L) = \begin{vmatrix} 10C - L & 3C & 0 \\ 3C & 2C - L & 0 \\ 0 & 0 & 6C - L \end{vmatrix} = 0 \tag{e}$$

where the approximation $\epsilon_{\alpha\beta} \approx e_{\alpha\beta}$ has been used.

Expanding Eq. (e), we obtain

$$\begin{aligned} F(L) &= (6C - L)(L^2 - 12CL + 11C^2) \\ &= (6C - L)(L - 11C)(L - C) = 0 \end{aligned}$$

Thus, the roots are

$$L_1 = e_1 \approx \epsilon_1 = 11C \qquad L_2 = e_2 \approx \epsilon_2 = 6C \qquad L_3 = e_3 \approx \epsilon_3 = C \tag{f}$$

where we have ordered the principal strains such that $\epsilon_1 > \epsilon_2 > \epsilon_3$.

The principal axes (eigenvectors) associated with the principal strains may be computed by the theory presented in Section 2-11. Thus, for $\epsilon_{11} \approx 11C$, we must solve Eqs. (2-11.2) subject to the condition that the direction cosines (say, N_1, N_2, N_3) of the principal axes satisfy the relationship $N_1^2 + N_2^2 + N_3^2 = 1$. Hence, in Eq. (2-11.2) we let (a_{11}, a_{12}, a_{13}) be denoted by (N_1, N_2, N_3). Then the principal direction associated with ϵ_1 is obtained from the equations (with $L_1 = \epsilon_1 = 11C$)

$$\begin{aligned} (10C - 11C)N_1 + 3CN_2 &= 0 \\ 3CN_1 + (2C - 11C)N_2 &= 0 \\ (6C - 11C)N_3 &= 0 \\ N_1^2 + N_2^2 + N_3^2 &= 1 \end{aligned} \tag{g}$$

The first and third of Eqs. (g) yield $N_1 = 3N_2$ and $N_3 = 0$. Then by the last of Eqs. (g), $N_1^2 + N_2^2 = 9N_2^2 + N_2^2 = 10N_2^2 = 1$, or $N_2 = \pm 1/\sqrt{10}$; therefore, $N_1 = \pm 3/\sqrt{10}$. Thus, the principal axis is defined by the direction cosines:

$$N_1 = \pm\frac{3}{\sqrt{10}} \qquad N_2 = \pm\frac{1}{\sqrt{10}} \qquad N_3 = 0 \tag{h}$$

The remaining two principal axes may be determined similarly with $L_1 = 6C$ and $L_3 = C$, respectively. In each case, because of the direction cosine requirement [the last of Eqs. (g)], each direction cosine will involve a plus and minus sign. For example, in the case $L_1 = 11C$ we could arbitrarily select the signs $N_1 = +3\sqrt{10} = 3N_2$. Hence, $N_2 = +1/\sqrt{10}$. Alternatively, if we take $N_1 = -3/\sqrt{10}$, we have $N_2 = N_1/3 = -1\sqrt{10}$. We can proceed to arbitrarily select the sign for a second principal axis. However, the signs of the direction cosines of the third (remaining) principal axis are selected so that the axes associated with principal strains L_1, L_2, L_3 form a right-hand coordinate system (see Chapter 1, Section 1-24).

With the principal strains determined, the octahedral shearing strain may be computed by means of Eq. (2-12.9). Thus,

$$\begin{aligned} \Gamma = \gamma_{\text{oct}} &= \tfrac{2}{3}[(L_1 - L_2)^2 + (L_1 - L_3)^2 + (L_2 - L_3)^2]^{1/2} \\ &= \tfrac{2}{3}[5^2 + 10^2 + 5^2]^{1/2}C = 8.165C \end{aligned}$$

Problem Set 2-15

1. Let $\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_{12} = \epsilon_{23} = \epsilon_{13} = 0$. Neglecting quadratic terms in the strain–displacement relations, solve the resulting equations for displacement components (u_1, u_2, u_3); that is, derive the displacement components for a linearized rigid-body displacement.

2. Show that $\epsilon_{\alpha\beta} = (u_{\alpha,\beta} + u_{\beta,\alpha})/2, \alpha, \beta = 1, 2, 3$, represent the components of a second-order tensor.

3. Given that $\epsilon_{11} = A(L - x_1), \epsilon_{22} = B(L - x_1), \epsilon_{12} = 0, u_1 = u_1(x_1, x_2), u_2 = u_2(x_1, x_2), u_3 = 0$, A, B, and L are constants. Use linearized strain–displacement relations to determine displacement components (u_1, u_2) for the case $u_1(0, 0) = u_2(0, 0) = \boldsymbol{\omega}$ (0, 0) = 0. ($\boldsymbol{\omega}$ = rotation vector.)

4. For two-dimensional small-displacement theory, the strains in a body are given by

$$\epsilon_x = Cy(L - x) \qquad \epsilon_y = Dy(L - x) \qquad \gamma_{xy} = -(C + D)(A^2 - y^2)$$

where C, D, L, and A are known constants (see Problem 2-11.8). The boundary conditions are $u(0, 0) = 0$, $v(0, 0) = 0$, $(\partial u/\partial y)(0, 0) = 0$. Determine the displacement components u and v as functions of x and y.

5. The classical small-displacement theory of elasticity yields the following displacement components for a beam subjected to pure bending: $u_1 = -kx_1x_2$, $u_2 = k_2(x_1^2 + \gamma x_2^2 - \gamma x_3^2)$, $u_3 = k_3\gamma x_2 x_3$. Compute the rotations of a volume element in the beam with respect to (x_1, x_2, x_3) axes, respectively.

6. The strains of a deformed body are given by the relations

$$\epsilon_x = \nu C(l - z) \qquad \epsilon_y = \nu C(l - z) \qquad \epsilon_z = -C(l - z) \qquad \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0$$

where ν, C, and l are constants. Assuming that strains and rotations are infinitesimally small, derive formulas for the displacement components (u, v, w) in the (x, y, z) directions, respectively. The boundary conditions are:

$$\text{For } (0, 0, 0),\ u = v = w = 0$$

$$\text{For } (0, 0, l),\ \omega_{23} = 0, \omega_{31} = 0 \qquad \omega_{12} = A$$

where A is a constant. Specialize the results for $A = 0$.

7. For a problem in small-displacement theory, the strain components are given by $\epsilon_x = A(x - z)$, $\epsilon_y = A(x - y)$, $\epsilon_z = A(y + z)$, $\gamma_{xy} = \gamma_{yz} = \gamma_{xz} = 0$, where $A =$ constant. Determine the (x, y, z) displacement components (u, v, w) in terms of (x, y, z), where $u = v = w = \omega_{23} = \omega_{31} = \omega_{12} = 0$, for $x = y = z = 0$.

8. Let the displacement u_α be defined by the equations $u_\alpha = C_{\alpha\beta}x_\beta$, where $C_{\alpha\beta}$ are constants and x_β denotes rectangular Cartesian coordinates. Is it possible to select the $C_{\alpha\beta}$ so that the components $\epsilon_{\alpha\beta}$ of the strain tensor consist only of quadratic terms in $C_{\alpha\beta}$? Explain. Assuming that it is possible, discuss the significance of this result in the process of approximating the strain components $\epsilon_{\alpha\beta}$ by their small-displacement approximations $e_{\alpha\beta}$.

9. Consider the following displacement field (u, v, w) relative to material (Lagrangian) coordinates (x, y, z):

$$u = -(1 - \cos\phi)x - y\sin\phi$$

$$v = x\sin\phi + (1 - \cos\phi)y$$

$$w = 0$$

where ϕ is a constant.

(a) Compute the small-displacement approximations $e_{\alpha\beta}$ of the strain tensor components $\epsilon_{\alpha\beta}$, and describe the associated deformation characterized by (u, v, w).

(b) Compute the strain tensor components $\epsilon_{\alpha\beta}$, and describe the associated deformation characterized by (u, v, w).

(c) Discuss the significance of the results obtained in parts (a) and (b).

10. A rectangular region (Fig. P2-15.10*a*) is deformed into a parallelogram region (Fig. P2-15.10*b*) under displacement $(u, v, 0)$ relative to (x, y, z) axes.

(a) Write expressions for u, v, as functions of x, y.

(b) Calculate the components $\epsilon_{\alpha\beta}$ of the strain tensor.

(c) Determine the initial direction cosines of the line element that in the deformed region lies parallel to the y axis.

(d) Determine the volumetric strain.

(e) Determine the volumetric rotation.

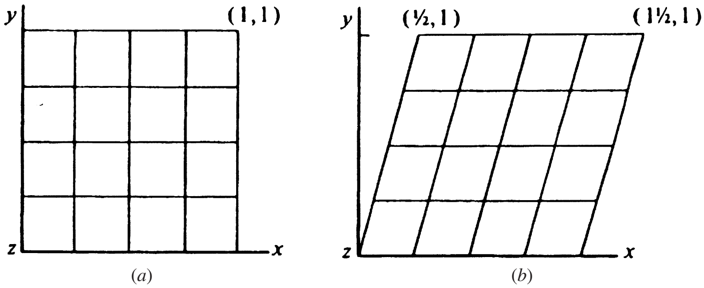

Figure P2-15.10

2-16 Compatibility Conditions of the Classical Theory of Small Displacements

The *six* strain components ϵ_{ij}, $i, j = 1, 2, 3$, cannot be given arbitrarily as functions of x_i, as they are determined completely by the three displacement components u_i. Hence, there must exist relations between the strain components because they are not independent functions. To obtain these relations, we eliminate (u_1, u_2, u_3) from the strain–displacement equations. We restrict ourselves to the case of small displacement and to simply connected regions.[15] Hence, for small displacements [Eqs. (2-15.14)], taking the second derivatives of ϵ_{11}, ϵ_{22}, and $2\epsilon_{12}$ with respect to (x_2, x_1) and (x_1, x_2), respectively, we obtain

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} \qquad \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}$$

$$2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2}$$

where we have used the notation $x = x_1$, $y = x_2$, $z = x_3$. Adding the first two of these equations and equating the results to the third equation, we obtain

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2} \qquad \text{(2-16.1a)}$$

[15]Region R is simply connected if for every closed curve scribed in R, the curve can be shrunk to a point without cutting region R apart or taking the curve outside R. Otherwise, for multiply connected regions (regions that are not simply connected), additional requirements as to the single-valued nature of u must be met.

Because Eqs. (2-15.15) are cyclically permutable in (x_1, x_2, x_3), the permutations $(1, 2) \rightarrow (1, 3)$ and $(1, 3) \rightarrow (2, 3)$ in Eq. (2-16.1a) yield

$$\frac{\partial^2 \epsilon_{11}}{\partial x_3^2} + \frac{\partial^2 \epsilon_{33}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{13}}{\partial x_1 \partial x_3} \tag{2-16.1b}$$

$$\frac{\partial^2 \epsilon_{22}}{\partial x_3^2} + \frac{\partial^2 \epsilon_{33}}{\partial x_2^2} = 2\frac{\partial^2 \epsilon_{23}}{\partial x_2 \partial x_3} \tag{2-16.1c}$$

Similarly, we also obtain from Eqs. (2-15.14)

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2\, \partial x_3} + \frac{\partial^2 \epsilon_{23}}{\partial x_1^2} = \frac{\partial^2 \epsilon_{13}}{\partial x_1\, \partial x_2} + \frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_3} \tag{2-16.1d}$$

$$\frac{\partial^2 \epsilon_{22}}{\partial x_1\, \partial x_3} + \frac{\partial^2 \epsilon_{13}}{\partial x_2^2} = \frac{\partial^2 \epsilon_{23}}{\partial x_1\, \partial x_2} + \frac{\partial^2 \epsilon_{12}}{\partial x_2\, \partial x_3} \tag{2-16.1e}$$

$$\frac{\partial^2 \epsilon_{33}}{\partial x_1\, \partial x_2} + \frac{\partial^2 \epsilon_{12}}{\partial x_3^2} = \frac{\partial^2 \epsilon_{13}}{\partial x_2\, \partial x_3} + \frac{\partial^2 \epsilon_{23}}{\partial x_1\, \partial x_3} \tag{2-16.1f}$$

In index notation, Eqs. (2-16.1) may be written concisely in the form

$$\frac{\partial^2 \epsilon_{mn}}{\partial x_i\, \partial x_j} + \frac{\partial^2 \epsilon_{ij}}{\partial x_m\, \partial x_n} = \frac{\partial^2 \epsilon_{im}}{\partial x_j\, \partial x_n} + \frac{\partial^2 \epsilon_{jn}}{\partial x_i\, \partial x_m} \tag{2-16.2}$$

or

$$\epsilon_{mn,ij} + \epsilon_{ij,mn} = \epsilon_{im,jn} + \epsilon_{jn,im}$$

where $i, j, m, n = 1, 2, 3$. Although many of Eqs. (2-16.2) are redundant, Eqs. (2-16.1) are contained in Eqs. (2-16.2).

The differential relations given in Eqs. (2-16.1) or (2-16.2) are called the *conditions of compatibility*. The above demonstration proves the *necessity* of the conditions of compatibility to ensure the existence of functions (u, v, w) related to ϵ_{ij} by Eqs. (2-15.14). Various proofs have been given that they are also *sufficient*. The simplest of these proofs introduces the components of rotation ω_{ij} [see Eqs. (2-5.3)].

Thus, for small displacements all the first differential coefficients of (u_1, u_2, u_3) may be expressed in terms of the nine quantities ω_{ij}, ϵ_{ij}. For example,

$$\frac{\partial u_1}{\partial x_1} = \epsilon_{11} \qquad \frac{\partial u_1}{\partial u_2} = \epsilon_{12} - \omega_{12} \qquad \frac{\partial u_1}{\partial x_3} = \epsilon_{13} - \omega_{13}$$

The conditions of compatibility of the nine equations that express the first derivatives of (u_1, u_2, u_3) with respect to x_i yield six equations of the type

$$\frac{\partial \epsilon_{11}}{\partial x_2} = \frac{\partial \epsilon_{12}}{\partial x_1} - \frac{\partial \omega_{12}}{\partial x_1}$$

and three equations of the type

$$\frac{\partial \epsilon_{13}}{\partial x_2} - \frac{\partial \epsilon_{12}}{\partial x_3} = \frac{\partial \omega_{13}}{\partial x_2} - \frac{\partial \omega_{12}}{\partial x_3} = \frac{\partial \omega_{23}}{\partial x_1}$$

All the first differential coefficients of ω_{ij} can thus be expressed in terms of the first differential coefficients of ϵ_{ij}. For example,

$$\frac{\partial \omega_{23}}{\partial x_1} = \frac{\partial \epsilon_{13}}{\partial x_2} - \frac{\partial \epsilon_{12}}{\partial x_3} \qquad \frac{\partial \omega_{23}}{\partial x_2} = \frac{\partial \epsilon_{23}}{\partial x_2} - \frac{\partial \epsilon_{22}}{\partial x_3}$$

$$\frac{\partial \omega_{23}}{\partial x_3} = \frac{\partial \epsilon_{33}}{\partial x_2} - \frac{\partial \epsilon_{23}}{\partial x_3}, \ldots,$$

The conditions of compatibility of these nine equations are the six equations of Eqs. (2-16.1).

Alternative Derivation of Compatibility Relations. The strain compatibility relations may be derived by an alternative technique based upon the assumed single-valued continuous nature of the displacement u_α. From this viewpoint, the compatibility relations are requirements that the deformation occur without discontinuities (Section 2-1). Accordingly the compatibility relations are sometimes called *equations of continuity* or *the conditions of continuity*.

For small-displacement theory, the linear parts of the strain components are [Eqs. (2-15.14)]

$$e_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{a}$$

Recall that specification of $e_{\alpha\beta}$ does not uniquely determine u_α, as u_α may involve a rigid-body displacement that does not affect $e_{\alpha\beta}$. In other words, in general we may write

$$u_\alpha = u_\alpha^{(d)} + u_\alpha^{(r)} \tag{b}$$

where $u_\alpha^{(d)}$ denotes the deformational part of u_α and $u_\alpha^{(r)}$ denotes the rigid-body displacement part of u. Furthermore, we may write

$$u_\alpha^{(r)} = u_\alpha^{(T)} + u_\alpha^{(R)} \tag{c}$$

where $u_\alpha^{(T)}$ denotes a rigid-body translation and $u_\alpha^{(R)}$ denotes a rigid-body rotation (Section 2-2).

We seek the requirement (restrictions) on $e_{\alpha\beta}$ for a continuous single-valued displacement u_α. Accordingly, consider the deformation of region R into region $\mathscr{R}$

(Fig. 2-16.1). We consider u_α known. Thus, $u_\alpha(P), \ldots, u_\alpha(B), \ldots, u_\alpha(Q)$ represent the displacements of points $P, \ldots, B, \ldots, Q$ of curve C in region R. Under the displacement, C is transformed into curve $\mathscr{C}$ in region $\mathscr{R}$. The requirements of single-valued continuous displacement field u_α imply that $u_\alpha(Q)$ may be obtained by integration of u_α from any point, say, P, along any curve C between points P and Q.

Accordingly,

$$u_\alpha(Q) = u_\alpha(P) + \int_P^Q du_\alpha \tag{d}$$

Because $u_\alpha = u_\alpha(x_1, x_2, x_3)$, $du_\alpha = u_{\alpha,\beta}\, dx_\beta$, therefore

$$u_\alpha(Q) = u_\alpha(P) + \int_P^Q u_{\alpha,\beta} dx_\beta \tag{e}$$

Employing the notation of Eq. (2-5.7), we may write

$$u_\alpha(Q) = u_\alpha(P) + \int_P^Q e_{\alpha\beta} dx_\beta + \int_P^Q \omega_{\beta\alpha} dx_\beta \tag{f}$$

Integration by parts of the second integral yields

$$\begin{aligned}\int_P^Q \omega_{\beta\alpha} dx_\beta &= \int_P^Q \omega_{\beta\alpha} d[x_\beta - x_\beta(Q)] \\ &= [x_\beta(Q) - x_\beta(P)]\omega_{\beta\alpha}(P) - \int_P^Q [x_\beta - x_\beta(Q)]\omega_{\beta\alpha,\gamma} dx_\gamma \end{aligned} \tag{g}$$

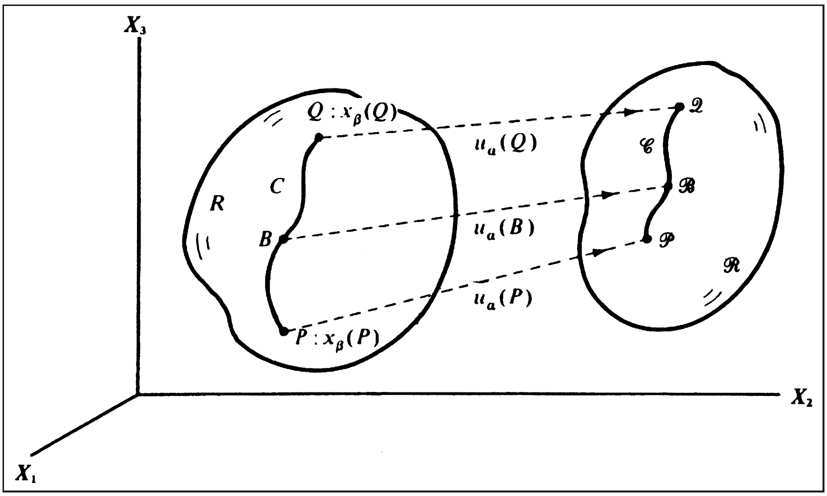

Figure 2-16.1

where the transformation $dx_\beta = d[x_\beta - x_\beta(Q)]$ allows the representation of the integral in terms of $x_\beta(Q)$, $x_\beta(P)$, $\omega_{\beta\alpha}(P)$ and a line integral from P to Q. Hence, Eqs. (f) and (g) yield

$$u_\alpha(Q) = u_\alpha(P) + [x_\beta(Q) - x_\beta(P)]\omega_{\beta\alpha}(P) + \int_P^Q \{e_{\alpha\gamma} - [x_\beta - x_\beta(Q)]\omega_{\beta\alpha,\gamma}\}dx_\gamma \tag{h}$$

For u_α to be continuous and single valued, the value $u_\alpha(Q)$ obtained by the integration process [Eq. (h)] must be the same no matter along which path we integrate from P to Q. That is, the line integral $\int_P^Q$ must be path independent. Before explicitly noting necessary and sufficient conditions that the line integral be path independent, we note the relation

$$\omega_{\beta\alpha,\gamma} = \frac{\partial}{\partial x_\gamma}\left[\frac{1}{2}(u_{\alpha,\beta} - u_{\beta,\alpha})\right] = e_{\alpha\gamma,\beta} - e_{\beta\gamma,\alpha} \tag{i}$$

Hence, substitution of Eq. (i) into Eq. (h) yields

$$u_\alpha(Q) = u_\alpha(P) + [x_\beta(Q) - x_\beta(P)]\omega_{\beta\alpha}(P) + \int_P^Q R_{\alpha\gamma}dx_\gamma \tag{j}$$

where

$$R_{\alpha\gamma} = e_{\alpha\gamma} - [x_\beta - x_\beta(Q)](e_{\gamma\alpha,\beta} - e_{\beta\gamma,\alpha}) \tag{k}$$

For $u_\alpha(Q)$ to be independent of path from P to Q, $R_{\alpha\gamma}\, dx_\gamma = dF_\alpha$ must be an exact differential such that $\int_P^Q dF_\alpha = F_\alpha(Q) - F_\alpha(P)$. Necessary and sufficient conditions for the exact differential dF_α to exist are (see Chapter 1, Section 1-19)

$$\frac{\partial R_{\alpha\delta}}{\partial x_\theta} - \frac{\partial R_{\alpha\theta}}{\partial x_\delta} = 0 \tag{2-16.3}$$

Substitution of Eq. (k) into Eq. (2-16.3) yields

$$e_{ij,kl} + e_{kl,ij} = e_{kj,il} + e_{il,kj} \qquad i,\ j,\ k,\ l = 1,\ 2,\ 3 \tag{2-16.4}$$

Hence, Eqs. (6-16.4) consist of $3^4 = 81$ equations. Because some of these equations are satisfied identically, and some are repeated as a result of the symmetry of indices *ij* and *kl*, the 81 equations may be shown to reduce to the six equations, Eqs. (2-16.1).

Example 2-16.1. Compatibility of Strain in a Region. The small-displacement components of the strain tensor in a region are given by

$$\begin{array}{lll} e_{11} = x_2^2 & e_{22} = x_1^2 & e_{33} = x_2^2 \\ e_{12} = 2x_1x_2 & e_{23} = x_2x_3 & e_{31} = x_1 + x_3 \end{array} \tag{a}$$

We wish to determine if these components are compatible for all values of rectangular Cartesian coordinates (x_1, x_2, x_3).

By Eqs. (a) and (2-15.14), Eqs. (2-16.1) yield

$$\begin{array}{ccc} 2+2=4 & 0+0=0 & 0+2=2 \\ 0+0=0+0 & 0+0=0+0 & 0+0=0+0 \end{array}$$

Hence, compatibility of strain is ensured for all x_i.

Example 2-16.2. Compatibility of Strain at a Point. For a plane strain state in the (x_1, x_2) plane, the small-displacement strain components about a region surrounding point $(x_1, x_2) = (0, 0)$ were calculated to be

$$e_{11} = x_1 x_2^2 \qquad e_{12} = x_1^3 + x_2^3 \qquad e_{22} = x_1^2 x_2 \tag{a}$$

Determine whether compatibility is satisfied (a) at the point (0, 0) and (b) for all values of (x_1, x_2).

By Eqs. (2-16.1) we obtain, with Eq. (a), the single condition $2x_1 + 2x_2 = 0$. For $x_1 = x_2 = 0$, we get $0 + 0 = 0$. Hence, compatibility is satisfied at this point. However, it cannot be satisfied for all points in the region but only for those points for which $x_1 = -x_2$. Accordingly, Eqs. (a) are generally incompatible, as the compatibility conditions must be satisfied throughout the region in question.

Problem Set 2-16

1. For small-displacement theory the displacement components (u, v, w) relative to rectangular Cartesian axes (x, y, z) are $u_1 = ax_2x_3$, $u_2 = bx_3x_1$, $u_3 = cx_1x_2$, where a, b, c are constants. Show that the deformation is compatible.

2. For the two-dimensional small-displacement theory, the strains for a cantilever beam of length L and depth $2a$, subject to a concentrated lateral load P at the free end, are given by

$$\epsilon_{11} = Ax_1x_2 \qquad \epsilon_{22} = -\nu Ax_1x_2 \qquad 2\epsilon_{12} = A(1+\nu)(a^2 - x_2^2)$$

where A, a, ν are constants and (x_1, x_2) denote plane rectangular Cartesian coordinates with origin at the free end of the beam, with x_1 axis coinciding with the central longitudinal axis of the beam and with axis x_2 coincident with the force P. Assume that the displacement (u_1, u_2) relative to axis (x_1, x_2) are functions of (x_1, x_2).

(a) Show that continuous single-valued displacements (u_1, u_2) are possible. Are they compatible?

(b) Hence, if possible, derive formulas for (u_1, u_2) as explicit functions of (x_1, x_2), with the conditions $u_1 = u_2 = u_{1,2} = 0$ for $x_1 = L, x_2 = 0$.

3. For a two-dimensional small-displacement problem, the strain components for a continuum are determined to be [see Eq. (2-8.6)]

$$\epsilon_x = Axy \qquad \epsilon_y = Bxy \qquad \gamma_{xy} = Cy^2$$

where A, B, C are known constants (see Problem 2-11.8). Determine the (x, y) displacement components (u, v) as functions of (x, y). The boundary conditions are

$$u(0,0) = 0 \qquad v(0,0) = 0 \qquad u_y(0,0) = D$$

where D is a known constant.

4. In a region about the origin of a Cartesian coordinate system (x_1, x_2, x_3), the small strain components are computed to be

$$e_{11} = x_1^2 \qquad e_{12} = x_1 x_2 \qquad e_{13} = x_1 x_3 \qquad e_{22} = x_2^2$$
$$e_{23} = x_2 x_3 \qquad e_{33} = x_3^2$$

Determine whether they are compatible in this region.

5. For a plane strain problem in the (x_1, x_2) plane, the strain components are given as

$$e_{11} = \cos(x_2/L) \qquad e_{12} = 0 \qquad e_{22} = \cos(x_1/L)$$

where L is a constant. Consider the compatibility of these strain components.

2-17 Additional Conditions Imposed by Continuity

In Section 2-4 certain properties of a physical deformation were discussed. In this section we consider several additional properties that are of importance in a complete description of the deformation of continuous media.

Transformations of Lines and Surfaces. Let a point P on a line L be denoted by the continuous functions

$$x_1 = x_1(s) \qquad x_2 = x_2(s) \qquad x_3 = x_3(s) \tag{2-17.1}$$

where s is some parameter, say, arc length. As s varies from a to b, we obtain successively all points on L; for $s = a$, we obtain one end; for $s = b$, we obtain the other end.

Letting time $t =$ constant and substituting Eqs. (2-17.1) into Eqs. (2-3.1), we obtain (ξ_1, ξ_2, ξ_3) as continuous functions of s. As s varies from a to b, the point $\mathscr{P}$: (ξ_1, ξ_2, ξ_3) passes through all points of the line $\mathscr{L}$ in the deformed medium; the two ends of line $\mathscr{L}$ correspond to $s = a$ and $s = b$, respectively. Hence, there is a one-to-one correspondence of points on lines L and $\mathscr{L}$. Thus, we have the following theorem:

Theorem 2-17.1. *If a set of points forms a continuous line L in the undeformed medium R, at any later instant t this set of points forms a continuous line* $\mathscr{L}$ *in the deformed medium* $\mathscr{R}$, *there being a one-to-one correspondence between points on L and* $\mathscr{L}$.

An immediate corollary to Theorem 2-17.1 follows:

Corollary 2-17.1. *If line L is closed in region R, the corresponding line* $\mathscr{L}$ *is closed in region* $\mathscr{R}$.

This corollary follows from Theorem 2-17.1 simply by letting the end points of line L, corresponding to $s = a$ and $s = b$, coincide. Hence, the end points of line $\mathscr{L}$ also coincide, and $\mathscr{L}$ is closed.

In a similar fashion, the following theorem and corollary may be proved:

Theorem 2-17.2. *If a set of points form a continuous surface S in the region R of an underformed medium, it also forms a continuous surface* $\mathscr{S}$ *in the region* $\mathscr{R}$ *of the deformed medium, there being a one-to-one correspondence between points in S and* $\mathscr{S}$.

Corollary 2-17.2. *If the surface S is closed, the surface* $\mathscr{S}$ *is also closed. Furthermore, there is a one-to-one correspondence between points interior to surface S and points interior to surface* $\mathscr{S}$*, and conversely.*

The proofs of Theorem 2-17.2 and Corollary 2-17.2 are left to the reader.

Material Form of the Continuity Equation. Consider an infinitesimal volume dV enclosing particle P: (x, y, z) of an initially undeformed medium. Let S be the surface containing dV. Consider the deformation of the medium during some time interval t_1. During time t_1, the volume dV is transformed into the volume $d\mathscr{V}$ enclosed by the surface $\mathscr{S}$. By Theorem 2-17.2, the surface $\mathscr{S}$ is a pointwise transformation of surface S. Furthermore, a particle that was initially enclosed in surface S is now enclosed in surface $\mathscr{S}$ (Corollary 2-17.2). Consequently, it follows that the mass enclosed in surface S is enclosed in surface $\mathscr{S}$ after the deformation; that is, mass is conserved in the transformation from S to $\mathscr{S}$.

Let ρ be the initial mass density of the medium enclosed in S. Let ρ^* be the mass density of the medium enclosed in $\mathscr{S}$. Hence, the conservation of mass of the volume element $dV = dx_1 dx_2 dx_3$ is expressed by the equation

$$\iiint \rho \; dx_1 \; dx_2 \; dx_3 = \iiint \rho^* \; d\xi_1 \; d\xi_2 \; d\xi_3 \tag{2-17.2}$$

where the asterisk denotes the deformed state.

By the theory of transformation of multiple integrals (Courant, 1992), the integral on the right side of Eq. (2-17.2) may be transformed into an integral in $dxdydz$; that is,

$$\iiint \rho^* \; d\xi_1 \; d\xi_2 \; d\xi_3 = \iiint \rho^* J \; dx_1 \; dx_2 \; dx_3 \tag{2-17.3}$$

where

$$J = \frac{\partial(\xi_2, \xi_2, \xi_3)}{\partial(x_1, x_2, x_3)}$$

is the functional determinant or Jacobian of (ξ_1, ξ_2, ξ_3) with respect to (x_1, x_2, x_3) [see Eqs. (2-4.1) or (2-4.2)].

Substituting Eq. (2-17.3) into Eq. (2-17.2), we obtain

$$\iiint \rho dx_1 dx_2 dx_3 = \iiint \rho^* J dx_1 dx_2 dx_3$$

or

$$\iiint (\rho^* J - \rho)\ dx_1\ dx_2\ dx_3 = 0 \qquad (2\text{-}17.4)$$

Because Eq. (2-17.4) applies for any infinitesimal volume $dV = dx_1 dx_2 dx_3$ in the medium, the integrand must vanish identically. Hence, it follows that

$$J = \frac{\rho}{\rho^*} \qquad (2\text{-}17.5)$$

Equation (2-17.5) is called the Lagrangian (or *material*) form of the *equation of continuity* of the medium. It expresses the conservation of mass in the volume element dV before and after the deformation of the medium. Because ρ^* is taken to be a continuous positive function of time t, J, which is a function of (x_1, x_2, x_3), varies with time. For $t = 0$, $\rho = \rho^*$, and $J = 1$. Hence, if ρ^* is an increasing function of time, $J < 1$. If ρ^* is a decreasing function of time, $J > 1$. For an incompressible medium $\rho^* = \rho$ for all time. Consequently, for an incompressible fluid, $J = 1$. Furthermore, as both ρ and ρ^* are always positive, $J > 0$. Thus, by the above argument, we have again arrived at the conclusion expressed in Theorem 2-4.1.

2-18 Kinematics of Deformable Media

In our previous discussion we treated deformation as a purely geometric concept; that is, we considered deformation that occurred during some small, fixed time interval. Consequently, concepts of velocity and acceleration of a particle did not enter into our formulation. In the following discussion we extend our treatment of deformation to encompass the time rate of deformation. However, as the kinematics of continuous media is an extensive subject, we consider only a few basic results that are required for our immediate needs. Within the range of validity of the continuous media approximation of physical matter, these results are applicable to rigid solids, elastic solids, nonelastic solids, and real or viscous fluids.

The basic displacement–time relation for continuous media is given by Eq. (2-3.1):

$$x^* = x^*(x, y, z; t) \qquad y^* = y^*(x, y, z; t) \qquad z^* = z^*(x, y, z; t) \qquad (2\text{-}3.1)$$

where

$$\xi_1 = x^* \qquad \xi_2 = y^* \qquad \xi_3 = z^* \qquad x_1 = x \qquad x_2 = y \qquad x_3 = z$$

To simplify the subsequent equations, we employ the following notations:

$$\begin{array}{lll} x = x_0 & y = y_0 & z = z_0 \\ x^* = x & y^* = y & z^* = z \end{array} \tag{a}$$

With the notation of Eqs. (a), Eqs. 2-3.1 are interpreted as follows. A particle of mass dm that is at point (x, y, z) at time t is located at point (x_0, y_0, z_0) at time $t = 0$. Thus, as noted in Section 2-3, the deformation process may be described in terms of either $(x_0, y_0, z_0; t)$ (material variables) or $(x, y, z; t)$ (spatial variables).

In the material point of view, we focus our attention on a particular particle initially at (x_0, y_0, z_0), and we follow its motion in space. In the spatial point of view, we focus our attention on a particular geometrical region fixed in space; that is, we observe changes that take place in this fixed region or space.

Velocity and Acceleration. Let a particle travel along a curved path C in space (Fig. 2-18.1). The velocity $\mathbf{v}$ of the particle at point (x, y, z) on its path is defined as

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{s}}{\Delta t} = \frac{d\mathbf{s}}{dt} \tag{2-18.1}$$

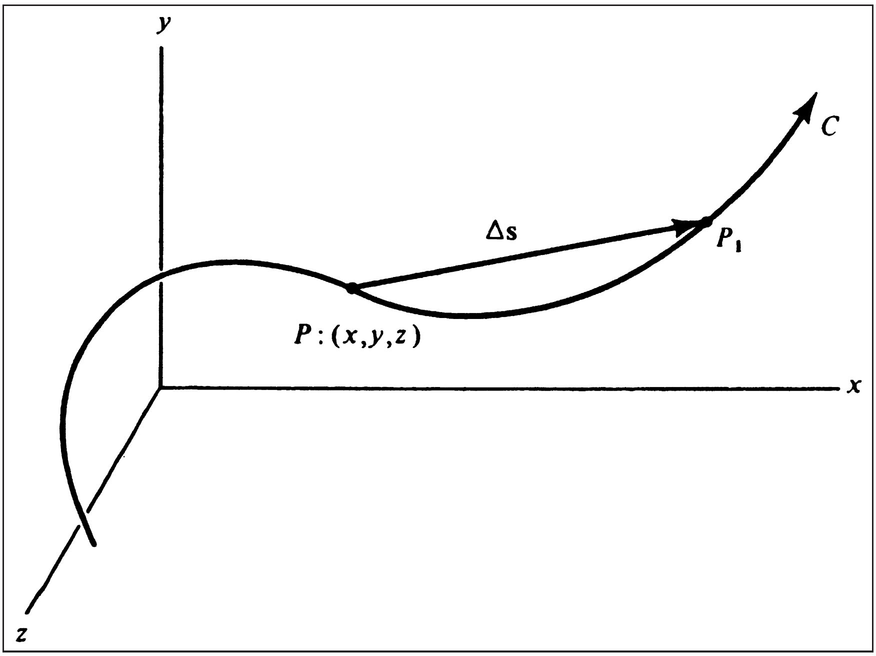

Figure 2-18.1

where $\Delta\mathbf{s}$ denotes the displacement vector between two different positions P, P_1 of the particle, and Δt denotes the time required for the particle to travel from P to P_1.

As $\Delta t \to 0$, $P_1 \to P$. Consequently, the direction of $\mathbf{v}$ coincides with the direction of the tangent to C at point P. Hence, the velocity vector $\mathbf{v}$ may be represented by three scalar equations:

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt} \qquad w = \frac{dz}{dt} \tag{2-18.2}$$

where $x = x(t)$, $y = y(t)$, $z = z(t)$. In Eq. (2-18.2), (u, v, w) denote the (x, y, z) projections of the *velocity vector* $\mathbf{v}$, *in contrast to their designation previously as projections of the displacement vector* (see Section 2-3).

Furthermore, the acceleration vector $\mathbf{a}$ is defined as (Boresi and Schmidt, 2000)

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} \tag{2-18.3}$$

In terms of (x, y, z) projections, the acceleration $\mathbf{a}$ is

$$a_x = \frac{du}{dt} = \frac{d^2x}{dt^2} \qquad a_y = \frac{dv}{dt} = \frac{d^2y}{dt^2} \qquad a_z = \frac{dw}{dt} = \frac{d^2z}{dt^2} \tag{2-18.4}$$

where (x, y, z) subscripts on $\mathbf{a}$ denote (x, y, z) projections, respectively.

Accordingly, it follows that when (u, v, w) are considered as functions of $(x_0, y_0, z_0; t)$, Eqs. (2-18.1) to (2-18.4) are the expressions for the velocity and the acceleration of a particle in the material point of view. To distinguish between the cases $u = u(x_0, y_0, z_0; t)$ and $u = u(x, y, z; t)$, we denote differentials of functions of the variables $(x_0, y_0, z_o; t)$ by an ordinary[16] d, whereas we denote differentials of functions of the variables $(x, y, z; t)$ by the symbol ∂. As noted in Eqs. (2-18.2), $dx/dt = u$ is the x projection of the velocity vector. However, $\partial x/\partial t = 0$, because here x denotes the x coordinate of a point fixed in space. Likewise, du/dt is the x projection of the acceleration vector, whereas $\partial u/\partial t$ denotes the difference in the x projections of the velocities of the two particles that coincide with the geometrical point (x, y, z) at times t and $t + dt$, respectively. Hence, for steady-state flow of water over a weir, $\partial u/\partial t = 0$ everywhere; however, $du/dt \neq 0$ (Fig. 2-18.2).

Regarding (u, v, w) as functions of $(x, y, z; t)$, the total differentials du, dv dw are

$$\begin{aligned}
du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz + \frac{\partial u}{\partial t}dt \\
dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz + \frac{\partial v}{\partial t}dt \\
dw &= \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz + \frac{\partial w}{\partial t}dt
\end{aligned} \tag{2-18.5}$$

[16]Some authors use the symbol D. See Chapter 3, Section 3A-2.

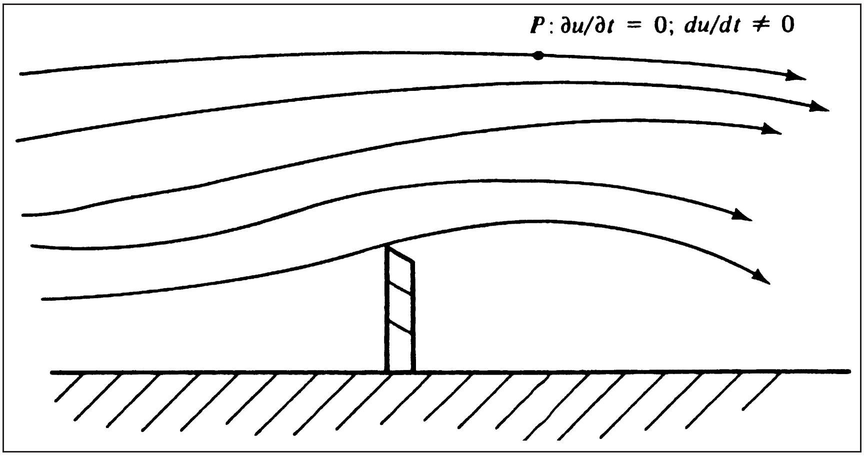

Figure 2-18.2

Dividing Eqs. (2-18.5) by dt, we obtain [with Eqs. (2-18.2) and (2-18.4)]

$$\begin{aligned}
\frac{du}{dt} &= a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \\
\frac{dv}{dt} &= a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \\
\frac{dw}{dt} &= a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}
\end{aligned} \tag{2-18.6}$$

or

$$\begin{aligned}
\frac{du}{dt} - \frac{\partial u}{\partial t} &= u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \\
\frac{dv}{dt} - \frac{\partial v}{\partial t} &= u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \\
\frac{dw}{dt} - \frac{\partial w}{\partial t} &= u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}
\end{aligned} \tag{2-18.7}$$

The terms on the right side of Eqs. (2-18.7) (i.e., the differences between the material acceleration $d\mathbf{v}/dt$ and the local acceleration $\partial\mathbf{v}/\partial t$) are called the *convective terms of the acceleration*.

In general, let $Q(x, y, z; t)$ be any scalar function (such as density, temperature, pressure, a projection of the velocity vector, etc.). With the earth as a reference frame, $\partial Q/\partial t$ denotes the rate of change of Q with the respect to t at a geometrical point (x, y, z) that is fixed with respect to the earth. If the process is a steady-state process, $\partial Q/\partial t = 0$; for example, a steady-state flow is characterized by the condition that the function Q remain constant for all times at point (x, y, z). Accordingly, for steady-state conditions all partial derivatives with respect to time vanish.

From the Lagrangian point of view, for a fixed (x_0, y_0, z_0), the point (x, y, z) is a function of time [see (Eqs. 2-3.1) and Eqs. (a)]. Accordingly, for a given particle $P: (x, y, z)$ of a medium, Q is a function of time t alone. Hence, the total derivative dQ/dt denotes the rate of change of Q with respect to t as we follow the particle along its path. Hence, analogous to Eqs. (2-18.6), we have

$$\frac{dQ}{dt} = u\frac{\partial Q}{\partial x} + v\frac{\partial Q}{\partial y} + w\frac{\partial Q}{\partial z} + \frac{\partial Q}{\partial t} \tag{2-18.8}$$

Equation (2-18.8) relates the material derivative dQ/dt to the local derivative $\partial Q/\partial t$.

In vector notation, Eq. (2-18.8) takes the form

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \mathbf{v} \cdot (\boldsymbol{\nabla} Q) \tag{2-18.9}$$

Equations (2-18.6) may be written in vector notation by first rewriting the terms on the right side. For example, a_x may be written in the form

$$a_x = \frac{1}{2}\frac{\partial}{\partial x}(u^2 + v^2 + w^2) + v\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial u}{\partial t}$$

Similar expressions hold for a_y and a_z. Consequently, in vector form, Eqs. (2-18.6) become [with Eqs. (1-8.5) and (1-13.1)]

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\nabla(\mathbf{v})^2 - (\mathbf{v}) \times (\text{curl } \mathbf{v}) \tag{2-18.10}$$

where $(\mathbf{v})^2 = u^2 + v^2 + w^2$ is a scalar.

For rectangular Cartesian coordiantes, Eq. (2-18.6) may be written in index notation as

$$a_\alpha = \frac{\partial u_\alpha}{\partial t} + u_\beta \frac{\partial u_\alpha}{\partial x_\beta} \qquad \alpha, \beta = 1,\ 2,\ 3 \tag{2-18.11}$$

where $a_1 = a_x, a_2 = a_y, a_3 = a_z$, and $u_1 = u, u_2 = v, u_3 = w$.

Spatial Equation of Continuity. In Section 2-17 we derived the continuity equation $\rho = J\rho^*$, where ρ denotes the initial mass density of a medium enclosed in a volume element dV, ρ^* denotes the mass density in the corresponding volume element $d\mathscr{V}$ of the deformed medium, and J denotes the Jacobian with respect to the material coordinates (x_0, y_0, z_0). Accordingly, $\rho = J\rho^*$ is called the *material equation of continuity*.

Alternatively, the conservation of mass concept referred to spatial coordinates yields the *spatial equation of continuity* in the form (see Chapter 1, Section 1-14)

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{2-18.12}$$

or, in index notation, in the forms

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_\alpha)}{\partial x_\alpha} = 0 \tag{2-18.13}$$

$$\frac{\partial \rho}{\partial t} + u_\alpha \frac{\partial \rho}{\partial x_\alpha} + \rho \frac{\partial u_\alpha}{\partial x_\alpha} = 0 \tag{2-18.14}$$

$$\frac{d\rho}{dt} + \rho \frac{\partial u_\alpha}{\partial x_\alpha} = 0 \tag{2-18.15}$$

where $x_1 = x, x_2 = y, x_3 = z, u_1 = u, u_2 = v, u_3 = w$, and where in Eq. (2-18.15) we have used the *material derivative* of ρ [Eq. (2-18.18)].

A vector field (u, v, w) that satisfies the relations

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \qquad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \tag{2-18.16}$$

is said to be *irrotational*. Alternatively, Eqs. (2-18.16) may be written in the form curl $\mathbf{v} = \nabla \times \mathbf{v} = 0$. Accordingly, for irrotational flow the curl of the velocity is zero. Consequently, by Eqs. (2-18.6) and (2-18.16) the acceleration components for an irrotational velocity field are given by the relations [also see Eq. (2-18.10)]

$$a_x = \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial V^2}{\partial x} \qquad a_y = \frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial V^2}{\partial y} \qquad a_z = \frac{\partial w}{\partial t} + \frac{1}{2}\frac{\partial V^2}{\partial z} \tag{2-18.17}$$

where

$$V^2 = \mathbf{v}^2 = u^2 + v^2 + w^2 \tag{2-18.18}$$

If the components (u, v, w) of a velocity field are the partial derivatives of a scalar function ϕ with respect to (x, y, z), respectively, Eq. (2-18.15) may be written in the form

$$\frac{d\rho}{dt} + \rho \nabla^2 \phi = 0 \tag{2-18.19}$$

where

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \tag{2-18.20}$$

and

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{2-18.21}$$

Furthermore, it follows by Eq. (2-18.16) that the vector field (u, v, w) is irrotational.

If the vector field is a velocity field for an incompressible fluid, $d\rho/dt = 0$. Then the continuity equation [Eq. (2-18.19)] reduces to Laplace's equation $\nabla^2\phi = 0$ (see also Chapter 1, Sections 1-13 and 1-14).

Problem Set 2-18

1. A velocity field is given by

$$u = yz + t \qquad v = zx - t \qquad w = xy$$

Determine if this flow is possible in an incompressible fluid. Determine if the flow is irrotational. Derive formulas for the acceleration components a_x, a_y, a_z.

APPENDIX 2A STRAIN–DISPLACEMENT RELATIONS IN ORTHOGONAL CURVILINEAR COORDINATES

2A-1 Geometrical Preliminaries

In Chapter 1, Sections 1-20 to 1-22, certain properties of orthogonal curvilinear coordinate systems in three-dimensional space are discussed. In this section we develop some additional properties prerequisite to the derivation of strain–displacement relations in orthogonal curvilinear coordinate systems. We employ notation that differs somewhat from that used in Chapter 1.

We let three independent scalar functions (X, Y, Z) be defined in terms of three independent variables (x, y, z), as follows:

$$X = X(x, y, z) \qquad Y = Y(x, y, z) \qquad Z = Z(x, y, z) \tag{2A-1.1}$$

If (X, Y, Z) denote rectangular Cartesian coordinates, then for any set of (X, Y, Z) the variables (x, y, z) are space coordinates (see Chapter 1, Section 1-20). By independent functions we mean that Eq. (2A-1.1) may be solved uniquely (in a region of regularity) for (x, y, z); that is,

$$x = x(X, Y, Z) \qquad y = y(X, Y, Z) \qquad z = z(X, Y, Z) \tag{2A-1.2}$$

If (x, y, z) are assigned constant values (x_0, y_0, z_0), Eq. (2A-1.2) yields

$$x(X, Y, Z) = x_0 \qquad y(X, Y, Z) = y_0 \qquad z(X, Y, Z) = z_0 \tag{2A-1.3}$$

The equation $x = x_0$ defines a surface, called a *coordinate surface*. Hence, corresponding to various values of x_0, there exists a family of coordinate surfaces, one surface for each value of x_0. Similarly, the equations $y = y_0, z = z_0$ yield two other families of coordinate surfaces. As noted in Chapter 1, Section 1-20, the intersection of two coordinate surfaces defines a *coordinate line*. For example, the intersection of the surface $y = y_0$ with the surface $z = z_0$ defines a coordinate line along which only x varies; it is called the x coordinate line. Similarly, the intersection of the surfaces $x = x_0$ and $z = z_0$ defines a y coordinate line; intersection of the surfaces $x = x_0$ and $y = y_0$ defines a z coordinate line. In general, the coordinate lines are curved. Hence, the variables (x, y, z) are called *curvilinear* coordinates.

Three coordinate surfaces in general intersect at a point in space. Hence, a point in space is associated with a triplet (x_i, y_i, z_i). If the curvilinear coordinate lines through any point (x, y, z) are mutually perpendicular, they are said to be *orthogonal*. Then, the curvilinear coordinates (x, y, z) are called *orthogonal curvilinear coordinates*. For example, cylindrical coordinates (r, θ, z) and spherical coordinates (r, θ, ϕ) are systems of orthogonal curvilinear coordinates.

Relative to rectangular Cartesian axes (X, Y, Z), the position vector $\mathbf{r}$ of a point (x, y, z) may be written $\mathbf{r} = \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z$, where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote unit vectors in the X, Y, Z directions, respectively. Hence, a system of curvilinear coordinates (x, y, z) may be defined by the single vector equation $\mathbf{r} = \mathbf{r}(x, y, z)$. Furthermore, $\mathbf{r}_x$, $\mathbf{r}_y$, $\mathbf{r}_z$ are tangent vectors to the (x, y, z) coordinate lines, respectively, where (x, y, z) subscripts on $\mathbf{r}$ denote partial derivatives relative to (x, y, z). This statement follows from the fact that $d\mathbf{r} = \mathbf{r}_x\, dx + \mathbf{r}_y dy + \mathbf{r}_z dz$, and from the fact that $d\mathbf{r} = \mathbf{r}_x dx$ for $dy = dz = 0$, $d\mathbf{r} = \mathbf{r}_y\, dy$ for $dx = dz = 0$, and $d\mathbf{r} = \mathbf{r}_z dz$ for $dx = dy = 0$. Accordingly, for an orthogonal curvilinear coordinate system

$$\mathbf{r}_x \cdot \mathbf{r}_y = \mathbf{r}_x \cdot \mathbf{r}_z = \mathbf{r}_y \cdot \mathbf{r}_z = 0 \tag{2A-1.4}$$

Noting that the distance ds between two neighboring points is defined by $ds^2 = d\mathbf{r} \cdot \mathrm{d}\mathbf{r}$, we find with Eq. (2A-1.4) that

$$ds^2 = \alpha^2\, dx^2 + \beta^2\, dy^2 + \gamma^2\, dz^2 \tag{2A-1.5}$$

where

$$\alpha^2 = \mathbf{r}_x \cdot \mathbf{r}_x \qquad \beta^2 = \mathbf{r}_y \cdot \mathbf{r}_y \qquad \gamma^2 = \mathbf{r}_z \cdot \mathbf{r}_z \tag{2A-1.6}$$

Accordingly, because $\mathbf{r}_x$, $\mathbf{r}_y$, $\mathbf{r}_x$, are tangent vectors to (x, y, z) coordinate lines, unit tangent vectors with respect to (x, y, z) coordinate lines are defined by

$$\mathbf{e}_1 = \frac{\mathbf{r}_x}{\alpha} \qquad \mathbf{e}_2 = \frac{\mathbf{r}_y}{\beta} \qquad \mathbf{e}_3 = \frac{\mathbf{r}_z}{\gamma} \tag{2A-1.7}$$

Because $\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_z$ (hence $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$) are orthogonal vectors relative to (x, y, z) coordinate lines, any other vector may be expressed linearly in terms of them. For example, the second derivative of R with respect to x may be expressed in the form

$$\mathbf{r}_{xx} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$$

To compute the coefficients (a, b, c), we form the scalar products of $\mathbf{r}_{xx}$ with $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Thus, we find

$$\mathbf{e}_1 \cdot \mathbf{r}_{xx} = a \qquad \mathbf{e}_2 \cdot \mathbf{r}_{xx} = b \qquad \mathbf{e}_3 \cdot \mathbf{r}_{xx} = c \tag{2A-1.8}$$

To evaluate the scalar products $\mathbf{e}_1 \cdot \mathbf{r}_{xx}$, and so on, we differentiate Eqs. (2A-1.4) and (2A-1.6) with respect to (x, y, z) and take into account Eqs. (2A-1.7). A few typical results of these differentiations are

$$\begin{gathered}\mathbf{r}_x \cdot \mathbf{r}_{xx} = \alpha\alpha_x \qquad \mathbf{r}_x \cdot \mathbf{r}_{xy} = \alpha\alpha_y \qquad \mathbf{r}_x \cdot \mathbf{r}_{xz} = \alpha\alpha_z \\ \mathbf{r}_z \cdot \mathbf{r}_{xx} + \mathbf{r}_x \cdot \mathbf{r}_{xz} = 0 \qquad \mathbf{r}_x \cdot \mathbf{r}_{xy} + \mathbf{r}_y \cdot \mathbf{r}_{xx} = 0\end{gathered} \tag{2A-1.9}$$

Equations (2A-1.7) to (2A-1.9) yield $a = \alpha_x, b = -\alpha\alpha_y/\beta, c = -\alpha\alpha_z/\gamma$. Similarly, the other second derivatives of $\mathbf{r}$ may be expressed as linear combinations of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The complete set of relations is

$$\begin{aligned}\mathbf{r}_{xx} &= \alpha\left(\frac{\alpha_x}{\alpha}\mathbf{e}_1 - \frac{\alpha_y}{\beta}\mathbf{e}_2 - \frac{\alpha_z}{\gamma}\mathbf{e}_3\right) \\ \mathbf{r}_{yy} &= \beta\left(-\frac{\beta_x}{\alpha}\mathbf{e}_1 + \frac{\beta_y}{\beta}\mathbf{e}_2 - \frac{\beta_z}{\gamma}\mathbf{e}_3\right) \\ \mathbf{r}_{zz} &= \gamma\left(-\frac{\gamma_x}{\alpha}\mathbf{e}_1 - \frac{\gamma_y}{\beta}\mathbf{e}_2 + \frac{\gamma_z}{\gamma}\mathbf{e}_3\right) \\ \mathbf{r}_{xy} &= \alpha_y\mathbf{e}_1 + \beta_x\mathbf{e}_2 \\ \mathbf{r}_{xz} &= \alpha_z\mathbf{e}_1 + \gamma_x\mathbf{e}_3 \\ \mathbf{r}_{yz} &= \beta_z\mathbf{e}_2 + \gamma_y\mathbf{e}_3\end{aligned} \tag{2A-1.10}$$

The preceding equations[17] are employed in the following section.

2A-2 Strain–Displacement Relations

Let (x, y, z) be orthogonal curvilinear coordinates relative to the undeformed state of a medium; that is, (x, y, z) are material coordinates. Let (u, v, w) be the projections of the displacement vector of a point (x, y, z) in the medium on tangents to the coordinate lines at point (x, y, z). Then, since the unit tangents $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ to the

[17]This development and the derivation in the following section follow closely the treatment given by H. L. Langhaar, *Foundations of Practical Shell Analysis* (Urbana, Ill.: University of Illinois, Theoretical and Applied Mechanics Dept., 1964).

coordinate lines are defined by Eqs. (2A-1.7), the displacement vector of a particle initially located at point (x, y, z) defined by the position vector $\mathbf{r} = \mathbf{r}(x, y, z)$ is

$$\Delta\boldsymbol{\rho} = \frac{u}{\alpha}\mathbf{r}_x + \frac{v}{\beta}\mathbf{r}_y + \frac{w}{\gamma}\mathbf{r}_z \tag{2A-2.1}$$

After the deformation, the particle that initially lies at point (x, y, z) is located at the point (x^*, y^*, z^*) defined by the position vector $\mathbf{r}^*(x^*, y^*, z^*) = \mathbf{r} + \Delta\boldsymbol{\rho}$. Accordingly, with Eq. (2A-2.1), the final position vector $\mathbf{r}^*$ of the particle that initially lies at point $\mathbf{r}$ is

$$\mathbf{r}^* = \mathbf{r} + \frac{u}{\alpha}\mathbf{r}_x + \frac{v}{\beta}\mathbf{r}_y + \frac{w}{\gamma}\mathbf{r}_z \tag{2A-2.2}$$

The initial length of a line element $(\alpha dx, \beta dy, \gamma dz)$ is determined by Eq. (2A-1.5). The final length ds^* of the line element is determined by the relation (see Section 2-3)

$$\left(\frac{ds^*}{ds}\right)^2 = \left(\frac{d\mathbf{r}^*}{ds}\right)^2 = \left(\mathbf{r}_x^*\frac{dx}{ds} + \mathbf{r}_y^*\frac{dy}{ds} + \mathbf{r}_z^*\frac{dz}{ds}\right)^2 \tag{2A-2.3}$$

The derivatives $(\mathbf{r}_x^*, \mathbf{r}_y^*, \mathbf{r}_z^*)$ may be evaluated by Eq. (2A-2.2), with the aid of Eqs. (2A-1.10); thus, we find

$$\begin{aligned}
\mathbf{r}_x^* &= \left(1 + \frac{u_x}{\alpha} + \frac{\alpha_y v}{\alpha\beta} + \frac{\alpha_z w}{\gamma\alpha}\right)\mathbf{r}_x + \left(\frac{v_x}{\beta} - \frac{\alpha_y u}{\beta^2}\right)\mathbf{r}_y + \left(\frac{w_x}{\gamma} - \frac{\alpha_z u}{\gamma^2}\right)\mathbf{r}_z \\
\mathbf{r}_y^* &= \left(\frac{u_y}{\alpha} - \frac{\beta_x v}{\alpha^2}\right)\mathbf{r}_x + \left(1 + \frac{\beta_x u}{\alpha\beta} + \frac{v_y}{\beta} + \frac{\beta_z w}{\beta\gamma}\right)\mathbf{r}_y + \left(\frac{w_y}{\gamma} - \frac{\beta_z v}{\gamma^2}\right)\mathbf{r}_z \\
\mathbf{r}_z^* &= \left(\frac{u_z}{\alpha} - \frac{\gamma_x w}{\alpha^2}\right)\mathbf{r}_x + \left(\frac{v_z}{\beta} - \frac{\gamma_y w}{\beta^2}\right)\mathbf{r}_y + \left(1 + \frac{\gamma_x u}{\gamma\alpha} + \frac{\gamma_y v}{\beta\gamma} + \frac{w_z}{\gamma}\right)\mathbf{r}_z
\end{aligned} \tag{2A-2.4}$$

Furthermore, the derivatives $dx/ds, dy/ds, dz/ds$ may be expressed in terms of the direction cosines (l, m, n) of the vector $d\mathbf{r}$ relative to local coordinate lines, because

$$l = \alpha\frac{dx}{ds} \qquad m = \beta\frac{dy}{ds} \qquad n = \gamma\frac{dz}{ds} \tag{2A-2.5}$$

The strain components $\epsilon_x, \epsilon_y, \ldots, \epsilon_{yz}$ are defined, as for rectangular coordinates, by Eq. (2-6.8). Hence, substitution of Eqs. (2A-2.4) and (2A-2.5) into Eq. (2A-2.3) yields, with Eqs. (2A-1.4) and (2-6.8) (see Problem 2-11.8),

$$\begin{aligned}
\epsilon_x = \frac{1}{\alpha}\Bigg[&u_x + \frac{\alpha_y v}{\beta} + \frac{\alpha_z w}{\gamma} + \frac{1}{2\alpha}\left(u_x + \frac{\alpha_y v}{\beta} + \frac{\alpha_z w}{\gamma}\right)^2 + \frac{1}{2\alpha}\left(v_x - \frac{\alpha_y u}{\beta}\right)^2 \\
&+ \frac{1}{2\alpha}\left(w_x - \frac{\alpha_z u}{\gamma}\right)^2\Bigg]
\end{aligned}$$

$$
\begin{aligned}
\epsilon_y &= \frac{1}{\beta}\left[v_y + \frac{\beta_z w}{\gamma} + \frac{\beta_x u}{\alpha} + \frac{1}{2\beta}\left(v_y + \frac{\beta_z w}{\gamma} + \frac{\beta_x u}{\alpha}\right)^2 + \frac{1}{2\beta}\left(w_y - \frac{\beta_z v}{\gamma}\right)^2 \right. \\
&\quad \left. + \frac{1}{2\beta}\left(u_y - \frac{\beta_x v}{\alpha}\right)^2\right] \\
\epsilon_z &= \frac{1}{\gamma}\left[w_z + \frac{\gamma_x u}{\alpha} + \frac{\gamma_y u}{\beta} + \frac{1}{2\gamma}\left(w_z + \frac{\gamma_x u}{\alpha} + \frac{\gamma_y v}{\beta}\right)^2 + \frac{1}{2\gamma}\left(u_z - \frac{\gamma_x w}{\alpha}\right)^2 \right. \\
&\quad \left. + \frac{1}{2\gamma}\left(v_z - \frac{\gamma_y w}{\beta}\right)^2\right] \\
\gamma_{xy} &= \frac{u_y}{\beta} + \frac{v_x}{\alpha} - \frac{\beta_x v}{\alpha\beta} - \frac{\alpha_y u}{\alpha\beta} + \frac{1}{\alpha\beta}\left(u_x + \frac{\alpha_y v}{\beta} + \frac{\alpha_z w}{\gamma}\right)\left(u_y - \frac{\beta_x v}{\alpha}\right) \\
&\quad + \frac{1}{\alpha\beta}\left(v_x - \frac{\alpha_y u}{\beta}\right)\left(v_y + \frac{\beta_x u}{\alpha} + \frac{\beta_z w}{\gamma}\right) + \frac{1}{\alpha\beta}\left(w_x - \frac{\alpha_z u}{\gamma}\right)\left(w_y - \frac{\beta_z v}{\gamma}\right) \\
\gamma_{xz} &= \frac{w_x}{\alpha} + \frac{u_z}{\gamma} - \frac{\alpha_z u}{\alpha\gamma} - \frac{\gamma_x w}{\alpha\gamma} + \frac{1}{\alpha\gamma}\left(w_z + \frac{\gamma_x u}{\alpha} + \frac{\gamma_y v}{\beta}\right)\left(w_x - \frac{\alpha_z u}{\gamma}\right) \\
&\quad + \frac{1}{\alpha\gamma}\left(u_z - \frac{\gamma_x w}{\alpha}\right)\left(u_x + \frac{\alpha_z w}{\gamma} + \frac{\alpha_y v}{\beta}\right) + \frac{1}{\alpha\gamma}\left(v_z - \frac{\gamma_y w}{\beta}\right)\left(v_x - \frac{\alpha_y u}{\beta}\right) \\
\gamma_{yx} &= \frac{v_z}{\gamma} + \frac{w_y}{\beta} - \frac{\gamma_y w}{\beta\gamma} - \frac{\beta_z v}{\beta\gamma} + \frac{1}{\beta\gamma}\left(v_y + \frac{\beta_z w}{\gamma} + \frac{\beta_x u}{\alpha}\right)\left(v_z - \frac{\gamma_y w}{\beta}\right) \\
&\quad + \frac{1}{\beta\gamma}\left(w_y - \frac{\beta_z v}{\gamma}\right)\left(w_z + \frac{\gamma_y v}{\beta} + \frac{\gamma_x u}{\gamma}\right) + \frac{1}{\beta\gamma}\left(u_y - \frac{\beta_x v}{\alpha}\right)\left(u_z - \frac{\gamma_x w}{\alpha}\right)
\end{aligned}
\tag{2A-2.6}
$$

Equations (2A-2.6) are exact geometric expressions for the strain components; that is, they are not quadratic approximations. For small-displacement theory the quadratic terms in (u, v, w) are discarded. Then Eqs. (2A-2.6) reduce to linear relations between the strain components and the displacement components.

The strain–displacement relations may be specialized for particular orthogonal curvilinear coordinate systems. For example, $\alpha = \beta = \gamma = 1$ for rectangular Cartesian coordinates, and then Eqs. (2A-2.6) reduce to Eqs. (2-6.9), with the equivalence $\epsilon_x = \epsilon_{11}, \ldots, \gamma_{xy} = 2\epsilon_{12}, \ldots$.

For small-displacement theory the following specializations of Eqs. (2A-2.6) are obtained:

Cylindrical Coordinate System (r, θ, z):

$$\alpha = 1 \qquad \beta = r \qquad \gamma = 1$$

$$\epsilon_r = \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \tag{2A-2.7}$$

$$\gamma_{\theta z} = \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}$$

Spherical Coordinate System (r, θ, ϕ):

$$\alpha = 1 \qquad \beta = r \qquad \gamma = r\sin\theta$$

$$\epsilon_r = \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\epsilon_\phi = \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} + \frac{u}{r} + \frac{v}{r}\cot\theta \tag{2A-2.8}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \qquad \gamma_{r\phi} = \frac{1}{r\sin\theta}\frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial r} - \frac{w}{r}$$

$$\gamma_{\theta\phi} = \frac{1}{r}\left(\frac{\partial w}{\partial \theta} - w\cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \phi}$$

Plane Polar Coordinates (r, θ):

$$\alpha = 1 \qquad \beta = r \qquad \gamma = 1 \qquad w = \frac{\partial}{\partial z} = 0 \qquad u = u(r,\theta) \qquad v = v(r,\theta)$$

$$\epsilon_r = \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{2A-2.9}$$

Similar results may be obtained for other orthogonal curvilinear coordinate systems by substitution of appropriate values for α, β, γ.

APPENDIX 2B DERIVATION OF STRAIN–DISPLACEMENT RELATIONS FOR SPECIAL COORDINATES BY CARTESIAN METHODS

2B-1 Cylindrical Coordinates

Cylindrical coordinates (r, θ, z) are related to rectangular Cartesian coordinates (x, y, z) (Fig. 2B-1.1) by

$$x = r\ \cos\ \theta \qquad y = r\ \sin\ \theta \qquad z = z \tag{2B-1.1}$$

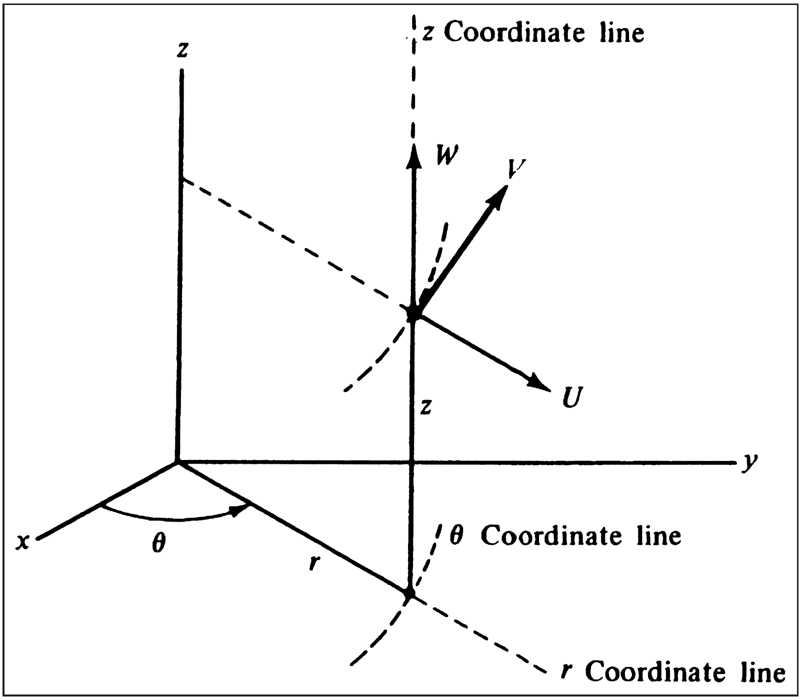

Figure 2B-1.1

Hence, the square of an infinitesimal line element ds emanating from point P is given by [Eq. (1-21.5)]

$$ds^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2\, d\theta^2 + dz^2 \qquad \text{(2B-1.2)}$$

Under deformation, point P undergoes a displacement (U, V, W), where U, V, W denote displacement components parallel to tangents along the (r, θ, z) coordinate lines, respectively. The final position $\mathscr{P}$ of P has coordinates

$$\begin{aligned} \xi_1 &= x + U \cos\theta - V \sin\theta \\ \xi_2 &= y + U \sin\theta + V \cos\theta \\ \xi_3 &= z + W \end{aligned} \qquad \text{(2B-1.3)}$$

where $U = U(r, \theta, z)$, $V = V(r, \theta, z)$, $W = W(r, \theta, z)$. The deformed line element ds now has length $d\mathscr{s}$, defined by

$$d\mathscr{s}^2 = d\xi_1^2 + d\xi_2^2 + d\xi_3^2 \qquad \text{(2B-1.4)}$$

By Eqs. (2B-1.1) to (2B-1.4), we find

$$
\begin{aligned}
d\mathscr{s}^2 &= [(1+U_r)^2 + V_r^2 + W_r^2]\,dr^2 \\
&\quad + \left[\left(\frac{U_\theta - V}{r}\right)^2 + \left(1 + \frac{U + V_\theta}{r}\right)^2 + \left(\frac{W_\theta}{r}\right)^2\right] r^2\,d\theta^2 \\
&\quad + [U_z^2 + V_z^2 + (1+W_z)^2]\,dz^2 \\
&\quad + \left[(1+U_r)\left(\frac{U_\theta - V}{r}\right) + V_r\left(1 + \frac{U + V_\theta}{r}\right) + \frac{W_r W_\theta}{r}\right] r\,dr\;d\theta \\
&\quad + [(1+U_r)U_z + V_r V_z + W_r(1+W_z)]\,dr\;dz \\
&\quad + \left[\frac{U_\theta - V}{r}U_z + \left(1 + \frac{U + V_\theta}{r}\right)V_z + \frac{W_\theta}{r}(1+W_z)\right] r\;d\theta\;dz \qquad \text{(2B-1.5)}
\end{aligned}
$$

where (r, θ, z) subscripts denote partial differentiation.

Hence, forming $\frac{1}{2}(d\mathscr{s} - ds^2)$, with Eqs. (2B-1.2) and (2B-1.5), in the form

$$
\begin{aligned}
\tfrac{1}{2}(d\mathscr{s}^2 - ds^2) &= \epsilon_{rr}\,dr^2 + \epsilon_{\theta\theta}(r\,d\theta)^2 + \epsilon_{zz}\,dz^2 \\
&\quad + 2\epsilon_{r\theta} r\;dr\;d\theta + 2\epsilon_{rz}\;dr\;dz + 2\epsilon_{\theta z} r\;d\theta\;dz \qquad \text{(2B-1.6)}
\end{aligned}
$$

we find relative to cylindrical coordinates (r, θ, z) the strain components

$$
\begin{aligned}
\epsilon_{rr} &= U_r + \tfrac{1}{2}(U_r^2 + V_r^2 + W_r^2) \\
\epsilon_{\theta\theta} &= \frac{U + V_\theta}{r} + \frac{1}{2}\left[\left(\frac{U_\theta - V}{r}\right)^2 + \left(\frac{U + V_\theta}{r}\right)^2 + \left(\frac{W_\theta}{r}\right)^2\right] \\
\epsilon_{zz} &= W_z + \tfrac{1}{2}(U_z^2 + V_z^2 + W_z^2) \qquad \text{(2B-1.7)} \\
2\epsilon_{r\theta} &= \frac{U_\theta - V}{r} + V_r + U_r\frac{U_\theta - V}{r} + V_r\frac{U + V_\theta}{r} + \frac{W_r W_\theta}{r} \\
2\epsilon_{rz} &= U_z + W_r + U_r U_z + V_r V_z + W_r W_z \\
2\epsilon_{\theta z} &= V_z + \frac{W_\theta}{r} + \frac{U_\theta - V}{r}U_z + \frac{U + V_\theta}{r}V_z + \frac{W_\theta W_z}{r}
\end{aligned}
$$

The method described above may be employed conveniently for any orthogonal curvilinear coordinate system. It may also be used directly for nonorthogonal coordinates with straight-line coordinate lines.

2B-2 Oblique Straight-Line Coordinates

Consider a plane set of oblique coordinate lines (y_1, y_2) that subtend angle θ (Fig. 2B-2.1).

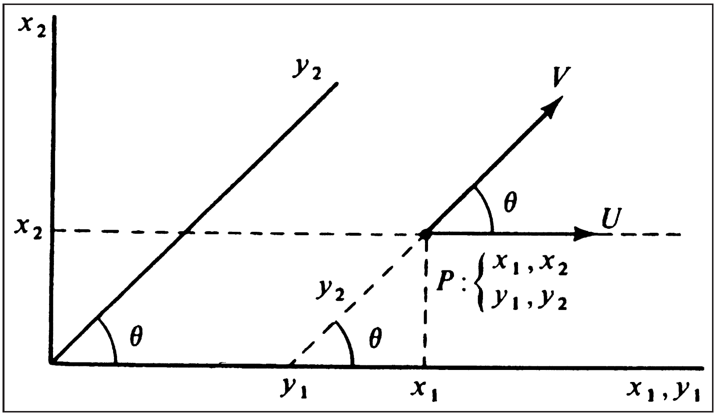

Figure 2B-2.1

The coordinates of a point P in the plane may be denoted by oblique coordinates (y_1, y_2) or rectangular coordinates (x_1, x_2), where

$$\begin{aligned} x_1 &= y_1 + y_2 \cos\theta \\ x_2 &= y_2 \sin\theta \end{aligned} \qquad \text{(2B-2.1)}$$

Hence, an infinitesimal line element (dx_1, dx_2) emanting from P has length ds defined by

$$ds^2 = dx_1^2 + dx_2^2 = dy_1^2 + dy_2^2 + 2\ \cos\ \theta\ dy_1\ dy_2 \qquad \text{(2B-2.2)}$$

Under a deformation, point P undergoes a displacement (U, V) along coordinate lines (y_1, y_2), respectively, and ds is transformed into $d\mathfrak{s}$, where

$$d\mathfrak{s}^2 = d\xi_1^2 + d\xi_2^2 \qquad \text{(2B-2.3)}$$

where

$$\begin{aligned} &\xi_1 = x_1 + U + V\ \cos\ \theta \qquad \xi_2 = x_2 + V\ \sin\ \theta \\ &U = U(y_1, y_2) \qquad V = V(y_1, y_2) \end{aligned} \qquad \text{(2B-2.4)}$$

Consequently, by Eqs. (2B-2.1) to (2B-2.3), we find

$$\tfrac{1}{2}(d\mathfrak{s}^2 - ds^2) = \epsilon_{11}\ dy_1^2 + \epsilon_{22}\ dy_2^2 + 2\epsilon_{12}\ dy_1\ dy_2 \qquad \text{(2B-2.5)}$$

where the strain components relative to (y_1, y_2) are

$$\begin{aligned} \epsilon_{11} &= \frac{\partial U}{\partial y_1} + \frac{\partial V}{\partial y_1}\cos\ \theta + \frac{1}{2}\left[\left(\frac{\partial U}{\partial y_1}\right)^2 + 2\frac{\partial U}{\partial y_1}\frac{\partial V}{\partial y_1}\cos\ \theta + \left(\frac{\partial V}{\partial y_1}\right)^2\right] \\ \epsilon_{22} &= \frac{\partial U}{\partial y_2}\cos\ \theta + \frac{\partial V}{\partial y_2} + \frac{1}{2}\left[\left(\frac{\partial U}{\partial y_2}\right)^2 + 2\frac{\partial U}{\partial y_2}\frac{\partial V}{\partial y_2}\cos\ \theta + \left(\frac{\partial V}{\partial y_2}\right)^2\right] \end{aligned} \qquad \text{(2B-2.6)}$$

$$2\epsilon_{12} = \frac{\partial V}{\partial y_1} + \frac{\partial U}{\partial y_2} + \left(\frac{\partial U}{\partial y_1} + \frac{\partial V}{\partial y_2}\right)\cos\ \theta + \frac{\partial U}{\partial y_1}\frac{\partial U}{\partial y_2}$$

$$+ \left(\frac{\partial U}{\partial y_1}\frac{\partial V}{\partial y_2} + \frac{\partial U}{\partial y_2}\frac{\partial V}{\partial y_1}\right)\cos\ \theta + \frac{\partial V}{\partial y_1}\frac{\partial V}{\partial y_2}$$

APPENDIX 2C STRAIN–DISPLACEMENT RELATIONS IN GENERAL COORDINATES

The topics treated in Chapter 1, Sections 1-23 to 1-27, and in Chapter 2 are preliminary to this appendix. See also Chapter 2 in Synge and Schild (1978).

2C-1 Euclidean Metric Tensor

The theory of deformation of a medium rests fundamentally upon the concept of the extension of an infinitesimal line element ds. Accordingly, the measurement of length ds is a basic requirement of the theory. Because the deformation of a medium is described in terms of a reference frame, measurement of length in a general coordinate system is required.

Consider a three-dimensional Euclidean space. Let the range of all indices be 1, 2, 3. Let

$$y_i = y_i(x_1, x_2, x_3) \tag{2C-1.1}$$

be an admissible transformation from the rectangular Cartesian coordinates (x_1, x_2, x_3) to general coordinates (y_1, y_2, y_3). Assume that the one-to-one inverse of Eqs. (2C-1.1) exists. Thus,

$$x_i = x_i(y_1, y_2, y_3) \tag{2C-1.2}$$

Consider the line elements ds with rectangular Cartesian projections dx^1, dx^2, dx^3. Then, because (x_1, x_2, x_3) are rectangular Cartesian coordinates, the length of the line element ds is determined by the Pythagorean rule in the form

$$ds^2 = dx^i\ dx^i = \delta_{ij}\ dx^i\ dx^j \tag{2C-1.3}$$

(Here, we employ i, j symbols for summation. If summation is not implied, a statement to that effect will be indicated.)

The distance ds may also be expressed in terms of the general coordinates (y_1, y_2, y_3) by replacing dx^i, dx^j in Eq. (2C-1.3) by appropriate expressions in terms of y_i. Accordingly, by the usual rules of differentiations we find

$$dx^i = x_{i,k}\ dy^k \tag{2C-1.4}$$

where (, k) denotes differentiation with respect to y_k. Substitution of Eq. (2C-1.4) into Eq. (2C-1.3) yields

$$ds^2 = x_{i,k}x_{i,m}dy^k dy^m \tag{2C-1.5}$$

$$= g_{km}dy^k dy^m \tag{2C-1.5a}$$

where the set of functions g_{km} is called the *Euclidean metric tensor* in y_i. By comparison of Eqs. (2C-1.5) and (2C-1.5a), we find

$$g_{km} = x_{i,k}x_{i,m} = g_{mk} \tag{2C-1.6}$$

Thus, the Euclidean metric tensor is symmetric in k, m. Written out in full, Eqs. (2C-1.5a) is

$$\begin{aligned} ds^2 = g_{11}\, dy^1\, dy^1 + g_{12}\, dy^1\, dy^2 + g_{13}\, dy^1\, dy^3 \\ + g_{21}\, dy^2\, dy^1 + g_{22}\, dy^2\, dy^2 + g_{23}\, dy^2\, dy^3 \\ + g_{31}\, dy^3\, dy^1 + g_{32}\, dy^3\, dy^2 + g_{33}\, dy^3\, dy^3 \end{aligned} \tag{2C-1.7}$$

or

$$\begin{aligned} ds^2 = g_{11}\, dy^1\, dy^1 + 2g_{12}\, dy^1\, dy^2 + 2g_{13}\, dy^1\, dy^3 \\ + g_{22}\, dy^2\, dy^2 + 2g_{23}\, dy^2\, dy^3 + g_{33}\, dy^3\, dy^3 \end{aligned} \tag{2C-1.7a}$$

The Euclidean metric tensor in any other general coordinate system (z_1, z_2, z_3) may be derived from the Euclidean metric tensor in the coordinate system (y_1, y_2, y_3) and the transformation from coordinates y_i to coordinates z_i. For example, let the transformation from y_i to z_i be given in the form

$$y_i = y_i(z_1, z_2, z_3) \tag{2C-1.8}$$

Hence,

$$dy^k = y_{k,p}dz^p \tag{2C-1.9}$$

Substitution of Eq. (2C-1.9) into Eq. (2C-1.5a) yields

$$ds^2 = g_{km}y_{k,p}y_{m,q}\; dz^p\; dz^q \tag{2C-1.10}$$

or

$$ds^2 = G_{pq}\; dz^p\; dz^q \tag{2C-1.10a}$$

where

$$G_{pq} = g_{km}y_{k,p}y_{m,q} \tag{2C-1.11}$$

is the Euclidean metric tensor in the coordinate system (z_1, z_2, z_3). Equation (2C-1.11) is the characteristic law of transformation between the Euclidean metric tensors g_{km}, G_{pq} in the general coordinates y_i, z_i, respectively. In the language of

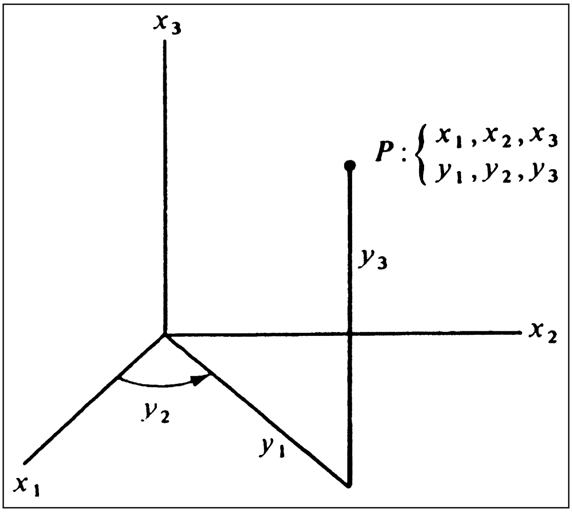

Figure 2C-1.1

tensor calculus, the transformation law of Eq. (2C-1.11) defines a *covariant tensor field of rank two*. That is, the set of components g_{km} form a *covariant tensor of rank two*. The components of the strain field and of the stress field transform according to the rule of Eq. (2C-1.11).

In summary, Eqs. (2C-1.3), (2C-1.5a), and (2C-1.10a) play a fundamental role in general studies of theories of deformation of a continuum.

Problem. Let (x_1, x_2, x_3) denote rectangular Cartesian coordinates (conventionally denoted by x, y, z). Let (y_1, y_2, y_3) denote cylindrical coordinates (conventionally denoted by r, θ, z) (Fig. 2C-1.1). Derive the relations between y_i and x_i in the form of Eqs. (2C-1.2). Then, derive the Euclidean metric tensor in terms of (y_1, y_2, y_3) (or r, θ, z). Write down the expression for ds^2 in terms of y_i.

2C-2 Strain Tensors

In the theory of deformation, we consider the transformation of a line element PA in the initial region R into a line element $\mathscr{P}\mathscr{A}$ in the deformed region $\mathscr{R}$ (see Chapter 2). We let the point P be denoted by $P(x_1, x_2, x_3)$ and point A by $A(x_1 + dx^1, x_2 + dx^2, x_3 + dx^3)$, where (x_1, x_2, x_3) are general coordinates. Similarly, in terms of general coordinates (y_1, y_2, y_3), we write $\mathscr{P}\,(y_1, y_2, y_3)$ and $\mathscr{A}\,(y_1 + dy^1, y_2 + dy^2, y_3 + dy^3)$. In the initial configuration the length of PA may be written

$$ds^2 = g_{ij}\, dx^i\, dx^j \tag{2C-2.1}$$

and the length of $\mathscr{P}\mathscr{A}$

$$d\mathscr{s}^2 = G_{ij}\, dy^i\, dy^j \tag{2C-2.2}$$

where g_{ij}, G_{ij} denote the Euclidean metric tensors relative to coordinates (x_1, x_2, x_3), (y_1, y_2, y_3), respectively.

Because the following relations between x_i and y_i exist,

$$\begin{aligned} x_i &= x_i(y_1, y_2, y_3) \\ y_i &= y_i(x_1, x_2, x_3) \end{aligned} \tag{2C-2.3}$$

we may also write [see Eq. (2C-1.10)]

$$ds^2 = g_{ij} x_{i,k}\ x_{j,m}\ dy^k\ dy^m \tag{2C-2.4}$$

$$d\mathscr{s}^2 = G_{ij} y_{i,p}\ y_{j,q}\ dx^p\ dx^q \tag{2C-2.5}$$

Accordingly, depending upon which set of coordinates we employ, we may express the difference $d\mathscr{s}^2 - ds^2$ in the forms

$$d\mathscr{s}^2 - ds^2 = 2\epsilon_{ij}\ dx^i\ dx^j \tag{2C-2.6}$$

$$d\mathscr{s}^2 - ds^2 = 2\mathscr{E}_{ij}\ dy^i\ dy^j \tag{2C-2.7}$$

where, in the modern theory of continuum, the symmetrical set of elements [see Eq. (2-6.13)]

$$\epsilon_{pq} = \epsilon_{qp} = \tfrac{1}{2}(G_{ij} y_{i,p} y_{j,q} - g_{pq}) \tag{2C-2.8}$$

is called *Green's strain tensor* (it was first introduced by Green and Saint-Venant) and the symmetrical set of elements [see Eq. (2-6.11)]

$$\mathscr{E}_{km} = \mathscr{E}_{mk} = \tfrac{1}{2}(G_{ij} - g_{ij} x_{i,k} x_{j,m}) \tag{2C-2.9}$$

is called *Almansi's strain tensor* (it was first used by Cauchy in small-displacement theory and by Almansi in large-displacement theory). Alternatively, because x_i are called material coordinates and y_i are called spatial coordinates, ϵ_{ij} and $\mathscr{E}_{ij}$ are referred to as strain in material coordinates and in spatial coordinates, respectively. The tensor character of ϵ_{ij} and $\mathscr{E}_{ij}$ follows from the fact that under a further transformation of coordinates they obey the rule of transformation of Eq. (2C-1.11).

Choice of Coordinates. If coordinates (x_1, x_2, x_3), (y_1, y_2, y_3) are rectangular Cartesian referred to a given rectangular Cartesian frame, Eqs. (2C-1.6) and (2C-1.11) yield the simple result

$$G_{ij} = g_{ij} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{2C-2.10}$$

The associated theory of deformation is discussed in Chapter 2.

For finite deformation it is often convenient to consider a spatial coordinate system (y_1, y_2, y_3) in the transformed region $\mathscr{R}$ so that point $\mathscr{P}$ has the same coordinates in the system y_i as does the point P in the original region R (in the system x_i). (For example, this procedure is used in the theory of finite deformation of shells.) Then, $x_1 = y_1, x_2 = y_2, x_3 = y_3$, and by Eqs. (2C-2.8) and (2C-2.9) it follows that

$$\epsilon_{ij} = \mathscr{E}_{ij} = \tfrac{1}{2}(G_{ij} - g_{ij}) \tag{2C-2.11}$$

Accordingly, the metric tensors G_{ij}, g_{ij} define the strain tensors ϵ_{ij}, $\mathscr{E}_{ij}$ directly. The coordinates x_i, y_i are then called *intrinsic* or *convected* coordinates.

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CHAPTER 3

THEORY OF STRESS

This chapter presents the three-dimensional theory of stress of a continuous medium. As in the theory of deformation, by a continuous medium we mean a material in which each volume of substance is sufficiently dense so that concepts such as mass density, temperature, stress, and so forth have meaning at every point in the region occupied by the material. The theory of stress rests upon Newton's laws, which are independent of the nature of materials that fall within the continuous-medium model. Consequently, the theory of stress developed here is applicable to all continuous media, regardless of their mechanical behavior of response to forces—that is, whether they behave elastically, plastically, viscoelastically, or in any other manner. The main part of the chapter is devoted to classical stress theory in which stress couples and body couples are neglected. A brief discussion of the concept of stress couples and body couples is presented in Appendix 3B.[1]

3-1 Definition of Stress

It is noted in elementary mechanics that point forces never really occur in nature; forces are always distributed throughout regions. Nevertheless, the point force is an indispensable concept in mechanics. For example, distributed forces that act on a rigid body are dynamically equivalent to a single point force and a couple.

To gain insight into the nature of distributed forces, we consider the forces that act inside a solid or a fluid. The theories of deformable bodies (fluid mechanics, elasticity, and plasticity) are based on the concept of action by direct contact. If we imagine a body to be partitioned intocells by fictitious surfaces, one cell

[1]See also Appendices 5A (Chapter 5) and 6A (Chapter 6) and Brown (1976).

does not exert a direct effect on another cell unless it is in contact with it. If two cells are in contact with each other along one of the fictitious surfaces of separation, a force may be transmitted from the first cell to the second cell and vice versa.

To elaborate on this idea, let us pass a fictitious plane Q through a body and mark an area A on the plane. One side of the plane Q will be designated as positive, the other side as negative (Fig. 3-1.1). The material on the positive side of the plane Q exerts a force upon the material on the negative side. This force is transmitted through the plane Q by direct contact of material on the two sides of the plane. The force that is transmitted through the area A is denoted by $\mathbf{F}$. Note that we do not use the notations ΔA, $\Delta \mathbf{F}$ as in some works, as use of these notations in the limiting process that defines stress may lead to confusion with the concept of derivative of a vector [see Eq. (a) below and Chapter 1, Section 1-10]. In general, $\mathbf{F}$ is not perpendicular to the plane Q. In accordance with Newton's law of reaction, the material on the negative side of plane Q transmits, through the area A, a force equal to $-\mathbf{F}$. The force $\mathbf{F}$ is an internal force, as its reaction is exerted within the body.

The force $\mathbf{F}$ may be resolved into components $\mathbf{F}_n$ and $\mathbf{F}_s$, such that the component $\mathbf{F}_n$ is perpendicular to plane Q, and the component $\mathbf{F}_s$ is tangent to plane Q (Fig. 3-1.1). The component $\mathbf{F}_n$ is called the normal force on the area A, and the component $\mathbf{F}_s$ is called the shearing force on the area A. The word "normal" has the same meaning as the word "perpendicular."

The foregoing concepts are equally applicable to stationary bodies and to deforming bodies (e.g., to flowing fluids). During a deformation process, $\mathbf{F}_n$ and

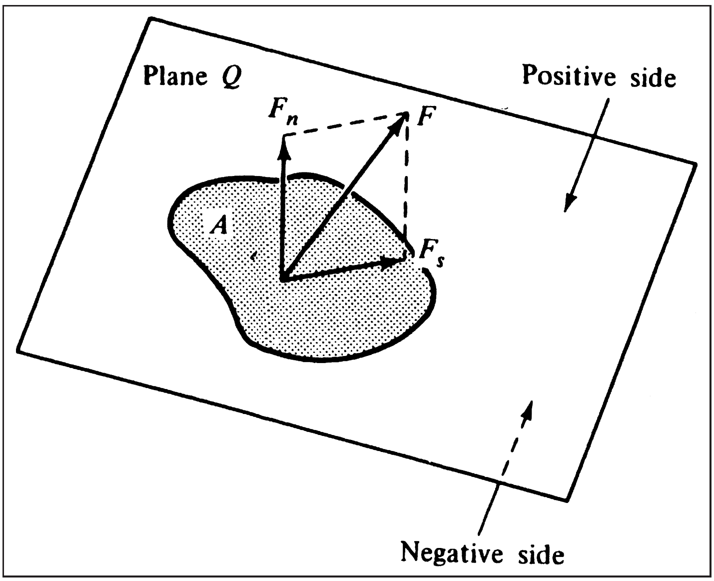

Figure 3-1.1

$\mathbf{F}_s$ ordinarily vary with time. The forces $\mathbf{F}_n$ and $\mathbf{F}_s$ naturally depend on the area A. The magnitudes of the average forces per unit area are F_n/A and F_s/A. These ratios are called the average normal stress and the average shearing stress on the area A. The concept of stress at a point is obtained by letting area A be infinitesimal. Then the forces $\mathbf{F}_n$ and $\mathbf{F}_s$ approach zero, but the ratios F_n/A and F_s/A usually approach limits different from zero. The limiting values of the ratios F_n/A and F_s/A are called the normal stress and the shearing stress on plane Q at the point where the infinitesimal area A is located. In general, these stresses depend not only on the coordinates of the infinitesimal area A but also on the plane in which the area A lies. The normal stress and the shearing stress may be regarded as normal and tangential projections of a stress vector that is associated with the infinitesimal area A. Accordingly, we may speak of the direction of the stress vector that acts at a given point on a given plane; it is the direction of the infinitesimal force that acts on the elemental area. Mathematically, the foregoing remarks may be summarized as follows:

$$\lim_{A \to 0} \frac{F_n}{A} = \sigma \qquad \lim_{A \to 0} \frac{F_s}{A} = \tau \tag{a}$$

where σ is the normal stress at a point in area A in plane Q and τ is the shearing stress at the same point in area A in plane Q.

There are significant differences between the internal forces in fluids and in solids. Solids frequently sustain large internal tensile forces. In contrast, normal forces in fluids are usually compressive. In other words, the normal force transmitted from the fluid on one side of a fictitious plane to the fluid on the other side is usually a push rather than a pull. In fluids, the reactions (pushes) measured per unit area are referred to as *pressures* (negative stresses). In the case of solids, we retain the terminology “stress” and consider pressures or *compressions* as negative stresses.

The materials that are known as fluids have another property that distinguishes them from solids. Fluid materials flow (i.e., they deform continuously) whenever shearing stresses exist. It is customary to designate this property as the definition of a fluid. Accordingly, shearing stresses cannot exist in a fluid that is at rest. This definition may be applied to ascertain whether a given material is a fluid. For example, clay does not flow unless the absolute value of the shearing stress exceeds a certain positive value. Consequently, clay is classified as a plastic solid rather than a fluid.

Intuitively, we should expect that the shearing stress in free-flowing fluids, such as air and water, must be small, even though the fluids are in motion. This observation has led to the concept of a frictionless fluid, that is, an ideal fluid. A frictionless fluid is defined to be a material in which shearing stresses cannot be developed. Much of classical hydrodynamics is concerned with frictionless fluids. However, the theory of frictionless fluids has not been so useful as it was originally expected to be, as significant shearing stresses always exist in a flowing fluid in the regions near solid boundaries.

A fluid in which shearing stresses are developed when flow occurs is said to be viscous. To some extent, all real fluids are viscous.

3-2 Stress Notation

In the theory of stress of continuous bodies, a distinction is made between the following two types of forces: (1) body forces, acting on the elements of volume (or mass) of the body, and (2) stresses, acting on surface elements inside or on the boundary of the body. Examples of body forces are gravitational forces, magnetic forces, and inertia forces. Examples of stresses (of surface forces) are contact forces between solid bodies, or hydrostatic pressure between a solid body and a fluid.

To establish a stress notation, we imagine a plane surface cutting through a body in a deformed state (stressed state) and consider the interaction between the two parts of the body across the surface of separation. For simplicity, we take the body to be a regular prism with sides parallel to axes (X, Y, Z) (Fig. 3-2.1), with the plane of separation perpendicular to the X axis. The two parts of the body are shown separated for clarity. A positive X plane in the part on the left is shaded. We define a positive X plane as one whose outward normal points in the positive X direction. The shaded positive X plane is considered to be a rectangle with sides ΔY, ΔZ. The X surface, which bounds the right part of the body and coincides with the positive X surface of the left part, is also shaded in Fig. 3-2.1. Because its outward normal points in the *negative* X direction, it is a *negative* X plane. As noted in Section 3-1, the force exerted by the negative X surface on the positive X surface is $\boldsymbol{\sigma}\ \Delta Y\ \Delta Z$, where $\boldsymbol{\sigma}$ is the stress vector. In general, $\boldsymbol{\sigma}$ is not perpendicular to the positive X plane. Hence, we may resolve the force $\boldsymbol{\sigma}\ \Delta Y\ \Delta Z$ and the associated stress into components along the positive (X, Y, Z) directions. The (X, Y, Z) components of stress are denoted by σ_{xx}, σ_{xy}, σ_{xz}, respectively. Hence, the notation σ_{xx} denotes the stress component normal to the positive X plane. Similarly, the notation σ_{xy}, σ_{xz} denotes the *shearing components* (or *tangential components*) of the stress vector that lies in the positive X plane, the components being directed in the positive Y, Z directions, respectively. We note that in the above notation the first subscript denotes the surface upon with $\boldsymbol{\sigma}$ acts, and the second subscript denotes the direction of the stress component.

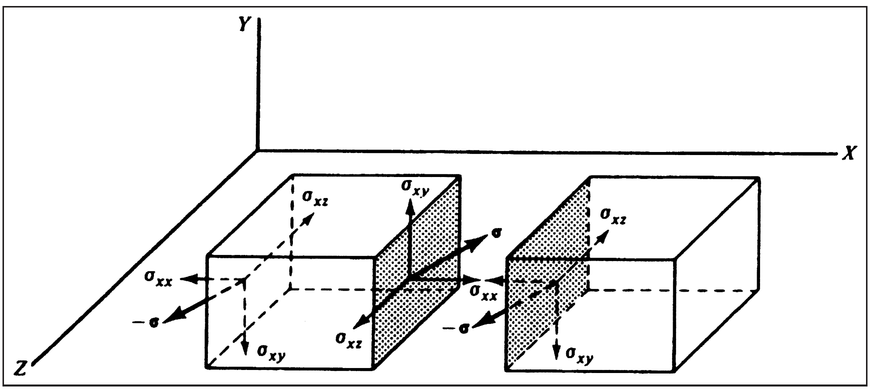

Figure 3-2.1

By Newton's third law (action and reaction), the stress components that act on the negative X surface (right part) must act in the opposite direction (Fig. 3-2.1). Thus, a positive component σ_{xj} relative to the negative X surface means a stress component in the negative j direction. Likewise, this holds for the negative X plane of the left part (the surface obtained by a translation of the negative X surface of the right part through a distance ΔX). In the theory of deformable solids, we will consider the above convention to define positive stress components. Negative components are shown schematically by reversing the direction of the arrow denoting positive components. For example, consider an infinitesimal cubic element at a point O in a body, with sides parallel to axes (X, Y, Z) (Fig. 3-2.2). The stress components acting on positive and negative planes are shown in the positive senses. Thus, on positive planes the arrowheads point in the positive senses of the corresponding axes, whereas on negative planes they point in the negative senses of the axes.

The axes (X, Y, Z) are attached to frame F (see Chapter 2, Section 2-2). Because the body (Fig. 3-2.1) is in a deformed state, the quantities $(\sigma_{xx}, \sigma_{xy}, \ldots, \sigma_{zz})$ are defined relative to the deformed state (stressed state) of the body. Thus, it follows that the equation of motion of the body is most simply written in terms of spatial coordinates (see Chapter 2, Section 2-3, and Section 3-8).

The stress notation illustrated in Fig. 3-2.2 is a conventional notation. However, other stress notations are common. The more frequent notations for components of the stress tensor are listed in Table 3-2.1.

Index Notation. The set of nine stress components associated with the cube of Fig. 3-2.2 (stress at point O) may be written in the index form σ_{ij}, $i, j = 1, 2, 3$.

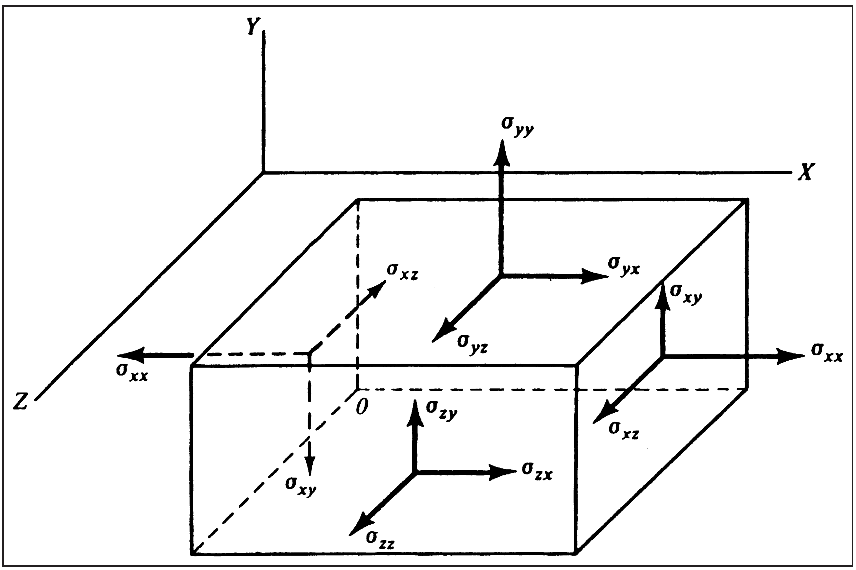

Figure 3-2.2

TABLE 3-2.1 Summary of Stress Notations[a]

Engineering	σ_x	σ_y	σ_z	$\tau_{xy} = \tau_{yx}$	$\tau_{xz} = \tau_{zx}$	$\tau_{yz} = \tau_{zy}$
Some American writers[b]	σ_{xx}	σ_{yy}	σ_{zz}	$\sigma_{xy} = \sigma_{yz}$	$\sigma_{xz} = \sigma_{zx}$	$\sigma_{yz} = \sigma_{zy}$
Love (also some Russian and English writers)	X_x	Y_y	Z_z	$X_y = Y_x$	$X_z = Z_x$	$Y_z = Z_y$
Planck	$-X_x$	$-Y_y$	$-Z_z$	$-X_y = -Y_x$	$-X_z = -Z_x$	$-Y_z = -Z_y$
Some English writers	P	Q	R	S	T	U

[a] $\sigma_{ij} = \sigma_{ji}$; see Section 3-3.
[b] Any other symbol may be used in place of σ; e.g., τ, s, t, etc.

Here we have employed the notation

$$\begin{aligned} &\sigma_{xx} = \sigma_{11}, \quad \sigma_{yy} = \sigma_{22}, \quad \sigma_{zz} = \sigma_{33} \\ &\sigma_{xy} = \sigma_{12}, \quad \sigma_{xz} = \sigma_{13}, \quad \sigma_{yz} = \sigma_{23} \end{aligned} \tag{3-2.1}$$

and so on.

In index form, axes (X, Y, Z) are referred to by the notation (X_1, X_2, X_3); see Chapter 1, Section 1-23.

3-3 Summation of Moments. Stress at a Point. Stress on an Oblique Plane

By the foregoing sign convention, the stress components with reference to rectangular spatial (x, y, z) may be tabulated in the following array:

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{3-3.1}$$

In this array, the stress components in the first row act on a plane perpendicular to the X axis, the stress components in the second row act on a plane perpendicular to the Y axis, and the stress components in the third row act on a plane perpendicular to the Z axis. Apparently, nine stress components are required to define stress at a point in a body. However, by simple consideration of summation of moments acting on a differential element, the number of stress components can be reduced by three (see Appendix 3B).

Returning to Fig. 3-2.2, we note that summation of moments with respect to point O yields the following equations[2] (in the absence of body moments and

[2] If we let σ_{xx}, σ_{xy}, σ_{xz} be the stress components on the left face of the element, then in general the stress components on the right face will be $\sigma_{xx} + (\partial\sigma_{xx}/\partial x)dx, \sigma_{xy} + (\partial_{xy}/\partial x)dx$, and $\sigma_{xz} + (\partial\sigma_{xz}/\partial x)dx$. Similar relations hold for other pairs of faces. However, the terms $(\partial\sigma_{xx}/\partial x)dx$ and so forth contribute only higher-order terms in dx, dy, dz to Eqs. (3-3.2). Also, the acceleration effects, being proportional to the mass moment of inertia of the element, are of higher degree in dx, dy, dz, hence do not contribute to Eqs. (3-3.2).

couple stresses due to other sources such as magnetic effects; see Appendix 3B):

$$
\begin{aligned}
M_x &= (\sigma_{yx}\ dx\ dz)\,dy - (\sigma_{zy}\ dx\ dy)\,dz = 0 \\
M_y &= (\sigma_{zx}\ dx\ dy)\,dz - (\sigma_{xz}\ dy\ dz)\,dx = 0 \\
M_z &= (\sigma_{xy}\ dy\ dz)\,dx - (\sigma_{yx}\ dx\ dz)\,dy = 0
\end{aligned}
\tag{3-3.2}
$$

where M_x, M_y, M_z denote moment components with respect to the (X, Y, Z) axes, respectively. Hence, by Eq. (3-3.2),

$$
\sigma_{yz} = \sigma_{zy}, \qquad \sigma_{zx} = \sigma_{xz}, \qquad \sigma_{xy} = \sigma_{yx} \tag{3-3.3}
$$

or, in index notation

$$
\sigma_{\alpha\beta} = \sigma_{\beta\alpha} \tag{3-3.3a}
$$

With Eqs. (3-3.3), Eq. (3-3.1) may be written in the form

$$
\begin{pmatrix}
\sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\
\sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\
\sigma_{xz} & \sigma_{zy} & \sigma_{zz}
\end{pmatrix}
\tag{3-3.4}
$$

This array of stress components is symmetrical with respect to the principal diagonal (running from the upper left corner to the lower right corner); that is, the shearing stress components (the off-diagonal components) are equal in pairs. Hence, in the absence of body moments and couple stresses, only six stress components are required to define *stress at a point in a body*. That is, Eq. (3-3.4) is a square symmetric array (see Chapter 1, Section 1-27).

The stress vectors $(\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3)$ on planes that are perpendicular, respectively, to the X_1, X_2, and X_3 axes are (in index notation)

$$
\begin{aligned}
\boldsymbol{\sigma}_1 &= \mathbf{i}_1\sigma_{11} + \mathbf{i}_2\sigma_{12} + \mathbf{i}_3\sigma_{13} \\
\boldsymbol{\sigma}_2 &= \mathbf{i}_1\sigma_{21} + \mathbf{i}_2\sigma_{22} + \mathbf{i}_3\sigma_{23} \\
\boldsymbol{\sigma}_3 &= \mathbf{i}_1\sigma_{31} + \mathbf{i}_2\sigma_{32} + \mathbf{i}_3\sigma_{33}
\end{aligned}
\tag{3-3.5}
$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are unit vectors in the (X_1, X_2, X_3) directions, respectively (see Fig. 3-3.1, where $\boldsymbol{\sigma}_1$ is illustrated).

Consider the stress vector $\boldsymbol{\sigma}_n$ on an oblique plane P with unit normal $\mathbf{n}$ through point O of a medium (Fig. 3-3.2). The unit normal vector to the plane is

$$
\mathbf{n} = \mathbf{i}_1 n_1 + \mathbf{i}_2 n_2 + \mathbf{i}_3 n_3 \tag{3-3.6}
$$

where n_i are the direction cosines of the unit vector $\mathbf{n}$ relative to axes (X_1, X_2, X_3).

By Fig. 3-3.2, by Newton's second law, summation of forces yields (because acceleration effects are of higher degree in dx_1, dx_2, dx_3)

$$
\boldsymbol{\sigma}_n = n_1\boldsymbol{\sigma}_1 + n_2\boldsymbol{\sigma}_2 + n_3\boldsymbol{\sigma}_3 \tag{3-3.7}
$$

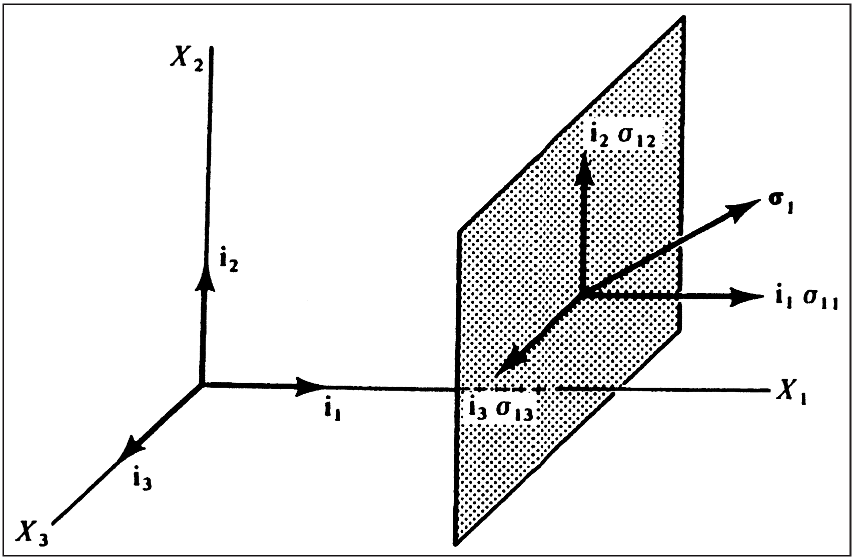

Figure 3-3.1

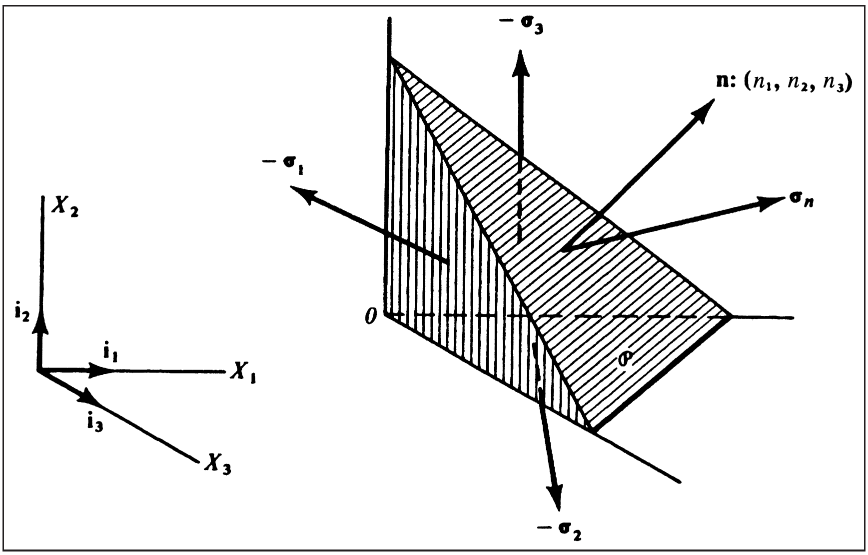

Figure 3-3.2

Substitution of Eqs. (3-3.5) into Eq. (3-3.7) yields

$$\boldsymbol{\sigma}_n = \mathbf{i}_1(n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31}) + \mathbf{i}_2(n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32}) + \mathbf{i}_3(n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33}) \tag{3-3.8}$$

where $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ are unit vectors along (X_1, X_2, X_3) axes. By definition, the stress vector $\boldsymbol{\sigma}_n$ may be represented in terms of its (X_1, X_2, X_3) projections. Hence,

$$\boldsymbol{\sigma}_n = \mathbf{i}_1\sigma_{n1} + \mathbf{i}_2\sigma_{n2} + \mathbf{i}_3\sigma_{n3} \tag{3-3.9}$$

where σ_{n1}, σ_{n2}, σ_{n3} are the (X_1, X_2, X_3) projections of the vector $\boldsymbol{\sigma}_n$. Consequently, equating Eqs (3-3.8) and (3-3.9), we obtain the scalar relations

$$\begin{aligned} \sigma_{n1} &= n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31} = n_\alpha\sigma_{\alpha 1} \\ \sigma_{n2} &= n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32} = n_\alpha\sigma_{\alpha 2} \\ \sigma_{n3} &= n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33} = n_\alpha\sigma_{\alpha 3} \end{aligned} \tag{3-3.10}$$

Equations 3-3.10 represent the components of stress at a point O on an oblique plane P (whose unit normal has direction cosines n_i) in terms of the six components of stress σ_{ij} ($\sigma_{ij} = \sigma_{ji}$). If point O is in the surface bounding a medium, and if plane P is tangent to the surface at point O, Eqs. (3-3.10) are the *stress boundary conditions at point O in terms of the stress components*.

Normal Stress and Shearing Stress on an Oblique Plane. The normal stress σ_{nn} on the plane P is the projection of the vector $\boldsymbol{\sigma}_n$ in the direction of the unit normal $\mathbf{n}$ of plane P; that is, $\sigma_{nn} = \mathbf{n} \cdot \boldsymbol{\sigma}_n$. Hence, by Eqs. (3-3.3) and (3-3.8),

$$\sigma_{nn} = n_1^2\sigma_{11} + n_2^2\sigma_{22} + n_3^2\sigma_{33} + 2n_1n_2\sigma_{12} + 2n_1n_3\sigma_{13} + 2n_2n_3\sigma_{23} \tag{3-3.11}$$

By Eq. (3-3.11) the normal stress σ_{nn} on an oblique plane P (with unit normal $\mathbf{n}$ whose direction cosines are n_i) through point O is expressed in terms of the six stress components σ_{ij}. In some problems the maximum value of σ_{nn} is of importance. We will return later to the problem of computing the maximum value of σ_{nn}.

The magnitude of σ_{nt} of the shearing stress on plane P is the magnitude of the orthogonal projection on plane P of vector $\boldsymbol{\sigma}_n$; that is, it is the magnitude of the projection of $\boldsymbol{\sigma}_n$ in the direction of a unit vector $\mathbf{t}$ that is perpendicular to $\mathbf{n}$ and that lies in plane P (Fig. 3-3.3).

Consequently, by geometry and Fig. 3-3.3,

$$\sigma_{nt}^2 = \sigma_n^2 - \sigma_{nn}^2$$

or

$$\sigma_{nt}^2 = \sigma_{n1}^2 + \sigma_{n2}^2 + \sigma_{n3}^2 - \sigma_{nn}^2 \tag{3-3.12}$$

When Eqs. (3-3.10) and (3-3.11) are substituted into Eq. (3-3.12), the shearing stress σ_{nt} is given in terms of stress components σ_{ij} and unit normal (n_1, n_2, n_3).

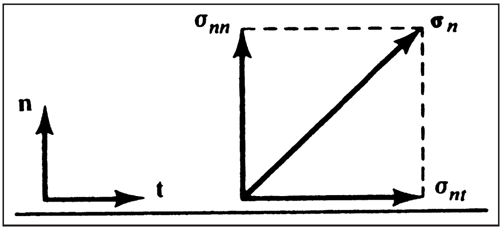

Figure 3-3.3

Having discussed the physical significance of stress on an oblique plane, we now present the results in abbreviated (index) notation. Accordingly, we let the unit normal vector **n** to plane P be denoted by n_i. Because n_i is a unit vector, $n_\alpha n_\alpha = 1$. Let the stress vector $\boldsymbol{\sigma}_n$ at point O on plane P be denoted by $\sigma_{n\alpha}$, with the understanding that we are considering plane P with normal **n**. Then Eqs. (3-3.9) or (3-3.10) may be written in the form [with $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$; Eq. (3-3.3a)]

$$\sigma_{n\alpha} = \sigma_{\beta\alpha} n_\beta = \sigma_{\alpha\beta} n_\beta \tag{3-3.13}$$

Hence, the normal stress σ_{nn} on the plane P is given by the relation (see Eq. (3-3.11)

$$\sigma_{nn} = \sigma_{n\alpha} n_\alpha = \sigma_{\beta\alpha} n_\beta n_\alpha = \sigma_{\alpha\beta} n_\alpha n_\beta \tag{3-3.14}$$

With the above notation and Eq. (3-3.12), we obtain the shearing stress σ_{nt} in the form

$$\sigma_{nt}^2 = \alpha_{n\alpha} \sigma_{n\alpha} - \sigma_{nn}^2 \tag{3-3.15}$$

Substituting Eqs. (3-3.13) and (3-3.14) into Eq. (3-3.15), we obtain σ_{nt} in terms of $\sigma_{\alpha\beta}$, n_β.

Example 3-3.1. The state of stress at a point P in a medium relative to rectangular Cartesian axes x_i is given by the stress array

$$(\sigma_{ij}) = \begin{pmatrix} 5 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \tag{a}$$

It is desired to compute the stress vectors on the positive faces of the planes perpendicular to the x_1 and x_2 axes and on the positive face of the plane with unit normal $(1/\sqrt{2}, 1/\sqrt{2}, 0)$.

Solution. The plane perpendicular to the x_1 axis has unit normal (1, 0, 0). Hence, by Eqs. (3-3.5) the associated stress vector on the positive face of the plane is

$$\boldsymbol{\sigma}_1 = 5\mathbf{i}_1 + 2\mathbf{i}_2 \tag{b}$$

and similarly, for the plane perpendicular to the x_2 axis (with unit normal 0, 1, 0), the stress vector is

$$\boldsymbol{\sigma}_2 = 2\mathbf{i}_1 + 3\mathbf{i}_2$$

For the plane with unit normal $(1/\sqrt{2}, 1/\sqrt{2}, 0)$, the stress vector is given by Eq. (3-3.8) [or by Eqs. (3-3.10)]. Thus, for this plane

$$\boldsymbol{\sigma}_n = \frac{7}{\sqrt{2}}\mathbf{i}_1 - \frac{1}{\sqrt{2}}\mathbf{i}_2 \tag{c}$$

Problem Set 3-3

1. The flat strip shown in Fig. P3-3.1 is subjected to tensile stress σ. Express the tensile stress σ' and the shearing stress τ on the oblique section AC in terms of θ and σ. (*Ans.:* $\sigma' = \sigma \cos^2\theta$; $\tau = \sigma \sin\theta \cos\theta$.)

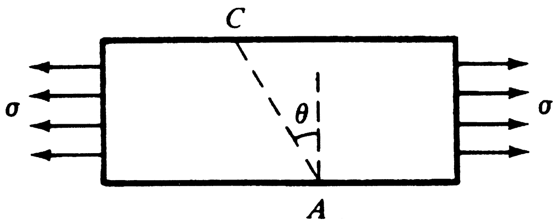

Figure P3-3.1

2. The square plate is subjected to shearing stress τ on its edges (see Fig. P3-3.2). Determine the shearing stresses and the tensile stresses on sections $A - A$ and $B - B$. (*Ans.*: Section $A - A$: $\sigma' = \tau$, $\tau' = 0$; section $B - B$: $\sigma' = -\tau, \tau' = 0$.)

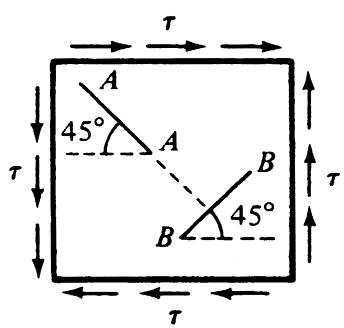

Figure P3-3.2

3. A rectangular plate is subjected to axial compression stress σ along two parallel edges and axial tension stress σ along the other two parallel edges, in the perpendicular direction.

(a) Determine the shearing stress and the tensile stress on a section $A - A$ that forms an angle θ with the direction of the axial tension. Express the results in terms of σ and θ. Evaluate for $\theta = 45^\circ$.

(b) Repeat for a rectangular plate loaded by axial tension σ along its four edges. [*Ans.*: (a) $-\sigma' = \sigma \cos 2\theta$; $\tau' = \sigma \sin 2\theta$. (b) $\sigma' = \sigma$; $\tau' = 0$.]

4. Compute τ for equilibrium (Fig. P3-3.4).

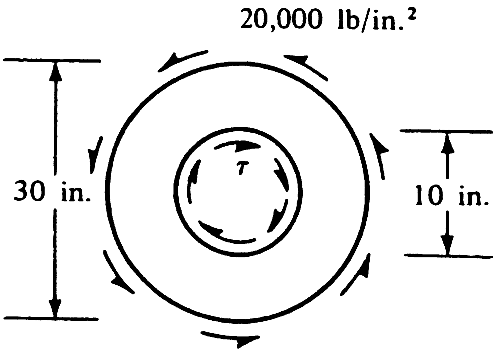

Figure P3-3.4

5. Compute τ for equilibrium (Fig. P3-3.5).

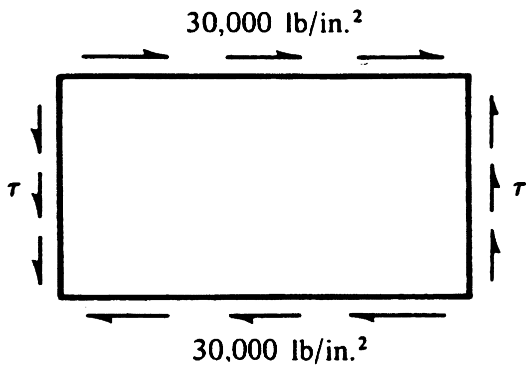

Figure P3-3.5

6. The square plate is loaded as shown in Fig. P3-3.6. Compute the shearing stresses and the tensile stresses on sections $A - A$ and $B - B$ in terms of σ_1, σ_2, τ, and θ. Specialize the result (a) for $\sigma_1 = \sigma_2$ and $\theta = 45^\circ$, and (b) for $\sigma_1 = \sigma_2 = 0$.

7. The stress tensor is

$$\begin{pmatrix} 3 & 1 & 4 \\ 1 & 2 & -5 \\ 4 & -5 & 0 \end{pmatrix}$$

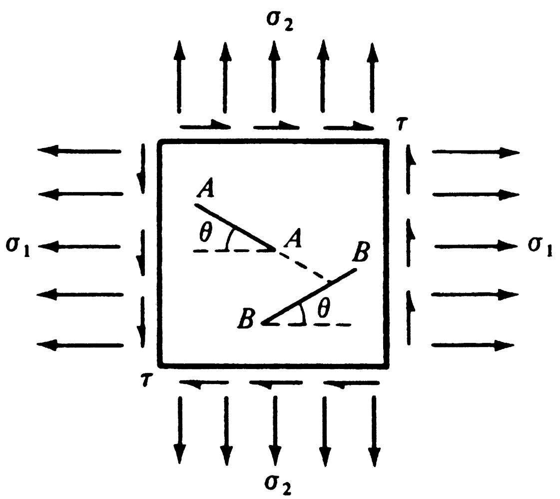

Figure P3-3.7

Determine σ_{n1}, σ_{n2}, σ_{n3}, σ_{nn}, σ_{nt} for the plane whose normal has direction cosines $(1\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$.

8. The skewed plate of unit thickness is loaded by uniformly distributed stresses of 100 and 200 psi along the sides of the plate in the directions as shown in Fig. P3-3.8.

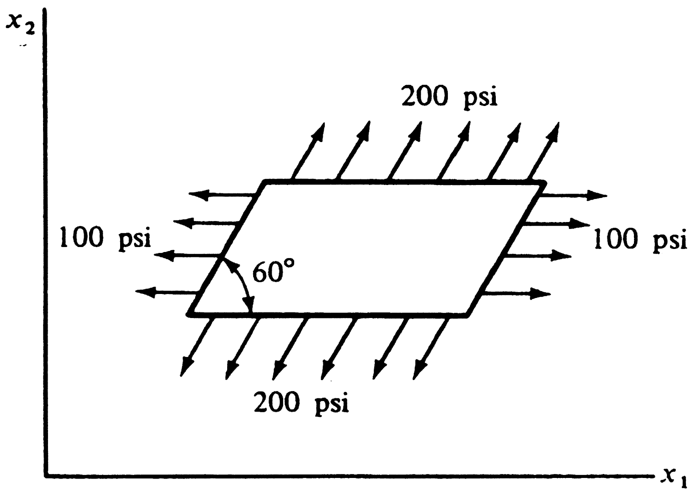

Figure P3-3.8

(a) Determine the stress matrix for this case ($\sigma_{33} = \sigma_{23} = \sigma_{31} = 0$).

(b) Determine the normal stress on a plane making angles of 45° with x_1 and x_2 axes.

9. On several planes through a point in a body, the stress vectors relative to axes (x_1, x_2, x_3) are tabulated as follows:

Unit Normal to Plane	Stress Vector on Plane
1, 0, 0	$\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
$1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$	$2\sqrt{3}\mathbf{i} + 2\sqrt{3}\mathbf{j}$
0, 1, 0	$2(\mathbf{i} + \mathbf{j} + \mathbf{k})$

Determine the components of the stress array referred to (x_1, x_2, x_3) axes; that is, determine the stress components $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23}$.

10. The stress tensor at a point P is defined by the rectangular Cartesian components $\sigma_{11} = 36 = -\sigma_{22}$, $\sigma_{12} = 27$, $\sigma_{23} = \sigma_{13} = 0$, and $\sigma_{33} = 18$, relative to axes (x_1, x_2, x_3).

(a) Determine the three rectangular (x_1, x_2, x_3) components of the stress vector acting on a plane (with unit normal $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$) passing through point P.

(b) Determine the magnitude of the stress vector of part (a).

(c) Determine the normal component of the stress vector of part (a).

(d) Determine the angle between the stress vector and the unit normal of part (a).

11. The stress components are $\sigma_{11} = 2$, $\sigma_{22} = 4$, $\sigma_{33} = 7$, $\sigma_{23} = -1$, $\sigma_{31} = 3$, and $\sigma_{12} = 5$. Compute the normal stress and the magnitude of the shearing stress on a plane whose normal makes equal angles with the (x_1, x_2, x_3) axes.

12. A crystal in a bar has the lattice directions indicated by the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$. The projections of these vectors on the x axis are $\sqrt{3}/2$, $\sqrt{2}/4$, and $\sqrt{2}/4$, respectively (Fig. P3-3.12). A plane in the crystal is normal to the vector $\mathbf{N}$ given by $\mathbf{N} = \mathbf{i}_1 + \mathbf{i}_2 + \mathbf{i}_3$. Determine the normal component of the stress on this plane, assuming that the bar is in a state of uniaxial tension of magnitude s (force/area) in the x direction.

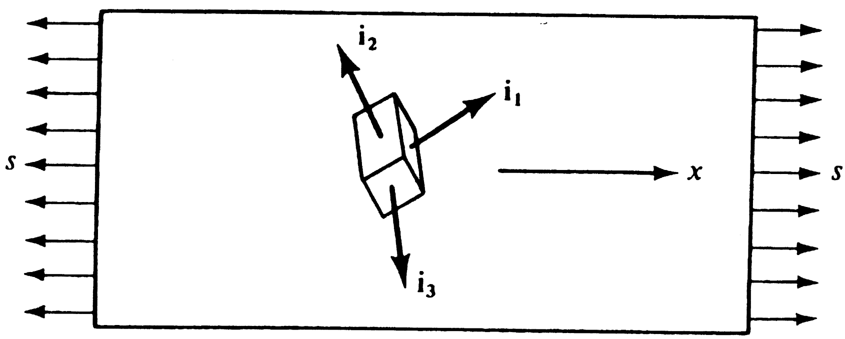

Figure P3-3.12

13. Consider $\sigma_x = 4$, $\sigma_y = 2$, $\sigma_z = -2$, $\tau_{yz} = 8$, $\tau_{zx} = -2$, and $\tau_{xy} = 3$. Compute the stress vectors on planes with unit normals $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ and $(1/\sqrt{14}, 3/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$. Compute the normal stresses on these planes; then compute the shearing stresses (see Tables 3-2.1).

14. A state of stress at a point in a continuous medium is defined by the stress $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, where α, $\beta = 1, 2, 3$, relative to axes x_i, consider a plane P with unit normal n_i and a plane Q with unit normal N_i. Denote the stress vector on plane P by σ_i and the stress vector on plane Q by Σ_i. Show that the projection of the stress vector σ_i on the unit normal N_i is equal to the projection of the stress vector Σ_i on the unit normal n_i.

15. The state of stress at a point in a body is defined by the stress components $\sigma_{11} = 1000$, $\sigma_{22} = -1000$, $\sigma_{12} = 500$, $\sigma_{23} = -200$, and $\sigma_{13} = \sigma_{33} = 0$. Consider a plane that passes through the point and has unit normal vector $(1/10, 3/10, 3/\sqrt{10})$ relative to axes (x_1, x_2, x_3).

(a) Determine the (x_1, x_2, x_3) components of the stress vector that acts on the plane.

(b) Determine the magnitude of the stress vector that acts on the plane.

(c) Determine the magnitude of the normal stress that acts on the plane.

(d) Determine the magnitude of the shearing stress that acts on the plane.

3-4 Tensor Character of Stress. Transformation of Stress Components under Rotation of Coordinate Axes

The stress array $\sigma_{\alpha\beta}$ possesses the same mathematical (tensor) character as the strain array $\epsilon_{\alpha\beta}$. However, physically, the stress array and the strain array represent different physical quantities. Accordingly, in this section we examine transformation of $\sigma_{\alpha\beta}$ from the physical viewpoint of stress.

Let (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) be two rectangular coordinate systems attached to frame F with common origin O (Fig. 3-4.1). Let $a_{\alpha\beta}$ denote the direction cosines between axes Y_α and X_β (Table 3-4.1). Each entry in Table 3-4.1 is the cosine of the angle between two coordinate axes designated at the top of its column and left of its row. For example, a_{23} is the cosine of the angle between axes X_3, Y_2. Let $\Sigma_{\alpha\beta}$ denote the stress components relative to axes Y_α. The symbol Σ (capital σ)

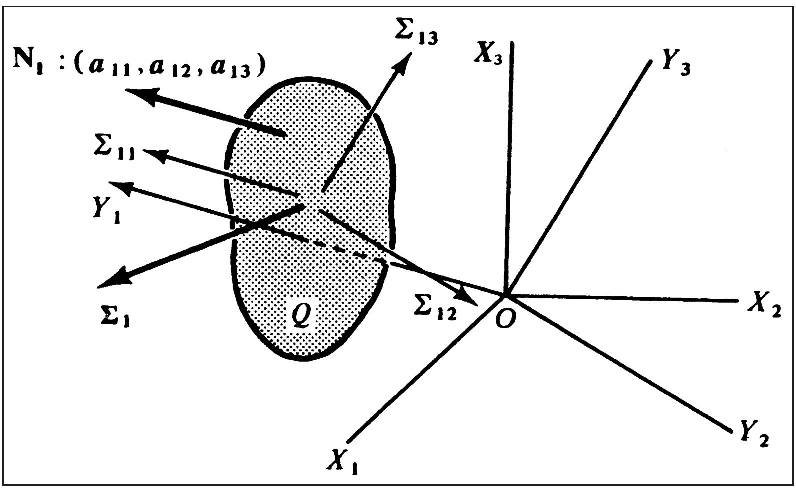

Figure 3-4.1

TABLE 3-4.1

	X_1	X_2	X_3
Y_1	a_{11}	a_{12}	a_{13}
Y_2	a_{21}	a_{22}	a_{23}
Y_3	a_{31}	a_{32}	a_{33}

in this text is used to denote stress components relative to final transformed axes (axes Y_α). It should not be confused with the summation symbol as used generally in mathematics. The stress components $\Sigma_{\alpha\beta}$ are defined relative to axes Y_α in the same manner as the stress components $\sigma_{\alpha\beta}$ are defined relative to axes X_α. For example, let Q be a plane perpendicular to axis Y_1 (Fig. 3-4.1). The unit normal to plane Q is $\mathbf{N}_1$: (a_{11}, a_{12}, a_{13}). The stress vector acting on plane Q is Σ_1, with components Σ_{11}, Σ_{12}, Σ_{13}. Hence, by Eq. (3-3.7) or (3-3.14),

$$\begin{aligned}\Sigma_{11} &= a_{11}^2\sigma_{11} + a_{12}^2\sigma_{22} + a_{13}^2\sigma_{33} + 2a_{12}a_{13}\sigma_{23} \\ &\quad + 2a_{11}a_{13}\sigma_{13} + 2a_{11}a_{12}\sigma_{12} = \sigma_{\alpha\beta}a_{1\alpha}a_{1\beta}\end{aligned} \tag{a}$$

Similarly, for Σ_{22}, Σ_{33} (components normal to planes perpendicular to axes Y_2, Y_3, respectively), we find

$$\Sigma_{22} = \sigma_{\alpha\beta}a_{2\alpha}a_{2\beta} \qquad \Sigma_{33} = \sigma_{\alpha\beta}a_{3\alpha}a_{3\beta} \tag{b}$$

The shearing stress component Σ_{12} is the component of $\boldsymbol{\sigma}_1$ in the Y_2 direction (Section 3-3). Hence, by vector algebra, it is the scalar product of the vector Σ_1 and the unit vector $\mathbf{N}_2$: (a_{12}, a_{22}, a_{23}) parallel to the Y_2 axis. Thus,

$$\Sigma_{12} = \boldsymbol{\Sigma}_1 \cdot \mathbf{N}_2 \tag{c}$$

where, by Eq. (3-3.8),

$$\begin{aligned}\boldsymbol{\Sigma}_1 &= \mathbf{i}_1(a_{11}\sigma_{11} + a_{12}\sigma_{12} + a_{13}\sigma_{13}) \\ &\quad + \mathbf{i}_2(a_{11}\sigma_{12} + a_{12}\sigma_{22} + a_{13}\sigma_{23}) \\ &\quad + \mathbf{i}_3(a_{11}\sigma_{13} + a_{12}\sigma_{23} + a_{13}\sigma_{33})\end{aligned} \tag{d}$$

and by Eq. (3-3.6),

$$\mathbf{N}_2 = \mathbf{i}_1 a_{21} + \mathbf{i}_2 a_{22} + \mathbf{i}_3 a_{23} \tag{e}$$

Consequently, Eqs. (c), (d), and (e) yield

$$\begin{aligned}\Sigma_{12} &= a_{11}a_{21}\sigma_{11} + a_{12}a_{22}\sigma_{22} + a_{13}a_{23}\sigma_{33} \\ &\quad + (a_{12}a_{23} + a_{22}a_{13})\sigma_{23} + (a_{11}a_{23} + a_{21}a_{13})\sigma_{13} \\ &\quad + (a_{11}a_{22} + a_{21}a_{12})\sigma_{12} \\ &= \sigma_{\alpha\beta}a_{1\alpha}a_{2\beta}\end{aligned} \tag{f}$$

With the definitions of Σ_{13}, Σ_{23}, similar calculations yield

$$\Sigma_{13} = \sigma_{\alpha\beta} a_{1\alpha} a_{3\beta} \qquad \Sigma_{23} = \sigma_{\alpha\beta} a_{2\alpha} a_{3\beta} \tag{g}$$

Equations (a), (b), (f), and (g) determine the stress components $\Sigma_{\alpha\beta}$ in any system of rectangular coordinates Y_α if the stress components $\sigma_{\alpha\beta}$ are known for a given system X_α. Summarizing the results, we may write, in index notation,

$$\Sigma_{\gamma\delta} = \sigma_{\alpha\beta} a_{\gamma\alpha} a_{\delta\beta} \qquad \alpha, \beta = 1, 2, 3 \tag{3-4.1}$$

Comparison of Eqs. (3-4.1) with Eqs. (2-9.1) shows that the stress components σ_{ij} transform according to the same rules as the strain components ϵ_{ij}. Hence, σ_{ij} transforms according to the rules of transformation of a *second-order tensor* [see Chapter 1, Eq. (1-23.14)]. Accordingly, the matrix of Eq. (3-3.1) is called the stress tensor. When $\sigma_{ij} = \sigma_{ji}$, the stress tensor is a *symmetrical* tensor of second order [see Eq. (3-3.4)].

In summary, the tensor nature of σ_{ij} may be established formally in index notation as follows. Let X_i and Y_i be two systems of rectangular coordinates (Fig. 3-4.1). Then, by Eq. (3-3.13), the stress vector $\mathbf{V}_\gamma$ on a plane perpendicular to axis Y_γ is given in index form as

$$V_{\gamma\beta} = \sigma_{\beta\alpha} a_{\gamma\alpha} = \sigma_{\alpha\beta} a_{\gamma\alpha} \tag{h}$$

The component $\Sigma_{\gamma\delta}$ of the stress vector $V_{\gamma\beta}$ in the direction of axis Y_δ (with direction cosines $a_{\delta\beta}$) is given by the scalar product of vectors in the form $V_{\gamma\beta} a_{\delta\beta}$. Hence, Eq. (h) yields

$$\Sigma_{\gamma\delta} = \sigma_{\alpha\beta} a_{\gamma\alpha} a_{\delta\beta} \tag{i}$$

There are many physical quantities that transform according to Eq. (3-4.1) besides stress and strain components; for example, moments and products of inertia of a rigid body, the vorticity of hydrodynamics, the magnetic field tensor, and others obey the transformation rule of Eq. (3-4.1).

Alternatively, it may be shown that the inverse transformation from axes Y_α to X_α yields

$$\sigma_{\alpha\beta} = a_{\gamma\alpha} a_{\delta\beta} \Sigma_{\gamma\delta} \tag{3-4.2}$$

Equation (3-4.2) is analogous to Eq. (2-9.3). Hence, Eqs. (3-4.1) and (3-4.2) represent the law of transformation of stress components from one system of rectangular Cartesian coordinate axes to another (see Chapter 1, Sections 1-23 and 1-24).

The stress tensor σ we discussed so far may be named as the Cauchy stress. We now introduce two more stress measures, the first Piola–Kirchhoff (PK1) stress tensor and the second Piola–Kirchhoff (PK2) stress tensor, as

$$T_{i\alpha} \equiv J F_{i\beta}^{-1} \sigma_{\beta\alpha} \tag{3-4.3}$$

$$S_{ij} \equiv J F_{i\alpha}^{-1} F_{j\beta}^{-1} \sigma_{\alpha\beta} \tag{3-4.4}$$

where the deformation gradient F and its inverse F^{-1} are defined as

$$F_{\alpha i} \equiv \frac{\partial \xi_\alpha}{\partial x_i} \qquad F^{-1}_{i\alpha} \equiv \frac{\partial x_i}{\partial \xi_\alpha} \tag{3-4.5}$$

and J is the Jacobian, the determinant of the deformation gradient. Recall that the relations among the displacements $\mathbf{u}$, material coordinates $\mathbf{x}$, and spatial coordinates $\boldsymbol{\xi}$ can be expressed as

$$\mathbf{x} + \mathbf{u} - \boldsymbol{\xi} = 0 \tag{3-4.6}$$

one may expand Eqs. (3-4.5) as

$$\mathbf{F} = \begin{Vmatrix} 1 + \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & 1 + \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & 1 + \frac{\partial u_3}{\partial x_3} \end{Vmatrix} \qquad \mathbf{F}^{-1} = \begin{Vmatrix} 1 - \frac{\partial u_1}{\partial \xi_1} & -\frac{\partial u_1}{\partial \xi_2} & -\frac{\partial u_1}{\partial \xi_3} \\ -\frac{\partial u_2}{\partial \xi_1} & 1 - \frac{\partial u_2}{\partial \xi_2} & -\frac{\partial u_2}{\partial \xi_3} \\ -\frac{\partial u_3}{\partial \xi_1} & -\frac{\partial u_3}{\partial \xi_2} & 1 - \frac{\partial u_3}{\partial \xi_3} \end{Vmatrix} \tag{3-4.7}$$

It should be emphasized that the displacements in the expression for the deformation gradient has to be put in Lagrangian (material) description, that is, $\mathbf{u} = \mathbf{u}(\mathbf{x})$; and the displacements in the expression for the inverse deformation gradient has to be put in Eulerian (spatial) description, that is, $\mathbf{u} = \mathbf{u}(\boldsymbol{\xi})$. It is straightforward to prove that Eqs. (3-4.3) and (3-4.4) can be written as

$$\sigma_{\alpha\beta} = J^{-1} F_{\alpha j} T_{j\beta} = J^{-1} F_{\alpha i} F_{\beta j} S_{ij} \tag{3-4.8}$$

The first-order and second-order Piola–Kirchhoff stresses are the keys to understanding large-strain theory. They are also essential in the study of biomechanics of soft tissues, as we will see later.

Example 3-4.1. Transformation of Stress Components. The state of stress at a point P in a medium is given by the array (relative to axes x_i)

$$(\sigma_{ij}) = \begin{pmatrix} 10 & 20 & 0 \\ 20 & 5 & 0 \\ 0 & 0 & 10 \end{pmatrix} \tag{a}$$

It is required to compute the stress components relative to axes y_i with direction cosines relative to axes x_i as follows: $a_{11} = \sqrt{3}/2, a_{12} = \frac{1}{2}, a_{13} = 0$; $a_{21} = -\frac{1}{2}, a_{22} = \sqrt{3}/2, a_{23} = 0$; and $a_{31} = a_{32} = 0, a_{33} = 1$ (see Table 3-4.1). By Eq. (3-4.1), we find

$$\Sigma_{ij} = \begin{pmatrix} 26.070 & 7.835 & 0 \\ 7.835 & -11.070 & 0 \\ 0 & 0 & 10 \end{pmatrix} \tag{b}$$

Problem Set 3-4

1. Let (Y_1, Y_2, Y_3) and (X_1, X_2, X_3) be two systems of rectangular coordinates with the same origin. The cosine of the angle between any two of the axes is designated in Table 3-4.1. For example, the cosine of the angle between the Y_2 axis and the X_3 axis is a_{23}. Show that the sum of the squares of the numbers in any row or any column of this table is 1. Show that the sum of the products of the numbers in any row (or column) with the corresponding numbers in any other row (or column) is zero. (*Hint*: See Section 1-17.)

2. Show by means of Eq. (3-4.1) that $\sigma_X + \sigma_Y + \sigma_Z = \sigma_x + \sigma_y + \sigma_z$; that is, the sum of the normal stress components is a constant in all rectangular coordinate systems (or, in other words, is invariant).

3. Show that for a body subjected to hydrostatic pressure p (see Table 3-2.1)

$$\sigma_X + \sigma_Y + \sigma_Z = \sigma_x + \sigma_y + \sigma_z = -3p$$

4. For Example 3-4.1, by means of Eq. (3-4.2) derive Eq. (a) by transforming the stress components $\Sigma_{\alpha\beta}$ from axes y_α to x_α.

5. A stress state at a point is defined by the stress components $\sigma_{11} = a$, $\sigma_{12} = b$, $\sigma_{22} = c$, and $\sigma_{13} = \sigma_{23} = \sigma_{33} = 0$ relative to axes x_α. Determine the stress components $\Sigma_{\alpha\beta}$ relative to axes y_α obtained by a rotation of 30° about axes x_2.

3-5 Principal Stresses. Stress Invariants. Extreme Values

As noted in Section 3-4, mathematically the stress tensor is entirely analogous to the strain tensor. Accordingly, by analogy to the theory of Section 2-11, for any general state of stress, through any point P in a medium, there exist three mutually perpendicular planes on which the shearing stresses σ_{ij} $(i \neq j)$ vanish identically. The resulting stresses on these planes are normal stresses; they are called *principal stresses* (*extreme values, characteristic values, or eigenvalues*; see Chapter 1, Sections 1-27 and 1-29). Axes through point P coincident with the principal stress direction are called *principal axes of stress*. Planes on which shearing stresses vanish are called *principal planes of stress*. Thus, by definition, principal stresses are perpendicular to the planes on which they act; that is, principal stresses are normal to principal planes of stress. A cube subjected to principal stresses is easily visualized, as the forces on the surface are normal to the faces.

Because the stress tensor is mathematically analogous to the strain tensor, the principal stresses are the roots of the following cubic equation in σ [see Chapter 2, Eq. (2-11.3)], provided $\sigma_{ij} = \sigma_{ji}$:

$$F(\sigma) = \begin{vmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma \end{vmatrix} = 0 \tag{3-5.1}$$

where the notation $\sigma_{11} = \sigma_{xx}$, $\sigma_{22} = \sigma_{yy}$, $\sigma_{33} = \sigma_{zz}$, $\sigma_{12} = \sigma_{xy}$, $\sigma_{13} = \sigma_{xz}$, and $\sigma_{23} = \sigma_{yz}$ has been used.

Expansion of Eq. (3-5.1) yields

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \tag{3-5.2}$$

where

$$\begin{aligned}
I_1 &= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
I_2 &= \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - \sigma_{12}^2 - \sigma_{13}^2 - \sigma_{23}^2 \\
&= \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix} \\
I_3 &= \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}
\end{aligned} \tag{3-5.3}$$

In index notation,

$$\begin{aligned}
I_1 &= \sigma_{\alpha\alpha} \\
I_2 &= \tfrac{1}{2}(I_1^2 - \sigma_{\alpha\beta}\sigma_{\beta\alpha}) \\
I_3 &= \det(\sigma_{\alpha\beta}) = \tfrac{1}{6}(2\sigma_{\alpha\beta}\sigma_{\beta\gamma}\sigma_{\gamma\alpha} - 3I_1\sigma_{\alpha\beta}\sigma_{\beta\alpha} + I_1^3)
\end{aligned} \tag{3-5.3a}$$

Equation (3-5.2) determines the three roots (principal stresses) σ_1, σ_2, and σ_3. The principal stresses $(\sigma_1, \sigma_2, \sigma_3)$ are physical invariants that are independent of choice of axes (X_1, X_2, X_3). Furthermore, by the theory of equations, if $(\sigma_1, \sigma_2, \sigma_3)$ are the three roots of Eq. (3-5.2), the following relations hold:

$$\begin{aligned}
I_1 &= \sigma_1 + \sigma_2 + \sigma_3 \\
I_2 &= \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 \\
I_3 &= \sigma_1\sigma_2\sigma_3
\end{aligned} \tag{3-5.4}$$

Accordingly, because $(\sigma_1, \sigma_2, \sigma_3)$ are invariants, it follows by Eq. (3-5.4) that I_1, I_2, and I_3 [Eq. (3-5.3)] are invariants; that is, (I_1, I_2, I_3) are independent of the choice of coordinate axes (X_1, X_2, X_3). Because (I_1, I_2, I_3) are determined by stress components σ_{ij}, in contradistinction with strain invariants (J_1, J_2, J_3), they are called *stress invariants* or *invariants of the stress tensor*. By definition, pressure in a fluid is $p = -I_1/3$; that is, pressure is also a stress invariant. For a tensor proof of invariance, see Section 2-12.

Principal Axes of Stress. Analogous to Eq. (2-11.2) in Chapter 2, we have three sets of three equations each (one set for each of the principal stresses σ_1, σ_2, σ_3) of the form

$$\begin{aligned}
(\sigma_{11} - \sigma)a_{11} + \sigma_{12}a_{12} + \sigma_{13}a_{13} &= 0 \\
\sigma_{12}a_{11} + (\sigma_{22} - \sigma)a_{12} + \sigma_{23}a_{13} &= 0 \\
\sigma_{13}a_{11} + \sigma_{23}a_{12} + (\sigma_{33} - \sigma)a_{13} &= 0
\end{aligned} \tag{3-5.5}$$

With the relation $a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$ and Eq. (3-5.5), the three sets of direction cosines of the *three principal axes* corresponding to extreme values of σ_{11}, σ_{22}, and σ_{33} (i.e., to σ_1, σ_2, and σ_3) and to zero values of σ_{12}, σ_{13}, and σ_{23} may be computed.

However, cases arise in which $\sigma_{11} = \sigma_{22} = \sigma_{33}$ (e.g., frictionless fluids). Then, all planes through point O are principal planes, and all stresses at point O are principal stresses. Accordingly, all shearing stresses vanish on all planes through point O. If two principal stresses are equal, the corresponding directions are not unique. Then all planes through the principal axis for which the stress has a different value from the other two principal stresses are principal planes.

Extreme Values of Shearing Stress. In terms of principal axes $(1, 2, 3)$ the shearing stress σ_{nt} on an oblique plane is, by Eqs. (3-3.10), (3-3.11), and (3-3.12),

$$\begin{aligned} \sigma_{nt}^2 &= \sigma_{n1}^2 + \sigma_{n2}^2 + \sigma_{n3}^2 + \sigma_{nn}^2 \\ &= (n_1\sigma_1)^2 + (n_2\sigma_2)^2 + (n_3\sigma_3)^2 - (n_1^2\sigma_1 - n_2^2\sigma_2 + n_3^2\sigma_3)^2 \end{aligned} \tag{3-5.6}$$

or, in index notation,

$$\sigma_{nt}^2 = (n_\alpha)^2(\sigma_\alpha)^2 - (n_\alpha^2\sigma_\alpha)^2 \tag{3-5.7}$$

To determine the maximum and minimum values of σ_{nt}, we may use the Lagrange multiplier technique to simplify the calculations. Accordingly, we consider the function (see Chapter 1, Section 1-29)

$$F = \sigma_{nt}^2 + \lambda^2(n_1^2 + n_2^2 + n_3^2 - 1) \tag{3-5.8}$$

where λ^2 is the Lagrange multiplier. The conditions for extreme values (stationary values) of F are

$$\frac{\partial F}{\partial n_1} = \frac{\partial F}{\partial n_2} = \frac{\partial F}{\partial n_3} = 0 \tag{3-5.9}$$

Equations (2-5.6), (3-5.8), and (3-5.9) yield

$$\begin{aligned} n_1\sigma_1^2 - (n_1^2\sigma_1 + n_2^2\sigma_2 + n_3^2\sigma_3)(2n_1\sigma_1) + \lambda^2 n_1 = 0 \\ n_2\sigma_2^2 - (n_1^2\sigma_1 + n_2^2\sigma_2 + n_3^2\sigma_3)(2n_2\sigma_2) + \lambda^2 n_2 = 0 \\ n_3\sigma_3^2 - (n_1^2\sigma_1 + n_2^2\sigma_2 + n_3^2\sigma_3)(2n_3\sigma_3) + \lambda^2 n_3 = 0 \end{aligned} \tag{3-5.10}$$

Equations (3-5.10) are necessary and sufficient conditions for F to take on stationary values. To seek extrema of σ_{nt}, we require further that $n_\alpha n_\alpha = 1$. Obvious solutions of Eqs. (3-5.10) are

$$n_1 = n_2 = 0 \qquad n_3 = \pm 1 \tag{a}$$

Accordingly, by Eqs. (3-5.10), $\lambda = \sigma_{33}$, and by Eq. (3-5.7), $\sigma_{nt} = 0$. However, because $n_1 = n_2 = 0, n_3 = \pm 1$ defines the principal axis 3, this result is known a priori. Similarly,

$$n_1 = n_3 = 0 \qquad n_2 = \pm 1 \tag{b}$$

yields $\lambda = \sigma_2$ and $\sigma_{nt} = 0$; and

$$n_2 = n_3 = 0 \qquad n_1 = \pm 1 \tag{c}$$

yields $\lambda = \sigma_1$ and $\sigma_{nt} = 0$.

For (n_1, n_2, n_3) all having nonzero values, that is, for $n_1 \neq 0, n_2 \neq 0$, and $n_3 \neq 0$, Eqs. (3-5.10) have the single solution $\sigma_1 = \sigma_2 = \sigma_3$. Hence, again Eq. (3-5.7) yields the result $\sigma_{nt} = 0$.

The remaining possibility is that only one of the direction cosines can be zero. For example, consider $n_1 = 0, n_2 \neq 0$, and $n_3 \neq 0$. Then the first of Eqs. (3-5.10) is satisfied identically. By the last two of Eqs. (3-5.10), we get after some simplification

$$(n_3^2 - n_2^2)(\sigma_2 - \sigma_3)^2 = 0 \tag{d}$$

For $\sigma_2 \neq \sigma_3$, Eq. (d) yields $n_2^2 = n_3^2$. Because $n_2^2 + n_3^2 = 1$, we have $n_2 = \pm 1/\sqrt{2}$ and $n_3 = \pm 1/\sqrt{2}$. With $n_1 = 0, n_2 = \pm 1\sqrt{2}$, and $n = \pm 1/\sqrt{2}$, Eq. (3-5.6) yields

$$\sigma_{nt} = \pm \tfrac{1}{2}(\sigma_2 - \sigma_3) \tag{3-5.11}$$

Similarly, for $n_2 = 0, n_1 = \pm 1/\sqrt{2}$, and $n_3 = \pm 1/\sqrt{2}$, we get

$$\sigma_{nt} = \pm \tfrac{1}{2}(\sigma_1 - \sigma_3) \tag{3-5.12}$$

and for $n_3 = 0, n_1 = \pm 1/\sqrt{2}$, and $n_2 = \pm 1/\sqrt{2}$,

$$\sigma_{nt} = \pm \tfrac{1}{2}(\sigma_1 - \sigma_2) \tag{3-5.13}$$

The derived results are tabulated in Table 3-5.1.

Example 3-5.1. Invariants of Stress Tensor. Let the stress tensor relative to axes (X_1, X_2, X_3) be given by the array

$$\begin{pmatrix} 4 & 1 & 2 \\ 1 & 6 & 0 \\ 2 & 0 & 8 \end{pmatrix} \tag{a}$$

TABLE 3-5.1 Extreme Values of Shear Stress

n_1	± 1	0	0	$\pm 1/\sqrt{2}$	$\pm 1/\sqrt{2}$	0
n_2	0	± 1	0	$\pm 1/\sqrt{2}$	0	$\pm 1/\sqrt{2}$
n_3	0	0	± 1	0	$\pm 1/\sqrt{2}$	$\pm 1/\sqrt{2}$
σ_{nt}	0	0	0	$\pm \frac{1}{2}(\sigma_1 - \sigma_2)$	$\pm \frac{1}{2}(\sigma_1 - \sigma_3)$	$\pm \frac{1}{2}(\sigma_2 - \sigma_3)$

TABLE 3-5.2

	X_1	X_2	X_3
Y_1	$1/\sqrt{2}$	$1/\sqrt{2}$	0
Y_2	$-1/\sqrt{2}$	$1/\sqrt{2}$	0
Y_3	0	0	1

Hence, by Eqs. (a) and (3.5.3), the stress invariants are

$$I_1 = 18 \qquad I_2 = 99 \qquad I_3 = 160 \tag{b}$$

Consider a rotation of the (X_1, X_2) axes 45° counterclockwise in the (X_1, X_2) plane to form axes (Y_1, Y_2). Let axes Y_3 and X_3 coincide. Then the transformation between (X_1, X_2, X_3) and (Y_1, Y_2, Y_3) is given by Table 3-5.2. By Eqs. (3-4.1) and Table 3-5.2, we find the following stress components relative to axes (Y_1, Y_2, Y_3):

$$\begin{array}{lll} \Sigma_{11} = 6 & \Sigma_{22} = 4 & \Sigma_{33} = 8 \\ \Sigma_{12} = 1 & \Sigma_{13} = \sqrt{2} & \Sigma_{23} = -\sqrt{2} \end{array} \tag{c}$$

Thus, relative to axes (Y_1, Y_2, Y_3) the stress tensor is defined by the array

$$\begin{pmatrix} 6 & 1 & \sqrt{2} \\ 1 & 4 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 8 \end{pmatrix} \tag{d}$$

By Eqs. (d) and (3-5.3), we find

$$I_1 = 18 \qquad I_2 = 99 \qquad I_3 = 160 \tag{e}$$

Equations (b) and (e) verify that I_1, I_2, I_3 are invariants under the transformation of Table 3-5.2. The invariants of I_1, I_2, I_3 can also be verified for the case of Example 3-4.1.

Problem Set 3-5

1. The stress tensor is defined by the array

$$\begin{pmatrix} 3 & 5 & 8 \\ 5 & 1 & 0 \\ 8 & 0 & 2 \end{pmatrix}$$

Determine the principal stresses and the principal directions. Write down the numerical values of the stress invariants.

2. Consider axes (x, y, z) and (X, Y, Z). Let the stress tensor of Problem 1 be taken relative to axes (x, y, z). Let axes (X, Y, Z) be defined relative to (x, y, z) by direction cosines:

$$l_1 = \frac{\sqrt{3}}{2} \qquad m_1 = \frac{1}{2} \qquad n_1 = 0 \qquad l_2 = -\frac{1}{2} \qquad m_2 = \frac{\sqrt{3}}{2}$$

$$n_2 = 0 \qquad l_3 = m_3 = 0 \qquad n_3 = 1$$

Compute the components of the stress relative to axes (X, Y, Z).

3. Consider a stress state characterized by the stress array $\overline{\sigma}_{ij} = \sigma_{ij} + \delta_{ij} p$, where $p = -(\sigma_{11} + \sigma_{22} + \sigma_{33})/3$. In terms of λ_1, λ_2, λ_3, the principal values of the array $\sigma_{ij}, i, j = 1, 2, 3$, derive expressions for the invariants $\overline{I}_1$, $\overline{I}_2$, $\overline{I}_3$ of the array $\overline{\sigma}_{ij}$:

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Determine the principal value of the array $\overline{\sigma}_{ij}$.

4. The stress components relative to rectangular coordinate axes (x_1, x_2, x_3) are, at a given point P in a medium, $\sigma_{11} = 0$, $\sigma_{22} = 7/2$, $\sigma_{33} = -7/2$, $\sigma_{23} = -1\sqrt{12}$, $\sigma_{31} = 7\sqrt{2}$, and $\sigma_{12} = -1/\sqrt{6}$.

(a) Determine the principal stresses and the maximum shearing stress. [*Hint:* The principal stresses are integers.] Consider axes y_α obtained by a rotation relative to axes x_α (see Table 3-4.1).

(b) Determine the principal stresses and maximum shearing stress relative to axes y_α.

3-6 Mean and Deviator Stress Tensors. Octahedral Stress

Experiments indicate that yielding and plastic deformation of many metals are essentially independent of the applied mean stress σ_m, where by definition

$$\sigma_m = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{1}{3} I_1 \tag{3-6.1}$$

Hence, most plasticity theories (Drucker, 1967) postulate that plastic behavior of materials is related primarily to that part of the stress tensor that is independent of σ_m. Certain behaviors of nonmetals are also independent of σ_m (Chen and Saleeb, 1994). Accordingly, the stress tensor [Eq. (3-3.4)] is rewritten in the form

$$\mathbf{T} = \mathbf{T}_m + \mathbf{T}_d \tag{3-6.2}$$

where $\mathbf{T}$ symbolically represents the stress tensor and where

$$\mathbf{T}_m = \begin{bmatrix} \sigma_m & 0 & 0 \\ 0 & \sigma_m & 0 \\ 0 & 0 & \sigma_m \end{bmatrix} = (\delta_{\alpha\beta}\sigma_m) \tag{3-6.3}$$

and

$$\mathbf{T}_d = \begin{bmatrix} \dfrac{2\sigma_{xx} - \sigma_{yy} - \sigma_{zz}}{3} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \dfrac{2\sigma_{yy} - \sigma_{xx} - \sigma_{zz}}{3} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \dfrac{2\sigma_{zz} - \sigma_{yy} - \sigma_{xx}}{3} \end{bmatrix} \tag{3-6.4}$$

$$= (\sigma_{\alpha\beta} - \sigma_m \delta_{\alpha\beta})$$

The validity of Eq. (3-6.2) follows from the definition of a tensor (see Section 1-24).

The tensor $\mathbf{T}_m$ is called the *mean stress tensor*. The tensor $\mathbf{T}_d$ is called the *deviator stress tensor*, as it is in a certain sense a measure of the deviation of the state of stress from a spherically symmetric state, that is, from the state of stress that exists in an ideal (frictionless) fluid.

If (X_1, X_2, X_3) are principal axes, then

$$\sigma_{11} = \sigma_1 \qquad \sigma_{22} = \sigma_2 \qquad \sigma_{33} = \sigma_3 \qquad \sigma_{12} = \sigma_{13} = \sigma_{23} = 0$$

and Eq. (3-6.2) is simplified accordingly. Application of Eqs. (3-5.3) to Eqs. (3-6.3) and (3-6.4) yields the following stress invariants for $\mathbf{T}_m$ and $\mathbf{T}_d$:

For $\mathbf{T}_m$:

$$\begin{aligned} I_{1m} &= I_1 = 3\sigma_m \\ I_{2m} &= \tfrac{1}{3} I_1^2 = 3\sigma_m^2 \\ I_{3m} &= \tfrac{1}{27} I_1^3 = \sigma_m^3 \end{aligned} \tag{3-6.5}$$

For $\mathbf{T}_d$:

$$\begin{aligned} I_{1d} &= 0 \\ I_{2d} &= I_2 - \frac{1}{3} I_1^2 = -\frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right] \\ I_{3d} &= I_3 - \frac{1}{3} I_1 I_2 + \frac{2}{27} I_1^3 = \frac{1}{27}(2\sigma_1 - \sigma_2 - \sigma_3)(2\sigma_2 - \sigma_3 - \sigma_1)(2\sigma_3 - \sigma_1 - \sigma_2) \end{aligned} \tag{3-6.6}$$

The principal values of the deviator tensor $\mathbf{T}_d$ are

$$\begin{aligned} S_1 = \sigma_1 - \sigma_m &= \frac{2\sigma_1 - \sigma_2 - \sigma_3}{3} = \frac{(\sigma_1 - \sigma_3) + (\sigma_1 - \sigma_2)}{3} \\ S_2 = \sigma_2 - \sigma_m &= \frac{(\sigma_2 - \sigma_3) + (\sigma_2 - \sigma_1)}{3} \\ &= \frac{(\sigma_2 - \sigma_3) + (\sigma_1 - \sigma_2)}{3} \\ S_3 = \sigma_3 - \sigma_m &= \frac{(\sigma_3 - \sigma_1) + (\sigma_3 - \sigma_2)}{3} \\ &= \frac{(\sigma_1 - \sigma_3) + (\sigma_2 - \sigma_3)}{3} \end{aligned} \tag{3-6.7}$$

Accordingly, because $S_1 + S_2 + S_3 = 0$, only two of the principal stresses (values) of $\mathbf{T}_d$ are independent. Many of the formulas of the mathematical theory of plasticity are often written in terms of the stress invariants of the deviators stress tensor (Sokolovski, 1955) $\mathbf{T}_d$.

Octahedral Shearing Stress. Another concept frequently employed in plasticity theory is that of octahedral shearing stress (Prager and Hodge, 1963). Octahedral shearing stress is defined as follows. Consider the directions defined by the conditions

$$m_1^2 = m_2^2 = m_3^2 = \frac{1}{3} \tag{a}$$

where m_1, m_2, m_3 are direction cosines *relative to principal axes*. There are eight planes through any point O in space whose direction cosines satisfy Eqs. (a). Consequently, the eight planes whose direction cosines satisfy Eqs. (a) are called the *octahedral planes*. These planes form equal angles with the principal directions. The shearing stress, say τ_0, which acts on the octahedral planes is called the *octahedral shearing stress*. By Eqs. (3-3.10), (3-3.11), (3-3.12), and (3-5.7), we find that

$$9\tau_0^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \tag{3-6.8}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ denote principal stresses. Because $(\sigma_1, \sigma_2, \sigma_3)$ are invariant under a transformation of coordinate axes, it follows that the octahedral shearing stress τ_0 is invariant.

Equation 3-6.8 may be written in the form

$$9\tau_0^2 = 2I_1^2 - 6I_2 = -6I_{2d} \tag{3-6.9}$$

where I_1, I_2 are the stress invariants defined by Eq. (3-5.4), and I_{2d} is defined by Eq. (3-6.6). Expressing I_1 and I_2 in terms of stress components taken relative to arbitrary (X_1, X_2, X_3) axes, by means of Eqs. (3-5.3) and (3-6.9), we obtain

$$\begin{aligned} 9\tau_0^2 = {} & (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \\ & + 6\sigma_{12}^2 + 6\sigma_{13}^2 + 6\sigma_{23}^2 \end{aligned} \tag{3-6.10}$$

Failure Criteria. To design a structural element or system to perform a given function, the designer must have a clear understanding of the possible ways or modes in which the part or system may fail to perform its function. In other words, the designer must determine possible *modes of failure* of the system and then establish suitable *failure criteria* that accurately predict the various modes of failure.

When a structural element or, more generally, a body is subject to loads, its response depends not only on the type of material from which it is made but also on the manner of loading and on environmental conditions. Depending on these factors, an element may fail to meet design requirements because of *excessive displacement, plastic deformation* (*yielding*), fracture, and so on (Boresi and Schmidt, 2002).

For each of these modes of failure, one or more criteria have been employed. For example, in the past at least six failure criteria have been proposed for initiation of yield, depending upon the material (Boresi and Schmidt, 2002). For most metals, either the *maximum shearing stress criterion* (Tresca–Saint–Venant–Coulomb–Guest criterion) or the *octahedral shearing stress criterion* [equivalent to the maximum strain energy of distortion criterion and also referred to as the von Mises–Hencky criterion; see Boresi and Schmidt (2002)] have been employed to predict yielding (i.e., departure from the elastic state).

The maximum shearing stress criterion states that inelastic action at any point in a body at which any state of stress exists *begins* when the maximum shearing stress reaches a value equal to the shearing stress in a tension specimen when yielding starts. In equation form the maximum shearing stress criterion states that yielding *begins* at any point in a body when the maximum shearing stress reaches the value

$$(\sigma_{nt})_{\max} = \tau_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = \frac{1}{2}Y \tag{3-6.11}$$

where Y is the yield stress of a tension specimen of the material, (σ_1, σ_3) are the maximum and minimum principal stresses at the point, and $\tau_{\max}$ is a shorthand notation for $(\sigma_{nt})_{\max}$; see Table 3-5.1.

The yielding criterion of failure according to the octahedral shearing stress concept states that inelastic action (yielding) *begins* at a point in the medium at which any state of stress exists when the octahedral shearing stress τ_0 reaches the value [see Eq. (3-6.10)]

$$\tau_0 = \left(\frac{\sqrt{2}}{3}\right) Y = 0.471Y \tag{3-6.12}$$

Thus, the octahedral shearing stress criterion may be stated as follows. Inelastic action at any point in a body under any combination of stresses *begins* when the octahedral shearing stress τ_0, given by Eq. (3-6.10), becomes equal to $0.471Y$, where Y is the tensile yield stress of the material as determined from a standard tension test.

Generally, it is found by experiments that initiation of yielding in many materials (especially ductile materials) is predicted fairly well by either the maximum shearing stress criterion or the octahedral stress criterion and the maximum distortional energy criterion, which is related to deviatorical stresses and leads to the same result as the octahedral shearing stress criterion. For geotechnical materials, the Mohr–Coulomb failure criterion, which accounts for differences in tensile and compressive strengths, gives better predictions (Chong and Smith, 1984). A special case of the Mohr–Coulomb criterion is the maximum shearing stress criterion (Boresi and Schmidt, 2002). The maximum principal stress criterion is also applicable for brittle materials, particularly metals.

***Example 3-6.1. Principal Directions (Eigenvectors) of the Stress Tensor* T *and the Deviator Stress Tensor* $\mathbf{T}_d$.** Let N_α^i denote the unit vector for the principal direction related to the principal stress σ_i. By Eq. (3-5.5) the unit vectors of the principal axes may be computed with the additional condition $N_\alpha^i N_\alpha^i = 1$; i is not summed. The equations that determine N_α^i [the equivalent of $a_{i\alpha}$ in Eq.(3-5.5)] are

$$\begin{aligned}(\sigma_{\alpha\beta} - \sigma_i \delta_{\alpha\beta}) N_\beta^i &= 0 \\ N_\alpha^i N_\alpha^i &= 1\end{aligned} \tag{a}$$

where i, α, β take on the values 1, 2, 3; i is not summed. Equation (a) may be written in the form

$$\begin{aligned}\sigma_{\alpha\beta} N_\beta^i &= \sigma_i \delta_{\alpha\beta} N_\beta^i \\ N_\alpha^i N_\alpha^i &= 1\end{aligned} \tag{b}$$

Equation (b) is in the form of the classical eigenproblem for the eigenvectors N_α^i associated with the principal stresses (eigenvalues) σ_i.

Now consider the deviator stress tensor [Eq. (3-6.2)], $\mathbf{T}_d = \mathbf{T} - \mathbf{T}_m = (\sigma_{\alpha\beta} - \sigma_m \delta_{\alpha\beta})$. By analogy with the stress tensor $\mathbf{T}$ and Eqs. (3-5.5) and (a), the principal axes of $\mathbf{T}_d$ [Eq. (3-6.4)] are determined by the equations

$$\begin{aligned}\left[(\sigma_{\alpha\beta} - \sigma_m \delta_{\alpha\beta}) - S_i \delta_{\alpha\beta}\right] M_\beta^i &= 0 \\ M_\alpha^i M_\alpha^i &= 1\end{aligned} \tag{c}$$

where M_α^i denotes the unit vector of principal axis i and where S_i are the principal stresses (eigenvalues) of the deviator stress tensor $\mathbf{T}_d$. Rewriting Eq. (c), we have

$$(\sigma_{\alpha\beta} - \sigma_m \delta_{\alpha\beta}) M_\beta^i = S_i \delta_{\alpha\beta} M_\beta^i \tag{d}$$

or

$$S_{\alpha\beta} M_\beta^i = S_i \delta_{\alpha\beta} M_\beta^i \tag{e}$$

where $S_{\alpha\beta} = \sigma_{\alpha\beta} - \sigma_m \delta_{\alpha\beta}$ are the stress components of the deviator tensor [Eq. (3-6.4)], and $S_i = \sigma - \sigma_m$ are the deviator tensor principal stresses [see Eq. (3-6.7) and Problem 3-6.2]. Hence, expansion of Eq. (e) yields

$$\sigma_{\alpha\beta} M_\beta^i = \sigma_i \delta_{\alpha\beta} M_\beta^i \tag{f}$$

Comparison of Eqs. (b) and (f) shows that the eigenvectors N_α^i of the stress tensor $\mathbf{T}$ are identical to the eigenvectors M_α^i of the deviator stress tensor $\mathbf{T}_d$.

More generally, it may be shown that any two symmetrical stress states $\sigma_{\alpha\beta}$, $S_{\alpha\beta}$ have a common principal axes system if and only if (Pearson, 1959)

$$\sigma_{\alpha\beta} S_{\beta\gamma} = S_{\alpha\beta} \sigma_{\beta\gamma} \tag{g}$$

The results of this example and in particular Eq. (g) hold for any symmetric tensors. For example, they apply to the strain tensor (Chapter 2, Section 2-11).

Problem Set 3-6

1. Show that $\mathbf{T}_d$ [Eq. (3-6.4)] may be written

$$\mathbf{T}_d = \begin{pmatrix} S_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & S_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & S_z \end{pmatrix}$$

where $S_x = \sigma_x - \sigma_m$, $S_y = \sigma_y - \sigma_m$, and $S_z = \sigma_z - \sigma_m$ (see Table 3-2.1).

2. Show that I_{2d} and I_{3d} [Eq. (3-6.6)] may be written $I_{2d} = -\frac{1}{6}[(S_1 - S_2)^2 + (S_2 - S_3)^2 + (S_3 - S_1)^2]$ and $I_{3d} = S_1 S_2 S_3$, where $S_1 = \sigma_1 - \sigma_m$, $S_2 = \sigma_2 - \sigma_m$, and $S_3 = \sigma_3 - \sigma_m$.

3. Show that the normal stress component σ_{oct} on the octahedral planes ($l^2 = m^2 = n^2 = \frac{1}{3}$ relative to principal axes) is given by the relation

$$\sigma_{\text{oct}} = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \sigma_m$$

4. The stress tensor is

$$\begin{pmatrix} 2 & 0 & \sqrt{3} \\ 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix}$$

(a) Determine the principal stresses.
(b) Determine the direction cosines of the normal to the plane on which $\sigma_{\max}$ acts.

5. The stress tensor is

$$\begin{pmatrix} -10 & 0 & -8 \\ 0 & 2 & 0 \\ -8 & 0 & 2 \end{pmatrix}$$

(a) Determine the principal stresses.
(b) Determine the octahedral shear stress.
(c) Determine the maximum shear stress.
(d) Determine the direction cosines of the normal to one of the planes on which the maximum shear stress acts.

6. At a certain point in a body, the stress components (see Table 3-2.1) are

$$\sigma_x = 8 \qquad \sigma_y = 6 \qquad \sigma_z = 2 \qquad \tau_{xy} = 2 \qquad \tau_{xz} = 4 \qquad \tau_{yz} = 1$$

(a) Determine the stress vector on a plane normal to the vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.
(b) Determine the principal stresses.
(c) Determine the maximum shear stress.
(d) Determine the octahedral shear stress.

7. **(a)** The stress tensor is

$$\begin{pmatrix} 4 & 0 & -4 \\ 0 & 3 & 0 \\ -4 & 0 & -2 \end{pmatrix}$$

(i) Determine the principal stresses.

(ii) Determine the direction of $\sigma_{\max}$.

(b) The three principal stresses for a body are $\sigma_1 = 4$, $\sigma_2 = 2$, and $\sigma_3 = 0$. They are in the x_1, x_2, and x_3 directions, respectively.

(i) Determine the octahedral shear stress.

(ii) Determine the maximum shear stress.

(iii) Determine the direction cosines of the normal to one of the planes on which the maximum shear stress acts.

8. The three principal stresses at a point in a body are $\sigma_1 = 6$, $\sigma_2 = 3$, and $\sigma_3 = -4$. They are in the (x_1, x_2, x_3) directions, respectively.

(a) Determine the octahedral shear stress.

(b) Determine the maximum shear stress.

(c) Determine the direction cosines of the normal to one of the planes on which the maximum shear stress acts.

9. The stress tensor with respect to (x_1, x_2, x_3) axes is

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

(a) Determine the octahedral shear stress.

(b) Determine the normal stress on the octahedral plane.

(c) Determine the maximum shear stress and plane on which it acts.

(d) Determine the normal stress on the plane of part (c).

10. The stress array for the torsion problem of a circular cross-sectional bar of radius a and with longitudinal axis coincident with the z axis of rectangular Cartesian axis (x, y, z) is

$$\begin{pmatrix} 0 & 0 & -Gy\beta \\ 0 & 0 & Gx\beta \\ -Gy\beta & Gx\beta & 0 \end{pmatrix}$$

where G, β are constants.

Compute the principal stresses at a point on the lateral surface of the bar. Determine the principal stress axes for a point on the lateral surface of the bar.

11. $\sigma_{11} = \sigma_{22} = \sigma_{33} = 0$ and $\sigma_{13} = \sigma_{32} = \sigma_{12} = \tau$.

(a) Calculate the principal stresses in terms of τ.

(b) Calculate the maximum shearing stress.

(c) Determine the directions of the axes of principal stress insofar as they are determinate.

12. At a certain point in a body, the stress components are $\sigma_{11} = 8, \sigma_{22} = 6, \sigma_{33} = 2, \sigma_{13} = 4, \sigma_{23} = 1$, and $\sigma_{12} = 2$.

(a) Determine the stress vector on a plane normal to the vector $\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(b) Determine the principal stresses.

(c) Determine the direction in which $\sigma_{\max}$ acts.

(d) Determine the maximum shear stress.

13. Let $\boldsymbol{\sigma}_x$, $\boldsymbol{\sigma}_y$, $\boldsymbol{\sigma}_z$ be stress vectors relative to surface elements that are perpendicular to rectangular Cartesian coordinate axes (x, y, z), respectively. Show that the sum of the squares of the magnitudes of these stress vectors is an invariant under any coordinate transformation.

14. The following array represents the state of stress at a point P in a medium relative to axes (x_1, x_2, x_3):

$$\begin{pmatrix} 19 & -5 & -\sqrt{6} \\ -5 & 19 & -\sqrt{6} \\ -\sqrt{6} & -\sqrt{6} & 10 \end{pmatrix}$$

(a) Determine the smallest principal stress value at the point P.

(b) Determine the principal axes directions.

(c) Determine the stress vector on a plane with unit normal $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ relative to axes (x_1, x_2, x_3).

15. Determine the stress component normal to the plane with unit normal vector $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ with respect to principal axes of stress. Express the results in terms of principal stresses $(\sigma_1, \sigma_2, \sigma_3)$. Hence, express the result in terms of stress components with respect to any rectangular Cartesian axes (x, y, z).

16. The components of the stress tensor (Table 3-2.1) are with respect to rectangular Cartesian axes (x, y, z)

$$\sigma_x = \sigma_y = 0 \qquad \sigma_z = -10 \qquad \tau_{yz} = -5 \qquad \tau_{zx} = 5 \qquad \tau_{xy} = 5$$

Determine the principal stresses. Determine the direction cosines of the principal axes. Use the cross-product relation to get the last principal axis after the other two are determined.

17. The following state of stress exists at a point in a body: $\sigma_x = 4, \sigma_y = 8, \sigma_z = -12, \tau_{xz} = 2$, and $\tau_{xy} = \tau_{yz} = 0$. Compute the magnitude of the maximum shearing stress.

18. Let $\sigma_1 > \sigma_2 > \sigma_3$ denote principal stresses. Let σ_1, σ_3 be given. Determine the values of σ_2 for which the octahedral shear stress τ_{oct} attains extreme values.

19. The stress at a point in a body is defined by the array

$$\begin{pmatrix} 10 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 7 & -3 \\ \sqrt{2} & -3 & 7 \end{pmatrix}$$

(a) Determine the defining cubic equation for the principal stresses.

(b) Determine the minimum stress at the point.

(c) Determine the direction of the principal axis for minimum stress.

20. A body is deformed under the action of forces. Strain components $\epsilon_{\alpha\beta}$ at a point are measured experimentally. Excluding a common factor, the components are given by the strain array

$$\begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}$$

(a) Determine the principal values $\epsilon_1 \geq \epsilon_2 \geq \epsilon_3$ of the strain tensor. Hence, determine the principal strain directions associated with ϵ_2.

(b) The stress components $\sigma_{\alpha\beta}$ of part (a) are given by the array (excluding a common factor)

$$\begin{pmatrix} 21 & 6 & 0 \\ 6 & 18 & -6 \\ 0 & -6 & 15 \end{pmatrix}$$

Determine the principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$. Determine the principal stress direction associated with σ_2.

21. Determine the magnitude and the direction of the principal stresses (Table 3-2.1), and the maximum shearing stresses, for the following cases:

	σ_x	σ_y	σ_z	τ_{xy}	τ_{xz}	τ_{yz}
(a)	15,000	−4,000	10,000	−3,000	0	1,000
(b)	10,000	−5,000	0	−5,000	0	0
(c)	−10,000	−5,000	10,000	2,000	3,000	4,000
(d)	10,000	−5,000	−5,000	2,000	2,000	0
(e)	10,000	0	0	0	0	0
(f)	0	0	−10,000	5,000	5,000	−5,000

22. Show that for the plane stress relating to the (x, y) plane, $\sigma_x + \sigma_y$ and $\begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{vmatrix}$ are invariants.

23. The nonzero stress components relative to axes x_α are $\sigma_{11} = -90$ MPa, $\sigma_{22} = 50$ MPa, and $\sigma_{12} = 6$ MPa.

(a) Determine the principal stresses ($\sigma_1 > \sigma_2 > \sigma_3$).

(b) Determine the maximum shearing stress.

(c) Determine the octahedral shearing stress.

(d) Determine the angle between the x_1 axis and axis X_1, where X_1 is in the direction of the largest principal stress σ_1.

24. Body A is loaded relative to x_α axes so that $\sigma_{11} = \sigma$ and the other stress components are zero. Body B is loaded so that $\sigma_{12} = \tau$ and the other stress components are zero. It is found that the octahedral shearing stress τ_0 has the same value for both bodies. Determine the ration σ/τ.

25. Indicate whether the following statements are true or false.

(a) Strain theory depends upon the material being considered. True ______ False ______

(b) Stress theory depends upon strain theory. True ______ False ______

(c) The mathematical theories of stress and strain are equivalent. True ______ False ______

(d) The correct strains of a strained continuum must be compatible. True ______ False ______

(e) If the stress components are directly proportional to the corresponding strain components, the values of the principal stresses are related to the values of the principal strains by the same proportional factor. True ______ False ______

3-7 Approximations of Plane Stress. Mohr's Circles in Two and Three Dimensions

Plane Stress. In a large class of important problems, certain approximations may be applied to simplify the three-dimensional stress tensor [see Eq. (3-3.4)]. For example, simplifying approximations can be made in analyzing the deformations that occur in a thin flat plate subjected to forces applied along its edge and directed so that they lie in the middle surface of the plate. We define a thin plate to be a prismatic member (such as a cylinder) of a very small length or *thickness* h. Accordingly, the middle surface of the plate, located halfway between its end (faces) and parallel to them, may be taken as the (X, Y) plane. The thickness direction is then coincident with the direction of the Z axis. Because the plate is not loaded on its faces, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$ on its lateral surfaces ($Z = \pm h/2$). Consequently, because the plate is thin, as a first approximation it may be assumed that

$$\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \tag{3-7.1}$$

throughout the plate thickness.[3]

Furthermore, it may be assumed that the remaining stress components σ_{xx}, σ_{yy}, σ_{xy} are independent of Z. With these approximations, the stress tensor σ_{ij} reduces to a function of the two variables (X, Y); then it is called a *plane stress tensor* or the *tensor of plane stress*. The corresponding stress condition in the plate is called a *state of plane stress with respect to the* (X, Y) *plane*.

Consider a transformation of coordinate axes from X_i to $Y_i : (Y_1, Y_2, Y_3)$. Let axes $Z = X_3$ and Y_3 remain coincident under the transformation. Then, for a state of plane stress in the (X, Y) plane, Table 3-7.1 gives the direction cosines between

[3]Actually, this restriction may be relaxed without increasing the complexity of the problem by letting the stress components σ_{zz}, σ_{xz}, σ_{yz} vary through the thickness of the plate. However, these stress components are taken to be symmetrical in z. See Chapter 5. [Again, here we consider (X, Y, Z) to be spatial coordinates; see Chapter 2, Section 2-3.]

TABLE 3-7.1

	X_1	X_2	X_3
Y_1	$\cos\theta$	$\sin\theta$	0
Y_2	$-\sin\theta$	$\cos\theta$	0
Y_3	0	0	1

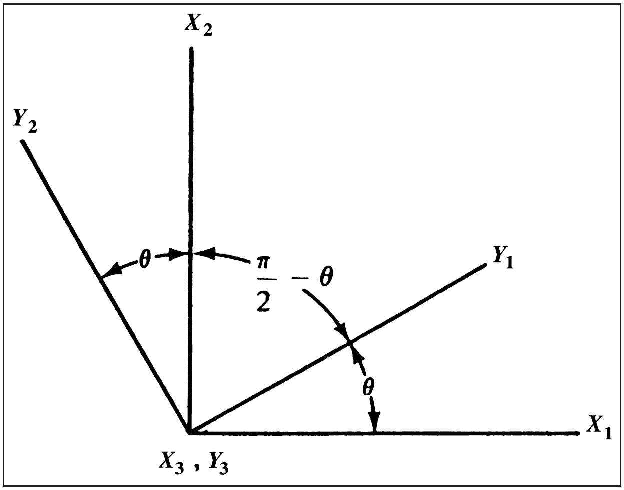

Figure 3-7.1

the axes in a transformation from axes X_i to axes Y_i, where $X_1 = X$, $X_2 = Y$ (see Fig. 3-7.1). Hence, with Table 3-7.1 and Fig. 3-7.1, Eqs. (3-4.1) yield

$$\begin{aligned}
\Sigma_{11} &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + 2\sigma_{12}\sin\theta\cos\theta \\
\Sigma_{22} &= \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - 2\sigma_{12}\sin\theta\cos\theta \\
\Sigma_{12} &= (\sigma_{22} - \sigma_{11})\sin\theta\cos\theta + (\cos^2\theta - \sin^2\theta)\sigma_{12}
\end{aligned} \tag{3-7.2}$$

where Σ_{ij} denotes stress components relative to axes Y_i and σ_{ij} denotes components relative to axes $X_i (\sigma_{11} = \sigma_{xx}, \sigma_{22} = \sigma_{yy}$, and $\sigma_{12} = \sigma_{21} = \sigma_{xy})$.

By means of trigonometric double-angle formulas, Eq. (3-7.2) may be written in the form

$$\begin{aligned}
\Sigma_{11} &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) + \frac{1}{2}(\sigma_{11} - \sigma_{22})\cos 2\theta + \sigma_{12}\sin 2\theta \\
\Sigma_{22} &= \frac{1}{2}(\sigma_{11} + \sigma_{22}) - \frac{1}{2}(\sigma_{11} - \sigma_{22})\cos 2\theta - \sigma_{12}\sin 2\theta \\
\Sigma_{12} &= \frac{1}{2}(\sigma_{22} - \sigma_{11})\sin 2\theta + \sigma_{12}\cos 2\theta
\end{aligned} \tag{3-7.3}$$

Equations (3-7.2) or (3-7.3) express the stress components Σ_{ij} in the coordinate system Y_i in terms of the corresponding stress components σ_{ij} in the coordinate system X_i for the plane transformation defined by Fig. 3-7.1 and Table 3-7.1.

Graphical Interpretation of Plane Stress. Mohr's Circle in Two Dimensions. In the form of Eqs. (3-7.3), the plane transformation of stress components is particularly suited for graphical interpretation. Furthermore, if we choose (X_1, X_2) axes to coincide with principal axes, Eqs. (3-7.3) are simplified further. Consequently, we let axes (X_1, X_2) be principal axes. Then $\sigma_{12} = 0$ and $\sigma_1 = \sigma_{11}$, $\sigma_2 = \sigma_{22}$. Accordingly, Eqs. (3-7.3) become

$$\begin{aligned}\Sigma_{11} &= \frac{1}{2}(\sigma_1 + \sigma_2) + \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\ \Sigma_{22} &= \frac{1}{2}(\sigma_1 + \sigma_2) - \frac{1}{2}(\sigma_1 - \sigma_2)\cos 2\theta \\ \Sigma_{12} &= -\frac{1}{2}(\sigma_1 - \sigma_2)\sin 2\theta\end{aligned} \tag{3-7.4}$$

where (σ_1, σ_2) denote principal stresses with $\sigma_1 > \sigma_2$ (see Section 3-5 and Fig. 3-7.2).

Recalling the physical significance of Σ_{11}, Σ_{22}, we note that the stress components on plane BE perpendicular to the Y_1 axis are Σ_{11}, Σ_{12}. The plane BE forms an angle θ in the positive direction of rotation (counterclockwise in Fig. 3-7.2) with the plane on which σ_1 acts. Similarly, the stress components Σ_{22}, Σ_{12}, $(\Sigma_{12} = \Sigma_{21})$ act on a plane forming an angle of 90° in the positive direction of rotation with plane BE. Accordingly, Eqs. (3-7.4) represent stress components on planes forming angles of θ and $(\pi/2) + \theta$ with the plane on which σ_1 acts. Hence, the variation of the stress components may be depicted graphically by constructing a diagram in which Σ_{11} and Σ_{12} (or Σ_{22} and Σ_{12}) are coordinates. For each plane BE, there is a point on the diagram whose coordinates correspond to values of Σ_{11} and Σ_{12}.

However, squaring the first and third of Eqs. (3-7.4) and adding, we obtain

$$\left[\Sigma_{11} - \frac{1}{2}(\sigma_1 + \sigma_2)\right]^2 + (\Sigma_{12})^2 = \frac{1}{4}(\sigma_1 - \sigma_2)^2 \tag{3-7.5}$$

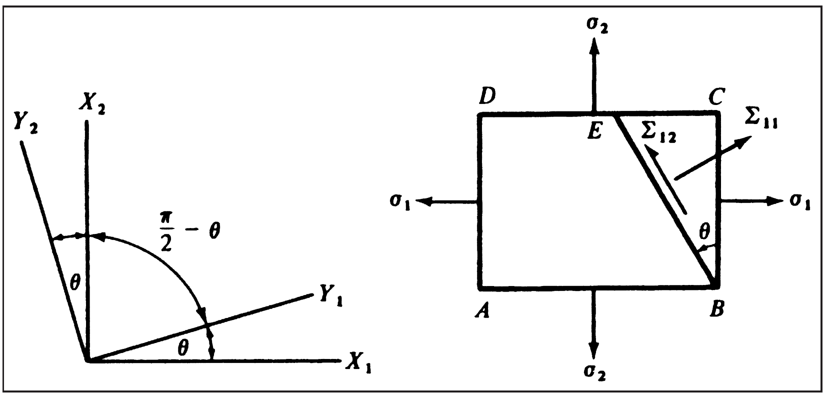

Figure 3-7.2

Equation 3-7.5 is the equation of a circle in the Σ_{11}, Σ_{12} plane with center at

$$\left[\frac{1}{2}(\sigma_1 + \sigma_2), 0\right] \tag{3-7.6}$$

and with radius

$$\frac{1}{2}(\sigma_1 - \sigma_2) \tag{3-7.7}$$

Consequently, the geometrical representation of Eqs. (3-7.4) is a circle (see Fig. 3-7.3). This stress circle is frequently called *Mohr's circle* in honor of O. Mohr, who first employed it to study plane stress problems (Mohr, 1882, 1914). (See Mohr's circles in three dimensions for maximum shear stress at a point.)

In the stress circle, we have taken the Σ_{12} axis positive downward. Hence, the point P, whose coordinates are the stress components on plane BE (Fig. 3-7.2), is obtained by rotating the radius $O'R$ of the circle counterclockwise (Fig. 3-7.3) through an angle of 2θ [see Eqs. (3-7.4)]; that is, to determine the stress components (coordinates of point P) on a plane that forms an angle θ (counterclockwise) with plane BC, we rotate the radius $O'R$ of the stress circle counterclockwise through

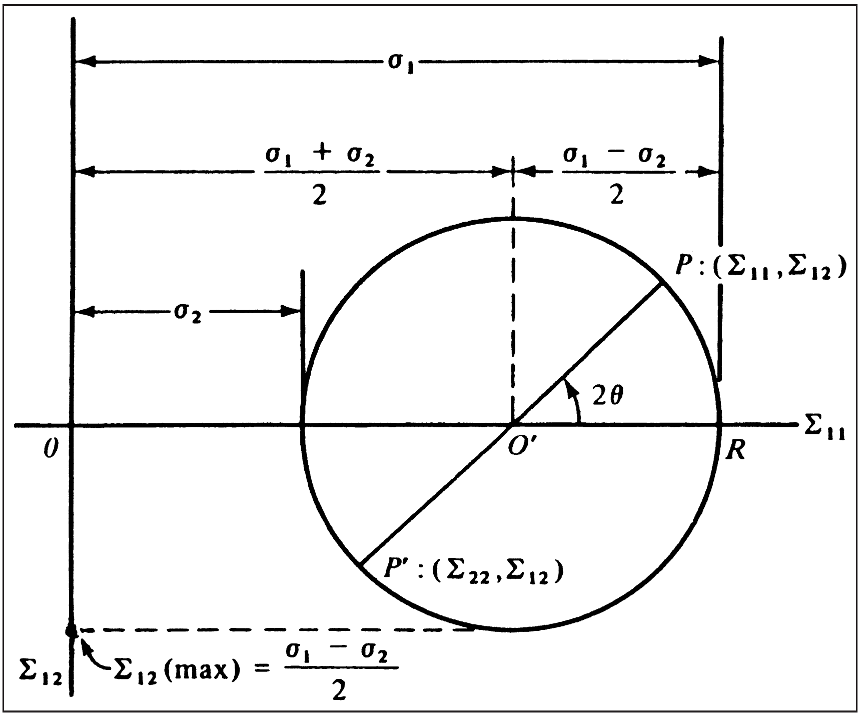

Figure 3-7.3 Mohr's circle (maximum stresses in x_1, x_2 plane). For maximum stresses in three dimensions, see the section on Mohr's circles below.

an angle of 2θ. Accordingly, stress components on a plane perpendicular to the Y_2 axis are given by the coordinates of a point P' obtained by rotating $O'P$ through 180° counterclockwise, as the plane perpendicular to the Y_2 axis forms an angle of 90° with plane BE (Fig. 3-7.2).

With the construction outlined above, the signs of the stress components agree with those given by Eqs. (3-7.4). Thus, the complete state of plane stress at a point in a medium is characterized by Mohr's circle, provided the principal stresses σ_1 and σ_2 are known. Alternatively, if the state of plane stress on two planes through a point is known, Mohr's circle may be constructed, and principal stresses may be determined.

Mohr's Circles in Three Dimensions. The stress circle construction of Mohr may be extended to three-dimensional problems. This construction is facilitated through the use of principal axes.

Relative to principal axes, the normal stress on a plane P with unit normal $\mathbf{n}$ relative to principal axes is, by Eq. (3-3.11),

$$\sigma_{nn} = n_1^2\sigma_1 + n_2^2\sigma_2 + n_3^2\sigma_3 \tag{3-7.8}$$

Similarly, the square of the shearing stress on plane P is, by Eqs. (3-3.13) and (3-3.15),

$$\sigma_{nt}^3 = n_1^2\sigma_1^2 + n_2^2\sigma_2^2 + n_3^2\sigma_3^2 - (n_1^2\sigma_1 + n_2^2\sigma_2 + n_3^2\sigma_3)^2 \tag{3-7.9}$$

Accordingly, Eqs. (3-7.8) and (3-7.9) yield

$$\begin{aligned} \sigma_{nn}^2 + \sigma_{nt}^2 &= n_1^2\sigma_1^2 + n_2^2\sigma_2^2 + n_3^2\sigma_3^2 \\ \sigma_{nn}^2 &= (n_1^2\sigma_1 + n_2^2\sigma_2 + n_3^2\sigma_3)^2 \end{aligned} \tag{3-7.10}$$

where

$$n_1^2 + n_2^2 + n_3^2 = 1$$

Solving Eqs. (3-7.10) for n_1^2, n_2^2, n_3^2 and noting that $n_1^2 \geq 0$, $n_2^2 \geq 0$, $n_3^2 \geq 0$, we obtain

$$\begin{aligned} n_1^2 &= \frac{\sigma_{nt}^2 + (\sigma_{nn} - \sigma_2)(\sigma_{nn} - \sigma_3)}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \geq 0 \\ n_2^2 &= \frac{\sigma_{nt}^2 + (\sigma_{nn} - \sigma_1)(\sigma_{nn} - \sigma_3)}{(\sigma_2 - \sigma_3)(\sigma_2 - \sigma_1)} \geq 0 \\ n_3^2 &= \frac{\sigma_{nt}^2 + (\sigma_{nn} - \sigma_1)(\sigma_{nn} - \sigma_2)}{(\sigma_3 - \sigma_1)(\sigma_3 - \sigma_2)} \geq 0 \end{aligned} \tag{3-7.11}$$

Ordering the principal stresses such that $\sigma_1 > \sigma_2 > \sigma_3$, we may write Eqs. (3-7.11) in the form

$$\sigma_{nt}^2 + (\sigma_{nn} - \sigma_2)(\sigma_{nn} - \sigma_3) \geq 0$$

$$\sigma_{nt}^2 + (\sigma_{nn} - \sigma_3)(\sigma_{nn} - \sigma_1) \leq 0$$

$$\sigma_{nt}^2 + (\sigma_{nn} - \sigma_1)(\sigma_{nn} - \sigma_2) \geq 0$$

These inequalities may be rewritten in the form

$$\begin{aligned} \sigma_{nt}^2 + \left(\sigma_{nn} - \frac{\sigma_2 + \sigma_3}{2}\right)^2 &\geq \left[\frac{1}{2}(\sigma_2 - \sigma_3)\right]^2 \\ \sigma_{nt}^2 + \left(\sigma_{nn} - \frac{\sigma_1 + \sigma_3}{2}\right)^2 &\leq \left[\frac{1}{2}(\sigma_3 - \sigma_1)\right]^2 \\ \sigma_{nt}^2 + \left(\sigma_{nn} - \frac{\sigma_1 + \sigma_2}{2}\right)^2 &\geq \left[\frac{1}{2}(\sigma_1 - \sigma_2)\right]^2 \end{aligned} \tag{3-7.12}$$

The inequalities of Eqs. (3-7.12) may be interpreted graphically as follows. Let $(\sigma_{nn}, \sigma_{nt})$ denote abscissa and ordinate, respectively, on a graph (Fig. 3-7.4). Then an admissible state of stress must lie within a region bounded by three circles obtained from Eqs. (3-7.12) where the equalities are taken.

Extreme Values of Normal Stress and Shear Stress. By Eqs. (3-7.4) and Fig. 3-7.3 for plane problems we note that the maximum value of the normal

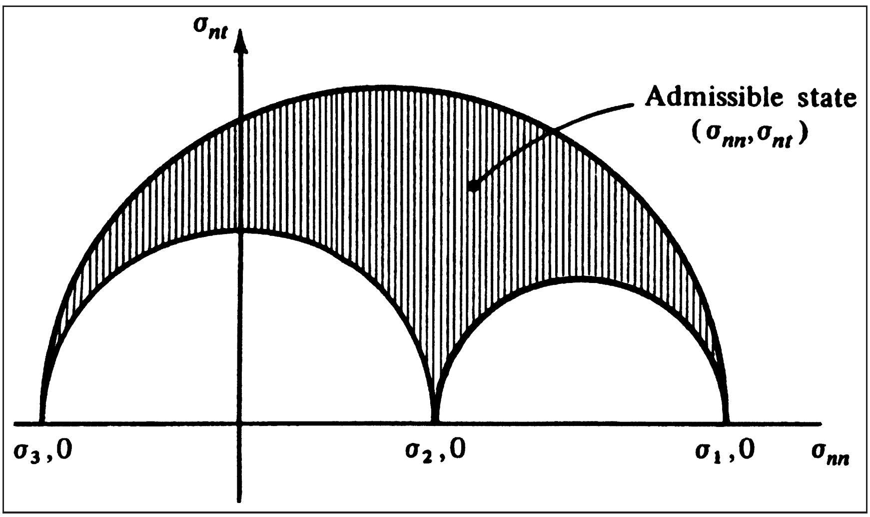

Figure 3-7.4

stress is (for $\theta = 0^\circ$)

$$(\sigma_{nn})_{\max} = \sigma_1 \tag{a}$$

Similarly, the minimum value of the normal stress is

$$(\sigma_{nn})_{\min} = \sigma_2 \tag{b}$$

For the shearing stress, we have, by Fig. 3-7.3,

$$(\sigma_{nt})_{\max} = \frac{1}{2}(\sigma_1 - \sigma_2)$$

Analogously, Eqs. (3-7.8) and (3-7.9) and Fig. 3-7.4 yield, for three-dimensional problems,

$$(\sigma_{nn})_{\max} = \sigma_1$$

$$(\sigma_{nn})_{\min} = \sigma_3$$

$$(\sigma_{nt})_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

In the early history of stress analysis, Mohr's circles were used extensively. However, today it is used principally as a heuristic device (Smith and Sidebottom, 1965).

Example 3-7.1. Stress Quantities at a Point in a Medium. The stress components at a point in a body are given as

$$\sigma_{11} = 50{,}000 \text{ psi} \qquad \sigma_{22} = 50{,}000 \text{ psi}$$
$$\sigma_{12} = \sigma_{13} = \sigma_{23} = \sigma_{33} = 0$$

Because there are no shearing stresses acting, the principal stresses at the point are

$$\sigma_1 = 50{,}000 \qquad \sigma_2 = 50{,}000 \qquad \sigma_3 = 0 \tag{a}$$

Because $\sigma_1 = \sigma_2 = 50{,}000$ and $\sigma_3 = 0$, the three Mohr circles reduce to a single circuit (Fig. 3-7.4) with origin at $\sigma_{nn} = 25{,}000$ and with radius $R = 25{,}000$. Accordingly, by Table 3-5.1, $(\sigma_{nt})_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3) = 25{,}000$ or, from Mohr's circle, $(\sigma_{nt})_{\max} = R = 25{,}000$. This is the largest value of shearing stress that exists at the point. For example, it is larger than τ_0 [see Eq. (b) below].

By Eqs. (a) and (3-6.8), the octahedral shearing stress τ_0 is given by

$$9\tau_0^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 = 5000 \times 10^6$$

or

$$\tau_0 = 23{,}570 \text{ psi} \tag{b}$$

Again, note that $(\sigma_{nt})_{\max}$ is larger than τ_0. The octahedral normal stress σ_{oct} (see Problem 3-6.3) is

$$\sigma_{oct} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} = \sigma_m = 33,333\,\text{psi} \qquad \text{(c)}$$

Problem Set 3-7

1. A thin skewed plate is acted on by tensile edge stresses (S_1, S_2) directed parallel to the sides of the plate (Fig. P3-7.1).

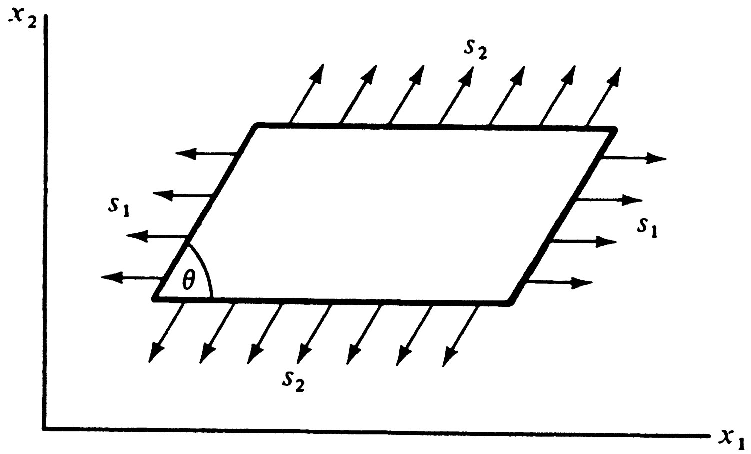

Figure P3-7.1

(a) In terms of S_1, S_2, and θ, derive expressions for stress components σ_{11}, σ_{22}, and σ_{12}.

(b) Sketch Mohr's circle for the stress state in the plate. Hence, derive expressions for the principal stress (σ_1, σ_2) in terms of σ_{11}, σ_{22}, and σ_{12}. Note that because σ_{11}, σ_{22}, and σ_{12} are known from part (a), these expressions may be used to express (σ_1, σ_2) in terms of S_1, S_2, and θ.

2. The stress tensor is

$$\begin{pmatrix} 9 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 6 \end{pmatrix}$$

Determine the numerical values of the principal stresses. Then determine the direction of the principal axes of stress.

3. The stress tensor with respect to (x_1, x_2, x_3) axes is

$$\begin{pmatrix} 2 & 0 & \sqrt{5} \\ 0 & 4 & 0 \\ \sqrt{5} & 0 & 6 \end{pmatrix}$$

Determine the principal stresses and their directions.

4. Show that in the case $\sigma_1 > \sigma_2 = \sigma_3$, Mohr's circles reduce to a single circle (Mohr's circle), Fig. P3-7.4. Then show that any admissible stress state $(\sigma_{nn}, \sigma_{nt})$ must lie on Mohr's circle.

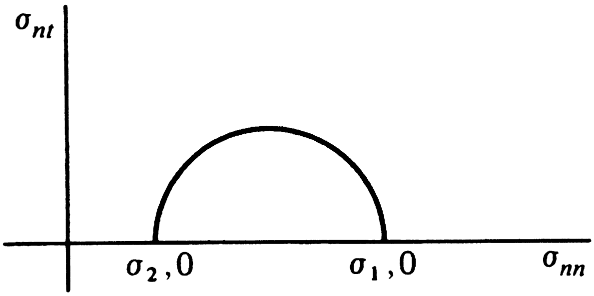

Figure P3-7.4

5. Show that when $\sigma_1 = \sigma_2 = \sigma_3$, Mohr's circles reduced to a single point in the $(\sigma_{nn}, \sigma_{nt})$ space.

6. Show that the maximum value of σ_{nn} is equal to σ_1, the minimum value of σ_{nn} is σ_3, and the maximum value of σ_{nt} is $(\sigma_1 - \sigma_3)/2$.

7. Construct Mohr's circles for Problem 3-6.7.

8. Construct Mohr's circles for Problem 3-6.17.

3-8 Differential Equations of Motion of a Deformable Body Relative to Spatial Coordinates

It is known from elementary mechanics that the resultant force that acts on any body is equal to the mass of the body times the acceleration of the mass center of the body, and the resultant moment that acts on the body is equal to the time rate of change of moment of momentum. In the case of rigid bodies (bodies that do not deform), these conditions lead to a system of six equations (three force equations and three moment equations), which with boundary conditions (initial conditions) completely specify the motion of the body. However, the motion of deformable media is not by any means completely defined by these conditions.

Nevertheless, the foregoing principles may be used to derive equations that (together with stress–strain relationships and boundary conditions derived later) describe the motion of a deformable medium. It is necessary not only to apply the principles to the medium as a whole but to each element of which the medium is composed.

Let $\mathscr{S}$ be an arbitrary closed surface within a deformed medium. Let $\mathscr{V}$ be the element of volume enclosed by $\mathscr{S}$. The external forces acting on the volume consist of two parts: (1) body force and (2) tractive or surface force.

Let (x_1, x_2, x_3) denote spatial coordinates (Chapter 2, Section 2-3). The projection of the resultant body force vector (Fig.3-8.1) on the x_1 axis is

$$\iiint\limits_{\text{through volume } \mathscr{V}} \mathscr{B}_1 \, d\mathscr{V} \tag{a}$$

where $\mathscr{B}_1$ denotes the x_1 projection of the body-force vector per unit volume relative to spatial coordinates (x_1, x_2, x_3). If the body force is entirely due to gravity, as is common, $\mathscr{B}_1 = \rho g_1$, where ρ is the mass density and g_1 is the x_1 projection of the vector acceleration of gravity (say, directed toward the center of the earth). Often the motion of a mass element is characterized by the introduction of inertia forces. In general, we may treat inertia forces as body forces; that is, they act on each element of mass in the body. However, because of their significance in the study of the motion of a medium, inertia forces are not usually included in the body-force vector. Rather, they are treated separately (Fig. 3-8.1).

Accordingly, the x_1 projection of the resultant inertia force vector is

$$\iiint\limits_{\text{through volume } \mathscr{V}} (-a_1) \, dm = \iiint\limits_{\text{through volume } \mathscr{V}} (-\rho a_1) \, d\mathscr{V} \tag{b}$$

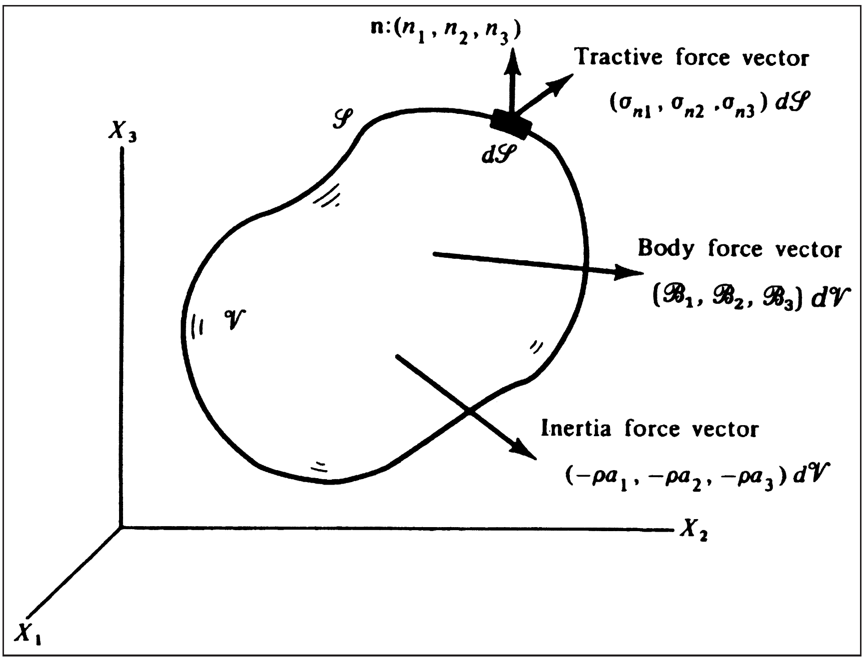

Figure 3-8.1

where a_1 is the x_1 projection of the acceleration vector of the mass increment dm, and $\rho\, d\mathbf{V} = dm$, where ρ denotes mass density of the volume element $d\mathscr{V}$.

The x_1 projection of the resultant traction vector (Fig. 3-8.1) exerted on surface $\mathscr{S}$ is

$$\iint\limits_{\text{over surface } \mathscr{S}} \sigma_{n1}\, d\mathscr{S} \tag{c}$$

where σ_{n1} denotes the x_1 projection of the stress vector on $d\mathscr{S}$. By Eq. (3-3.10), Eq. (c) may be written in the form

$$\iint\limits_{\text{over surface } \mathscr{S}} (n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31})\, d\mathscr{S} \tag{d}$$

where n_i is the unit normal vector of the surface, positive outward.

Summing the x_1 projections of the resultant force that acts on the mass element dm, by Eqs. (a), (b), and (d) we obtain

$$\iiint\limits_{\text{through volume } \mathscr{V}} (\mathscr{B}_1 - \rho a_1)\, d\mathscr{V} + \iint\limits_{\text{over surface } \mathscr{S}} (n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31})\, d\mathscr{S} = 0 \tag{e}$$

Applying the divergence theorem (Chapter 1, Section 1-16) to the surface integral of Eq. (e) and regrouping, we obtain

$$\iiint \left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \mathscr{B}_1 - \rho a_1 \right) d\mathscr{V} = 0 \tag{f}$$

Equation (f) applies to any volume element $d\mathscr{V}$ in the body. Consequently, for Eq. (f) to be satisfied for all parts of the body, the integrand must vanish identically; that is,

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \mathscr{B}_1 = \rho a_1 \tag{g}$$

In a similar manner, summations of force projections in the x_2 and x_3 directions yield two more equations. Thus, the following set of three equations is obtained:

$$\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + \mathscr{B}_1 &= \rho a_1 \\
\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + \mathscr{B}_2 &= \rho a_2 \\
\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \mathscr{B}_3 &= \rho a_3
\end{aligned} \tag{3-8.1}$$

where (a_1, a_2, a_3) are the (x_1, x_2, x_3) projections of the acceleration vector of mass element dm.

In index notation (Section 1-23), Eqs. (3-8.1) may be written ($\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$)

$$\sigma_{\beta\alpha,\beta} + \mathscr{B}_\alpha = \rho a_\alpha \qquad \alpha, \beta = 1, 2, 3 \tag{3-8.2}$$

Equations (3-8.1) or (3-8.2) are called the *differential equations of motion* of a deformable medium. Alternatively, they may be derived by summation of forces that act on the faces of a cubic element, considering the variation of stress through the cube. For incompressible media, $\rho =$ constant. If the medium is in statical equilibrium, the right-hand terms in Eqs. (3-8.1) and (3-8.2) are zero. Then Eqs. (3-8.1) or (3-8.2) are called the *differential equations of equilibrium*.

If the body force ($\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3$) is derivable from a potential function F, for example, gravity potential, Eqs. (3-8.1) may be written in the form

$$\begin{aligned}
\frac{\partial(\sigma_{11} - F)}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} &= \rho a_1 \\
\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial(\sigma_{22} - F)}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} &= \rho a_2 \\
\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial(\sigma_{33} - F)}{\partial x_3} &= \rho a_3
\end{aligned} \tag{3-8.3}$$

where

$$\mathscr{B}_1 = -\frac{\partial F}{\partial x_1} \qquad \mathscr{B}_2 = -\frac{\partial F}{\partial x_2} \qquad \mathscr{B}_3 = -\frac{\partial F}{\partial x_3} \tag{3-8.4}$$

or, in index notation,

$$\frac{\partial(\sigma_{\beta\alpha} - \delta_{\beta\alpha} F)}{\partial x_\beta} = \rho a_\alpha \qquad \alpha, \beta = 1,\ 2,\ 3 \tag{3-8.5}$$

where $\delta_{\beta\alpha}$ is the Kronecker delta (Chapter 1, Section 1-24).

The equations of this chapter summarize the general theory of stress. For the approximation of plane stress in the (x_1, x_2) plane, $\sigma_{33} = \sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0$ (Chapter 5). If body forces are negligible, we may set $\mathscr{B}_1 = \mathscr{B}_2 = \mathscr{B}_3 = 0$. The general equations are simplified accordingly.

For small-deformation (displacement) theory, Eqs. (3-8.1), (3-8.2), (3-8.3), *and* (3-8.5) *hold approximately if* $\sigma_{\beta\alpha}, \mathscr{B}_\alpha, F, \rho,$ *and* a_α *are considered functions of material coordinates* (*see Appendix* 3C).

It is worthwhile to note that so far the material body in question is considered as a continuum, that is, a continuous collection of uncountably infinite mass points each with vanishing size. That is why the size and mass of the entire material body are finite; only the mass density can be defined as

$$\rho \equiv \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \tag{3-8.6}$$

while both Δm and ΔV reduce to zero in the limiting process. Also, it is noticed that the spatial equation of continuity, Eq. (2-8.12), and the equation of motion, Eq. (3-8.2), are expressed as *partial differential equations in space and time*. These are the fundamental characteristics of any continuum theory. On the diametrically opposite side, molecular dynamics (MD) views a material body as a collection of finite number of atoms, which exert interatomic forces on each other. The governing equations are just Newton's second law, which can be simply expressed as

$$m^i \ddot{\mathbf{r}}^i = \mathbf{f}^i + \boldsymbol{\varphi}^i \qquad i = 1, 2, 3, \ldots, n \tag{3-8.7}$$

where n is the total number of atoms in the system; m^i, $\mathbf{r}^i$, and $\ddot{\mathbf{r}}^i$ are the mass, position vector, and acceleration vector of atom i, respectively; $\mathbf{f}^i$ and $\boldsymbol{\varphi}^i$ are the interatomic force and body force acting on atom i, respectively. Here, we put m^i, instead of m, as the mass of the ith atom to emphasize that the system consists of not just many atoms but also many kinds of atoms. The material and spatial equations of continuity are recalled as

$$\rho = \rho^* J \tag{2-17.5}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{2-8.12}$$

which are just statements of *law of conservation of mass* in different descriptions. The counterpart in MD is simply that m^i $(i = 1, 2, 3, \ldots, n)$ *are constants in time*. Equation 3-8.1 or (3-8.2) is the statement of the *law of balance of linear momentum* in continuum mechanics, including elasticity, while its counterpart in MD is Eq. (3-8.7), which is a set of *ordinary differential equations in time*. The *law of balance of angular momentum* for classical continuum mechanics implies that stress stensor is symmetric

$$\sigma_{\alpha\beta} = \sigma_{\beta\alpha} \tag{3-8.8}$$

In MD the *law of balance of angular momentum* is automatically satisfied since each atom is considered as a mathematical point with no finite size and, hence, no intrinsic angular momentum. We will discuss the law of conservation of energy for both elasticity and molecular dynamics in next chapter.

Problem Set 3-8

1. Using the principle that the resultant moment with respect to any fixed axis of all forces acting on volume $\mathscr{V}$ is equal to the time rate of change of moment of momentum of the volume with respect to the axis, derive the equations that result from consideration of moments with respect to the x axis. Simplify the equations for equilibrium. How would these equations be altered in the presence of a body moment resulting from an electric or magnetic field, that is, in the presence of a moment proportional to a mass element?

2. Derive Eq. (3-8.1) by applying the principle $\mathbf{F} = m\mathbf{a}$ to a cubic element in the body. [*Hint*: If the normal stress on the plane perpendicular to the x axis at a point x is σ_x, then the normal stress that acts on the plane perpendicular to the x axis at the point $x + dx$ is $\sigma_x + (\partial\sigma_x/\partial x)\,dx$, and so on.]

3. A body is in a state of equilibrium. The stress components $\sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}$ are zero. Body forces are zero. Derive the most general formulas for the stress components σ_x, τ_{xy}.

4. The stress vectors $\boldsymbol{\sigma}_x, \boldsymbol{\sigma}_y$ act on planes $x =$ constant, $y =$ constant, where (x, y) are plane oblique coordinates (Fig. P3-8.4). Let $(\mathbf{e}_x, \mathbf{e}_y)$ be unit vectors along axes (x, y). Then

$$\sigma_x = p_{xx}\mathbf{e}_x + p_{xy}\mathbf{e}_y$$

$$\sigma_y = p_{yx}\mathbf{e}_x + p_{yy}\mathbf{e}_y$$

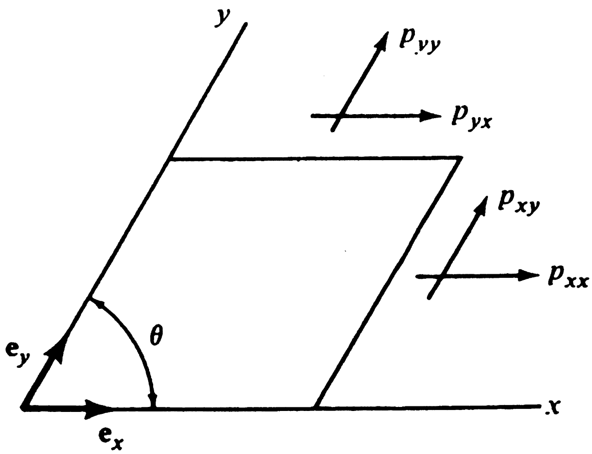

Figure P3-8.4

(a) Compute the normal stress (σ_x, σ_y) and the shear stresses (τ_{xy}, τ_{yx}) on planes $x =$ constant, $y =$ constant in terms of $p_{xx}, p_{xy}, p_{yx}, p_{yy}$, and θ.

(b) Derive the relation between τ_{xy} and τ_{yx}.

(c) Derive the differential equilibrium equations for the element in terms of $p_{xx}, p_{xy}, p_{yy}, p_{yx}$ and $\mathbf{e}_x, \mathbf{e}_y$.

5. For an axially symmetric state of stress in cylindrical coordinates, $\tau_{r\theta} = \tau_{\theta z} = 0$, and the other stress components are independent of θ (Appendix 3A). Show that if there is no body force, the equilibrium equations for an axially symmetric state of stress are satisfied automatically if the stresses are derived from two arbitrary functions, $F(r, z)$, $H(r, z)$, as follows:

$$\sigma_r = F_{zz} + \frac{1}{r}H_r \qquad \sigma_\theta = F_{zz} + H_{rr} \qquad \sigma_z = F_{rr} + \frac{1}{r}F_r \qquad \tau_{rz} = -F_{rz}$$

(Subscripts on F and H denote partial derivatives.)

6. The state of stress in a continuum is defined relative to axes x_α by the components $\sigma_{11} = x_1^2 + x_2^2$, $\sigma_{33} = x_1^2 + x_3^2$, $\sigma_{12} = x_1x_2$, and $\sigma_{22} = \sigma_{23} = \sigma_{13} = 0$. Determine the body forces that act in the continuum for the case of equilibrium.

APPENDIX 3A DIFFERENTIAL EQUATIONS OF EQUILIBRIUM IN CURVILINEAR SPATIAL COORDINATES

3A-1 Differential Equations of Equilibrium in Orthogonal Curvilinear Spatial Coordinates

Let $\mathscr{S}$ be a closed surface within a deformed medium, and let $\mathscr{V}$ denote the volume enclosed by $\mathscr{S}$ (Fig. 3-8.1). Let (l, m, n) be the direction cosines of the outwardly directed unit normal to $\mathscr{S}$ with respect to orthogonal curvilinear spatial coordinates (x, y, z) in the deformed region [see Eq. (2A-2.5) in Chapter 2].

As noted in Chapter 3, the stress vector is defined by the equilibrium conditions in terms of the stress components $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})$ defined relative to (x, y, z) axes. Defining the coordinate system by a vector function $\mathbf{r} = \mathbf{r}(x, y, z)$ and noting that unit vectors relative to (x, y, z) coordinate lines are defined by Eq. (2A-1.7) in Chapter 2, we express the stress vector on surface $\mathscr{V}$ as

$$(l\sigma_{xx}+m\sigma_{yx}+n\sigma_{zx})\frac{\mathbf{r}_x}{\alpha}+(l\sigma_{xy}+m\sigma_{yy}+n\sigma_{zy})\frac{\mathbf{r}_y}{\beta}+(l\sigma_{xz}+m\sigma_{yz}+n\sigma_{zz})\frac{\mathbf{r}_z}{\gamma} \qquad \text{(a)}$$

The body forces acting on the body (see Section 3-8) may be written

$$\rho\left(\mathscr{B}_x\frac{\mathbf{r}_x}{\alpha}+\mathscr{B}_y\frac{\mathbf{r}_y}{\beta}+\mathscr{B}_z\frac{\mathbf{r}_z}{\gamma}\right)\alpha\beta\gamma\;dx\;dy\;dz \qquad \text{(b)}$$

where here $(\mathscr{B}_x, \mathscr{B}_y, \mathscr{B}_z)$ denotes the body force per unit mass, ρ denotes mass density, and $\sigma\beta\gamma\, dx\, dy\, dz$ represents the volume in curvilinear coordinates (x, y, z) (see Chapter 1, Section 1-22).

With Eqs. (a) and (b) the equilibrium of forces acting on the material in $\mathscr{V}$ requires

$$\iint_{\mathscr{S}}\left[(l\sigma_{xx}+m\sigma_{yx}+n\sigma_{zx})\frac{\mathbf{r}_x}{\alpha}+(l\sigma_{xy}+m\sigma_{yy}+n\sigma_{zy})\frac{\mathbf{r}_y}{\beta}+(l\sigma_{xz}+m\sigma_{yz}+n\sigma_{zz})\frac{\mathbf{r}_z}{\gamma}\right]d\mathscr{S}$$
$$+\iiint_{\mathscr{V}}\rho\left(\mathscr{B}_x\frac{\mathbf{r}_x}{\alpha}+\mathscr{B}_y\frac{\mathbf{r}_y}{\beta}+\mathscr{B}_x\frac{\mathbf{r}_z}{\gamma}\right)\alpha\beta\gamma\,dx\,dy\,dz=0 \qquad \text{(c)}$$

Transforming the surface integral in Eq. (c) into a volume integral by means of the divergence theorem [Eqs. (1-22.8) and (1-22.12) in Chapter 1], we find

$$\iiint_{\mathscr{V}}\left\{\frac{\partial}{\partial x}\left[\beta\gamma\left(\sigma_{xx}\frac{\mathbf{r}_x}{\alpha}+\sigma_{xy}\frac{\mathbf{r}_y}{\beta}+\sigma_{xz}\frac{\mathbf{r}_z}{\gamma}\right)\right]+\frac{\partial}{\partial y}\left[\gamma\alpha\left(\sigma_{yx}\frac{\mathbf{r}_x}{\alpha}+\sigma_{yy}\frac{\mathbf{r}_y}{\beta}+\sigma_{yz}\frac{\mathbf{r}_z}{\gamma}\right)\right]\right.$$
$$+\frac{\partial}{\partial z}\left[\alpha\beta\left(\sigma_{zx}\frac{\mathbf{r}_x}{\alpha}+\sigma_{zy}\frac{\mathbf{r}_y}{\beta}+\sigma_{zz}\frac{\mathbf{r}_z}{\gamma}\right)\right]$$
$$\left.+\rho(\beta\gamma\mathscr{B}_x\mathbf{r}_x+\gamma\alpha\mathscr{B}_y\mathbf{r}_y+\alpha\beta\mathscr{B}_z\mathbf{r}_z)\right\}\;dx\;dy\;dz=0 \qquad \text{(d)}$$

Equation (d) must hold for arbitrary volume element; hence, the integrand must vanish identically. Accordingly, setting the integrand equal to zero and performing the indicated differentiations of products, we obtain the vector equilibrium equation ($\sigma_{xy} = \sigma_{yx}$, etc.):

$$\begin{aligned}
&\left[\frac{\partial}{\partial x}\left(\frac{\beta\gamma}{\alpha}\sigma_{xx}\right) + \frac{\partial}{\partial y}(\gamma\sigma_{yx}) + \frac{\partial}{\partial z}(\beta\sigma_{zx}) + \rho\beta\gamma\mathscr{B}_x\right]\mathbf{r}_x \\
&\quad + \left[\frac{\partial}{\partial x}(\gamma\sigma_{xy}) + \frac{\partial}{\partial y}\left(\frac{\gamma\alpha}{\beta}\sigma_{yy}\right) + \frac{\partial}{\partial z}(\alpha\sigma_{zy}) + \rho\gamma\alpha\mathscr{B}_y\right]\mathbf{r}_y \\
&\quad + \left[\frac{\partial}{\partial x}(\beta\sigma_{xz}) + \frac{\partial}{\partial y}(\alpha\sigma_{yz}) + \frac{\partial}{\partial z}\left(\frac{\alpha\beta}{\gamma}\sigma_{zz}\right) + \rho\alpha\beta\mathscr{B}_z\right]\mathbf{r}_z \\
&\quad + \frac{\beta\gamma}{\alpha}\sigma_{xx}\mathbf{r}_{xx} + \frac{\gamma\alpha}{\beta}\sigma_{yy}\mathbf{r}_{yy} + \frac{\alpha\beta}{\gamma}\sigma_{zz}\mathbf{r}_{zz} \\
&\quad + 2\alpha\sigma_{yz}\mathbf{r}_{yz} + 2\beta\sigma_{xz}\mathbf{r}_{xz} + 2\gamma\sigma_{xy}\mathbf{r}_{xy} = 0
\end{aligned} \tag{e}$$

The three scalar equations of equilibrium with respect to axes (x, y, z) are obtained by taking the scalar products of Eq. (e) with $\mathbf{r}_x, \mathbf{r}_y, \mathbf{r}_z$, respectively. Expressing the scalar products $\mathbf{r}_x \cdot \mathbf{r}_x, \mathbf{r}_x \cdot \mathbf{r}_y, \ldots, \mathbf{r}_x \cdot \mathbf{r}_{xy}$, and so on, in terms of α, β, γ by means of Eqs. (2A-1.4), (2A-1.6), and (2A-1.10) in Chapter 2, we obtain the three scalar equations ($\sigma_{xy} = \sigma_{yx}$, etc.):

$$\begin{aligned}
&\frac{\partial}{\partial x}(\beta\gamma\sigma_{xx}) + \frac{\partial}{\partial y}(\gamma\alpha\sigma_{yx}) + \frac{\partial}{\partial z}(\alpha\beta\sigma_{zx}) + \gamma\alpha_y\sigma_{xy} + \beta\alpha_z\sigma_{xz} \\
&\qquad - \gamma\beta_x\sigma_{yy} - \beta\gamma_x\sigma_{zz} + \rho\alpha\beta\gamma\mathscr{B}_x = 0 \\
&\frac{\partial}{\partial x}(\beta\gamma\sigma_{xy}) + \frac{\partial}{\partial y}(\gamma\alpha\sigma_{yy}) + \frac{\partial}{\partial z}(\alpha\beta\sigma_{zy}) + \alpha\beta_z\sigma_{yz} + \gamma\beta_x\sigma_{xy} \\
&\qquad - \alpha\gamma_y\sigma_{zz} - \gamma\alpha_y\sigma_{xx} + \rho\alpha\beta\gamma\mathscr{B}_y = 0 \\
&\frac{\partial}{\partial x}(\beta\gamma\sigma_{xz}) + \frac{\partial}{\partial y}(\gamma\alpha\sigma_{yz}) + \frac{\partial}{\partial z}(\alpha\beta\sigma_{zz}) + \beta\gamma_x\sigma_{xz} + \alpha\gamma_y\sigma_{yz} \\
&\qquad - \beta\alpha_z\sigma_{xx} - \alpha\beta_z\sigma_{yy} + \rho\alpha\beta\gamma\mathscr{B}_z = 0
\end{aligned} \tag{3A-1.1}$$

Equations (3A-1.1) represent the three scalar equilibrium equations relative to orthogonal curvilinear coordinates (x, y, z). Because they are purely statical in nature, they apply to all continuous-media materials. They may be extended to include dynamical problems, provided the body force $(\rho\mathscr{B}_x, \rho\mathscr{B}_y, \rho\mathscr{B}_z)$ is considered to include inertial forces. Love (2009) has derived Eq. (3A-1.1) without employing vector algebra.

3A-2 Specialization of Equations of Equilibrium

Commonly employed orthogonal curvilinear coordinate systems in three-dimensional problems are the cylindrical coordinate system (r, θ, z) and the spherical coordinate system (r, θ, ϕ); in plane problems the plane polar coordinate system (r, θ) is frequently used. Specialization of Eqs. (3A-1.1) for these systems follows:

a) *Cylindrical coordinate system* (r, θ, z). In Eqs. (3A-1.1) we let $x = r$, $y = \theta$, and $z = z$. Then the differential length ds is defined by the relation

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \tag{3A-2.1}$$

Comparison of Eqs. (2A-1.5) in Chapter 2 and 3A-2.1) yields

$$\alpha = 1 \qquad \beta = r \qquad \gamma = 1 \tag{3A-2.2}$$

Substituting Eq. (3A-2.2) into Eqs. (3A-1.1), we obtain the equilibrium equations ($\sigma_{r\theta} = \sigma_{\theta r}$, etc.):

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho \mathscr{B}_r = 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + \rho \mathscr{B}_\theta = 0 \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + \rho \mathscr{B}_z = 0 \end{aligned} \tag{3A-2.3}$$

where $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{r\theta}, \sigma_{rz}, \sigma_{\theta z})$ represent stress components defined relative to cylindrical coordinates (r, θ, z).

b) *Spherical Coordinate System* (r, θ, ϕ). In Eqs. (3A-1.1), we let $x = r$, $y = \theta$, and $z = \phi$, where r is the radial coordinate, θ is the latitude, and ϕ is the longitude. Because the differential length ds is defined by

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \tag{3A-2.4}$$

comparison of Eqs. (2A-1.5) and (3A-2.4) yields

$$\alpha = 1 \qquad \beta = r \qquad \gamma = r \sin \theta \tag{3A-2.5}$$

Substituting Eq. (3A-2.5) into Eqs. (3A-1.1), we obtain the equilibrium equations ($\sigma_{r\theta} = \sigma_{\theta r}$, etc.):

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta) + \rho \mathscr{B}_r = 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{1}{r}[(\sigma_{\theta\theta} - \sigma_{\theta\theta})\cot\theta + 3\sigma_{r\theta}] + \rho \mathscr{B}_\theta = 0 \\ \frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{1}{r}(3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta) + \rho \mathscr{B}_\phi = 0 \end{aligned} \tag{3A-2.6}$$

where $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{\phi\phi}, \sigma_{r\theta}, \sigma_{r\phi}, \sigma_{\theta\phi})$ are defined relative to spherical coordinates (r, θ, ϕ).

c) *Plane Polar Coordinate System* (r, θ). In plane-stress problems relative to (x, y) coordinates, $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$, and the remaining stress components are functions of (x, y) only (see Chapter 5, Section 5-2). Letting $x = r$, $y = \theta$, and $z = z$ in Eqs. (3A-2.3) and noting that $\sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = (\partial/\partial z) = 0$, we obtain from Eqs. (3A-2.3) ($\sigma_{r\theta} = \sigma_{\theta r}$, etc.):

$$\begin{aligned} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho \mathscr{B}_r = 0 \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\frac{\sigma_{r\theta}}{r} + \rho \mathscr{B}_\theta = 0 \end{aligned} \quad \text{(3A-2.7)}$$

where $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})$ are stress components defined relative to polar coordinates (r, θ). Equations (3A-2.7) hold also for plane-strain problems (Chapter 5, Section 5-1), and they apply to generalized plane-stress problems, provided $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})$ are defined as mean stress components relative to coordinate z (Chapter 5, Section 5-2). In addition to Eqs. (3A-2.6), we require $\mathscr{B}_z = 0$.

The form of Eqs. (3A-1.1), (3A-2.3), (3A-2.6), *and* (3A-2.7) *holds relative to material coordinates for small-displacement theory* (*see Appendix* 3*C*).

3A-3 Differential Equations of Equilibrium in General Spatial Coordinates

In tensor notation it may be shown that the differential equations of equilibrium relative to spatial coordinates x_α may be written (Green and Zerna, 1992)

$$\sigma^{\beta\alpha}||_\beta + \rho \mathscr{B}^\alpha = \rho a^\alpha \quad \text{(3A-3.1)}$$

where $\sigma^{\beta\alpha}$ denotes a *contravariant tensor* (stress tensor) *of the second kind*, $\mathscr{B}^\alpha$ and a^α denote *contravariant tensors of the first kind* (body force and acceleration vectors, respectively), ρ denotes mass density, and the symbol || denotes *covariant differentiation*. Alternatively, Eqs. (3A-3.1) may be written

$$\frac{\partial \sigma^{\beta\alpha}}{\partial x^\beta} + \sigma^{\beta\gamma}\Gamma^\alpha_{\beta\gamma} + \sigma^{\beta\alpha}\Gamma^\gamma_{\beta\gamma} + \rho \mathscr{B}^\alpha = \rho a^\alpha \quad \text{(3A-3.2)}$$

where $\Gamma^\alpha_{\beta\gamma}$ denote *Christoffel symbols of the second kind*. In turn, the Christoffel symbols are related to *covariant and contravariant metric tensors* $g_{\alpha\beta}$, $g^{\alpha\beta}$ by the formulas

$$\begin{aligned} \Gamma_{\alpha\beta\gamma} &= \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x_\beta} + \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x_\gamma}\right) \\ \Gamma^\gamma_{\alpha\beta} &= g^{\gamma\delta}\Gamma_{\alpha\beta\delta} \end{aligned} \quad \text{(3A-3.3)}$$

where $\Gamma_{\alpha\beta\gamma}$ denote *Christoffel symbols of the first kind*.

In turn, $g_{\alpha\beta}$, $g^{\alpha\beta}$ are related to the element ds of length squared by

$$ds^2 = g_{\alpha\beta} dy^\alpha dy^\beta = g^{\alpha\beta} dy_\alpha dy_\beta \tag{3A-3.4}$$

where dy^α, dy_α denote contravariant and covariant tensors, respectively.[4] In general,

$$g_{\alpha\beta} g^{\beta\gamma} = \delta_\alpha^\gamma \tag{3A-3.5}$$

where $\delta_\alpha^\gamma = 0$ for $\gamma \neq \alpha$, and $\delta_\alpha^\gamma = 1$ for $\gamma = \alpha$ is the mixed Kronecker delta. For orthogonal curvilinear coordinates

$$\begin{aligned} g^{\alpha\beta} &= g_{\alpha\beta} = 0 \qquad \alpha \neq \beta \\ g^{ii} &= 1/g_{ii} \qquad i = 1, 2, 3 \quad (i \text{ not summed}) \end{aligned} \tag{3A-3.6}$$

In terms of metric coefficients (α, β, γ) (see Chapter 2, Appendix 2A), for orthogonal coordinates

$$\begin{aligned} & g_{11} = \alpha^2 \qquad g_{22} = \beta^2 \qquad g_{33} = \gamma^2 \\ & g^{11} = \frac{1}{\alpha^2} \qquad g^{22} = \frac{1}{\beta^2} \qquad g^{33} = \frac{1}{\gamma^2} \\ & g_{12} = g_{13} = g_{23} = g^{12} = g^{13} = g^{23} = 0 \end{aligned} \tag{3A-3.7}$$

Also, for orthogonal coordinates, $\Gamma_{\alpha\beta}^\gamma = 0$, for $\alpha \neq \beta \neq \gamma \neq \alpha$, and $\Gamma_{\alpha\beta}^\alpha = \Gamma_{\beta\alpha}^\alpha$. Hence, Eqs. (3A-3.3) and (3A-3.7) yield

$$\begin{aligned} & \Gamma_{11}^1 = \frac{\alpha_x}{\alpha} \qquad \Gamma_{22}^2 = \frac{\beta_y}{\beta} \qquad \Gamma_{33}^3 = \frac{\gamma_z}{\gamma} \\ & \Gamma_{12}^1 = \frac{\alpha_y}{\alpha} \qquad \Gamma_{13}^1 = \frac{\alpha_z}{\alpha} \qquad \Gamma_{11}^2 = -\frac{\alpha\alpha_y}{\beta^2} \\ & \Gamma_{11}^3 = -\frac{\alpha\alpha_z}{\gamma^2} \qquad \Gamma_{12}^2 = \frac{\beta_x}{\beta} \qquad \Gamma_{32}^2 = \frac{\beta_z}{\beta} \\ & \Gamma_{22}^1 = -\frac{\beta\beta_x}{\alpha^2} \qquad \Gamma_{22}^3 = -\frac{\beta\beta_z}{\gamma^2} \qquad \Gamma_{33}^1 = -\frac{\gamma\gamma_x}{\alpha^2} \\ & \Gamma_{33}^2 = -\frac{\gamma\gamma_y}{\beta^2} \qquad \Gamma_{23}^3 = \frac{\gamma_y}{\gamma} \qquad \Gamma_{13}^3 = \frac{\gamma_x}{\gamma} \\ & \Gamma_{23}^1 = \Gamma_{31}^2 = \Gamma_{12}^3 = 0 \end{aligned} \tag{3A-3.8}$$

Substitution of Eqs. (3A-3.8) into Eqs. (3A-3.2) yields the differential equation of motion for orthogonal curvilinear coordinates [see Eqs. (3A-1.1), where $a^\alpha = 0$].

APPENDIX 3B EQUATIONS OF EQUILIBRIUM INCLUDING COUPLE STRESS AND BODY COUPLE

In this appendix the equations of equilibrium including couple stresses and body couple are derived. The effects of the introduction of couple stresses and body

[4]See Appendix 2C, where formulas for $g_{\alpha\beta}$ are derived. See also Green and Zerna (1992).

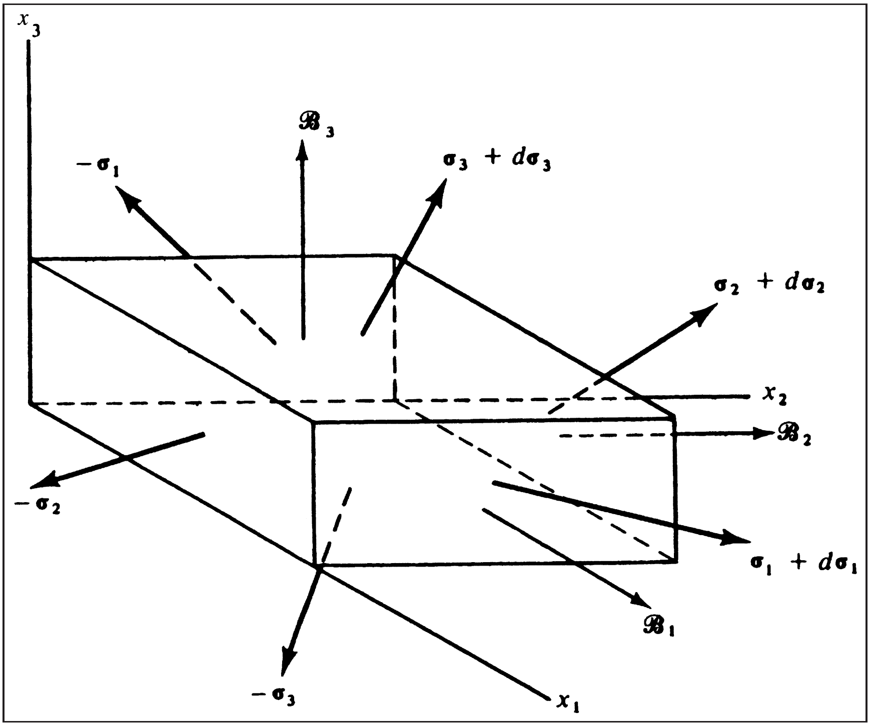

Figure 3B-1

couple on classical elasticity solutions have been studied by a number of investigators. Our purpose here, however, is to note the nature of the equations including stress couples and body couple. Questions of solutions of the associated boundary value problems, questions of uniqueness and completeness, and applications are left to the literature.[5]

Consider a regular parallelepiped volume element $\mathscr{V}$ of a deformed region $\mathscr{R}$ of mass density ρ. Let the stress vectors $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$ act on planes perpendicular to spatial axes (x_1, x_2, x_3). These stress vectors undergo changes $d\boldsymbol{\sigma}_1, d\boldsymbol{\sigma}_2, d\boldsymbol{\sigma}_3$ under changes dx_i of coordinates x_i (Fig. 3B-1). Let $\mathscr{B}_\alpha$ be the body-force vector per unit mass acting on $\mathscr{V}$ (see Section 3-8).

In addition to the stress vectors $\boldsymbol{\sigma}_i$ and body force vector $\mathscr{B}_\alpha$, let the element $\mathscr{V}$ be subjected to surface couples $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2$, $\boldsymbol{\mu}_3$ per unit area that undergo changes $d\boldsymbol{\mu}_i$ under changes dx_i in x_i. Also, let $\mathscr{V}$ be acted upon by a body-couple vector $\mathbf{C}_\alpha$ per unit volume. Hence, $\boldsymbol{\mu}_i dA_i$ and $\mathbf{C}_\alpha d\mathscr{V}$ represent couples acting upon planes of area $dA_i = dx_j dx_k (i \neq j \neq k \neq i$, and $i, j, k = 1, 2, 3)$, perpendicular to axes x_i and upon volume $d\mathscr{V}$, respectively (Fig. 3B-2). Then the stress equation of

[5]See footnote 1; see also D. E. Carlson, "Stress Functions for Couple and Dipolar Stresses," *24 Quart. J. Appl. Mech.*, 1: 29–35 (1966).

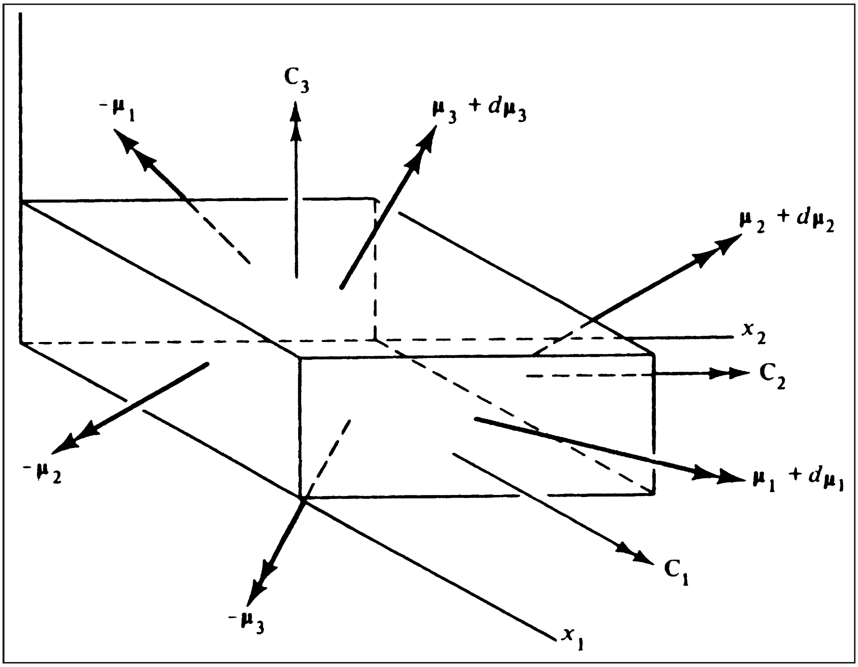

Figure 3B-2

equilibrium, obtained by summing forces along and moments about axes x_i, are

$$
\begin{aligned}
\sigma_{\alpha\beta,\alpha} + \rho\mathscr{B}_\beta &= 0 \\
m_{\alpha\beta,\alpha} + e_{\beta\gamma\delta}\sigma_{\gamma\delta} + C_\beta &= 0
\end{aligned}
\tag{3B-1}
$$

where $e_{\beta\gamma\delta}$ is the alternating tensor (Chapter 1, Section 1-26), $\sigma_{\alpha\beta}$ is the stress tensor (Sections 3-2, 3-3, and 3-8), $m_{\alpha\beta}$ are the components of $\boldsymbol{\mu}_\alpha$ along axes x_β, and subscript $(,\alpha)$ denotes partial differentiation relative to coordinate x_α (Chapter 1, Section 1-23). The set of components $m_{\alpha\beta}$ is the *couple stress tensor*.[6]

A direct consequence of the introduction of $C_\alpha, m_{\alpha\beta}$ into the theory of stress is that the symmetry of the stress tensor $\sigma_{\alpha\beta}$ is lost ($\sigma_{\alpha\beta} \neq \sigma_{\beta\alpha}$). The exploration of the effects of the introduction of stress couples and body couple into the theory of elasticity has been studied by Mindlin (1963), Mindlin and Tiersten (1962), and Sternberg (1968).[7]

[6]The introduction of body-couple C_α and couple stresses $m_{\alpha\beta}$ has its origin in the works of W. Voigt, Theoretische Studien über die Elasticitätsverhältnisse der Krystalle, *Abhandl. Ges. Wiss. Göttingen* 34 (1887); Über Medien ohne innere Kräfte une eine durch sie gelieferte mechanische Deutung der Maxwell-Hertzschen Gleichunge, *Abhandl. Ges. Wiss. Göttingen* 72–79 (1894). The theory was amplified by E. Cosserat and F. Cosserat, *Théorie des corps déformables* (Paris,Hermann & Cie,1909).

[7]See footnotes 1 and 5.

In addition, various related broader generalizations of conventional elasticity theory have been proposed in studies that include hyperstresses of ever-increasing generality and abstractness (physical elusiveness). Here, we take the reader no further but refer him or her to the ever-increasing related litarature of current technical journals.

APPENDIX 3C REDUCTION OF DIFFERENTIAL EQUATIONS OF MOTION FOR SMALL-DISPLACEMENT THEORY

3C-1 Material Derivative. Material Derivative of a Volume Integral

The concept of material derivative and derivative of a volume integral plays a role in the theory of continuum mechanics, particularly in the application of *balance laws*.

Let $Q(x_1, x_2, x_3; t)$ be a scalar point function (such as density, temperature, pressure, a projection of a velocity vector, etc.). With respect to a Newtonian reference frame (Boresi and Schmidt, 2001), $\partial Q/\partial t$ denotes the change of Q with respect to t at a geometrical point fixed with respect to the frame. If the process associated with the function Q is a steady-state process $\partial Q/\partial t = 0$; for example, a steady-state flow is characterized by the condition that Q remain constant for all time at each point (x_1, x_2, x_3). Accordingly, for steady-state conditions all partial derivatives with respect to time t vanish.

In the material (Lagrangian) description of a deformable medium, let (x_1, x_2, x_3) be material coordinates. Then, because (x_1, x_2, x_3) is considered to be the geometrical position of a material particle at a given time, say, $t = 0$, the time rate of change of a function Q in the material description is given by

$$\frac{dQ}{dt} = \left.\frac{\partial Q}{\partial t}\right|_{\mathbf{x}=\text{const}} \tag{3C-1.1}$$

where $\mathbf{x}$ stands for the set of coordinates (x_1, x_2, x_3).

In the spatial description, with coordinates (y_1, y_2, y_3), the coordinates y_i are considered to be the location at time t of a material particle originally at x_i at time $t = 0$. Accordingly, the time rate of change of the function Q is given by

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial y_1}\frac{dy_1}{dt} + \frac{\partial Q}{\partial y_2}\frac{dy_2}{dt} + \frac{\partial Q}{\partial y_3}\frac{dy_3}{dt} + \frac{\partial Q}{\partial t} \tag{3C-1.2}$$

Because $y_i = x_i + u_i$, $dy_i/dt = du_i/dt = \dot{u}_i$, as x_i is time independent. Hence, in the spatial description the time rate of change of Q is

$$\frac{dQ}{dt} = \dot{u}_1\frac{\partial Q}{\partial y_1} + \dot{u}_2\frac{\partial Q}{\partial y_2} + \dot{u}_3\frac{\partial Q}{\partial y_3} + \frac{\partial Q}{\partial t} \tag{3C-1.3}$$

In modern continuum mechanics, the time rate of change of a function Q is denoted by DQ/Dt and is called the *material derivative* (more fully, the time rate of change of physical property associated with a material point function). Hence, for the material description we have the material derivative

$$\frac{DQ}{Dt} = \left.\frac{\partial Q}{\partial t}\right|_{\mathbf{x}=\text{const}} \qquad Q = Q(\mathbf{x}, t) \tag{3C-1.4}$$

where $\mathbf{x}$ denotes the coordinates (x_1, x_2, x_3), and for the spatial description

$$\frac{dQ}{Dt} = \dot{u}_1 \frac{\partial Q}{\partial y_1} + \dot{u}_2 \frac{\partial Q}{\partial y_2} + \dot{u}_3 \frac{\partial Q}{\partial y_3} + \frac{\partial Q}{\partial t} \qquad Q = Q(\mathbf{y}, t) \tag{3C-1.5}$$

where $\mathbf{y}$ denotes the coordinates (y_1, y_2, y_3).

By analogy to the material derivative of a scalar point function Q, the time rate of change of a volume integral is called the *material derivative of a volume integral*.

In material coordinates, the material derivative of a volume integral is defined to be

$$\frac{D}{Dt}\int F\,dV = \frac{d}{dt}\int F\,dV = \frac{\partial}{\partial t}\left(\int F dV\right)\Bigg|_{\mathbf{x}=\text{const}} = \int \left.\frac{\partial F}{\partial t}\right|_{\mathbf{x}=\text{const}} dV \tag{3C-1.6}$$

where F is a continuous differentiable function of $(\mathbf{x}, t)$. In spatial coordinates, the material derivative of a volume integral is defined by

$$\frac{D}{Dt}\int F dV = \frac{d}{dt}\int F dV \tag{3C-1.7}$$

where F is a continuous differentiable function of $(\mathbf{x}, t)$. To derive the material derivative in terms of the spatial coordinates $\mathbf{y} = (y_1, y_2, y_3)$, we follow a procedure similar to that employed in the derivation of the momentum principle in fluid mechanics.[8]

We consider the chance in the volume integral for a given region $\mathscr{R}$ in space (the spatial viewpoint). For clarity, we consider the medium in $\mathscr{V}$ to be a flowing fluid. However, the argument applies in general. Hence, at time t the fluid is considered to occupy the fixed region $\mathscr{R}$ of volume $\mathscr{V}$ in space. The change of momentum that the fluid experiences may be regarded as a sum of two parts, the *local* change and the *convective* change. The local change results from the time variation of the momentum in region $\mathscr{R}$. Thus, the local change of momentum in region $\mathscr{R}$ is

$$\int_{\mathscr{V}} \frac{\partial(\rho \dot{u}_i)}{\partial t} d\mathscr{V} \tag{3C-1.8}$$

[8]See Section 16.9, Boresi and Schmidt (2001).

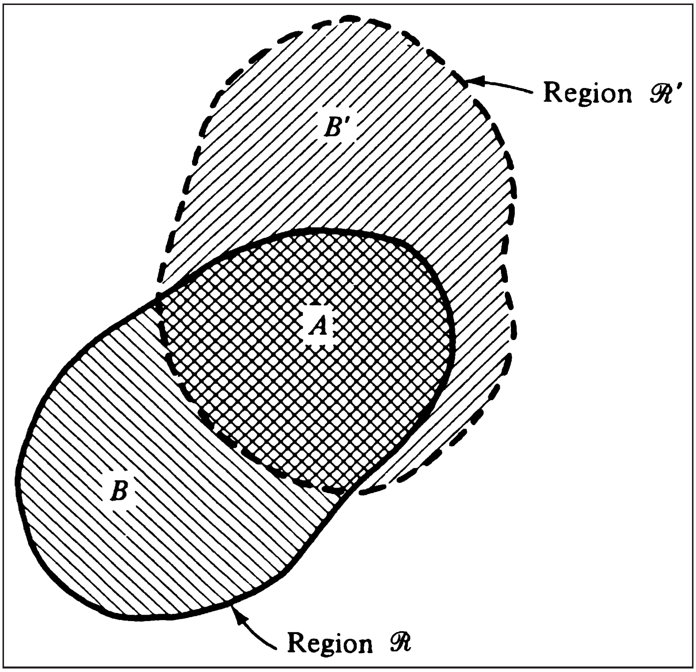

Figure 3C-1.1

where $\rho \dot{u}_1$ is the momentum of region $\mathscr{R}$ per unit volume as a function of x_i and t. For *steady flow*, the velocity $\dot{u}_1$ and density ρ at any point remain constant. Hence, the local change of momentum is zero for steady flow.

The convective change of momentum occurs because the fluid that lies in region $\mathscr{R}$ at time t passes into other regions where other flow conditions exist. To get a clear picture of this phenomenon, we represent the region $\mathscr{R}$ by the area enclosed by the solid line in Fig. 3C-1.1.

The fluid that lies in region $\mathscr{R}$ at time t occupies region $\mathscr{R}'$ at time $t + \Delta t$. The region $\mathscr{R}'$ is represented by the area enclosed by the dashed line in Fig. 3C-1.1. If the time interval Δt is small, regions $\mathscr{R}$ and $\mathscr{R}'$ overlap. The region of overlapping, that is, the region that belongs to both $\mathscr{R}$ and $\mathscr{R}'$, is denoted by A. The region that belongs to $\mathscr{R}$ but not to $\mathscr{R}'$ is denoted by B. The region that belongs to $\mathscr{R}'$ but not to $\mathscr{R}$ is denoted by B'. Then $\mathscr{R} = A + B$, and $\mathscr{R}' = A + B'$, where the $+$ sign denotes the union of the two regions (Fig. 3C-1.1).

We initially consider the case of steady flow. Then the vector momentum $G_i = \int \rho \dot{u}_i \, d\mathscr{V}$ of fluid in a given spatial region does not vary with time. The notation $G_i(\mathscr{R})$ denotes the momentum of fluid in region $\mathscr{R}$. The increment of momentum that the specified quantity of fluid experiences during time Δt is $\Delta G_i = G_i(\mathscr{R}') - G_i(\mathscr{R})$. Hence, because $\mathscr{R}' = A + B'$ and $\mathscr{R} = A + B$, $\Delta G_i = G_i(A + B') - G_i(A + B)$. Because the momentum of the fluid is the vector sum of its parts, $G_i(A + B) = G_i(A) + G_i(B)$ and $G_i(A + B') = G_i(A) + G_i(B')$.

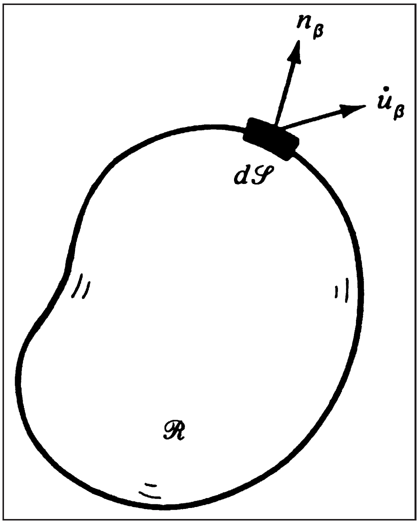

Figure 3C-1.2

Hence, $\Delta G_i = G_i(B') - G_i(B)$. Now the fluid in region B' has flowed out of region $\mathscr{R}$ during the time Δt, and the fluid in region B has flowed into the region $\mathscr{R}$ during time Δt. Hence, for steady flow the rate of increase of momentum of the quantity of fluid that occupies $\mathscr{R}$ is equal to the net rate at which momentum is convected out of region $\mathscr{R}$. Thus, letting $\Delta t \to 0$, $\mathscr{R}' \to \mathscr{R}$, and noting that the volume swept out of $\mathscr{R}$ by particles occupying an element of area $d\mathscr{S}$ on the boundary $\mathscr{S}$ of $\mathscr{V}$ (Fig. 3C-1.2) is $\dot{u}_\alpha n_\alpha \, d\mathscr{S} dt$, where summation notation holds (Chapter 1, Section 1-23), we obtain that part of the rate of change of momentum due to convection as

$$\begin{aligned} \lim_{\Delta t \to 0} \frac{\Delta G_\alpha}{\Delta t} &= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[\int_{\mathscr{S}} (\rho \dot{u}_\alpha) \dot{u}_\beta n_\beta d\mathscr{S} \Delta t \right] \\ &= \int_{\mathscr{S}} (\rho \dot{u}_\alpha) \dot{u}_\beta n_\beta d\mathscr{S} \end{aligned} \tag{3C-1.9}$$

Consequently, the time rate of change of momentum (including the local rate) is, by Eqs. (3C.1.8) and (3C.1.9),

$$\begin{aligned} \frac{D}{Dt} \int_{\mathscr{V}} (\rho \dot{u}_\alpha) d\mathscr{V} &= \int_{\mathscr{V}} \frac{\partial (\rho \dot{u}_\alpha)}{\partial t} d\mathscr{V} + \int_{\mathscr{S}} (\rho \dot{u}_\alpha) \dot{u}_\beta n_\beta d\mathscr{S} \\ &= \int_{\mathscr{V}} \left[\frac{\partial (\rho \dot{u}_\alpha)}{\partial t} + \frac{\partial}{\partial y_\beta} (\rho \dot{u}_\alpha \dot{u}_\beta) \right] d\mathscr{V} \end{aligned} \tag{3C-1.10}$$

where we have employed the divergence (Gauss) theorem (Chapter 1, Section 1-15) to transform the surface integral into a volume integral.

Because the above argument applies in general to any point function F of the fluid (such as temperature, density, etc.), we may replace the function $\rho \dot{u}_i$ with a general function F. Thus, for the material derivative of an integral with continuous differentiable integrand $F(y_1, y_2, y_3; t)$ defined over the region $\mathscr{R}$, we have

$$\begin{aligned}\frac{D}{Dt}\int_{\mathscr{V}} F(y_1, y_2, y_3; t)d\mathscr{V} &= \int_{\mathscr{V}} \frac{\partial F}{\partial t}d\mathscr{V} + \int_{\mathscr{S}} F\dot{u}_\beta n_\beta d\mathscr{S} \\ &= \int_{\mathscr{V}} \left[\frac{\partial F}{\partial t} + \frac{\partial(F\dot{u}_\beta)}{\partial y_\beta}\right]d\mathscr{V} \\ &= \int_{\mathscr{V}} \left[\frac{DF}{Dt} + F\frac{\partial \dot{u}_\beta}{\partial y_\beta}\right]d\mathscr{V}\end{aligned} \tag{3C-1.11}$$

where we have employed the spatial expression for the material derivative [Eq.(3C-1.5)].

A direct application of the material derivative of a volume integral yields the balance of mass law (conservation of mass). Thus, because in region $\mathscr{R}$ the mass m is constant, we have, by Eq. (3C.1.11),

$$\frac{Dm}{Dt} = \frac{D}{Dt}\int_{\mathscr{V}} \rho d\mathscr{V} = \int_{\mathscr{V}} \left(\frac{D\rho}{Dt} + \rho\frac{\partial \dot{u}_\beta}{\partial y_\beta}\right)d\mathscr{V} = 0$$

Accordingly, because this result must hold for every volume element in $\mathscr{R}$, we have

$$\frac{D\rho}{Dt} + \rho\frac{\partial \dot{u}_\beta}{\partial y_\beta} = 0 \tag{3C-1.12}$$

Equation (3C-1.12) is the spatial form of the conservation of mass (see Chapter 1, Section 1-14, and Chapter 2, Section 2-18). Substitution of Eq. (3C-1.12) into Eq. (3C-1.10) yields the result [employing Eq. (3C-1.5)]

$$\frac{D}{Dt}\int_{\mathscr{V}} (\rho\dot{u}_\alpha)d\mathscr{V} = \int_{\mathscr{V}} \frac{D\dot{u}_\alpha}{Dt}\rho\, d\mathscr{V} = \int_{\mathscr{V}} \rho a_\alpha d\mathscr{V} \tag{3C-1.13}$$

where

$$a_\alpha = \frac{D\dot{u}_a}{Dt} \tag{3C-1.14}$$

is the acceleration vector of a particle in volume $\mathscr{V}$ (see Chapter 2, Section 2-18).

3C-2 Differential Equations of Equilibrium Relative to Material Coordinates

Equations (3-8.1) or (3-8.2) are the equations of motion of a deformable body relative to spatial coordinates x_α. We wish to express them in terms of material coordinates. For simplicity we discard acceleration effects. Also, here we let ξ_α denote spatial coordinates and x_α denote material coordinates (Chapter 2, Section 2-3). In the development, we indicate the reduction of the equations for small-displacement theory (small strains, small rotations compared to 1).

The necessary transformations are simplified if the equation of force equilibrium is written initially in vector form. Accordingly, let an infinitesimal regular parallelepiped having edges of length $d\xi_1, d\xi_2, d\xi_3$ parallel to axes (y_1, y_2, y_3), respectively, be acted upon by stress vectors $-\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_1 + d\boldsymbol{\Sigma}_1, \ldots$, and body forces $\boldsymbol{\mathscr{B}} : (\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3)$ per unit volume, as shown in Fig. 3C-2.1 (Section 3-2).

Because $d\boldsymbol{\Sigma}_1 = (\partial \boldsymbol{\Sigma}_1/\partial \xi_1) d\xi_1$, summation of vector forces yields the equilibrium equation

$$\frac{\partial \boldsymbol{\Sigma}_1}{\partial \xi_1} + \frac{\partial \boldsymbol{\Sigma}_2}{\partial \xi_2} + \frac{\partial \boldsymbol{\Sigma}_3}{\partial \xi_3} + \boldsymbol{\mathscr{B}} = 0 \tag{3C-2.1}$$

where $\boldsymbol{\mathscr{B}} = (\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3)$ and where $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_2$, $\boldsymbol{\Sigma}_3$, and $\boldsymbol{\mathscr{B}}$ are functions of the spatial coordinates ξ_α.

By the chain rule of partial differentiation, differentiation with respect to the material coordinates x_α requires

$$\frac{\partial \boldsymbol{\Sigma}_\alpha}{\partial x_\beta} \frac{\partial x_\beta}{\partial \xi_\alpha} + \boldsymbol{\mathscr{B}} = 0 \qquad \alpha, \beta = 1, 2, 3 \tag{3C-2.2}$$

where the partial derivatives $\partial x_\beta / \partial \xi_\alpha$ may be obtained from the relationship between x_α, ξ_α, and u_α, the displacement vector [Eq. (2-3.4)]. Thus,

$$d\xi_\alpha = (\delta_{\alpha\beta} + u_{\alpha,\beta}) dx_\beta = [\delta_{\alpha\beta} + e_{\alpha\beta} + \omega_{\beta\alpha}] dx_\beta \tag{3C-2.3}$$

where Eq. (2-5.3) has been employed.

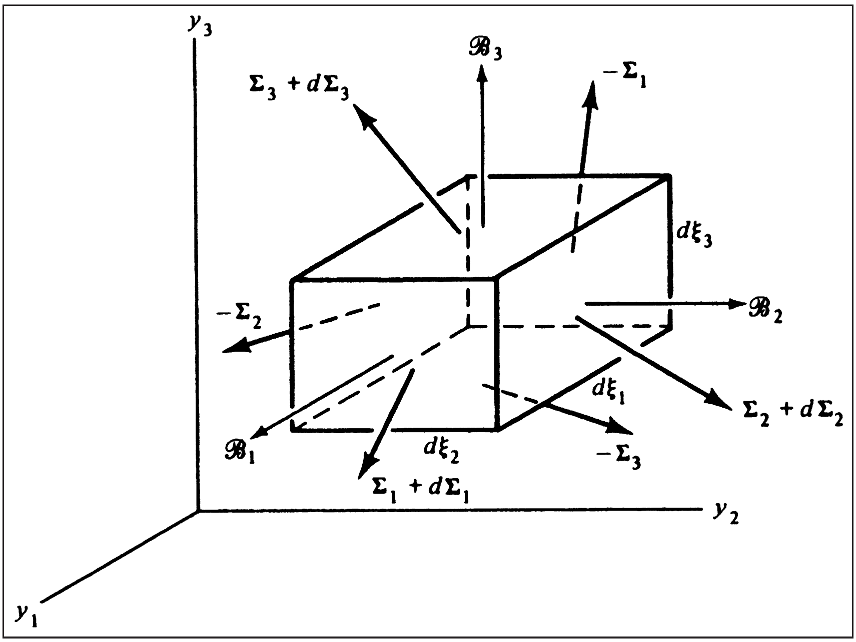

Figure 3C-2.1

Inversion of Eqs. (3C-2.3) yields

$$dx_\alpha = \frac{1}{J} C_{\alpha\beta}\, d\xi_\beta \tag{3C-2.4}$$

where J is the Jacobian of Eq. (2-3.3) and is given by Eq. (2-4.2):

$$J = \det(\delta_{\alpha\beta} + u_{\alpha,\beta}) = \begin{vmatrix} 1+u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & 1+u_{2,2} & u_{2,3} \\ u_{3,1} & u_{3,2} & 1+u_{3,3} \end{vmatrix} > 0 \tag{3C-2.5}$$

Or, using the notation of Eqs. (2-4.3) and (2-5.3), we write

$$\begin{aligned} J &= \det(\delta_{\alpha\beta} + e_{\alpha,\beta} + \omega_{\beta\alpha}) \\ &= \begin{vmatrix} 1+e_{11} & e_{12}+\omega_{21} & e_{13}+\omega_{31} \\ e_{21}+\omega_{12} & 1+e_{22} & e_{23}+\omega_{32} \\ e_{31}+\omega_{13} & e_{23}+\omega_{23} & 1+e_{33} \end{vmatrix} > 0 \end{aligned} \tag{3C-2.6}$$

The coefficients $C_{\alpha\beta}$ of Eq. (3C-2.4) are the elements of the transpose of the determinant consisting of the signed minors of J. Accordingly, writing $J_{ij} = \delta_{ij} + u_{ij} = \delta_{ij} + e_{ij} + \omega_{ji}$, we find

$$\begin{aligned} C_{ii} &= J_{jj} J_{kk} - J_{kj} J_{jk} \\ C_{ij} &= (-1)^{i+j} (J_{ij} J_{kj} - J_{ij} J_{kk}) \end{aligned} \tag{3C-2.7}$$

where $i \neq j \neq k \neq i$ take on the values 1, 2, 3. It then follows by Eqs. (3C-2.2) and (3C-2.4) that

$$\frac{\partial \Sigma_\alpha}{\partial x_\beta} C_{\beta\alpha} + \mathbf{B} = 0$$

or

$$\frac{\partial [C_{\beta\alpha}(\mathbf{x}) \Sigma_\alpha(\mathbf{x})]}{\partial x_\beta} + \mathbf{B}(\mathbf{x}) = 0 \tag{3C-2.8}$$

as $\partial C_{\beta\alpha}/\partial x_\beta = 0$, and $J\mathscr{B}(\xi) = \mathbf{B}(\mathbf{x})$. The condition $C_{\beta\alpha,\beta} = 0$ follows readily from representation of $C_{\beta\alpha}$ in terms of u_α. The reduction to $\mathbf{B(x)}$ follows from the fact that by the definition of the Jacobian, $dV = d\xi_1 d\xi_2 d\xi_3 = J dx_1 dx_2 dx_3 = J dV$; hence, $\mathscr{B} d\mathbf{V} = J \mathscr{B} d\mathbf{V} = \mathbf{B} dV$, where $J = d\mathscr{V}/dV = \mathbf{B}/\mathscr{B}$. Written in full, Eq. (3C-3.8) is

$$\begin{aligned} &\frac{\partial (C_{11}\mathbf{\Sigma}_1 + C_{12}\mathbf{\Sigma}_2 + C_{13}\mathbf{\Sigma}_3)}{\partial x_1} + \frac{\partial (C_{21}\mathbf{\Sigma}_1 + C_{22}\mathbf{\Sigma}_2 + C_{23}\mathbf{\Sigma}_3)}{\partial x_2} \\ &\qquad + \frac{\partial (C_{31}\mathbf{\Sigma}_1 + C_{32}\mathbf{\Sigma}_2 + C_{33}\mathbf{\Sigma}_3)}{\partial x_3} + \mathbf{B} = 0 \end{aligned} \tag{3C-2.9}$$

Novozhilov (1953) has shown that the terms in parentheses in Eq. (3C-2.9) have a definite physical meaning. Here, we follow a somewhat analogous procedure. Consider a rectangular area perpendicular to axes X_3 (of axes X_α), and with sides dx_1, dx_2, isolated from the body before deformation. Under a deformation this area becomes a parellogram with sides dy_1, dy_2 in the direction of unit vectors $\mathbf{i}_1, \mathbf{i}_2$, respectively. Consequently, the unit vector $\mathcal{N}_3$ in the direction of the normal to the paralleogram is given by the vector product of $\mathbf{i}_1$, $\mathbf{i}_2$ (see Chapter 1, Section 1-7).

$$\mathcal{N}_3 \sin(\mathbf{i}_1, \mathbf{i}_2) = \mathbf{i}_1 \times \mathbf{i}_2 \tag{3C-2.10}$$

Similarly, for unit normals $\mathcal{N}_1$, $\mathcal{N}_2$ perpendicular to those areas of the deformed body, which before the deformation were perpendicular to X_1 and X_2 axes, respectively, we have

$$\begin{aligned} \mathcal{N}_1 \sin(\mathbf{i}_2, \mathbf{i}_3) &= \mathbf{i}_2 \times \mathbf{i}_3 \\ \mathcal{N}_2 \sin(\mathbf{i}_3, \mathbf{i}_1) &= \mathbf{i}_3 \times \mathbf{i}_1 \end{aligned} \tag{3C-2.11}$$

Accordingly, by Eq. (2-8.2) for $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ we obtain Table 3C-2.1 of direction cosines relative to axes X_α [using Eq. (a) in Chapter 2, Section 2-10], where the general element a_{ij} (direction cosine between X_1 and $\mathbf{i}_j$) is given by

$$a_{ij} = \frac{\delta_{ij} + e_{ij} + \omega_{ji}}{1 + e_j} \tag{3C-2.12}$$

Accordingly, the projections of $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ (i.e., $\mathcal{N}_{11}, \mathcal{N}_{12}, \mathcal{N}_{13}, \mathcal{N}_{21}, \ldots, \mathcal{N}_{32}$, $\mathcal{N}_{33}$,) along axes (X_1, X_2, X_3) are given by

$$\begin{aligned} \mathcal{N}_{31} \sin(\mathbf{i}_1, \mathbf{i}_2) &= a_{21}a_{32} - a_{22}a_{31} \\ \mathcal{N}_{32} \sin(\mathbf{i}_1, \mathbf{i}_2) &= a_{12}a_{31} - a_{11}a_{32} \\ \mathcal{N}_{33} \sin(\mathbf{i}_1, \mathbf{i}_2) &= a_{11}a_{22} - a_{12}a_{21} \\ \mathcal{N}_{21} \sin(\mathbf{i}_3, \mathbf{i}_1) &= a_{23}a_{31} - a_{21}a_{33} \\ \mathcal{N}_{22} \sin(\mathbf{i}_3, \mathbf{i}_1) &= a_{11}a_{33} - a_{13}a_{31} \\ \mathcal{N}_{23} \sin(\mathbf{i}_3, \mathbf{i}_1) &= a_{13}a_{21} - a_{11}a_{23} \\ \mathcal{N}_{11} \sin(\mathbf{i}_2, \mathbf{i}_3) &= a_{22}a_{33} - a_{32}a_{23} \\ \mathcal{N}_{12} \sin(\mathbf{i}_2, \mathbf{i}_3) &= a_{32}a_{13} - a_{12}a_{33} \\ \mathcal{N}_{13} \sin(\mathbf{i}_2, \mathbf{i}_3) &= a_{12}a_{23} - a_{13}a_{22} \end{aligned} \tag{3C-2.13}$$

We note that, by Eq. (2-8.4) in Chapter 2, we may write $\cos(\mathbf{i}_1, \mathbf{i}_2) = \cos[(\pi/2) - \phi_{12}] = \sin\phi_{12} = 2\epsilon_{12}/\sqrt{(1 + 2\epsilon_{11})(1 + 2\epsilon_{22})}$. Hence,

TABLE 3C-2.1

	$\mathbf{i}_1$	$\mathbf{i}_2$	$\mathbf{i}_3$
X_1	a_{11}	a_{12}	a_{13}
X_2	a_{21}	a_{22}	a_{23}
X_3	a_{31}	a_{32}	a_{33}

$$\begin{aligned} \sin(\mathbf{i}_1, \mathbf{i}_2) &= \sqrt{1 - \frac{4\epsilon_{12}^2}{(1+2\epsilon_{11})(1+2\epsilon_{22})}} \\ &= \sqrt{\frac{(1+2\epsilon_{11})(1+2\epsilon_{22}) - 4\epsilon_{12}^2}{(1+2\epsilon_{11})(1+2\epsilon_{22})}} \end{aligned} \tag{3C-2.14}$$

Consequently, Eqs. (3C-2.7), (3C-2.12), (3C-2.14), and Table 3C-2.1 yield

$$\mathscr{N}_{ii} = \frac{C_{ii}}{\sqrt{(1+2\epsilon_{jj})(1+2\epsilon_{kk}) - 4\epsilon_{jk}^2}} \qquad \mathscr{N}_{ij} = \frac{C_{ij}}{\sqrt{(1+2\epsilon_{jj})(1+2\epsilon_{kk}) - 4\epsilon_{jk}^2}} \tag{3C-2.15}$$

where $i \neq j \neq k \neq i$ take on the values 1, 2, 3 (where i, j, k are not summed; see Chapter 1, Section 1-23).

Equations (3C-2.15) represent the projections of the unit vectors $\mathscr{N}_1 : (\mathscr{N}_{11}, \mathscr{N}_{12}, \mathscr{N}_{13}), \mathscr{N}_2 : (\mathscr{N}_{21}, \mathscr{N}_{22}, \mathscr{N}_{23}), \mathscr{N}_3 : (\mathscr{N}_{31}, \mathscr{N}_{32}, \mathscr{N}_{33})$, along axes ($X_1$, X_2, X_3). These unit vectors are in the directions of the normals to those areas of the deformed body, which before the deformation were perpendicular to the (X_1, X_2, X_3) axes.

Now the area $dx_1\,dx_2$ becomes a parallelogram with sides $dy_1 = (1+e_1)\,dx_1\mathbf{i}_1$, $dy_2 = (1+e_2)\,dx_1\mathbf{i}_2$. Hence, with Eq. (3C-2.14), the ratio of its area A_3 before deformation to the area $\mathscr{A}_3$ after deformation is [see Eq. (2-7.5) in Chapter 2]

$$\frac{\mathscr{A}_3}{A_3} = (1+e_1)(1+e_2)(\sin \mathbf{i}_1, \mathbf{i}_2) = \sqrt{(1+2\epsilon_{11})(1+2\epsilon_{22}) - 4\epsilon_{12}^2} \tag{3C-2.16}$$

Analogously, we obtain

$$\begin{aligned} \frac{\mathscr{A}_2}{A_2} &= \sqrt{(1+2\epsilon_{11})(1+2\epsilon_{33}) - 4\epsilon_{13}^2} \\ \frac{\mathscr{A}_1}{A_1} &= \sqrt{(1+2\epsilon_{22})(1+2\epsilon_{33}) - 4\epsilon_{23}^2} \end{aligned} \tag{3C-2.17}$$

Accordingly, the square roots entering into the denominator of Eqs. (3C-2.15) are equal to the ratios of the areas of the rectangular elements that are perpendicular to the X_i axes before the deformation and become parallelograms after the deformation.

Noting Eqs. (3C-2.16) and (3C-2.17), we write Eq. (3C-2.9) in the form

$$\frac{\partial}{\partial x_1}\left(\frac{\mathscr{A}_1}{A_1}\boldsymbol{\sigma}_{n1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\mathscr{A}_2}{A_2}\boldsymbol{\sigma}_{n2}\right)+\frac{\partial}{\partial x_3}\left(\frac{\mathscr{A}_3}{A_3}\boldsymbol{\sigma}_{n3}\right)+\mathbf{B}=0 \qquad \text{(3C-2.18)}$$

as, by Eq. (3-3.7), $\boldsymbol{\sigma}_{n1}=\mathscr{N}_{11}\boldsymbol{\Sigma}_1+\mathscr{N}_{12}\boldsymbol{\Sigma}_2+\mathscr{N}_{13}\boldsymbol{\Sigma}_3$, with analogous results holding for $\boldsymbol{\sigma}_{n2}, \boldsymbol{\sigma}_{n3}$. Also, by Eqs. (3-3.5), we may write

$$\begin{aligned}\boldsymbol{\sigma}_{n1}&=\mathbf{i}_1\sigma_{11}+\mathbf{i}_2\sigma_{22}+\mathbf{i}_3\sigma_{13}\\ \boldsymbol{\sigma}_{n2}&=\mathbf{i}_1\sigma_{21}+\mathbf{i}_2\sigma_{22}+\mathbf{i}_3\sigma_{23}\\ \boldsymbol{\sigma}_{n3}&=\mathbf{i}_1\sigma_{31}+\mathbf{i}_2\sigma_{32}+\mathbf{i}_3\sigma_{33}\end{aligned} \qquad \text{(3C-2.19)}$$

Hence, Eqs. (3C-2.16), (3C-2.17), and (3C-2.18) yield

$$\begin{aligned}&\frac{\partial}{\partial x_1}\left[\frac{\mathscr{A}_1}{A_1}(\sigma_{11}\mathbf{i}_1+\sigma_{12}\mathbf{i}_2+\sigma_{13}\mathbf{i}_3)\right]+\frac{\partial}{\partial x_2}\left[\frac{\mathscr{A}_2}{A_2}(\sigma_{21}\mathbf{i}_1+\sigma_{22}\mathbf{i}_2+\sigma_{23}\mathbf{i}_3)\right]\\ &\quad+\frac{\partial}{\partial x_3}\left[\frac{\mathscr{A}_3}{A_3}(\sigma_{31}\mathbf{i}_1+\sigma_{32}\mathbf{i}_2+\sigma_{33}\mathbf{i}_3)\right]+\mathbf{B}=0\end{aligned} \qquad \text{(3C-2.20)}$$

Writing Eq. (3C-2.20) in scalar form with the aid of Table 3C-2.1 and Eqs. (3C-2.12), we have in summation notation ($\alpha,\ \beta=1,\ 2,\ 3$):

$$\frac{\partial}{\partial x_1}\left[(\delta_{i\beta}+e_{i\beta}+\omega_{\beta i})\sigma^*_{\alpha\beta}\right]+B_i=0 \qquad i=1,2,3 \qquad \text{(3C-2.21)}$$

where

$$\sigma^*_{ij}=\frac{\mathscr{A}_i}{A_i}\left(\frac{\sigma_{ij}}{1+e_i}\right) \qquad \text{(3C-2.22)}$$

$(i,\ j)$ not summed (Chapter 1, Section 1-23).

Equations (3C-2.21) are the scalar equations of equilibrium in terms of material coordinates x_i. However, the quantities σ^*_{ij} are not stress components. They may only be "interpreted" as stresses referring to the elements of the volume element *before* deformation.

Small.Displacement Approximations. For sufficiently small deformation, the factor

$$\frac{\mathscr{A}_i}{A_i}\left(\frac{1}{1+e_i}\right)\approx 1$$

Hence, then $\sigma^*_{ij}\approx\sigma_{ij}$ for small deformation (Chapter 2, Section 2-15); that is, then σ^*_{ij} serve as approximations to the stress components σ_{ij}, and the asterisk may be removed from Eqs. (3C-2.21). In addition, for small displacements $\mathscr{V}/V\approx 1$; hence, the body forces are such that $\mathscr{B}\approx\mathbf{B}$.

We recall that for small rotations e_{ij} differ from $\epsilon_{\alpha\beta}$ only by terms of the same order as the squares of the angles of rotation (Chapter 2, Section 2-13). Hence, under such situations $e_{\alpha\beta}$ may be discarded from Eq. (3C-2.21). If, in addition, the angles of rotation are themselves so small that $\omega_{\alpha\beta}$ may be regarded as angles of mean volume rotation relative to axes X_i (Chapter 2, Section 2-13), and the angles of rotation are so small that their products with $\sigma_{\alpha\beta}$ may be neglected compared to $\sigma_{\alpha\beta}$, we may write

$$\frac{\partial \sigma_{\alpha\beta}}{\partial x_\alpha} + B_\beta = 0 \qquad \text{(3C-2.23)}$$

Accordingly, for small-displacement theory, the equations of equilibrium in material coordinates reduce to the same form as in spatial coordinates. Then there is no need to distinguish between the spatial and material forms of the equilibrium equations [Eqs. (3C-2.1) and Eq. (3C-2.23), respectively].

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CHAPTER 4

THREE-DIMENSIONAL EQUATIONS OF ELASTICITY

In Chapters 2 and 3 we developed certain geometrical and dynamical concepts in the general theory of continuous media. In this chapter the concepts of geometry and dynamics are related through the introduction of material response (kinematics) to applied forces (dynamics). The treatment is restricted principally to bodies that respond in a linearly elastic manner. However, certain parts of the theory are applicable to more general responses. To simplify the mathematical development, the major part of the theory is restricted to small strains and small rotations (small displacements).[1] First, we discuss briefly the principal effects resulting from more general conditions.

4-1 Elastic and Nonelastic Response of a Solid

Initially, we review the results of the simple tension test of a circular cylindrical material bar, which is clamped at one end and is subjected to an axially directed tensile load (pull) P at the other end. It is assumed that the load is increased slowly from its initially zero value, as the material response depends not only upon the magnitude of the load but also upon the rate of loading as well. It is customary to plot the tensile stress σ in the bar with increasing values of P as a function of the strain ϵ of the bar (see Chapter 2, Section 2-7). In engineering practice, the tensile stress σ is usually approximated by $\sigma \approx P/A_0$, where A_0 is the original cross-sectional area of the bar. Then σ is proportional to load P. However, strictly speaking, according to the definition of stress (Chapter 3, Section 3-2), the

[1]For a fuller treatment of the large-displacement theory, see Novozhilov (1953) and Green and Adkins (1960).

stress is P/A, where A is the actual cross-sectional area of the bar when the load P acts. (The bar undergoes lateral contraction everywhere as it is loaded.) For a material such as mild steel, the stress–strain curve of a tensile test takes the form shown in Fig. 4-1.1. For load P, which produces sufficiently small strain, the strain disappears upon removal of load. Then the body is said to be strained within the limit of *perfect elasticity*. If the strain is proportional to the load, the body is said to be strained within the limit of *linear elasticity*. The limit of perfect elasticity is frequently referred to simply as the *elastic limit* σ_{EL}, whereas the limit of linear elasticity is referred to as the *proportional limit* σ_{PL}. These limits are usually different for steels (Fig. 4-1.1). The response of a body or material to load is said to be perfectly elastic so long as the deformation (strain) does not exceed the strain associated with the elastic limit.

For sufficiently small strains, the curve differs little whether area A_0 or area A is used for the cross-sectional area of the bar. Beyond the proportional limit, the stress–strain curve reaches a local maximum, called the upper yield, flow, or plastic limit, σ_{YL}, after which it drops to a local minimum (the lower yield point) and runs approximately parallel (in a wavy fashion) to the strain axis for

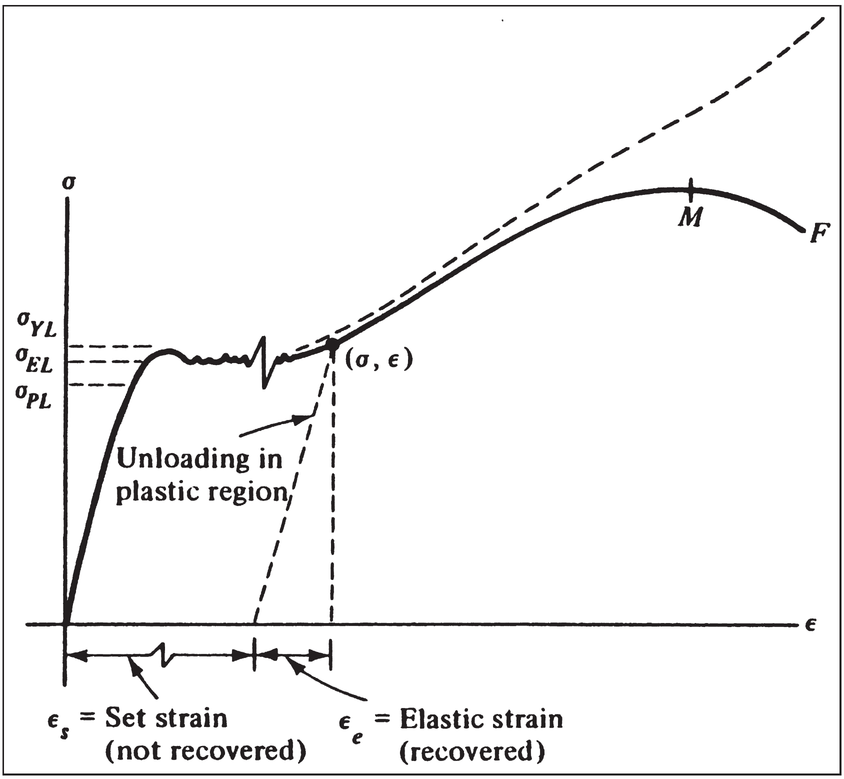

Figure 4-1.1 Stress–strain curve for mild steel.

a short distance. Often, no distinction is made between the elastic limit and the yield limit.

This region is followed by a plastic or flow region in which considerable deformation occurs. In the plastic or flow region, a relatively small change in load causes a large change in strain. In this region, also, there is considerable difference in the stress–strain curve, depending upon whether A_0 or A is used in the definition of stress. With A_0, the curve first rises slowly, turning with concave side down and attaining a maximum value M (the ultimate strength) before turning downward to fracture (point F). Physically, after point M is reached, the *necking down* of the steel bar occurs. This necking down is a drastic reduction of the cross-sectional area of the bar in the neighborhood of the place where the fracture of the bar ultimately occurs. If the load P is referred to the cross section A, $\sigma = P/A$, the stress–strain curve obtained in the plastic region (dashed in Fig. 4-1.1) differs considerably from the stress–strain curve relation to area A_0.

When the strain does not disappear after removal of load, the strain ϵ_s that remains is called *set*. To produce a set, the load must be sufficiently large to produce a stress σ that exceeds the elastic limit. Thus, for a stress–strain state (σ, ϵ), Fig. 4-1.1, removal of load results in a set. The strain ϵ_e that is recovered upon removal of load is called the *elastic strain*. Hence, beyond the elastic limit the strain ϵ is a sum of the set ϵ_s and the elastic strain ϵ_e or $\epsilon = \epsilon_s + \epsilon_e$. The condition of an occurrence of a set may be used as a definition of *nonelastic material response*.

Steel is one of the most important ductile (tough) materials. Other less tough (brittle) materials undergo different response to tensile load. For example, with cast iron there is no plastic range, and fracture F follows almost immediately at the end of the sudden ending elastic range (Fig. 4-1.2). There are also materials

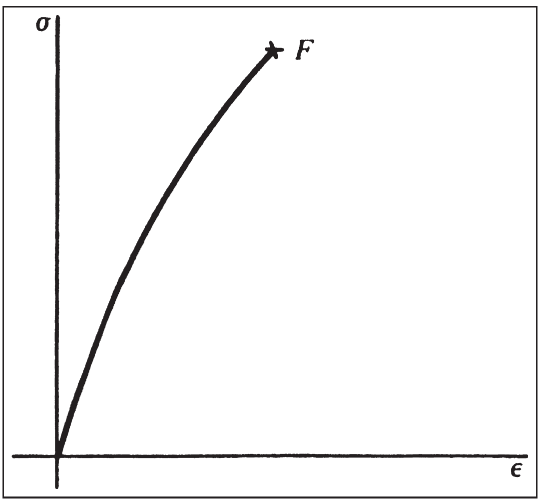

Figure 4-1.2 Stress–strain curve for cast iron.

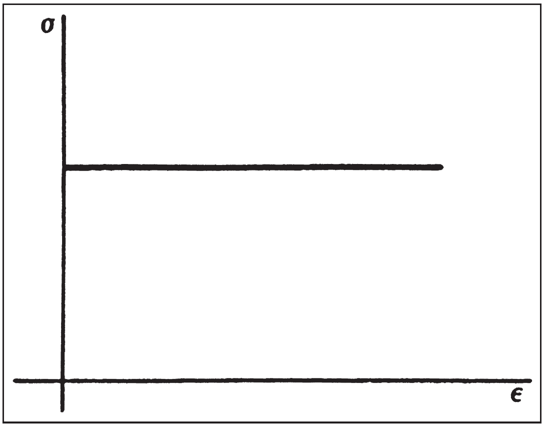

Figure 4-1.3 Perfectly plastic material.

that respond to tensile load almost entirely in a plastic manner, for example, lead and clay (Fig. 4-1.3). This response is referred to as *perfectly plastic* (Prager and Hodge, 1951).

Somewhat analogous results are obtained for compression, bending, and torsion tests (Smith and Sidebottom, 1965). However, pure compression causes fracture only in brittle materials. In the following, we are concerned primarily with perfect elastic response. Studies of viscoelasticity, plasticity, and the like generally lie outside the scope of this treatment.

Concept of Elasticity. We assume that the stress at every point P in the body depends at all times solely on the simultaneous deformation in the immediate neighborhood of the point P. In general, the stress in a solid body depends more or less not only on the force that acts at any instant but also on the previous history of deformation of the body. For example, the stress at point P may depend on residual stresses due to cold working or cold forming of the body. However, we concern ourselves with the study of the behavior of those solid bodies (i.e., those bodies composed of materials that possess large cohesive forces, in contrast to fluids, which can sustain only relatively small tension forces), which have the ability to recover their original size and shape instantly when the forces producing the deformation are removed. This property of instant recovery of initial size and shape upon removal of load is called *perfectly elasticity.*

Generally, a physical body is acted on continuously by forces. For example, in the vicinity of Earth a body is acted on by Earth's gravitational force even in the absence of other forces. Only in interstellar space does a body approach being free of the action of forces, although even there it is acted on by the gravitational attractions of the distance stars. Accordingly, the *zero state* or the *zero configuration* from which the deformations of the body are measured is arbitrary. However, once the zero configuration is specified, the strain of the body measured from the zero state determines the body's internal configuration.

Whenever a body exhibits the phenomenon of *hysteresis*—that is, of returning to its original size and shape only slowly or not at all—its behavior is not perfectly elastic. The study of bodies that recover their sizes and shapes only gradually after load is removed is treated in the theory of viscoelasticity. The study of bodies that do not return to their original sizes after removal of load is generally considered in the theory of plasticity.

Any body may be regarded as perfectly elastic provided it is not strained beyond a certain limiting value, called the *elastic limit.* Accordingly, the *theory of elasticity* may be applied to any body provided the deformations do not exceed the elastic limit.

Finally, the complete description of the initial state of a body requires the specification of the temperature at every point in the body, as well as its initial configuration; for, in general, a change in temperature will produce a change in configuration. In turn, a change in configuration may or may not be accompanied by a change in temperature.

4-2 Intrinsic Energy Density Function (Adiabatic Process)

The problem of equilibrium of a deformed solid body remains indeterminate until six equations, supplemental to the differential equations of motion and the strain–displacement equations, are established. These supplemental equations relate the components of the displacement vector to the components of the stress tensor, and they express the law according to which the material of a given body resists various forms of deformation. A theoretical explanation of this law would require an insight into the nature of the intermolecular forces that seek to keep the particles of a solid body at definite distances from one another—that is, an insight into the components of stress and strain within a solid body. This objective has been achieved only in the case of gases in states that are far removed from unstable states, and in the case of elastic solids the present state of scientific development offers no solution to this difficult problem. If relations between stress and strain interior to a body are found by experiment, it is always by inference from measurement of quantities that in general are not components of stress or strain (such as average strains, cubic compression, extension of a line element on the surface of the body, etc.). Hence, at the present time, the relation between stress and strain is established mainly by direct experiment. However, some general properties inherent in this relation can be explained theoretically. The law of conservation of energy forms the basis for the theoretical treatment of stress–strain relations.

Let us assume initially that the process of deformation is adiabatic[2]; that is, no heat is lost or gained in the system during the deformation. Furthermore, let the work expended on changing the volume and the form of an arbitrary infinitesimal element of the body be independent of the manner in which the transition from

[2]The subsequent analysis holds approximately, however, for isothermal processes. See, for example, Love (1944), Sections 62 and 65.

the zero state (undeformed state) to the final state (strained state) is realized. This condition is an alternative definition of elasticity. In other words, we assume that the role of dissipative (nonconservative) forces in the process of interaction of the particles of the body is negligible compared to the role of conservative forces.[3] A body that satisfies this assumption must return to its initial dimensions and form after the load is removed; that is, the body is *perfectly elastic.*

Under the above conditions, the work required to deform an initially undeformed differential element $d\mathbf{V}$ of an elastic body can be expressed in the form $F(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23})\, dV$; that is, it is equal to the product of the initial volume dV of the element and a certain function F of the six strain components. This function is called the *intrinsic strain energy density*; it depends on the physical properties of the material, but it is independent of the form and the size of the body. It should be noted that the strain energy density depends only on the six strain components (Green's assumption) and that it is independent of rigid-body motions (principle of material indifference; see Eringen, 1980, and Marsden and Hughes, 1994).

Alternatively, the strain components $\epsilon_{\alpha\beta}$ may be expressed in terms of the three principal strain components $(\epsilon_1, \epsilon_2, \epsilon_3)$ and the direction cosines $N_{\alpha\beta}$ of the principal axes of strain $(1, 2, 3)$ relative to axes (X_1, X_2, X_3) (see Chapter 2, Sections 2-10 and 2-11). Furthermore, because the principal axes are mutually perpendicular and the $N_{\alpha\beta}$ are components of unit vectors, the direction cosines may be expressed as functions of three independent angles, say, the Euler angles (θ, ϕ, ψ),[4] which determine the principal axes relative to axes (X_1, X_2, X_3). Then the work δI required to deform the incremental element dV may be written

$$\delta I = F(\epsilon_1, \epsilon_2, \epsilon_3, \theta,\ \phi, \psi)\, dV \tag{4-2.1}$$

Accordingly, it is clear from Eq. (4-2.1) that the work expended to deform an element of volume (say, a parallelepiped) depends not only upon the magnitude of the principal strain components $(\epsilon_1, \epsilon_2, \epsilon_3)$ but also upon the principal directions (directions of the sides of the parallelepiped) of the fibers of the volume element subjected to $(\epsilon_1, \epsilon_2, \epsilon_3)$.

The above argument implies that a volume element (body) responds to deformations differently in different directions. A body that behaves in this manner is said to be *anisotropic.* Or, more completely, we say that the material of which the body is composed is anisotropic. It exhibits different properties (responses to a given force) in different directions. If, in contrast, the material response is the same (for a given force) in all directions, we say that the material (body) is *isotropic.* For a body with material properties identical in all directions, the work required to deform a volume element does not depend upon the orientation of the element (i.e., it does not depend upon the Euler angles that locate principal directions). Then δI is a function only of the principal strains $(\epsilon_1, \epsilon_2, \epsilon_3)$. Thus, for isotropic material,

$$\delta I = F(\epsilon_1, \epsilon_2, \epsilon_3)\, dV \tag{4-2.2}$$

[3]The role of dissipative forces is of major importance in the study of plasticity (Hill, 1983).

[4]Different authors may define Euler angles differently. See Synge and Griffith (2008, Section 10.6).

Furthermore, because the principal strains $(\epsilon_1, \epsilon_2, \epsilon_3)$ may be expressed as functions of the strain invariants (J_1, J_2, J_3) (Chapter 2, Section 2-12), Eq. (4-2.2) may be written

$$\delta I = \mathscr{I}(J_1, J_2, J_3)\, dV \tag{4-2.3}$$

For general theories of deformation, Eq. (4-2.3) is more useful than Eq. (4-2.2), as in order to represent the principal strains $(\epsilon_1, \epsilon_2, \epsilon_3)$ in terms of strain components $\epsilon_{\alpha\beta}$, it is necessary to obtain the roots of a cubic equation [Eq. (2-11.3) in Chapter 2], whereas the strain invariants are expressible directly in terms of $\epsilon_{\alpha\beta}$ [Eq. (2-12.2)]. For small-displacement theory, however, the form of Eq. (4-2.2) is useful, as then $(\epsilon_1, \epsilon_2, \epsilon_3)$ have simple physical meanings (see Chapter 2, Section 2-8).

The work expended in deforming the entire body is, by Eq. (4-2.3),

$$I = \int_V \mathscr{I}(J_1, J_2, J_3)\, dV \tag{4-2.4}$$

The function $\mathscr{I}(J_1, J_2, J_3)$ as well as the function F of Eqs. (4-2.1) and (4-2.2) is referred to as the *intrinsic energy density* function. It represents the work of deformation referred to a unit volume of the body in the undeformed state. In the following section we show that the differential of the intrinsic energy density function may be interpreted as a strain energy function.

4-3 Relation of Stress Components to Strain Energy Density Function

We limit the following analysis to strains small compared to 1. Also, the major results are restricted to small rotations. In other words, the treatment is applicable primarily for small-displacement theory (Chapter 2, Section 2-15). A more general treatment is presented by Novozhilov (1953), where the reduction to small-displacement results is indicated.

Because we restrict our study here to small-displacement theory, we need not distinguish between material (Lagrangian) and spatial (Eulerian) coordinates. Hence, let S be a closed surface within the body, and let V be the volume enclosed by S. Suppose that the body is in a deformed equilibrium state. We might also suppose that the body is in a process of deformation (Love, 1944, Sections 61 and 62). However, it may be shown that the resulting relation between the stress components and the strain energy density function (the intrinsic energy density function) remains unchanged. Let W denote the work that the external forces perform on volume V during the deformation. The change or variation of internal energy of the volume resulting from the deformation is denoted by δI. If the deformation is adiabatic, the law of conservation of energy yields (Pippard, 1960; Wark, 1994) $\delta W = \delta I$. Now $I = \iiint \mathscr{I} dV$, where $\mathscr{I}$ is the intrinsic energy density. Hence,

$$\delta I = \delta \left[\iiint \mathscr{I} dV \right]$$

or

$$\delta W = \delta \left[\iiint \mathscr{I}\, dV \right] \tag{a}$$

The work δW is the sum of the work δW_B of the body forces that act on volume V and the work δW_S of the surface forces that act on surface S. With the notation introduced in the theory of stress (Chapter 3), the work δW_B is

$$\begin{aligned} dW_B &= \iiint (B_1\, \delta u_1 + B_2\, \delta u_2 + B_3\, \delta u_3)\, dV \\ &= \iiint (B_\alpha\, \delta u_\alpha)\, dV \qquad \alpha = 1, 2, 3 \end{aligned} \tag{b}$$

where (u_1, u_2, u_3) are components of the displacement vector, and (B_1, B_2, B_3) are components of body force relative to volume dV. Similarly, the work δW_S is

$$\begin{aligned} \delta W_S &= \iint (\sigma_{n1}\, \delta u_1 + \sigma_{n2}\, \delta u_2 + \sigma_{n3}\, \delta u_3)\, dS \\ &= \iint (\sigma_{n\alpha}\, \delta u_\alpha)\, dS \qquad \alpha = 1, 2, 3 \end{aligned}$$

With Eq. (3-3.10) in Chapter 3, this equation may be written in the form

$$\begin{aligned} \delta W_S = \iint &[(\sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3)\, \delta u_1 \\ &+ (\sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3)\, \delta u_2 \\ &+ (\sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3)\, \delta u_3]\, dS \\ = \iint &(\sigma_{\beta\alpha}\, n_\beta\, \delta u_\alpha)\, dS = \iint (\sigma_{\beta\alpha}\, \delta u_\alpha)\, n_\beta\, dS \end{aligned}$$

By the divergence theorem (Chapter 1, Section 1-15), this surface integral may be transformed into the volume integral

$$\begin{aligned} \delta W_s = \iiint &\left[\frac{\partial}{\partial x_1}(\sigma_{11}\delta u_1) + \frac{\partial}{\partial x_2}(\sigma_{21}\delta u_1) + \frac{\partial}{\partial x_3}(\sigma_{31}\delta u_1) \right. \\ &+ \frac{\partial}{\partial x_1}(\sigma_{12}\delta u_2) + \frac{\partial}{\partial x_2}(\sigma_{22}\delta u_2) + \frac{\partial}{\partial x_3}(\sigma_{32}\delta u_2) \\ &\left. + \frac{\partial}{\partial x_1}(\sigma_{13}\delta u_3) + \frac{\partial}{\partial x_2}(\sigma_{23}\delta u_3) + \frac{\partial}{\partial x_3}(\sigma_{33}\delta u_3) \right] dV \\ = \iiint &\frac{\partial}{\partial x_\beta}(\sigma_{\beta\alpha}\, \delta u_\alpha)\, dV \qquad \alpha,\ \beta = 1, 2, 3 \end{aligned} \tag{c}$$

Performing the differentiations indicated in Eq. (c), by Eqs. (a), (b), (c), (3-8.4), and (2-12.17), we obtain (because $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}$)

$$\begin{aligned}\iiint (\delta\mathscr{I})\, dV &= \iiint (\sigma_{11}\,\delta\epsilon_{11} + \sigma_{22}\,\delta\epsilon_{22} + \sigma_{33}\,\delta\epsilon_{33} \\ &\qquad + 2\sigma_{12}\,\delta\epsilon_{12} + 2\sigma_{13}\,\delta\epsilon_{13} + 2\sigma_{23}\,\delta\epsilon_{23})\, dV \\ &= \iiint \sigma_{\alpha\beta}\,\delta\epsilon_{\alpha\beta}\, dV \end{aligned} \tag{d}$$

where (small-displacement theory, Chapter 2, Section 2-12)

$$\epsilon_{\alpha\beta} = \epsilon_{\beta\alpha} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \tag{e}$$

In Eq. (d) the differential quantity $\delta\mathscr{I}$ is the differential of the intrinsic energy density function $\mathscr{I}$, which is, in turn, a single-valued function of temperature and the quantities $\epsilon_{\alpha\beta}$ that determine the body configuration. Corresponding to any state of deformation, the value of $\mathscr{I}$ is the measure of internal energy δI [Eq. (a)]. In the zero state (Sections 4-1 and 4-2) the value of $\mathscr{I}$ is zero.

Because Eq. (d) represents the statement of the first law of thermodynamics (with no heat input; adiabatic process), we have

$$\delta\mathscr{I} = s_\alpha\,\delta e_\alpha \qquad \alpha = 1, 2, \ldots, 6 \tag{f}$$

where, temporarily, we employ the notation

$$\begin{array}{lll} s_1 = \sigma_{11} & s_2 = \sigma_{22} & s_3 = \sigma_{33} \\ s_4 = \sigma_{12} & s_5 = \sigma_{13} & s_6 = \sigma_{23} \\ e_1 = \epsilon_{11} & e_2 = \epsilon_{22} & e_3 = \epsilon_{33} \\ e_4 = 2\epsilon_{12} & e_5 = 2\epsilon_{13} & e_6 = 2\epsilon_{23} \end{array} \tag{g}$$

Accordingly, the expression on the right side of Eq. (f) is, in the adiabatic case, an exact differential of the strains, and there exists a function U that has the property expressed by the relations (Chapter 1, Section 1-19)

$$s_\alpha = \frac{\partial U}{\partial e_\alpha} \qquad \alpha = 1, 2, \ldots, 6 \tag{4-3.1}$$

The function U represents potential energy per unit volume stored up in the body by the deformation (strain). When the body is strained adiabatically, the variations δU of U are the same as the variations $\delta\mathscr{I}$ of the intrinsic energy $\mathscr{I}$ of the body. A function U that satisfies the relations of Eq. (4-3.1) is called a *strain energy density function.* The existence of a strain energy density function may also be demonstrated for an isothermal (constant temperature) process, as noted in Section 4-2. Practically speaking, an adiabatic process may be approximated by changes that take place in bodies undergoing small, rapid vibrations, whereas an isothermal process may be approximated by changes that occur in a body that is strained slowly by gradually increasing load and that is continually in temperature equilibrium with surrounding bodies.

In terms of engineering notations $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$ and $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$ (see Sections 3-2 and 2A-2 and Table 3-2.1), we may write Eq. (4-3.1) in the form

$$\begin{aligned} \sigma_x &= \frac{\partial U}{\partial \epsilon_x} \qquad \sigma_y = \frac{\partial U}{\partial \epsilon_y} \qquad \sigma_z = \frac{\partial U}{\partial \epsilon_z} \\ \tau_{xy} &= \frac{\partial U}{\partial \gamma_{xy}} \qquad \tau_{xz} = \frac{\partial U}{\partial \gamma_{xz}} \qquad \tau_{yz} = \frac{\partial U}{\partial \gamma_{yz}} \end{aligned} \tag{4-3.2}$$

Equations (4-3.2) provide a great simplification of the problem of determining stress components in the small-deflection theory of elasticity because instead of seeking six unknown functions $(\sigma_x, \ldots, \tau_{yz})$, we need seek only one function U. In general, U is a function of six strain components (Section 4-2). However, a further simplification results if the material is isotropic (see also Section 4-5). Then, because the directions of principal strains have no bearing on the strain energy density, U is a function of the principal strains $(\epsilon_1, \epsilon_2, \epsilon_3)$. Then, by Eqs. (4-3.2), the principal stresses are

$$\sigma_1 = \frac{\partial U}{\partial \epsilon_1} \qquad \sigma_2 = \frac{\partial U}{\partial \epsilon_2} \qquad \sigma_3 = \frac{\partial U}{\partial \epsilon_3} \tag{4-3.3}$$

The principal stresses and strains are not affected by rotations of particles of the medium, and therefore Eq. (4-3.3) is valid, *even though the displacements are large, provided that the strains are small compared to 1.*

Constitutive Relation in Molecular Dynamics. We now recall the governing equations in molecular dynamics previously stated

$$m^i \ddot{\mathbf{r}}^i = \mathbf{f}^i + \boldsymbol{\varphi}^i \qquad i = 1, 2, 3, \ldots, n \tag{3-8.7}$$

where n is the total number of atoms in the system; m^i, $\mathbf{r}^i$, and $\ddot{\mathbf{r}}^i$ are the mass, position vector, and acceleration vector of atom i, respectively; $\mathbf{f}^i$ and $\boldsymbol{\phi}^i$ are the interatomic force and body force acting on atom i, respectively. Here we emphasize that $\mathbf{f}^i$ is the interatomic force acting on atom i due to the interaction between atom i and all the other atoms in the system. Similar to Eq. (4-3.1) or (4-3.2), which means stress tensor is the derivative of a scale-valued function, named potential energy per unit volume or strain energy density function, with respect to strain tensor, in MD the interatomic force vector $\mathbf{f}^i$ can also be expressed as the derivative of potential energy V with respect to the position vector $\mathbf{r}^i$ as

$$V = V(\mathbf{r}^1, \mathbf{r}^2, \mathbf{r}^3, \ldots, \mathbf{r}^n) \equiv V(\mathbf{r}) \tag{4-3.4}$$

$$\mathbf{f}^i = -\frac{\partial V}{\partial \mathbf{r}^i} \tag{4-3.5}$$

The total energy of the system, T, is equal to the sum of kinetic energy, K, and potential energy, V, that is,

$$\begin{aligned} T &= K + V \\ &= \sum_{i=1}^{n} \frac{1}{2} m^i \dot{\mathbf{r}}^i \cdot \dot{\mathbf{r}}^i + V(\mathbf{r}) \end{aligned} \tag{4-3.6}$$

The work done by the body force can be calculated as an integral:

$$W(t) = \sum_{i=1}^{n} \int_0^t \boldsymbol{\varphi}^i(\tau) \cdot \dot{\mathbf{r}}^i(\tau)\, d\tau \tag{4-3.7}$$

It is seen that the integrand is the inner product of the body force on atom i and velocity of atom i, which is equal to the rate of work done on atom i. One may readily show that

$$\begin{aligned} \frac{dT}{dt} &= \frac{d(K+V)}{dt} = \sum_{i=1}^{n} m^i \ddot{\mathbf{r}}^i \cdot \dot{\mathbf{r}}^i + \sum_{i=1}^{n} \frac{dV(\mathbf{r})}{dt} \\ &= \left\{ \sum_{i=1}^{n} m^i \ddot{\mathbf{r}}^i + \sum_{i=1}^{n} \frac{\partial V(\mathbf{r})}{\partial \mathbf{r}^i} \right\} \cdot \dot{\mathbf{r}}^i \\ &= \left\{ \sum_{i=1}^{n} m^i \ddot{\mathbf{r}}^i - \sum_{i=1}^{n} \mathbf{f}^i \right\} \cdot \dot{\mathbf{r}}^i \end{aligned} \tag{4-3.8}$$

$$\begin{aligned} \frac{dW(t)}{dt} &= \sum_{i=1}^{n} \frac{d}{dt} \int_0^t \boldsymbol{\varphi}^i(\tau) \cdot \dot{\mathbf{r}}^i(\tau)\, d\tau \\ &= \sum_{i=1}^{n} \boldsymbol{\varphi}^i(t) \cdot \dot{\mathbf{r}}^i(t) \end{aligned} \tag{4-3.9}$$

Because of Eq. (3-8.6), we obtain

$$\frac{dT}{dt} - \frac{dW}{dt} = \sum_{i=1}^{n} (m^i \ddot{\mathbf{r}}^i - \mathbf{f}^i - \boldsymbol{\varphi}^i) \cdot \dot{\mathbf{r}}^i = 0 \tag{4-3.10}$$

If the body force is zero, that is, the system is isolated from its environment, then Eq. (4-3.10) says that the total energy is a constant. Equation (4-3.10) is actually the *law of conservation of energy* in MD. Also, one may see that the first problem in *Problem Set 4-3* states the *law of conservation of energy* in elasticity. Therefore, in a way, Eqs. (3-8.7) and (4-3.5) define *molecular dynamics.*

Constitutive Equation for Soft Biological Tissue.

Pseudoelasticity. A typical stress–stretch relation of a soft biological tissue exhibits a nonlinear behavior over large strains [cf. Fig. 6.2 of Humphrey and Delange (2004)]. Fung (1990) reported that the behavior of soft tissue tends not to depend strongly on strain rate. In other words, one may treat the soft tissue as an elastic solid. However, hysteresis is also found as a characteristic of soft tissue. Fung suggested that one might treat the behaviors in loading and in unloading as elastic with the same form of the constitutive equation but different numerical values for the associated material parameters and named this approach *pseudoelasticity* to remind us that this material behavior is not truly elastic. Fung's concept of pseudoelasticity is particularly applicable to tissues that are subjected in vivo to consistent loading and unloading, such as the arteries, diastolic heart, and lungs. Fung also performed one-dimensional extension tests on excised strips of mesentery, a thin collagenous membrane located in the abdomen and found an almost linear relation between the stiffness and the stress, which can be described by (Humphrey and Delange, 2004)

$$\frac{dT}{d\Lambda} = a + bT \tag{4-3.11}$$

where T is a component in the first-order Piola–Kirchhoff (PK1) stress tensor, associated with the axis of the one-dimensional specimen; Λ is the stretch ratio that is a component of the deformation gradient; a and b are material parameters. Then the one-dimensional constitutive relation of the soft tissue is obtained as

$$T = \frac{a}{b}\left[e^{b(\Lambda-1)} - 1\right] \tag{4-3.12}$$

Motivated by, but not directly derived from, the one-dimensional exponential result, Eq. (4-3.12), Fung (1990) postulated a strain energy density function:

$$U = \tfrac{1}{2}c[e^{Q} - 1] \tag{4-3.13}$$

with

$$S_{ij} = \frac{\partial U}{\partial \epsilon_{ij}} \tag{4-3.14}$$

where $\mathbf{S}$ is the second-order Piola–Kirchhoff (PK2) stress tensor; $\boldsymbol{\epsilon}$ is the Green–Saint-Venant strain tensor; and

$$Q = C_{ijkl}\epsilon_{ij}\epsilon_{kl} \tag{4-3.15}$$

Then the stress–strain relation for soft biological tissue with large strain is obtained as

$$S_{ij} = ce^{Q}C_{ijkl}\epsilon_{kl} \tag{4-3.16}$$

It is noticed that, in case of small strain, practically there is no difference among the Cauchy stress, first-order Piola–Kirchhoff stress, and second-order Piola–Kirchhoff stress; there is no difference among the infinitesimal strain, Green–Saint-Venant strain, and Almansi strain.

Incompressible and Nearly Incompressible Soft Biological Tissues. If a biological tissue is modeled as an incompressible material, then the mass density remains unchanged during any deformation process. *Remember that mass is conserved.* Now incompressibility implies that volume of the material body is unchanged, that is, $J = 1$, which implies [cf. Eq. (2-4.18)] $\mathrm{III}_c = 1$. This condition places a restriction on the Green's deformation tensor $\mathbf{C}$ as well as on the Green–Saint-Venant strain tensor $\boldsymbol{\epsilon}$ [cf. Eq. (2-7.13)]. This means all components C_{ij}, and ϵ_{ij} alike, are not independent. Hence we must take proper caution in evaluating the partial derivatives $\frac{\partial U}{\partial \boldsymbol{\epsilon}}$ in Eq. (4-3.14). This can be achieved by the *method of Lagrange's multipliers.* Thus, the strain energy density function U is replaced by

$$U \leftarrow U - p(\mathrm{III}_C - 1)/2 \tag{4-3.17}$$

where p is the unknown Lagrange multiplier. Now the stress–strain relation is changed to

$$\begin{aligned} S_{ij} &= \frac{\partial\{-p(\mathrm{III}_C - 1)/2 + U\}}{\partial \epsilon_{ij}} \\ &= -p\frac{\partial \mathrm{III}_C}{\partial C_{ij}} + \frac{\partial U}{\partial \epsilon_{ij}} \\ &= -pJ^2 C_{ij}^{-1} + \frac{\partial U}{\partial \epsilon_{ij}} \\ &= -pC_{ij}^{-1} + ce^Q C_{ijkl}\epsilon_{kl} \end{aligned} \tag{4-3.18}$$

Recall Eq. (3-4.8); the Cauchy stress is obtained as

$$\begin{aligned} \sigma_{\alpha\beta} &= J^{-1} F_{\alpha i} F_{\beta j} S_{ij} \\ &= F_{\alpha i} F_{\beta j}\big\{ - pC_{ij}^{-1} + ce^Q C_{ijkl}\epsilon_{kl}\big\} \\ &= F_{\alpha i} F_{\beta j}\big\{ - pF_{i\xi}^{-1} F_{j\xi}^{-1} + ce^Q C_{ijkl}\epsilon_{kl}\big\} \\ &= -p\delta_{ij} + ce^Q C_{ijkl}\epsilon_{kl} F_{\alpha i} F_{\beta j} \end{aligned} \tag{4-3.19}$$

It is emphasized that the Lagrange multiplier p, usually referred to as hydrostatic pressure, is used to enforce the constraint of incompressibility ($J = 1$ or $\mathrm{III}_C = 1$). If the biological tissue is modeled as a nearly incompressible material, then the following form is suggested

$$U = d(J \ln J - J + 1) + c(e^Q - 1)/2 \tag{4-3.20}$$

where d is another material constant. Vetter and McCulloch (2000) found that the nearly incompressible model gives more accurate numerical results than the incompressible formulation without significantly affecting tissue volume.

Muscle Mechanics and Active Stress. As far as mechanics is concerned, *is there any fundamental difference between living biological tissues and lifeless materials?* Of course, there should be. Is the existence of large strain or incompressibility a fundamental difference? Obviously not. Is Fung's (1990) concept of *pseudoelasticity*, including his famous constitutive relation, Eq. (4-3.13), counted as the fundamental difference? Not exactly. All lifeless materials are passive. Put simply, a passive material deforms according to the loading to which it is subjected; namely, it elongates when there is a tensile stress and contracts when there is a compressive stress. But the muscle, when it is activated, may exert active tensile stress even in the state of contraction. The full constitutive relation has been suggested to have the form (Humphrey, 2002)

$$\sigma_{ij} = -p\delta_{ij} + \sigma_{ij}^{p} + T\left(\mathrm{Ca}^{2+}, \alpha\right)m_i m_j \tag{4-3.21}$$

where $\boldsymbol{\sigma}$ is the total Cauchy stress (active plus passive); p is the Lagrange multiplier (hydrostatic pressure) enforcing incompressibility; $\boldsymbol{\sigma}^p$ is the passive contribution to the stress and may be equal to $ce^Q C_{ijkl}\epsilon_{kl}F_{\alpha i}F_{\beta j}$ if a Fung-type elastic relation is adopted; $T(\mathrm{Ca}^{2+}, \alpha)$ is the muscle tension in the direction $\mathbf{m}$, which is a unit vector in the direction of a muscle fiber in a deformed state (Eulerian description). The muscle tension $T(\mathrm{Ca}^{2+}, \alpha)$ is often assumed to depend on the intracellular calcium Ca^{2+} and the stretch α of the muscle fiber relative to its reference sarcomere length, that is,

$$\alpha\mathbf{m} = \mathbf{F}\cdot\mathbf{M} \quad \text{or} \quad \alpha m_i = F_{i\beta}M_\beta \tag{4-3.22}$$

where $\mathbf{F}$ is the deformation gradient tensor, defined in Eq. (2-4.4); $\mathbf{M}$ is a unit vector in the original muscle fiber direction (Lagrangian description). Various forms of $T(\mathrm{Ca}^{2+}, \alpha)$ have been suggested in the literature as, for example, for vascular smooth muscle (Rachev and Hayashi, 1999; Humphrey, 2002),

$$T\left(\mathrm{Ca}^{2+}, \alpha\right) = A\left(\mathrm{Ca}^{2+}\right)\alpha\left[1 - \left(\frac{\alpha^m - \alpha}{\alpha^m - \alpha^o}\right)^2\right] \tag{4-3.23}$$

where $A\left(\mathrm{Ca}^{2+}\right)$ is a so-called activation function; α^m is the stretch at which the Piola–Kirchhoff stress is maximum; α^o is the stretch at which force generation is vanishing. Notice that $A\left(\mathrm{Ca}^{2+}\right) = 0$ if the muscle is not activated irrespective of the stretch. The values of α^m and α^o vary from one blood vessel type to another blood vessel type; the typical values are $\alpha^o \in (0.5, 0.8)$ and $\alpha^m \in (1.4, 1.8)$. For illustrative purpose, here we choose $\alpha^o = 0.5$ and $\alpha^m = 1.5$ and plot the normalized active Cauchy and first-order Piola–Kirchhoff stresses versus muscle stretch ratio α in Fig. 4-3.1. The normalized active Cauchy stress is defined as

$$\bar{t} \equiv \frac{T\left(\mathrm{Ca}^{2+}, \alpha\right)}{A\left(\mathrm{Ca}^{2+}\right)} = \alpha\left[1 - \left(\frac{\alpha^m - \alpha}{\alpha^m - \alpha^o}\right)^2\right] \tag{4-3.24}$$

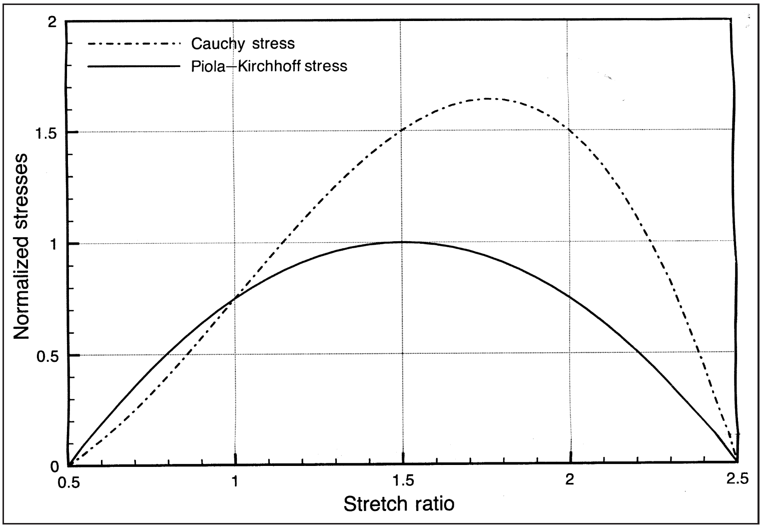

Figure 4-3.1 Normalized active Cauchy stress and first-order Piola–Kirchhoff stress are shown as functions of stretch ratio.

The normalized active first-order Piola–Kirchhoff stress in this case is

$$\overline{T} = \frac{\overline{t}}{\alpha} = 1 - \left(\frac{\alpha^m - \alpha}{\alpha^m - \alpha^o}\right)^2 \tag{4-3.25}$$

Notice that (1) $\alpha = 1$ means that the length of the muscle fiber is equal to its reference sarcomere length, that is, there is no elongation or shortening, (2) $\alpha < 1$ means the muscle fiber is shortened, (3) $\alpha > 1$ means the muscle fiber is elongated. Figure 4-3.1 indicates that, as long as the muscle is activated, the active stress is always a tensile stress along the direction of the muscle fiber irrespective of whether the fiber is shortened or elongated. This is the fundamental difference between a living biological tissue and a lifeless material. The effect of active stress will be demonstrated through a solved problem in Chapter 6.

Problem Set 4-3

1. Show that Eqs. (4-3.2) are valid if we take $dW/dt = dK/dt + dI/dt$, where $K = \frac{1}{2}\iiint \rho(\dot{u}_1^2 + \dot{u}_2^2 + \dot{u}_3^2)\,dV$ denotes the kinetic energy of the system and W and I are defined as in Section 4-2. Dots above u_1, u_2, u_3 denote time derivatives, and t denotes time.

2. For isotropic soft tissues, the material property tensor in Eq. (4-3.15) is reduced to

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + G(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Find the second-order Piola–Kirchhoff stresses from Eq. (4-3.16).

4-4 Generalized Hooke's Law

In its most general form, Hooke's law asserts that each of the stress components is a linear function of the components of the strain tensor, that is,

$$\begin{aligned}
\sigma_x &= C_{11}\epsilon_x + C_{12}\epsilon_y + C_{13}\epsilon_z + C_{14}\gamma_{xy} + C_{15}\gamma_{xz} + C_{16}\gamma_{yz} \\
\sigma_y &= C_{21}\epsilon_x + C_{22}\epsilon_y + C_{23}\epsilon_z + C_{24}\gamma_{xy} + C_{25}\gamma_{xz} + C_{26}\gamma_{yz} \\
\sigma_z &= C_{31}\epsilon_x + C_{32}\epsilon_y + C_{33}\epsilon_z + C_{34}\gamma_{xy} + C_{35}\gamma_{xz} + C_{36}\gamma_{yz} \\
\tau_{xy} &= C_{41}\epsilon_x + C_{42}\epsilon_y + C_{43}\epsilon_z + C_{44}\gamma_{xy} + C_{45}\gamma_{xz} + C_{46}\gamma_{yz} \\
\tau_{xz} &= C_{51}\epsilon_x + C_{52}\epsilon_y + C_{53}\epsilon_z + C_{54}\gamma_{xy} + C_{55}\gamma_{xz} + C_{56}\gamma_{yz} \\
\tau_{yz} &= C_{61}\epsilon_x + C_{62}\epsilon_y + C_{63}\epsilon_z + C_{64}\gamma_{xy} + C_{65}\gamma_{xz} + C_{66}\gamma_{yz}
\end{aligned} \tag{4-4.1}$$

where the 36 coefficients, $C_{11}, \ldots, C_{66}$, are called elastic coefficients (stiffnesses) (see Table 3-2.1 in Chapter 3).

In general, the coefficients C_{ij} are not constants but may depend on location in the body as well as on temperature. Ordinarily, the C_{ij} decrease with increasing temperature. In index notation (Chapter 1, Section 1-23), Eq. (4-4.1) may be written in the form

$$\sigma_\alpha = C_{\alpha\beta}\epsilon_\beta \qquad \alpha, \beta = 1, 2, \ldots, 6 \tag{4-4.2}$$

where

$$\begin{aligned}
\sigma_1 = \sigma_x \qquad & \sigma_2 = \sigma_y, \ldots, \sigma_6 = \tau_{yz} \\
\epsilon_1 = \epsilon_x \qquad & \epsilon_2 = \epsilon_y, \ldots, \epsilon_6 = \gamma_{yz}
\end{aligned} \tag{4-4.3}$$

In reality, Eq. (4-4.1) is no law but merely an approximation that is valid for small strains, as any continuous function is approximately linear in a sufficiently small range of the variables. For a given temperature and location in the body, the coefficients $C_{\alpha\beta}$ in Eq. (4-4.1) are constants that are characteristics of the material.

Equations (4-3.2) and (4-4.1) yield

$$\begin{aligned}
\frac{\partial U}{\partial \epsilon_x} &= \sigma_x = C_{11}\epsilon_x + \cdots + C_{16}\gamma_{yz} \\
\frac{\partial U}{\partial \epsilon_y} &= \sigma_y = C_{21}\epsilon_x + \cdots + C_{26}\gamma_{yz}
\end{aligned} \tag{4-4.4}$$

Hence, differentiation of Eq. (4-4.4) yields

$$\begin{aligned}
&\frac{\partial^2 U}{\partial \epsilon_x \partial \epsilon_y} = C_{12} = C_{21} \qquad \frac{\partial^2 U}{\partial \epsilon_x \partial \epsilon_z} = C_{13} = C_{31}, \ldots, \\
&\frac{\partial^2 U}{\partial \epsilon_x \gamma_{yz}} = C_{16} = C_{61}, \ldots
\end{aligned} \tag{4-4.5}$$

These equations show that $C_{12} = C_{21}, C_{13} = C_{31}, \ldots, C_{ik} = C_{ki}, \ldots,$ and $C_{56} = C_{65}$; that is, the elastic coefficients $C_{\alpha\beta}$ are symmetrical. Accordingly, there are only

21 distinct C's. In other words, the general anisotropic linearly elastic material has 21 elastic coefficients. In view of the preceding relation, the strain energy density of a general anisotropic material is [by integration[5] of Eqs. (4-4.4)]

$$\begin{aligned} U = {} & \tfrac{1}{2}C_{11}\epsilon_x^2 + \tfrac{1}{2}C_{12}\epsilon_x\epsilon_y + \cdots + \tfrac{1}{2}C_{16}\epsilon_x\gamma_{yz} \\ & + \tfrac{1}{2}C_{12}\epsilon_x\epsilon_y + \tfrac{1}{2}C_{22}\epsilon_y^2 + \cdots + \tfrac{1}{2}C_{26}\epsilon_y\gamma_{yz} \\ & + \tfrac{1}{2}C_{13}\epsilon_x\epsilon_z + \tfrac{1}{2}C_{23}\epsilon_y\epsilon_z + \cdots + \tfrac{1}{2}C_{36}\epsilon_z\gamma_{yz} \\ & + \cdots \\ & + \tfrac{1}{2}C_{16}\epsilon_x\gamma_{yz} + \tfrac{1}{2}C_{26}\epsilon_y\gamma_{yz} + \cdots + \tfrac{1}{2}C_{66}\gamma_{yz}^2 \end{aligned} \tag{4-4.6}$$

In index notation, Eq. (4-4.6) may be written

$$U = \tfrac{1}{2}C_{\alpha\beta}\epsilon_\alpha\epsilon_\beta = \tfrac{1}{2}\sigma_\alpha\epsilon_\alpha \qquad \alpha, \beta = 1, 2, \ldots, 6 \tag{4-4.7}$$

In its general form, Eq. (4-4.6) is important in the study of crystals (Planck, 1949; Love, 1944; Nye, 1987).

Symmetry Conditions. The elastic coefficients $C_{\alpha\beta}$ of Eqs. (4-4.1) may be denoted by the array (matrix)

$$\begin{pmatrix} C_{11} & C_{12} & \cdots & C_{16} \\ C_{12} & C_{22} & \cdots & C_{26} \\ \vdots & \vdots & \ddots & \vdots \\ C_{16} & C_{26} & \ldots & C_{66} \end{pmatrix} \tag{4-4.8}$$

In certain structural materials, special kinds of symmetry may exist. For example, the elastic coefficients may remain invariant under a coordinate transformation $x \to x$, $y \to y$, and $z \to -z$. Such a transformation is called a reflection with respect to the (x, y) plane. The direction cosines of this transformation are defined by

$$\begin{aligned} & a_{11} = a_{22} = 1 \qquad a_{33} = -1 \\ & a_{21} = a_{31} = a_{12} = a_{32} = a_{13} = a_{23} = 0 \end{aligned} \tag{4-4.9}$$

(see Table 3-4.1 in Chapter 3). Substitution of Eqs. (4-4.9) into Eqs. (3-4.1) and (2-9.1) reveals that for a reflection with respect to the (x, y) plane

$$\begin{aligned} & \Sigma_{11} = \sigma_x' = \sigma_{11} = \sigma_x && \Sigma_{22} = \sigma_y' = \sigma_{22} = \sigma_y \\ & \Sigma_{33} = \sigma_z' = \sigma_{33} = \sigma_z && \Sigma_{12} = \tau_{xy}' = \sigma_{12} = \tau_{xy} \\ & \Sigma_{23} = \tau_{yz}' = -\sigma_{23} = -\tau_{yz} && \Sigma_{13} = \tau_{xz}' = -\sigma_{13} = -\tau_{xz} \end{aligned} \tag{4-4.10}$$

[5]Here we discard an arbitrary function of (x, y, z), as we are interested in derivatives of U with respect to ϵ_α. Furthermore, in agreement with Eq. (4-4.1), we assume that the stress components σ_α [see Eq. (4-4.3)] vanish identically with the strain components. Accordingly, linear terms in ϵ_β are discarded from Eq. (4-4.6). If the σ_α do not vanish with the ϵ_β (e.g., as in the case of residual stresses), arbitrary functions of (x, y, z) must be added to Eq. (4-4.1). In turn, these functions lead to linear terms in ϵ_β in Eq. (4-4.6).

and

$$\begin{aligned} E_{11} &= \epsilon'_x = \epsilon_{11} = \epsilon_x & E_{22} &= \epsilon'_y = \epsilon_{22} = \epsilon_y \\ E_{33} &= \epsilon'_z = \epsilon_{33} = \epsilon_z & E_{12} &= \tfrac{1}{2}\gamma'_{xy} = \epsilon_{12} = \tfrac{1}{2}\gamma_{xy} \\ E_{23} &= \tfrac{1}{2}\gamma_{yz} = -\epsilon_{23} = -\tfrac{1}{2}\gamma_{yz} & E_{13} &= \tfrac{1}{2}\gamma'_{xz} = -\epsilon_{13} = -\tfrac{1}{2}\gamma_{xz} \end{aligned} \tag{4-4.11}$$

Hence, under the transformation of Eq. (4-4.9), the first of Eqs. (4-4.1) yields

$$\sigma'_x = C_{11}\epsilon'_x + C_{12}\epsilon'_y + C_{13}\epsilon'_z + C_{14}\gamma'_{xy} + C_{15}\gamma'_{xz} + C_{16}\gamma'_{yz} \tag{4-4.12}$$

Substitution of Eqs. (4-4.10) and (4-4.11) into Eq. (4-4.12) yields

$$\sigma_x = \sigma'_x = C_{11}\epsilon_x + C_{12}\epsilon_y + C_{13}\epsilon_z + C_{14}\gamma_{xy} - C_{15}\gamma_{xz} - C_{16}\gamma_{yz} \tag{4-4.13}$$

Comparison of the first of Eqs. (4-4.1) with Eq. (4-4.13) yields the conditions $C_{15} = -C_{15}$, $C_{16} = -C_{16}$, or $C_{15} = C_{16} = 0$. Similarly, considering $\sigma'_y, \ldots, \tau'_{yz}$, we find

$$C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0$$

Accordingly, the elastic coefficients for a material whose elastic properties are invariant under a reflection with respect to the (x, y) plane (i.e., the body possesses a plane of elastic symmetry) are summarized by the matrix

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{pmatrix} \tag{4-4.14}$$

If the material has two mutually orthogonal planes of elastic symmetry, it may be shown that $C_{14} = C_{24} = C_{34} = C_{56} = 0$. Then Eq. (4-4.14) reduces to

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix} \tag{4-4.15}$$

Equation (4-4.1) is simplified accordingly.

The conditions under which the elastic coefficients $C_{\alpha\beta}$ remain invariant under a rotation through an angle θ about an axis are more complicated to derive. For example, a rotation θ about the z axis is defined by the relations $x \to x\cos\theta +$

$y \sin\theta$, $y \to -x \sin\theta + y \cos\theta$, $z \to z$. Under this transformation, Eqs. (2-9.1) and (3-4.1) yield

$$\begin{aligned}
E_{11} &= \epsilon_{11} \cos^2\theta + \epsilon_{22} \sin^2\theta + \epsilon_{12} \sin 2\theta \\
E_{22} &= \epsilon_{11} \sin^2\theta + \epsilon_{22} \cos^2\theta - \epsilon_{12} \sin 2\theta \\
E_{33} &= \epsilon_{33} \\
E_{23} &= \epsilon_{23} \cos\theta - \epsilon_{13} \sin\theta \\
E_{13} &= \epsilon_{23} \sin\theta + \epsilon_{13} \cos\theta \\
E_{12} &= (\epsilon_{22} - \epsilon_{11}) \sin\theta \cos\theta + \epsilon_{12} \cos 2\theta
\end{aligned} \tag{4-4.16}$$

and

$$\begin{aligned}
\Sigma_{11} &= \sigma_{11} \cos^2\theta + \sigma_{22} \sin^2\theta + \sigma_{12} \sin 2\theta \\
\Sigma_{22} &= \sigma_{11} \sin^2\theta + \sigma_{22} \cos^2\theta - \sigma_{12} \sin 2\theta \\
\Sigma_{33} &= \sigma_{33} \\
\Sigma_{23} &= \sigma_{23} \cos\theta - \sigma_{13} \sin\theta \\
\Sigma_{13} &= \sigma_{23} \sin\theta + \sigma_{13} \cos\theta \\
\Sigma_{12} &= (\sigma_{22} - \sigma_{11}) \sin\theta \cos\theta + \sigma_{12} \cos 2\theta
\end{aligned} \tag{4-4.17}$$

Proceeding as in the above examples, after some lengthy calculations we may show that (for θ not equal to π, $\frac{1}{2}\pi$, $\frac{2}{3}\pi$)

$$C_{14} = C_{24} = C_{34} = C_{45} = C_{46} = C_{56} = C_{16} = C_{26} = C_{36} = C_{15} = C_{25} = C_{35} = 0$$

and furthermore,

$$C_{11} = C_{22} \qquad C_{13} = C_{23} \qquad C_{55} = C_{66} \qquad C_{44} = \tfrac{1}{2}(C_{11} - C_{12})$$

For $\theta = \pi$,

$$C_{16} = C_{26} = C_{15} = C_{25} = C_{46} = C_{45} = C_{36} = C_{35} = 0$$

with no additional relations between the C_{ij}. For $\theta = \frac{1}{2}\pi$,

$$C_{34} = C_{46} = C_{45} = C_{56} = C_{16} = C_{26} = C_{15} = C_{25} = C_{36} = C_{35} = 0$$

and, furthermore,

$$C_{11} = C_{22} \qquad C_{13} = C_{23} \qquad C_{55} = C_{66} \qquad C_{14} = -C_{24}$$

Finally, for $\theta = \frac{2}{3}\pi$,

$$C_{14} = C_{24} = C_{34} = C_{36} = C_{56} = C_{35} = 0$$

and

$$C_{11} = C_{22} \qquad C_{13} = C_{23} \qquad C_{55} = C_{66} \qquad C_{44} = \tfrac{1}{2}(C_{11} - C_{12})$$
$$C_{16} = -C_{26} = C_{45} \qquad -C_{15} = C_{25} = C_{46}$$

For certain special material behavior, the anisotropic coefficients may be expressed readily in terms of engineering coefficients such as Young's moduli, shear moduli, and Poisson's ratios (Lekhnitskii, 1963). For example, transversely isotropic materials are characterized by five material coefficients. These materials exhibit isotropic behavior in a plane (say, the x-y plane) and anisotropic behavior perpendicular to this plane (in the direction of the z axis, the symmetry axis). In other words, the elastic coefficients are unaltered (remain invariant) under a rotation through any angle θ about the axis of symmetry. Thus, as derived above, a transversely isotropic material has the characteristic array of elastic coefficients:

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{pmatrix} \tag{4-4.15a}$$

where $C_{44} = \frac{1}{2}(C_{11} - C_{12})$. In terms of engineering representations,

$$\begin{aligned} C_{11} &= \frac{(1 - n\nu_{zx}^2)E_x}{AB} \qquad C_{12} = \frac{(\nu_{xy} + n\nu_{zx}^2)E_x}{AB} \\ C_{13} &= \frac{\nu_{zx}E_x}{B} \qquad C_{33} = \frac{(1 - \nu_{xy})E_z}{B} \qquad C_{55} = G_{xz} = G_{yz} \end{aligned} \tag{4-4.15b}$$

where

$$A = 1 + \nu_{xy} \qquad B = 1 - \nu_{xy} - 2n\nu_{zx}^2 \qquad n = E_x/E_z$$

The ratio n is a measure of the degree of anisotropy. The symbols E, G, and ν with appropriate subscripts denote Young's moduli, shear moduli, and Poisson's ratios associated with the corresponding axes (see Section 4-7). The coefficient C_{44}, which is determined in terms of C_{11}, C_{12} is given by $C_{44} = E_x/(2A)$; see Eq. (4-4.15b).

The remaining engineering (elastic) coefficients are related to the four coefficients (E_x, E_z, ν_{xy}, ν_{zx}) as follows:

$$G_{xy} = \frac{E_x}{2(1 + \nu_{xy})} \qquad \nu_{xz} = n\nu_{zx} \tag{4-4.15c}$$

The engineering coefficients E_x, E_z, and so on have been determined by Chong et al. (1980a, 1980b) and Chong (1983) for oil shale, which is a transversely isotropic material. Like many other materials, oil shale is sensitive to loading or strain rate. The effect of strain rate on the elastic coefficients of oil shale has been discussed by Chong et al. (1981) in terms of a nonlinear solid model. The experimental determination of engineering coefficients is treated by Schreiber et al. (1973).

General Form of Transformation Law of Elastic Coefficients. In terms of double-subscript notation for the stress components $\sigma_{\alpha\beta}$ and for the strain components $\epsilon_{\alpha\beta}$, we may write Hooke's generalized law (also called elastic constitutive relations) in the following form [Eq. (4-4.1)] relative to axes (x_1, x_2, x_3):

$$\sigma_{\alpha\beta} = C_{\alpha\beta\gamma\delta}\epsilon_{\gamma\delta} \qquad \alpha, \beta, \gamma, \delta = 1, 2, 3 \tag{4-4.18}$$

where the elastic coefficients $C_{\alpha\beta}$ of Eq. (4-4.1) have been modified appropriately to account for the fact that $2\epsilon_{12} = \gamma_{xy}$, $2\epsilon_{13} = \gamma_{xz}$, and $2\epsilon_{23} = \gamma_{yz}$ relative to axes (x, y, z) or (x_1, x_2, x_3). In Eq. (4-4.18) the elastic coefficients $C_{\alpha\beta\gamma\delta}$ require four indexes to accommodate indexing of $\epsilon_{\gamma\delta}$.

We wish to consider the rules under which the elastic coefficients $C_{\alpha\beta\gamma\delta}$ transform, when the stress components $\sigma_{\alpha\beta}$ and the strain components $\epsilon_{\gamma\delta}$ are referred to axes (y_1, y_2, y_3) defined relative to axes x_i by direction cosines $a_{\alpha\beta}$ (see Table 3-4.1). By Eq. (3-4.1), we have the stress components relative to axes y_α:

$$\Sigma_{mn} = \sigma_{\alpha\beta}a_{m\alpha}a_{n\beta} \tag{a}$$

Hence, Eqs. (4-4.18) and (a) yield

$$\Sigma_{mn} = C_{\alpha\beta\gamma\delta}a_{m\alpha}a_{n\beta}\epsilon_{\gamma\delta} \tag{b}$$

By Eq. (2-9.3) we obtain the strain components $\epsilon_{\alpha\beta}$ relative to axes x_α in terms of the components E_{pq} relative to axes y_α (here we sum on p and q):

$$\epsilon_{\gamma\delta} = E_{pq}a_{p\gamma}a_{q\delta} \tag{4-4.19}$$

Substitution of Eq. (4-4.19) into Eq. (b) yields

$$\Sigma_{mn} = C_{\alpha\beta\gamma\delta}a_{m\alpha}a_{n\beta}a_{p\gamma}a_{q\delta}E_{pq} \tag{c}$$

Writing Eq. (c) in the form

$$\Sigma_{mn} = C'_{mnpq}E_{pq} \tag{d}$$

where C'_{mnpq} denote the elastic coefficients relative to axes y_α, we obtain, upon comparison of Eqs. (c) and (d),

$$C'_{mnpq} = C_{\alpha\beta\gamma\delta}a_{m\alpha}a_{n\beta}a_{p\gamma}a_{q\delta} \qquad \alpha, \beta, \gamma, \delta, m, n, p, q = 1, 2, 3 \tag{4-4.20}$$

Equation (4-4.20) is the general rule for transformation of the elastic coefficients $C_{\alpha\beta\gamma\delta}$ under a rotation of axes described by Table 3-4.1. Because m, n, p, q (or $\alpha, \beta, \gamma, \delta$) take on values $1, 2, 3$, there are seemingly $3^4 = 81 C_{\alpha\beta\gamma\delta}$. However, because $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, Eq. (4-4.18) yields the result that $C_{\alpha\beta\gamma\delta} = C_{\beta\alpha\gamma\delta}$. Also, we assume that symmetry exists in the second pair of indices, without loss of generality. For, if symmetry did not exist, we could attain symmetry by replacing each $\epsilon_{\alpha\beta}$ by the identity

$$\epsilon_{\alpha\beta} = \tfrac{1}{2}(\epsilon_{\alpha\beta} + \epsilon_{\beta\alpha}) \tag{e}$$

Then

$$C_{\alpha\beta\gamma\delta} = C_{\alpha\beta\delta\gamma}$$

Finally, we observe that symmetry exists between the two pairs of indexes (α, β) and (γ, δ). This result may be shown with the help of the theory of Section 4-3, as follows:

By Eq. (d) of Section 4-3, we may write, by the argument following Eq. (d),

$$\delta U = \sigma_{\alpha\beta}\, \delta\epsilon_{\alpha\beta} \tag{f}$$

where δU denotes the increment of strain energy U, where U is a function of the strain components $\epsilon_{\alpha\beta}$ (Section 4-2). Accordingly, if *U is written symmetrically in terms* of $\epsilon_{\alpha\beta}$ [which can be done by the substitution of Eq. (e) into U], Eq. (f) yields

$$\frac{\partial U}{\partial \epsilon_{\alpha\beta}}\delta\epsilon_{\alpha\beta} = \sigma_{\alpha\beta}\, \delta\epsilon_{\alpha\beta} \tag{g}$$

Because of the symmetry of U in $\epsilon_{\alpha\beta}$, Eq. (g) yields

$$\sigma_{\alpha\beta} = \frac{\partial U}{\partial \epsilon_{\alpha\beta}} \tag{4-4.21}$$

Accordingly, Eqs. (4-4.18) and (4-4.21) yield

$$\frac{\partial^2 U}{\partial \epsilon_{\alpha\beta}\, \partial \epsilon_{\gamma\delta}} = \frac{\partial^2 U}{\partial \epsilon_{\gamma\delta}\, \partial \epsilon_{\alpha\beta}}$$

or

$$C_{\alpha\beta\gamma\delta} = C_{\gamma\delta\alpha\beta}$$

In summary, the coefficients $C_{\alpha\beta\gamma\delta}$ possess the following symmetry properties:

$$\begin{aligned} C_{\alpha\beta\gamma\delta} &= C_{\beta\alpha\gamma\delta} \\ C_{\alpha\beta\gamma\delta} &= C_{\alpha\beta\delta\gamma} \\ C_{\alpha\beta\gamma\delta} &= C_{\gamma\delta\alpha\beta} \end{aligned} \tag{4-4.22}$$

Consequently, there are 21 independent $C_{\alpha\beta\gamma\delta}$ in the most general anisotropic material.

Finally, we caution that Eq. (4-4.21) is valid if and only if the strain energy density function is written symmetrically in the $\epsilon_{\alpha\beta}, \alpha$, and $\beta = 1, 2, 3$. It may be shown that Eq. (4-4.21) holds for orthogonal coordinates in general.

Example 4-4.1. Orthotropic Elastic Coefficients for a Plane Stress Region in the (x, y) Plane. A state of plane stress relative to the (x, y) plane is defined by the conditions (Chapter 3, Section 3-7)

$$\begin{aligned} &\sigma_x = \sigma_x(x, y) \qquad \sigma_y = \sigma_y(x, y) \qquad \tau_{xy} = \tau_{xy}(x, y) \\ &\sigma_z = \tau_{xz} = \tau_{yz} = 0 \end{aligned} \tag{a}$$

Show that the stress–strain relations for this plane stress region in which the material is orthotropic relative to axes (x, y) takes on the form

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{b}$$

and express the coefficients P_{ij} in terms of the coefficients C_{ij} [see Eqs. (4-4.1) and (4-4.15)].

By Eqs. (4-4.1), (4-4.15), and (a), we have

$$\begin{aligned} \sigma_x &= C_{11}\epsilon_x + C_{12}\epsilon_y + C_{13}\epsilon_z \\ \sigma_y &= C_{12}\epsilon_x + C_{22}\epsilon_y + C_{23}\epsilon_z \\ \sigma_z &= C_{13}\epsilon_x + C_{23}\epsilon_y + C_{33}\epsilon_z = 0 \\ \tau_{xy} &= C_{44}\gamma_{xy} \\ \tau_{xz} &= C_{55}\gamma_{xz} = 0 \\ \tau_{yz} &= C_{66}\gamma_{yz} = 0 \end{aligned} \tag{c}$$

If C_{55}, C_{66} are nonzero, the last two of Eqs. (c) require that $\gamma_{xz} = \gamma_{yz} = 0$. By the third of Eqs. (c),

$$\epsilon_z = -\frac{(C_{13}\epsilon_x + C_{23}\epsilon_y)}{C_{33}} \tag{d}$$

Substitution of Eq. (d) into the first two of Eqs. (c) yields

$$\begin{aligned} \sigma_x &= P_{11}\epsilon_x + P_{12}\epsilon_y \\ \sigma_y &= P_{12}\epsilon_x + P_{22}\epsilon_y \\ \tau_{xy} &= P_{33}\gamma_{xy} \end{aligned} \tag{e}$$

where

$$P_{11} = \frac{C_{11}C_{33} - C_{13}^2}{C_{33}} \qquad P_{12} = \frac{C_{12}C_{33} - C_{13}C_{23}}{C_{33}}$$
$$P_{22} = \frac{C_{22}C_{33} - C_{23}^2}{C_{33}} \qquad P_{33} = C_{44} \tag{f}$$

Thus, by Eq. (e) we have

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & 0 \\ P_{12} & P_{22} & 0 \\ 0 & 0 & P_{33} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} \tag{g}$$

where Eq. (f) relates the P_{ij} to the C_{ij}.

General Form of Potential Energy in Molecular Dynamics. The general interatomic potential in MD can be written as

$$V = \sum_{i,j=1}^{n} \frac{1}{2!} V^{ij}(\mathbf{r}^i, \mathbf{r}^j) + \sum_{i,j,k=1}^{n} \frac{1}{3!} V^{ijk}(\mathbf{r}^i, \mathbf{r}^j, \mathbf{r}^k) + \sum_{i,j,k,l=1}^{n} \frac{1}{4!} V^{ijkl}(\mathbf{r}^i, \mathbf{r}^j, \mathbf{r}^k, \mathbf{r}^l) + \cdots \tag{4-4.23}$$

where one recognizes that the terms on the right-hand side may be referred to as 2-body potential, 3-body potential, 4-body potential, and so on. It should be emphasized that for the summation all indices must be distinct. For example, $j \neq i$ for the 2-body potential; $j \neq i$, $k \neq j$, and $k \neq i$ for the 3-body potential. In other words, $V^{ijkl\ldots.} = 0$ if any two indices are equal.

Before we introduce a few potential energies popoularly used in MD simulation, we define the relative position vector, $\mathbf{r}^{ij}$, and separation distance, r^{ij}, between atom i and atom j as follows:

$$\mathbf{r}^{ij} \equiv \mathbf{r}^i - \mathbf{r}^j$$
$$r^{ij} \equiv \|\mathbf{r}^i - \mathbf{r}^j\| = \sqrt{\left(r_x^i - r_x^j\right)^2 + \left(r_y^i - r_y^j\right)^2 + \left(r_z^i - r_z^j\right)^2} \tag{4-4.24}$$

It is worthwhile to show that

$$\begin{aligned} \frac{\partial r^{ij}}{\partial r_x^i} &= \frac{\partial \left[\left(r_x^i - r_x^j\right)^2 + \left(r_y^i - r_y^j\right)^2 + \left(r_z^i - r_z^j\right)^2\right]^{1/2}}{\partial r_x^i} \\ &= \left[\left(r_x^i - r_x^j\right)^2 + \left(r_y^i - r_y^j\right)^2 + \left(r_z^i - r_z^j\right)^2\right]^{-1/2} \left(r_x^i - r_x^j\right) \\ &= \frac{r_x^i - r_x^j}{r^{ij}} \end{aligned} \tag{4-4.25}$$

Then it is easy to verify that

$$\frac{\partial r^{ij}}{\partial \mathbf{r}^i} = \frac{\mathbf{r}^{ij}}{r^{ij}} \qquad \frac{\partial r^{ij}}{\partial \mathbf{r}^j} = -\frac{\mathbf{r}^{ij}}{r^{ij}} \tag{4-4.26}$$

Lennard-Jones Potential. This potential is a 2-body potential, or pair potential, that has been most frequently used in MD simulation for nanoscale systems (Jones, 1924a, 1924b; Kittel, 2005). It can be expressed as

$$V(r) = 4e\left[\left(\frac{a}{r}\right)^{12} - \left(\frac{a}{r}\right)^{6}\right] \tag{4-4.27}$$

The interatomic force can be calculated as

$$\mathbf{f}^i = -\mathbf{f}^j = 24e\left(2\frac{a^{12}}{r^{13}} - \frac{a^6}{r^7}\right)\frac{\mathbf{r}^{ij}}{r} \tag{4-4.28}$$

It is seen that (1) when the separation distance is very small, the first term dominates and the force is repulsive because $\mathbf{f}^i$ is in the direction of $\mathbf{r}^{ij} = \mathbf{r}^i - \mathbf{r}^j$; (2) when the separation distance is large, the second term dominates and the force is attractive; (3) the equilibrium position is at where $\mathbf{f}^i = 0$ and it is obtained as $r^o = 2^{1/6}a$. However, readers should not be misled by the last statement, which was obtained for the case that only two atoms are considered. Actually, one should consider all the atoms in the system in principle, at least all the nearest neighbors for reasonably accurate solution. Notice that there are 12 nearest neighbors in the face-centered-cubic crystal structure and Kittel (2005) obtained $r^o/a = 1.09$.

Coulomb–Buckingham Potential. The Coulomb–Buckingham potential is also a pair potential and can be expressed as

$$V^{ij}\left(r^{ij}\right) = \frac{q^i q^j}{r^{ij}} + A^{ij}\exp\left(-\frac{r^{ij}}{B^{ij}}\right) - \frac{C^{ij}}{\left(r^{ij}\right)^6} \tag{4-4.29}$$

where q^i and q^j are the electric charge of atom i and atom j, respectively; A^{ij}, B^{ij}, and C^{ij} are the material constants associated with the ith kind atom and the jth kind atom. The first term on the right-hand side of Eq. (4-4.29) is the famous Coulomb potential between two charges, and the next two terms specify the Buckingham potential. The Coulomb–Buckingham potential is suitable to describe the atomic interactions of ionic crystals, such as rocksalt-type and perovskite-type crystals. Here we put indices i and j even as superscripts on the potential function V to emphasize that not only the material constants but also the function form depend on the type of atoms involved. The interatomic force can be calculated as

$$\mathbf{f}^i = -\mathbf{f}^j = \left[q^i q^j \left(r^{ij}\right)^{-2} + \frac{A^{ij}}{B^{ij}}e^{-r^{ij}/B^{ij}} - 6C^{ij}\left(r^{ij}\right)^{-7}\right]\frac{\mathbf{r}^{ij}}{r^{ij}} \tag{4-4.30}$$

Tersoff Potential. The Tersoff (1988) potential may be expressed as

$$V^{ij} = f_C\left(r^{ij}\right)\left[f_A\left(r^{ij}\right) - b^{ij} f_B\left(r^{ij}\right)\right] \tag{4-4.31}$$

where $f_C\left(r^{ij}\right)$ is a cutoff function given by

$$f_C\left(r^{ij}\right) = \begin{cases} 1 & r^{ij} < D^{ij} \\ \frac{1}{2}\{1 + \cos[\pi\left(r^{ij} - D^{ij}\right)/\left(S^{ij} - D^{ij}\right)]\} & D^{ij} < r^{ij} < S^{ij} \\ 0 & r^{ij} > S^{ij} \end{cases} \tag{4-4.32}$$

and S^{ij} is called the cutoff radius, a material constant for atom i and atom j. This simply means if the separation distance between atom i and atom j is greater than the cutoff radius S^{ij}, then the potential energy V^{ij} is vanishing; in other words, Tersoff potential represents a short-range interaction between atoms. The other parameters in Eq. (4-4.31) are explicitly specified as

$$\begin{aligned} f_A\left(r^{ij}\right) &= A^{ij} \exp\left(-\lambda^{ij} r^{ij}\right) \\ f_B\left(r^{ij}\right) &= B^{ij} \exp\left(-\mu^{ij} r^{ij}\right) \\ b^{ij} &= \left[1 + \beta_i^{n_i}\left(\xi^{ij}\right)^{n_i}\right]^{-1/2n_i} \\ \xi^{ij} &= \sum_{k \neq i,j} f_C\left(r^{ij}\right) g\left(\theta^{ijk}\right) \\ g(\theta_{ijk}) &= 1 + \frac{c_i^2}{d_i^2} - \frac{c_i^2}{d_i^2 + (h_i - \cos\theta^{ijk})^2} \end{aligned} \tag{4-4.33}$$

where θ^{ijk} is the bond angle between the two vectors $\mathbf{r}^{ij}$ and $\mathbf{r}^{ik}$; A^{ij}, B^{ij}, λ^{ij}, μ^{ij}, D^{ij}, β^i, n_i, c_i, d_i, and h_i are material constants. It is noticed that b^{ij} involves the existence of all the other atoms k within the cutoff radius, and hence it represents the bond strength due to *many-body effects*. Usually, the material constants related to atom i and atom j can be simplified as

$$\begin{aligned} A^{ij} &= \sqrt{A^i A^j} & B^{ij} &= \sqrt{B^i B^j} \\ S^{ij} &= \sqrt{S^i S^j} & D^{ij} &= \sqrt{D^i D^j} \\ \lambda^{ij} &= \frac{\lambda^i + \lambda^j}{2} & \mu^{ij} &= \frac{\mu^i + \mu^j}{2} \end{aligned} \tag{4-4.34}$$

where A^i, B^i, S^i, D^i, λ^i, and μ^i are the corresponding material constants associated with atom i. To calculate the interatomic forces acting on atom i, atom j, and atom k, it is worthwhile to find the following derivatives to begin with:

$$
\begin{aligned}
f_C'\left(r^{ij}\right) &\equiv \frac{d f_C\left(r^{ij}\right)}{d r^{ij}} = \begin{cases} 0 & r^{ij} < D^{ij} \\ -\dfrac{\pi}{2\left(S^{ij}-D^{ij}\right)} \sin\left(\dfrac{\pi\left(r^{ij}-D^{ij}\right)}{S^{ij}-D^{ij}}\right) & D^{ij} < r^{ij} < S^{ij} \\ 0 & r^{ij} > S^{ij} \end{cases} \\
g'\left(\theta^{ijk}\right) &\equiv \frac{dg}{d\theta^{ijk}} = 2c_i^2\left[d_i^2 + \left(h_i - \cos\theta^{ijk}\right)^2\right]^{-2}\left(h_i - \cos\theta^{ijk}\right)\sin\theta^{ijk} \\
\frac{dg}{d\cos\theta^{ijk}} &= -2c_i^2\left[d_i^2 + \left(h_i - \cos\theta^{ijk}\right)^2\right]^{-2}\left(h_i - \cos\theta^{ijk}\right)
\end{aligned}
\tag{4-4.35}
$$

Then the atomic forces can be obtained as

$$
\begin{aligned}
\mathbf{F}^i = &-\frac{1}{2}\left[f_C'\left(f_A - b^{ij} f_B\right) + f_C\left(-\lambda^{ij} f_A + \mu^{ij} b^{ij} f_B\right)\right]\frac{\mathbf{r}^{ij}}{r^{ij}} \\
&+ \frac{1}{2}c^{ij} f_C\left(r^{ij}\right) f_B\left(r^{ij}\right)\frac{\partial \xi^{ij}}{\partial \mathbf{r}^i}
\end{aligned}
\tag{4-4.36}
$$

$$
\begin{aligned}
\mathbf{F}^j = &\frac{1}{2}\left[f_C'\left(f_A - b^{ij} f_B\right) + f_C\left(-\lambda^{ij} f_A + \mu^{ij} b^{ij} f_B\right)\right]\frac{\mathbf{r}^{ij}}{r^{ij}} \\
&+ \frac{1}{2}c^{ij} f_C\left(r^{ij}\right) f_B\left(r^{ij}\right)\frac{\partial \xi^{ij}}{\partial \mathbf{r}^j}
\end{aligned}
\tag{4-4.37}
$$

$$
\mathbf{F}^k = \frac{1}{2}c^{ij} f_C\left(r^{ij}\right) f_B\left(r^{ij}\right)\frac{\partial \xi^{ij}}{\partial \mathbf{r}^k}
\tag{4-4.38}
$$

with

$$
\begin{aligned}
\frac{\partial \xi^{ij}}{\partial \mathbf{r}^i} &= \left[g f_C'\left(r^{ik}\right)\frac{\mathbf{r}^{ik}}{r^{ik}} + d^{ij}\frac{\partial\cos\theta^{ijk}}{\partial \mathbf{r}^i}\right] \\
\frac{\partial \xi^{ij}}{\partial \mathbf{r}^j} &= d^{ij}\frac{\partial\cos\theta^{ijk}}{\partial \mathbf{r}^j} \\
\frac{\partial \xi^{ij}}{\partial \mathbf{r}^k} &= \left[-g f_C'\left(r^{ik}\right)\frac{\mathbf{r}^{ik}}{r^{ik}} + d^{ij}\frac{\partial\cos\theta^{ijk}}{\partial \mathbf{r}^k}\right]
\end{aligned}
\tag{4-4.39}
$$

where

$$
\begin{aligned}
c^{ij} &\equiv -0.5\beta_i\left(\beta_i \xi^{ij}\right)^{n_i-1}\left[1 + \left(\beta_i \xi^{ij}\right)^{n_i}\right]^{-1/2n_i-1} \\
d^{ij} &\equiv f_C\left(r^{ik}\right)\frac{-2c_i^2\left(h_i - \cos\theta^{ijk}\right)}{\left[d_i^2 + \left(h_i - \cos\theta^{ijk}\right)^2\right]^2}
\end{aligned}
$$

It is lengthy but straightforward to obtain

$$\frac{\partial \cos\theta^{ijk}}{\partial \mathbf{r}^i} = \left(\frac{1}{r^{ik}} - \frac{\cos\theta^{ijk}}{r^{ij}}\right)\frac{\mathbf{r}^{ij}}{r^{ij}} + \left(\frac{1}{r^{ij}} - \frac{\cos\theta^{ijk}}{r^{ik}}\right)\frac{\mathbf{r}^{ik}}{r^{ik}}$$
$$\frac{\partial \cos\theta^{ijk}}{\partial \mathbf{r}^j} = \frac{\cos\theta^{ijk}}{r^{ij}}\frac{\mathbf{r}^{ij}}{r^{ij}} - \frac{1}{r^{ij}}\frac{\mathbf{r}^{ik}}{r^{ik}} \qquad (4\text{-}4.40)$$
$$\frac{\partial \cos\theta^{ijk}}{\partial \mathbf{r}^k} = -\frac{1}{r^{ik}}\frac{\mathbf{r}^{ij}}{r^{ij}} + \frac{\cos\theta^{ijk}}{r^{ik}}\frac{\mathbf{r}^{ik}}{r^{ik}}$$

It is seen that

$$\frac{\partial \cos\theta^{ijk}}{\partial \mathbf{r}^i} + \frac{\partial \cos\theta^{ijk}}{\partial \mathbf{r}^j} + \frac{\partial \cos\theta^{ijk}}{\partial \mathbf{r}^k} = 0 \qquad (4\text{-}4.41)$$

which implies, among any three atoms, i, j, and k. Tersoff potential yields

$$\mathbf{F}^i + \mathbf{F}^j + \mathbf{F}^k = 0 \qquad (4\text{-}4.42)$$

One more time, it says the total interatomic forces are vanishing, and it verifies that Newton's third law is automatically satisfied.

Stiffness Matrix in Molecular Dynamics. Here, for simplicity, we only consider pair potentials. For atom i and atom j, we define

$$\begin{aligned} \mathbf{r} &\equiv \mathbf{r}^i - \mathbf{r}^j \\ r &\equiv \|\mathbf{r}^i - \mathbf{r}^j\| \\ V^{ij}(r) &= V^{ji}(r) \triangleq V(r) \\ \phi(r) &\equiv -\frac{1}{r}\frac{dV}{dr} \\ \Phi(r) &\equiv \frac{1}{r}\frac{d\phi}{dr} \end{aligned} \qquad (4\text{-}4.43)$$

We now rewrite the governing equations in molecular dynamics using tensor notations as

$$\begin{aligned} m^i \ddot{u}^i_\alpha &= f^i_\alpha + \varphi^i_\alpha \\ &= \phi r^{ij}_\alpha + \varphi^i_\alpha \end{aligned} \qquad (4\text{-}4.44)$$

where $\mathbf{u}^i \equiv \mathbf{r}^i - \mathbf{R}^i$ is the displacement vector of atom i with $\mathbf{R}^i$ being its initial position and notice that $\ddot{\mathbf{r}}^i = \ddot{\mathbf{u}}^i$. To illustrate a point, we assume that all the atomic displacements involved are small. Now we do the Taylor series expansion of the interatomic force about the initial position

$$
\begin{aligned}
f_\alpha^i &= \phi r_\alpha^{ij}\Big|_{\mathbf{r}^i=\mathbf{R}^i,\mathbf{r}^j=\mathbf{R}^j} + \sum_\beta \frac{\partial\left(\phi r_\alpha^{ij}\right)}{\partial r_\beta^i}\Bigg|_{\substack{\mathbf{r}^i=\mathbf{R}^i\\ \mathbf{r}^j=\mathbf{R}^j}} \left(r_\beta^i - R_\beta^i\right) \\
&\quad + \sum_\beta \frac{\partial\left(\phi r_\alpha^{ij}\right)}{\partial r_\beta^j}\Bigg|_{\substack{\mathbf{r}^i=\mathbf{R}^i\\ \mathbf{r}^j=\mathbf{R}^j}} \left(r_\beta^j - R_\beta^j\right) + \cdots \\
&= f_\alpha^i(0) + \sum_\beta \frac{\partial\left(\phi r_\alpha^{ij}\right)}{\partial r_\beta^i}\Bigg|_{\substack{\mathbf{r}^i=\mathbf{R}^i\\ \mathbf{r}^j=\mathbf{R}^j}} u_\beta^i + \sum_\beta \frac{\partial\left(\phi r_\alpha^{ij}\right)}{\partial r_\beta^j}\Bigg|_{\substack{\mathbf{r}^i=\mathbf{R}^i\\ \mathbf{r}^j=\mathbf{R}^j}} u_\beta^j + \cdots
\end{aligned}
\tag{4-4.45}
$$

and notice that

$$
\begin{aligned}
\frac{\partial\left(\phi r_\alpha^{ij}\right)}{\partial r_\beta^i} &= \frac{\partial\phi}{\partial r_\beta^i} r_\alpha^{ij} + \phi\frac{\partial\left(r_\alpha^{ij}\right)}{\partial r_\beta^i} = \frac{d\phi}{dr}\frac{r_\beta^{ij}}{r} r_\alpha^{ij} + \phi\delta_{\alpha\beta} \equiv \Phi r_\alpha^{ij} r_\beta^{ij} + \phi\delta_{\alpha\beta} \\
\frac{\partial\left(\phi r_\alpha^{ij}\right)}{\partial r_\beta^j} &= \frac{\partial\phi}{\partial r_\beta^j} r_\alpha^{ij} + \phi\frac{\partial\left(r_\alpha^{ij}\right)}{\partial r_\beta^j} = -\frac{d\phi}{dr}\frac{r_\beta^{ij}}{r} r_\alpha^{ij} - \phi\delta_{\alpha\beta} \equiv -\left\{\Phi r_\alpha^{ij} r_\beta^{ij} + \phi\delta_{\alpha\beta}\right\}
\end{aligned}
\tag{4-4.46}
$$

The stiffness matrix can be obtained as

$$
-K_{\alpha\beta}^{ij} \equiv \left[\Phi r_\alpha^{ij} r_\beta^{ij} + \phi\delta_{\alpha\beta}\right]\Big|_{\mathbf{r}^i=\mathbf{R}^i,\mathbf{r}^j=\mathbf{R}^j} \tag{4-4.47}
$$

Then Eq. (4-4.45) can be rewritten as

$$
f_\alpha^i = f_\alpha^i(0) - \sum_\beta K_{\alpha\beta}^{ij} u_\beta^i + \sum_\beta K_{\alpha\beta}^{ij} u_\beta^j + \cdots \tag{4-4.48}
$$

This means that, for a system of N atoms with pair potential and small displacements, the governing equation can be expressed in the following matrix form:

$$
\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{f}^o + \boldsymbol{\varphi} \tag{4-4.49}
$$

which is exactly equivalent to the governing equations in classical mechanical vibration.

Remark: The above-mentioned molecular dynamics (MD) may be considered classical molecular dynamics in which the interatomic potential functions (predefined or fixed) are based on either empirical data or independent electronic structure calculations. These potential functions may not be accurate enough, especially for “chemically complex” system, of which the electronic structure and thus the bonding pattern changes qualitatively in the course of the simulation. The classical MD has been greatly extended by the family of techniques, named as ab initio MD or quantum MD. The basic idea underlying all ab initio MD methods is to compute the forces acting on the nuclei from electronic structure calculations that are performed "on-the-fly" as the MD trajectory is generated during the simulation. In other words, the electronic variables are not integrated out beforehand but are considered as active degrees of freedom. It also means ab initio MD is focusing on selecting a particular approximation in order to solve the Schrödinger equation. The most popular electronic structure theory implemented within ab initio molecular dynamics is the *density functional theory* (Car and Parrinello, 1985). Obviously, the ab initio molecular dynamics is much more computationally demanding than the classical one; the bright side is that new phenomena, especially of chemically complex systems, can be predicted or explained. Interested readers are referred to the comprehensive work of Marx and Hutter (2000).

Problem Set 4-4

1. A strain gage rosette provides the following data: $\epsilon_x = 0.01, \epsilon_y = 0.02$, and $\epsilon_{30^\circ} = 0$, where ϵ_{30° is the strain of a line element at 30° to the x axis. Compute ϵ_{60°. Is the result valid if the material is anisotropic? If it is inelastic?

2. Consider a solid body that has three mutually orthogonal planes of symmetry. Derive the matrix for the elastic coefficients C_{mn}.

3. The stress–strain relations for an elastic body are

$$
\begin{aligned}
\sigma_x &= C_{11}\epsilon_x + C_{12}\epsilon_y \\
\sigma_y &= C_{12}\epsilon_x + C_{22}\epsilon_y \\
\tau_{xy} &= C_{33}\gamma_{xy} \\
\sigma_z &= \tau_{xz} = \tau_{yz} = 0
\end{aligned}
$$

relative to axes (x, y, z), where C_{ij} are elastic coefficients. Consider axes (x', y', z') obtained by a rotation of axes (x, y, z) through angle θ about the z axis. Express the stress components $\sigma'_x, \sigma'_y, \tau'_{xy}$ relative to axes (x', y', z') in terms of strain components $\epsilon'_x, \epsilon'_y, \epsilon'_{xy}$ relative to axes (x', y', z'), elastic coefficients C_{ij}, and angle θ.

4. In Example 4-4.1 it is desired, because of shape considerations, to represent stress components and strain components relative to axes (x_1, y_1), which are obtained by a rotation of θ about axis z, which is perpendicular to the (x, y) plane. (Axes x, y, and z form a right-handed coordinate system.) Determine the elastic coefficients relative to axis (x_1, y_1, z_1) for values of $\theta = 30^\circ$ and 45°.

4-5 Isotropic Media. Homogeneous Media

If the orientations of crystals and grains constituting the material of a solid body are distributed sufficiently randomly, any part of the body will display essentially the same material properties in all directions. If a solid body is composed of such randomly oriented crystals and grains, it is said to be *isotropic*. Thus, isotropy may be considered a directional property of the material. Accordingly, if a material body is isotropic, its physical properties at a point P in the body are invariant under a rotation with respect to axes with origin at P. A medium is said to be *elastically isotropic* if its characteristic elastic coefficients are invariant under any rotation of coordinates.

If the material properties at a point in a medium depend on the direction considered, the body (material) is said to be *nonisotropic, anisotropic*, or *aeolotropic.* Wood is an example of an anisotropic material.

If the material properties are identical for every point in a body, the body is said to be *homogeneous.* In other words, homogeneity implies that the physical properties of a body are invariant under a translation. Alternatively, a body whose material properties change from point to point is said to be *nonhomogeneous.* For example, because in general the elastic coefficients are functions of temperature, a body subjected to a nonuniform temperature distribution is nonhomogeneous. Accordingly, the property of nonhomogeneity is a scalar property, that is, it depends only on the location of a point in the body, not on any direction at the point. Consequently, a body may be nonhomogeneous but isotropic. For example, consider a flat plate sandwich formed by a layer of aluminum bounded by layers of steel. If the point considered is in a steel layer, the material properties have certain values that are generally independent of direction. That is, the steel is essentially isotropic. Furthermore, if the temperature is approximately constant throughout the plate, the material properties do not change greatly from point to point. If the point considered is in the aluminum, the material properties differ from those of steel. Accordingly, taken as a complete body, the sandwich plate exhibits nonhomogeneity. However, at any point in the body, the properties are essentially independent of direction.[6]

Analogously, a body may be nonisotropic but homogeneous. For example, the physical properties of a crystal depend on direction in the crystal, but the properties vary little from one point to another.[7]

If an elastic body is composed of isotropic material, the strain energy density depends only on the principal strains (which are invariants), since for isotropic materials the elastic coefficients are invariants under arbitrary rotations [see Eq. (4-6.2)].

4-6 Strain Energy Density for Elastic Isotropic Medium

The strain energy density of an elastic isotropic material depends only on the principal strains $(\epsilon_1, \epsilon_2, \epsilon_3)$. Accordingly, if the elasticity is linear, Eq. (4-4.6) yields

$$\begin{aligned} U = {} & \tfrac{1}{2}C_{11}\epsilon_1^2 + \tfrac{1}{2}C_{12}\epsilon_1\epsilon_2 + \tfrac{1}{2}C_{13}\epsilon_1\epsilon_3 + \tfrac{1}{2}C_{12}\epsilon_1\epsilon_2 + \tfrac{1}{2}C_{22}\epsilon_2^2 \\ & + \tfrac{1}{2}C_{23}\epsilon_2\epsilon_3 + \tfrac{1}{2}C_{13}\epsilon_1\epsilon_3 + \tfrac{1}{2}C_{33}\epsilon_3^2 \end{aligned} \tag{4-6.1}$$

[6]An exception occurs at the boundaries (interfaces) between the aluminum layer and the steel layers. Here, the body is nonisotropic in nature.

[7]For an extensive discussion of anisotropy and nonhomogeneity in crystals, see Nye (1987).

As noted in Section 4-4, a strain energy density function U exists for either adiabatic or isothermal deformations. However, the numerical values of the elastic coefficients $C_{\alpha\beta}$ differ in these two cases (Nye, 1987).

By symmetry, the naming of the principal axis is arbitrary. Hence, $C_{11} = C_{22} = C_{33} = C_1$ and $C_{23} = C_{13} = C_{12} = C_2$. Consequently, Eq. (4-6.1) contains only two distinct coefficients. Hence, for a linearly elastic isotropic material, the strain energy density may be expressed in the form

$$U = \tfrac{1}{2}\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3)^2 + G\left(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2\right) \tag{4-6.2}$$

where $\lambda = C_2$ and $G = (C_1 - C_2)/2$ are elastic coefficients called Lamé's elastic coefficients. If the material is homogeneous, λ and G are constants at all points. In terms of the strain invariants [see Eq. (2-12.3)], Eq. (4-6.2) may be written in the following form:

$$U = \left(\tfrac{1}{2}\lambda + G\right)J_1^2 - 2GJ_2 \tag{4-6.3}$$

Returning to arbitrary rectangular coordinates (x_1, x_2, x_3) and introducing the general definitions of J_1 and J_2 from Eq. (2-12.2), we obtain

$$U = \tfrac{1}{2}\lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})^2 + G\left(\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 + 2\epsilon_{23}^2 + +2\epsilon_{13}^2 + 2\epsilon_{12}^2\right) \tag{4-6.4}$$

Equations (4-3.2) and (4-6.4) now yield Hooke's law for a linearly elastic isotropic material in the form

$$\begin{aligned} &\sigma_{11} = \lambda e + 2G\epsilon_{11} \qquad \sigma_{22} = \lambda e + 2G\epsilon_{22} \qquad \sigma_{33} = \lambda e + 2G\epsilon_{33} \\ &\sigma_{12} = 2G\epsilon_{12} \qquad \sigma_{13} = 2G\epsilon_{13} \qquad \sigma_{23} = 2G\epsilon_{23} \end{aligned} \tag{4-6.5}$$

where $e = u_{1,1} + u_{2,2} + u_{3,3} = J_1$ is the classical small-displacement cubical strain [see Eq. (2-12.6)]. Thus, we have shown that for isotropic linearly elastic media, the stress–strain law involves only two elastic coefficients. An analytic proof of the fact that no further reduction is possible on a theoretical basis can be constructed (Jeffreys, 1987; Love, 1944, Section 69).

In index notation, Eqs. (4-6.5) may be written in the form

$$\sigma_{\alpha\beta} = \lambda e\,\delta_{\alpha\beta} + 2G\epsilon_{\alpha\beta} \tag{4-6.6}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta (Chapter 1, Section 1-24). By means of Eqs. (4-6.5), we find, with Eqs. (2-12.3) and (3-5.4), that

$$\begin{aligned} I_1 &= (3\lambda + 2G)J_1 \\ I_2 &= \lambda(3\lambda + 4G)J_1^2 + 4G^2J_2 \\ I_3 &= \lambda^2(\lambda + 2G)J_1^3 + 4\lambda G^2J_1J_2 + 8G^3J_3 \end{aligned} \tag{4-6.7}$$

which relate the stress invariants I_1, I_2, I_3 to the strain invariants J_1, J_2, J_3.

Inverting Eqs. (4-6.5), we obtain

$$\begin{aligned}
&\epsilon_{11} = \frac{1}{E}[(1+\nu)\sigma_{11} - \nu I_1] \qquad \epsilon_{22} = \frac{1}{E}[(1+\nu)\sigma_{22} - \nu I_1] \\
&\epsilon_{33} = \frac{1}{E}[(1+\nu)\sigma_{33} - \nu I_1] \\
&\epsilon_{12} = \frac{1}{2G}\sigma_{12} = \frac{1+\nu}{E}\sigma_{12} \qquad \epsilon_{13} = \frac{1}{2G}\sigma_{13} = \frac{1+\nu}{E}\sigma_{13} \\
&\epsilon_{23} = \frac{1}{2G}\sigma_{23} = \frac{1+\nu}{E}\sigma_{23}
\end{aligned} \tag{4-6.8}$$

where

$$E = \frac{G(3\lambda + 2G)}{\lambda + G} \qquad \nu = \frac{\lambda}{2(\lambda + G)} \tag{4-6.9}$$

are elastic coefficients called Young's modulus and Poisson's ratio, respectively. (For a physical interpretation of E and ν, see Section 4-7.) In addition,

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad G = \frac{E}{2(1+\nu)} \qquad k = \frac{E}{3(1-2\nu)} \tag{4-6.9a}$$

where k is the bulk modulus (see Problem 4-6.3).

Alternatively, Eqs. (4-6.5) may be written in terms of E and ν as

$$\begin{aligned}
&\sigma_{11} = \frac{E}{(1+\nu)(1-2\nu)}[(1-2\nu)\epsilon_{11} + \nu J_1] \\
&\sigma_{22} = \frac{E}{(1+\nu)(1-2\nu)}[(1-2\nu)\epsilon_{22} + \nu J_1] \\
&\sigma_{33} = \frac{E}{(1+\nu)(1-2\nu)}[(1-2\nu)\epsilon_{33} + \nu J_1] \\
&\sigma_{12} = \frac{E}{1+\nu}\epsilon_{12} \qquad \sigma_{13} = \frac{E}{1+\nu}\epsilon_{13} \qquad \sigma_{23} = \frac{E}{1+\nu}\epsilon_{23}
\end{aligned} \tag{4-6.10}$$

In index notation, we may write Eqs. (4-6.8) and (4-6.10) as

$$\epsilon_{\alpha\beta} = \frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{\nu}{E} I_1 \delta_{\alpha\beta} \tag{4-6.11}$$

and

$$\sigma_{\alpha\beta} = \frac{E}{1+\nu}\epsilon_{\alpha\beta} + \frac{\nu E}{(1+\nu)(1-2\nu)} J_1\, \delta_{\alpha\beta} \tag{4-6.12}$$

Substitution of Eqs. (4-6.8) into Eq. (4-6.4) yields the strain energy density U in terms of stress quantities. Thus, we obtain

$$\begin{aligned} U &= \frac{1}{2E}\big[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - 2v(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}) \\ &\qquad + 2(1+v)(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)\big] \\ &= \frac{1}{2E}\big[I_1^2 - 2(1+\nu)I_2\big] \end{aligned} \tag{4-6.13}$$

If the axes (x_1, x_2, x_3) are directed along the principal axes of *strain*, then $\epsilon_{12} = \epsilon_{13} = \epsilon_{23} = 0$. Hence, by Eqs. (4-6.5), $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$. Therefore, the axes (x_1, x_2, x_3) must lie along the principal axes of *stress*. Consequently, for an isotropic material the principal axes of stress are coincident with the principal axes of strain. *Hence, when we deal with isotropic material, no distinction need be made between principal axes of stress and principal axes of strain. Such axes are simply called principal axes.*

Similarly, in terms of G, v, Eq. (4-6.4) may be written

$$U = G\left[\frac{\nu}{1-2\nu}J_1^2 + \epsilon_{\alpha\beta}\epsilon_{\beta\alpha}\right] \qquad \alpha, \beta = 1, 2, 3 \tag{4-6.14}$$

Example 4-6.1. Stress–Strain Relations for Fiber-Reinforced Material. In modern technology, materials called *composites* are designed with different properties in different directions by embedding fiberlike elements in a matrix base. Usually, the matrix properties differ from those of the fibers. The base matrix may be a plastic, a metal, concrete, and so on, and the fibers may be fiberglass or polymer filaments, metal strands, steel reinforcing bars, and the like. The fibers are often laid in a particular pattern. For example, in Fig. E4-6.1, fiber reinforcements are laid parallel to the x_2 direction and at angle $\pm\alpha$ relative to the x_1 direction.

To compute the strain energy density function of the composite, we require the strains of the fibers as well as the strains of the matrix [Eqs. (4-6.4)]. Because the strain energy density function is a scalar quantity, we may compute the strain energy density for the fibers and the matrix separately and add the energies to obtain the strain energy density for the composite. We assume that the pattern of fibers does not change in the direction perpendicular to the plane of Fig. E4-6.1.

The strain energy of a fiber may be computed by considering the fiber to act as a tension member. Hence, the strain energy of oblique bars per unit volume (per unit area of surface in x_1, x_2 plane, per unit thickness perpendicular to the x_1, x_2 plane) is

$$U_0 = \tfrac{1}{2}E_0 a_0 \epsilon_0^2 \tag{E4-6.1}$$

where E_0 is the modulus of elasticity of a fiber, a_0 is the cross-sectional area of a fiber per unit width perpendicular to the fiber direction, per unit thickness, and ϵ_0 is the strain of an oblique fiber. By Eq. (2-7.2), the strain ϵ_0 in terms of strain

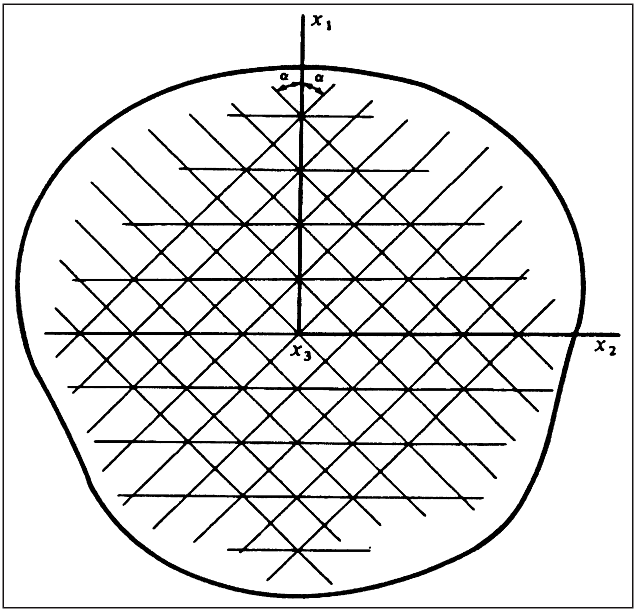

Figure E4-6.1

components $\epsilon_{11}, \epsilon_{12}, \epsilon_{22}$ relative to axes x_1, x_2 (we neglect the effects of strains $\epsilon_{13}, \epsilon_{23}, \epsilon_{33}$ on the fibers) is

$$\epsilon_0 = \epsilon_{11} \cos^2 \alpha + \epsilon_{22} \sin^2 \alpha + \epsilon_{12} \sin 2\alpha \tag{E4-6.2}$$

To obtain the strain energy of the oblique fibers, we substitute Eq. (E4-6.2) into Eq. (E4-6.1) with $\alpha = +\alpha$ (fibers of positive slope) and with $\alpha = -\alpha$ (fibers of negative slope), and add the results. Thus the strain energy U_0 of oblique fibers is

$$U_0 = E_0 a_0 [(\epsilon_{11} \cos^2 \alpha + \epsilon_{22} \sin^2 \alpha)^2 + \epsilon_{12}^2 \sin^2 2\alpha] \tag{E4-6.3}$$

Similarly, the strain energy of the horizontal fibers is

$$U_h = \tfrac{1}{2} E_0 a_h \epsilon_{22}^2 \tag{E4-6.4}$$

where a_h for the horizontal fibers is equivalent to a_0 for the oblique fibers.

Assuming a condition of plane stress exists in the matrix (i.e., $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$; see Chapter 3, Section 3-7), by Eqs. (4-6.13) and (4-6.5) we obtain the strain energy of the matrix as

$$U_m = \frac{E_m}{2(1 - \nu^2)} \left[\epsilon_{11}^2 + \epsilon_{22}^2 + 2\nu \epsilon_{11} \epsilon_{22} + 2(1 - \nu) \epsilon_{12}^2\right] \tag{E4-6.5}$$

Hence, the total strain energy density is $U = U_m + U_0 + U_h$. Accordingly, by Eq. (4-3.1), we obtain the stress–strain relations

$$\begin{aligned} \sigma_{11} &= C_{11}\epsilon_{11} + C_{12}\epsilon_{22} \\ \sigma_{22} &= C_{12}\epsilon_{11} + C_{22}\epsilon_{22} \\ \sigma_{12} &= C_{33}\epsilon_{12} \end{aligned} \tag{E4-6.6}$$

where

$$\begin{aligned} C_{11} &= \frac{E_m}{1-\nu^2} + 2E_0 a_0 \cos^4\alpha \\ C_{12} &= \frac{\nu E_m}{1-\nu^2} + 2E_0 a_0 \sin^2\alpha\ \cos^2\alpha \\ C_{22} &= \frac{E_m}{1-\nu^2} + 2E_0 a_0 \sin^4\alpha + E_0 a_h \\ C_{33} &= \frac{E_m}{1+\nu} + 4E_0 a_0 \sin^2\alpha\ \cos^2\alpha \end{aligned} \tag{E4-6.7}$$

The elastic coefficients C_{ij} of Eq. (E4-6.7) define the anisotropic nature of the composite, as four coefficients are required to describe the material response.

Example 4-6.2. Further Restrictions on the Elastic Coefficients. The strain energy of a material must be positive, otherwise the material would be unstable. This fact means that the quadratic form that represents the strain energy density U in terms of strain components, Eq. (4-4.6), must be positive definite (Langhaar, 1989). [See Section 4.16, Eq. (4-16.1) for a definition of positive definite.] In order for U to be positive definite, the discriminants of the matrix of coefficients $C_{\alpha\beta}$, Eq. (4-4.8), must all be positive (Hildebrand, 1992). This condition places further restrictions on the elastic coefficients.

For example, for the case of Example 4-6.1, the coefficients C_{ij} form the array

$$\begin{pmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{pmatrix} \tag{a}$$

Hence, they must satisfy the conditions

$$C_{11} > 0 \qquad C_{11}C_{22} - C_{12}^2 > 0 \qquad C_{33}(C_{11}C_{22} - C_{12}^2) > 0 \tag{b}$$

for U to be positive definite. Because ν, E_m, E_0, a_0, a_h are positive quantities and $\nu < \frac{1}{2}$, Eq. (b) is satisfied with the definition of C_{ij} given by Eq. (E4-6.7).

Problem Set 4-6

1. Displacement components are given by the formulas $u_1 = 0$, $u_2 = C_1 x_3$, and $u_3 = C_2 x_2$, where C_1 and C_2 are nonzero constants. What restrictions must be placed on C_1, C_2 in order that these displacement components may exist for a real body? Derive the strain components of the strain tensor. For small deflections, determine the state of stress that exists in the body if the body is linearly elastic and isotropic. Locate the principal axes of strain and the principal axes of stress.

2. The coordinate axes x and y on the free unloaded surface of a linearly elastic isotropic body are principal directions. At what angle θ relative to the x axis must a strain gage be place in order that direct measurement of the principal stress σ_x be made with this gage, that is, so that $\sigma_x = K\epsilon_g$? Assume that the elastic coefficients of the material are known.

3. For an isotropic elastic medium subjected to a hydrostatic state of stress, $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$, $\sigma_{12} = \sigma_{13} = \sigma_{23} = 0$, where p denotes pressure. Show that for this state of stress $p = -ke$, where e is the cubical strain and $k = E/[3(1-2\nu)]$ is called the bulk modulus. Discuss the case $\nu > \frac{1}{2}$, $\nu < -1$. [See also Love (1944), Section 70.]

4. Three strain gages are located on the free surface of a deformed body as shown in Fig. P4-6.4. The extensional strains measured by gages a, b, and c are ϵ_a, ϵ_b, and ϵ_c, respectively.

(a) Derive an expression for the strain components ϵ_{11}, ϵ_{22}, and ϵ_{12} in terms of ϵ_a, ϵ_b, and ϵ_c.

(b) Assume that the material is linearly elastic and isotropic. Express the stress components σ_x, σ_y, and τ_{xy} on the surface in terms of ϵ_a, ϵ_b, and ϵ_c and elastic constants v and E.

(c) Assume that the direction 1 is a principal direction of strain. Express the angle θ in terms of ϵ_a, ϵ_b, and ϵ_c.

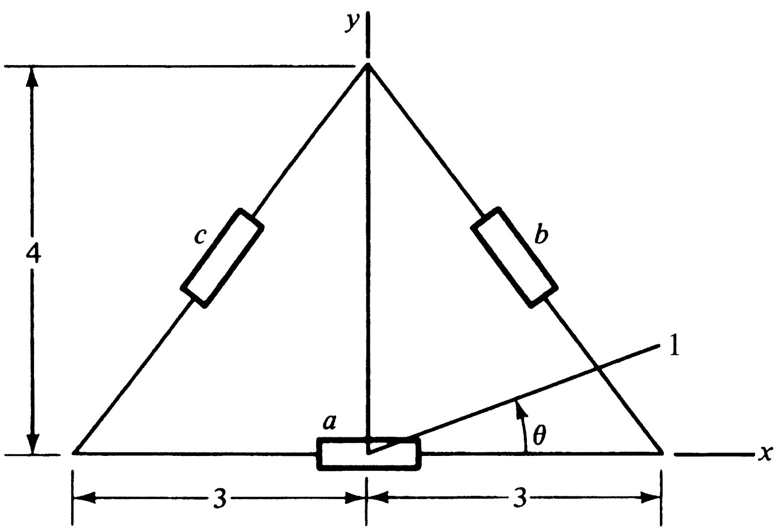

Figure P4-6.4

5. An elastic medium subjected to a state of stress σ_{ij} deforms incompressibly ($J_1 = 0$). Consider rectangular Cartesian coordinate axes (x, y, z).

(a) Assume that $\sigma_{11} + \sigma_{22} + \sigma_{33} \neq 0$. Determine the value of ν, Poisson's ratio, for the material.

(b) Assume additionally that $\epsilon_{33} = 0$. Show that two values of ν are theoretically possible. Determine these two values of ν.

6. A strain gage rosette is attached on a point of the free unloaded surface of an elastic isotropic body (Fig. P4-6.6). Under deformation of the body, the strain gages in arms 1, 2, 3 record strains $\epsilon_1, \epsilon_2, \epsilon_3$, respectively. In terms of $\epsilon_1, \epsilon_2, \epsilon_3$ and angle θ, derive expressions for the strain components $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$, and derive expressions for the stress components $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$ at the point.

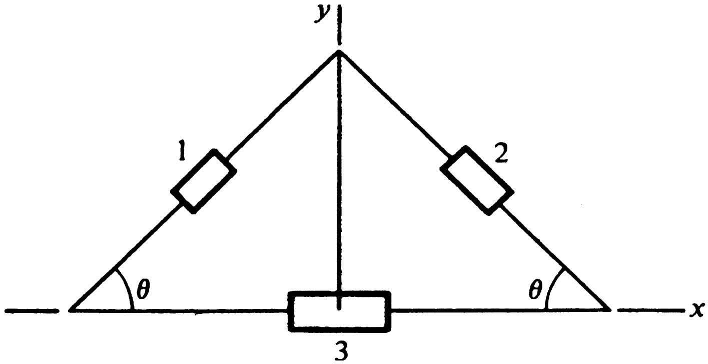

Figure P4-6.6

7. The skewed plate of unit thickness is loaded by uniformly distributed stress S_1 and S_2 applied perpendicularly to the sides of the plate (see Fig. P4-6.7).

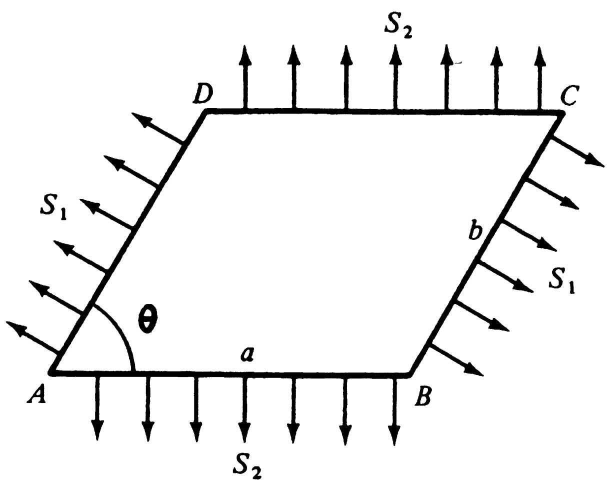

Figure P4-6.7

(a) Determine all possible conditions of equilibrium for the plate in terms of S_1, S_2, a, b, and θ.

(b) For $\theta = 90^\circ$, derive an expression for the elongation of the diagonal AC under the action of the stresses S_1 and S_2, assuming that the material is linearly elastic and isotropic.

8. In Fig. P4-6.7 let the stresses S_1 and S_2 be applied so that they are directed parallel to the edges AB (or DC) and AD (or BC) of the skewed plate. Derive expressions for the principal stresses and the principal strains in terms of S_1, S_2, a, b, θ, E, and ν, where E and ν denote Young's modulus and Poisson's ratio, respectively.

9. The following strains have been measured at a point on the *free* (*unloaded*) surface of an elastic isotropic body:

Direction	Angle ϕ	Strain ϵ
1	0°	0.002
2	120°	0.002
3	120°	-0.001

Determine the principal strains in the plane of the surface and the principal directions of strain.

10. A strain rosette is used to determine the strain on a free (unloaded) surface of an isotropic elastic body (Fig. P4-6.10). The modulus of elasticity is $E = 10^7$ psi. Poisson's ratio is $\nu = 0.25$. The measured strains in directions (a, b, c) are $\epsilon_a = 0.0002$, $\epsilon_b = 0.0001$, and $\epsilon_c = 0.0004$. Determine the principal stresses and their directions relative to direction a.

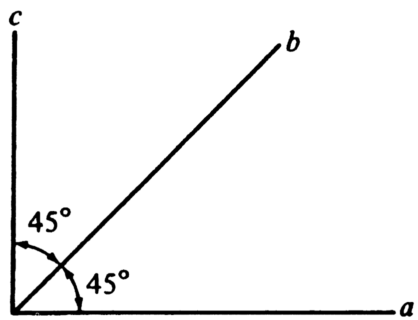

Figure P4-6.10

11. The following strains have been measured at a point on the *free* (*unloaded*) surface of a body

Direction	Angle ϕ	Strain ϵ
1	0°	0.002
2	120°	0.002
3	240°	-0.001

Determine the principal strains in the plane of the surface and the principal directions of strain. Assume the body is isotropic but not necessarily elastic. Discuss the solution of this problem for the elastic anisotropic body.

12. Strains have been measured by three strain gages at a point P on the *free surface* of a plate that lies in the (x, y) plane. The measured strains are $C, 3C$, and $2C$ along the three directions that form the angles $0^\circ, 60^\circ$, and 120°, respectively, with the positive x axis. With the assumption that the material is isotropic, it may be shown that at P the direction normal to the plane of the plate (the x, y plane) is a principal strain direction. Let this principal strain be denoted by ϵ_3. Determine the strain components $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ relative to the rectangular Cartesian axes (x, y, z), where z is perpendicular to the plate.

To check the strain measurements, a fourth gage is placed in the direction forming the angle 270° with the positive x axis, and the plate is reloaded. The measured strains are now $C, 3C, 2C$, and $2C$, respectively, in the directions 0, 60, 120, and 270°. Calculate the strain components and compare them to the previously determined strain components. Comment on this result.

13. Let y_i denote rectangular Cartesian coordinate lines that coincide at any point P (in an isotropic, elastic medium) with the tangent lines to orthogonal curvilinear coordinate lines x_i (Fig. P4-6.13). Let $\sigma_{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ denote components of the stress tensor and the strain tensor, respectively, relative to planes perpendicular to the tangent lines at P to coordinate lines (x_1, x_2, x_3). Noting the invariant form of the strain energy density function U (Section 4-6), employ the relation between stress components and the strain energy density function (Section 4-3) to derive the stress–strain relations for axes x_i. For cylindrical coordinates $(x_1 \equiv r, x_2 \equiv \theta, x_3 \equiv z)$, show that $\sigma_r = \lambda(\epsilon_r + \epsilon_\theta + \epsilon_z) + 2G\epsilon_r$, with similar relations for σ_θ, σ_z, and $\tau_{r\theta} = G\gamma_{r\theta}, \tau_{rz} = G\gamma_{rz}, \tau_{\theta z} = G\gamma_{\theta z}$.

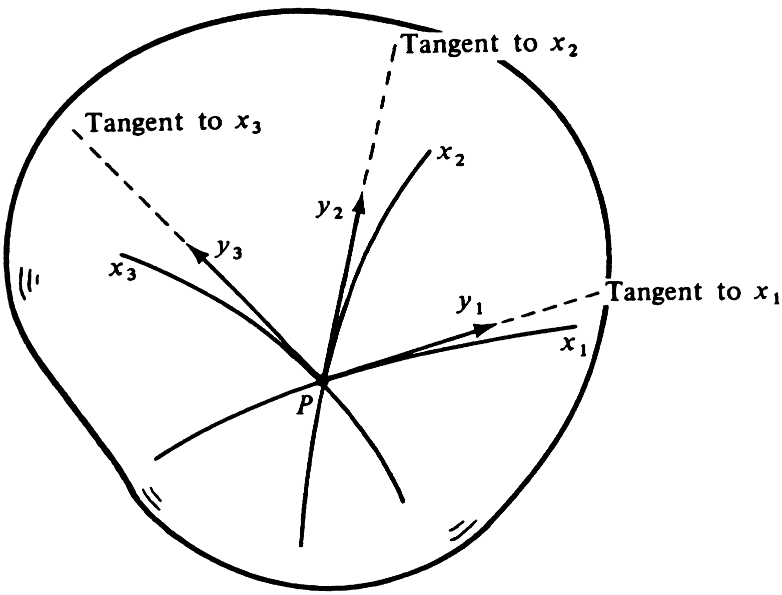

Figure P4-6.13

14. A state of *plane strain* relative to the (x, y) plane is defined by $u = u(x, y), v = v(x, y), w = 0$. The strain energy density U_0 of a certain crystal undergoing plane strain is given by

$$U_0 = \tfrac{1}{2}\left(b_{11}\epsilon_x^2 + b_{22}\epsilon_y^2 + b_{33}\gamma_{xy}^2 + 2b_{12}\epsilon_x\epsilon_y + 2b_{13}\epsilon_x\gamma_{xy} + 2b_{23}\epsilon_y\gamma_{xy}\right)$$

where $b_{ij}, i, j = 1, 2, 3$, are elastic coefficients. For small-displacement theory, derive the differential equations of equilibrium in terms of (u, v) for plane strain of the crystal, including the effects of body forces.

4-7 Special States of Stress

Simple Tension. To interpret the Lamé constants λ and G, we consider a body in the following state of stress relative to axes (x, y, z):

$$\sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = \tau_{yz} = 0 \qquad \sigma_z = \text{const} = \sigma$$

When this state of stress exists in a cylindrical or prismatic bar whose axis is parallel to the z axis, the stress on the lateral boundary vanishes. On the ends, the normal stress is σ, and the shearing stress is zero. Hence, this is the state of stress in a bar under simple tension.

Equations (4-6.5) yield $\lambda e + 2G\epsilon_x = \lambda e + 2G\epsilon_y = \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$ and $\lambda e + 2G\epsilon_z = \sigma$, where $\epsilon_x = \epsilon_{11}, \epsilon_x = \epsilon_{11}$, and so on. Solving the equations for the ϵ's, we obtain

$$\begin{aligned} \epsilon_x = \epsilon_y &= -\frac{\lambda\sigma}{2G(3\lambda + 2G)} \\ \epsilon_z &= \frac{(\lambda + G)\sigma}{G(3\lambda + 2G)} \end{aligned} \tag{4-7.1}$$

It follows that

$$-\frac{\epsilon_x}{\epsilon_z} = -\frac{\epsilon_y}{\epsilon_z} = \frac{\lambda}{2(\lambda + G)} = \nu \tag{4-7.2}$$

The quantities

$$E = \frac{G(3\lambda + 2G)}{\lambda + G} \quad \text{and} \quad \nu = \frac{\lambda}{2(\lambda + G)} \tag{4-7.3}$$

are called *Young's modulus of elasticity* and *Poisson's ratio*, respectively. In terms of ν and E, Eq. (4-7.1) becomes

$$\epsilon_x = \epsilon_y = -\frac{\nu\sigma}{E} \qquad \epsilon_z = \frac{\sigma}{E} \tag{4-7.4}$$

Solving Eqs. (4-7.3) for λ and G in terms of the readily measurable quantities E and ν, we obtain

$$G = \frac{E}{2(1 + \nu)} \qquad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \tag{4-7.5}$$

Integrating Eqs. (4-7.4) and disregarding an arbitrary rigid-body displacement, we obtain the displacement vector

$$u = -\frac{v\sigma}{E}x \qquad v = -\frac{v\sigma}{E}y \qquad w = \frac{\sigma}{E}z \tag{4-7.6}$$

Because (u, v, w) are linear functions of the coordinates (x, y, z), this type of strain is homogeneous (see Chapter 2, Section 2-14).

Pure Shear. Consider the state of pure shear characterized by the stress components $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = 0$, $\tau_{yz} = \tau =$ constant. For this state of stress, Eqs. (4-6.5) yield

$$\lambda e + 2G\epsilon_x = \lambda e + 2G\epsilon_y = \lambda e + 2G\epsilon_z = \gamma_{xy} = \gamma_{xz} = 0 \qquad \gamma_{yz} = \frac{\tau}{G}$$

Solving these equations for the strain components, we obtain

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = \gamma_{xz} = 0 \qquad \gamma_{yz} = \frac{\tau}{G}$$

These formulas show that a rectangular parallelepiped *ABCD* (Fig. 4-7.1) whose faces are parallel to the coordinate planes is sheared in the *yz* plane so that the right angle between the edges of the parallelepiped parallel to the *y* and *z* axes decreases by the amount γ_{yz}. For this reason, *G* is called the *shear modulus of elasticity.*

Example 4-7.1. Elimination of Friction Effect in a Uniaxial Compression Test. In a uniaxial compression test, the effect of friction between the test specimen and the testing machine platens may be eliminated by a properly designed experiment. One way to eliminate the effect of friction is to design the specimen and/or machine

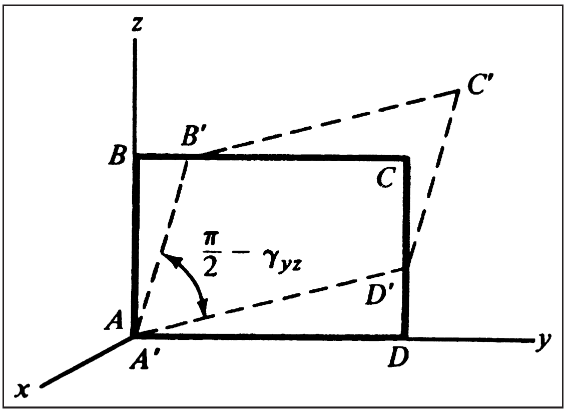

Figure 4-7.1

platens so that (1) the cross sections of the specimen and the end platens are the same, and (2) a certain relation exists between the material properties of the specimen and the platens.

To illustrate this point, let quantities associated with the specimen be denoted by subscript s and those associated with the platens by subscript p. Let P be the load applied to the specimen through the end platens. Because the cross-sectional shapes of the specimen and the platens are the same, we denote the areas by A. Let coordinate z be taken along the longitudinal axis of the specimen, and coordinate x be perpendicular to axis z. Then, under a machine load P, the longitudinal strains in the specimen and platens, respectively, are

$$(\epsilon_z)_s = \frac{P}{E_s A} \qquad (\epsilon_z)_p = \frac{P}{E_p A} \tag{a}$$

The associated lateral strains are

$$(\epsilon_x)_s = -\nu_s(\epsilon_z)_s = -\frac{\nu_s P}{E_s A} \qquad (\epsilon_x)_p = -\nu_P(\epsilon_z)_p = -\frac{\nu_p P}{E_P A} \tag{b}$$

If the lateral strains in the specimen and the platens are equal, they will expand laterally the same amount, thus eliminating friction that might be induced by the tendency of the specimen to move laterally relative to the platens. Hence, by Eq. (b) the requirement for friction to be nonexistent is that $(\epsilon_x)_s = (\epsilon_x)_p$, or

$$\frac{\nu_s}{E_s} = \frac{\nu_p}{E_p} \tag{c}$$

Therefore, in addition to identical cross sections of specimen and platens, the moduli of elasticity and Poisson's ratios must satisfy Eq. (c). Thus, in order to reduce the effect of friction on the test results, it is essential to select the material properties of the platens to satisfy Eq. (c) as closely as possible (Cook, 1962). In addition, to ensure uniform axial stress at the interfaces of the ends of the specimen and the platens, the length of a platen should be about the same as the maximum dimension of its cross section.

Problem Set 4-7

1. Let isotropic elastic material in the (x, y) plane be subjected to the stress components $\sigma_{11} = 0, \sigma_{22} = \sigma, \sigma_{12} = \tau$. Let $u_1 = u_2 = \omega = 0$ for $x_1 = x_2 = 0$, where (u_1, u_2) denote (x_1, x_2) displacement components and ω denotes volumetric rotation.

(a) Show that the circle $x_1^2 + x_2^2 = a^2$ is deformed into an ellipse.

(b) For the case $\tau = 0$, show that the major and minor axes of the ellipse coincide with the (x_1, x_2) axes, and express their lengths in terms of a and the elastic properties of the material.

2. The strain energy density U of a linearly elastic material is given by the relation $U = \left(\frac{1}{2}\lambda + G\right)J_1^2 - 2GJ_2$, where (λ, G) are the Lamé elastic constants and (J_1, J_2) are the first and second strain invariants.

(a) Employing the relation between U and the stress components σ_α, derive the stress–strain relations for a state of plane strain $\epsilon_{33} = \epsilon_{23} = \epsilon_{13} = 0$ relative to the (x_1, x_2) plane.

(b) Repeat part (a) for a state of plane stress $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$.

3. Derive results equivalent to Eqs. (4-7.1) through (4-7.6) for the case of hydrostatic compression $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \sigma_{12} = \sigma_{23} = \sigma_{31} = 0$, where p is pressure.

4. Relative to axes (x, y, z), let $u = w = 0$ and $v = kz$, where k is a constant. Show that this case (*simple shear*) differs from that of pure shear by a rigid-body rotation.

5. The volumetric strain or dilatation e is equal to the strain invariant J_1 [Eq. (2-12.6)]. What is the dilatation for the cases of (a) simple tension and (b) pure shear?

4-8 Equations of Thermoelasticity

The classical study of thermoelasticity is concerned with the distribution of stress (or strain) in a solid subjected to a nonuniform temperature distribution $T(x, y, z)$—that is, a temperature distribution that is not linear in (x, y, z)—or in a solid that is physically or geometrically constrained and then subjected to a uniform or nonuniform temperature change.

The subject was initially formulated by Duhamel (1837, 1838) when he derived equations for the distribution of strain in an elastic medium containing temperature gradients. Duhamel's results were subsequently reformulated by a number of authors. Finally, Neumann (1885) presented the theory of thermal stress in the following way.

Consider an *isotropic* elastic solid in an arbitrary state of stress. Let a small element of the solid be detached from its surroundings. Let the element be subjected to a temperature change T. The additional straining in the element is given by the components $\epsilon'_{ij} = kT\,\delta_{ij}$, where k is the coefficient of linear thermal expression for the solid and δ_{ij} is the Kronecker delta (see Chapter 1, Section 1-26). Consequently, if the net strain in the body is denoted by the components of the strain tensor δ_{ij}, then the portion of the strain produced by the stress is characterized by the components $\epsilon_{ij} - kT\delta_{ij}$. To arrive at a stress–strain–temperature relation, one then replaces the components ϵ_{ij} by the components $\epsilon_{ij} - kT\delta_{ij}$ in the generalized Hooke's law. In Section 4-12 we will see that by modifying the stress–strain relations in this manner and expressing the equilibrium conditions in terms of displacement components, we obtain the result that the usual displacement equilibrium equations are modified by a body force $Ek\,\nabla T/(1-2\nu)$ per unit volume, where E is Young's modulus of elasticity and ν is Poisson's ratio. Furthermore, we must superimpose on the load stress a "hydrostatic stress" equal to $-EkT/(1-2\nu)$; and finally, we must superimpose a surface traction with (x, y, z) components $EkTl/(1-2\nu)$, $EkTm/(1-2\nu)$, and $EkTn/(1-2\nu)$, where l, m, n denote the (x, y, z) components of the unit normal vector to the surface. In other words, if T is a known function (found by solving the heat conduction equation; see Section 4-9), the thermoelasticity problem reduces to determination of the displacement field from a determinate set of equations.

Duhamel applied the basic theory outlined above to a number of specific problems. Later Neumann and others used the theory to study the double refracting property of nonuniformly heated glass plates. In these investigations a number of techniques were developed for the solution of thermoelasticity boundary value problems. Although for a long time these methods appeared to be more or less academic, recently the study of thermal stress has been stimulated by practical problems, and it has become an increasingly important factor in the design of components of modern structures and machines that undergo heating.

As implied in the foregoing, the classical Duhamel–Neumann theory of thermal stress assumes that although the state of strain of an elastic solid is affected by a nonuniform temperature distribution, the heat conduction process is unaffected by a deformation. This assumption is true if the system is in mechanical and thermal equilibrium. However, it is an approximation in the time-dependent thermal problem because then the acceleration terms cannot vanish identically as implied by the Duhamel–Neumann theory. Ordinarily, the thermal acceleration effects are small. Nevertheless, application of the Duhamel–Neumann theory to the transient thermal problem leads to inconsistencies that cannot be resolved within the scope of this theory because elastic and thermal constants that appear in the Duhamel–Neumann formulation are defined under conditions that are not met in the transient case.

An extensive discussion of the dynamical theory of thermoelasticity has been given by Chadwick (1960). The theory has been generalized to encompass the transient problem by Biot (1956) through application and further development of the methods of irreversible thermodynamics, including the thermoelastic potential and the minimum entropy production principle. A survey of mathematical methods and techniques of treating thermoelastic problems has been presented by Parkus (1963).

In the following articles we restrict ourselves to classical thermoelasticity theory as developed by Duhamel and Neumann. Comprehensive treatises on thermoelasticity have been written by Boley and Weiner (1997) and by Nowacki (1986).

4-9 Differential Equation of Heat Conduction

For a large class of problems, temperature distribution in a solid may be calculated by solving the heat conduction equation, subject to the geometrical and temperature boundary conditions (Carslaw and Jaeger, 1986). In general, the temperature distribution T will depend on time t. However, unless the inertia effects that arise as a result of a suddenly applied temperature change are significant, time t enters the thermal-stress problem only as a passive parameter. The problem may then be treated as a quasi-static one, as the temperature distribution enters into the thermal-stress calculation as an integral load function. Accordingly, for a large class of problems, the temperature distribution may be expressed functionally as

$$T = T(x, y, z; t)$$

where (x, y, z) denote rectangular Cartesian coordinates and t denotes time. For a given time $t = t_1$, the above equation defines the temperature distribution as a function of coordinates (x, y, z).

By the theory of heat, the temperature $T(x, y, z; t)$ that exist at a point (x, y, z) of a thermally isotropic homogeneous body referred to rectangular Cartesian coordinates is determined by the partial differential equation

$$\frac{\partial T}{\partial t} = \kappa \nabla^2 T + \frac{Q}{c\rho} \tag{4-9.1}$$

where ∇^2 denotes the Laplacian operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{4-9.2}$$

In Eq. (4-9.1), ρ denotes the mass density; c denotes the specific heat, that is, the quantity of heat that is necessary to raise the temperature of a unit mass 1°C. The term κ is the *temperature diffusivity.* It is defined by the ratio

$$\kappa = \frac{\alpha}{c\rho} \tag{4-9.3}$$

where α is the *thermal conductivity.* In turn, α is related to the quantity of heat dq that flows through a surface element ΔS with normal n during the time dt by the relation

$$dq = -\alpha \frac{dT}{dn} \Delta S \, dt \tag{4-9.4}$$

In the following, we consider mainly thermally isotropic and homogeneous bodies. Hence, ordinarily α depends on neither direction nor location in the body. Additionally, if it is assumed that α and c do not depend on temperature or stress level, they remain constant. If the temperature gradient is not too great, this last assumption is permissible. However, if large temperature gradients occur, it may be necessary to consider variations of α and c with temperature.

The term Q in Eq. (4-9.1) represents the quantity of heat per unit time and unit volume that is produced by heat sources that lie in the interior of the volume element. A unit volume dV produces accordingly the quantity of heat $Q\, dV\, dt$ during time dt.

If the temperature distribution is independent of time, we speak of a *stationary* or *steady-state temperature distribution.* Equation (4-9.1) then reduces to the Poisson equation of potential theory:

$$\nabla^2 T + \frac{Q}{\alpha} = 0 \tag{4-9.5}$$

In the absence of heat sources in the body, $Q = 0$. Hence, for steady-state heat flow in the absence of heat sources, the temperature distribution in the body must satisfy the equation

$$\nabla^2 T = 0 \tag{4-9.6}$$

Equation (4-9.6) is subject to the temperature conditions on the surface of the body.

The temperature distribution is not determined completely by Eqs. (4-9.1) to (4-9.6). For nonsteady heat flow, an initial temperature distribution (at $t = 0$) must be specified. The initial temperature distribution may be a continuous or a discontinuous function of the coordinates; that is, $T(x, y, z; 0) = f(x, y, z)$.

The boundary conditions of the problem depend on the effect that the environment of the body exerts on its surface. The equations describing this effect must be known at each point on the surface. The boundary conditions are in their simplest form when the temperature T_0 on the surface is given as a function of position and time. However, they can also be specified in terms of heat flow, that is, in terms of the quantity of heat that flows through the surface as a function of time [Eq. (4-9.4)]. Finally, as is common, but also mathematically more difficult, the boundary conditions may be represented in terms of the temperature θ of the environment by the law of heat exchange between the surface of the body and its environment. To formulate the problem mathematically, an approximation attributed to Newton is often used:

$$\frac{\partial T}{\partial n_0} = \frac{e}{\alpha}(\theta - T_0) \tag{4-9.7}$$

Equation (4-9.7) relates the temperature gradient on the surface of the body to the temperature difference between the surface of the body and its environment. The ratio e/α is called the *relative emissivity*, and e is called the *emissivity* of the surface.

4-10 Elementary Approach to Thermal-Stress Problem in One and Two Variables

Consider an infinitesimal element dx of a solid body. Initially, let the temperature of the element be T_0. The temperature T_0 is considered to be that temperature for which the length of the element is dx. For simplicity, let us take $T_0 = 0$ because the elongation of the element depends on differences between existing temperature in the element and temperature T_0. Let the element be subjected to temperature T. Then, the element will undergo an infinitesimal elongation de (provided T is positive; for negative T, a contraction occurs). By the theory of heat, the elongation de is related to the temperature T by the equation

$$de = kT\,dx \tag{a}$$

where k is the coefficient of thermal expansion for the material of the element. In general k is a function of temperature T. For example, for crystals the following relation between k and T is often used:

$$k = a + bT + cT^2 \tag{b}$$

where T is temperature in degrees Celsius and where a, b, c are constants with magnitudes of the order 10^{-6}, 10^{-8}, 10^{-11}, respectively. Theoretically, the coefficient of thermal expansion for a material is defined by the relation

$$k = \frac{1}{L_T}\frac{dL}{dT} \tag{c}$$

where L_T denotes the length of the element for temperature T. Usually, k is determined experimentally by employing the relation

$$\bar{k} = \frac{1}{L_0}\frac{\Delta L}{\Delta T} \tag{d}$$

where $\bar{k}$ denotes an average value of k, ΔL denotes the finite change in length of the element for the finite change in temperature ΔT, and L_0 denotes the length of the element at some temperature, say, room temperature. Accordingly, the constants a, b, c of Eq. (b) are not generally well defined, although average values for certain materials are often employed in practice. Furthermore, in the development of the theory of thermoelasticity, we find that k enters into the equations only in the product form kT. Consequently, we may account for variations of k with T by setting $\overline{T} = kT$ and considering $\overline{T}$ as a pseudotemperature parameter for the body; that is, the variation of k with T may be accounted for by replacing the product kT by the parameter $\overline{T}$.

By Eq. (a), the strain in the fiber resulting from T is

$$\epsilon = \frac{de}{dx} = kT \tag{e}$$

In general, a temperature change in the element will not produce stress in the element unless either the element is physically prevented by forces from expanding or, if physically free to expand, it is unable to expand in a manner compatible with the temperature distribution in the element. For example, if the element is restrained so that its length is unchanged under a temperature increase of T, forces P must act at its ends (Fig. 4-10.1). Figuratively speaking, we imagine that first the element is allowed to elongate a distance de. Then by application of forces P, the element is returned to its initial length dx. Hence, to compute the stress induced in the element by the temperature change T when its ends are restrained from moving, we compute the stress σ induced in the element by forces P under compression de. Hence,

$$\sigma = \frac{P}{A} = -E\epsilon \tag{f}$$

where A is the cross-sectional area of the element and E is the modulus of elasticity of the material.

By Eqs. (e) and (f) we obtain

$$\sigma = -EkT \tag{g}$$

where the minus sign denotes compression.

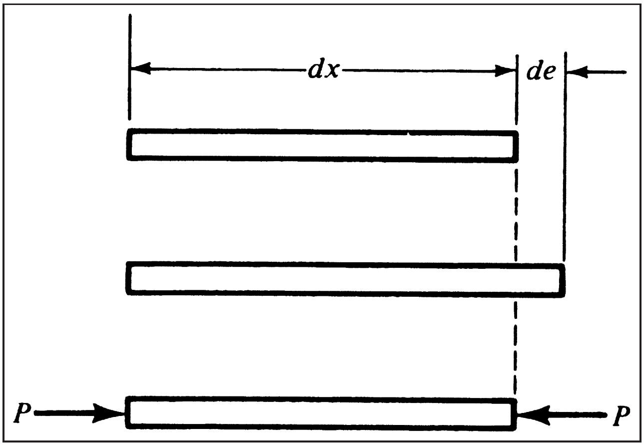

Figure 4-10.1

Thermal-Stress Problems in Beams. The above argument may be applied to elementary beam (strip) problems. For example, consider a rectangular strip in the (x, y) plane (Fig. 4-10.2). Let the strip be subjected to a temperature distribution $T = T(y)$, even in y. Then the resulting elongation of the strip is symmetrical with respect to axis (x). Now on each infinitesimal longitudinal element dx of the strip, we imagine that the stress

$$\sigma_x' = -EkT \tag{h}$$

acts. If the strip is prevented from elongating, Eq. (h) determines the normal stress component in the x direction. In elementary beam theory, the stress components $\sigma_y, \sigma_z, \tau_{xz}, \tau_{yz}$ are neglected. Furthermore, if the weight of the beam is neglected

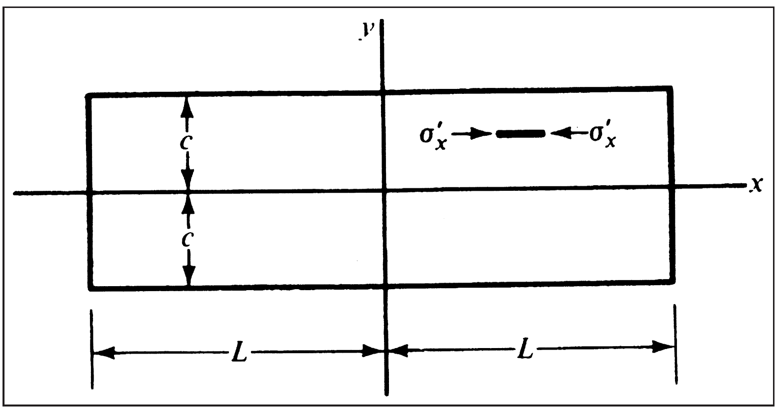

Figure 4-10.2

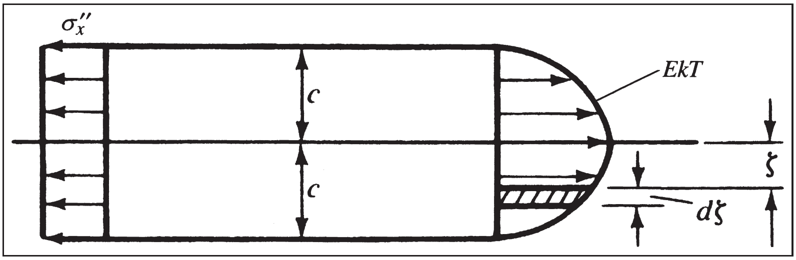

Figure 4-10.3

and no other loads act, there is no shear stress τ_{xy}. Hence, the stress of elementary beam theory is given completely in this case by Eq. (h).

If the beam is free to expand, that is, if it is free of forces at its ends, the stress σ_x must vanish on the ends. For a beam with free ends, the end stress due to σ_x' [Eq. (h)] is $\sigma_x = -EkT$, where T is evaluated at the ends. Accordingly, to eliminate the boundary forces, we need to add a distributed stress EkT over the ends of the beam. It follows by Saint-Venant's principle (see Section 4-15) that the stress σ_x'' at some distance from the ends (a distance several times the depth $2c$ of the strip) resulting from the distributed load EkT is for a beam of unit thickness (Fig. 4-10.3),

$$\sigma_x'' = \frac{1}{2c}\int_{-c}^{c} EkT\,d\zeta \tag{i}$$

Hence, except near the ends, the stress σ_x in the beam is obtained by superposition of σ_x' and σ_x''; that is,

$$\sigma_x = \sigma_x' + \sigma_x'' = \frac{1}{2c}\int_{-c}^{c} EkT\,d\zeta - EkT \tag{j}$$

Equation (j) is valid for variable k and E; that is, k and E may be functions of temperature T or coordinate y. If EkT is a constant, Eq. (j) yields $\sigma_x = 0$. Hence, if the free strip is subjected to a constant temperature change and if E and k are constant, no stress is induced. The beam simply elongates with no stress.

If the temperature distribution is nonsymmetrical with respect to the longitudinal axis of the beam, $T = T(y)$ is an odd function of y. Accordingly, the end forces EkT give rise to a resultant moment M that alters the stress σ_x in the beam.

By theory of moments (Fig. 4-10.3),

$$M = \int_{-c}^{c} EkT\zeta\,d\zeta \tag{k}$$

The moment M produces stress σ_x''' in the beam. If we assume that this stress varies linearly with respect to y, as is done in elementary beam theory, we may write

$$\sigma_x''' = \frac{\sigma y}{c} \tag{l}$$

where σ denotes the value of σ_x''' at $y = c$. The moment M may also be expressed in terms of σ_x''' as

$$M = \int_{-c}^{c} \sigma_x''' \zeta \, d\zeta \tag{m}$$

Equations (k), (l), and (m) yield

$$\sigma_x''' = \frac{y}{I} \int_{-c}^{c} EkT\zeta \, d\zeta \tag{n}$$

where $I = 2c^3/3$ denotes the moment of inertia of the area of the cross section of the beam with respect to the z axis. Equation (n) is restricted to a linear distribution of stress across the cross section of the beam (the beam has unit thickness).

The net stress in the beam may be obtained by superposition of the stresses $\sigma_x', \sigma_x'', \sigma_x'''$. Thus, for a beam with free ends and with rectangular cross section, the stress resulting from a temperature distribution $T(y)$ is given by the relation

$$\sigma_x = -EkT + \frac{1}{2c} \int_{-c}^{c} EkT \, d\zeta + \frac{3y}{2c^3} \int_{-c}^{c} EkT\zeta \, d\zeta \tag{o}$$

Equation (o) holds for the plane-stress problem ($\sigma_z = 0$) of rectangular strips. For plane-strain problems ($\epsilon_z = 0$), it may be shown that

$$\sigma_x = \frac{-EkT}{1-\nu} + \frac{1}{2c} \int_{-c}^{c} \frac{EkT}{1-\nu} \, d\zeta + \frac{3y}{2c^3} \int_{-c}^{c} \frac{EkT}{1-\nu} \zeta \, d\zeta \tag{p}$$

where ν denotes Poisson's ratio for the material.

Problem Derive Eq. (p).

4-11 Stress–Strain–Temperature Relations

The equilibrium equations and the strain–displacement relations remain valid in thermal-stress problems, as they are independent of material properties. However, the stress–strain relations are altered by temperature.

If a body is subjected to a temperature change T, and if the body is allowed to expand freely, a line element of length ds in the body is elongated to a length $(1 + kT)\,ds$, where k is the coefficient of thermal expansion. For thermal isotropic bodies, k is independent of the direction of ds. Additionally, for a number of structural materials, k remains fairly constant for a wide range of temperature (see Section 4-10). Hence, unless large temperature gradients occur, k may be taken as a constant.

It has been observed that k also depends on the stress level (Rosenfeld and Averbach, 1956). Although this effect is not included in the subsequent discussion, it may be of considerable importance, as the experiments of Rosenfeld and Averbach (1956) show that k for steel may increase as much as 10% in the elastic range for a tensile-stress change of 40,000 psi (13.8 MPa). They also observed that the coefficient of thermal expansion of Invar in the elastic range *decreased* with increasing tensile stress. Accordingly, for certain temperature ranges the variation of k with stress may be more significant than variations of material properties with temperature. Variations in k can be treated numerically. Further study of the dependency of k on stress level is needed.

Under the above assumptions, for a thermally isotropic body the angles of an infinitesimal rectangular parallelepiped remain unchanged. Hence, the strains in perpendicular directions are equal, and the shearing strains are zero. Consequently, the strain–temperature relations for a body subjected to the temperature change T are, with respect to rectangular Cartesian coordinates (x_1, x_2, x_3),

$$\epsilon'_{11} = \epsilon'_{22} = \epsilon'_{33} = kT \qquad \epsilon'_{12} = \epsilon'_{13} = \epsilon'_{23} = 0 \tag{4-11.1}$$

or, in index notation,

$$\epsilon'_{ij} = kT\,\delta_{ij} \tag{4-11.2}$$

where k denotes the linear coefficient of thermal expansion of the material and δ_{ij} denotes the Kronecker delta (Chapter 1, Section 1-26). For a nonhomogeneous body, k may be a function of coordinates and of temperature; that is, $k = k(x_1, x_2, x_3; T)$.

Now let the body be subjected to forces that induce stresses δ_{ij} in the body. Accordingly, if δ_{ij} denote the strain components in the body after the application of the forces, the net change in strain produced by the forces is represented by the equations

$$\epsilon'_{ij} = \epsilon_{ij} - kT\delta_{ij} \tag{4-11.3}$$

In general, T may depend on the location in the body and on time t. Hence, $T = T(x_1, x_2, x_3; t)$.

Substitution of Eqs. (4-11.3) into Eqs. (4-6.5) yields

$$\begin{aligned} &\sigma_{11} = \lambda e + 2G\epsilon_{11} - cT \qquad \sigma_{22} = \lambda e + 2G\epsilon_{22} - cT \\ &\sigma_{33} = \lambda e + 2G\epsilon_{33} - cT \qquad \sigma_{12} = 2G\epsilon_{12} \qquad \sigma_{13} = 2G\epsilon_{13} \\ &\sigma_{23} = 2G\epsilon_{23} \end{aligned} \tag{4-11.4}$$

where

$$c = (3\lambda + 2G)k = \frac{E}{1 - 2\nu}k \tag{4-11.5}$$

Similarly, substitution of Eqs. (4-11.3) into Eq. (4-6.8) yields

$$
\begin{aligned}
\epsilon_{11} &= \frac{1}{E}[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})] + kT \\
\epsilon_{22} &= \frac{1}{E}[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})] + kT \\
\epsilon_{33} &= \frac{1}{E}[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})] + kT \\
\epsilon_{12} &= \frac{1+\nu}{E}\sigma_{12} \\
\epsilon_{13} &= \frac{1+\nu}{E}\sigma_{13} \\
\epsilon_{23} &= \frac{1+\nu}{E}\sigma_{23}
\end{aligned}
\qquad (4\text{-}11.6)
$$

or, in index notation [see Eq. (4-6.1)],

$$
\epsilon_{ij} = \frac{1}{E}[(1+\nu)\sigma_{ij} - (\nu I_1 - EkT)\delta_{ij}] \qquad i, j = 1, 2, 3 \qquad (4\text{-}11.7)
$$

where $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}$ is the first stress invariant and δ_{ij} is the Kronecker delta. Finally, substituting Eqs. (4-11.6) into Eqs. (4-6.3) and (4-6.4), we find

$$
U = \left(\tfrac{1}{2}\lambda + G\right)J_1^2 - 2GJ_2 - cJ_1T + \tfrac{3}{2}ckT^2 \qquad (4\text{-}11.8)
$$

In terms of the strain components [see Eqs. (2-12.2) in Chapter 2 and (4-6.4)], we obtain

$$
\begin{aligned}
U = {} & \tfrac{1}{2}\lambda(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})^2 + G(\epsilon_{11}^2 + \epsilon_{22}^2 + \epsilon_{33}^2 + 2\epsilon_{12}^2 + 2\epsilon_{13}^2 + 2\epsilon_{23}^2) \\
& - c(\epsilon_{11} + \epsilon_{22} + \epsilon_{33})T + \tfrac{3}{2}ckT^2
\end{aligned}
\qquad (4\text{-}11.9)
$$

Equations (4-11.4) and (4-11.6) are the basic stress–strain relations of classical thermoelasticity for isotropic materials. For temperature changes T, the strain energy density is modified by a temperature-dependent term that is proportional to the volumetric strain $e = J_1 = \epsilon_{11} + \epsilon_{22} + \epsilon_{33}$ and by a term proportional to T^2 [Eqs. (4-11.8) and (4-11.9)].

We find by Eqs. (4-11.6) and (4-11.9) [or Eqs. (4-11.7) and (4-11.8)] that

$$
U = \frac{1}{2E}\left[I_1^2 - 2(1+\nu)I_2\right] \qquad (4\text{-}11.10)
$$

and

$$
\begin{aligned}
U = \frac{1}{2E}\big[& \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - 2\nu(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}) \\
& + 2(1+\nu)(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)\big]
\end{aligned}
\qquad (4\text{-}11.11)
$$

in terms of stress components. Equation (4-11.11) does not contain T explicitly. However, the temperature distribution may affect the stresses.

In index notation, we may write Eqs. (4-11.9) and (4-11.11) in the forms

$$U = G\epsilon_{\alpha\beta}\epsilon_{\beta\alpha} + \frac{\nu G}{1-2\nu}(\epsilon_{\alpha\alpha})^2 - \frac{E}{1-2\nu}kT\epsilon_{\alpha\alpha} + \frac{3Ek^2T^2}{2(1-2\nu)} \tag{4-11.12}$$

and

$$U = \frac{1}{4G}\sigma_{\alpha\beta}\sigma_{\beta\alpha} - \frac{\nu}{2E}(\sigma_{\alpha\alpha})^2 \tag{4-11.13}$$

where we have written U as a symmetric function of $\epsilon_{\alpha\beta}, \sigma_{\alpha\beta}$ [see Section 4-4 and Eq. (4-4.21)].

Example 4-11.1. Stress–Strain–Temperature Relations for Beryllium. The Cartesian stress and strain tensors are denoted by $(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})$ or $(s_1, s_2, s_3, s_4, s_5, s_6)$ and $(\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{12}, \epsilon_{13}, \epsilon_{23})$ or $(e_1, e_2, e_3, \frac{1}{2}e_4, \frac{1}{2}e_5, \frac{1}{2}e_6)$. The strain–energy density of a Hookean body is (discarding quadratic terms in T since they produce no stresses)

$$U = \tfrac{1}{2}C_{\alpha\beta}e_\alpha e_\beta - Tc_\alpha e_\alpha \tag{E4-11.1}$$

where repeated indexes are summed from 1 to 6 [Section 4-4, Eq. (4-4.2)]. In Eq. (E4-11.1), the effects of temperature T have been included (Section 4-11), in which the material is considered to have different thermal properties, denoted by c_i, relative to axes x_i [Eq. (4-11.3)]. By Eqs. (4-3.1), the stress–strain relations are given by $s_i = \partial U/\partial e_i$.

The beryllium crystal belongs to the close-packed hexagonal system. It has an axis of symmetry such that a rotation of the crystal through 60° about that axis brings the space lattice into coincidence with its original configuration. The base plane of the crystal is perpendicular to that axis. The notations (x, y, z) and (x_1, x_2, x_3) are used interchangeably. The base plane of the crystal is taken to be the (x, y) plane; hence, the z axis is the axis of the crystal. Love [1944, Eqs. (5) and (6), p. 154] shows that for the type of symmetry exhibited by the beryllium crystal (see Section 4-4),

$$\begin{aligned}
&C_{14} = C_{24} = C_{34} = C_{46} = C_{45} = C_{56} = C_{16} = C_{26} = C_{15} = C_{25} = 0\\
&C_{36} = C_{35} = 0 \qquad C_{11} = C_{22} \qquad C_{13} = C_{23} \qquad C_{66} = C_{55}\\
&C_{44} = \tfrac{1}{2}(C_{11} - C_{12})
\end{aligned} \tag{E4-11.2}$$

Love's method also leads to the following relations for the thermal constants:

$$c_1 = c_2 \qquad c_4 = c_5 = c_6 = 0 \tag{E4-11.3}$$

Accordingly, the beryllium crystal has five elastic constants (say B_1, B_2, B_3, B_4, B_5) and two thermoelastic constants (call them C_1, C_2), such that

$$\begin{aligned} U_0 = \tfrac{1}{2}B_1(e_1^2 + e_2^2) + B_2 e_1 e_2 + B_3 e_3(e_1 + e_2) + \tfrac{1}{2}B_4 e_3^2 \\ + \tfrac{1}{2}B_5(e_5^2 + e_6^2) + \tfrac{1}{2}B_6 e_4^2 - C_1 T(e_1 + e_2) - C_2 T e_3 \end{aligned} \tag{E4-11.4}$$

where we have denoted

$$\begin{aligned} &C_{11} = C_{22} = B_1 \\ &C_{12} = B_2 \qquad C_{13} = C_{23} = B_3 \\ &C_{33} = B_4 \qquad C_{55} = C_{66} = B_5 \\ &C_{44} = B_6 = \tfrac{1}{2}(B_1 - B_2) \end{aligned}$$

The relations $\partial U/\partial e_i = s_i$ yield

$$\begin{aligned} s_1 &= B_1 e_1 + B_2 e_2 + B_3 e_3 - C_1 T \\ s_2 &= B_2 e_1 + B_1 e_2 + B_3 e_3 - C_1 T \\ s_3 &= B_3 e_1 + B_3 e_2 + B_4 e_3 - C_2 T \\ s_6 &= B_5 e_6 \qquad s_5 = B_5 e_5 \qquad s_4 = B_6 e_4 \end{aligned} \tag{E4-11.5}$$

Equation (E4-11.5) reduces to the proper relations for an isotropic body (Section 4-6) if

$$B_1 = B_4 = \lambda + 2G \qquad B_2 = B_3 = \lambda \qquad B_5 = B_6 = G \qquad C_1 = C_2 \tag{E4-11.6}$$

where (λ, G) are Lamé's constants. Alternatively, Eq. (E4-11.5) may be written as

$$\begin{aligned} \sigma_{11} &= B_1\epsilon_{11} + B_2\epsilon_{22} + B_3\epsilon_{33} - C_1 T \\ \sigma_{22} &= B_2\epsilon_{11} + B_1\epsilon_{22} + B_3\epsilon_{33} - C_1 T \\ \sigma_{33} &= B_3\epsilon_{11} + B_3\epsilon_{22} + B_4\epsilon_{33} - C_2 T \\ \sigma_{23} &= 2B_5\epsilon_{23} \qquad \sigma_{31} = 2B_5\epsilon_{31} \qquad \sigma_{12} = 2B_6\epsilon_{12} \end{aligned} \tag{E4-11.7}$$

The determinant of Eq. (E4-11.7) is

$$\Delta = (B_1 - B_2)[(B_1 + B_2)B_4 - 2B_3^2] \tag{E4-11.8}$$

Inversion of Eq. (E4-11.7)

$$\begin{aligned} \epsilon_{11} &= A_1\sigma_{11} + A_2\sigma_{22} + A_3\sigma_{33} + K_1 T \\ \epsilon_{22} &= A_2\sigma_{11} + A_1\sigma_{22} + A_3\sigma_{33} + K_1 T \\ \epsilon_{33} &= A_3\sigma_{11} + A_3\sigma_{22} + A_4\sigma_{33} + K_2 T \\ \epsilon_{23} &= 2A_5\sigma_{23} \qquad \epsilon_{31} = 2A_5\sigma_{31} \qquad \epsilon_{12} = 2A_6\sigma_{12} \end{aligned} \tag{E4-11.9}$$

where

$$
\begin{aligned}
A_1 &= \frac{B_1 B_4 - B_3^2}{\Delta} \qquad A_2 = -\frac{B_2 B_4 - B_3^2}{\Delta} \qquad A_3 = -\frac{(B_1 - B_2)B_3}{\Delta} \\
A_4 &= \frac{B_1^2 - B_2^2}{\Delta} \qquad A_5 = \frac{1}{4B_5} \qquad A_6 = \frac{1}{4B_6} \\
K_1 &= \frac{B_1 - B_2}{\Delta}(B_4 C_1 - B_3 C_2) \\
K_2 &= \frac{B_1 - B_2}{\Delta}[-2B_3 C_1 + (B_1 + B_2)C_2]
\end{aligned}
\tag{E4-11.10}
$$

With the aid of tensor properties of stress and strain, it can be shown that Eqs. (E4-11.7) and (E4-11.9) are invariant if the crystal is rotated through any angle about the z axis. Consequently, insofar as the elastic constants are concerned, the crystal has the same type of symmetry as a circular cylinder that is coaxial with the z axis.

Stress–Strain Relations Relative to Axes Inclined to Crystal Axis. It is also of interest to determine the stress–strain–temperature relations with reference to axes (x', y', z') inclined to the z axis (Fig. 4-11.1). The simplest procedure is to express $\sigma'_{11}, \sigma'_{22}, \sigma'_{33}, \sigma'_{23}, \sigma'_{31}, \sigma'_{12}$ in terms of $\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{31}, \sigma_{12}$ by the equations of stress transformations [Chapter 3, Eq. (3-4.1)]. The latter stresses may be expressed in terms of the strains by Eq. (E4-11.7). Finally, $\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, \epsilon_{23}, \epsilon_{31}, \epsilon_{12}$ may be expressed in terms of $\epsilon'_{11}, \epsilon'_{22}, \epsilon'_{33}, \epsilon'_{23}, \epsilon'_{31}, \epsilon'_{12}$ by the equations of strain transformation [Chapter 2, Eq. (2-9.3)]. The results are

$$
\begin{aligned}
\sigma'_{11} &= B_1 \epsilon'_{11} + (B_2 \cos^2\theta + B_3 \sin^2\theta)\epsilon'_{22} + (B_2 \sin^2\theta + B_3 \cos^2\theta)\epsilon'_{33} \\
&\quad + 2(B_2 - B_3)\epsilon'_{23} \sin\theta \cos\theta - C_1 T \\
\sigma'_{22} &= (B_2 \cos^2\theta + B_3 \sin^2\theta)\epsilon'_{11} \\
&\quad + [B_1 \cos^4\theta + 2(B_3 + 2B_5) \sin^2\theta \cos^2\theta + B_4 \sin^4\theta]\epsilon'_{22} \\
&\quad + [(B_1 + B_4 - 4B_5) \sin^2\theta \cos^2\theta + B_3(\sin^4\theta + \cos^4\theta)]\epsilon'_{33} \\
&\quad + 2[(B_1 - B_3 - 2B_5) \cos^2\theta + (B_3 - B_4 + 2B_5) \sin^2\theta]\epsilon'_{23} \sin\theta \cos\theta \\
&\quad - T(C_1 \cos^2\theta + C_2 \sin^2\theta) \\
\sigma'_{33} &= (B_2 \sin^2\theta + B_3 \cos^2\theta)\epsilon'_{11} \\
&\quad + [(B_1 + B_4 - 4B_5) \sin^2\theta \cos^2\theta + B_3(\sin^4\theta + \cos^4\theta)]\epsilon'_{22} \\
&\quad + [B_1 \sin^4\theta + B_4 \cos^4\theta + 2(B_3 + 2B_5) \sin^2\theta \cos^2\theta]\epsilon'_{33} \\
&\quad + 2[(B_1 - B_3 - 2B_5) \sin^2\theta + (B_3 - B_4 + 2B_5) \cos^2\theta]\epsilon'_{23} \sin\theta \cos\theta \\
&\quad - T(C_1 \sin^2\theta + C_2 \cos^2\theta)
\end{aligned}
$$

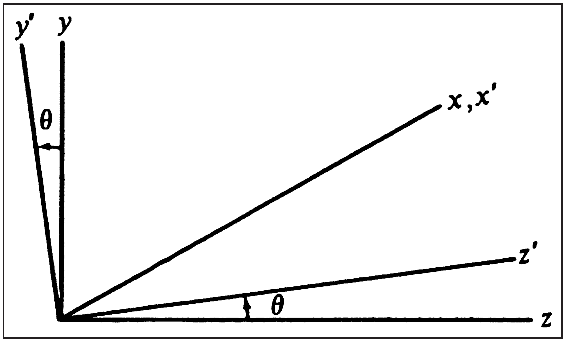

Figure 4-11.1

$$\begin{aligned}
\sigma'_{23} &= (B_2 - B_3)\epsilon'_{11} \sin\theta\cos\theta \\
&\quad + [(B_1 - B_3 - 2B_5)\cos^2\theta + (B_3 - B_4 + 2B_5)\sin^2\theta]\epsilon'_{22} \sin\theta\cos\theta \\
&\quad + [(B_1 - B_3 - 2B_5)\sin^2\theta + (B_3 - B_4 + 2B_5)\cos^2\theta]\epsilon'_{33} \sin\theta\cos\theta \\
&\quad + 2[(-2B_3 + B_1 + B_4 - 2B_5)\sin^2\theta\cos^2\theta + B_5(\sin^4\theta + \cos^4\theta]\epsilon'_{23} \\
&\quad + (C_2 - C_1)T \sin\theta\cos\theta \\
\sigma'_{31} &= 2(B_6\sin^2\theta + B_5\cos^2\theta)\epsilon'_{31} + 2(B_6 - B_5)\epsilon'_{12}\sin\theta\cos\theta \\
\sigma'_{12} &= 2(B_6 - B_5)\epsilon'_{31}\sin\theta\cos\theta + 2(B_6\cos^2\theta + B_5\sin^2\theta)\epsilon'_{12} \qquad (4\text{-}11.14)
\end{aligned}$$

It is possible to solve Eq. (4-11.14) for the strain components, but it is easier to proceed in the same way that Eq. (4-11.14) was derived, interchanging the roles of stress and strain tensors and transforming in the same way. In the process, Eq. (E4-11.9) is used instead of Eq. (E4-11.7). Since Eqs. (E4-11.7) and (E4-11.9) are of the same form, the desired results are obtained by interchanging σ'_{ij} by ϵ'_{ij}, and replacing B_i by A_i in Eq. (4-11.14). Also, C_i is replaced by $-K_i$. Thus, by an obvious transformation of Eq. (4-11.14), we obtain

$$\begin{aligned}
\epsilon'_{11} &= A_1\sigma'_{11} + (A_2\cos^2\theta + A_3\sin^2\theta)\sigma'_{22} + (A_2\sin^2\theta + A_3\cos^2\theta)\sigma'_{33} \\
&\quad + 2(A_2 - A_3)\sigma'_{23}\sin\theta\cos\theta + K_1 T \\
\epsilon'_{22} &= (A_2\cos^2\theta + A_3\sin^2\theta)\sigma'_{11} \\
&\quad + [A_1\cos^4\theta + 2(A_3 + 2A_5)\sin^2\theta\cos^2\theta + A_4\sin^4\theta]\sigma'_{22} \\
&\quad + [(A_1 + A_4 - 4A_5)\sin^2\theta\cos^2\theta + A_3(\sin^4\theta + \cos^4\theta)]\sigma'_{33} \\
&\quad + 2[(A_1 - A_3 - 2A_5)\cos^2\theta + (A_3 - A_4 + 2A_5)\sin^2\theta]\sigma'_{23}\sin\theta\cos\theta \\
&\quad + T(K_1\cos^2\theta + K_2\sin^2\theta)
\end{aligned}$$

$$
\begin{aligned}
\epsilon'_{33} &= (A_2 \sin^2\theta + A_3 \cos^2\theta)\sigma'_{11} \\
&\quad + [(A_1 + A_4 - 4A_5)\sin^2\theta\cos^2\theta + A_3(\sin^4\theta + \cos^4\theta)]\sigma'_{22} \\
&\quad + [A_1 \sin^4\theta + A_4 \cos^4\theta + 2(A_3 + 2A_5)\sin^2\theta\cos^2\theta]\sigma'_{33} \\
&\quad + 2[(A_1 - A_3 - 2A_5)\sin^2\theta + (A_3 - A_4 + 2A_5)\cos^2\theta]\sigma'_{23}\sin\theta\cos\theta \\
&\quad + T(K_1 \sin^2\theta + K_2 \cos^2\theta) \\
\epsilon'_{23} &= (A_2 - A_3)\sigma'_{11}\sin\theta\cos\theta \\
&\quad + [(A_1 - A_3 - 2A_5)\cos^2\theta + (A_3 - A_4 + 2A_5)\sin^2\theta]\sigma'_{22}\sin\theta\cos\theta \\
&\quad + [(A_1 - A_3 - 2A_5)\sin^2\theta + (A_3 - A_4 + 2A_5)\cos^2\theta]\sigma'_{33}\sin\theta\cos\theta \\
&\quad + 2[(-2A_3 + A_1 + A_4 - 2A_5)\sin^2\theta\cos^2\theta + A_5(\sin^4\theta + \cos^4\theta)]\sigma'_{23} \\
&\quad + (K_1 - K_2)T\sin\theta\cos\theta \\
\epsilon'_{31} &= 2(A_6 \sin^2\theta + A_5 \cos^2\theta)\sigma'_{31} + 2(A_6 - A_5)\sigma'_{12}\sin\theta\cos\theta \\
\epsilon'_{12} &= 2(A_6 - A_5)\sigma'_{31}\sin\theta\cos\theta + 2(A_6 \cos^2\theta + A_5 \sin^2\theta)\sigma'_{12}
\end{aligned}
\tag{4-11.15}
$$

Example 4-11.1 demonstrates the application of rules for transformation of stress and strain components, as well as the derivation of stress–strain relations through use of the strain energy density function.

Problem Set 4-11

1. Show that Eqs. (E4-11.7) and (E4-11.9) are invariant if the crystal is rotated through any angle about the z axis.

2. The values of the elastic coefficients B_i of Example 4-11.1 in pounds per square inch are

$$
\begin{aligned}
B_1 &= 43{,}420{,}000 & B_2 &= 4{,}003{,}000 \\
B_3 &= 1{,}595{,}000 & B_4 &= 49{,}630{,}000 \\
B_5 &= 24{,}100{,}000 & B_6 &= 19{,}710{,}000
\end{aligned}
$$

By Eqs. (E4-11.10),

$$
\begin{aligned}
A_1 &= 2.325 \times 10^{-8}\ \text{in.}^2/\text{lb} & A_2 &= -0.2118 \times 10^{-8}\ \text{in.}^2/\text{lb} \\
A_3 &= -0.0679 \times 10^{-8}\ \text{in.}^2/\text{lb} & A_4 &= 2.019 \times 10^{-8}\ \text{in.}^2/\text{lb} \\
A_5 &= 1.037 \times 10^{-8}\ \text{in.}^2/\text{lb} & A_6 &= 1.268 \times 10^{-8}\ \text{in.}^2/\text{lb}
\end{aligned}
$$

Using these values, verify the following results.

For stretching in the z direction, Poisson's ratio is $\nu = -A_3/A_4 = 0.0336$. For stretching in the x direction, the lateral contractions in the y and z directions are different; $\nu_y = -A_2/A_1 = 0.0911$ and $\nu_z = -A_3/A_1 = 0.0292$. For stretching in the z' direction with $\theta = 45^\circ$, $\nu'_y = -0.0072$, $\nu'_x = 0.0669$. For stretching in the z direction, $E = 1/A_4 = 49{,}530{,}000$ psi. For stretching in any direction perpendicular to the z axis, $E = 1/A_1 = 43{,}010{,}000$ psi. The maximum value of Young's modulus occurs for stretching in the z' direction with $\theta = 11.4^\circ$. Its value is 49,540,000 psi.

Evidently, Poisson's ratio is small for stretching in any direction. As may be seen from Eqs. (E4-11.5) or (E4-11.6), B_5 and B_6 are shear moduli corresponding to shears in the coordinate planes.

From Eq. (E4-11.12) the shear modulus for shear in the (y', z') plane is

$$\frac{\sigma'_{23}}{2\epsilon'_{23}} = \tfrac{1}{4}[(-2A_3 + A_1 + A_4 - 2A_5)\sin^2\theta\cos^2\theta + A_5(\sin^4\theta + \cos^4\theta)]^{-1}$$

This attains a minimum value for $\theta = 45^\circ$. The corresponding value of the shear modulus is 22,320,000 psi. Likewise, for $\theta = 45^\circ$,

$$\frac{\sigma'_{12}}{2\epsilon'_{12}} = \frac{\sigma'_{31}}{2\epsilon'_{31}} = 21{,}700{,}000 \text{ psi}$$

Accordingly, the crystal is roughly isotropic with respect to shear moduli; an average value is about 22,000,000 psi.

3. Consider the problem of small-displacement plane strain thermoelasticity for which

$$\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

(a) Derive an expression for σ_z in terms of stress components σ_x and σ_y, material properties k (thermal coefficient of linear expansion) and E (modulus of elasticity), and temperature change T measured from an arbitrary zero.

(b) Assume the additional conditions that stress components $\sigma_x = \sigma_y = \tau_{xy} = 0$. Hence, derive expressions for the strain components ϵ_x, ϵ_y, and γ_{xy}.

(c) Show that under the combined conditions of parts (a) and (b) the compatibility conditions reduce to $\nabla^2 T = 0$ for constant E and k.

(d) Using the results of part (b), show that the rotation of a volume element in the xy plane is

$$\omega_z = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Hence, show that

$$\frac{\partial \epsilon'}{\partial x} = \frac{\partial \omega_z}{\partial y} \qquad \frac{\partial \epsilon'}{\partial y} = -\frac{\partial \omega_z}{\partial x}$$

where $\epsilon' = (1+\nu)kT$. That is, show that ϵ' and ω_z satisfy the Cauchy–Riemann equations. (Consequently, the theory associated with the Cauchy–Riemann equations may be applied to ϵ' and ω_z.)

4. Consider a plane element of rectangular plan form (Fig. P4-11.4) ($b \ll L, b \ll h$, and $h \ll L$). A known temperature variation (measured from an arbitrary zero) $T = T(x)$ through the depth $2h$ of the element exists. Because the flat element is thin (b small), it is reasonable to assume that a state of *plane stress* ($\sigma_z = \tau_{zx} = \tau_{yz} = 0$) exists. Assume also that $\sigma_x = \tau_{xy} = 0$ and that $\sigma_y = \sigma_y(x)$.

(a) Show that in the absence of body forces and acceleration, the equations of motion are satisfied.

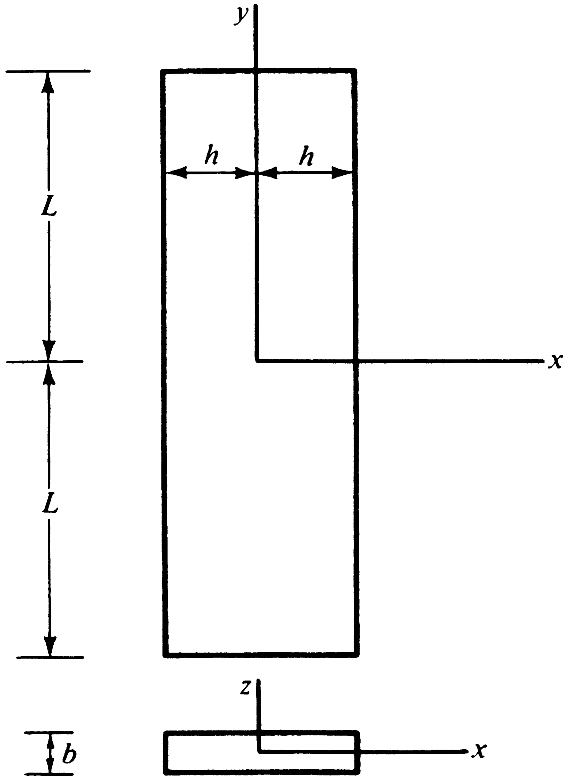

Figure P4-11.4

(b) Determine the equations of compatibility for this case. Hence, solve these equations to obtain the most general expression for $\sigma_y(x)$.

(c) To evaluate arbitrary constants in σ_y, employ the boundary conditions that the resultant force and the resultant moment at $y = \pm L$ vanish; that is, for $y = \pm L$

$$\sum F_y = \int_{-h}^{h} \sigma_y \, dx = 0 \qquad \sum M_z = \int_{-h}^{h} x\sigma_y \, dx = 0$$

Hence, express σ_y as a function of x.

4-12 Thermoelastic Equations in Terms of Displacement

As noted in Section 4-8, by modifying the stress–strain–temperature relations appropriately and expressing the equilibrium equations in terms of displacement components, we may reduce the thermoelasticity problem to one of determining the displacement field from a determinant set of equations. In this section, this transformation is carried out.

Introducing the notations $I_1 = \sigma_x + \sigma_y + \sigma_z$, $E = 2(1+\nu)G$, we may write Eqs. (4-11.6) in the form (using x, y, z notation)

$$\begin{aligned}
\epsilon_x &= \frac{1}{2G}\left(\sigma_x - \frac{\nu}{1+\nu}I_1\right) + kT \\
\epsilon_y &= \frac{1}{2G}\left(\sigma_y - \frac{\nu}{1+\nu}I_1\right) + kT \\
\epsilon_z &= \frac{1}{2G}\left(\sigma_z - \frac{\nu}{1+\nu}I_1\right) + kT \\
\gamma_{xy} &= \frac{1}{G}\tau_{xy}, \ldots, \ldots
\end{aligned} \tag{4-12.1}$$

or briefly, in index notation,

$$\begin{aligned}
\epsilon_{\alpha\beta} &= \frac{1}{2G}\left(\sigma_{\alpha\beta} - \frac{\nu\delta_{\alpha\beta}}{1+\nu}I_1\right) + kT\delta_{\alpha\beta} \qquad \alpha, \beta = 1, 2, 3 \\
\delta_{\alpha\beta} &= \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}
\end{aligned} \tag{4-12.2}$$

Adding the first three of Eqs. (4-12.1), we obtain

$$e = \frac{1-2\nu}{1+\nu}\frac{I_1}{2G} + 3kT = \frac{1-2\nu}{E}I_1 + 3kT \tag{4-12.3}$$

where $e = \epsilon_x + \epsilon_y + \epsilon_z$ is the volume dilatation (or the strain invariant J_1).

The temperature–displacement relations may be determined as follows: Solving Eqs. (4-12.1) for stresses, and utilizing Eq. (4-12.3), we obtain the stress–strain–temperature relations [see also Eq. (4-11.4)]:

$$\begin{aligned}
\sigma_x &= \lambda e + 2G\epsilon_x - \frac{kET}{1-2v}, \ldots, \ldots \\
\tau_{xy} &= G\gamma_{xy}, \ldots, \ldots
\end{aligned} \tag{4-12.4}$$

where the ellipses denote similar equations in (σ_y, σ_z) and in (τ_{xz}, τ_{yz}), and

$$\lambda = \frac{vE}{(1+v)(1-2v)}$$

Substituting Eqs. (4-12.4) into the equilibrium equations, that is,

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + X = 0, \ldots, \ldots \tag{a}$$

we obtain [with Eqs. (2-15.14)]

$$\begin{aligned}
(\lambda + G)\frac{\partial e}{\partial x} + G\nabla^2 u + \left(X - \frac{E}{1-2\nu}\frac{\partial kT}{\partial x}\right) &= 0 \\
(\lambda + G)\frac{\partial e}{\partial y} + G\nabla^2 v + \left(Y - \frac{E}{1-2\nu}\frac{\partial kT}{\partial y}\right) &= 0 \\
(\lambda + G)\frac{\partial e}{\partial z} + G\nabla^2 w + \left(Z - \frac{E}{1-2\nu}\frac{\partial kT}{\partial z}\right) &= 0
\end{aligned} \tag{4-12.5}$$

Equations (4-12.5) are the displacement–temperature–equilibrium relations. They reduce to the usual displacement–equilibrium relations if T = constant.

Boundary Conditions. The boundary conditions in terms of stress components are

$$\begin{aligned}
\sigma_{Px} &= \sigma_x l + \tau_{xy} m + \tau_{xz} n \\
\sigma_{Py} &= \tau_{xy} l + \sigma_y m + \tau_{yz} n \\
\sigma_{Pz} &= \tau_{xz} l + \tau_{yz} m + \sigma_z n
\end{aligned} \tag{4-12.6}$$

Substituting Eqs. (4-12.4) into Eq. (4-12.6), we obtain

$$\begin{aligned}
\sigma_{Px} + \frac{kET}{1-2\nu} l = \lambda e l &+ G\left(\frac{\partial u}{\partial x} l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n\right) \\
&+ G\left(\frac{\partial u}{\partial x} l + \frac{\partial v}{\partial x} m + \frac{\partial w}{\partial x} n\right), \ldots, \ldots
\end{aligned} \tag{4-12.7}$$

where the ellipses denote similar equations in σ_{Py} and σ_{Pz}. Equations (4-12.7) reduce to the usual displacement boundary conditions if the terms in T are discarded. Consequently, by the above equations, the problem of thermal stress is reduced to the problem of determining displacement components (u, v, w) that satisfy the temperature–displacement relations [Eq. (4-12.5)] and the boundary conditions [Eq. (4-12.7)]. With (u, v, w) known, the strain components may be computed by the strain–displacement relations. Then by Eq. (4-12.4) the stress components may be determined. Compatibility is automatically satisfied.

Physical Interpretation of the Thermal-Stress Problem. Referring to Eq. (4-12.4), we note that the stress components consist of two parts: (1) a part related directly to the strain components in the usual manner and (2) a part proportional to the temperature at each point. The latter part may be imagined to be due to a "hydrostatic" pressure equal in magnitude to $kET/(1 - 2\nu)$.

Referring to Eqs. (4-12.5) and (4-12.7), we note that the body forces (X, Y, Z) and the surface stresses $(\sigma_{Px}, \sigma_{Py}, \sigma_{Pz})$ are modified by the terms

$$-\frac{E}{1-2\nu}\frac{\partial kT}{\partial x} \qquad -\frac{E}{1-2\nu}\frac{\partial kT}{\partial y} \qquad -\frac{E}{1-2\nu}\frac{\partial kT}{\partial z} \tag{4-12.8}$$

and

$$\frac{kET}{1-2\nu}l \qquad \frac{kET}{1-2\nu}m \qquad \frac{kET}{1-2\nu}n \tag{4-12.9}$$

respectively. Equation (4-12.9) represents a normal tension on the surface equal to $kET/(1-2\nu)$. Hence, the total stress produced in a body subjected to temperature distribution $T(x, y, z; t)$ is obtained by superimposing on the load stress the "hydrostatic stress" $-kET/(1-2\nu)$, the stress produced by the equivalent body forces [Eq. (4-12.8)], and the stress produced by the equivalent surface stresses [Eq. (4-12.9)]. When the thermal-stress problem is formulated in terms of stress components, the solution must satisfy the compatibility equations [Eqs. (4-14.2)] as well as the equations of equilibrium [Eqs. (a)] and the boundary conditions [Eqs. (4-12.6)].

Problem. Derive the most general temperature distribution $T(x, y, z)$ for which an unrestrained isotropic homogeneous elastic body may undergo stress-free thermal expansion, that is, for which the stresses $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$ are zero.

Thermomechanical Coupling. To understand the physical phenomena of thermomechanical coupling, we now take a broader view. First, we recall *the differential equations of motion of a deformable medium* [cf. Eqs. (2-18.4) and (3-8.2)]:

$$\sigma_{\beta\alpha,\beta} + B_\alpha = \rho \dot{v}_\alpha \tag*{(3-8.2)*}$$

Equation (4-11.1) is the strain–temperature relation for thermally isotropic material. We may generalize it for anisotropic material as

$$\epsilon'_{ij} = k_{ij} T \tag{4-12.10}$$

Now the total strain can be decomposed into two parts: (1) due to mechanical deformation and (2) due to temperature rise [cf. Eq. (4-11.3)]. Then the stresses can be obtained as [cf. Eq. (4-4.18)]

$$\begin{aligned} \sigma_{\beta\alpha} &= C_{\beta\alpha\gamma\delta}(\epsilon_{\gamma\delta} - k_{\gamma\delta} T) \\ &\equiv C_{\beta\alpha\gamma\delta}\epsilon_{\gamma\delta} - \kappa_{\beta\alpha} T \end{aligned} \tag{4-12.11}$$

Substitution of Eq. (4-12.11) into Eq. (3-8.2) leads to

$$\rho \ddot{u}_\alpha = C_{\beta\alpha\gamma\delta}\epsilon_{\gamma\delta,\beta} - (\kappa_{\beta\alpha} T)_{,\beta} + B_\alpha \tag{4-12.12}$$

If we adopt a small-strain assumption, Eq. (4-12.12) is reduced to

$$\rho \ddot{u}_\alpha = C_{\beta\alpha\gamma\delta} u_{\gamma,\delta\beta} - (\kappa_{\beta\alpha} T)_{,\beta} + B_\alpha \tag{4-12.13}$$

For isotropic material

$$C_{\beta\alpha\gamma\delta} \rightarrow \lambda\delta_{\alpha\beta}\,\delta_{\gamma\delta} + G(\delta_{\beta\gamma}\,\delta_{\alpha\delta} + \delta_{\beta\delta}\,\delta_{\alpha\gamma}) \tag{4-12.14}$$

$$\begin{aligned}\kappa_{\beta\alpha} &\equiv C_{\beta\alpha\gamma\delta}k_{\gamma\delta} \rightarrow \{\lambda\delta_{\alpha\beta}\,\delta_{\gamma\delta} + G(\delta_{\beta\gamma}\,\delta_{\alpha\delta} + \delta_{\beta\delta}\,\delta_{\alpha\gamma})\}k\,\delta_{\gamma\delta} \\ &\rightarrow (3\lambda + 2G)k\end{aligned} \tag{4-12.15}$$

Then Eq. (4-12.12) is reduced to [notice that $3\lambda + 2G = E/(1-2\nu)$]

$$\rho\ddot{u}_\alpha = (\lambda + G)u_{\beta,\beta\alpha} + Gu_{\beta,\alpha\alpha} - (3\lambda + 2G)(kT)_{,\alpha} + B_\alpha \tag{4-12.16}$$

which is essentially Eqs. (4-12.5). Now we recall the equation of heat conduction, Eq. (4-9.1), as

$$\begin{aligned}\rho c\dot{T} &= \rho c\kappa\nabla^2 T + Q \\ &= \alpha\nabla^2 T + Q\end{aligned} \tag{4-9.1}$$

Actually, we should generalize the equation of heat conduction to the level corresponding to thermoelasticity as represented by Eq. (4-12.11). The equation of heat conduction in thermoelasticity can be expressed as (Eringen, 1980)

$$\rho c\dot{T} = \alpha\nabla^2 T - (3\lambda + 2G)kT^o\nabla\cdot\dot{\mathbf{u}} + Q \tag{4-12.17}$$

From Eqs. (4-12.16) and (4-12.17), it is noticed that the displacement field **u** and the temperature field T are fully coupled. In fact, Eq. (4-12.16) is *the law of balance of linear momentum* for isotropic thermoelastic solid; Eq. (4-12.17) is *the law of conservation of energy* for isotropic thermoelastic solid. Also, it is seen that if $k = 0$, namely, the effect of thermal expansion is not counted, then the thermal and mechanical processes are completely decoupled. At this moment, it should be emphasized that in *continuum physics*, of which thermoelasticity is just a special case, temperature is an independent variable and has the same ranking as the displacements. This point is clearly indicated in Eqs. (4-12.16) and (4-12.17).

Temperature in Molecular Dynamics. If one wishes to understand the role of temperature not just in continuum theory but also in atomistic theory, to be precise, in molecular dynamics, then at this moment one should ask the question: Is temperature an independent variable in molecular dynamics?

The answer is no. We recall the governing equations in molecular dynamics, Eq. (3-8.7), as

$$m^i\ddot{\mathbf{r}}^i = \mathbf{f}^i + \boldsymbol{\varphi}^i \tag{3-8.7}$$

which can be rewritten as

$$m^i\ddot{\mathbf{u}}^i = \mathbf{f}^i + \boldsymbol{\varphi}^i \tag{4-12.18}$$

In this equation, $\boldsymbol{\varphi}^i$ is the body force, equivalent to **B** in eq. (4-12.18); $\mathbf{f}^i$ is the interatomic force, which is a function of the positions of all the atoms in the system.

On the other hand, the law of conservation of energy is automatically satisfied [cf. Eq. (4-3.10)]. Now the question becomes: How does the temperature come into the picture?

In molecular dynamics, temperature for a group of N atoms is calculated as

$$3Nk_B T(t;\Delta t) = \frac{1}{\Delta t}\int_{\tau=t}^{\tau+\Delta t}\sum_{i=1}^{N} m^i(\dot{\mathbf{u}}^i(\tau) - \overline{\dot{\mathbf{u}}}^i)\cdot(\dot{\mathbf{u}}^i(\tau) - \overline{\dot{\mathbf{u}}}^i)\,d\tau \tag{4-12.19}$$

where $\overline{\dot{\mathbf{u}}}^i$ is the time-interval averaged velocity of atom i defined as

$$\overline{\dot{\mathbf{u}}}^i(t;\Delta t) \equiv \frac{1}{\Delta t}\int_{\tau=t}^{\tau+\Delta t}\dot{\mathbf{u}}^i(\tau)\,d\tau \tag{4-12.20}$$

and k_B is the Boltzmann constant; m^i is the mass of atom i. It is seen that the temperature T according to this definition, Eq. (4-12.19), is the space–time average of the velocities of N atoms for a time period between t and $t + \Delta t$. It is worthwhile to note that the definition of temperature can be reduced to two special cases:

Space-averaged temperature (Haile, 1992)

$$3Nk_B T = \sum_{i=1}^{N} m^i(\dot{\mathbf{u}}^i - \hat{\dot{\mathbf{u}}})\cdot(\dot{\mathbf{u}}^i - \hat{\dot{\mathbf{u}}})$$
$$\hat{\dot{\mathbf{u}}} \equiv \frac{\sum_{i=1}^{N} m^i\dot{\mathbf{u}}^i}{M} \qquad M \equiv \sum_{i=1}^{N} m^i \tag{4-12.21}$$

Time-averaged temperature (Irvine and Kirkwood, 1950; Hardy, 1982)

$$3k_B T(t;\Delta t) = \frac{1}{\Delta t}\int_{\tau=t}^{\tau+\Delta t} m\left[\dot{\mathbf{u}}(\tau) - \overline{\dot{\mathbf{u}}})\cdot(\dot{\mathbf{u}}(\tau) - \overline{\dot{\mathbf{u}}}\right]d\tau$$
$$\overline{\dot{\mathbf{u}}}(t;\Delta t) \equiv \frac{1}{\Delta t}\int_{\tau=t}^{\tau+\Delta t}\dot{\mathbf{u}}(\tau)\,d\tau \tag{4-12.22}$$

Of course, Eq. (4-12.19) is a much more reasonable definition for temperature. On the other hand, no matter which equation is adopted as the definition for temperature, it is seen that, in molecular dynamics, temperature is not an independent variable; instead it is derivable from the velocities of atoms. One may specify temperature as boundary conditions or consider the temperature of the whole system as a given function of time. Molecular dynamics simulation, with the consideration of temperature, is to determine the trajectories of a system of atoms subjected to given constraints. In the following, we introduce several methods to implement MD simulation with given constraints due the consideration of temperature.

1. *Velocity Upgrade* (Haile, 1992)

Suppose, in the numerical procedure, at the nth time step ($t = n\,\Delta t$) for a group of N atoms, it is found that

$$\overline{\mathbf{v}} = \frac{1}{M}\sum_{i=1}^{N} m^i \mathbf{v}^i$$
$$T = \frac{1}{3k_B N}\sum_{i=1}^{N} m^i (\mathbf{v}^i - \overline{\mathbf{v}})^2 \qquad (4\text{-}12.23)$$

It is understood that this group of N atoms may or may not be the whole specimen. This freedom enables us to specify different kinds of boundary conditions. If the desired (specified) temperature is $T^* \neq T$, then we simply modify the velocities of the N atoms as

$$\mathbf{v}^{i*} = \sqrt{T^*/T}(\mathbf{v}^i - \overline{\mathbf{v}}) + \overline{\mathbf{v}} \qquad (4\text{-}12.24)$$

while keeping the positions of the atoms unchanged. It is straightforward to check:

$$\overline{\mathbf{v}}^* = \frac{1}{M}\sum_{i=1}^{N} m^i \mathbf{v}^{i*} = \frac{1}{M}\sum_{i=1}^{N} m^i \left[\sqrt{T^*/T}(\mathbf{v}^i - \overline{\mathbf{v}}) + \overline{\mathbf{v}}\right] = \overline{\mathbf{v}}$$
$$\frac{1}{3k_B N}\sum_{i=1}^{N} m^i (\mathbf{v}^{i*} - \overline{\mathbf{v}}^*)^2 = \frac{1}{3k_B N}\sum_{i=1}^{N} m^i \left[\sqrt{T^*/T}(\mathbf{v}^i - \overline{\mathbf{v}}) + \overline{\mathbf{v}} - \overline{\mathbf{v}}\right]^2 = T^*$$
$$(4\text{-}12.25)$$

This means that the temperature has been upgraded to T^* and, although the velocities have been changed, the total momentum is conserved because

$$M\overline{\mathbf{v}}^* = \sum_{i=1}^{N} m^i \mathbf{v}^{i*} = M\overline{\mathbf{v}} = \sum_{i=1}^{N} m^i \mathbf{v}^i \qquad (4\text{-}12.26)$$

Since the positions of the atoms are kept unchanged, the total potential energy is unchanged. On the other hand, the kinetic energy has been changed from

$$E_{\text{kinetic}} = \frac{1}{2}\sum_{i=1}^{N} m^i \mathbf{v}^i \cdot \mathbf{v}^i \qquad (4\text{-}12.27)$$

to

$$
\begin{aligned}
E^*_{\text{kinetic}} &= \frac{1}{2}\sum_{i=1}^{N} m^i \mathbf{v}^{i*}\cdot\mathbf{v}^{i*} \\
&= \frac{1}{2}\sum_{i=1}^{N} m^i \left[\sqrt{T^*/T}(\mathbf{v}^i - \overline{\mathbf{v}}) + \overline{\mathbf{v}}\right]^2 \\
&= \frac{T^*}{T}E_{\text{kinetic}} + \left(1 - \frac{T^*}{T}\right)\frac{1}{2}M\overline{\mathbf{v}}\cdot\overline{\mathbf{v}}
\end{aligned}
\tag{4-12.28}
$$

2. Random Number Generation

This method stipulates that, at $t = n\,\Delta t$, we randomly created a set of N vectors, namely, $\tilde{\mathbf{v}}^i$ such that

$$
\begin{aligned}
&\sum_{i=1}^{N} m^i \tilde{\mathbf{v}}^i = 0 \\
&\frac{1}{3k_B N}\sum_{i=1}^{N} m^i \tilde{\mathbf{v}}^i \cdot \tilde{\mathbf{v}}^i = T^*
\end{aligned}
\tag{4-12.29}
$$

Then let the velocities of the N atoms change to

$$
\mathbf{v}^{i*} = \tilde{\mathbf{v}}^i + \overline{\mathbf{v}} \tag{4-12.30}
$$

where $\overline{\mathbf{v}}$ is the mass-weighted average of the original velocities $\mathbf{v}^i$. It can be easily verified that

$$
\begin{aligned}
\frac{1}{M}\sum_{i=1}^{N} m^i \mathbf{v}^{i*} &= \frac{1}{M}\sum_{i=1}^{N} m^i(\tilde{\mathbf{v}}^i + \overline{\mathbf{v}}) = \overline{\mathbf{v}} \\
\frac{1}{3k_B N}\sum_{i=1}^{N} m^i(\mathbf{v}^{i*} - \overline{\mathbf{v}})^2 &= \frac{1}{3k_B N}\sum_{i=1}^{N} m^i(\tilde{\mathbf{v}}^i + \overline{\mathbf{v}} - \overline{\mathbf{v}})^2 = T^*
\end{aligned}
\tag{4-12.31}
$$

which means the new set of velocities keeps the total momentum unchanged and yields the desired temperature.

3. Nose–Hoover Thermostat

In this algorithm (Hoover, 1985), the governing equations, Eq. (3-8.7), are modified to

$$
\begin{aligned}
\dot{\mathbf{r}}^i &= \mathbf{v}^i \\
\dot{\mathbf{v}}^i &= \frac{\mathbf{f}^i + \boldsymbol{\varphi}^i}{m^i} - \chi(t)\mathbf{v}^i
\end{aligned}
\tag{4-12.32}
$$

and the so-called *friction coefficient*, χ, is a scalar-valued function controlled by the first-order differential equation

$$\dot{\chi} = \frac{1}{T^* \tau^2}[T(t) - T^*] \tag{4-12.33}$$

where τ is a specified time constant, normally in the range [0.5, 2] picoseconds.

4. Berendsen Thermostat

The Berendsen algorithm (Berendsen et al., 1984) can be better understood by rewriting the governing equations as

$$\begin{aligned}
\mathbf{v}^i\left(t + \frac{1}{2}\Delta t\right) &= \left\{\mathbf{v}^i\left(t - \frac{1}{2}\Delta t\right) + \frac{\Delta t[\mathbf{f}^i(t) + \boldsymbol{\varphi}^i(t)]}{m^i}\right\}\chi(t) \\
\mathbf{v}^i(t) &= \frac{\mathbf{v}^i(t + \frac{1}{2}\Delta t) + \mathbf{v}^i(t - \frac{1}{2}\Delta t)}{2} \\
\chi(t) &= \sqrt{1 + \frac{\Delta t}{\tau}\left(\frac{T^*}{T} - 1\right)} \\
\mathbf{r}^i(t + \Delta t) &= \mathbf{r}^i(t) + \Delta t\, \mathbf{v}^i(t + \tfrac{1}{2}\Delta t)
\end{aligned} \tag{4-12.34}$$

It is seen that if $T = T^* \rightarrow \chi = 1$, then Eqs. (4-12.34) are essentially Eq. (3-8.7) in finite difference form.

5. Gaussian Constraints

In this algorithm (Smith and Forester, 1994; Smith et al., 2008), the governing equations are the same as Eqs. (4-12.32). However, the idea behind it is quite different. Rewrite the temperature of N atoms as

$$T = \frac{1}{3k_B N}\sum_{i=1}^{N}\frac{(m^i \mathbf{v}^i)^2}{m^i} \tag{4-12.35}$$

If the temperature reaches equilibrium, that is, $T = T^*$, or we wish it is true, then

$$\frac{dT}{dt} \sim \frac{d}{dt}\left[\sum_{i=1}^{N}(m^i\mathbf{v}^i)^2\right] \sim \sum_{i=1}^{N} m^i\mathbf{v}^i \cdot (\mathbf{f}^i + \boldsymbol{\varphi}^i) = 0 \tag{4-12.36}$$

Therefore χ is chosen to be

$$\chi(t) = \frac{\sum_{i=1}^{N} m^i\mathbf{v}^i \cdot (\mathbf{f}^i + \boldsymbol{\varphi}^i)}{\sum_{i=1}^{N} (m^i\mathbf{v}^i)^2} \tag{4-12.37}$$

It is noticed that by writing the temperature as in Eq. (4-12.35) one implicitly assume the total momentum of this group of atoms is zero.

Remark: In using velocity upgrade or random number generation, one modifies the velocities once in a while without changing the governing equations. It is noticed that these governing equations are actually Newton's second law for those atoms. Modification of the velocities is essentially an action to implement the boundary condition, which reflects the interaction between the material body and its environment. On the contrary, in using Nose–Hoover thermostat, Berendsen thermostat, or Gaussian constraints, one has to modify the governing equations.

4-13 Spherically Symmetrical Stress Distribution (The Sphere)

Let a hollow sphere with inner radius a and outer radius b be subjected to a temperature T that is a function only of the radial coordinate R. Then the displacement of each point in the sphere is radial. Hence, the displacement vector is $U = U(R)$; that is, the deformation is symmetrical with respect to the center of the sphere. Consequently, the equations of equilibrium reduce to the single equation [see Eqs. (3A-26) in Chapter 3]

$$\frac{d\sigma_R}{dR} + \frac{2}{R}(\sigma_R - \sigma_T) = 0 \tag{4-13.1}$$

where the radial component of the stress vector is σ_R and the tangential components of the stress vector are equal to σ_T. The components (σ_R, σ_T) are functions of R only. The stress–strain–temperature relations [see Eqs. (4-12.4)] reduce to

$$\begin{aligned} \sigma_R &= \lambda e + 2G\epsilon_R - \frac{EkT}{1-2\nu} \\ \sigma_T &= \lambda e + 2G\epsilon_T - \frac{EkT}{1-2\nu} \end{aligned} \tag{4-13.2}$$

where

$$e = \epsilon_R + 2\epsilon_T \tag{4-13.3}$$

The strain–displacement relations are

$$\epsilon_R = \frac{dU}{dR} \qquad \epsilon_T = \frac{U}{R} \tag{4-13.4}$$

Substitution of Eqs. (4-13.4) into Eqs. (4-13.3) and (4-13.2) yields

$$\begin{aligned} \sigma_R &= (\lambda + 2G)\frac{dU}{dR} + 2\lambda\frac{U}{R} - \frac{EkT}{1-2\nu} \\ \sigma_T &= 2(\lambda + G)\frac{U}{R} + \lambda\frac{dU}{dR} - \frac{EkT}{1-2\nu} \end{aligned} \tag{4-13.5}$$

Substituting Eqs. (4-13.5) into Eqs. (4-13.1), we obtain

$$\frac{d^2U}{dR^2} + \frac{2}{R}\frac{dU}{dR} - \frac{2U}{R^2} = \frac{1+\nu}{1-\nu}\frac{d(kT)}{dR} \tag{4-13.6}$$

Rewriting Eq. (4-13.6), we obtain

$$\frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}(R^2U)\right] = \frac{1+\nu}{1-\nu}\frac{d(kT)}{dR} \tag{4-13.7}$$

Integration of Eq. (4-13.7) yields

$$U = \frac{1+\nu}{1-\nu}\frac{1}{R^2}\int_a^R \rho^2 kT\, d\rho + AR + \frac{B}{R^2} \tag{4-13.8}$$

In Eq. (4-13.8) the coefficient of thermal expansion k may vary with temperature; that is, it may vary with ρ. The constants A and B are determined by boundary conditions.

Substitution of Eq. (4-13.8) into Eqs. (4-13.5) yields the following expressions for the stress components:

$$\begin{aligned} \sigma_R &= -\frac{2E}{1-\nu}\frac{1}{R^3}\int_a^R \rho^2 kT\, d\rho + \frac{EA}{1-2\nu} - \frac{2EB}{1+\nu}\frac{1}{R^3} \\ \sigma_T &= \frac{E}{1-\nu}\frac{1}{R^3}\int_a^R \rho^2 kT d\rho + \frac{EA}{1-2\nu} + \frac{EB}{1+\nu}\frac{1}{R^3} - \frac{kTE}{1-\nu} \end{aligned} \tag{4-13.9}$$

Equations (4-13.9) are the general formulas for the stress components in a sphere subjected to temperature symmetrically distributed with respect to the center of the sphere. For the special cases of the solid sphere and the hollow sphere subjected to a temperature distribution $T = T(R; t)$, the constants of integration are determined from the following boundary conditions:

$$\begin{array}{lllll} \text{Solid sphere:} & \sigma_R = 0 \quad \text{at} \quad R = b & & U = 0 \quad \text{at} \quad R = 0 \\ \text{Hollow sphere:} & \sigma_R = 0 \quad \text{at} \quad R = a & \text{and} & R = b \end{array} \tag{4-13.10}$$

Intracranial Saccular Aneurysm. An intracranial saccular aneurysm (Fig. 4-13.1) is a localized blood-filled balloonlike dilatation of the arterial wall that often occurs in or near bifurcations in the circle of Willis (the primary network of arteries that supply blood to the brain; see Fig. 4-13.2). A subclass of intracranial saccular aneurysms can be treated reasonably well as thin-walled pressurized hollow spheres. Although the rupture potential of these aneurysms is very low, less than 0.1 to 1.0% per year. However, when they rupture, 50% of the patients die and 50% of the survivors will have severe, lasting neurological deficits. Therefore, knowing that the rupture appears to occur when wall stresses, $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$, exceed

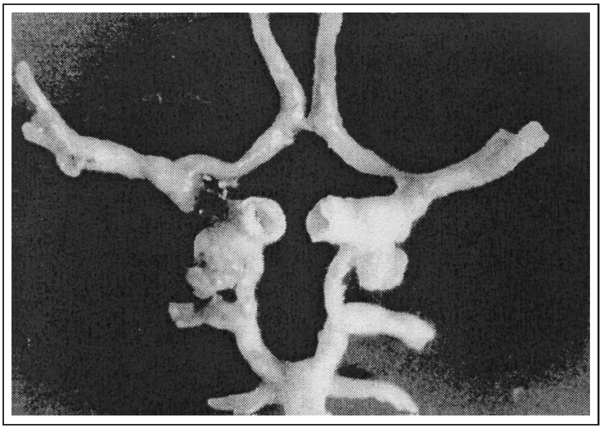

Figure 4-13.1 Schema of a subclass of intracranial saccular aneurysms (cf. Fig. 4-13.2) that can be modeled, to a first approximation, as a thin-walled pressurized sphere of radius a. Although pressure gradients are associated with the blood flow within the lesion, these gradients tend to be small in comparison to the mean blood pressure ($\sim$ 93 mm Hg); hence, we can often assume a uniform internal pressure p. Also shown is a picture of a human circle of Willis with bilateral aneurysms, one ruptured and one not—the rupture being the cause of death. This reminds us that biomechanics is not just intellectually challenging and fun, it has potential to affect the lives of individuals and families. From Humphrey and Delange (2004), with permission from Springer.

strength locally, the question is reduced to: How can we better predict the rupture potential of a given aneurysm (Humphrey and Delange, 2004)? Now consider an aneurysm is a hollow sphere with inner radius a and outer radius b subjected to an internal pressure p. Because the displacement vector of each point in the sphere is radial, one may write in spherical coordinates

$$\mathbf{u} = [u_r, u_\theta, u_\phi] = [U(r), 0, 0] \tag{4-13.11}$$

From the strain–displacement relations [Eqs. (2A-2.8)], one may obtain

$$\begin{aligned} &\epsilon_r = \frac{dU}{dr} \qquad \epsilon_\theta = \frac{U}{r} \qquad \epsilon_\phi = \frac{U}{r} \\ &\gamma_{r\theta} = 0 \qquad \gamma_{r\phi} = 0 \qquad \gamma_{\theta\phi} = 0 \end{aligned} \tag{4-13.12}$$

The general stress–strain relation for linear isotropic elastic solid can be expressed as (Sadd, 2009)

$$\begin{aligned} \sigma_{rr} &= \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi}) + 2G\epsilon_{rr} \\ \sigma_{\theta\theta} &= \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi}) + 2G\epsilon_{\theta\theta} \\ \sigma_{\phi\phi} &= \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{\phi\phi}) + 2G\epsilon_{\phi\phi} \\ \sigma_{r\theta} &= G\gamma_{r\theta} \\ \sigma_{r\phi} &= G\gamma_{r\phi} \\ \sigma_{\theta\phi} &= G\gamma_{\theta\phi} \end{aligned} \tag{4-13.13}$$

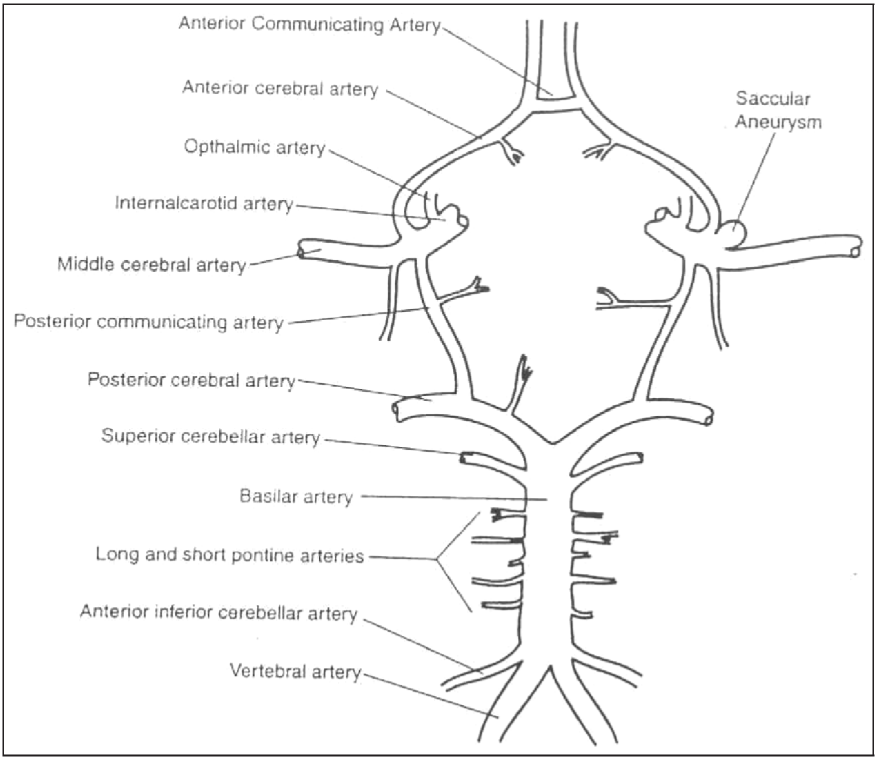

Figure 4-13.2 Schema of the circle of Willis, the primary network of arteries that supplies blood to the brain. Note the intracranial secular aneurysm, which is a focal dilatation of the arterial wall on the left middle cerebral artery (with the circle viewed from the base of the brain). Such lesions tend to be thin walled and susceptible to rupture. From Humphrey and Delange (2004), with permission from Springer.

Substituting Eqs. (4-13.12) into Eqs. (4-13.13) yields

$$\begin{aligned}\sigma_{rr} &= \lambda\left(\frac{dU}{dr} + 2\frac{U}{r}\right) + 2G\frac{dU}{dr} \\ \sigma_{\theta\theta} = \sigma_{\phi\phi} &= \lambda\left(\frac{dU}{dr} + 2\frac{U}{r}\right) + 2G\frac{U}{r} \\ \sigma_{r\theta} = \sigma_{r\phi} &= \sigma_{\theta\phi} = 0\end{aligned} \tag{4-13.14}$$

Substituting Eqs. (4-13.14) in Eqs. (3A-2.6) and assuming that there is no body force, the only nontrivial equilibrium equation is obtained as

$$\frac{d^2U}{dr^2} + \frac{2}{r}\frac{dU}{dr} - \frac{2}{r^2}U = 0 \tag{4-13.15}$$

The boundary conditions are

$$\begin{aligned}\sigma_{rr}(r=a) &= (\lambda+2G)\frac{dU(a)}{dr}+2\lambda\frac{U(a)}{a}=-p \\ \sigma_{rr}(r=b) &= (\lambda+2G)\frac{dU(b)}{dr}+2\lambda\frac{U(b)}{b}=0\end{aligned} \tag{4-13.16}$$

The solution for Eq. (4-13.15) is

$$U = Ar + Br^{-2} \tag{4-13.17}$$

The coefficients A and B are determined by the boundary conditions, Eqs. (4-13.16), as

$$\begin{aligned}(3\lambda+2G)A-4GBa^{-3} &= -p \\ (3\lambda+2G)A-4GBb^{-3} &= 0\end{aligned} \tag{4-13.18}$$

which implies

$$A=\frac{pa^3}{(3\lambda+2G)(b^3-a^3)} \qquad B=\frac{pa^3b^3}{4G(b^3-a^3)} \tag{4-13.19}$$

Then the stresses are obtained as

$$\begin{aligned}\sigma_{rr} &= \frac{pa^3}{b^3-a^3}\left(1-\frac{b^3}{r^3}\right)\le 0 \\ \sigma_{\theta\theta}=\sigma_{\phi\phi} &= \frac{pa^3}{b^3-a^3}\left(1+\frac{b^3}{2r^3}\right)>0\end{aligned} \tag{4-13.20}$$

It is seen that the normal stress in the radial direction σ_{rr} is a compressive stress; the normal stresses in the tangential directions (referred as wall stresses), $\sigma_{\theta\theta}$ and $\sigma_{\phi\phi}$, are tensile stresses; the maximum wall stress occurs at the inner radius, that is,

$$(\sigma_{\theta\theta})_{\max}=(\sigma_{\phi\phi})_{\max}=p\frac{2a^3+b^3}{b^3-a^3} \tag{4-13.21}$$

The ratio between the maximum wall stress and the inner pressure is $(2a^3+b^3)/(b^3-a^3)$, which may become a huge value if the wall becomes thinner.

Problem Set 4-13

1. Evaluate the constants A, B of Eqs. (4-13.9) for the solid sphere using the conditions of Eqs. (4-13.10). Repeat for the hollow sphere.

2. Let $kT = CR^2$, where C is a constant. For this temperature distribution, express (in terms of E, ν, C, R) the stress components σ_R, τ_T for a hollow sphere with inner radius a and outer radius b. Repeat for a solid sphere.

3. For the intracranial saccular aneurysm problem, let the mean blood pressure be 93 mm Hg and 1.5 mm $\leq a \leq$ 5 mm and the wall thickness $h \equiv b - a$ is in the order of 25 to 250 μm. Find the range of the maximum wall stress.

4-14 Thermoelastic Compatibility Equations in Terms of Components of Stress and Temperature. Beltrami–Michell Relations

Using the stress–strain relations [Eqs. (4-11.6)], we may write the strain compatibility relations [Eqs. (2-16.1) in Chapter 2] in terms of stress components. Rewriting Eqs. (4-11.6), we obtain Hooke's law in the form (with x, y, z notation)

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}[(1+\nu)\sigma_x - \nu I_1] + kT \\
\epsilon_y &= \frac{1}{E}[(1+\nu)\sigma_y - \nu I_1] + kT \\
\epsilon_z &= \frac{1}{E}[(1+\nu)\sigma_z - \nu I_1] + kT \\
\gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} \\
\gamma_{xz} &= \frac{2(1+\nu)}{E}\tau_{xz} \\
\gamma_{yz} &= \frac{2(1+\nu)}{E}\tau_{yz}
\end{aligned} \tag{4-14.1}$$

where $I_1 = \sigma_x + \sigma_y + \sigma_z$, is the first stress invariant [see Eq. (3-5.3) in Chapter 3], and where $T = T(x, y, z)$. Consider the first of Eqs. (2-16.1a) in Chapter 2:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{a}$$

Substitution of Eqs. (4-14.1) into Eq. (a) yields, for ν and E constant,

$$\begin{aligned}
&(1+\nu)\left(\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2}\right) - \nu\left(\frac{\partial^2 I_1}{\partial x^2} + \frac{\partial^2 I_1}{\partial y^2}\right) \\
&\quad = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y} - \frac{E\partial^2(KT)}{\partial x^2} - \frac{E\partial^2(KT)}{\partial y^2}
\end{aligned} \tag{b}$$

By Eq. (3-8.1) in Chapter 3, with the right-hand terms set equal to zero, we obtain

$$\frac{\partial \tau_{xy}}{\partial y} = -\frac{\partial \sigma_x}{\partial x} - \frac{\partial \tau_{xz}}{\partial z} - X$$

$$\frac{\partial \tau_{xy}}{\partial x} = -\frac{\partial \tau_{yz}}{\partial z} - \frac{\partial \sigma_y}{\partial y} - Y$$

where (X, Y) denote body-force components $(\mathscr{B}_1, \mathscr{B}_2)$, respectively. Differentiation of the first of these equations by x and of the second by y yields

$$\frac{\partial^2 \tau_{xy}}{\partial x\, \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \tau_{xz}}{\partial x\, \partial z} - \frac{\partial X}{\partial x}$$

$$\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \tau_{yz}}{\partial x\, \partial z} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial Y}{\partial y}$$

Adding these equations, we obtain

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial}{\partial z}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y}\right) - \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y}$$

With the last of Eqs. (3-8.1), with $a_3 = 0$, the above equation yields

$$2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = \frac{\partial^2 \sigma_x}{\partial z^2} - \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial Z}{\partial z} - \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y}$$

where $Z \equiv \mathscr{B}_3$. Substitution of this last equation into Eq. (b) yields, after simplification by the use of Eq. (3-5.3),

$$\nabla^2(I_1 + EkT) - \frac{\partial^2(I_1 + EkT)}{\partial z^2} - (1+\nu)\nabla^2\sigma_z = (1+\nu)\left(\frac{\partial Z}{\partial z} - \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y}\right) \tag{c}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

In a similar manner, by Eqs. (2-16.1b) and (2-16.1c) in Chapter 2, we obtain

$$\nabla^2(I_1 + EkT) - \frac{\partial^2(I_1 + EkT)}{\partial x^2} - (1+\nu)\nabla^2\sigma_x = (1+\nu)\left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right)$$

$$\nabla^2(I_1 + EkT) - \frac{\partial^2(I_1 + EkT)}{\partial y^2} - (1+\nu)\nabla^2\sigma_y = (1+\nu)\left(\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x} - \frac{\partial Z}{\partial z}\right) \tag{d}$$

Adding Eqs. (c) and (d), we get

$$(1-\nu)\nabla^2 I_1 = -(1+\nu)\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) - 2\nabla^2 EkT \tag{e}$$

Substitution of Eqs. (e) into Eqs. (c) and (d) yields

$$\begin{aligned}
\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2(I_1+EkT)}{\partial x^2} &= -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x} - \frac{\nabla^2 EkT}{1-\nu}\\
\nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2(I_1+EkT)}{\partial y^2} &= -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) - 2\frac{\partial Y}{\partial y} - \frac{\nabla^2 EkT}{1-\nu}\\
\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2(I_1+EkT)}{\partial z^2} &= -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+\frac{\partial Z}{\partial z}\right) - 2\frac{\partial Z}{\partial z} - \frac{\nabla^2 EkT}{1-\nu}
\end{aligned} \tag{4-14.2a}$$

In a similar manner, Eqs. (2-16.1d), (2-16.1e), and (2-16.1f) yield

$$\begin{aligned}
\nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2(I_1+EkT)}{\partial x\partial y} &= -\left(\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}\right)\\
\nabla^2\tau_{xz} + \frac{1}{1+\nu}\frac{\partial^2(I_1+EkT)}{\partial x\partial z} &= -\left(\frac{\partial Z}{\partial x}+\frac{\partial X}{\partial z}\right)\\
\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2(I_1+EkT)}{\partial y\partial z} &= -\left(\frac{\partial Y}{\partial z}+\frac{\partial Z}{\partial y}\right)
\end{aligned} \tag{4-14.2b}$$

If the body forces are constant throughout the body, the body-force terms on the right side of Eqs. (4-14.2) are zero. We note that although Eqs. (4-14.2) were derived utilizing the equilibrium equations, they hold for dynamical problems, provided inertial forces are included in the body-force terms.

Replacing $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$ by $(\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})$ and (X, Y, Z) by $(\mathscr{B}_1, \mathscr{B}_2, \mathscr{B}_3)$, we may write Eqs. (4-14.2) in the form

$$\nabla^2\sigma_{ij} + \frac{1}{1+v}\frac{\partial^2(I_1+EkT)}{\partial x_i\partial x_j} = -\frac{v}{1-v}\frac{\partial \mathscr{B}_\alpha}{\partial x_\alpha}\delta_{ij} - \left(\frac{\partial \mathscr{B}_i}{\partial x_j}+\frac{\partial \mathscr{B}_j}{\partial x_i}\right) - \frac{\nabla^2 EkT}{1-v}\delta_{ij} \tag{4-14.3}$$

where $i, j, \alpha = 1, 2, 3$ and $\nabla^2 = (\partial^2/\partial x_1^2) + (\partial^2/\partial x_2^2) + (\partial^2/\partial x_3^2) = \delta_{\alpha\beta} \cdot (\partial^2/\partial x_\alpha\, \partial x_\beta)$, where $\delta_{\alpha\beta}$ is the Kronecker delta.

Equations (4-14.2) represent the thermoelastic strain compatibility conditions in terms of stress components and temperature. Because Hooke's law is used in their derivation, they are restricted to linearly elastic material. Furthermore, they are restricted to isotropic homogeneous materials, as it has been assumed that E and ν are constants and that the material is isotropic. In the absence of temperature T, Eqs. (4-14.2) are known as the *Beltrami–Michell compatibility relations.*

Example 4-14.1. Thermoelastic Equations for Axially Symmetrical Stress Distribution. In cylindrical coordinates, the axially symmetric state of stress is characterized by the conditions $\tau_{r\theta} = \tau_{z\theta} = 0$, $\partial/\partial\theta = 0$. Then the general equations of equilibrium reduce to the form [Eqs. (3A-2.7) in Chapter 3]

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} &= 0 \end{aligned} \tag{E4-14.1}$$

Body forces are not included in Eqs. (E4-14.1). The strain–displacement relations are [because $v = 0$ and $\partial/\partial\theta = 0$; see Eqs. (2A-2.7) in Chapter 2]

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{u}{r} \qquad \epsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \end{aligned} \tag{E4-14.2}$$

The stress–strain–temperature relations are

$$\begin{aligned} \epsilon_r &= E^{-1}[\sigma_r - \nu(\sigma_\theta + \sigma_z)] + kT \\ \epsilon_\theta &= E^{-1}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] + kT \\ \epsilon_z &= E^{-1}[\sigma_z - \nu(\sigma_\theta + \sigma_r)] + kT \\ \gamma_{rz} &= G^{-1}\tau_{rz} \end{aligned} \tag{E4-14.3}$$

where k is the coefficient of thermal expansion and $T = T(r, z)$ is the temperature. Solving Eqs. (E4-14.3) for the stresses, we obtain

$$\begin{aligned} \sigma_r &= \lambda e + 2G\epsilon_r - \frac{EkT}{1 - 2\nu} \\ \sigma_\theta &= \lambda e + 2G\epsilon_\theta - \frac{EkT}{1 - 2\nu} \\ \sigma_z &= \lambda e + 2G\epsilon_z - \frac{EkT}{1 - 2\nu} \\ \tau_{rz} &= G\gamma_{rz} \end{aligned} \tag{E4-14.4}$$

where

$$\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)} \quad \text{and} \quad G = \frac{E}{2(1 + \nu)} \tag{E4-14.5}$$

Substitution of Eqs. (E4-14.2) into Eqs. (E4-14.4) and substitution of the results into Eqs. (E4-14.1) yield

$$\begin{aligned} \nabla^2 u - \frac{u}{r^2} + \frac{1}{1 - 2\nu}\frac{\partial e}{\partial r} - \frac{2(1 + \nu)}{1 - 2\nu}\frac{\partial (kT)}{\partial r} &= 0 \\ \nabla^2 w + \frac{1}{1 - 2\nu}\frac{\partial e}{\partial z} - \frac{2(1 + \nu)}{1 - 2\nu}\frac{\partial (kT)}{\partial z} &= 0 \end{aligned} \tag{E4-14.6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{E4-14.7}$$

A particular solution of Eqs. (E4-14.6) may be obtained through the concept of the displacement potential. Accordingly, we let

$$u = \frac{\partial \psi}{\partial r} \qquad w = \frac{\partial \psi}{\partial z} \qquad e = \nabla^2 \psi \tag{E4-14.8}$$

where $\psi = \psi(r, z)$ is the displacement-potential function.

Noting that

$$\begin{aligned} \nabla^2\left(\frac{\partial \psi}{\partial r}\right) &= \frac{\partial}{\partial r}(\nabla^2 \psi) + \frac{1}{r^2}\frac{\partial \psi}{\partial r} \\ \nabla^2\left(\frac{\partial \psi}{\partial z}\right) &= \frac{\partial}{\partial z}(\nabla^2 \psi) \end{aligned} \tag{E4-14.9}$$

by Eqs. (E4-14.6) and (E4-14.8), we obtain

$$\begin{aligned} \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\psi - (1+\nu)kT\right] &= 0 \\ \frac{\partial}{\partial z}\left[(1-\nu)\nabla^2\psi - (1+\nu)kT\right] &= 0 \end{aligned} \tag{E4-14.10}$$

Accordingly, a particular integral of Eq. (E4-14.10) is

$$\nabla^2 \psi = \frac{1+v}{1-v} kT \tag{E4-14.11}$$

For a prescribed temperature T, Eq. (E4-14.11) defines the displacement-potential function ψ.

By Eqs. (E4-14.3), (E4-14.4), and (E4-14.8), we find the stress components associated with the particular solution ψ:

$$\begin{aligned} \sigma_r' &= 2G\left(\frac{\partial^2 \psi}{\partial r^2} - \nabla^2 \psi\right) \qquad & \sigma_\theta' &= 2G\left(\frac{1}{r}\frac{\partial \psi}{\partial r} - \nabla^2 \psi\right) \\ \sigma_z' &= 2G\left(\frac{\partial^2 \psi}{\partial z^2} - \nabla^2 \psi\right) \qquad & \tau_{rz}' &= 2G\frac{\partial^2 \psi}{\partial r \partial z} \end{aligned} \tag{E4-14.12}$$

The complementary solution of Eqs. (E4-14.1) or (E4-14.6) expressed in terms of a stress function is (Timoshenko and Goodier, 1970)

$$\begin{aligned}
\sigma_r'' &= \frac{\partial}{\partial z}\left[\nu\nabla^2 F - \frac{\partial^2 F}{\partial r^2}\right] \\
\sigma_\theta'' &= \frac{\partial}{\partial z}\left[\nu\nabla^2 F - \frac{1}{r}\frac{\partial F}{\partial r}\right] \\
\sigma_z'' &= \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 F - \frac{\partial^2 F}{\partial z^2}\right] \\
\tau_{rz}'' &= \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2 F - \frac{\partial^2 F}{\partial z^2}\right]
\end{aligned} \tag{E4-14.13}$$

provided that the stress function F satisfies the relation $\nabla^2\nabla^2 F = 0$, where ∇^2 is defined by Eq. (E4-14.7). A general solution of the axially symmetric thermal-stress problem is given by the sum of Eqs. (E4-14.12) and (E4-14.13) (see Chapter 8).

Problem Set 4-14

1. Consider the equations of equilibrium in cylindrical coordinates (r, θ, z). Assume that the stress components and the displacement components u, w are independent of θ and that $v = 0$. For the linearly elastic isotropic body, perform the following:

(a) Specialize the equilibrium equations and the strain–displacement relations for this case (see appendices in Chapters 2 and 3).

(b) Assume that the stress components are defined in terms of a function ϕ as

$$\begin{aligned}
\sigma_r &= \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right] \\
\sigma_\theta &= \frac{\partial}{\partial z}\left[\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right] \\
\sigma_z &= \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \\
\tau_{rz} &= \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right]
\end{aligned}$$

where

$$\phi = \phi(r, z) \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Derive the defining equation for ϕ in the absence of body forces and temperature field.

2. The stress components $\sigma_x = \sigma_z = \tau_{yz} = \tau_{xz} = 0$. Body forces are zero. The material is not necessarily elastic. Derive the most general formulas for the stress components σ_y, τ_{xy}.

3. Determine whether or not the following stress components are a possible solution of an elasticity problem in the absence of body forces and temperature effects:

$$\begin{aligned} \sigma_x &= ayz & \tau_{xy} &= dz^2 \\ \sigma_y &= bxz & \tau_{xy} &= ey^2 \\ \sigma_z &= cxy & \tau_{yz} &= fx^2 \end{aligned}$$

where a, b, c, d, e, f are constants.

4. Derive Eqs. (4-14.2b).

5. Let an isotropic homogeneous body be subjected to nonuniform temperature distribution. Assume that the body is free to expand thermally; that is, it is not subjected to geometrical constraints. Show that the most general temperature distribution T for a "stress-free" expansion of the body is given by the relation

$$kT = ax + by + cz + d$$

where k denotes the linear coefficient of thermal expansion and a, b, c, d are arbitrary constants.

6. Consider a hollow right-circular cylinder subjected to the temperature distribution $kT = (Ar^2 + B)e^{-\beta z}$, where A, B, and β are constants and r denotes the radial coordinate of the cylinder. Consider a particular solution of the form $\psi = f(r)e^{-\beta z}$, where $f(r)$ is a function of r. Derive the explicit form of ψ. Derive the stress components associated with the particular solution.

7. A right-circular hollow cylinder of inner radius a and outer radius b is free to expand laterally, but it is constrained at its ends to prevent axial displacements. It is subjected to a steady-state heat source Q specified by the relation $Q = Az$, where A is a constant and z is the axial coordinate measured from one end of the cylinder. Discuss the temperature distribution in the cylinder, specifying required quantities where needed. Discuss the stress distribution in the cylinder. Perform appropriate analyses to aid your discussion.

8. An aluminum bar is 300 mm long, has a constant cross section of 800 mm^2, a modulus of elasticity $E = 105$ GPa, and a coefficient of thermal expansion $k = 23 \times 10^{-6}$ per °C. It is supported at each end so that its length remains constant, but it is free to expand laterally. It is subjected to a temperature increase $T = 30$°C.

(a) Determine the longitudinal stress and strain due to T.

(b) A longitudinal load $P = 500$ kN is applied at the midlength of the bar, in addition to the temperature increase T. Determine the longitudinal stress and strain in each half of the bar.

4-15 Boundary Conditions

In addition to the equilibrium equations [Eqs. (3-8.1) in Chapter 3] and the compatibility conditions [Eqs. (4-14.2)], the stress components on the surface of the body must be in equilibrium with the external forces acting on the surface (boundary). The equilibrium conditions at the boundary may be obtained from the theory of stress at a point. Equation (3-3.8) gives the stress $\boldsymbol{\sigma}_n$ on an oblique plane P (with the unit normal $\mathbf{n}$) through a point in a body. If the plane P is tangent to the surface of the body, $\boldsymbol{\sigma}_n$ is a stress on the boundary of the body. Hence, by Eqs. (3-3.10),

$$\begin{aligned}
\sigma_{n1} &= n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31} \\
\sigma_{n2} &= n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32} \\
\sigma_{n3} &= n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33}
\end{aligned} \tag{4-15.1}$$

where $\sigma_{n1}, \sigma_{n2}, \sigma_{n3}$ denote the components of the surface stress vector at a point on the boundary, and n_1, n_2, n_3 denote the direction cosines of the normal (positive outward) to the surface at this point. When Eqs. (3-3.10) pertain to a point on the boundary [Eqs. (4-15.1)], they are called *stress boundary conditions.*

The solutions of elasticity problems require that the stress components satisfy equilibrium conditions [Eqs. (3-8.1)], compatibility conditions [Eqs. (4-14.2)], and boundary conditions [Eqs. (4-15.1)]. In general, these conditions are usually sufficient to determine the stress components uniquely. However, if the body is in equilibrium, one cannot prescribe the body force (X, Y, Z) and the surface stress $(\sigma_{n1}, \sigma_{n2}, \sigma_{n3})$ in a perfectly arbitrary way. For example, if a solution of the problem is to exist, the distribution of body forces and surface forces acting on the body must be such that the resultant force and the resultant moment vanish.

Because the basic equations of classical linear elasticity may be formulated either in terms of stresses or in terms of strains (through the use of stress–strain relations), instead of prescribing stresses acting on the boundary surface, we could prescribe displacements (u, v, w). Consequently, we may formulate the following fundamental boundary value problems of elasticity:

1. *Determine the stress and the displacement in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of forces acting on the surface of the body is known.*
2. *Determine the stress and the displacement in the interior of an elastic body in equilibrium when the body forces are prescribed and the displacements on the surface of the body are known.*

In addition to problems 1 and 2, a third boundary value problem of elasticity occurs when part of the boundary is subjected to prescribed forces and the remaining part of the boundary is subjected to prescribed displacements (the *mixed* boundary value problem). Thus,

3. *Determine the stress and the displacement in the interior of an elastic body in equilibrium when the body forces are prescribed and the distribution of forces acting on S_1 and the distribution of displacements on S_2 are known, where $S_1 + S_2 = S$ denotes the bounding surface of the body.*

The general three-dimensional problem of elasticity presents formidable complications because of the difficulty of satisfying boundary conditions precisely.

The majority of the general solutions of the boundary value problems of three-dimensional elasticity amounts to proofs of the existence of solutions. However, effective general methods have been developed for the solution of two-dimensional problems of elasticity (Muskhelishvili, 1975). Furthermore, solutions are often obtained in approximate form by employing *Saint-Venant's principle*, that is, by satisfying certain boundary conditions in integral form rather than in the pointwise manner required by the theory of elasticity.

Roughly speaking, the usual engineering interpretation of Saint-Venant's principle may be summarized as follows:

Two statically equivalent force systems that act over a given small portion S on the surface of a body produce approximately the same stress and displacement at a point in the body sufficiently far removed from the region S over which the force systems act.[8]

Saint-Venant's principle as stated above is open to criticism. More complete interpretations of the principle have been given. Some aspects of the implications of Saint-Venant's principle are of considerable importance to the engineer. Accordingly, we summarize some further aspects of the principle. For more complete details, refer to the literature.[9]

Saint-Venant's Principle. A meaningful statement of Saint-Venant's principle should contain estimates of the difference between the actual stress and displacement in a body and the stress and displacement obtained under the approximate satisfaction of the boundary conditions implied by the principle. von Mises has given such statements for several problems of elasticity (von Mises, 1945; Sternberg, 1954). We summarize these results here.

von Mises (1945) stated Saint-Venant's principle more correctly as follows (p. 555):

> If the forces acting upon a body are restricted to several small parts of the surface, each included in a sphere of radius ϵ, then the strains and stresses produced in the interior of the body at a finite distance from all those parts are smaller in order of magnitude when the forces for each single part are in equilibrium than when they are not.

von Mises noted that if this statement is true, it must be a consequence of the fundamental differential equations of elasticity theory. He examined particularly the cases of (1) the infinite half-space $z \geq 0$ subjected to a system of forces $X_\alpha, Y_\alpha, Z_\alpha$ acting on the plane $z = 0$ at (x, y) coordinates $\xi_\alpha, \eta_\alpha, \alpha = 1, 2, 3, \ldots$, and (2) the circular disk. Following Boussinesq's approach, von Mises notes that for the infinite

[8]That is, two force systems that have the same resultant, but not necessarily the same distribution over S.

[9]For further discussion of the implications of Saint-Venant's principle, see von Mises (1945) and Sternberg (1954). For a discussion of Saint-Venant's principle as often employed in engineering practice, see Timoshenko and Goodier (1970). For its application in anisotropic elasticity, see Choi and Horgan (1977).

half-space under the loads indicated the mean stress σ_m is given by (Boussinesq, 1885; Love, 1944, p. 191)

$$\begin{aligned}\frac{6\pi}{\kappa - 3} r^3 \sigma_m = x \sum X_\alpha + y \sum Y_\alpha + z \sum Z_\alpha \\ + \frac{1}{r^2}\Big[(3x^2 - r^2)\sum \xi_\alpha X_\alpha + 3xy \sum \xi_\alpha Y_\alpha + 3xz \sum \xi_\alpha Z_a \\ + 3xy \sum \eta_\alpha X_\alpha + (3y^2 - r^2)\sum \eta_\alpha Y_\alpha + 3yz \sum \eta_\alpha Z_\alpha\Big] \\ + \text{higher-order terms in } \xi_\alpha, \eta_\alpha \end{aligned} \tag{4-15.2}$$

Similar expressions hold for other stress values. In Eq. (4-15.2), $r^2 = x^2 + y^2 + z^2$, and $\kappa = 1 - 2\nu$, where ν is Poisson's ratio (Section 4.7).

Accordingly, von Mises notes that if all ξ_α and η_α are of the order of magnitude ϵ (some positive number), we can conclude as follows. The stresses (and strains) at a point (x, y, z) are of the order ϵ, if the sums of force components $\sum X_\alpha$, $\sum Y_\alpha$, $\sum Z_\alpha$ are zero; they are of the order ϵ^2 if and only if the six linear moments $\sum \xi_\alpha X_\alpha$, $\sum \xi_\alpha Y_\alpha$, $\sum \xi_\alpha Z_\alpha$ and $\sum \eta_\alpha X_\alpha$, $\sum \eta_\alpha Y_\alpha$, $\sum \eta_\alpha Z_\alpha$ also vanish. The case of a system in equilibrium, that is, $\sum \xi_\alpha Z_\alpha = \sum \eta_\alpha Z_\alpha = \sum(\xi_\alpha Y_\alpha - \eta_\alpha X_\alpha) = 0$, is, in general, in no way distinguished. Only if all forces are parallel to each other, either normal to the boundary surface or inclined under an angle different from zero, do the three equilibrium conditions entail the other three conditions. In general, the order of magnitude of the inner stresses is reduced to ϵ^2 if and only if the external forces acting upon a small part of the surface are such as to remain in equilibrium when turned through an arbitrary angle (*astatic equilibrium*).

The results are illustrated in the four simple examples of Figure 4-15.1. All forces are here parallel to the x direction ($Y = Z = 0$). In the case of (Figure 4-15.1a) we have a single force that provides a finite stress value according to the first term of Eq. (4-15.2). In Figs. 4-15.1b and 4-15.1c the sum of forces is zero, but in 4-15.1b the sum of $\eta_\alpha X_\alpha$ and in 4-15.1c the sum of $\xi_\alpha X_\alpha$ are different from zero. It follows from Eq. (4-15.2) that in either case the stress has the order of magnitude ϵ. If the Saint-Venant principle is correct, the stress should be small of higher order in Fig. 4-15.1c, where all equilibrium conditions are fulfilled. In fact, in 4-15.1d, only

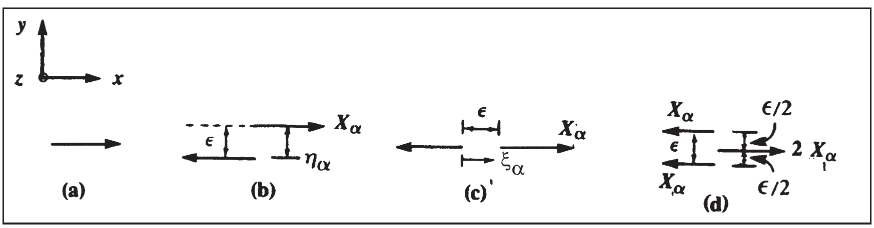

Figure 4-15.1 Finite stress (a), stress of order of magnitude ϵ (b, c), and stress of order ϵ^2 (d).

where the three forces form a system in astatic equilibrium with all linear moments zero does the stress have the order of magnitude ϵ^2.

Noting that the above results derived for an infinite half-space may be in question when considering a finite body, von Mises shows that for a circular (finite) disk, results comparable to those of the infinite half-space also apply to the finite body. In conclusion, von Mises (1945) states (p. 561):

> In order to obtain a precise and sufficiently general statement let us consider a finite simply connected body, supported at an adequate number of distinct surface points $S_1, S_2, S_3, \ldots$. Let P_1 be a point of the surface where the load F_1 is applied and P an inner point of the body at finite distances from P_1 and from $S_1, S_2, S_3, \ldots$. Let, finally, σ be some well defined strain or stress quantity in P, for instance, the normal stress in x-direction, or any component of the distortion. Then, with constant F_1, this σ will be a function of the coordinates of P_1. If P_1 is a regular surface point (tangential plane, finite curvature) the function will have finite derivatives. That means, if P_1 moves through a small distance ϵ the change in σ will be of the order of magnitude ϵ. Consequently, two equal and opposite forces attacking at points of distance ϵ will produce a σ-value of the order ϵ. On the other hand, the load F_1 can be replaced by several loads that have the vector sum F_1, all attacking at P_1. Each of them can be shifted to the neighborhood and then reversed. The system of these reversed forces combined with original F_1 will still produce a σ-value of order ϵ. Thus our first statement reads [see quote above]:
>
> (a) If a system of loads on an adequately supported body, all applied at surface points within a sphere of diameter ϵ, have the vector sum zero, they produce in an inner point P of the body a strain or stress value σ of the order of magnitude ϵ.
>
> To this statement we add the results reached in the preceding sections by way of direct computation for two particular cases, the infinite half-space and the circular disk. The general proof following the same lines can be given without difficulty.
>
> (b) If the loads, in addition to having the vector sum zero, fulfill three further conditions so as to form an equilibrium system within the sphere of diameter ϵ, the σ-value produced in P will, in general, still be of the order of magnitude ϵ.
>
> (c) If the loads, in addition to being an equilibrium system satisfy three more conditions so as to form a system in astatic equilibrium, then the σ-value produced in P will be of the order of magnitude ϵ^2 or smaller. In particular, if loads applied to a small area are parallel to each other and not tangential to the surface and if they form an equilibrium system, they are also in astatic equilibrium and thus lead to a σ of the order ϵ^2.

In this whole argument the loads as well as the supporting reactions were supposed to be concentrated, finite forces acting at distinct points of the surface. No difficulty arises if, instead, continually distributed surface stresses are assumed with the provision that all integrals of such stresses over finite regions (and the regions that tend to zero) remain finite.

A final remark is in order about the legitimate application of Saint-Venant's principle (or some equivalent statement) in cases of thin rods, shells, and so on. The only precise and consistent way to deal with thin elastic rods is the theory of the so-called one-dimensional elastica. In this theory the forces acting on the ends of the rod enter the computation only with their resultant vector and resultant moment. This implies, evidently, a principle of "statically equivalent loads." What Saint-Venant originally had in mind was doubtlessly the case of a long cylinder with infinite ratio of length to diameter. The purpose of the present discussion was to show that an extension of the principle to bodies of finite dimensions is not legitimate.

For additional studies of the Saint-Venant principle, refer to the technical literature (Horvay and Born, 1957; Keller, 1965; Knowles and Horgan, 1969; Choi and Horgan, 1977).

Problem Set 4-15

1. On the basis of the principle of astatic equilibrium, state the nature of the stress produced in the plane circular disk by the forces acting on the boundary (Fig. P4-15.1). Magnitudes of all forces are $|F|$.

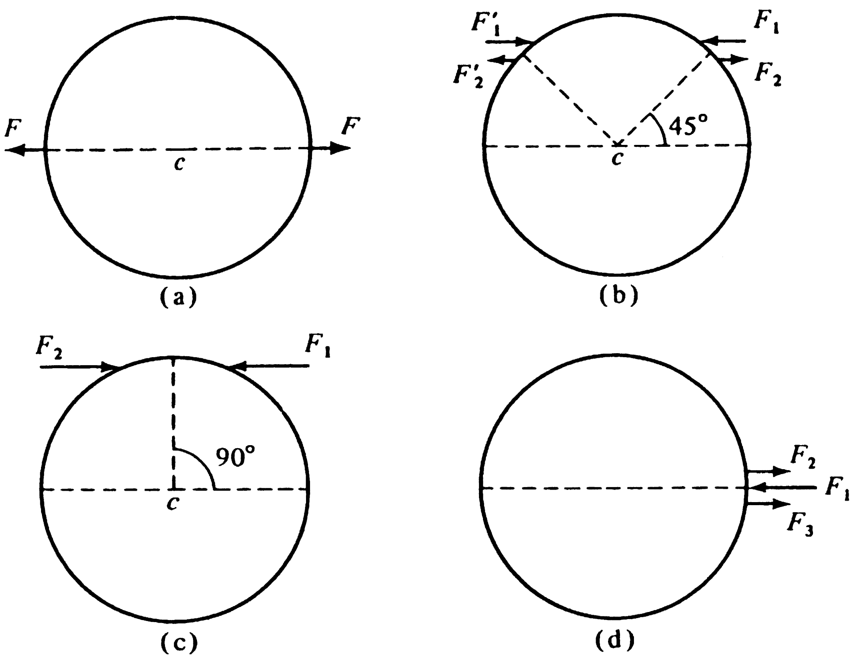

Figure P4-15.1

4-16 Uniqueness Theorem for Equilibrium Problem of Elasticity

In the following sections we seek solutions to the equilibrium problem of elasticity. However, before doing so, we prove the following theorem.[10]

Theorem 4-16.1. *If either the surface displacements or the surface stresses are given, the solution of the equilibrium problem is unique for the small-displacement theory of elasticity; that is, the state of stress (and strain) is determinable unequivocally.*

In terms of principal axes and for isotropic material, we observe that the strain energy density function U may be written in the form [Eq. (4-6.2)]

$$U = \tfrac{1}{2}\lambda(\epsilon_1 + \epsilon_2 + \epsilon_3)^2 + G\left(\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2\right)$$

where $(\epsilon_1, \epsilon_2, \epsilon_3)$ denote principal strains and (λ, G) are Lamé coefficients. First, we observe that if $\lambda > 0$, $G > 0$,

$$U \begin{cases} > 0 & \text{for nonzero } \epsilon_1, \epsilon_2, \epsilon_3 \\ = 0 & \text{if and only if } \epsilon_1 = \epsilon_2 = \epsilon_3 = 0 \end{cases} \tag{4-16.1}$$

In other words, U is positive definite (Langhaar, 1989).

Alternatively, if we assume that the strain energy density function U is positive definite, we may show that $G > 0, \lambda > 0$ for known materials. The basis for this result follows from the fact that necessary and sufficient conditions that U, given in matrix form (Chapter 1, Section 1-28) by

$$2U = [\epsilon_1, \epsilon_2, \epsilon_3]\begin{bmatrix} \lambda + 2G & \lambda & \lambda \\ \lambda & \lambda + 2G & \lambda \\ \lambda & \lambda & \lambda + 2G \end{bmatrix}\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{bmatrix}$$

be positive definite, are that the eigenvalues of the array (matrix)

$$\begin{bmatrix} \lambda + 2G & \lambda & \lambda \\ \lambda & \lambda + 2G & \lambda \\ \lambda & \lambda & \lambda + 2G \end{bmatrix}$$

be positive. The eigenvalues of the array are the three roots S_1, S_2, S_3 of the determinantal equation

$$\begin{bmatrix} (\lambda + 2G) - S & \lambda & \lambda \\ \lambda & (\lambda + 2G) - S & \lambda \\ \lambda & \lambda & (\lambda + 2G) - S \end{bmatrix} = 0$$

[10]This theorem is attributed to Kirchhoff (1858). See also Love (1944), pp. 170, 176.

Hence,

$$\begin{aligned} S_1 &= S_2 = 2G > 0 \\ S_3 &= 3\lambda + 2G > 0 \end{aligned} \tag{a}$$

We note that λ and G, the Lamé coefficients, are related to Young's modulus of elasticity E and Poisson's ratio ν by Eqs. (4-7.5). Further, for known materials, Poisson's ratio ν is never negative. In reality, for elastic materials Poisson's ratio satisfies the condition $0 < \nu < \frac{1}{2}$. Consequently, with Eqs. (4-7.5) the condition $G > 0$ implies $E > 0$; hence, for $0 < \nu < \frac{1}{2}$, $\lambda > 0$. It thus follows that for known materials, $\lambda > 0$, $G > 0$ are necessary and sufficient conditions for U to be positive definite.

More generally, for anisotropic materials the strain energy may be written in the form [see Eqs. (4-4.7)]

$$2U = C_{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} \tag{b}$$

where the elastic coefficients consist of the elements of the sixth-order symmetric matrix $(C_{\alpha\beta\gamma\delta})$. It may be shown that U is positive definite if and only if the six discriminants of $(C_{\alpha\beta\gamma\delta})$ are positive (Hildebrand, 1992). For example, for isotropic materials

$$(C_{\alpha\beta\gamma\delta}) = \begin{bmatrix} \lambda + 2G & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2G & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2G & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{bmatrix} \tag{c}$$

The six discriminants of Eq. (c) are

$$\begin{aligned} &\lambda + 2G > 0 \qquad (\lambda + 2G)^2 - \lambda^2 = 4G(\lambda + G) > 0 \\ &(\lambda + 2G)^3 + 2\lambda^3 - 3\lambda^2(\lambda + 2G) = 4G^2(3\lambda + 2G) > 0 \\ &8G^3(3\lambda + 2G) > 0 \qquad 16G^4(3\lambda + 2G) > 0 \qquad 32G(3\lambda + 2G) > 0 \end{aligned} \tag{d}$$

Equations (d) are satisfied if $G > 0$ and $3\lambda + 2G > 0$, which agree with Eqs. (a).

In terms of Young's modulus E and Poisson's ratio ν, the conditions $G > 0$, $3\lambda + 2G > 0$ are mathematically equivalent to the conditions (see Problem 4-6.3)

$$E > 0 \qquad -1 < \nu < \tfrac{1}{2} \tag{e}$$

which for known materials becomes $E > 0$, $0 < \nu < \frac{1}{2}$.

For transversely isotropic materials with an axis of symmetry z [Eqs. (4-4.15a), (4-4.15b), and (4-4.15c)], U is positive definite, provided the moduli are positive and

$$1 - \nu_{xy} > 2\nu_{xz}\nu_{zx} \tag{f}$$

Equation (f) is equivalent to the second of Eqs. (e) for the isotropic case $\nu_{xy} = \nu_{xz} = \nu_{zx} = \nu$.

Now let us assume that (u', v', w') and (u'', v'', w'') are two possible systems of *nonsingular* (x, y, z) displacement components that satisfy the equilibrium equations [Eqs. (3-8.1) in Chapter 3, with $a_1 = a_2 = a_3 = 0$] and the boundary conditions [Eqs. (4-15.1)]. We may express the equilibrium equations in terms of the function U by employing the relations given in Eqs. (4-3.2). For example, the first of Eqs. (3-8.1) may be written in (x, y, z) notation as

$$\frac{\partial}{\partial x}\frac{\partial U}{\partial \epsilon_x} + \frac{\partial}{\partial y}\frac{\partial U}{\partial \gamma_{xy}} + \frac{\partial}{\partial z}\frac{\partial U}{\partial \gamma_{xz}} + B_x = 0 \tag{4-16.2}$$

with similar expressions holding for the last two of Eqs. (3-8.1). Then, if we set

$$u = u' - u'' \qquad v = v' - v'' \qquad w = w' - w'' \tag{4-16.3}$$

(u, v, w) is a system of displacements that satisfy the equations

$$\frac{\partial}{\partial x}\frac{\partial U}{\partial \epsilon_x} + \frac{\partial}{\partial y}\frac{\partial U}{\partial \gamma_{xy}} + \frac{\partial}{\partial z}\frac{\partial U}{\partial \gamma_{xz}} = 0, \ldots, \ldots, \tag{4-16.4}$$

where the ellipses denote two similar equations. Because Eqs. (4-16.4) hold at every point in the body, we may write

$$\iiint \left[u \left(\frac{\partial}{\partial x}\frac{\partial U}{\partial \epsilon_x} + \frac{\partial}{\partial y}\frac{\partial U}{\partial \gamma_{xy}} + \frac{\partial}{\partial z}\frac{\partial U}{\partial \gamma_{xz}} \right) + v \left(\frac{\partial}{\partial x}\frac{\partial U}{\partial \gamma_{xy}} + \frac{\partial}{\partial y}\frac{\partial U}{\partial \epsilon_y} + \frac{\partial}{\partial z}\frac{\partial U}{\partial \gamma_{yz}} \right) \right.$$
$$\left. + w \left(\frac{\partial}{\partial x}\frac{\partial U}{\partial \gamma_{xz}} + \frac{\partial}{\partial y}\frac{\partial U}{\partial \gamma_{yz}} + \frac{\partial}{\partial z}\frac{\partial U}{\partial \epsilon_z} \right) \right] dx\,dy\,dz = 0 \tag{4-16.5}$$

By the divergence theorem [Eq. (1-15.3) in Chapter 1] we may transform Eq. (4-16.5) into the form [utilizing Eqs. (2-15.14) in Chapter 2]

$$\iint \left[u \left(l\frac{\partial U}{\partial \epsilon_x} + m\frac{\partial U}{\partial \gamma_{xy}} + n\frac{\partial U}{\partial \gamma_{xz}} \right) + v \left(l\frac{\partial U}{\partial \gamma_{xy}} + m\frac{\partial U}{\partial \epsilon_y} + n\frac{\partial U}{\partial \gamma_{yz}} \right) \right.$$
$$\left. + w \left(l\frac{\partial U}{\partial \gamma_{xz}} + m\frac{\partial U}{\partial \gamma_{yz}} + n\frac{\partial U}{\partial \epsilon_z} \right) \right] dS$$
$$- \iiint \left(\frac{\partial U}{\partial \epsilon_x}\epsilon_x + \frac{\partial U}{\partial \epsilon_y}\epsilon_y + \frac{\partial U}{\partial \epsilon_z}\epsilon_z + \frac{\partial U}{\partial \gamma_{yz}}\gamma_{yz} + \frac{\partial U}{\partial \gamma_{xz}}\gamma_{xz} + \frac{\partial U}{\partial \gamma_{xy}}\gamma_{xy} \right) dx\,dy\,dz = 0 \tag{4-16.6}$$

If boundary conditions are of the displacement type, $u = v = w = 0$ on S. If boundary conditions are of the stress type, the stress components calculated from (u, v, w) vanish on S [because each of sets (u', v', w') and (u'', v'', w'') yields the same stress components on S]. In either case, the double integral of Eq. (4-16.6) vanishes.

By Eqs. (4-4.4) and (4-4.6), we observe that the volume integral may be written in the form $\iiint 2U\,dx\,dy\,dz$. Hence, in order that Eq. (4-16.6) be satisfied, it follows that U must vanish. Because by Eq. (4-16.1) U is either positive or zero, in order that U be zero, $\epsilon_1 = \epsilon_2 = \epsilon_3 = 0$. Hence, it follows that (u, v, w) represents a rigid-body displacement at most, and the assumed displacement sets (u', v', w') and (u'', v'', w'') can differ by a rigid-body displacement at most. However, when the boundary conditions are of the displacement type, (u, v, w) must vanish everywhere, as they vanish at all points on S.

Accordingly, we conclude that the solution to the equilibrium problem is unique; that is, the stress and strain components are unique. In general, the displacement is unique to within an arbitrary rigid-body displacement (Chapter 2, Section 2-15).

Uniqueness has been discussed for certain cases in which singularities are allowed (Sternberg and Eubanks, 1955).

4-17 Equations of Elasticity in Terms of Displacement Components

In certain types of elasticity problems it is desirable to represent the equations of motion in terms of the displacement vector $\mathbf{q} = \mathbf{i}_1 u_1 + \mathbf{i}_2 u_2 + \mathbf{i}_3 u_3$, where $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ denote unit vectors directed along (x_1, x_2, x_3) axes, respectively.

Accordingly, substitution of Eqs. (4-6.5) into the equations of motion [Eqs. (3-8.1) in Chapter 3] yields with Eqs. (2-15.14) in Chapter 2 (with $a_1 = \ddot{u}_1$, $a_2 = \ddot{u}_2, a_3 = \ddot{u}_3$, where dots denote differentiation with respect to time):

$$
\begin{aligned}
(\lambda + G)\frac{\partial e}{\partial x_1} + G\nabla^2 u_1 + B_1 &= \rho \ddot{u}_1 \\
(\lambda + G)\frac{\partial e}{\partial x_2} + G\nabla^2 u_2 + B_2 &= \rho \ddot{u}_2 \\
(\lambda + G)\frac{\partial e}{\partial x_3} + G\nabla^2 u_3 + B_3 &= \rho \ddot{u}_3 \\
e &= u_{1,1} + u_{2,2} + u_{3,3}
\end{aligned}
\tag{4-17.1}
$$

Multiplying the first of Eqs. (4-17.1) by $\mathbf{i}_1$, the second by $\mathbf{i}_2$, the third by $\mathbf{i}_3$, and adding, we obtain in vector form

$$
(\lambda + G)\nabla\nabla \cdot \mathbf{q} + G\nabla^2 \mathbf{q} + \mathbf{B} = \rho \ddot{\mathbf{q}} \tag{4-17.2}
$$

where

$$
\nabla = \mathbf{i}_1 \frac{\partial}{\partial x_1} + \mathbf{i}_2 \frac{\partial}{\partial x_2} + \mathbf{i}_3 \frac{\partial}{\partial x_3}
$$

denotes the vector gradient operator, ∇^2 denotes the Laplace operator (see Chapter 1, Section 1-22), and

$$
\mathbf{B} = \mathbf{i}_1 B_1 + \mathbf{i}_2 B_2 + \mathbf{i}_3 B_3
$$

denotes the body-force vector per unit volume. For $\ddot{\mathbf{q}} = 0$, in the absence of body forces we obtain

$$\nabla\nabla \cdot \mathbf{q} + \frac{G}{\lambda + G}\nabla^2\mathbf{q} = 0 \tag{4-17.3}$$

Because ∇ and ∇^2 are invariants under coordinate transformations, Eq. (4-17.3) holds for general coordinate systems, the quantities ∇ and ∇^2 being expressed in the coordinate system of interest.

Boundary Conditions. The boundary conditions at a point P on the surface of a body in terms of stress components are given by Eqs. (3-3.10) in Chapter 3. Substitution of Eqs. (4-6.5) into Eqs. (3-3.10) yields with Eqs. (2-15.14)

$$\begin{aligned}
\sigma_{n1} &= \lambda e n_1 + G\left(n_1\frac{\partial u_1}{\partial x_1} + n_2\frac{\partial u_1}{\partial x_2} + n_3\frac{\partial u_1}{\partial x_3}\right)\\
&\quad + G\left(n_1\frac{\partial u_1}{\partial x_1} + n_2\frac{\partial u_2}{\partial x_1} + n_3\frac{\partial u_3}{\partial x_1}\right) = \lambda e n_1 + G\left[n_\alpha\left(\frac{\partial u_1}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_1}\right)\right]\\
\sigma_{n2} &= \lambda e n_2 + G\left(n_1\frac{\partial u_2}{\partial x_1} + n_2\frac{\partial u_2}{\partial x_2} + n_3\frac{\partial u_2}{\partial x_3}\right)\\
&\quad + G\left(n_1\frac{\partial u_1}{\partial x_2} + n_2\frac{\partial u_2}{\partial x_2} + n_3\frac{\partial u_3}{\partial x_2}\right) = \lambda e n_2 + G\left[n_\alpha\left(\frac{\partial u_2}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_2}\right)\right]\\
\sigma_{n3} &= \lambda e n_3 + G\left(n_1\frac{\partial u_3}{\partial x_1} + n_2\frac{\partial u_3}{\partial x_2} + n_3\frac{\partial u_3}{\partial x_3}\right)\\
&\quad + G\left(n_1\frac{\partial u_1}{\partial x_3} + n_2\frac{\partial u_2}{\partial x_3} + n_3\frac{\partial u_3}{\partial x_3}\right) = \lambda e n_3 + G\left[n_\alpha\left(\frac{\partial u_3}{\partial x_\alpha} + \frac{\partial u_\alpha}{\partial x_3}\right)\right]
\end{aligned} \tag{4-17.4}$$

or

$$\sigma_{n\alpha} = \lambda e n_\alpha + G\left[n_\beta\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right)\right] \qquad \alpha, \beta = 1, 2, 3$$

Multiplying the first of Eqs. (4-17.4) by $\mathbf{i}_1$, the second by $\mathbf{i}_2$, the third by $\mathbf{i}_3$, and adding, we obtain

$$\boldsymbol{\sigma}_n = \lambda\mathbf{n}\nabla \cdot \mathbf{q} + G(n_1\nabla u_1 + n_2\nabla u_2 + n_3\nabla u_3 + \mathbf{n}\cdot\nabla\mathbf{q}) \tag{4-17.5}$$

where $\mathbf{n} = \mathbf{i}_1 n_1 + \mathbf{i}_2 n_2 + \mathbf{i}_3 n_3$ is the unit normal to the surface at P.

By the above equations, the classical elasticity problem is transformed into a problem of determining the displacement vector $\mathbf{q} = \mathbf{i}_1 u_1 + \mathbf{i}_2 u_2 + \mathbf{i}_3 u_3$ that satisfies the equations of motion, Eqs. (4-17.1) or (4-17.2), and the boundary conditions, Eqs. (4-17.4) or (4-17.5). With $\mathbf{q}$ determined, we may compute the strain components by means of Eqs. (2-15.14) and then the stress components by Eqs. (4-6.5). Because we are dealing with the displacement, compatibility is automatically ensured (Chapter 2, Section 2-16).

Differentiation of Eqs. (4-17.1), the first with respect to x, the second with respect to y, and the third with respect to z, and addition of the results leads to an interesting property of the volumetric strain e. Thus, for the equilibrium problem,

$$(\lambda + 2G)\nabla^2 e + \nabla \cdot \mathbf{B} = 0 \tag{4-17.6}$$

Hence, in the absence of body force or with constant body force, Eq. (4-17.6) yields

$$\nabla^2 e = 0 \tag{4-17.7}$$

In other words, in this case the volumetric strain e satisfies Laplace's equation or is harmonic. Accordingly, the vast literature of Laplace's equation (potential theory) may be applied to seek solutions of the elasticity problem.

Problem Set 4-17

1. Consider an isotropic linearly elastic body subjected to small displacements. Note that if Poisson's ratio has the value $\nu = \frac{1}{2}$, the shear modulus $G = E/3$, the bulk modulus $k = \infty$ (see Problem 4-6.3), and the volumetric strain $e = 0$.

(a) Interpret the physical situation described by these conditions.

(b) Show that in this case, the displacement components (u, v, w) relative to (x, y, z) axes and the first stress invariant $I_1 = \sigma_x + \sigma_y + \sigma_z$ are determined by the four equations

$$G\nabla^2 u = \frac{1}{3}\frac{\partial I_1}{\partial x} + B_1 = 0$$

$$G\nabla^2 v = \frac{1}{3}\frac{\partial I_1}{\partial y} + B_2 = 0$$

$$G\nabla^2 w = \frac{1}{3}\frac{\partial I_1}{\partial z} + B_3 = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

where $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$, (B_1, B_2, B_3) denotes the body force relative to (x, y, z) axes, and G is the shear modulus.

2. The Helmholtz transformation relates the displacement vector $\mathbf{q}$ to the gradient of a scalar potential function ϕ (curl $\nabla\phi = 0$) and to the curl of a solenoidal vector potential function $\mathbf{S}$ (div $\mathbf{S} = 0$) in the form

$$\mathbf{q} = \text{grad } \phi + \text{curl } \mathbf{S}$$

(a) Show that grad ϕ ($= \nabla\phi$; see Chapter 1, Section 1-14) results in a dilatation only (Section 2-15), whereas curl $\mathbf{S}$ ($= \nabla \times \mathbf{S}$; Section 1-13) produces a rotation only (Section 2-15).

(b) Hence, show that the equations of motion (Section 4-17) may be written in the form

$$G\nabla^2[\alpha \text{ grad } \phi + \text{curl } \mathbf{S}] = -\mathbf{B}$$

where

$$\alpha = \frac{2(1-\nu)}{1-2\nu}$$

Thus, the three-dimensional elasticity problem is transformed into the problem of seeking a scalar potential function ϕ and a solenoidal vector potential function **S**.

4-18 Elementary Three-Dimensional Problems of Elasticity. Semi-Inverse Method

The solutions of an elasticity problem must satisfy not only the equations of motion [Eqs. (3-8.1) in Chapter 3] and the boundary conditions [Eqs. (4-15.1)], but also the compatibility conditions [Eqs. (2-16.1) or (2-16.2) in Chapter 2]. With respect to rectangular Cartesian coordinate axes (x, y, z), when expressed in terms of stress components, the compatibility equations contain only second derivatives of stress components and first derivatives of body forces [see Eqs. (4-14.2)]. If the body forces are constant, the compatibility equations contain only terms in second derivatives of stress components. Consequently, in a particular problem, if the equations of equilibrium and the boundary conditions are satisfied by stress components that are linear functions of (x, y, z), or constants, the compatibility equations are satisfied identically. Hence, these stress components are a solution to the elasticity problem. Furthermore, by the uniqueness theorem of elasticity (Section 4-16), it follows that this solution is the only solution to the problem.

Semi-Inverse Method. Often a solution to an elasticity problem may be obtained without seeking simultaneous solutions to the equations of equilibrium, the compatibility conditions, and the boundary conditions. For example, one may attempt to seek solutions by making certain assumptions (guesses) about the components of stress, the components of strain, or the components of displacement, while leaving sufficient freedom in these assumptions so that the equations of elasticity may be satisfied. If the assumptions allow us to satisfy the elasticity equations, then, by the uniqueness theorem, we have succeeded in obtaining the solution to the problem. This method was employed by Saint-Venant in his treatment of the torsion problem (Section 7-2). Hence, it is often referred to as the Saint-Venant semi-inverse method.

Example 4-18.1. Hydrostatic State of Stress. In the absence of body forces, let a medium be subjected to the hydrostatic state

$$\sigma_x = \sigma_y = \sigma_z = -p \qquad \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \tag{a}$$

where the constant p denotes pressure. Equations (a) automatically satisfy compatibility:

$$\begin{aligned} \epsilon_x = \epsilon_y = \epsilon_z = \frac{-p}{3\lambda + 2G} = -(1-2\nu)\frac{p}{E} \\ \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0 \end{aligned} \tag{b}$$

Equation (b) yields

$$e = \epsilon_x + \epsilon_y + \epsilon_z = -3(1-2\nu)\frac{p}{E} = -\frac{p}{k} \tag{c}$$

where e denotes the volumetric strain, and $k = E/[3(1-2\nu)] =$ bulk modulus. Substitution of Eqs. (b) into Eq. (2-15.14) in Chapter 2 yields after integration

$$\begin{aligned} u &= -\frac{p}{3\lambda + 2G}x + ay + bz + c \\ v &= -\frac{p}{3\lambda + 2G}y - ax + dz + f \\ w &= -\frac{p}{3\lambda + 2G}z - bx - dy + g \end{aligned} \tag{d}$$

where a, b, c, d, f, g are constants, which define a rigid-body displacement (see Chapter 2, Sections 2-2 and 2-15). If we specify at the point $x = y = z = 0$ that $u = v = w = \omega_x = \omega_y = \omega_z = 0$, where $(\omega_x, \omega_y, \omega_z)$ denotes the rotation vector (see Section 2-13), we obtain $a = b = c = d = f = g = 0$. Then Eqs. (d) reduce to

$$u = -\frac{p}{3\lambda + 2G}x \qquad v = -\frac{p}{3\lambda + 2G}y \qquad w = -\frac{p}{3\lambda + 2G}z \tag{e}$$

Equations (e) represent a simple dilatation (see Section 2-14).

If the medium is incompressible, $e = 0$. Then Eq. (c) yields $\nu = \frac{1}{2}$. Accordingly, for an incompressible medium, Poisson's ratio is one-half.

Example 4-18.2. Pure Bending of Prismatic Bars. In the elements of strength of materials, simplifying approximations are employed to study problems of beam bending. In this example we take the initial assumptions of elementary beam theory and consider them in the light of the theory of elasticity. Consider an initially straight bar with rectangular cross section subject at its ends to couples, of moment M, which lie in the plane $y = 0$ for (x, y, z) axes shown in Fig. E4-18.1.

Initially the (x, y, z) axes coincide with the principal axes of the beam. Taking the results of elementary beam theory, we assume (semi-inverse) the system of stress components

$$\sigma_z = -\frac{M}{I}x \qquad \sigma_x = \sigma_y = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0 \tag{a}$$

where I denotes the moment of inertia of the rectangular cross section relative to the y axis. By Eqs. (3-8.1) in Chapter 3 we note that Eqs. (a) satisfy the equations of equilibrium (in the absence of body force). Furthermore, because the stress

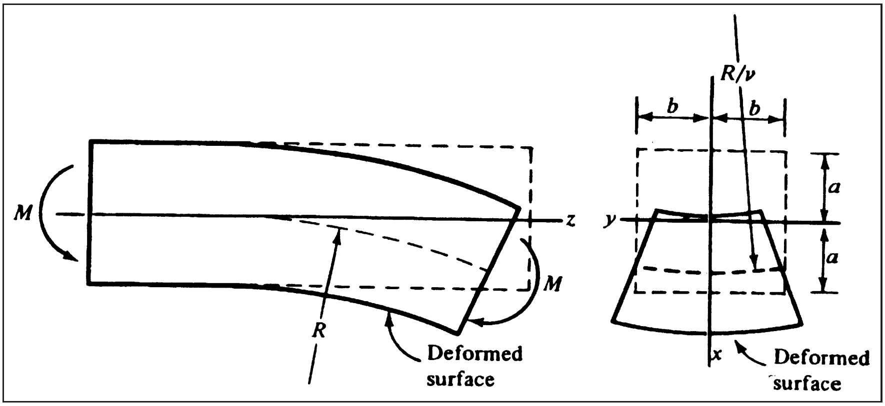

Figure E4-18.1

components are linear functions of (x, y, z), the equations of compatibility [Eqs. (4-14.2)] are automatically satisfied.

By means of Eqs. (4-6.8) and (2-15.15), and using (x, y, z) notations for stress and strain components, we obtain with Eqs. (a) (assuming that small-displacements theory holds)

$$\frac{\partial u}{\partial x} = \frac{\nu M}{EI}x \qquad \frac{\partial v}{\partial y} = \frac{\nu M}{EI}x \qquad \frac{\partial w}{\partial z} = -\frac{M}{EI}x$$
$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = 0 \qquad \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0 \tag{b}$$

where ν and E denote Poisson's ratio and Young's modulus, respectively. Because compatibility is ensured, we are guaranteed that integration of Eqs. (b) yields admissible displacement components (u, v, w) (see Chapter 2, Section 2-16). Accordingly, we proceed to integrate Eqs. (b) as follows:

By the third of Eqs. (b), we find

$$w = -\frac{M}{EI}xz + f_1(x, y) \tag{c}$$

where f_1 is an unknown function of (x, y).

With Eq. (c), the fifth and sixth of Eqs. (b) yield

$$u = \frac{M}{2EI}z^2 - z\frac{\partial f_1}{\partial x} + f_2(x, y)$$
$$v = -z\frac{\partial f_1}{\partial y} + f_3(x, y) \tag{d}$$

where $f_2(x, y)$ and $f_3(x, y)$ are unknown functions of (x, y). Substitution of Eqs. (d) into the first two of Eqs. (b) yields

$$-z\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial f_2}{\partial x} = \frac{\nu M}{EI}x$$
$$-z\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial f_3}{\partial y} = \frac{\nu M}{EI}x$$

Consequently, by Eqs. (b),

$$\frac{\partial^2 f_1}{\partial x^2} = \frac{\partial^2 f_1}{\partial y^2} = 0$$

Accordingly, integration yields

$$\begin{aligned} f_2(x, y) &= \frac{\nu M}{2EI}x^2 + g_1(y) \\ f_3(x, y) &= \frac{\nu M}{EI}xy + g_2(x) \\ f_1(x, y) &= axy + bx + cy + e \end{aligned} \tag{e}$$

where g_1, g_2 are unknown functions of y and x, respectively, and a, b, c, e are constants. Substitution of Eqs. (d) and (e) into the fourth of Eqs. (b) yields

$$\frac{dg_1}{dy} + \frac{dg_2}{dx} + \frac{\nu My}{EI} = 2az \tag{f}$$

Because the terms on the left side of Eq. (f) are independent of z, Eq. (f) requires that the constant a be zero.

Hence

$$\frac{dg_1}{dy} + \frac{\nu My}{EI} = -\frac{dg_2}{dx} \tag{g}$$

Because $g_1 = g_1(y)$ and $g_2 = g_2(x)$, Eq. (g) implies that at most

$$\frac{dg_1}{dy} + \frac{\nu My}{EI} = -\frac{dg_2}{dx} = \alpha \tag{h}$$

where α is a constant. Accordingly, integration of Eq. (h) yields

$$\begin{aligned} g_2 &= -\alpha x + \beta \\ g_1 &= -\frac{\nu My^2}{2EI} + \alpha y + \gamma \end{aligned} \tag{i}$$

where α, β, γ are arbitrary constants. Thus, with the preceding conditions, the displacement components become [Eqs. (i), (e), and (d)]:

$$u = \frac{M}{2EI}(z^2 + \nu x^2 - \nu y^2) + \alpha y - bz + \gamma$$
$$v = \frac{\nu M}{EI} xy - \alpha x - cz + \beta \qquad \text{(j)}$$
$$w = -\frac{M}{EI} xz + bx + cy + e$$

In Eqs. (j) the linear terms in (x, y, z) represent a rigid-body displacement [Eqs. (2-15.19) in Chapter 2]. The constants $\alpha, \beta, \gamma, b, c, e$ depend on the manner in which the beam is constrained. For example, by fixing the centroid of the left face of the beam $(x = y = z = 0)$, and by fixing an element of the z axis at the origin $x = y = z = 0$, and an element of area in the xz plane at the origin, we ensure that there is no rigid-body displacement of translation or rotation relative to the origin. These constraints imply that, for $x = y = z = 0$,

$$u = v = w = \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = \frac{\partial u}{\partial y} \left(\text{or } \frac{\partial v}{\partial x} \right) = 0$$

Hence, $\alpha = \beta = \gamma = b = c = e = 0$. Thus, for these constraints, the displacement (u, v, w) are reduced to

$$u = \frac{M}{2EI}(z^2 + \nu x^2 - \nu y^2)$$
$$v = \frac{M}{EI} \nu xy \qquad \text{(k)}$$
$$w = -\frac{M}{EI} xz$$

Inspection of Eqs. (k) shows that the beam fibers lying in the plane $x = 0$ undergo no displacement in the z direction; that is, these fibers do not elongate (or contract). Consequently, the plane $x = 0$ is called the *neutral plane* of the beam. The fiber that coincided with the z axis before deformation is called the *neutral axis* of the beam.

The longitudinal fibers for which $x > 0$ contract, and the fibers for which $x < 0$ elongate. Because the point (x, y, z) in the beam passes to the point $\xi = x + u$, $\eta = y + v$, $\zeta = z + w$, points that were originally on the neutral axis pass into the points

$$\zeta = z \qquad \xi = \frac{M}{2EI} z^2 = \frac{M}{2EI} \zeta^2 \qquad \eta = 0 \qquad \text{(l)}$$

The plane $y = 0$ contains the deformed neutral axis of the beam. Thus, the deformed neutral axis of the beam lies in the plane of the couples applied at the ends.

Because we have assumed small-displacement theory, the radius of curvature R of the deformed neutral axis is approximated as

$$\frac{1}{R} = \frac{d^2\xi/d\zeta^2}{[1 + (d\xi/d\zeta)^2]^{3/2}} \approx \frac{d^2\xi}{d\zeta^2}$$

Hence, by Eqs. (l),

$$\frac{1}{R} = \frac{M}{EI} \tag{m}$$

Equation (m) is the Bernoulli–Euler equation of elementary beam theory. It states that in pure bending of prismatic bars, the bending moment M is proportional to the curvature $1/R$ of the neutral axis. The Bernoulli–Euler equation forms the basis for elementary beam theory.

Deformed Shape of Cross Section. Consider the cross section of the beam at some arbitrary point $z = z_1$. After deformation, points in this section lie in the cross section $\zeta = z_1 + w$. By Eq. (k), $w = -Mxz_1/EI$. Hence,

$$\zeta = z_1\left(1 - \frac{Mx}{EI}\right) = z_1\left(1 - \frac{x}{R}\right) \tag{n}$$

Equation (n) is the equation of the plane normal to the deformed neutral axis at $z = z_1$. Hence, the assumption made in the elements of strength of material is valid; that is, planes originally perpendicular to the neutral axis remain plane and normal to the deformed neutral axis.

To examine the deformation of the cross section in its plane (Fig. E4-18.1), we note that the sides $y = \pm b$ deform into the lines

$$\eta = y + v = \pm b \pm \frac{\nu Mxb}{EI} = \pm b\left[1 + \frac{\nu Mx}{EI}\right] = \pm b\left[1 + \left(\frac{\nu x}{R}\right)\right]$$

Thus, the vertical lines become inclined. Similarly, the lines $x = \pm a$ at section $z = z_1$ deform into the lines $\xi = \pm a + [z_1^2 + \nu(a^2 - y^2)]/2R$. Hence, for small values of $1/R$, the x lines deform into a parabola with curvature of magnitude $|d^2\xi/d\eta^2| \approx |d^2\xi/dy^2| = \nu/R$. The center of curvature of the x lines of the cross section $z = z_1$ lie on the opposite side of the neutral plane from the center of curvature of the neutral axis (Fig. E4-18.1). Consequently, the deformed neutral surface of the beam is an *anticlastic surface.*

The general case of pure bending in which couples act in planes that do not coincide with a principal plane may be treated by considering individual components of the couples with moments directed along principal axes. The foregoing analysis is then applicable to each component. The solution may then be obtained by superposition.

Problem Set 4-18

1. The following stress array is proposed as a solution to a certain *equilibrium* problem of a plane body bounded in the region $-L/2 \leq x \leq L/2$, $-h/2 \leq y \leq h/2$:

$$\sigma_x = Ay + Bx^2y + Cy^3 \qquad \sigma_y = Dy^3 + Ey + F$$
$$\tau_{xy} = (G + Hy^2)x \qquad \sigma_x = \tau_{xz} = \tau_{yz} = 0$$

where (x, y, z) are rectangular Cartesian coordinates and $A, B, \ldots, H$ are nonzero constants. Determine the conditions under which this array is a possible equilibrium solution. It is proposed that the region be loaded such that $\tau_{xy} = 0$ for $y = \pm h/2$, $\sigma_y = 0$ for $y = h/2$, $\sigma_y = -\sigma\,(\sigma = \text{constant})$ for $y = -h/2$, and $\sigma_x = 0$ for $x = \pm L/2$. Determine whether the proposed stress array may satisfy these conditions.

2. In bending of a straight bar with rectangular cross section, the stress state is given by

$$\sigma_x = C_1 y + C_2 xy \qquad \tau_{xy} = C_3(c^2 - y^2)$$
$$\sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

where (x, y, z) are right-hand rectangular Cartesian axes, with axis x coincident with the axis of the bar and bending occurring in the x, y plane.

(a) Determine under what conditions these stresses satisfy the equations of equilibrium. Neglect body forces.

(b) In a sketch, show what boundary stresses must exist on the lateral surfaces and on the end faces of a cantilever beam of length L and depth $2c$. Take the origin of (x, y, z) axes at the free end of the beam, with y axis in the direction of the depth dimension.

3. A beam of rectangular cross section is composed of a material whose stress–strain relation for one-dimensional loading is $\sigma_x = k\epsilon_x^{1/3}$, where k is a known constant (Fig. P4-18.3). The beam is bent by couples into a circular arc of radius ρ at the middle surface. The middle surface does not strain in the x direction and all stresses are zero except σ_x. Derive an expression for the strain energy per unit length of the beam in terms of a, b, k, and ρ.

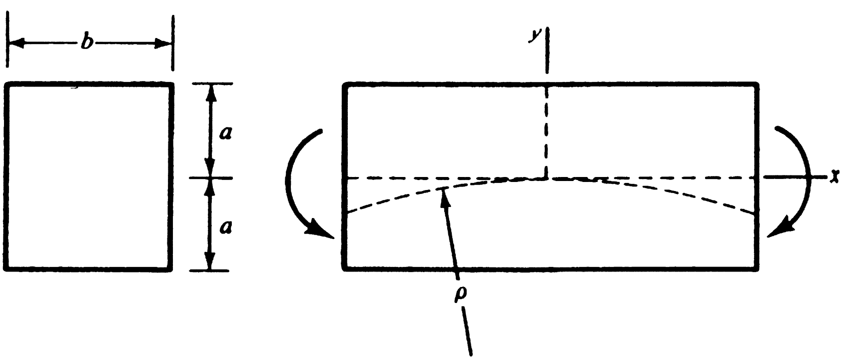

Figure P4-18.3

4. Show that the stress components

$$\sigma_y = \frac{E}{R} z + K \qquad \sigma_x = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$$

where E, R, and K are constants, satisfy the equilibrium equations and the boundary conditions for pure bending in the y, z plane of a prismatic bar. Derive expressions for (x, y, z) displacement components (u, v, w) relative to one end of the bar. Are the compatibility conditions satisfied?

5. A semi-infinite space is subjected to a uniformly distributed pressure over its entire bounding plane (Fig. P4-18.5). Consider an infinitesimal volume element *ABCD* at some distance from the bounding plane. The normal stress on surface *AB* is $\sigma_y = \sigma$. In terms of the appropriate material properties and σ, derive expressions for the normal stress components σ_x, σ_z that act on the volume element (axis z is perpendicular to the x, y plane). (*Hint*: What are the values of the strain components ϵ_x, ϵ_z?)

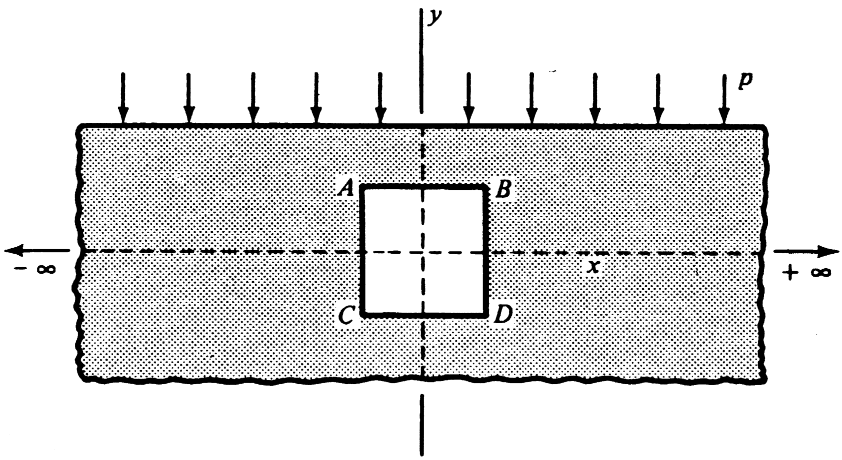

Figure P4-18.5

6. For the cantilever beam, it is assumed that the elementary beam formulas hold; that is [see Fig. P4-18.6],

$$\sigma_x = \frac{12Pxy}{a^3 b} \qquad \tau_{xy} = \frac{3P(a^2 - 4y^2)}{2a^3 b}$$

$$\sigma_y = \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

The material is elastic and deflections are small. Determine whether all the requirements of three-dimensional elasticity theory are satisfied by this solution. If not, what is violated?

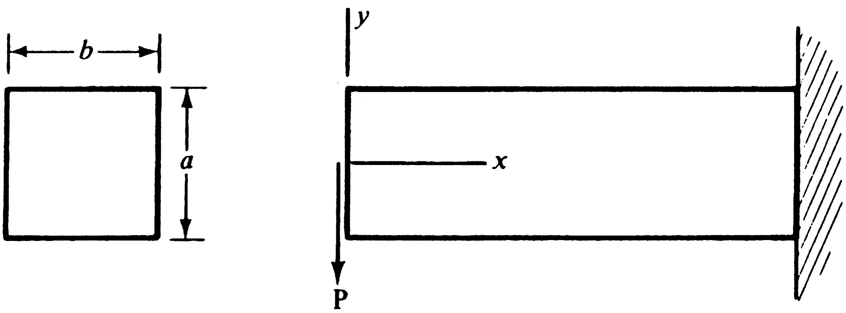

Figure P4-18.6

7. By the small-displacement theory of pure bending of a three-dimensional beam, we find that the (x, y, z) displacement components (u, v, w) are

$$
\begin{aligned}
u &= \frac{1}{R}xy + Dy + Ez + F \\
v &= -\frac{1}{2R}[x^2 + \nu(y^2 - z^2)] - Dx + Az + C \\
w &= -\frac{1}{R}\nu yz - Ay - Ex + B
\end{aligned} \tag{a}
$$

where ν is Poisson's ratio, R is the radius of curvature of the beam centerline in the deformed position, and A, B, C, D, E, F are constants. Initially, the undeformed beam occupies the region $0 \le x \le L(L = \text{beam length})$, $-c \le y \le c(2c = \text{beam depth})$, $-b \le z \le b(2b = \text{beam width})$.

(a) Derive expressions for the (x, y, z) strain components of the strain tensor ϵ_{ij} and for the rotation tensor (vector) ω as functions of (x, y, z).

(b) Impose the condition that $u = v = w = 0$ at $x = y = z = 0$, and establish the required restrictions on Eqs. (a).

(c) Impose the condition that each component of $\boldsymbol{\omega}$ is zero at $x = y = z = 0$ and compute the components of $\boldsymbol{\omega}$ at $x = L, y = z = 0$.

(d) By means of Eqs. (a), compute the slope of the centerline at $x = L, y = z = 0$, and show that the result corresponds to one component of $\boldsymbol{\omega}$.

8. Given the following stress state:

$$
\begin{aligned}
\sigma_x &= C[y^2 + \nu(x^2 - y^2)] \qquad & \tau_{xy} &= -2C\nu xy \\
\sigma_y &= C[x^2 + \nu(y^2 - x^2)] \qquad & \tau_{xy} &= \tau_{xz} = 0 \\
\sigma_z &= C\nu(x^2 + y^2)
\end{aligned}
$$

discuss possible reasons for which this stress state may not be a solution of a problem in elasticity.

9. For the case of spherical symmetry in which stress, strain, and displacement are functions of only the radial coordinate r of spherical coordinates (r, θ, ϕ), reduce the general three-dimensional equilibrium equations to a single differential equation in terms of stress components [Section 3A-2, Eq. (3A-2.6)]. Express this equation in terms of displacement and solve the resulting differential equation to obtain the Lamé solution for the spherical container.

10. When one represents the equation of motion for an elastic body in terms of displacement, one obtains the Lamé equation:

$$(\lambda + G)\nabla(\nabla \cdot \mathbf{q}) + G\nabla^2\mathbf{q} + \mathbf{B} = \frac{\partial^2 \mathbf{q}}{\partial t^2}$$

where (λ, G) are the Lamé constants, t is time, $\nabla = \mathbf{i}(\partial/\partial x) + \mathbf{j}(\partial/\partial y) + \mathbf{k}(\partial/\partial z)$, $\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2)$, (x, y, z) are rectangular Cartesian coordinates, $\mathbf{B}$ is the body-force vector, and $\mathbf{q}$ is the displacement vector. Consider the unbounded elastic medium (infinite region):

(a) Applying the inverse method, let the displacement field be $u = u(x, t)$, $v = w = 0$, and let $\mathbf{B} = 0$. For this case, determine the mathematical problem to which the elasticity problem reduces.

(b) Repeat for the case $u = v = 0$, $w = w(x, t)$.

(c) Show that $u = f(x - ct) + g(x + ct)$, and $w = f(x - ct) + g(x + ct)$, respectively, are solutions of the mathematical problems of parts (a) and (b), where c is a constant. Determine c in each case.

(d) Consider the special solution $u = u_0 \sin 2\pi[(x/\lambda) - (t/T)]$, where u_0, λ, T are parameters (called the amplitude, wavelength, and period of vibration, respectively). The ratio $\lambda/T = V$ is velocity of wave propagation. Derive a formula for the ratio $\lambda/T = V$, and show that for a particular medium $V =$ constant for the assumed motion. Compare V to c.

11. The stress–strain relations for a general anisotropic body may be written either in the form $\sigma_\alpha = C_{\alpha\beta}\epsilon_\beta, \sigma, \beta = 1, 2, \ldots, 6$ or in the form $\epsilon_\alpha = S_{\alpha\beta}\sigma_\beta = 1, 2, \ldots, 6$, where the stiffness $C_{\alpha\beta} = C_{\beta\alpha}$ are related to the compliances $S_{\alpha\beta} = S_{\beta\alpha}$ by the relations

$$C_{\alpha\beta} = \frac{\text{cofactor of } S_{\alpha\beta}}{\det S_{\alpha\beta}}$$

where $\sigma_1 = \sigma_x, \sigma_2 = \sigma_y, \sigma_3 = \sigma_z, \ldots, \sigma_6 = \tau_{yz}$ are the stress components relative to rectangular Cartesian axes $x_1 = x, x_2 = y, x_3 = z$, and similarly for strain components $\epsilon_1 = \epsilon_x, \epsilon_2 = \epsilon_y, \epsilon_3 = \epsilon_z, \ldots, \epsilon_6 = \gamma_{yz}$.

Consider the simply supported anisotropic beam (Fig. P4-18.11) bent by couples M in the (x_2, x_3) plane. Assume that the stresses are given by elementary beam theory; that is,

$$\sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0 \qquad \sigma_3 = \frac{Mx_2}{I}$$

where I is the moment of inertia of the cross section relative to axis x_1 (perpendicular to plane x_2, x_3).

Derive formulas for the displacement components (u_1, u_2, u_3) relative to (x_1, x_2, x_3) axes, respectively, evaluating arbitrary constants from end conditions at $x_3 = 0$ and $x_3 = L$.

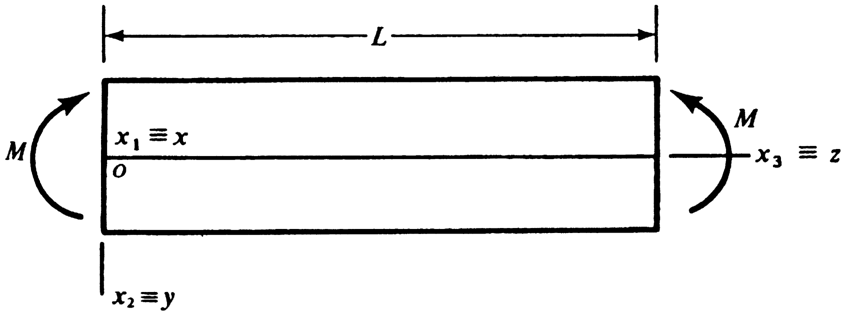

Figure P4-18.11

4-19 Torsion of Shaft with Constant Circular Cross Section

Consider a solid cylinder with constant circular cross section A and with length L. Let the cylinder be subjected to axial twisting couples **M** applied at its ends; the vector that represents the couple is directed along the z axis, the axis of the shaft (see Fig. 4-19.1).

Under the action of **M**, an originally straight generator of the cylinder will deform into a helical curve. However, because of the axial symmetry of the cross section, it is reasonable to assume that plane cross sections of the cylinder normal to the z axis remain plane after the deformation. Furthermore, for small displacements a radius of a given section remains essentially straight and inextensible. In other words, the couple **M** causes each section to rotate approximately as a rigid body about the axis of the couple, that is, the *axis of twist, z*. Furthermore, if we measure the rotation θ of each section relative to the plane $z = 0$, the rotation θ of a given section will depend on its distance from the plane $z = 0$. For small deformations,

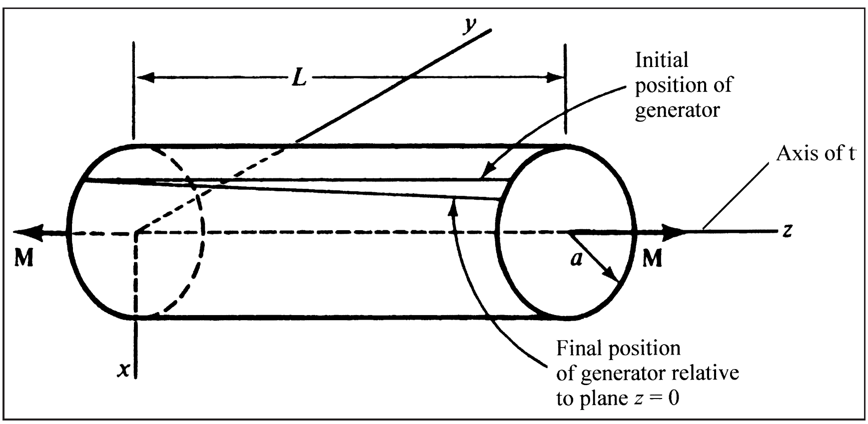

Figure 4-19.1

a reasonable assumption is that the amount of rotation of a given section depends linearly on its distance z from the plane $z = 0$. Thus, the rotation θ of a section relative to the plane $z = 0$ is

$$\theta = \beta z \tag{4-19.1}$$

where β is the twist per unit length of the shaft. Under the assumption that plane sections remain plane and that Eq. (4-19.1) holds, we now seek to satisfy the equations of elasticity; that is, we employ the semi-inverse method of seeking the elasticity solution.

Because plane sections are assumed to remain plane, the displacement component w, parallel to the z axis, is taken to be zero. To calculate the (x, y) components u and v, consider a cross-section distance z from the plane $z = 0$. Consider a point in the circular cross section (Fig. 4-19.2) with radial distance OP.

Under the deformation, radius OP rotates into the radius OP^*. In terms of the angular displacement θ of the radius, the displacement components (u, v) are

$$\begin{aligned} u &= x^* - x = OP\,[\cos(\theta + \phi) - \cos\phi] \\ v &= y^* - y = OP\,[\sin(\theta + \phi) - \sin\phi] \end{aligned} \tag{4-19.2}$$

Expanding $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$ and noting that $x = OP\cos\phi$, $y = OP\sin\phi$, we may write Eq. (4-19.2) in the form

$$\begin{aligned} u &= x(\cos\theta - 1) - y\sin\theta \\ v &= x\sin\theta + y(\cos\theta - 1) \end{aligned} \tag{4-19.3}$$

Restricting the deformation to be small, we obtain (as then $\sin\theta \approx \theta$, $\cos\theta \approx 1$)

$$u = -y\theta \qquad v = x\theta \tag{4-19.4}$$

to first-degree terms in θ.

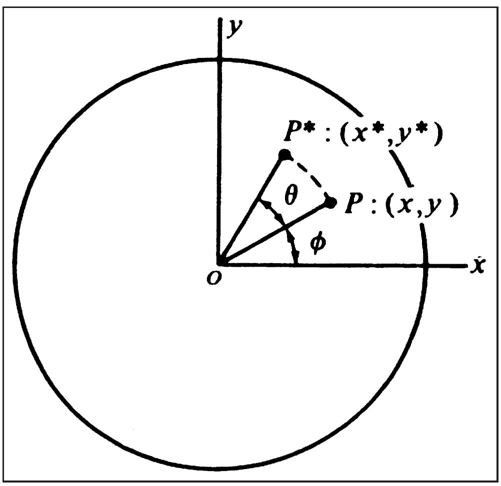

Figure 4-19.2

Substitution of Eq. (4-19.1) into Eqs. (4-19.4) yields

$$u = -\beta yz \qquad v = \beta xz \qquad w = 0 \tag{4-19.5}$$

On the basis of the foregoing assumptions, Eqs. (4-19.5) represent the displacement components of a circular shaft subjected to a twisting couple M.

Substitution of Eqs. (4-19.5) into Eqs. (2-15.14) in Chapter 2 yields the strain components

$$\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{yz} = 0 \qquad \gamma_{xz} = -\beta y \qquad \gamma_{yz} = \beta x \tag{4-19.6}$$

With Eqs. (4-19.6), Eqs. (4-6.5) yield the stress components

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \qquad \tau_{xz} = -\beta G y \qquad \tau_{yz} = \beta G x \tag{4-19.7}$$

Because Eqs. (4-19.7) are linear in (x, y), they automatically satisfy compatibility Eqs. (4-14.2) in the absence of body forces and temperature. Furthermore, they satisfy equilibrium provided the body forces are zero [Eqs. (3-8.4) in Chapter 3].

To satisfy the boundary conditions, Eqs. (4-19.7) must yield no forces on the lateral boundary; on the ends they must yield stresses such that the net moment is equal to M and the resultant force vanishes. Because the direction cosines of the unit normal to the lateral surface are $(l, m, 0)$, the first two of Eqs. (4-15.1) are satisfied identically. The last of Eqs. (4-15.1) yields

$$l\tau_{xz} + m\tau_{yz} = 0 \tag{4-19.8}$$

By Fig. 4-19.3,

$$l = \cos\phi = \frac{x}{a} \qquad m = \sin\phi = \frac{y}{a} \tag{4-19.9}$$

Substitution of Eqs. (4-19.7) and (4-19.9) into Eqs. (4-19.8) yields

$$-\frac{xy}{a} + \frac{xy}{a} = 0$$

Accordingly, the boundary conditions on the lateral boundary are satisfied.

On the ends the stresses must be distributed so that the net moment is M.

Because all stress components except τ_{yz}, τ_{xz} vanish, summation of forces on the end planes yield $F_x = F_y = F_z = 0$. Also, summation of moments with respect to the z axis yields (Fig. 4-19.4)

$$\sum M_z = M = \int_A (x\tau_{yz} - y\tau_{xz})\, dA \tag{4-19.10}$$

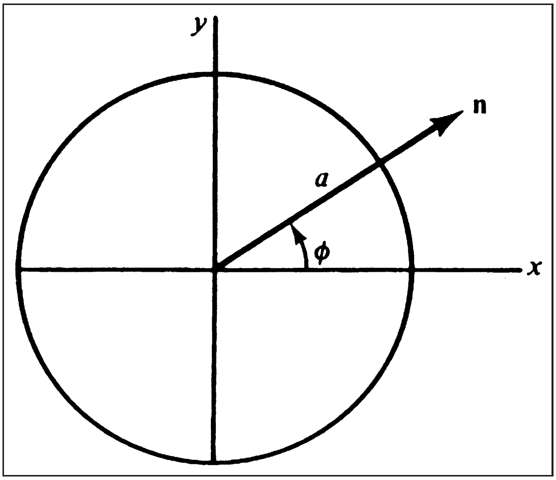

Figure 4-19.3

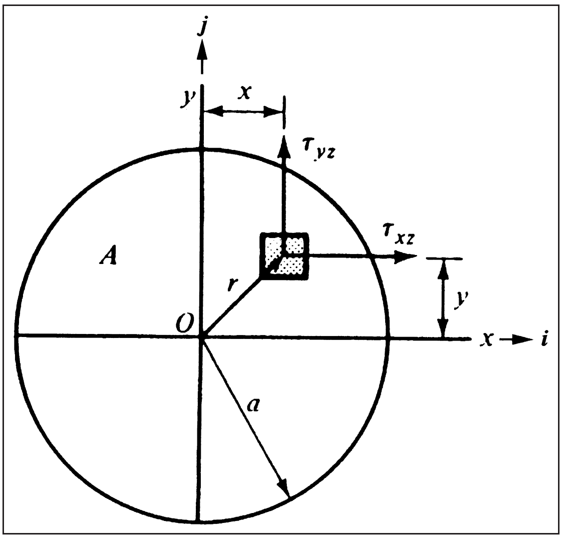

Figure 4-19.4

Substitution of Eqs. (4-19.7) into Eq. (4-19.10) yields

$$M = G\beta \int_A \left(x^2 + y^2\right) dA = G\beta \int_A r^2 dA \tag{4-19.11}$$

Integration over the circular area yields

$$M = G\beta I_0 \tag{4-19.12}$$

where

$$I_0 = \frac{\pi}{2} a^4 \tag{4-19.12a}$$

is the polar moment of inertia of the circular cross section. Equation (4-19.12) relates the angular twist β per unit length of shaft to the applied moment M.

Because compatibility and equilibrium are satisfied, Eqs. (4-19.7) represent the solution of the elasticity problem, provided the stress components τ_{xz}, τ_{yz} are distributed over the end planes according to Eqs. (4-19.7). Because τ_{xy}, τ_{yz} are independent of z, the stress distribution is the same for all cross sections. Thus, the stress vector $\boldsymbol{\sigma}$ for any point P in a cross section is given by the relation

$$\boldsymbol{\sigma} = -\mathbf{i}\beta G y + \mathbf{j}\beta G x \tag{4-19.13}$$

It lies in the plane of the section and is perpendicular to the radius vector $\mathbf{r}$ joining point P to the origin O.

By Eq. (4-19.13), the magnitude of $\boldsymbol{\sigma}$ is

$$\sigma = \beta G \sqrt{x^2 + y^2} = \beta G r \tag{4-19.14}$$

Hence, σ is a maximum for $r = a$; that is, σ attains a maximum value of $\beta G a$ on the lateral boundary of the shaft.

Problem Set 4-19

1. When we twist a slender prismatic bar with noncircular cross section and with generators parallel to the z axis by means of couples of magnitude M applied to the ends of the bar, we find the only nonzero stress components are τ_{xz} and τ_{yz}.

(a) Write the governing differential equations of equilibrium for the bar and the boundary conditions for the lateral stress-free surfaces of the bar.

(b) Determine the principal stresses and the corresponding principal stress directions in terms of τ_{xz}, τ_{yz}, and τ, where $\tau^2 = \tau_{xz}^2 + \tau_{yz}^2$.

2. Consider a shaft with circular cross section. Let the shaft be subjected to a system of forces such that every cross section of the shaft rotates as a rigid body through an angle $\theta = \beta z$, where β is a constant and z is the axial distance measured from one end of the shaft. Also, the axial displacement of the shaft is zero.

For large rotation, express the displacement components (u, v) of a point in the direction of (x, y) axes in the plane of the cross section in terms of coordinates (x, y, z). (Axes x, y, z form a right-handed coordinate system.) Hence, derive expressions for the strain components $(\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz})$ and the stress components $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz})$. Is the lateral surface of the shaft free of boundary stress?

3. Consider a particle initially at the point (x, y, z) in a cylindrical shaft (Fig. P4-19.3). When the bar is subjected to a couple M directed along the z axis, the radius to point (x, y) rotates about the z axis through an angle βz, where β is the angle of twist per unit length of the bar. Assuming that the displacement is small, in terms of $\beta, x, y,$

and z, derive the displacement components $u = x^* - x$, $v = y^* - y$. Assume that $w = \beta f(x, y)$, where w is the z component of displacement of the particle and $f(x, y)$ is a function of x and y. Derive formulas for the six strain components $\epsilon_x, \ldots, \gamma_{yz}$ in terms of β, f, x, and y. Derive the corresponding expressions for the six stress components in terms of f, β, x, y, and G (the shear modulus). What is the equilibrium equation in terms of f? Neglect body forces.

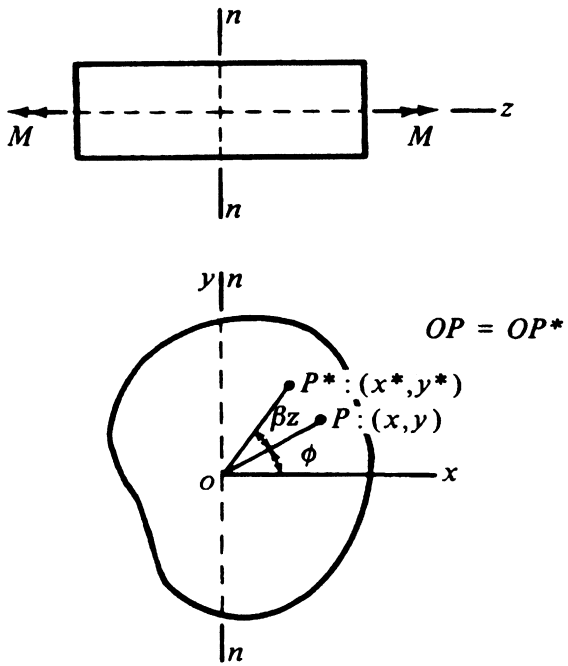

Figure P4-19.3

4. The stress array for the torsion problem of a circular cross section bar of radius a and with longitudinal axis coincident with the z axis of rectangular Cartesian axes (x, y, z) is

$$\begin{pmatrix} 0 & 0 & -Gy\beta \\ 0 & 0 & Gx\beta \\ -Gy\beta & Gx\beta & 0 \end{pmatrix}$$

where G, β are constants.

Compute the principal stresses at a point on the lateral surface of the bar. Determine the principal stress axes for a point on the lateral surface of the bar.

4-20 Energy Principles in Elasticity

As noted in Section 4-2, the linear elastic problem of equilibrium of a deformable solid body requires in general the solution of 6 first-order linear partial differential equations of motion, 6 compatibility relations in the form of second-order linear partial differential equations, and 6 stress–strain relations, with stress components and/or displacement components subject to appropriate boundary conditions. If stress couples and body couples are absent, the 6 differential equations of motion are reduced to 3 under the supplemental conditions that $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$ (Chapter 3, Section 3-3). Under certain restrictive assumptions, the 15 equations that then define the linear elasticity problem may be further reduced (such as plane problems, Chapters 5 and 6, or problems of plates and shells, etc.). However, the direct approach of seeking exact solutions to this problem (i.e., solving directly for the displacement vector or the stress tensor; see Sections 4-18 and 4-19) is usually extremely difficult. Consequently, solutions are often sought by alternative methods. Frequently, these alternative methods are based on energy concepts in conjunction with minimum principles of the calculus of variation (Chapter 1, Section 1-29).

Because energy is a scalar quantity, energy methods are sometimes referred to as *scalar methods.* However, we refer to the coupling of energy concepts with minimum principles briefly as *energy principles.*

Because the equations of elasticity may be solved alternatively by energy principles, we might anticipate that the equations of elasticity may be in part derived from energy concepts (see Sections 4-3 and 4-4). Energy principles of various kinds may be employed, depending on the nature of the variations (Chapter 1, Section 1-29). For example, when displacements are varied, we are led to the principle of virtual work (virtual displacement), Section 4-21; when stresses are varied, we obtain principles of virtual stress (Castigliano's theorem), Section 4-22; and when both stress and strain are varied, we obtain mixed minimum principles (Reissner's theorem), Section 4-23. Conceptually, the energy approaches in elasticity may be interpreted as replacements for the various equations of elasticity. For example, the principle of virtual displacement may be interpreted as a replacement of the equation of motion (because displacement components are employed, compatibility is ensured), whereas principles of virtual stress may be considered a replacement for the compatibility requirements.

Although in a certain sense the use of energy principles in seeking elasticity solutions may be thought of as an alternative method of attack, their use is important from a number of other viewpoints. For example, energy principles may be used to simplify the derivation of the governing differential equations of various structural problems of plates and shells, as well as for obtaining the associated boundary conditions (Langhaar, 1989). Furthermore, certain features of the solution of the problem may be clarified without a complete knowledge of the solution (Langhaar, 1989, Section 4-10). With the widespread use of high-speed electronic computers, the adaptability of energy principles in seeking approximate solutions has become one of their most important features. Particularly, the principles of virtual work and virtual stress form the basis for much of the modern work in finite element methods.

4-21 Principle of Virtual Work

We present the principle of virtual work for conservative unchecked systems. The law of kinetic energy (Boresi and Schmidt, 2002) when applied to conservative unchecked systems leads immediately to the general static principle known as the principle of virtual work (virtual displacement). We consider the development of the principle initially for a system of particles before discussing elastic bodies (conservative systems).

Principle of Virtual Work (Displacement) for Particles. In the case of a single particle, when the particle begins to move from a state of rest, it is gaining kinetic energy. Therefore, by the law of kinetic energy, the forces F_j that act on the particle are performing net positive work. Hence, the particle does not move unless it can undergo some arbitrarily small displacement, say δu_i, for which the corresponding increment of work δW of the forces is positive, that is, for which $\delta W > 0$. Alternatively, spontaneous motion of the particle is not possible if $\delta W \leq 0$ for all small displacements imaginable for the particle. Because the small displacements δu_i need not necessarily be realized physically, they are said to be *virtual displacements*; similarly the work δW is called the *virtual work.*

If the forces that act on the particle are in equilibrium, then under a virtual displacement δu_i: $(\delta u_1, \delta u_2, \delta u_3)$ relative to axes x_i, *in which the forces acting on the particle are unchanged*, we have

$$\delta u_1 \sum F_{x1} = 0 \qquad \delta u_2 \sum F_{x2} = 0 \qquad \delta u_3 \sum F_{x3} = 0 \tag{4-21.1}$$

as, by definition of equilibrium,

$$\sum F_{x1} = 0 \qquad \sum F_{x2} = 0 \qquad \sum F_{x3} = 0 \tag{4-21.2}$$

Hence

$$\delta W = \delta u_1 \sum F_{x1} + \delta u_2 \sum F_{x2} + \delta u_3 \sum F_{x3} = 0 \tag{4-21.3}$$

In other words, *for a particle in equilibrium, the virtual work of all forces acting on the particle in a virtual displacement is identically zero.*

Similarly, for a *system* of mass particles in equilibrium under external forces and internal forces (say, mutual attractions between the masses), the resultant of all external and internal forces of the system that act on any particle must vanish. Hence, under a virtual displacement of each particle, with all forces (external and internal) remaining unchanged, the virtual work done by the resultant force acting on each particle is zero [Eq. (4-21.3)]: thus, the total virtual work that is performed by all forces on all particles is zero. The virtual work is performed in part by the internal forces and in part by the external forces. For the elastic system, we consider all forces to be conservative. For internal forces of a system of particles, we therefore require the internal energy $\overline{U}$ to be a function of particle position only (see Section 4-2). Accordingly, the principle of virtual work may be stated for a system of particles in the form

$$\delta W_e + \delta W_i = 0 \tag{4-21.4}$$

where δW_e, δW_i are the virtual works of the external forces and of internal forces, respectively. Hence, we may write

$$\delta W_e = -\delta W_i = \delta \overline{U} \tag{4-21.5}$$

where $\delta \overline{U}$ is the first variation of the total internal energy $\overline{U}$ of the particles, that is, the first-order terms in the virtual displacements (see Chapter 1, Section 1-29, and Section 4-3). Equation (4-21.5) is the symbolic representation of the principle of virtual work for a system of material particles. In other words,

> if a system of particles is in a state of equilibrium under the action of conservative unchecked external and internal forces, then under virtual displacements given to each particle the virtual work performed by the external forces (which remain unchanged in the virtual displacement) is equal to the first variation of the internal energy.

Principle of Virtual Work (Displacement) for Elastic Bodies. We proceed in a manner analogous to that for a system of particles, noting, however, that for an elastic body we require a virtual displacement to satisfy the conditions of an admissible displacement (Chapter 2, Section 2-4) that satisfies any prescribed displacement boundary conditions (Sections 4-15 and 4-16). For example, in the case of the beam of Example 4-18.2, the virtual displacement components and certain of their derivatives must vanish at the point $x = y = z = 0$.

Let the state of stress of an elastic body be denoted by the stress components $\sigma_{\alpha\beta}$, and the deformation state by the displacement components u_α relative to axes x_α. Let the body be in a state of small dynamic motion or in a state of equilibrium. Then analogous to the work performed by external forces on a particle, we have the virtual work due to surface stress $\sigma_{n\alpha}$ [Eq. (3-3.10) in Chapter 3],

$$\delta W_S = \iint_S \sigma_{n\alpha}\, \delta u_\alpha\, dS \tag{4-21.6}$$

and the virtual work due to body forces B_α (as well as inertial forces $-\rho \ddot{u}_\alpha$):

$$\delta W_B = \iiint_V (B_\alpha - \rho \ddot{u}_\alpha)\, \delta u_\alpha\, dV \tag{4-21.7}$$

By Eqs. (3-3.10) and (4-21.6), we obtain

$$\delta W_S = \iint_S [(\sigma_{\beta\alpha} n_\beta)\delta u_\alpha]\, dS \tag{4-21.8}$$

or, by use of the divergence theorem (Chapter 1, Section 1-15),

$$\delta W_s = \iiint\limits_V \frac{\partial}{\partial x_\beta}(\sigma_{\beta\alpha}\delta u_\alpha)\,dV$$
$$= \iiint\limits_V \left(\frac{\partial \sigma_{\beta\alpha}}{\partial x_\beta}\delta u_\alpha + \sigma_{\beta\alpha}\frac{\partial \delta u_\alpha}{\partial x_\beta}\right) dV$$
$$= \iiint\limits_V (\sigma_{\beta\alpha,\beta}\delta u_\alpha + \sigma_{\beta\alpha}\delta\varepsilon_{\alpha\beta})\,dV \tag{4-21.9}$$

where we have employed the linearity (hence interchangeability) of $\partial\delta u_\alpha/\partial x_\beta = (\delta u_\alpha)_{,\beta} = \delta(u_{\alpha\beta})$, and where $\delta\epsilon_{\alpha\beta} = \delta[(u_{\alpha,\beta} + u_{\beta,\alpha})/2]$ (see Section 4-3) denotes the virtual strain components associated with δu_α.

Introducing Eq. (3-8.1) into Eq. (4-21.9), we obtain with Eq. (4-21.7)

$$\delta\overline{W} = \delta W_S + \delta W_B = \iiint\limits_V \sigma_{\beta\alpha}\delta\epsilon_{\alpha\beta}\,dV \tag{4-21.10}$$

However, by Eq. (d), Section 4-3, we have for the conservative elastic system

$$\delta I = \delta\overline{U}(\epsilon_{\alpha\beta}) = \iiint\limits_V \sigma_{\beta\alpha}\delta\epsilon_{\alpha\beta}\,dV \tag{4-21.11}$$

where we consider $\overline{U}$, hence $\sigma_{\beta\alpha}$, functions of the strain components $\epsilon_{\alpha\beta}$. Consequently, Eqs. (4-21.10) and (4-21.11) yield

$$\delta\overline{W} = \delta\overline{U}(\epsilon_{\alpha\beta}) \tag{4-21.12}$$

where

$$\delta\overline{W} = \delta\left[\iint\limits_S \sigma_{n\alpha}u_\alpha\,dS + \iiint\limits_V (B_\alpha - \rho\ddot{u}_\alpha)u_\alpha\,dV\right] \tag{4-21.13}$$

and

$$\delta\overline{U}(\epsilon_{\alpha\beta}) = \delta\iiint\limits_V \sigma_{\beta\alpha}\epsilon_{\alpha\beta}\,dV \tag{4-21.14}$$

where we place the variation sign outside the integral, recalling that under a virtual displacement (virtual strain) the surface stress $\sigma_{n\alpha}$, body force B_α, stress components $\sigma_{\beta\alpha}$, and inertial forces $-\rho\ddot{u}_\alpha$ are not changed.

For an equilibrium state, $\rho\ddot{u} = 0$. Then Eq. (4-21.12) yields

$$\iint\limits_S \sigma_{n\alpha}\,\delta u_\alpha\,dS + \iiint\limits_V B_\alpha\,\delta u_\alpha\,dV = \iiint\limits_V \sigma_{\beta\alpha}\,\delta\epsilon_{\alpha\beta}\,dV \tag{4-21.15}$$

Equation (4-21.15) is a statement of the principle of virtual work (displacement) for an elastic body.

Recalling from elementary mechanics that for a conservative system of forces the work $\delta\overline{W}$ done by forces undergoing a virtual displacement is equal to the negative of the potential energy change $\delta\overline{\Omega}$ of the forces, we may write Eq. (4-21.12) in the form

$$\delta\pi = 0 \tag{4-21.16}$$

where $\delta\pi$ is the variation of the total potential energy $\pi = \overline{\Omega} + \overline{U}(\epsilon_{\alpha\beta})$ of the system, and $\overline{\Omega} = -\overline{W}$ is the potential energy of external forces. Hence, for a conservative system, the principle of virtual work is equivalent to the *principle of stationary potential energy* $\delta\pi = 0$ (see Chapter 1, Section 1-29). Alternatively, we may note that if a displacement state u_α satisfies the condition $\delta\pi = 0$ for all δu_α, it can be shown that u_α is the solution of the elasticity problem.

Because in the development of the principle of virtual work ($\delta\pi = 0$) we have employed admissible displacements u_α (and δu_α), the compatibility conditions of strain are ensured (Chapter 2, Section 2-16). In addition, we require that u_α satisfy all displacement boundary conditions prescribed on boundary S'' (then $\delta u_\alpha = 0$ on S''). Accordingly, to ensure that displacement u_α is the solution of the elasticity problem, the condition $\delta\pi = 0$ must be shown equivalent to the equations of equilibrium (motion) and to appropriate boundary conditions. This fact may be shown as follows. Let stress boundary conditions be prescribed on S'' and displacement boundary conditions be prescribed on S'', where $S = S' + S''$ denotes the surface bounding an elastic region R.

Then, by Eq. (4-21.15) [or Eq. (4-21.16)], we obtain (because $\delta u_\alpha = 0$ on S'')

$$\iint\limits_{S'} \sigma'_\alpha \, \delta u_\alpha \, dS + \iiint\limits_{R} B_\alpha \delta u_\alpha \, dV = \delta\overline{U} \tag{4-21.17}$$

where σ'_α denotes the stress vector on surface S',

$$\delta\overline{U} = \iiint\limits_{R} \sigma_{\beta\alpha} \, \delta\epsilon_{\alpha\beta} \, dV = \iiint\limits_{R} \sigma_{\beta\alpha} \, \delta u_{\alpha,\beta} \, dV \tag{4-21.18}$$

and where we have used (δu_α), $_\beta = \delta(u_{\alpha,\beta})$ [see Eq. (4-21.9)]. Again, because $\delta u_\alpha = 0$ on S'', by Eqs. (4-21.18) and (1-15.3) in Chapter 1, we write

$$\delta\overline{U} = -\iiint\limits_{R} \sigma_{\beta\alpha,\beta} \, \delta u_\alpha \, dV + \iint\limits_{S'} \sigma_{\beta\alpha} \, \delta u_\alpha \, n_\beta \, dS \tag{4-21.19}$$

Consequently, by Eqs. (4-21.17) and (4-21.19), we find

$$\iint\limits_{S'} (\sigma_{\beta\alpha} n_\beta - \sigma'_\alpha) \, \delta u_\alpha \, dS - \iiint\limits_{R} (\sigma_{\beta\alpha,\beta} + B_\alpha) \, \delta u_\alpha \, dV = 0 \tag{4-21.20}$$

for any δu_α that vanishes on S''. Consider the particular case $\delta u_\alpha = 0$ everywhere on S. Then the first integral in Eq. (4-21.20) is identically zero, and we obtain

$$\iiint_R (\sigma_{\beta\alpha,\beta} + B_\alpha)\, \delta u_\alpha \, dV = 0 \tag{4-21.21}$$

However, by the fundamental lemma of the calculus of variations (Langhaar, 1989, Section 3-2), if the integrand of Eq. (4-21.21) is not identically zero, then a δu_α vanishing on S may be found such that the integral of Eq. (4-21.21) *does not* vanish. Consequently, a necessary and sufficient condition that Eq. (4-21.21) be satisfied is that $\sigma_{\beta\alpha,\beta} + B_\alpha = 0$; that is, the equations of equilibrium must be satisfied by stress components $\sigma_{\beta\alpha}$ associated with u_α. Hence, the second integrand of Eq. (4-21.20) vanishes regardless of the increment (virtual displacement) δu_α. Thus, Eq. (4-21.20) reduces to

$$\iint_{S'} (\sigma_{\beta\alpha} n_\beta - \sigma'_\alpha)\, \delta u_\alpha \, dS = 0 \tag{4-21.22}$$

and, by the fundamental lemma of the calculus of variations, the integrand of Eq. (4-21.22) vanishes. In other words, the stress components $\sigma_{\beta\alpha}$ associated with displacement components u_α must satisfy the stress boundary condition $\sigma'_\alpha = \sigma_{\beta\alpha} n_\beta$ on S'. Accordingly, the displacement component u_α satisfies all the conditions of the elasticity problem and hence is unique (Section 4-16).

Theorem of Minimum Strain Energy (Elastic Energy). In the absence of body forces ($B_\alpha = 0$), and the case where the displacement is prescribed everywhere on the bounding surface S of the elastic body, $\delta u_\alpha = 0$ on S. Then the equilibrium state of the system ($\rho \ddot{u}_\alpha = 0$) as defined by Eq. (4-21.12) reduces to

$$\delta \iiint_V \sigma_{\beta\alpha} \epsilon_{\alpha\beta} \, dV = \delta \overline{U}(\epsilon_{\alpha\beta}) = 0 \tag{4-21.23}$$

Equation (4-21.23) is called the *theorem of minimum strain energy* (*elastic energy*). We note that $\overline{U}$ is the total strain energy $\iiint U \, dV$, where explicit forms for the strain energy density U as functions of strain components $\epsilon_{\alpha\beta}$ are given in Sections 4-6 and 4-11.

Problem Set 4-21

1. An isotropic Hookean body is subjected to plane strain; that is, $u = u(x, y)$, $v = v(x, y)$, and $w = 0$. There is no body force, but there is a temperature field $T(x, y)$. Using small-displacement theory, derive the differential equations for u, v by the principle of stationary potential energy.

2. A uniform Hookean rod of cross-sectional area A and length L is suspended vertically in a gravity field from one end. The specific weight of the material is s. The rod is loaded by its own weight. Set up the formula for the total potential energy in terms of the downward displacement $u(x)$. State the forced boundary condition at the support. By the principle of stationary potential energy, derive the natural boundary condition at the free end and the differential equation for u. The strain energy of the bar is

$$U = \frac{A}{2E} \int_0^L \sigma_x^2 \, dx$$

where E is Young's modulus and $\sigma_x = E\epsilon_x = E(du/dx)$.

3. A rectangular plate of length a and width b is free on the edge $x = a$ and is simply supported on the other three edges. It is loaded by a uniformly distributed load $p = p_0$. Using small-deflection theory, assume the lateral bending deflection w to be of the form

$$w = Axy(y - b)$$

and determine the constant A by the principle of stationary potential energy.

Note that this deflection pattern satisfies the forced boundary conditions on the supported edges. Does it satisfy the natural boundary conditions on the free edge also? The formula for the strain energy of bending is

$$U_b = \frac{1}{2} D \iint \left[\left(w_{xx} + w_{yy} \right)^2 - 2(1 - \nu)\left(w_{xx} w_{yy} - w_{xy}^2 \right) \right] dx\, dy$$

Neglect the strain energy due to stretching of the plate. Subscripts on w denote partial derivatives.

4-22 Principle of Virtual Stress (Castigliano's Theorem)

As noted in Chapters 2 and 3, the mathematical character of the strain tensor $\sigma_{\alpha\beta}$ and the stress tensor $\epsilon_{\alpha\beta}$ are identical. Accordingly, from a mathematical viewpoint, we may expect that results somewhat analogous to those derived in Section 4-21 may be obtained by considering virtual changes $\delta\sigma_{\beta\alpha}$ (variations) in the stress tensor $\sigma_{\beta\alpha}$.

Accordingly, if we proceed in the manner of Section 4-21, we find

$$\delta\overline{U}(\epsilon_{\alpha\beta}) = \iint_S u_\alpha \delta\sigma_{n\alpha} \, dS + \iiint_B u_\alpha \delta B_\alpha \, dV \qquad (4\text{-}22.1)$$

where

$$\delta\sigma_{n\alpha} = (\delta\sigma_\alpha) n_\beta$$
$$\delta B_\alpha = -\delta(\sigma_{\alpha\beta,\beta})$$
$$\delta\sigma_{\alpha\beta} = \delta\sigma_{\beta\alpha} \qquad \sigma_{\alpha\beta} = \sigma_{\beta\alpha}$$

If we assume that the body forces are identical (unchanged) under the virtual changes of stress, $\delta B_\alpha = 0$, and then

$$\begin{aligned} \delta(\sigma_{\alpha\beta,\beta}) &= 0 \\ \delta(\sigma_{\alpha\beta} n_\beta) &= \delta\sigma_{n\alpha} \end{aligned} \tag{4-22.2}$$

Equation (4-22.1) or (4-22.2) is referred to as the *principle of virtual stress.* This principle states that the variations of the solution $\sigma_{\alpha\beta}$ of the equilibrium elasticity problem must satisfy the equations of equilibrium and the stress boundary conditions. Furthermore, the converse statement, that "given $\sigma_{\alpha\beta}$ such that $\delta\sigma_{\beta\alpha}$ satisfies Eq. (4-22.1), then it follows that $\sigma_{\alpha\beta}$ is the solution of the elasticity problem," may be proved (Pearson, 1959, p. 147), with the result that $\sigma_{\alpha\beta}$ may be shown to satisfy the equations of compatibility [see Eqs. (4-14.2)].

Alternatively, we may also proceed to demonstrate the principle of virtual stress as follows: We have noted that the equations of equilibrium are (Chapter 3, Section 3-8) with $\sigma_{\beta\alpha} = \sigma_{\alpha\beta}$,

$$\sigma_{\alpha\beta,\beta} + B_\alpha = 0 \tag{4-22.3}$$

and the stress boundary conditions are (Section 4-15)

$$\sigma_{n\alpha} = \sigma_{\alpha\beta} n_\beta \tag{4-22.4}$$

In general, Eqs. (4-22.3) and (4-22.4) are not adequate to determine the stress components $\sigma_{\alpha\beta}$ uniquely (Section 4-16). Hence, it is possible to obtain any number of stress states that satisfy these equations. Accordingly, with the introduction of virtual stress components $\delta\sigma_{\alpha\beta}$ and body-force changes δB_α, let us consider another system of stress components:

$$\sigma_{\alpha\beta} + \delta\sigma_{\alpha\beta} \qquad B_\alpha + \delta B_\alpha \tag{4-22.5}$$

Because of the linear nature of the equations of equilibrium and the boundary conditions, in order that Eqs. (4-22.5) satisfy Eqs. (4-22.3) and (4-22.4) we must have (principle of superposition)

$$\begin{aligned} (\delta\sigma_{\alpha\beta})_{,\beta} + \delta B_\alpha &= 0 \\ \delta(\sigma_{n\alpha}) &= (\delta\sigma_{\alpha\beta}) n_\beta \end{aligned} \tag{4-22.6}$$

In other words, the variations $\delta\sigma_{\alpha\beta}, \delta B_\alpha$ must themselves satisfy the equations of equilibrium and the boundary conditions. For simplicity, let $\delta B_\alpha = 0$. Then, proceeding as in Section 4-21, we obtain (now giving virtual changes to $\sigma_{\alpha\beta}$, while maintaining the displacement and hence the strain fixed under these changes)

$$\iint_S (\delta\sigma_{n\alpha}) u_\alpha \, dS = \iiint_R (\delta\sigma_{\alpha\beta}) \epsilon_{\alpha\beta} \, dV \tag{4-22.7}$$

For the isotropic case, because $\epsilon_{\alpha\beta}$ is kept constant, by Eqs. (4-6.10) and (4-6.13), we obtain

$$\begin{aligned}
\iiint_R (\delta\sigma_{\alpha\beta})\,\epsilon_{\alpha\beta}\,dV &= \iiint_R \delta[\sigma_{11}\epsilon_{11} + \sigma_{22}\epsilon_{22} + \sigma_{33}\epsilon_{33} + 2\sigma_{12}\epsilon_{12} \\
&\qquad + 2\sigma_{13}\epsilon_{13} + 2\sigma_{23}\epsilon_{23}]\,dV \\
&= \iiint_R \frac{1}{E}[\sigma_{11}\delta\sigma_{11} + \sigma_{22}\delta\sigma_{22} + \sigma_{33}\delta\sigma_{33} \\
&\qquad - \nu(\sigma_{22}\delta\sigma_{11} + \sigma_{11}\delta\sigma_{22} + \sigma_{33}\delta\sigma_{11} + \sigma_{11}\delta\sigma_{33} \\
&\qquad + \sigma_{33}\delta\sigma_{22} + \sigma_{22}\delta\sigma_{33}) + 2(1+\nu)(\sigma_{12}\delta\sigma_{12} \\
&\qquad + \sigma_{13}\delta\sigma_{13} + \sigma_{23}\delta\sigma_{23})]\,dV \\
&= \delta\left\{\iiint_R \frac{1}{2E}[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - 2\nu(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}) \right. \\
&\qquad \left. + 2(1+\nu)(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2)]\,dV\right\} \\
&= \delta \iiint_R U\,dV
\end{aligned} \tag{4-22.8}$$

where we have used the equivalent notation

$$\begin{aligned}
\delta\left(\sigma_{11}^2\right) &= 2\sigma_{11}\delta\sigma_{11}, \ldots, \ldots \\
\delta(\sigma_{11}\sigma_{22}) &= \sigma_{22}\,\delta\sigma_{11} + \sigma_{11}\,\delta\sigma_{22}, \ldots, \ldots
\end{aligned} \tag{4-22.9}$$

Consequently, Eq. (4-22.7) may be written in the form

$$\iint_S [(\delta\sigma_{n1})u_1 + (\delta\sigma_{n2})u_2 + (\delta\sigma_{n3})u_3]\,dS = \delta\overline{U}(\sigma_{\alpha\beta}) \tag{4-22.10}$$

where $\overline{U}(\sigma_{\alpha\beta}) = \iiint U dV$ and where explicit forms for U as a function of stress components $\sigma_{\alpha\beta}$ are given in Sections 4-6 and 4-8. Accordingly, Eq. (4-22.10) states that for the true state of stress $\sigma_{\alpha\beta}$, the work done by the variations of the surface tractions on the actual displacement u_α equals the variation of the total elastic strain energy $\overline{U}$ of the body resulting from the variation of the actual state of stress, where the variations $\delta\sigma_{\alpha\beta}$ satisfy Eqs. (4-22.6) or Eqs. (4-22.2) when $\delta B_\alpha = 0$.

The result denoted by Eq. (4-22.10) is referred to as the *principle of virtual stress or Castigliano's variational equation.*

Castigliano's Theorem on Deflections. A special case of Eq. (4-22.10) is called *Castigliano's theorem on deflections.* For example, let an elastic body be supported so that rigid-body displacements of the body are not possible. Let the body be in a state of equilibrium under the action of surface forces and body forces, with certain point forces being included in the external loads. Then Eq. (4-22.10) yields the result (Langhaar, 1989, Sections 4-10 and 4-11)

$$q_i = \frac{\partial \overline{U}(F_1, F_2, \ldots)}{\partial F_i} \tag{4-22.11}$$

where q_i is the component in the direction of force F_i of the displacement of the point of application of force F_i, and $\overline{U}$ is considered to be a function of the forces F_i. Castigliano's theorem of deflection may also be shown to apply to point moments if the q_i are interpreted to be rotations in the sense of the moments, the F_i are interpreted to be moments, and $\overline{U}$ is considered to be a function of the moments (Langhaar, 1989).

4-23 Mixed Virtual Stress–Virtual Strain Principles (Reissner's Theorem)

In the principle of virtual work, strain compatibility equations are ensured because displacement components are employed in the calculations, and the principle $\delta\pi = 0$ replaces the equations of equilibrium and the associated stress boundary conditions. In the principle of virtual stress, equilibrium equations are satisfied a priori, and the virtual stress principle is analogous to satisfaction of the compatibility conditions.

Reissner (1950, 1953, 1958), with the objective of giving more equal weight to the equilibrium and compatibility conditions, has formulated an alternative principle in which variations in *both* stress and strain components are admitted. Following Reissner, we consider the variation of the quantity

$$Q = \iiint_R \sigma_{\alpha\beta}\epsilon_{\alpha\beta}\, dV - \overline{U}(\sigma_{\alpha\beta}) \tag{4-23.1}$$

for arbitrary virtual stresses $\delta\sigma_{\alpha\beta}$ and strains $\delta\epsilon_{\alpha\beta}$ to obtain

$$\delta Q = \iiint_R [(\delta\sigma_{\alpha\beta})\epsilon_{\alpha\beta} + \sigma_{\alpha\beta}(\delta\epsilon_{\alpha\beta})]\, dV - \iiint_R F(\sigma_{\alpha\beta})\delta\sigma_{\alpha\beta}\, dV \tag{4-23.2}$$

where $F(\sigma_{\alpha\beta})$ is the functional expression for $\epsilon_{\alpha\beta}$ in terms of $\sigma_{\alpha\beta}$. Letting $\sigma_{\alpha\beta}$, $\epsilon_{\alpha\beta}$ be the solution to the elasticity problem, the first and third integrands of Eq. (4-23.1) vanish identically. Hence, Eqs. (4-23.1) and (4-23.2) yield

$$\delta\left[\iiint_R \sigma_{\alpha\beta}\epsilon_{\alpha\beta}\, dV - \overline{U}(\sigma_{\alpha\beta})\right] = \iiint_R \sigma_{\alpha\beta}(\delta\epsilon_{\alpha\beta})\, dV = \iiint_R \sigma_{\alpha\beta}(\delta u_\alpha)_{,\beta}\, dV \tag{4-23.3}$$

or, with Eq. (1-15.3) in Chapter 1 and Eqs. (3-8.1) in Chapter 3, with $\rho\ddot{u}_\alpha = 0$ and $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$,

$$\delta\left[\iiint\limits_R \sigma_{\alpha\beta}\epsilon_{\alpha\beta}\,dV - \overline{U}(\sigma_{\alpha\beta})\right] = \iint\limits_S \sigma_{n\alpha}(\delta u_\alpha)\,dS + \iiint\limits_R B_\alpha(\delta u_\alpha)\,dV \tag{4-23.4}$$

Reissner shows that both compatibility and equilibrium follow from Eq. (4-23.4) when one assumes that $\sigma_{n\alpha} = \sigma_{\alpha\beta}n_\beta$ on S' and $u_\alpha = u_\alpha$ (prescribed) on S'', where $S = S' + S''$ is the surface bounding region R.

Mixed variational principles of the type derived by Reissner play an important role in the development of finite element methods.

The variational principles discussed in Sections 4-20 through 4-23 also form the basis for other kinds of approximation methods of numerical stress analysis.

APPENDIX 4A APPLICATION OF THE PRINCIPLE OF VIRTUAL WORK TO A DEFORMABLE MEDIUM (NAVIER–STOKES EQUATIONS)

As noted in Section 4-20, the principle of virtual work may be employed directly to derive the governing differential equations of a continuum. In this appendix we derive the differential equations of motion of a deformable medium (see Section 3-8). The technique is applicable to a particular structural system, a fluid system, or a general solid, provided appropriate expressions are used for the various components of virtual work.

The general form of the principle of virtual work is [(Langhaar, 1989, Section 4-10); also Eqs. (4-21.4) and (4-21.10)]

$$\delta W = 0 \tag{4A-1}$$

where δW denotes the virtual work of all forces acting on a system that undergoes an infinitesimal virtual displacement δu_α from a given configuration. Considering forces acting on the system to be grouped into internal forces F_I, external forces F_E, and inertial forces F_A, we may write Eq. (4A-1) in the form

$$\delta W_I + \delta W_E + \delta W_A = 0 \tag{4A-2}$$

where δW_I is the virtual work of internal forces $F_I, \delta W_E$ is the virtual work of external forces F_E, and δW_A is the virtual work of the inertial forces F_A. By Eq. (d), Section 4-3, we have for a medium contained in region R bounded by surface S

$$\delta W_I = -\iiint\limits_R (\delta\mathscr{I})\,dV = -\iiint\limits_R \sigma_{\alpha\beta}\delta\epsilon_{\alpha\beta}\,dV \tag{4A-3}$$

because the work performed by the internal forces is the negative of the intrinsic (internal) energy, and where $\sigma_{\alpha\beta}$ is the stress tensor and $\epsilon_{\alpha\beta}$ is the strain tensor.

Let δu_α be the infinitesimal virtual displacement vector. Then the virtual strain components associated with δu_α are

$$\delta\epsilon_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial\,\delta u_\alpha}{\partial x_\beta} + \frac{\partial\,\delta u_\beta}{\partial x_\alpha}\right) \tag{4A-4}$$

Hence, because $\sigma_{\alpha\beta}$ and $\delta\epsilon_{\alpha\beta}$ are symmetric tensors,

$$\delta W_I = -\iiint\limits_R -2\sigma_{\alpha\beta}\frac{\partial(\delta u_\alpha)}{\partial x_\beta}\,dV \tag{4A-5}$$

Noting that

$$\sigma_{\alpha\beta}\frac{\partial(\delta u_\alpha)}{\partial x_\beta} = \frac{\partial}{\partial x_\beta}(\sigma_{\alpha\beta}\delta u_\alpha) - \delta u_\alpha\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta} \tag{4A-6}$$

we write Eq. (4A-5) in the form

$$\delta W_I = -\iiint\limits_R \frac{\partial}{\partial x_\beta}(\sigma_{\alpha\beta}\,\delta u_\alpha)\,dV + \iiint\limits_R \delta u_\alpha\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}\,dV \tag{4A-7}$$

Applying the divergence theorem (Chapter 1, Section 1-15) to Eq. (4A-7), we obtain

$$\delta W_I = \iiint\limits_R \delta u_\alpha\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}\,dV - \iint\limits_S \sigma_{\alpha\beta}\,\delta u_\alpha n_\beta\,dS \tag{4A-8}$$

where n_β is the outward unit normal to surface S.

Now by Eqs. (4-15.1), $\sigma_{n\alpha} = \sigma_{\alpha\beta}n_\beta$, where $\sigma_{n\alpha}$ is the stress vector on S. Hence,

$$\delta W_I = \iiint\limits_R \delta u_\alpha\frac{\partial\sigma_{\alpha\beta}}{\partial x_\beta}\,dV - \iint\limits_S \sigma_{n\alpha}\delta u_\alpha\,dS \tag{4A-9}$$

The virtual work due to external forces is

$$\delta W_E = \iiint\limits_R B_\alpha\,\delta u_\alpha\,dV + \iint\limits_S \sigma_{n\alpha}\,\delta u_\alpha\,dS \tag{4A-10}$$

where B_α denotes body-force components per unit volume.

The virtual work due to inertial forces $-\rho a_\alpha$, where ρ denotes mass density and a_α denotes the acceleration components, is

$$\delta W_A = -\iiint\limits_R \rho a_\alpha\,\delta u_\alpha\,dV \tag{4A-11}$$

Accordingly, by Eqs. (4A-2), (4A-9), (4A-10), and (4A-11), the principle of virtual work yields

$$\iiint\limits_R \left(\frac{\partial \sigma_{\alpha\beta}}{\partial x_\beta} + B_\alpha - \rho a_\alpha \right) \delta u_\alpha \, dV = 0 \tag{4A-12}$$

Because Eq. (4A-12) must hold for any virtual displacement δu_α (Langhaar, 1989), the integrand of Eq. (4A-12) must vanish identically. Thus,

$$\sigma_{\alpha\beta,\beta} + B_\alpha = \rho a_\alpha \tag{4A-13}$$

Equation (4A-13) is the equation of motion for a deformable medium [Eqs. (3-8.1) in Chapter 3]. In deriving it, we have not used stress–strain relations, nor have we employed thermodynamics concepts.

For a Newtonian incompressible fluid, the stress $\sigma_{\alpha\beta}$ is related to the velocity field v_α by the relations (Goldstein, 1965, Chapter 3)

$$\begin{aligned} \sigma_{\alpha\beta} &= -p\delta_{\alpha\beta} + \mu \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right) \\ 3p &= -(\sigma_{11} + \sigma_{22} + \sigma_{33}) = -I_1 \end{aligned} \tag{4A-14}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta, μ is the Newtonian viscosity coefficient, and v_α is the velocity.

Equations (4A-13) and (4A-14) yield

$$\sigma_{\alpha\beta,\beta} = -p_{,\alpha} + \mu \nabla^2 v_\alpha \tag{4A-15}$$

as for an incompressible fluid, $v_{\beta,\beta} = 0$ (Chapter 2, Section 2-18). Substitution of Eqs. (4A-15) into Eq. (4A-13) yields

$$-\frac{\partial p}{\partial x_\alpha} + \mu \nabla^2 v_\alpha + B_\alpha = \rho \left(\frac{\partial v_\alpha}{\partial t} + \frac{\partial v_\alpha}{\partial x_\beta} v_\beta \right) \tag{4A-16}$$

where

$$\rho a_\alpha = \rho \frac{d v_\alpha}{dt} = \rho \left(\frac{\partial v_\alpha}{\partial t} + \frac{\partial v_\alpha}{\partial x_\beta} v_\beta \right) \tag{4A-17}$$

Equations (4A-16) are the Navier–Stokes equations of viscous fluid flow.

APPENDIX 4B NONLINEAR CONSTITUTIVE RELATIONSHIPS

Many isotropic and anisotropic materials possess linear stress–strain (constitutive) relations below their elastic limits. The generalized Hooke's law (Section 4-4) is applicable to these materials. However, for certain other materials—for example, geomaterials such as concrete, soil, and rock—the constitutive relationships are nonlinear (Chen and Saleeb, 1994). Generally, three different methods are employed to model nonlinear materials:

1. The use of variable stress–strain coefficients
2. The introduction of higher-order strain energy terms
3. Hypoelastic formulations

4B-1 Variable Stress–Strain Coefficients

Nonlinear stress–strain relations may be characterized by tangent moduli that are dependent on magnitudes of stress and/or strain (Chong et al., 1980b; Boresi et al 2002). Models that employ bilinear, piecewise linear and curve-fitting techniques have been used to approximate stress–strain relations (Desai and Christian, 1977). These approaches are adaptable especially to computer applications.

The main disadvantage of the variable coefficient approach is that the constitutive relationship is independent of path (Chen, 1984). Hence, the main applications of the variable stress–strain coefficients approach are limited to monotonic or proportional loading situations (Desai and Siriwardane, 1984). This method is not applicable to cyclic loading cases. Chen (1984) has referred to the variable coefficient approach as the Cauchy elastic formulation.

4B-2 Higher-Order Relations

In this modeling technique, higher-order strains or strain invariants are included in the strain energy density function U. The stress–strain relations are obtained by differentiation [Eqs. (4-3.2)]. Materials modeled with higher-order invariants are called hyperelastic materials or Green-type materials (Eringen, 1962; Chen and Saleeb, 1994). The main disadvantage of this modeling technique is that the number of required elastic coefficients increases tremendously for each higher-order invariant introduced. For example, for isotropic materials, the third-order hyperelastic model contains 9 material coefficients (compared to 2 material coefficients in Hooke's law, which is a special case of hyperelasticity), and a fifth-order hyperelastic model requires 14 coefficients (Chen, 1984). For transversely isotropic materials, a third-order hyperelastic model requires 35 coefficients (Cleary, 1978), compared to 5 coefficients for the Cauchy elastic formulation (Chong et al., 1980b). To accurately determine experimentally 5 coefficients is extremely difficult; to accurately determine 35 coefficients is impractical, if not impossible (Chong et al., 1980b).

Nevertheless, hyperelastic models can account for various effects not included in the Cauchy elastic formulation. For example, the change of volume resulting from shear can be accounted for in higher-order hyperelasticity laws (Desai and Siriwardane, 1984).

4B-3 Hypoelastic Formulations

The hyperelastic and Cauchy elastic materials (Sections 4B-1 and 4B-2) are independent of loading path, and upon unloading, they return to their original unstrained state. To account for the past history of loading and permanent set, Truesdell (1955) proposed a hypoelastic formulation in which the basic relation is

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}\dot{\boldsymbol{\epsilon}} \tag{4B-3.1}$$

where the matrices $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\epsilon}}$ are functions of rates of stress and strain, respectively, and the matrix $\mathbf{C}$ is a function of stress.

Integration of Eq. (4B-3.1) with respect to time t yields a relationship for the stress matrix $\boldsymbol{\sigma}$ in the form

$$\boldsymbol{\sigma} = \int_0^t \mathbf{C}\frac{\partial \boldsymbol{\epsilon}}{\partial \tau}\, d\tau + \boldsymbol{\sigma}_0 \tag{4B-3.2}$$

where the matrix $\boldsymbol{\sigma}_0$ contains appropriate functions of integration.

The main disadvantage of the hypoelastic model is that anisotropic behavior is induced by stress (Chen, 1984), resulting in different principal axes for stress and strain. As a result, coupling between normal stresses and shearing strains occurs. In addition, the hypoelastic model requires 21 material coefficients, making it difficult to apply.

4B-4 Summary

Each of the methods described above have merit for certain applications. At present, the most frequently used nonlinear approach for practical problems is the Cauchy elastic formulation (Section 4B-1). However, considerable research is being conducted currently on modeling of nonlinear elastic material behavior.

APPENDIX 4C MICROMORPHIC THEORY

Micromorphic theory envisions a material body as a continuous collection of deformable particles; each possesses finite size and inner structure. On the other hand, classical continuum mechanics, including elasticity, envisions a material body as a continuous collection of material points, each with infinitesimal size and no inner structure. The purpose to go beyond the classical continuum mechanics is to take into account the microstructure of the material body in question while still keeping the advantages of continuum theory intact. Micromorphic theory is considered as the most successful top-down formulation of a two-level continuum model in which the deformation is expressed as a sum of macroscopic continuous deformation and microscopic deformation of the inner structure.

4C-1 Introduction

Microcontinuum field theories constitute extensions of the classical field theories concerned with deformations, motions, and electromagnetic interactions of material media, as continua, in miniaturized space and time scales. In terms of a physical picture, a microcontinuum may be envisioned as a continuous collection of deformable point particles, each with finite size and inner structure. It is emphasized that in the classical continuum theory a point particle is represented by a geometrical point, which is infinitesimal in size and hence has no inner structure. Then the question arises: *How can one represent the intrinsic deformation of a point particle in microcontinuum?* Eringen settled this question by replacing the deformable point particle with a geometric point P and some vectors attached to P, which denote the orientations and intrinsic deformations of all the material points in the deformable point particle (Eringen and Suhubi, 1964a, 1964b; Eringen, 1964, 1999, 2001, 2002). This is compatible with the classical picture where a material point in a continuum is endowed with physical properties such as mass density, displacement vector, electric field, stress tensor, and so forth. Therefore, the vectors assigned to P represent the additional degrees of freedom arising from the motions, relative to P, of all the material points in the particle. Geometrically, a particle P is identified by its position vector $\mathbf{X}$, in the reference (Lagrangian or material) state B, and vectors $\boldsymbol{\Xi}^{\alpha}(\alpha = 1, 2, 3, \ldots, N)$ attached to P, representing the inner structure of P. Here N is the number of discrete material points in the particle. The motions may be expressed as [cf. Eq. (2-3.3a)]

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{4C-1.1}$$

$$\boldsymbol{\xi}^{\alpha} = \boldsymbol{\xi}^{\alpha}(\mathbf{X}, \boldsymbol{\Xi}^{\alpha}, t) \qquad \alpha \in (1, 2, 3, \ldots, N) \tag{4C-1.2}$$

where t is the time; $\mathbf{x}$ and $\boldsymbol{\xi}^{\alpha}$, corresponding to $\mathbf{X}$ and $\boldsymbol{\Xi}^{\alpha}$, respectively, are the position vectors in the deformed (Eulerian or spatial) state b. A medium with such general motions is named *microcontinuum of grade* N by Eringen. In the two-level continuum model, first let the position vector of a material point be decomposed as the sum of the position vector of the centroid (mass center) of the particle and the position vector of a material point relative to the centroid (cf. Fig. 4C-1.1):

$$\mathbf{x}' = \mathbf{x} + \boldsymbol{\xi} \qquad \mathbf{X}' = \mathbf{X} + \boldsymbol{\Xi} \tag{4C-1.3}$$

and then let the motions be reduced to

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \qquad \boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{X}, \boldsymbol{\Xi}, t) \tag{4C-1.4}$$

If the micromotion $\boldsymbol{\xi} = \boldsymbol{\xi}(\mathbf{X}, \boldsymbol{\Xi}, t)$ is further reduced to an affine motion, that is,

$$\boldsymbol{\xi} = \boldsymbol{\chi}_K(\mathbf{X}, t)\Xi_K \quad \text{or} \quad \xi_k = \chi_{kK}(\mathbf{X}, t)\Xi_K \tag{4C-1.5}$$

we arrive at the doorstep of the *micromorphic theory*. It is seen that the macromotion $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ accounts for the motion of the centroid of the particle; the micromotion $\xi_k = \chi_{kK}(\mathbf{X}, t)\Xi_K$ accounts for the intrinsic motions of the particle; and χ_{kK} is called the microdeformation tensor. If the Jacobians of the macromotion and the micromotion are strictly positive, that is,

$$\begin{aligned} J &\equiv \det\left(\frac{\partial x_k}{\partial X_K}\right) \triangleq \det(x_{k,K}) > 0 \\ j &\equiv \det(\chi_{kK}) > 0 \end{aligned} \tag{4C-1.6}$$

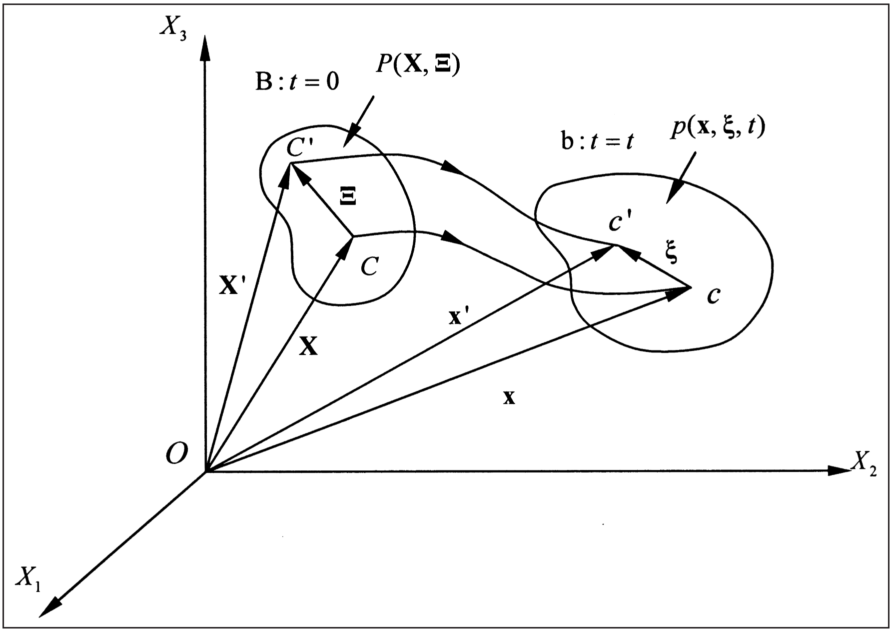

Figure 4C-1.1

then there exist unique inverse motions

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{4C-1.7}$$

$$\boldsymbol{\Xi} = \overline{\boldsymbol{\chi}}_k(\mathbf{x}, t)\xi_k \quad \text{or} \quad \Xi_K = \overline{\chi}_{Kk}(\mathbf{x}, t)\xi_k \tag{4C-1.8}$$

with

$$\begin{aligned} x_{k,K}X_{K,l} &= \delta_{kl} \qquad & X_{K,k}x_{k,L} &= \delta_{KL} \\ \chi_{kK}\overline{\chi}_{Kl} &= \delta_{kl} \qquad & \overline{\chi}_{Kk}\chi_{kL} &= \delta_{KL} \end{aligned} \tag{4C-1.9}$$

Because χ_{kK} is a second-order tensor, the particle has 9 independent degrees of freedom in addition to the 3 classical translational degrees of freedom of the centroid. A unit cell or a polyatomic molecule may be viewed as a point particle in micromorphic theory (Eringen, 1999, 2001).

The microgyration tensor is defined as

$$\omega_{kl} \equiv \dot{\chi}_{kK}\overline{\chi}_{Kl} \tag{4C-1.10}$$

then it is straightforward to prove that

$$\dot{\chi}_{kK} = \omega_{kl}\chi_{lK} \tag{4C-1.11}$$

$$\dot{\xi}_k = \omega_{kl}\xi_l \tag{4C-1.12}$$

Let $\rho^o(\rho)$ and $\Delta V(\Delta v)$ denote the mass density and the volume of the deformable point particle in the Lagrangian (Eulerian) state and let primed quantities refer to those of the point in the particle. This leads to

$$\begin{aligned}
&\rho^o \Delta V = \int_{\Delta V} (\rho^o)' dV' && \rho \Delta v = \int_{\Delta v} \rho' dv' \\
&\int_{\Delta V} (\rho^o)' \Xi \, dV' = 0 && \int_{\Delta v} \rho' \boldsymbol{\xi} dv' = 0 \\
&\rho^o I_{KL} \Delta V \equiv \int_{\Delta V} (\rho^o)' \Xi_K \Xi_L dV' && \rho i_{kl} \Delta v \equiv \int_{\Delta v} \rho' \xi_k \xi_l dv'
\end{aligned} \tag{4C-1.13}$$

where I_{KL} and i_{kl} are the microinertia of the deformable point particle in the Lagrangian (undeformed) state and Eulerian (deformed) state, respectively.

4C-2 Balance Laws of Micromorphic Theory

Eringen and Suhubi (1964a, 1964b) and Eringen (1964) derived the laws of conservation of mass, conservation of microinertia, balance of linear momentum, balance of momentum moments, and conservation of energy for micromorphic theory by means of a *microscopic space-averaging process.* Later Eringen (1999) derived the balance laws in a more elegant way: Balances of linear momentum and momentum moments are the consequences of the objectivity of conservation of energy. The balance laws of micromorphic theory, including the Clausius–Duhem inequality, can be expressed as

$$\frac{d\rho}{dt} + \rho v_{k,k} = 0 \tag{4C-2.1}$$

$$\frac{di_{kl}}{dt} = \phi_{kl} + \phi_{lk} \tag{4C-2.2}$$

$$\rho \frac{dv_i}{dt} = \sigma_{ji,j} + B_i \tag{4C-2.3}$$

$$\rho \frac{d\phi_{ij}}{dt} = m_{kij,k} + \sigma_{ji} - s_{ij} + \rho i_{mn} \omega_{im} \omega_{jn} + L_{ij} \tag{4C-2.4}$$

$$\rho \frac{de}{dt} = m_{ijk} \omega_{jk,i} + \sigma_{ij}(v_{j,i} - \omega_{ji}) + s_{ij}\omega_{ij} - q_{i,i} + h \tag{4C-2.5}$$

$$-\rho(\dot{\psi} + \eta\dot{\theta}) + m_{ijk}\omega_{jk,i} + \sigma_{ij}(v_{j,i} - \omega_{ji}) + s_{ij}\omega_{ij} - \frac{q_i \theta_{,k}}{\theta} \geq 0 \tag{4C-2.6}$$

where $\mathbf{v}$ is the velocity vector; $\boldsymbol{\sigma}$ is the Cauchy stress; $\mathbf{s} = \mathbf{s}^T$ is the microstress; e is the internal energy density; $\mathbf{q}$ is the heat flux; h is the heat source; and the generalized spin tensor is defined as

$$\boldsymbol{\varphi} \equiv \boldsymbol{\omega} \cdot \mathbf{i} \quad \text{or} \quad \phi_{kl} \equiv \omega_{km} i_{ml} \tag{4C-2.7}$$

the body-force density and the body-couple density are defined as

$$\mathbf{B}\,\Delta v \equiv \int_{\Delta v} \mathbf{B}' dv'$$
$$\mathbf{L}\,\Delta v \equiv \int_{\Delta v} \mathbf{B}' \otimes \boldsymbol{\xi} dv' \tag{4C-2.8}$$

the moment stress, a third-order tensor, is defined as

$$m_{kij}\Delta a_k \equiv \int_{\Delta a} \sigma'_{ki}\xi_j da'_k \tag{4C-2.9}$$

with da'_k being the differential surface area with outward normal n_k. It is noticed that in micromorphic theory the Cauchy stress is not symmetric, that is, $\sigma_{kl} \neq \sigma_{lk}$. If the size of the point particle is reduced to zero, that is, $\|\boldsymbol{\xi}\| \to 0$, then it leads to $\mathbf{i} = \boldsymbol{\omega} = \boldsymbol{\varphi} = \mathbf{m} = \mathbf{L} = 0$. This means Eqs. (4C-2.1) and (4C-2.3) remain unchanged; Eq. (4C-15) becomes a trial statement, that is, $0 = 0$; Eqs. (4C-2.4) and (4C-2.5) are reduced to

$$\sigma_{ji} = s_{ij} \Rightarrow \sigma_{ij} = \sigma_{ji} \tag{4C-2.10}$$

$$\rho \frac{de}{dt} = \sigma_{ij} v_{j,i} - q_{i,i} + h \tag{4C-2.11}$$

Of course, under this limiting situation (size of the point particle is vanishing), micromorphic theory is identical to classical continuum theory.

4C-3 Constitutive Equations of Micromorphic Elastic Solid

First, we recall the Cauchy strain tensor as defined in Eq. (2-4.6) and rewrite it in the notation used in Appendix 4C as

$$C_{KL} \equiv x_{k,K} x_{k,L}$$

In micromorphic theory, we need three strain measures, which may be named as the generalized Lagrangian strain tensors (Eringen, 1999; Lee et al., 2004):

$$\begin{aligned} \alpha_{KL} &\equiv x_{k,K}\overline{\chi}_{Lk} - \delta_{KL} \\ \beta_{KL} &\equiv \chi_{kK}\chi_{kL} - \delta_{KL} \\ \gamma_{KLM} &\equiv \overline{\chi}_{Kk}\chi_{kL,M} \end{aligned} \tag{4C-3.1}$$

One may verify that the strain rates can be obtained as

$$\begin{aligned} \dot{\alpha}_{KL} &= (v_{l,k} - \omega_{lk}) x_{k,K}\overline{\chi}_{Ll} \equiv a_{kl} x_{k,K}\overline{\chi}_{Ll} \\ \dot{\beta}_{KL} &= (\omega_{kl} + \omega_{lk})\chi_{kK}\chi_{lL} \equiv b_{kl}\chi_{kK}\chi_{lL} = \dot{\beta}_{LK} \\ \dot{\gamma}_{KLM} &= \omega_{kl,m}\overline{\chi}_{Kk}\chi_{lL}x_{m,M} \equiv c_{klm}\overline{\chi}_{Kk}\chi_{lL}x_{m,M} \end{aligned} \tag{4C-3.2}$$

A micromorphic elastic solid is defined as a micromorphic material of which the dependent constitutive variables $\{\mathbf{m}, \boldsymbol{\sigma}, \mathbf{s}, \mathbf{q}, \psi, \eta\}$ are functions of independent constitutive variables $\{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \theta, \nabla\theta\}$, where η is the entropy density, θ is the absolute temperature, and $\psi \equiv e - \eta\theta$ is the Helmholtz's free-energy density. Then the constitutive equations for micromorphic elastic solid can be derived to be (Eringen, 1999; Lee et al., 2004)

$$\psi = \psi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \theta) \tag{4C-3.3}$$

$$\eta = -\frac{\partial\psi}{\partial\theta} \tag{4C-3.4}$$

$$\sigma_{kl} = \rho\frac{\partial\psi}{\partial\alpha_{KL}}x_{k,K}\overline{\chi}_{Ll} \tag{4C-3.5}$$

$$s_{kl} = 2\rho\frac{\partial\psi}{\partial\beta_{KL}}\chi_{kK}\chi_{lL} \tag{4C-3.6}$$

$$m_{klm} = \rho\frac{\partial\psi}{\partial\gamma_{LMK}}x_{k,K}\overline{\chi}_{Ll}\chi_{mM} \tag{4C-3.7}$$

$$-q_k\theta_{,k} \geq 0 \tag{4C-3.8}$$

If, for whatever reason, we consider that the temperature gradient, $\nabla\theta$, is not in the list of the independent constitutive variables, then, from Eq. (4C-3.8), we have to conclude that $\mathbf{q} = 0$, which implies that the material considered is an insulator.

It is seen that (1) although we started with $\psi = \psi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \theta, \nabla\theta)$ as a constitutive equation, it ends up with $\psi = \psi(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \theta)$, that is, the Helmholtz's free energy density will not depend on temperature gradient; (2) Eq. (4C-3.4) says the entropy density is derivable from the Helmholtz's free energy density—this is one of the Gibbs equations in thermodynamics; (3) the three stress tensors are also derivable from the Helmholtz's free energy density—it can be viewed as a generalization of the stress–strain relation in Eq. (4-3.1) or in Eq. (4-3.2); (4) Eq. (4C-3.8) actually is an inequality that stipulates that heat flows from the high-temperature region to the low-temperature region—a conclusion drawn from the second law of thermodynamics.

One may derive constitutive equations for micromorphic electromagnetic solids and fluids, anisotropic fluids and suspensions, liquid crystals, blood, micromorphic theory of turbulence, micromorphic thermoplasticity, and so forth. (Eringen 1999, 2001, 2002; Chen and Lee 2003; Lee and Chen 2004; Lee et al., 2004).

APPENDIX 4D ATOMISTIC FIELD THEORY

In this appendix, we are going to introduce an atomistic field theory that bridges the gap between continuum and atomistic descriptions of mechanics of materials and is capable of describing the dynamics features of *multicomponent crystalline systems*, that is, crystalline systems that have more than one kind of atom in the primitive unit cell (Chen and Lee, 2005; Chen, 2006, 2009).

4D-1 Introduction

In recent years, intensive research effort in physics, chemistry, and engineering has focused on the development of coupled atomistic/continuum material modeling from the nanoscale to the macroscale. To date, however, progress has been very slow. The most formidable challenge is interfacing molecular dynamics (MD) with continuum mechanics (Cai et al., 2000; E and Huang, 2001; E et al., 2003; To and Li, 2005; Li et al., 2006). Fundamentally, this is because continuum mechanics cannot represent the dynamics of material systems at atomistic scale. Elastic distortions give rise to both acoustic and optical waves with dispersive *frequency–wave vector relations* in the atomistic region. Across this interface, however, only the acoustic waves with nondispersive *frequency-wave vector relations* can exist in the continuum mechanics region. This mismatch in phonon representation gives rise to unphysical phonon scattering/wave reflections, which in turn fundamentally alter the dynamic behavior of materials, rendering most of the current multiscale methods powerless in simulations of dynamic material behavior. It is clear that overcoming this barrier requires fundamental understanding of the connection between atomistic and continuum representations at the atomic length/time scale.

Historically, statistical mechanics has provided a theoretical link between atomistic models and macroscopic continuum mechanics. Assuming a single-component single-phase fluid system consisting of molecules interacting under central forces, conservation equations were derived by means of classical statistical mechanics, and expressions were determined for stress tensor and heat flux vector in terms of molecular variables (Irving and Kirkwood, 1950).

The past decades have seen an exponential growth of interest in linking atomistic models to continuum mechanics (Hardy, 1982; Lutsko, 1988; Zhou and McDowell, 2002; Zimmerman et al., 2004; Delph, 2005). Among these efforts, it is worthwhile to mention the formalism developed by Hardy: Analytical formulas for local properties and balance equations at atomic scale were obtained as exact consequences of Newton's equations of motion with neither an ensemble average nor a time average being required. However, the above-mentioned works can only describe the dynamics features of *single-component systems*, that is, crystalline systems that have only one atom per primitive unit cell.

4D-2 Phase-Space and Physical-Space Descriptions

From the lattice dynamics point of view, classical continuum mechanics is the long-wavelength limit of *monatomic* lattices, since the classical continuum mechanics only reproduces acoustic waves, while elastic distortion gives rise to wave propagation of both acoustic and optical types in any *multiatomic* lattices. Therefore, to link an atomistic description with continuum field description of general material systems at atomic length and time scales, we need an integrated theory, not just a numerical procedure to treat the interface between atomic region and continuum region.

Microscopic dynamic quantities in classical N-body dynamics are functions of phase-space coordinates $(\mathbf{r}, \mathbf{p})$, that is, the positions and momenta of atoms. Any single crystal can be treated as a multicomponent system (or multielement system). The phase-space coordinates are defined as

$$\mathbf{r} = \{\mathbf{R}^{k\alpha} = \mathbf{R}^k + \Delta\mathbf{r}^{k\alpha} | k = 1, 2, 3, \ldots, N_l; \alpha = 1, 2, 3, \ldots, N_a\} \tag{4D-2.1}$$

$$\mathbf{p} = \{m^\alpha \mathbf{V}^{k\alpha} = m^\alpha(\mathbf{V}^k + \Delta\mathbf{v}^{k\alpha}) | k = 1, 2, 3, \ldots, N_l; \alpha = 1, 2, 3, \ldots, N_a\} \tag{4D-2.2}$$

where the superscript $k\alpha$ refers to the αth atom in the kth unit cell; m^α is the mass of the αth atom; $\mathbf{R}^{k\alpha}$ and $\mathbf{V}^{k\alpha}$ are the position and velocity vector of the $k\alpha$ atom, respectively; $\mathbf{R}^k$ and $\mathbf{V}^k$are the position and velocity of the mass center of the kth unit cell, respectively; $\Delta\mathbf{r}^{k\alpha}$ and $\Delta\mathbf{v}^{k\alpha}$ are the atomic position and velocity of the αth atom relative to the mass center of the kth unit cell, respectively; N_l is the total number of unit cells in the system; N_a is the number of atoms in a unit cell.

Note that a crystalline material can be viewed as a collection of lattice cells and a group of discrete and distinct atoms embedded in each lattice cell, depicted in Figs. 4D-2.1 and 4D-2.2. For physical space coordinates, distinguishing from the standard treatment in statistical mechanics, let $\mathbf{x}$ represent the coordinate of the continuously distributed lattice point, $\mathbf{y}^\alpha$ the coordinate of the discrete αth atom relative to $\mathbf{x}$. Thus, the link between a dynamic function in phase space and its corresponding density function in the physical space can be established through a localization function δ and a Kronecker delta function $\tilde{\delta}$ as (Chen, 2009)

$$\mathbf{a}(\mathbf{x}, \mathbf{y}^\alpha, t) = \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \mathbf{A}[\mathbf{r}(t), \mathbf{p}(t)]\delta(\mathbf{R}^k - \mathbf{x})\tilde{\delta}(\Delta\mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \tag{4D-2.3}$$

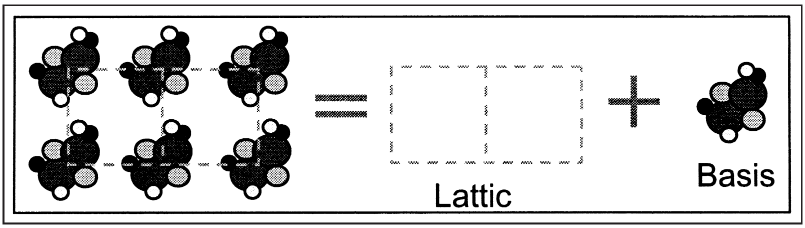

Figure 4D-2.1

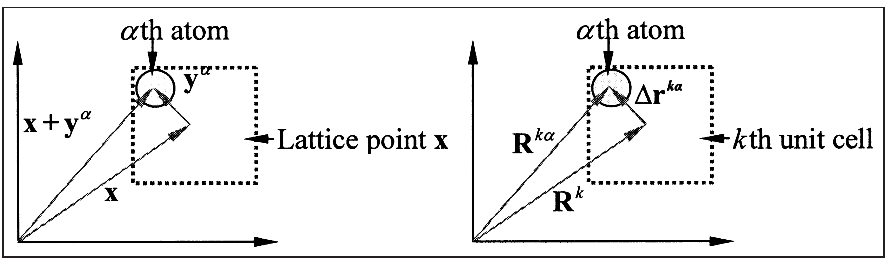

Figure 4D-2.2

with the following normalization conditions:

$$\int_V \delta(\mathbf{R}^k - \mathbf{x})d\mathbf{x} = 1 \tag{4D-2.4}$$

$$\tilde{\delta}(\Delta\mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \equiv \int_{V(\mathbf{y}^\alpha)} \delta(\Delta\mathbf{r}^{k\xi} - \mathbf{y})d\mathbf{y} \equiv \begin{cases} 1 & \text{if } \xi = \alpha \text{ and } \Delta\mathbf{r}^{k\alpha} = \mathbf{y}^\alpha \\ 0 & \text{otherwise} \end{cases} \tag{4D-2.5}$$

This density function $\mathbf{a}(\mathbf{x}, \mathbf{y}^\alpha, t)$, in Eq. (4D-2.3), defines an atomic-scale local density of $\mathbf{A}[\mathbf{r}(t), \mathbf{p}(t)]$. It is a smooth and continuous function in $\mathbf{x}$, while discontinuous in $\alpha, \alpha = 1, 2, 3, \ldots, N_a$, with $\mathbf{y}^\alpha$ being an subscale internal variable. Similarly one may define a local density function $\mathbf{b}(\mathbf{x}, t)$ on cell level:

$$\mathbf{b}(\mathbf{x}, t) = \sum_{k=1}^{N_l} \mathbf{B}(\mathbf{r}, \mathbf{p})\delta(\mathbf{R}^k - \mathbf{x}) \tag{4D-2.6}$$

With the sifting properties of localization function δ and Kronecker delta function $\tilde{\delta}$ (McLennan, 1989), the time evolution law of physical quantities can be obtained as (Chen, 2006, 2009)

$$\begin{aligned} \left.\frac{\partial \mathbf{a}(\mathbf{x}, \mathbf{y}^\alpha, t)}{\partial t}\right|_{\mathbf{x}, \mathbf{y}^\alpha} &= \sum_{k=1}^{N_l}\sum_{\xi=1}^{N_a} \dot{\mathbf{A}}(\mathbf{r}, \mathbf{p})\delta(\mathbf{R}^k - \mathbf{x})\tilde{\delta}(\Delta\mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \\ &- \nabla_{\mathbf{x}} \cdot \left(\sum_{k=1}^{N_l}\sum_{\xi=1}^{N_a} \mathbf{V}^k \otimes \mathbf{A}(\mathbf{r}, \mathbf{p})\delta(\mathbf{R}^k - \mathbf{x})\tilde{\delta}(\Delta\mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \right) \\ &- \nabla_{\mathbf{y}^\alpha} \cdot \left(\sum_{k=1}^{N_l}\sum_{\xi=1}^{N_a} \Delta\mathbf{v}^{k\xi} \otimes \mathbf{A}(\mathbf{r}, \mathbf{p})\delta(\mathbf{R}^k - \mathbf{x})\tilde{\delta}(\Delta\mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \right) \end{aligned} \tag{4D-2.7}$$

4D-3 Definitions of Atomistic Quantities in Physical Space

Following the pattern of Eq. (4D-2.3), mass density ρ^α, linear momentum density $\rho^\alpha(\mathbf{v} + \Delta\mathbf{v}^\alpha)$, angular momentum density $\rho^\alpha \boldsymbol{\phi}^\alpha$, total energy density $\rho^\alpha E^\alpha$, internal energy density $\rho^\alpha e^\alpha$, interatomic force density $\mathbf{f}^\alpha$, external force density $\mathbf{f}^\alpha_{ext}$, homogeneous part and inhomogeneous part of stress tensor ($\mathbf{t}^\alpha$ and $\boldsymbol{\tau}^\alpha$), homogeneous part and inhomogeneous part of heat flux ($\mathbf{q}^\alpha$ and $\mathbf{j}^\alpha$), and heat source h^α are defined as

$$\rho^\alpha = \sum_{k=1}^{N_l}\sum_{\xi=1}^{N_a} m^\xi \delta(\mathbf{R}^k - \mathbf{x})\tilde{\delta}(\Delta\mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \tag{4D-3.1}$$

$$\rho^\alpha(\mathbf{v} + \Delta\mathbf{v}^\alpha) = \sum_{k=1}^{N_l}\sum_{\xi=1}^{N_a} m^\xi (\mathbf{V}^k + \Delta\mathbf{v}^{k\xi})\delta(\mathbf{R}^k - \mathbf{x})\tilde{\delta}(\Delta\mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \tag{4D-3.2}$$

$$\rho^\alpha \boldsymbol{\varphi}^\alpha = \rho^\alpha (\mathbf{v} + \Delta \mathbf{v}^\alpha) \times (\mathbf{x} + \mathbf{y}^\alpha)$$
$$= \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \left(m^\xi \mathbf{V}^{k\xi} \times \mathbf{R}^{k\xi} \delta(\mathbf{R}^{\mathbf{k}} - \mathbf{x}) \delta(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha \right) \tag{4D-3.3}$$

$$\rho^\alpha E^\alpha = \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \left[\tfrac{1}{2} m^\xi \left(\mathbf{V}^{k\xi}\right)^2 + U^{k\xi} \right] \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \tag{4D-3.4}$$

$$\rho^\alpha e^\alpha = \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \left[\tfrac{1}{2} m^\xi \left(\tilde{\mathbf{V}}^{k\xi}\right)^2 + U^{k\xi} \right] \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \tag{4D-3.5}$$

$$\mathbf{f}^\alpha = \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \mathbf{F}^{k\xi} \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \tag{4D-3.6}$$

$$\mathbf{f}^\alpha_{\text{ext}} = \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \mathbf{F}^{k\xi}_{\text{ext}} \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \tag{4D-3.7}$$

$$\mathbf{t}^\alpha = - \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} m^\xi \tilde{\mathbf{V}}^k \otimes \tilde{\mathbf{V}}^{k\xi} \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha)$$
$$- \frac{1}{2} \sum_{k,l=1}^{N_l} \sum_{\xi,\eta=1}^{N_a} \left(\mathbf{R}^k - \mathbf{R}^l\right) \otimes \mathbf{F}^{k\xi} B(k, \xi, l, \eta, \mathbf{x}, \mathbf{y}^\alpha) \tag{4D-3.8}$$

$$\boldsymbol{\tau}^\alpha = - \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} m^\xi \Delta \tilde{\mathbf{v}}^{k\xi} \otimes \tilde{\mathbf{V}}^{k\xi} \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha)$$
$$- \frac{1}{2} \sum_{k,l=1}^{N_l} \sum_{\xi,\eta=1}^{N_a} \left(\Delta \mathbf{r}^{k\xi} - \Delta \mathbf{r}^{l\eta}\right) \otimes \mathbf{F}^{k\xi} B(k, \xi, l, \eta, \mathbf{x}, \mathbf{y}^\alpha) \tag{4D-3.9}$$

$$\mathbf{q}^\alpha = - \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \tilde{\mathbf{V}}^k \left[\tfrac{1}{2} m^\xi \left(\tilde{\mathbf{V}}^{k\xi}\right)^2 + U^{k\xi} \right] \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha)$$
$$- \frac{1}{2} \sum_{k,l=1}^{N_l} \sum_{\xi,\eta=1}^{N_a} \left(\mathbf{R}^k - \mathbf{R}^l\right) \tilde{\mathbf{V}}^{k\xi} \cdot \mathbf{F}^{k\xi} B(k, \xi, l, \eta, \mathbf{x}, \mathbf{y}^\alpha) \tag{4D-3.10}$$

$$\mathbf{j}^\alpha = - \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \Delta \tilde{\mathbf{v}}^{k\xi} \left[\tfrac{1}{2} m^\xi \left(\tilde{\mathbf{V}}^{k\xi}\right)^2 + U^{k\xi} \right] \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha)$$
$$- \frac{1}{2} \sum_{k,l=1}^{N_l} \sum_{\xi,\eta=1}^{N_a} \left(\Delta \mathbf{r}^{k\xi} - \Delta \mathbf{r}^{l\eta}\right) \tilde{\mathbf{V}}^{k\xi} \cdot \mathbf{F}^{k\xi} B(k, \xi, l, \eta, \mathbf{x}, \mathbf{y}^\alpha) \tag{4D-3.11}$$

$$h^\alpha = \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} \tilde{\mathbf{V}}^{k\xi} \cdot \mathbf{F}^{k\xi}_{ext} \delta(\mathbf{R}^k - \mathbf{x}) \tilde{\delta}(\Delta \mathbf{r}^{k\xi} - \mathbf{y}^\alpha) \tag{4D-3.12}$$

where

$$B(k, l, \xi, \eta, \mathbf{x}, \mathbf{y}^\alpha) \equiv \int_0^1 \delta(\mathbf{R}^k\lambda + \mathbf{R}^l(1-\lambda) - \mathbf{x})\tilde{\delta}(\Delta\mathbf{r}^{k\xi}\lambda + \Delta\mathbf{r}^{l\eta}(1-\lambda) - \mathbf{y}^\alpha)\, d\lambda \tag{4D-3.13}$$

the differences between phase-space velocities and physical-space velocities are defined as

$$\begin{aligned} \tilde{\mathbf{V}}^{k\xi} &\equiv \mathbf{V}^{k\xi} - (\mathbf{v} + \Delta\mathbf{v}^\xi) \\ \tilde{\mathbf{V}}^k &\equiv \mathbf{V}^k - \mathbf{v} \\ \Delta\tilde{\mathbf{v}}^{k\xi} &\equiv \Delta\mathbf{v}^{k\xi} - \Delta\mathbf{v}^\xi \end{aligned} \tag{4D-3.14}$$

$\mathbf{F}^{k\xi}_{\text{ext}}$ is the body force, such as gravitational force and Lorentz force, acting on the $k\xi$ atom; $U^{k\xi}$ is the potential energy of the $k\xi$ atom. Notice that the potential energy is additive; therefore, the total potential energy of the system is obtained as

$$U = \sum_{k=1}^{N_l} \sum_{\xi=1}^{N_a} U^{k\xi} \tag{4D-3.15}$$

and the interatomic force acting on the $k\xi$ atom is calculated through

$$\mathbf{F}^{k\xi} = -\frac{\partial U}{\partial \mathbf{R}^{k\xi}} \tag{4D-3.16}$$

4D-4 Conservation Equations

Based on Eq. (4D-2.7), a lengthy but straightforward process leads to the local conservation laws of mass, linear momentum, angular momentum, and energy for each atom $\alpha \in [1, 2, 3, \ldots, N_a]$ at any point in the field $(\mathbf{x}, t)$ as follows (Chen and Lee, 2005):

$$\frac{d\rho^\alpha}{dt} + \rho^\alpha \nabla_{\mathbf{x}} \cdot \mathbf{v} + \rho^\alpha \nabla_{\mathbf{y}^\alpha} \cdot \Delta\mathbf{v}^\alpha = 0 \tag{4D-4.1}$$

$$\rho^\alpha \frac{d(\mathbf{v} + \Delta\mathbf{v}^\alpha)}{dt} = \nabla_{\mathbf{x}} \cdot \mathbf{t}^\alpha + \nabla_{\mathbf{y}^\alpha} \cdot \boldsymbol{\tau}^\alpha + \mathbf{f}^\alpha_{\text{ext}} \tag{4D-4.2}$$

$$\mathbf{t}^\alpha + \boldsymbol{\tau}^\alpha = (\mathbf{t}^\alpha + \boldsymbol{\tau}^\alpha)^T \tag{4D-4.3}$$

$$\rho^\alpha \frac{de^\alpha}{dt} = \mathbf{t}^\alpha : \nabla_{\mathbf{x}}(\mathbf{v} + \Delta\mathbf{v}^\alpha) + \boldsymbol{\tau}^\alpha : \nabla_{\mathbf{y}^\alpha}(\mathbf{v} + \Delta\mathbf{v}^\alpha) + \nabla_{\mathbf{x}} \cdot \mathbf{q}^\alpha + \nabla_{\mathbf{y}^\alpha} \cdot \mathbf{j}^\alpha + h^\alpha \tag{4D-4.4}$$

where

$$\nabla_{\mathbf{x}} \equiv \frac{\partial}{\partial \mathbf{x}} \qquad \nabla_{\mathbf{y}^\alpha} \equiv \frac{\partial}{\partial \mathbf{y}^\alpha} \tag{4D-4.5}$$

and the material time rate of A^α is defined as

$$\frac{dA^\alpha}{dt} \equiv \frac{\partial A^\alpha}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} A^\alpha + \Delta\mathbf{v}^\alpha \cdot \nabla_{\mathbf{y}^\alpha} A^\alpha \tag{4D-4.6}$$

Similar to the situation in classical continuum theory, the balance of angular momentum leads to the symmetry of the total stress tensor $\mathbf{t}^\alpha + \boldsymbol{\tau}^\alpha$. It is seen that, from Eqs. (4D-3.8) and (4D-3.9), the symmetry of the total stress tensor is automatically satisfied.

We may define mass density, microinertia density, linear momentum density, moment of momentum density, and total energy density at cell level as

$$\rho = \sum_{k=1}^{N_l} \sum_{\alpha=1}^{N_a} m^\alpha \delta(\mathbf{R}^k - \mathbf{x}) \equiv \sum_{k=1}^{N_l} m\delta(\mathbf{R}^k - \mathbf{x}) \tag{4D-4.7}$$

$$\rho\mathbf{i} = \sum_{k=1}^{N_l} \sum_{\alpha=1}^{N_a} m^\alpha \Delta\mathbf{r}^{k\alpha} \otimes \Delta\mathbf{r}^{k\alpha} \delta(\mathbf{R}^k - \mathbf{x}) \tag{4D-4.8}$$

$$\rho\mathbf{v} = \sum_{k=1}^{N_l} \sum_{\alpha=1}^{N_a} m^\alpha \mathbf{V}^{k\alpha} \delta(\mathbf{R}^k - \mathbf{x}) = \sum_{k=1}^{N_l} m\mathbf{V}^k \delta(\mathbf{R}^k - \mathbf{x}) \tag{4D-4.9}$$

$$\rho\boldsymbol{\varphi} = \sum_{k=1}^{N_l} \sum_{\alpha=1}^{N_a} m^\alpha \Delta\mathbf{v}^{k\alpha} \otimes \Delta\mathbf{r}^{k\alpha} \delta(\mathbf{R}^k - \mathbf{x}) \tag{4D-4.10}$$

$$\rho E = \sum_{k=1}^{N_l} \left\{ \tfrac{1}{2} m(\mathbf{V}^k)^2 + \sum_{\alpha=1}^{N_a} \left[\tfrac{1}{2} m^\alpha (\Delta\mathbf{v}^{k\alpha})^2 + U^{k\alpha} \right] \right\} \delta(\mathbf{R}^k - \mathbf{x}) \tag{4D-4.11}$$

If one assumes the micromotion is affine as in the micromorphic theory (cf. Appendix 4C), that is,

$$\Delta\mathbf{v}^{k\alpha} = \boldsymbol{\omega}^k \cdot \Delta\mathbf{r}^{k\alpha} \tag{4D-4.12}$$

then one obtains almost the same set of balance laws as in micromorphic theory except [cf. Eqs. (4C-2.2) and (4C-2.4)]:

$$\rho \frac{d\mathbf{i}}{dt} = \rho\boldsymbol{\varphi} + \rho\boldsymbol{\varphi}^T - \nabla_{\mathbf{x}} \cdot \boldsymbol{\gamma} \tag{4D-4.13}$$

$$\rho \frac{d\boldsymbol{\varphi}}{dt} = \nabla_{\mathbf{x}} \cdot \mathbf{m} + \boldsymbol{\omega} \cdot \rho\mathbf{i} \cdot \boldsymbol{\omega}^T + \mathbf{t}^T - \mathbf{s} + \mathbf{L} - \nabla_{\mathbf{x}} \cdot (\boldsymbol{\omega} \cdot \boldsymbol{\gamma}) \tag{4D-4.14}$$

where

$$\boldsymbol{\gamma} = \sum_{k=1}^{N_l} (\mathbf{V}^k - \mathbf{v}) \otimes \sum_{\alpha=1}^{N_a} m^\alpha \Delta\mathbf{r}^{k\alpha} \otimes \Delta\mathbf{r}^{k\alpha} \delta(\mathbf{R}^k - \mathbf{x}) \tag{4D-4.15}$$

It is seen that $\mathbf{V}^k - \mathbf{v} \equiv \tilde{\mathbf{V}}^k$ is the difference between phase-space velocity and physical-space velocity. It is easy to understand why $\boldsymbol{\gamma}$ does not appear in the balance laws of micromorphic theory. Because they were obtained through the *microscopic space-averaging* process, which is different from the *statistical Mechanics* process pioneered by Irving and Kirkwood (1950) and Hardy (1982).

Notice that the balance laws [Eqs. (4D-4.1) to (4D-4.4)] and the constitutive relations [Eqs. (4D-3.5), (4D-3.8) to (4D-3.11), and (4D-3.16)] are obtained through the atomistic formulation, which naturally leads to a generalized continuum field theory. The atomistic field theory is identical to molecular dynamics at atomic scale and can be reduced to classical continuum field theory at macroscopic scale.

A corresponding finite element method to implement this atomistic field theory has been formulated (Lee and Chen 2008; Lee et al., 2009a, 2009b). This theory has been applied to different materials and/or different cases by Lei et al. (2008) and Xiong et al. (2007a, 2007b, 2008, 2009a, 2009b, 2009c).

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CHAPTER 5

PLANE THEORY OF ELASTICITY IN RECTANGULAR CARTESIAN COORDINATES

If a problem of elasticity is reducible to a two-dimensional problem, we say that it is a *plane problem of elasticity*. The corresponding theory is referred to as the *plane theory of elasticity*.

The equations of the plane theory of elasticity apply to the following two cases of equilibrium of elastic bodies, which are of considerable interest in practice: (1) *plane strain* (Section 5-1) and (2) *deformation of a thin plate under forces applied to its boundary and acting in its plane* (Section 5-2).

In the past decade or so, a considerable literature on the application of complex variables to the analytical solution of plane problems has evolved. In fact, the complex-variable method has been developed to the extent that it is currently considered a routine approach to the plane problem of elasticity. However, in many plane problems, the complex-variable method is now being superseded by numerical methods such as finite element methods, which lend themselves to the treatment of difficult boundary value problems in engineering. The complex-variable method has been expounded extensively and authoritatively by Muskhelishvili (1975) and also by Sokolnikoff (1983). Consequently, the method is treated only briefly in this book (see Appendix 5B).

In Appendix 5A we discuss briefly the problem of plane elasticity with couple stresses.

5-1 Plane Strain

The plane strain approximation, which serves to represent a three-dimensional problem by a two-dimensional one, may be applicable to a prismatic body whose length is large compared to its cross-sectional dimensions and which is loaded uniformly along its length. An example of such a body is a long hollow cylinder subjected to lateral pressure. In such bodies the longitudinal displacement component—say, w in the z direction—is often very small compared to the displacement components in the cross section—say, in the (x, y) plane—and under certain conditions may be ignored. A formal definition of plane strain is given below, and the equations of elasticity are simplified accordingly.

For convenience, we employ (x, y, z) notation.

Definition. *A body is in a state of plane strain, parallel to the* (x, y) *plane, if the displacement* w *is zero, and if the components* (u, v) *are functions of* (x, y) *only.*

In view of this definition, the cubical strain for plane strain is

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{5-1.1}$$

Hence, Eqs. (4-6.5) reduce to (isotropic material) (see Table 3-2.1)

$$\begin{aligned} &\sigma_x = \lambda e + 2G\epsilon_x \qquad \sigma_y = \lambda e + 2G\epsilon_y \qquad \sigma_z = \lambda e \\ &\tau_{xy} = G\gamma_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \tau_{xz} = \tau_{yz} = 0 \end{aligned} \tag{5-1.2}$$

Equations (5-1.2) show that the stress components are functions of (x, y) only, because (u, v) hence e, are functions of (x, y) only.

The equilibrium equations for plane strain [see Eqs. (3-8.1)] are

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y &= 0 \\ Z &= 0 \end{aligned} \tag{5-1.3}$$

Consequently, in plane strain with respect to the (x, y) plane, the component of body force perpendicular to the (x, y) plane must vanish. Also, because $\sigma_x, \sigma_y, \tau_{xy}$ are functions of (x, y) only, the components (X, Y) of the body force are independent of z.

The strain–displacement relations [Eqs. (2-15.14)] reduce to the following form for plane strain:

$$\begin{aligned} &\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \epsilon_z = 0 \\ &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \gamma_{yz} = 0 \end{aligned} \tag{5-1.4}$$

Hence, by Eqs. (4-6.6) and (5-1.4),

$$\sigma_z = \nu(\sigma_x + \sigma_y) \tag{5-1.5}$$

Thus, the static equations of elasticity for a body in plane strain with respect to the (x, y) plane reduce to

$$\begin{gathered}\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \\ \sigma_x = \lambda e + 2G\epsilon_x \qquad \sigma_y = \lambda e + 2G\epsilon_y \\ \sigma_z = \lambda e = \nu(\sigma_x + \sigma_y) \qquad \tau_{xy} = G\gamma_{xy}\end{gathered} \tag{5-1.6}$$

In Eqs. (5-1.6) it should be noted that σ_z is deduced from σ_x and σ_y [Eq. (5-1.5)]. Hence, the problem is reduced to determining three stress components $\sigma_x, \sigma_y, \tau_{xy}$.

With Eq. (5-1.5), the stress–strain relations [Eqs. (4-6.8)] may be written in the form

$$\begin{aligned}\epsilon_x &= \frac{1+\nu}{E}[\sigma_x - \nu(\sigma_x + \sigma_y)] \\ \epsilon_y &= \frac{1+\nu}{E}[\sigma_y - \nu(\sigma_x + \sigma_y)] \\ \gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy}\end{aligned} \tag{5-1.7}$$

A state of plane strain can be maintained in a cylindrically shaped body by suitably applied forces. For example, by Eq. (5-1.5), we see that σ_z does not vanish in general. Hence, for a state of plane strain in a cylindrical body with the generators of the body parallel to the z axis, a tension or compression σ_z must be applied over the terminal sections formed by planes perpendicular to the z axis. Thus, the effect of σ_z is to keep constant the length of all longitudinal fibers of the body. In addition, the stress components σ_x and σ_y must attain values on the lateral surface of the body that are consistent with Eqs. (5-1.2) or (5-1.6).

The solution of the plane strain problem of the cylindrical body may be used in conjunction with the auxiliary problem of a cylindrical body subjected to longitudinal terminal forces to solve the problem of deformation of a cylindrical body with terminal sections free of force. If the longitudinal terminal forces are equal in magnitude but opposite in sign to σ_z, the superposition of the results clears the terminal sections of the cylinder of force. However, the resulting deformation of the body is not necessarily a plane deformation. In general, the solution of the auxiliary problem involves the deformation of a cylinder by longitudinal end forces that produce a net axial force and a net couple (pure bending); see Chapter 7.

Example 5-1.1. Plane State of Strain. A region in the (x, y) plane is subjected to a state of plane strain such that the (x, y) displacement components (u, v) are linear functions of (x, y), namely (with $w = 0$ in the z direction)

$$u = a_1 x + b_1 y + c_1 \qquad v = a_2 x + b_2 y + c_2 \qquad w = 0 \tag{a}$$

Measurements indicate that for $x = 0$, $y = 1$ m, $u = -3$ mm, and $v = 2.5$ mm; for $x = 1$ m, $y = 0$, $u = -2$ mm, and $v = 1$ mm; for $x = 1$ m, $y = 1$ m, $u = -5$ mm, and $v = 3.5$ mm; and for $x = y = 0, u = v = 0$. Substitution of these conditions into Eqs. (a) yields the result

$$u = -0.002x - 0.003y \qquad v = 0.001x + 0.0025y \tag{b}$$

With Eqs. (b), Eqs. (2-15.14) in Chapter 2 yield, for small-displacement theory,

$$\begin{aligned}
\epsilon_{11} &= \epsilon_x = \frac{\partial u}{\partial x} = -0.002 \\
\epsilon_{22} &= \epsilon_y = \frac{\partial v}{\partial y} = 0.0025 \\
\epsilon_{12} &= \epsilon_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = -0.001 \\
\epsilon_{33} &= \epsilon_z = 0 \qquad \epsilon_{13} = \epsilon_{xz} = 0 \qquad \epsilon_{23} = \epsilon_{yz} = 0
\end{aligned} \tag{c}$$

With Eqs. (c) and Eqs. (2-9.1) in Chapter 2, strain components relative to any other set of axes (say, X, Y) may be computed. For example, let axes (X, Y) be obtained by a rotation in the (x, y) plane of 30° such that the direction cosines between axes (X, Y) and axes (x, y) are

$$a_{11} = \sqrt{3}/2 \qquad a_{12} = 1/2 \qquad a_{21} = -1/2 \qquad a_{22} = \sqrt{3}/2$$

(see Table 1-24.1 in Chapter 1). Then by Eqs. (c) and (2-9.1), we obtain the strain components E_X, E_Y, E_{XY} relative to axes (X, Y) as (noting that $E_Z = E_{XZ} = E_{YZ} = 0$):

$$\begin{aligned}
E_{11} &= E_X = \epsilon_{11}a_{11}^2 + \epsilon_{22}a_{12}^2 + 2\epsilon_{12}a_{11}a_{12} = -0.001741 \\
E_{22} &= E_Y = \epsilon_{11}a_{21}^2 + \epsilon_{22}a_{22}^2 + 2\epsilon_{12}a_{21}a_{22} = 0.002241 \\
E_{12} &= E_{XY} = \epsilon_{11}a_{11}a_{12} + \epsilon_{22}a_{12}a_{22} + \epsilon_{12}(a_{11}a_{22} + a_{12}a_{21}) = 0.001449
\end{aligned} \tag{d}$$

Note that $J_1 = \epsilon_x + \epsilon_y = E_X + E_Y = 0.0005$ and $J_2 = \epsilon_x\epsilon_y - \epsilon_{xy}^2 = E_X E_Y - E_{XY}^2 = -0.000006$.

Example 5-1.2. Stress–Strain–Strain Energy Density Relations: Plane Strain. The strain energy density for an anisotropic (crystalline) material subjected to a state of plane strain is given by

$$U = \tfrac{1}{2}(C_{11}\epsilon_{11}^2 + C_{22}\epsilon_{22}^2 + 2C_{33}\epsilon_{12}^2 + 2C_{12}\epsilon_{11}\epsilon_{22} + 4C_{13}\epsilon_{11}\epsilon_{12} + 4C_{23}\epsilon_{22}\epsilon_{12}) \tag{a}$$

We wish to determine the stress–strain relations for the material. By Eq. (4-4.21) in Chapter 4, $\sigma_{\alpha\beta} = \partial U/\partial\epsilon_{\alpha\beta}$, where U must be expressed symmetrically in terms of $\epsilon_{12}, \epsilon_{21}$. Thus, let

$$\epsilon_{12} = \tfrac{1}{2}(\epsilon_{12} + \epsilon_{21}) \tag{b}$$

Then substitution of Eq. (b) into Eq. (a) yields

$$U = \tfrac{1}{2}\left[C_{11}\epsilon_{11}^2 + C_{22}\epsilon_{22}^2 + 2C_{33}\left(\frac{\epsilon_{12}+\epsilon_{21}}{2}\right)^2 + 2C_{12}\epsilon_{11}\epsilon_{22} + 4C_{13}\epsilon_{11}\left(\frac{\epsilon_{12}+\epsilon_{21}}{2}\right) + 4C_{23}\epsilon_{22}\left(\frac{\epsilon_{12}+\epsilon_{21}}{2}\right)\right] \quad \text{(c)}$$

Then

$$\sigma_{11} = \frac{\partial U}{\partial \epsilon_{11}} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + C_{13}(\epsilon_{12}+\epsilon_{21})$$

or, as $\epsilon_{12} = \epsilon_{21}$,

$$\sigma_{11} = C_{11}\epsilon_{11} + C_{12}\epsilon_{22} + 2C_{13}\epsilon_{12} \quad \text{(d)}$$

Similarly,

$$\begin{aligned}\sigma_{22} &= \frac{\partial U}{\partial \epsilon_{22}} = C_{12}\epsilon_{11} + C_{22}\epsilon_{22} + 2C_{23}\epsilon_{12} \\ \sigma_{21} = \sigma_{12} &= \frac{\partial U}{\partial \epsilon_{12}} = C_{13}\epsilon_{11} + C_{23}\epsilon_{22} + C_{33}\epsilon_{12}\end{aligned} \quad \text{(e)}$$

Example 5-1.3. Integration of Plane Strain–Displacement Relations. The plane strain–displacement relations as given by Eqs. (5-1.4) are

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \text{(a)}$$

where $u = u(x, y)$ and $v = v(x, y)$. Elimination of (u, v) from Eqs. (a) yields the strain compatibility relationship for plane strain

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \text{(b)}$$

In a plane strain problem, the strain components were determined as

$$\epsilon_x = Ax^2 + By^2 \qquad \epsilon_y = -Bx^2 - Ay^2 \qquad \gamma_{xy} = 0 \quad \text{(c)}$$

Substitution of Eqs. (c) into Eq. (b) shows that the strain components are compatible. By Eqs. (a) and (c),

$$\epsilon_x = \frac{\partial u}{\partial x} = Ax^2 + By^2 \qquad \epsilon_y = \frac{\partial v}{\partial y} = -Bx^2 - Ay^2 \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \quad \text{(d)}$$

Integration of the first two of Eqs. (d) yields

$$u = \frac{1}{3}Ax^3 + Bxy^2 + f_1(y) \qquad v = -Bx^2y - \frac{1}{3}Ay^3 + f_2(x) \tag{e}$$

where $f_1(y)$, $f_2(x)$ are y and x functions of integration, respectively.

Substitution of Eqs. (e) into the third of Eqs. (d) yields $f_1'(y) + f_2'(x) = 0$, or

$$f_1'(y) = C \qquad f_2'(x) = -C \tag{f}$$

where C is a constant. Hence, integration yields

$$f_1(y) = Cy + D \qquad f_2(x) = -Cx + F \tag{g}$$

where C, D, and F are constants of integration that must be determined by specification of the rigid-body displacement (Section 2-15 in Chapter 2). Equations (e) and (g) yield the displacement components

$$\begin{aligned} u &= \tfrac{1}{3}Ax^3 + Bxy^2 + Cy + D \\ v &= -Bx^2y - \tfrac{1}{3}Ay^3 - Cx + F \end{aligned} \tag{h}$$

Problem Set 5-1

1. The two-dimensional body $OABC$ is held between two rigid frictionless walls as shown in Fig. P5-1.1. The region under the body is filled with a fluid at uniform pressure p. What are the boundary conditions required to solve for the stresses in body $OABC$? Neglect gravity.

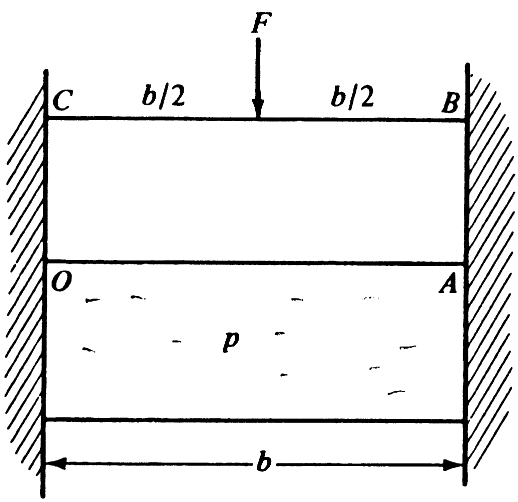


Figure P5-1.1

2. For a state of plane strain, $\sigma_x = f(y)$. Neglecting body forces, derive the most general equations for σ_x, σ_y, σ_z, and τ_{xy}.

3. The strain energy density U of a linearly elastic material is given by the relation $U = (\frac{1}{2}\lambda + G)J_1^2 - 2GJ_2$, where (λ, G) are the Lamé elastic constants and (J_1, J_2) are the first and second strain invariants.
   Employing the relationship between U and the stress components σ_α, derive the stress–strain relations for a state of plane strain relative to the (x, y) plane.

4. For a state of plane strain in an isotropic body, $\sigma_x = ay^2, \sigma_y = -ax^2$, and $\tau_{xy} = 0$. The body forces and temperature are zero. Using small-displacement elasticity theory, compute the displacement components $u(x, y)$ and $v(x, y)$ (a is a constant). (See also Section 5-6.)

5. For a state of plane strain in an isotropic body,

$$\sigma_x = ay^2 + bx \qquad \sigma_y = -ax^2 + by \qquad \tau_{xy} = -b(x + y)$$

   The body forces and temperature are zero. Using small-displacement elasticity theory, compute the displacement components $u(x, y)$ and $v(x, y)$ (a and b are constants). (See Section 5-6.)

6. Consider a rectangular region in the (x, y) plane subjected to a uniform stress σ in the x direction along the edges parallel to the y axis. The (x, y) axes have origin at the center of the region.
   (a) For an isotropic, homogeneous elastic material in this region, derive expressions for the (x, y) displacement components (u, v) in terms of (x, y) and arbitrary constants of integration by the theory of elasticity.
   (b) Employ appropriate conditions at the origin ($x = y = 0$) to eliminate rigid-body displacements of the region and evaluate the arbitrary constants of integration.
   (c) By elementary means of mechanics of materials, derive the displacement components and show that the results obtained in part (b) agree with these results.

5-2 Generalized Plane Stress

As described in Section 3-7 in Chapter 3, for certain kinds of loading, the equations of plane theory of elasticity apply to thin plates. We define a thin plate to be a prismatic member (e.g., a cylinder) of very small length or *thickness h.* The middle surface of the plate, located halfway between its ends and parallel to them, is taken as the (x, y) plane (see Fig. 5-2.1).

We assume that the faces (upper and lower ends) are free from external stresses and that the stresses that act on the edges of the plate are parallel to the faces and are distributed symmetrically with respect to the middle surface. Similar restrictions apply to the body forces. By symmetry, note that points that are originally in the middle surface of the plate lie in the middle surface after deformation. Also, because the plate is assumed thin, the displacement component w is small, and variations of the displacement components (u, v) through the

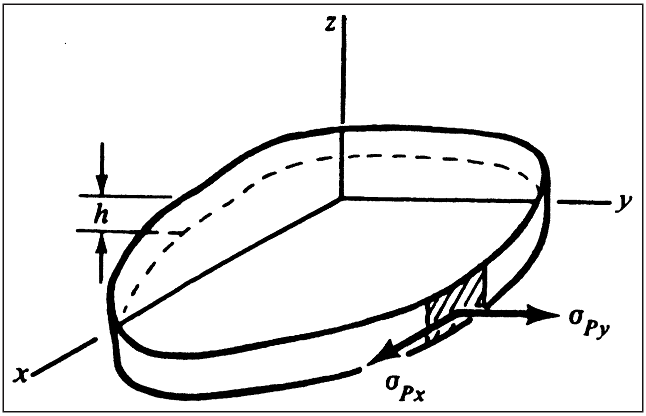


Figure 5-2.1

thickness are small. Consequently, satisfactory results are obtained, if we treat the equilibrium problem of the plate in terms of mean values $\overline{u}, \overline{v}$, and $\overline{w}$ of displacement components (u, v, w) defined as follows:

$$\overline{u}(x, y) = \frac{1}{h}\int_{-h/2}^{h/2} u(x, y, z)\,dz \qquad \overline{v}(x, y) = \frac{1}{h}\int_{-h/2}^{h/2} v(x, y, z)\,dz$$
$$\overline{w}(x, y) = \frac{1}{h}\int_{-h/2}^{h/2} w(x, y, z)\,dz \tag{5-2.1}$$

where bars over letters denote mean values. In turn, substitution of Eqs. (5-2.1) into Eqs. (2-15.14) in Chapter 2 yields mean strains $\overline{\epsilon}_x, \overline{\epsilon}_y, \overline{\epsilon}_z, \overline{\gamma}_{xy}, \overline{\gamma}_{yz}, \overline{\gamma}_{xz}$.

Because it is assumed that $\tau_{xz} = \tau_{yz} = 0$ on the ends, that is, for $z = \pm h/2$ in the absence of body forces, it follows from the last of Eqs. (3-8.1) in Chapter 3 that for $z = \pm h/2$, $\partial\sigma_z/\partial z = 0$. This follows from the fact that because $\tau_{xz} = 0$ for $z = \pm h/2$, $\partial\tau_{xz}/\partial x = 0$ for $z = \pm h/2$, and because $\tau_{yz} = 0$ for $z = \pm h/2$, $\partial\tau_{yz}/\partial y = 0$ for $z = \pm h/2$.

Hence, not only is σ_z zero for $z = \pm h/2$, but also its derivative with respect to z vanishes. Therefore, as the plate is thin, σ_z is small throughout the plate. These observations lead us naturally to the *approximation* that $\sigma_z = 0$ everywhere.

Analogously, we define mean values of stress components $(\sigma_x, \sigma_y, \tau_{xy})$ as follows:

$$\overline{\sigma}_x = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_x\,dz \qquad \overline{\sigma}_y = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_y\,dz$$
$$\overline{\tau}_{xy} = \frac{1}{h}\int_{-h/2}^{h/2} \tau_{xy}\,dz \tag{5-2.2}$$

Accordingly, the mean values of $(\sigma_x, \sigma_y, \tau_{xy})$ are independent of z.

Furthermore,

$$\frac{1}{h}\int_{-h/2}^{h/2}\frac{\partial}{\partial z}(\tau_{xz})\,dz = \frac{1}{h}\tau_{xz}\Big|_{-h/2}^{h/2} = 0$$

$$\frac{1}{h}\int_{-h/2}^{h/2}\frac{\partial}{\partial z}(\tau_{yz})\,dz = \frac{1}{h}\tau_{yz}\Big|_{-h/2}^{h/2} = 0 \tag{5-2.3}$$

Also, mean values of body forces are defined as

$$\overline{X} = \frac{1}{h}\int_{-h/2}^{h/2} X\,dz \qquad \overline{Y} = \frac{1}{h}\int_{-h/2}^{h/2} Y\,dz \qquad \overline{Z} = \frac{1}{h}\int_{-h/2}^{h/2} Z\,dz = 0 \tag{5-2.4}$$

Substitution of Eqs. (5-2.2), (5-2.3), and (5-2.4) into the first two of Eqs. (3-8.1) yields, after integration with respect to z (neglecting acceleration effects),

$$\frac{\partial\overline{\sigma}_x}{\partial x} + \frac{\partial\overline{\tau}_{xy}}{\partial y} + \overline{X} = 0 \qquad \frac{\partial\overline{\tau}_{xy}}{\partial x} + \frac{\partial\overline{\sigma}_y}{\partial y} + \overline{Y} = 0 \tag{5-2.5}$$

From the stress–strain relations [Eqs. (4-6.5) in Chapter 4], it follows from $\sigma_z = \lambda e + 2G\epsilon_z = 0$ that (isotropic material)

$$\epsilon_z = -\frac{\lambda}{\lambda + 2G}(\epsilon_x + \epsilon_y) = -\frac{\nu}{1-\nu}(\epsilon_x + \epsilon_y) \tag{5-2.6}$$

Substituting Eq. (5-2.6) into the first and second of Eqs. (4-6.5), we obtain

$$\sigma_x = \frac{2\lambda G}{\lambda + 2G}(\epsilon_x + \epsilon_y) + 2G\epsilon_x \qquad \sigma_y = \frac{2\lambda G}{\lambda + 2G}(\epsilon_x + \epsilon_y) + 2G\epsilon_y \tag{5-2.7}$$

The fourth of Eqs. (4-6.5) is

$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{5-2.8}$$

Taking the mean values of Eqs. (5-2.7) and (5-2.8), we obtain

$$\begin{aligned}
\overline{\sigma}_x &= \overline{\lambda}(\overline{\epsilon}_x + \epsilon_y) + 2G\overline{\epsilon}_x = \overline{\lambda}\overline{e} + 2G\overline{\epsilon}_x \\
\overline{\sigma}_y &= \overline{\lambda}(\overline{\epsilon}_x + \overline{\epsilon}_y) + 2G\overline{\epsilon}_y = \overline{\lambda}\overline{e} + 2G\overline{\epsilon}_y \\
\overline{\tau}_{xy} &= G\left(\frac{\partial\overline{u}}{\partial y} + \frac{\partial\overline{v}}{\partial x}\right)
\end{aligned} \tag{5-2.9}$$

where

$$\begin{aligned}
\overline{\lambda} &= \frac{2\lambda G}{\lambda + 2G} = \frac{\nu E}{1-\nu^2} \qquad \overline{\epsilon}_x = \frac{1}{h}\int_{-h/2}^{h/2}\epsilon_x\,dz \\
\overline{\epsilon}_y &= \frac{1}{h}\int_{-h/2}^{h/2}\epsilon_y\,dz
\end{aligned} \tag{5-2.10}$$

Comparison of Eqs. (5-2.5) and (5-2.9) with Eqs. (5-1.6) shows that the mean values of displacement components (u, v) and the mean values of the stress components $(\sigma_x, \sigma_y, \tau_{xy})$ satisfy the same equations that govern the case of plane strain, the only difference being that λ is replaced by $\overline{\lambda}$ defined by Eq. (5-2.10). Additionally, the stress components σ_{nx}, σ_{ny} on the boundary of the plate are replaced by their mean values $\overline{\sigma}_{nx}, \overline{\sigma}_{ny}$ [see Eqs. (4-15.1) in Chapter 4].

Taking note of these facts, we may write equations of generalized plane stress without bars over the symbols. We keep in mind that components of stress, strain, and displacement are mean values and that λ is replaced by

$$\overline{\lambda} = \frac{2\lambda G}{\lambda + 2G} = \frac{\nu E}{1 - \nu^2}$$

Thus, we see that for plane strain and for generalized plane stress, we are led to the study of the following system of equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \tag{5-2.11}$$

$$\sigma_x = \lambda e + 2G\epsilon_x \qquad \sigma_y = \lambda e + 2G\epsilon_y \qquad \tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = G\gamma_{xy} \tag{5-2.12}$$

where

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \tag{5-2.13}$$

Equations (5-2.11) may be written entirely in terms of strain components by substitution of Eqs. (5-2.12) into Eqs. (5-2.11). Equations (5-2.11) may also be written entirely in terms of displacement components by substitution of Eqs. (5-2.13) and (2-15.14) into Eqs. (5-2.12) and substitution of the result into Eqs. (5-2.11).

A more specialized state of stress, called *plane stress*, is obtained if we set $\sigma_z = \tau_{xz} = \tau_{yz} = Z = 0$ everywhere. Then the equilibrium equations are given by Eqs. (5-1.3).

Although in generalized plane stress, the mean values of the displacement components are independent of z, in a state of plane stress the displacement components (u, v, w) are not, in general, independent of z. In particular, we note that ϵ_z does not vanish and that it is defined by Eq. (5-2.6).

Furthermore, we observe that in a plate, a state of plane stress requires the body forces and the tractions at the edges to be distributed in certain special ways. It does not, however, require tractions on the faces of the plate.

Finally, we also remark that the average values of displacement in any problem of plane stress are the same as if the problem were one of generalized plane stress. Accordingly, the solution of problems of plane stress may be employed to examine effects produced by certain distributions of forces that do not produce plane stress states, as any such problem can be solved by treating it as a plane stress problem and by replacing λ by $\overline{\lambda}$ in the results. For example, this technique may be employed in problems of equilibrium of a thin plate deformed by forces applied in the plane of the plate. Although the actual values of stress and displacement are not determined by this procedure (unless the forces actually produce a state of plane stress), the average values across the thickness of the plate are obtained. Moreover, average values are the usual quantities measured experimentally.

Example 5-2.1. Plane Stress. Consider a plane stress problem relative to the (x, y) plane, that is,

$$\begin{aligned} &\sigma_x = \sigma_x(x, y) \qquad \sigma_y = \sigma_y(x, y) \qquad \tau_{xy} = \tau_{xy}(x, y) \\ &\sigma_z = \tau_{xz} = \tau_{yz} = 0 \end{aligned} \tag{a}$$

The corresponding strain components [Eqs. (4-6.8) or (5-3.6)] for constant temperature T are

$$\begin{aligned} \epsilon_{11} &= \epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \epsilon_{22} = \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ 2\epsilon_{12} &= \gamma_{xy} = \frac{2(1+\nu)}{E}\sigma_{12} = \frac{2(1+\nu)}{E}\tau_{xy} \\ \epsilon_{33} &= \epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \end{aligned} \tag{b}$$

For a particular plane stress problem, it has been found that the (x, y) displacement components (u, v) are given by the equations

$$\begin{aligned} u &= a_1 + a_2 x + a_3 y + a_4 xy \\ v &= b_1 + b_2 x + b_3 y + b_4 xy \end{aligned} \tag{c}$$

where a_i, b_i are constants. We wish to determine the corresponding small-displacement nonzero strain components [Eqs. (b)] and the stress components [Eqs. (a)] as functions of (x, y).

By Eqs. (2-15.4) and (c),

$$\begin{aligned} \epsilon_{11} &= \epsilon_x = \frac{\partial u}{\partial x} = a_2 + a_4 y \qquad \epsilon_{22} = \epsilon_y = \frac{\partial v}{\partial y} = b_3 + b_4 x \\ 2\epsilon_{12} &= \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_3 + b_2 + a_4 x + b_4 y \end{aligned} \tag{d}$$

To determine ϵ_z we need σ_x, σ_y. Hence, substitution of Eqs. (d) into Eqs. (b) and solution for (σ_x, σ_y) yields

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}(a_2 + \nu b_3 + \nu b_4 x + a_4 y) \\ \sigma_y &= \frac{E}{1-\nu^2}(\nu a_2 + b_3 + b_4 x + \nu a_4 y) \end{aligned} \tag{e}$$

Then, by the last two of Eqs. (b) and Eqs. (d) and (e),

$$\begin{aligned} \tau_{xy} &= \frac{E}{2(1+\nu)}(a_3 + b_2 + a_4 x + b_4 y) \\ \epsilon_z &= -\frac{\nu}{1-\nu}(a_2 + b_3 + b_4 x + a_4 y) \end{aligned} \tag{f}$$

Equations (d), (e), and (f) determine the nonzero stress and strain components.

Problem Set 5-2

1. Repeat Problem 5-1.3 for the case of plane stress.
2. A material is isotropic and elastic. Body forces and temperature are zero. All stress components are zero except τ_{xy}. Using small-displacement theory, determine the most general form for τ_{xy}.
3. Consider a plane stress problem relative to the (x, y) plane. At a point P in the (x, y) plane the normal stresses on three planes perpendicular to the (x, y) plane and forming angles 120° relative to each other are $4C, 3C$, and $2C$, respectively, in the counterclockwise direction, with the direction of the stress $4C$ coincident with the positive x axis. Determine the principal stresses at P.
4. The following stress array is proposed as a solution to a certain *equilibrium* problem of a plane body bounded in the region $-L/2 \le x \le L/2$, $-h/2 \le y \le h/2$:

$$\sigma_x = Ay + Bx^2y + Cy^3 \qquad \sigma_y = Dy^3 + Ey + F$$
$$\tau_{xy} = (G + Hy^2)x \qquad \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

   where (x, y, z) are rectangular Cartesian coordinates and $A, B, \ldots, H$ are nonzero constants. Determine the conditions under which this array is a possible equilibrium solution.

   It is proposed that the region be loaded such that $\tau_{xy} = 0$ for $y = \pm h/2$, $\sigma_y = 0$ for $y = h/2$, $\sigma_y = -\sigma$ (σ = constant) for $y = -h/2$, and $\sigma_x = 0$ for $x = \pm L/2$. Determine whether the proposed stress array may satisfy these conditions.

5. A flat plate is in a state of biaxial tension. The principal stresses are σ_x and σ_y (see Fig. P5-2.5). Two electrical strain gages are located as shown. The angle α is given by

$$\cos\alpha = \sqrt{\frac{1}{1+\nu}} \qquad \sin\alpha = \sqrt{\frac{\nu}{1+\nu}}$$

   Assume that the material is linearly elastic and isotropic. Prove that the principal stresses may be read directly (except for a constant factor) as the strains in the direction of the two strain gages 1 and 2.

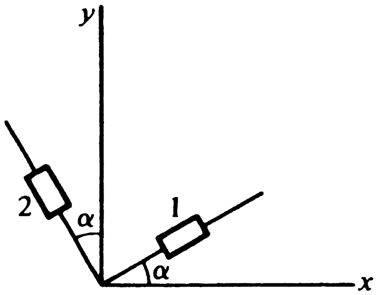


Figure P5-2.5

6. A semi-infinite space is subjected to a uniformly distributed pressure over its entire bounding plane (Fig. P5-2.6). Consider an infinitesimal volume element *ABCD* at some distance from the bounding plane. The normal stress on surface *AB* is $\sigma_y = \sigma$. In terms of the appropriate material properties and σ, derive expressions for the normal stress components σ_x, σ_z that act on the volume element (axis z is perpendicular to the x, y plane). *Hint*: What are the values of the strain components ϵ_x, ϵ_z?

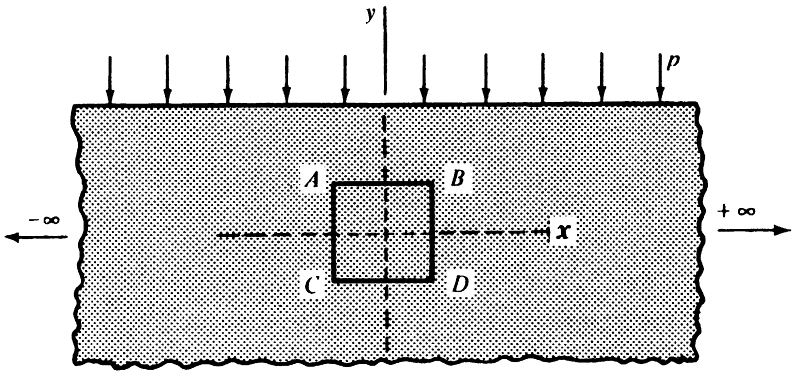


Figure P5-2.6

5-3 Compatibility Equation in Terms of Stress Components

Equations (5-2.11) and one supplementary condition (the compatibility condition), which ensures that there exist two displacement components (u, v) related to the three stress components $(\sigma_x, \sigma_y, \tau_{xy})$ through Eqs. (5-2.12), comprise the equations of plane elasticity. The compatibility equation may be derived from Eqs. (4-14.2) or from Eqs. (2-16.1).

Plane Strain. Consider the state of plane strain. Such a state is defined by the conditions that $\epsilon_x, \epsilon_y, \gamma_{xy}$ are independent of z, and $\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$. Hence, the compatibility conditions reduce to the single equation [Eq. (2-16.1) in Chapter 2]:

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \tag{5-3.1}$$

Also, Eqs. (4-6.8) with Eqs. (5-1.5) become

$$\begin{aligned} \epsilon_x &= \frac{1}{2G}\left[\sigma_x - \frac{\lambda}{2(\lambda + G)}(\sigma_x + \sigma_y)\right] \\ \epsilon_y &= \frac{1}{2G}\left[\sigma_y - \frac{\lambda}{2(\lambda + G)}(\sigma_x + \sigma_y)\right] \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy} \end{aligned} \tag{5-3.2}$$

Substitution of Eqs. (5-3.2) into Eq. (5-3.1) yields

$$\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu \nabla^2(\sigma_x + \sigma_y) = 2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} \tag{a}$$

Equations (5-1.3) yield

$$-2\frac{\partial^2 \tau_{xy}}{\partial x\, \partial y} = \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \tag{b}$$

Substitution of Eq. (b) into Eq. (a) yields after simplification

$$\begin{aligned} \nabla^2(\sigma_x + \sigma_y) &= -\frac{2(\lambda + G)}{\lambda + 2G}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \\ &= -\frac{1}{1 - \nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) \end{aligned} \tag{5-3.3}$$

Equations (5-2.11) and (5-3.3) represent the equations of plane strain. The equations of generalized plane stress are obtained from these equations if mean values of stress and body force are used and if λ is replaced by

$$\overline{\lambda} = \frac{2\lambda G}{\lambda + 2G} = \frac{\nu E}{1 - \nu^2}$$

Generalized Plane Stress. For generalized plane stress, $\sigma_z = 0$. Hence, by the third of Eqs. (4-6.5) in Chapter 4,

$$\epsilon_z = -\frac{\nu}{1 - \nu}(\epsilon_x + \epsilon_y) = -\frac{\nu}{1 - \nu}(u_x + v_y) \tag{5-3.4}$$

By Eqs. (5-3.4), we may eliminate ϵ_z from the first two of Eqs. (4-6.5). Then, on taking mean values, we obtain

$$\overline{\sigma}_x = \overline{\lambda}\overline{e} + 2G\overline{\epsilon}_x \qquad \overline{\sigma}_y = \overline{\lambda}\overline{e} + 2G\overline{\epsilon}_y \qquad \overline{\tau}_{xy} = G\overline{\gamma}_{xy}$$
$$\overline{e} = \overline{\epsilon}_x + \overline{\epsilon}_y \qquad \overline{\lambda} = \frac{2G\nu}{1-\nu} \tag{5-3.5}$$

or, alternatively, in terms of E, ν,

$$\begin{aligned} \overline{\sigma}_x &= \frac{E}{1-\nu^2}(\overline{\epsilon}_x + \nu\overline{\epsilon}_y) \\ \overline{\sigma}_y &= \frac{E}{1-\nu^2}(\overline{\epsilon}_y + \nu\overline{\epsilon}_x) \\ \overline{\tau}_{xy} &= \frac{E}{2(1+\nu)}\overline{\gamma}_{xy} \end{aligned} \tag{5-3.5a}$$

The inverse relations are

$$E\overline{\epsilon}_x = \overline{\sigma}_x - \nu\overline{\sigma}_y \qquad E\overline{\epsilon}_y = \overline{\sigma}_y - \nu\overline{\sigma}_x \qquad G\overline{\gamma}_{xy} = \overline{\tau}_{xy} \tag{5-3.6}$$

The mean strain components evidently satisfy the compatibility conditions [Eqs. (2-16.1) or (5-3.1)]. With Eq. (5-3.6), Eq. (5-3.1) may be expressed in terms of stress as

$$\nabla^2(\overline{\sigma}_x + \overline{\sigma}_y) = -(1+\nu)\left(\frac{\partial \overline{X}}{\partial x} + \frac{\partial \overline{Y}}{\partial y}\right) \tag{5-3.7}$$

In view of the principle of superposition, body forces can be eliminated from consideration if a particular solution is found. We must then solve a problem with no body forces but with altered boundary conditions. For constant body forces or centrifugal body forces, particular solutions are easily found. Consequently, let us consider cases in which body forces are absent. Then the compatibility conditions for generalized plane stress and strain [Eqs. (5-3.7) and (5-3.3)] are identical. The stress–strain relations are the same in both cases, except that $\overline{\lambda}$ replaces λ in problems of generalized plane stress.

In terms of the Airy stress function F (see Section 5-4), the problem, in either case, reduces to the solution $\nabla^2\nabla^2 F = 0$ in the absence of body forces. Furthermore, by the principle of superposition, any solution of an axial stress problem may be superimposed on a plane strain solution. For the general plane orthogonal curvilinear coordinate system the defining equation for F is obtained by specializing the expression for ∇^2 for the plane, that is, setting $h_3 = 1$ and $\partial/\partial w = 0$ [see Section 1-22 and Eq. (1-22.13) in Chapter 1].

Summary of Equations of Plane Elasticity. For convenience we summarize the equations of the plane theory of elasticity for an isotropic, homogeneous material. Also, for completeness we include the effects of body force (X, Y) and temperature T. All quantities are considered to be functions of (x, y) coordinates.

Plane Strain. The stress–strain–temperature relations are

$$
\begin{aligned}
\sigma_x &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_x + \nu\epsilon_y - (1+\nu)kT] \\
\sigma_y &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\epsilon_x + (1-\nu)\epsilon_y - (1+\nu)kT] \\
\tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \\
\sigma_z &= \frac{E}{(1+\nu)(1-2\nu)}[\nu(\epsilon_x + \epsilon_y) - (1+\nu)kT] \\
e &= u_x + v_y = \epsilon_x + \epsilon_y \\
\epsilon_z &= \gamma_{xz} = \gamma_{yz} = \tau_{xz} = \tau_{yz} = 0
\end{aligned}
\tag{5-3.8}
$$

The compatibility relation in terms of stress components is

$$
\nabla^2(\sigma_x + \sigma_y) + \frac{E}{1-\nu}\nabla^2(kT) + \frac{1}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0 \tag{5-3.9}
$$

where E (Young's modulus) and ν (Poisson's ratio) are constants and where

$$
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
$$

Plane Stress. The stress–strain–temperature relations are

$$
\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}[\epsilon_x + \nu\epsilon_y - (1+\nu)kT] \\
\sigma_y &= \frac{E}{1-\nu^2}[\nu\epsilon_x + \epsilon_y - (1+\nu)kT] \\
\tau_{xy} &= G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \\
\epsilon_z &= -\frac{1}{1-\nu}[\nu(\epsilon_x + \epsilon_y) - (1+\nu)kT] \\
e &= \epsilon_x + \epsilon_y + \epsilon_z = \frac{1}{1-\nu}[(1-2\nu)(\epsilon_x + \epsilon_y) + (1+\nu)kT] \\
\sigma_z &= \tau_{xz} = \tau_{yz} = \gamma_{xz} = \gamma_{yz} = 0
\end{aligned}
\tag{5-3.10}
$$

The compatibility relation in terms of stress components is

$$
\nabla^2(\sigma_x + \sigma_y) + E\nabla^2(kT) + (1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0 \tag{5-3.11}
$$

Equations (5-3.9) and (5-3.11), subject to appropriate boundary conditions, constitute the equations from which the sum of stress components σ_x, σ_y is determined.

Mathematically speaking, Eqs. (5-3.9) and (5-3.11) are equivalent, as we may write

$$\nabla^2(\sigma_x + \sigma_y) + K_1\nabla^2(kT) + K_2\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = 0 \tag{5-3.12}$$

where for plane strain $K_1 = E/(1-\nu)$, $K_2 = 1/(1-\nu)$, and for plane stress $K_1 = E$, $K_2 = 1+\nu$. In other words, Eq. (5-3.9) is obtained from Eq. (5-3.11) by the substitutions

$$E \to \frac{E}{1-\nu} \qquad 1+\nu \to \frac{1}{1-\nu} \tag{5-3.13}$$

Accordingly, the mathematical problems of plane strain and plane stress are equivalent.

Example 5-3.1. Compatibility Conditions for Plane Problems. Relation to Three-Dimensional Compatibility Relations. A Caution. In Section 5-3 we noted that the strain compatibility equation may be represented in terms of stress components. In particular, in the absence of body forces and temperature, the compatibility relation for plane problems reduces to [Eqs. (5-3.9) and (5-3.12)]

$$\nabla^2(\sigma_x + \sigma_y) = 0 \tag{a}$$

whereas for the three-dimensional problem, the compatibility relations in terms of stress components are given by Eqs. (4-14.2) in Chapter 4. It is possible that a two-dimensional state of stress may satisfy Eq. (a) but may not satisfy all of Eqs. (4-14.2). For example, a two-dimensional solution for a cantilever beam [see Eq. (b), Example 5-7.1] is given by the stress state

$$\begin{aligned} \sigma_x &= A - 2Bxy \qquad \sigma_y = 0 \\ \tau_{xy} &= -B(c^2 - y^2) \end{aligned} \tag{b}$$

where A, B, and c are constants. Equation (a) is satisfied by Eqs. (b). However, if Eqs. (b) are substituted in Eqs. (4-14.2), it is found that all the equations are satisfied identically except for the equation

$$\nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2 I_1}{\partial x\,\partial y} = 0 \tag{c}$$

Equations (b) and (c) yield the result

$$1 + \nu = 1 \tag{d}$$

Thus, Eq. (d) cannot be satisfied unless Poisson's ratio $\nu = 0$, which is not possible for known materials. Hence, a solution may be compatible in the two-dimensional state but not in the three-dimensional state.

Problem Set 5-3

1. Consider a wedge hanging vertically in a gravity field of acceleration g (Fig. P5-3.1). The following elasticity solution for the stress problem of the wedge is proposed: $\sigma_x = \sigma_y = \tau_{xy} = \tau_{yz} = 0$, $\sigma_z = \frac{1}{2}\rho g z$, and $\tau_{xz} = \frac{1}{2}\rho g x$. Discuss this proposed solution.

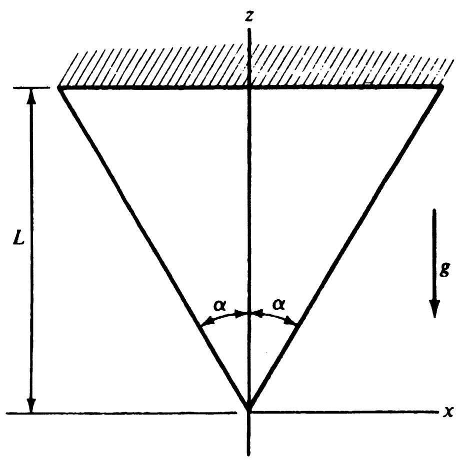


Figure P5-3.1

2. Consider a beam in the region $-h/2 \leq y \leq h/2$, $-b/2 \leq z \leq b/2$, and $0 \leq x \leq L$. Assume plane stress in the (x, y) plane, with zero body forces. The stress component normal to the plane perpendicular to the x axis is $\sigma_x = -My/I$, where $M = M(x)$ is a function of x only, and $I = bh^3/12$. Derive expressions for σ_y and τ_{xy} subject to the boundary conditions $\tau_{xy} = 0$ for $y = \pm h/2$ and $\sigma_y = 0$ for $y = h/2$. What restriction, if any, must be placed on M in order that the derived state of stress be compatible? What can be said about σ_y at $y = -h/2$?

3. Given the following stress state:

$$\sigma_x = C[y^2 + \nu(x^2 - y^2)] \qquad \tau_{xy} = -2C\nu xy$$

$$\sigma_y = C[x^2 + \nu(y^2 - x^2)] \qquad \tau_{yz} = \tau_{xz} = 0$$

$$\sigma_z = C\nu(x^2 + y^2)$$

Discuss the possible reasons for which this stress state may not be a solution of a problem in elasticity.

5-4 Airy Stress Function

Simply Connected Regions. For the plane theory of elasticity, the equilibrium equations [Eqs. (3-8.1) in Chapter 3] reduce to two equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0 \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y = 0 \tag{5-4.1}$$

As noted in Section 5-3, we may initially ignore body forces (X, Y) and seek solutions to Eqs. (5-4.1) modified accordingly. Then the effects of body forces may be superimposed. However, in the case of body forces derivable from a potential function $V(\nabla^2 V = 0)$, such that

$$X = -\frac{\partial V}{\partial x} \qquad Y = -\frac{\partial V}{\partial y} \tag{5-4.2}$$

we may incorporate the effects of body force directly. Thus, Eqs. (5-4.1) and (5-4.2) yield

$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \qquad \frac{\partial \sigma_y'}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \tag{5-4.3}$$

where

$$\sigma_x' = \sigma_x - V \qquad \sigma_y' = \sigma_y - V \tag{5-4.4}$$

Now, for simply connected regions, we note that the first of Eqs. (5-4.3) represents the necessary and sufficient condition that there exist a function $\phi(x, y)$ such that (see Section 1-19 in Chapter 1)

$$\frac{\partial \phi}{\partial y} = \sigma_x' \qquad \frac{\partial \phi}{\partial x} = -\tau_{xy} \tag{5-4.5}$$

The second of Eqs. (5-4.3) represents the necessary and sufficient condition that there exist a function $\theta(x, y)$ such that

$$\frac{\partial \theta}{\partial x} = \sigma_y' \qquad \frac{\partial \theta}{\partial y} = -\tau_{xy} \tag{5-4.6}$$

Comparison of the two expressions for τ_{xy} shows that

$$\frac{\partial \phi}{\partial x} = \frac{\partial \theta}{\partial y} \tag{5-4.7}$$

In turn, Eq. (5-4.7) is the necessary and sufficient condition that there exist a function $F(x, y)$ such that

$$\phi = \frac{\partial F}{\partial y} \qquad \theta = \frac{\partial F}{\partial x} \tag{5-4.8}$$

Substitution of Eq. (5-4.8) into Eqs. (5-4.5) and (5-4.6) shows that there always exists a function F such that for body forces represented by Eqs. (5-4.6), stress components in the plane theory of elasticity may be expressed in the form

$$\sigma'_x = \frac{\partial^2 F}{\partial y^2} \qquad \sigma'_y = \frac{\partial^2 F}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y}$$

Alternatively, by Eqs. (5-4.4) we have

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} + V \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} + V \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{5-4.9}$$

The function F is called the *Airy stress function* in honor of G. B. Airy, who first noted this relation.

Because it was assumed that the stresses $\sigma_x, \sigma_y, \tau_{xy}$ are single valued and continuous together with their second-order derivatives [note the compatibility equations in terms of stress components Eq. (5-3.3)], the function F must possess continuous derivatives up to and including fourth order. These derivatives, from the second order on up, must be single-valued functions throughout the region occupied by the body [see Eqs. (5-4.9)].

Conversely, if F has these properties, the functions $\sigma_x, \sigma_y, \tau_{xy}$ defined in terms of F by Eqs. (5-4.9) will satisfy Eq. (5-4.1), provided body forces are defined by Eqs. (5-4.2). Additionally, to ensure that the stresses so determined correspond to an actual deformation, the compatibility conditions for the plane theory of elasticity must be satisfied. For body forces defined by Eq. (5-4.2) (or for constant body forces), this condition becomes [see Eq. (5-3.3) or (5-3.7)]

$$\nabla^2(\sigma_x + \sigma_y) = 0 \tag{5-4.10}$$

Adding the first two of Eqs. (5-4.9), we note that

$$\sigma_x + \sigma_y = \nabla^2 F + 2V \tag{5-4.11}$$

Substitution of Eq. (5-4.11) into Eq. (5-4.10) yields (because $\nabla^2 V = 0$)

$$\nabla^2\nabla^2 F = \frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \tag{5-4.12}$$

Equation (5-4.12) is the compatibility condition of the plane theory of elasticity with constant body forces or body forces derivable from a potential function [Eq. (5-4.2)] in terms of the stress function F.

Equations of the form of Eq. (5-4.12) are called *biharmonic.* Solutions of Eq. (5-4.12) are called biharmonic functions (Brown and Churchill, 2008). Some well-known solutions to Eq. (5-4.12) are, in rectangular coordinates,

$$\begin{gathered} y,\ y^2,\ y^3,\ x,\ x^2,\ x^3,\ xy,\ x^2y,\ xy^2,\ x^3y,\ \text{and}\ xy^3 \\ x^2 - y^2, \qquad x^4 - y^4, \qquad x^2y^2 - \tfrac{1}{3}y^4, \ldots \\ \cos\lambda y \cosh\lambda x \qquad \cosh\lambda y \cos\lambda x \qquad y\cos\lambda y \cosh\lambda x \\ y\cosh\lambda y \cos\lambda x \qquad x\cos\lambda y \cosh\lambda x \qquad x\cosh\lambda y \cos\lambda x \end{gathered} \tag{5-4.13}$$

By the above analysis, the problem of plane elasticity has been reduced to seeking solutions to Eq. (5-4.12) such that the stress components [Eqs. (5-4.9)] satisfy the boundary conditions. A number of problems may be solved by using simple linear combinations of polynomials in x and y (see Section 5-7).

Airy Stress Function with Body Forces and Temperature Effects. More generally, Eq. (5-4.12) may be written to include temperature effects [see Eqs. (5-3.9), (5-3.11), and (5-3.12)]. Body forces derivable from a potential function [Eq. (5-4.2)] do not affect Eq. (5-4.12). Hence, for potential body forces, Eq. (5-4.12) generalized to include temperature effects is [see Eq. (5-3.12)]

$$\nabla^2\nabla^2 F + C\nabla^2(kT) = 0 \tag{5-4.12a}$$

where $C = E$ for plane stress and $C = E/(1-\nu)$ for plane strain. Cases of more general body forces ordinarily must be treated individually.

Problem. Verify that the functions listed in Eq. (5-4.13) satisfy Eq. (5-4.12).

Boundary Conditions. It is frequently convenient to have the stress boundary conditions [Eqs. (4-15.1) in Chapter 4] expressed in terms of the Airy stress function. For simply connected regions, Eqs. (4-15.1) may be transformed as follows.

Consider region G: (x, y) bounded by the curve $\mathbf{\Gamma}$ (Fig. 5-4.1). The unit normal vector (+outward) is

$$\mathbf{n} = (l, m, n) = \left(\frac{dy}{ds}, -\frac{dx}{ds}, 0\right) \tag{5-4.14}$$

where s denotes arc length measured from some arbitrary point P on $\mathbf{\Gamma}$. The unit tangent vector to $\mathbf{\Gamma}$ is denoted by $\mathbf{t}$, the positive direction of $\mathbf{t}$ being such that $(\mathbf{n},\mathbf{t})$ form a right-handed system.

For the plane theory of elasticity with respect to the (x, y) plane, the boundary conditions [Eqs. (4-15.1)] reduce to

$$\sigma_{nx} = l\sigma_x + m\tau_{xy} \qquad \sigma_{ny} = l\tau_{xy} + n\sigma_y \tag{5-4.15}$$

Substitution of Eqs. (5-4.14) into Eqs. (5-14.5) yields

$$\sigma_{nx} = \sigma_x \frac{dy}{ds} - \tau_{xy}\frac{dx}{ds} \qquad \sigma_{ny} = \tau_{xy}\frac{dy}{ds} - \sigma_y \frac{dx}{ds} \tag{5-4.16}$$

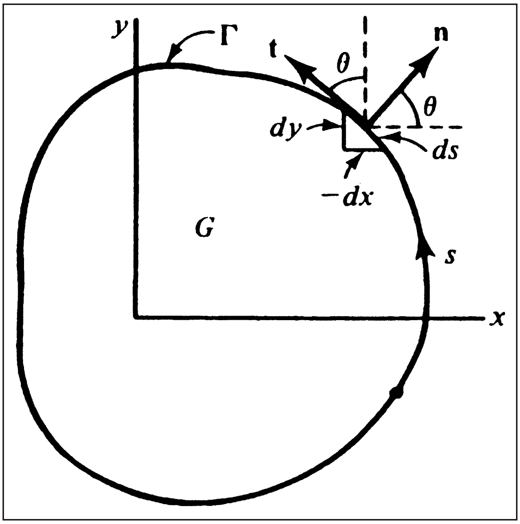


Figure 5-4.1

By Eqs. (5-4.16), (5-4.5), and (5-4.6), we eliminate $\sigma_x, \sigma_y, \tau_{xy}$ to obtain

$$\sigma_{nx} = \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial x}\frac{dx}{ds} = \frac{\partial \phi}{ds}$$

$$\sigma_{ny} = -\frac{\partial \theta}{\partial y}\frac{dy}{ds} - \frac{\partial \theta}{\partial x}\frac{dx}{ds} = -\frac{d\theta}{ds}$$

or, multiplying by ds, we get

$$\begin{aligned} \sigma_{nx}\, ds &= \frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy = d\phi \\ -\sigma_{ny}\, ds &= \frac{\partial \theta}{\partial x}\, dx + \frac{\partial \theta}{\partial y}\, dy = d\theta \end{aligned} \tag{5-4.17}$$

Integration of Eq. (5-4.17) yields [with Eq. (5-4.8)]

$$\begin{aligned} \phi &= \frac{\partial F}{\partial y} = \int \sigma_{nx}\, ds = \int_0^l \sigma_{nx}\, ds + C_1 = R_x + C_1 \\ \theta &= \frac{\partial F}{\partial x} = -\int \sigma_{ny}\, ds = -\int_0^l \sigma_{ny}\, ds + C_2 = -R_y + C_2 \end{aligned} \tag{5-4.18}$$

where (R_x, R_y) denote the (x, y) projections of the total force acting on $\boldsymbol{\Gamma}$ from 0 to l, and (C_1, C_2) are constants. Equations (5-4.18) express the stress boundary conditions [Eqs. (4-15.1)] in terms of derivatives of the Airy stress function F.

The stress boundary conditions may be interpreted physically in terms of the net force and net moment at $s = l$ resulting from the stress distributed on the boundary from $s = 0$ to $s = l$. For example, recall that by definition the total differential dF of F is

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy \tag{5-4.19}$$

Substitution of Eqs. (5-4.18) into Eq. (5-4.19) yields, after integration,

$$F(l) = \int_0^l dF = \int_0^l (-R_y\, dx + R_x\, dy) + C_1(y - y_0) + C_2(x - x_0) + C_3$$

Because linear terms in F do not contribute to the stress components [Eq. (5-4.9) with $V = 0$], we take $C_1 = C_2 = C_3 = 0$. Then integration by parts yields [with Eqs. (5-4.18)]

$$\begin{aligned}
F(l) &= \int_0^l (-R_y\, dx + R_x\, dy) \\
&= (-xR_y + yR_x)|_0^l - \int_0^l (-x\, dR_y + y\, dR_x) \\
&= -x_1 R_y(l) + y_1 R_x(l) + \int_0^l (x\sigma_{ny} - y\sigma_{nx})\, ds \\
&= -\int_0^l (x_l - x)\sigma_{ny}\, ds + \int_0^l (y_l - y)\sigma_{nx}\, ds = M_l
\end{aligned} \tag{5-4.20}$$

where M_l denotes the moment with respect to P: $(s = l)$ of boundary forces on $\boldsymbol{\Gamma}$ from the point P: $(s = 0)$ to the point P: $(s = l)$. Thus, Eq. (5-4.20) shows that the value $F(l)$ of the Airy stress function at $s = l$ relative to its value at $s = 0$, is equal to the net moment of the boundary forces on $\boldsymbol{\Gamma}$ from the point $s = 0$ to the point $s = l$.

Equation (5-4.20) replaces one of the boundary conditions [Eqs. (5-4.18)]. To obtain a second equation, consider the directional derivative of the Airy stress function in the direction of $\mathbf{n}$ (Fig. 5-4.1). We have [see Section 1-8 in Chapter 1 and Eqs. (5-4.14) and (5-4.18)]

$$\begin{aligned}
\frac{dF(l)}{dn} &= \mathbf{n} \cdot \text{grad } F \\
&= \left(\frac{dy}{ds}, -\frac{dx}{ds}\right) \cdot [-R_y(l), R_x(l)] \\
&= -\left(\frac{dx}{ds}, \frac{dy}{ds}\right) \cdot \left(\int_0^l \sigma_{nx}\, ds, \int_0^l \sigma_{ny}\, ds\right) \\
&= -\mathbf{t} \cdot \mathbf{R}
\end{aligned} \tag{5-4.21}$$

where **R** denotes the resultant external force acting on $\boldsymbol{\Gamma}$ from the point $s = 0$ to the point $s = l$. Hence, the normal derivative of F at point $s = l$ is equal to the negative of the projection **R** on the tangent **t** to the curve $\boldsymbol{\Gamma}$ at point $s = l$.

Equations (5-4.20) and (5-4.21) serve as boundary conditions in terms of the Airy stress function F. If the boundary $\boldsymbol{\Gamma}$ is free of external forces, Eqs. (5-4.20) and (5-4.21) yield

$$F(l) = 0 \qquad \frac{dF(l)}{dn} = 0 \tag{5-4.22}$$

Multiply Connected Regions. The above argument assumes that derivatives $G = G(x, y)$ of second order or higher of the Airy stress function are single-valued functions of (x, y). Hence, it is restricted to simply connected regions for which

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \qquad P = \frac{\partial G}{\partial x} \qquad Q = \frac{\partial G}{\partial y} \tag{5-4.23}$$

pcare necessary and sufficient conditions for the existence of G (see Section 1-19 in Chapter 1). For multiply connected regions with bounding contours L_k (Fig. 5-4.2), the condition (5-4.23) is only a necessary condition for the existence of the single-valued functions $G(x, y)$. For a multiply connected region, in addition to Eq. (5-4.23), the conditions

$$J_k = \int_{L_k} P\,dx + Q\,dy = 0 \qquad k = 1, 2, 3, \ldots, m \tag{5-4.24}$$

are also required.

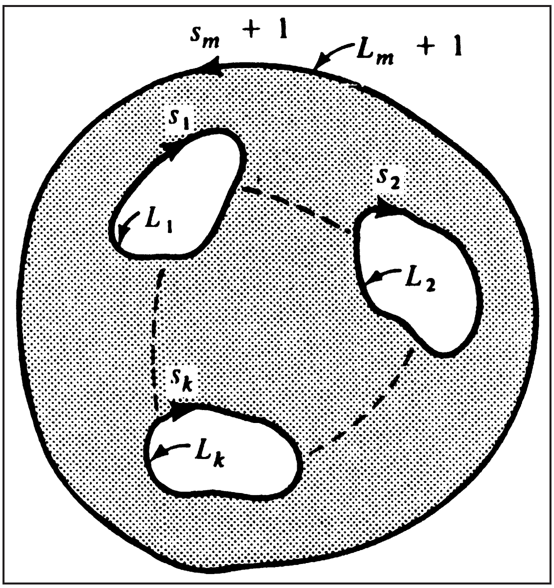


Figure 5-4.2

Accordingly, in order that the derivatives $G(x, y)$ of second order or higher of the Airy stress function $F(x, y)$ be single valued, it is necessary and sufficient that in addition to Eq. (5-4.23), the following conditions (Muskhelishvili, 1975) hold:

$$J_1 = J_2 = \cdots = J_k = \cdots = J_m = 0 \tag{5-4.25}$$

where J_k is defined by Eq. (5-4.24).

The defining equations for the Airy stress function [Eqs. (5-4.12), (5-4.20), (5-4.21), and (5-4.25)] may be expressed for the general plane orthogonal curvilinear coordinate system by specializing the results of Section 1-22 in Chapter 1 for the plane.

Equations (5-4.23) and (5-4.25) ensure the single valuedness of the stress components $\sigma_x, \sigma_y, \tau_{xy}$. However, they do not assure the existence of single-valued displacement components (u, v), as these components are obtained by an integration of stress (or strain) quantities, this integration process possibly yielding multivalued terms. Accordingly, if we require single-valued displacement, we must select the arbitrary functions (or constants) that result in the expressions for (u, v) in such a fashion that the single valuedness of displacement is ensured. Although we ordinarily require that the displacement be single valued, the concept of multivalued displacement components may be interpreted in a physical sense and finds an application through Volterra's theory of dislocation (see Love, 2009, pp. 221–228).

Example 5-4.1. Plane Theory of Thermoelasticity. Concept of Displacement Potential. In the absence of body forces, the plane theory of thermoelasticity may be reduced to the problem of determining a stress function F such that [Eq. (5-4.12a)]

$$\nabla^2\nabla^2 F = -C\nabla^2(kT) \tag{E5-4.1}$$

where $C = E$ for the plane stress state and $C = E/(1-\nu)$ for the plane strain state. In addition to Eq. (E5-4.1), the stress function F must satisfy appropriate boundary conditions (see Section 5-4). In general, the solution of Eq. (E5-4.1) subject to specific boundary conditions is a difficult mathematical problem, although in certain special cases simple solutions may be obtained. A general solution of Eq. (E5-4.1) may be obtained by adding a particular solution, for which the right-hand side of Eq. (E5-4.1) is satisfied identically, to the solution (complementary solution) of $\nabla^2\nabla^2 F = 0$. A method of obtaining a particular integral of Eq. (E5-4.1) has been outlined by Goodier (1937). The method is frequently referred to as the method of displacement potential because displacement representations and certain concepts from potential theory are employed.

Following Goodier (1937), we represent the plane theory of thermoelasticity in terms of displacement components. Initially, we consider the case of plane stress, the results for plane strain being obtained by a simple transformation of material constants.

Let (x, y) denote rectangular Cartesian coordinates. Let (u, v) denote displacement components in the (x, y) directions, respectively. In terms of (u, v), the stress components for plane stress are (see Section 4-12 in Chapter 4)

$$\sigma_x = \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} - (1+\nu)kT\right]$$

$$\sigma_y = \frac{E}{1-\nu^2}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x} - (1+\nu)kT\right] \tag{E5-4.2}$$

$$\tau_{xz} = \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

Substitution of Eqs. (E5-4.2) into the equilibrium equations for plane stress yields, in the absence of body force [see Eq. (5-4.1)],

$$\frac{\partial e}{\partial x} + \frac{1-\nu}{1+\nu}\nabla^2 u = 2k\frac{\partial T}{\partial x}$$

$$\frac{\partial e}{\partial y} + \frac{1-\nu}{1+\nu}\nabla^2 v = 2k\frac{\partial T}{\partial y} \tag{E5-4.3}$$

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Let

$$u = \frac{\partial \psi}{\partial x} \qquad v = \frac{\partial \psi}{\partial y} \tag{E5-4.4}$$

where $\psi = \psi(x, y)$ is called the displacement potential function. Substitution of Eqs. (E5-4.4) into Eqs. (E5-4.3) yields

$$\frac{\partial}{\partial x}\left(\frac{1}{1+\nu}\nabla^2\psi - kT\right) = 0$$

$$\frac{\partial}{\partial y}\left(\frac{1}{1+\nu}\nabla^2\psi - kT\right) = 0$$

These equations are satisfied identically if

$$\nabla^2\psi = (1+\nu)kT \tag{E5-4.5}$$

Accordingly, the solution of Eq. (E5-4.5) represents a particular solution of Eqs. (E5-4.3). To obtain a general solution of Eqs. (E5-4.3), we must add to the solution of Eq. (E5-4.5) the complementary solution of Eqs. (E5-4.3); that is, we must add the solution of Eqs. (E5-4.3) for the case $T = 0$. This general solution must then be made to satisfy the boundary conditions of the problem.

By Eqs. (E5-4.2), (E5-4.4), and (E5-4.5), the stress components corresponding to the particular solution ψ are

$$\sigma_x' = -2G\frac{\partial^2\psi}{\partial y^2}$$
$$\sigma_y' = -2G\frac{\partial^2\psi}{\partial x^2} \qquad \text{(E5-4.6)}$$
$$\tau_{xy}' = 2G\frac{\partial^2\psi}{\partial x \partial y}$$

In the absence of temperature T, the complementary solution of the plane problem is expressed in terms of the Airy stress function F [Eqs. (5-4.9)]. Accordingly, the stress components for a general solution of the plane stress thermoelastic problem are, by Eqs. (5-4.9) and (E5-4.6),

$$\sigma_x = \frac{\partial^2}{\partial y^2}(F - 2G\psi)$$
$$\sigma_y = \frac{\partial^2}{\partial x^2}(F - 2G\psi) \qquad \text{(E5-4.7)}$$
$$\tau_{xy} = -\frac{\partial^2}{\partial x \partial y}(F - 2G\psi)$$

Similarly, for the case of plane strain, we have *Stress–Displacement Relations*:

$$\sigma_x = \lambda e + 2G\frac{\partial u}{\partial x} - \frac{EkT}{1-2\nu}$$
$$\sigma_y = \lambda e + 2G\frac{\partial v}{\partial y} - \frac{EkT}{1-2\nu}$$
$$\sigma_z = \nu(\sigma_x + \sigma_y) - EkT = \lambda e - \frac{EkT}{1-2\nu} \qquad \text{(E5-4.8)}$$
$$\tau_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad G = \frac{E}{2(1+\nu)} \qquad e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

Equilibrium Equations in Terms of Displacement:

$$\frac{\partial e}{\partial x} + (1-2\nu)\nabla^2 u = 2(1+\nu)\frac{\partial (kT)}{\partial x}$$
$$\frac{\partial e}{\partial y} + (1-2\nu)\nabla^2 v = 2(1+\nu)\frac{\partial (kT)}{\partial y} \qquad \text{(E5-4.9)}$$

Displacement Potential–Temperature Relation:

$$\nabla^2 \psi = \frac{1+\nu}{1-\nu} kT \tag{E5-4.10}$$

With the displacement potential function ψ defined by Eq. (E5-4.10), the stress components $(\sigma_x, \sigma_y, \tau_{xy})$ are again given by Eqs. (E5-4.7). Then σ_z is determined by Eq. (E5-4.8).

In the preceding method of integration of the stress equations, we have used a stress function or a displacement potential. In a certain class of problems the thermal-stress equations may be integrated more directly by other methods (Sen, 1939; Sharma, 1956; McDowell and Sternberg, 1957).

Problem Set 5-4

1. In a state of plane strain relative to the (x, y) plane, the displacement component $w = 0$ and the displacement components (u, v) are functions of (x, y) only. Hence, the components of rotation $\omega_x = \omega_y = 0$ and $\omega = \omega_z$. For zero body forces (set $V = 0$), we note that the equations of equilibrium are satisfied by Eqs. (5-4.9). Show that

$$\sigma_x + \sigma_y = 2(\lambda + G)e$$

where e is the volumetric strain or dilatation and where λ, G are the Lamé constants. Hence, show that in terms of dilatation and rotation the equations of equilibrium are

$$(\lambda + 2G)\frac{\partial e}{\partial x} - 2G\frac{\partial \omega}{\partial y} = 0 \qquad (\lambda + 2G)\frac{\partial e}{\partial y} + 2G\frac{\partial \omega}{\partial x} = 0$$

Thus, show that e and ω are plane harmonic functions.

2. Because the dilatation e and rotation ω are plane harmonic functions (see Problem 1), $(\lambda + 2G)e + i2G\omega$ is a function of the complex variable $x+$ iy, where i is $\sqrt{-1}$. Also, the Airy stress function F is related to e by $\nabla^2 F = 2(\lambda + G)e$, where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Introduce the new function $\xi + i\eta$ of $x + iy$ as follows:

$$\xi + i\eta = \int [(\lambda + 2G)e + i2G\omega]\, d(x + iy)$$

so that

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = (\lambda + 2G)e = \frac{\lambda + 2G}{2(\lambda + G)}\nabla^2 F \qquad -\frac{\partial \xi}{\partial y} = \frac{\partial \eta}{\partial x} = 2G\omega$$

where F is Airy's stress function. Hence, show that

$$2G\frac{\partial u}{\partial x} = \frac{\partial^2 F}{\partial y^2} - \frac{\lambda}{2(\lambda + G)}\nabla^2 F = -\frac{\partial^2 F}{\partial x^2} + \frac{\partial \xi}{\partial x}$$

$$2G\frac{\partial v}{\partial y} = \frac{\partial^2 F}{\partial x^2} - \frac{\lambda}{2(\lambda + G)}\nabla^2 F = -\frac{\partial^2 F}{\partial y^2} + \frac{\partial \eta}{\partial y}$$

and that

$$2G\frac{\partial u}{\partial y} = -\frac{\partial^2 F}{\partial x \partial y} - 2G\omega = -\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial \xi}{\partial y}$$

$$2G\frac{\partial v}{\partial x} = -\frac{\partial^2 F}{\partial x \partial y} + 2G\omega = -\frac{\partial^2 F}{\partial x \partial y} + \frac{\partial \eta}{\partial x}$$

and that there follows

$$2Gu = -\frac{\partial F}{\partial x} + \xi \qquad 2Gv = -\frac{\partial F}{\partial y} + \eta$$

These equations define the displacement components (u, v) when F is known.

3. We recall that

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \qquad 2\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

These equations with the definitions of ξ, η given in Problem 2 yield, after integration,

$$u = \frac{\partial}{\partial x}\left[\frac{y\eta}{2(\lambda + 2G)}\right] + \frac{\partial}{\partial y}\left[\frac{y\xi}{2G}\right] + u'$$

$$v = \frac{\partial}{\partial y}\left[\frac{y\eta}{2(\lambda + 2G)}\right] + \frac{\partial}{\partial x}\left[\frac{y\xi}{2G}\right] + v'$$

where $v' + iu'$ is a function of $x + iy$. Let $u' = \partial f/\partial x$, $v' = \partial f/\partial y$, $\nabla^2 f = 0$. Show that

$$u = \frac{\xi}{2G} + \frac{\lambda + G}{2G(\lambda + 2G)} y \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial x}$$

$$v = \frac{\eta}{2(\lambda + 2G)} - \frac{\lambda + G}{2G(\lambda + 2G)} y \frac{\partial \eta}{\partial y} + \frac{\partial f}{\partial y}$$

These equations define (u, v) when e and ω are known.

4. With the information given in Problems 2 and 3, show that

$$F = -2Gf + \frac{\lambda + G}{\lambda + 2G} y\eta$$

and that the formulas for (u, v) given in Problems 2 and 3 are thus equivalent.

5. Let a thin plate with constant thickness and with mass density ρ rotate with constant angular velocity ω above the y axis (Fig. P5-4.5). Neglecting gravity, write an expression for the inertia force X per unit volume (body force per unit volume) that acts on an arbitrary mass element of the plate. Write the differential equations of equilibrium for the plate. Write the general solution of these equations in terms of Airy's stress function F. Show that the equation of compatibility is $\nabla^4 F = (1-\nu)\rho\omega^2$, where ν is Poisson's ratio.

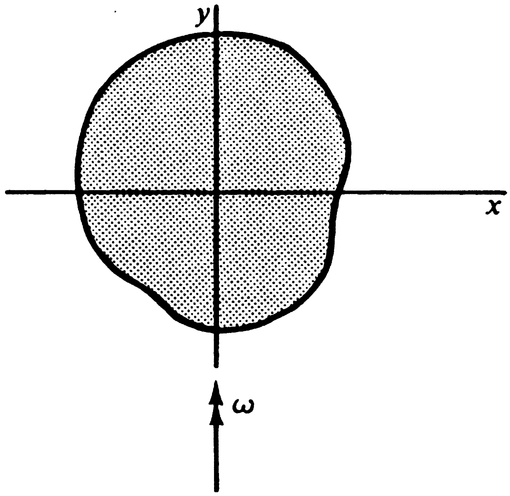


Figure P5-4.5

6. An infinite plane strip is bounded by the lines $y = \pm 1$. The stresses on the lines $y = \pm 1$ are $\sigma_y = \cos x$, $\tau_{xy} = 0$. There is no body force. By assuming an Airy stress function of the form $f(y)\cos x$, determine $\sigma_x, \sigma_y, \tau_{xy}$ as functions of (x, y).

7. The following stress–strain relations pertain to the anisotropic flat thin plate subjected to a state of generalized plane stress:

$$\epsilon_x = S_{11}\sigma_x + S_{12}\sigma_y$$

$$\epsilon_y = S_{12}\sigma_x + S_{22}\sigma_y$$

$$\gamma_{xy} = S_{33}\tau_{xy} \qquad (x, y) = \text{rectangular Cartesian coordinates}$$

where $S_{11}, S_{22}, S_{33}, S_{12}$ are elastic constants and where $(\sigma_x, \sigma_y, \tau_{xy})$ and $(\epsilon_x, \epsilon_y, \gamma_{xy})$ are average values of stress and strain through the thickness. Let $(\sigma_x, \sigma_y, \tau_{xy})$ be defined in terms of an Airy stress function F. Show that the defining equation for the Airy stress function F is of the form

$$\left(\frac{\partial^2}{\partial x^2} + \alpha_1 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 F}{\partial x^2} + \alpha_2 \frac{\partial^2 F}{\partial y^2}\right) = 0 \qquad \text{(a)}$$

where α_1, α_2 are constants. For the case $S_{11} = S_{22} = 1/E$, $S_{12} = -\nu/E$, and $S_{33} = 1/G$, show that Eq. (a) reduces to the biharmonic equation.

8. Let

$$F = ax^2 + by^3 + \sum_{n=1}^{\infty} A_n(y)\cos\left(\frac{n\pi x}{L}\right)$$

be an Airy stress function for a plane, isotropic problem, where a, b, L are constants, and $A_n(y)$ are functions of y, Derive the defining differential equation for the coefficients A_n.

Consider a plane rectangular region $-L \le x \le L$, $-C \le y \le C$. Assume that no net force or no net couple acts on the sections $x = \pm L$. Discuss how the arbitrary constants in the solution of the differential equation for $A_n(y)$ may be evaluated.

9. Consider a case of plane stress without body forces in the region $-c \le y \le c, 0 \le x \le \ell$ (see Fig. P5-4.9). If the resultant of the stresses in the x direction is zero, the elementary beam formula yields $\sigma_x = My/I$; that is, σ_x is a linear function of y.

(a) Let $\sigma_x = F_{yy}$, $\sigma_y = F_{xx}$, and $\tau_{xy} = -F_{xy}$. Write the most general expression for $F(x, y)$ that satisfies the equations of equilibrium and yields σ_x as linear function of y in the form $\sigma_x = yf(x)$.

(b) Assuming that the material is isotropic and linearly elastic, write the equation of compatibility for $F(x, y)$ as determined in part (a).

(c) Determine the most general form of $F(x, y)$ that satisfies the equations of equilibrium and compatibility, and yields σ_x linear in y.

(d) Derive expressions for the stress components using the stress function derived in part (c).

(e) Assume that no load is applied along the line $y = c$. Show that the elementary formula can be correct, strictly speaking, only if the stresses are those produced in a cantilever with a concentrated vertical load at the end and/or a moment applied at the end.

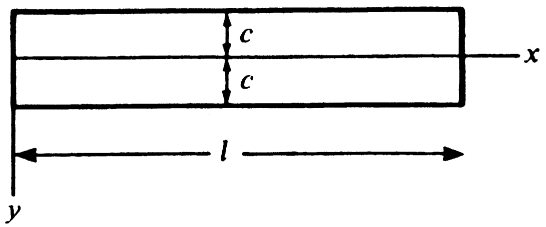


Figure P5-4.9

10. The general stress–strain–temperature relationship for an isotropic material is

$$\epsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z + kT$$

$$\epsilon_y = -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y - \frac{\nu}{E}\sigma_z + kT$$

$$\epsilon_z = -\frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y + \frac{1}{E}\sigma_z + kT$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

Consider a body that is in a state of plane strain.

(a) Derive the "two-dimensional" Hooke's law expressing the strains $\epsilon_x, \epsilon_y, \ldots$ as functions of $\sigma_x, \sigma_y, \tau_{xy}$, and $T = T(x, y)$.

(b) Assuming that body forces are negligible, let $\sigma_x = F_{yy}$, $\sigma_y = F_{xx}$, and $\tau_{xy} = -F_{xy}$, where F is a stress function. Derive the compatibility conditions in terms of T and F. Thus, show that $F(x, y)$ must be biharmonic if $T(x, y)$ is harmonic.

11. For a plane problem, the stress components in the (x, y) rectangular region $0 \le x \le L$, $-c \le y \le c$, where L and c are constants, are given by the relations (q = constant)

$$\sigma_x = \frac{qx^3y}{4c^3} + \frac{q}{4c^3}\left(-2xy^3 + \frac{6}{5}c^2xy\right)$$

$$\sigma_y = -\frac{qx}{2} + qx\left(\frac{y^3}{4c^3} - \frac{3y}{4c}\right)$$

$$\tau_{xy} = \frac{3qx^2}{8c^3}(c^2 - y^2) - \frac{q}{8c^3}(c^4 - y^4) + \frac{q}{4c^3} \cdot \frac{3c^2}{5}(c^2 - y^2)$$

(a) Show that these stress components satisfy the equations of equilibrium in the absence of body forces.

(b) Derive the Airy stress function from which these stress components are derivable.

(c) Show that the stress state is compatible.

(d) Determine the problem that the stress components represent.

12. The stress function for a cantilever beam loaded by a shear force P at the free end is

$$F = C_1xy^3 + C_2xy$$

(a) Evaluate the constants C_1 and C_2.

(b) Derive the expressions for the displacements u and v.

(c) Compare v with the expression derived for displacement y from elementary beam theory, $EI(d^2y/dx^2) = M$.

13. Apply the stress function $F = -(P/d^3)xy^2(3d - 2y)$ to the region $0 \le y \le d, 0 \le x$. Determine what kind of problem is solved by this stress function.

14. The stress–strain relationship for a certain orthotropic material may be written as

$$\epsilon_\alpha = C_{\alpha\beta}\sigma_\beta \qquad \alpha, \beta = 1, 2, \ldots, 6$$

where

$$C_{\alpha\beta} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix}$$

and

$$\sigma_1 = \sigma_x \qquad \sigma_2 = \sigma_y \qquad \sigma_3 = \sigma_z \qquad \sigma_4 = \tau_{xy} \qquad \sigma_5 = \tau_{xz} \qquad \sigma_6 = \tau_{yz}$$

$$\epsilon_1 = \epsilon_x \qquad \epsilon_2 = \epsilon_y \qquad \epsilon_3 = \epsilon_z \qquad \epsilon_4 = \gamma_{xy} \qquad \epsilon_5 = \gamma_{xz} \qquad \epsilon_6 = \gamma_{yz}$$

(a) For this material derive the fourth-order partial differential equation that a stress function must satisfy in order to meet equilibrium and compatibility requirements for plane stress in the *xy* plane. Neglect body forces.

(b) Show that the equation derived in part (a) reduces to $\nabla^4 F = 0$ for an isotropic material.

15. Consider the Airy stress function $F = Ax^3y$, where A is a constant and (x, y) are rectangular Cartesian coordinates. Determine the plane elasticity problem that is solved by this function for the region $-a \le x \le a, -b \le y \le b$.

16. Show that the function

$$F = \frac{q}{20c^3}\left[10x^2(2y^3 - 3cy^2) - 2y^2\left(2y^3 - 5cy^2 + 4c^2y - c^3\right)\right]$$

may be employed as a stress function. For the plane region $0 \le x \le L, 0 \le y \le c$, determine the stress boundary conditions, and describe fully the plane problem for which the stress function serves as the solution for equilibrium.

17. Show that the three-dimensional equilibrium equations without body force are satisfied, if the stresses are derived from any six functions A, B, C, L, M, N as follows

$$\sigma_x = B_{zz} + C_{yy} - 2L_{yz} \qquad \tau_{yz} = -A_{yz} + (M_y + N_z - L_x)_x$$

$$\sigma_y = C_{xx} + A_{zz} - 2M_{zx} \qquad \tau_{zx} = -B_{zx} + (N_z + L_x - M_y)_y$$

$$\sigma_z = A_{yy} + B_{xx} - 2N_{xy} \qquad \tau_{xy} = -C_{xy} + (L_x + M_y - N_z)_z$$

Subscripts on A, B, C, L, M, N denote partial derivatives.

By discarding some of the above functions, obtain Airy's solution to the equilibrium equations of plane stress theory relative to the *yz* plane.

18. A dam or retaining wall is subjected to a linearly varying pressure $p = p_0 y$. The slice shown in Fig. P5-4.18 is assumed to be in a plane state, with all quantities functions of (x, y) only.

(a) Write down the stress boundary conditions for the faces of AO, BO.

(b) On the basis of part (a), write the simplest Airy stress function that will ensure satisfaction of the boundary conditions on AO, BO. Explain your choice.

(c) Let the body force of the dam be ρg in the y direction, where ρ is the mass density and g is the gravity acceleration. Including the effect of body forces, determine explicitly in terms of known quantities the complete expressions for $\sigma_x, \sigma_y, \tau_{xy}$. (Hint: Note that the body force ρg is derivable from the potential function $V = -\rho g y$.)

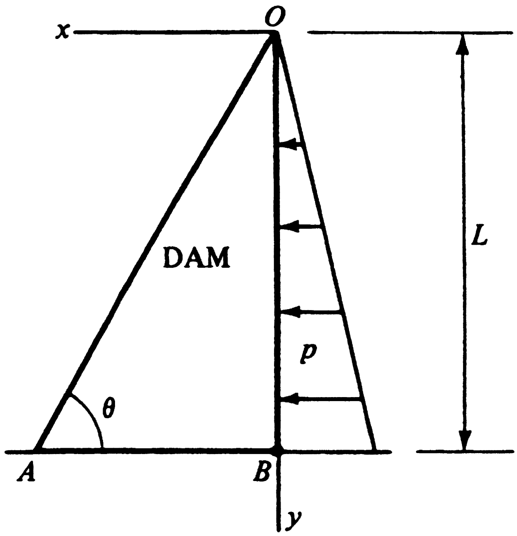


Figure P5-4.18

19. A solution to a plane strain equilibrium problem in the absence of body forces is generated by the Airy stress function Ayx^3.

(a) Determine whether this solution is compatible for a three-dimensional problem.

(b) With this Airy stress function, derive expressions for the stress components; hence, for a linearly elastic isotropic material, derive the corresponding strain components.

20. For homogeneous orthotropic plane stress problems, the stress–strain relations relative to (x, y) axes are

$$\epsilon_x = \left(\frac{\sigma_x}{E_x}\right) - \nu_{xy}\left(\frac{\sigma_y}{E_y}\right) \qquad \epsilon_y = \left(\frac{\sigma_y}{E_y}\right) - \nu_{yx}\left(\frac{\sigma_x}{E_x}\right)$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G} = \left[\frac{E_x + (1 + 2v_{yx})E_y}{E_x E_y}\right]\tau_{xy} \tag{a}$$

where the symbols are self-explanatory. The strain energy density U is given by the formula

$$U + A\epsilon_x^2 + B\epsilon_y^2 + 2C\epsilon_x\epsilon_y + D\gamma_{xy}^2 \tag{b}$$

(a) By the relations $\sigma_\alpha = \partial U/\partial\epsilon_\alpha$, derive the stress–strain relations.

(b) With the result of part (a) and Eq. (a), derive a relationship among E_x, ν_{xy}, E_y, and ν_{yx}.

(c) As in the isotropic case, assume that a stress function $F(x, y)$ exists such that

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{c}$$

Derive the defining equation for the stress function $F(x, y)$ in the form

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) = 0 \tag{d}$$

where k^2 is expressed in terms of E_x, E_y.

(d) Let $F = (P/6I)(3b^2xy - xy^3)$, where P, B, and I are constants. Show that F satisfies the equation $\nabla^2\nabla^2 F = 0$ for isotropic materials and also satisfies Eq. (d). Hence, F is an appropriate stress function for both isotropic and orthotropic materials.

5-5 Airy Stress Function in Terms of Harmonic Functions

In this section we consider the problem of representation of the Airy stress function in terms of a pair of suitably chosen conjugate harmonic functions and a third harmonic function. Such a representation allows us to express the general solution of the biharmonic equation in terms of harmonic functions.

Let ϕ be a harmonic function in (x, y); that is, $\nabla^2\phi = 0$, where ∇^2 is the two-dimensional Laplacian. It may be shown that a solution of the biharmonic equation $\nabla^2\nabla^2 F = 0$ may be expressed in terms of ϕ by any one of the following forms:

$$x\phi \qquad y\phi \qquad (x^2 + y^2)\phi \tag{5-5.1}$$

We note that a function Q_1 defined by

$$Q_1 = \nabla^2 F = \sigma_x + \sigma_y \tag{5-5.2}$$

where F is the Airy stress function, is harmonic in the absence of body forces and temperature, as $\nabla^2 Q_1 = \nabla^2\nabla^2 F = 0$. The function Q_2 related to Q_1 by the Cauchy–Riemann equations (Brown and Churchill, 2008)

$$\frac{\partial Q_1}{\partial x} = \frac{\partial Q_2}{\partial y} \qquad \frac{\partial Q_1}{\partial y} = -\frac{\partial Q_2}{\partial x} \tag{5-5.3}$$

is the conjugate harmonic of Q_1. By Eqs. (5-5.2) and (5-5.3), we note that

$$\nabla^2 Q_1 = \nabla^2 Q_2 = 0 \tag{5-5.4}$$

That is, Q_2 is harmonic.

By the Cauchy integral theorem (Brown and Churchill, 2008) of complex variables, the integral of the analytic function

$$f(z) = Q_1 + iQ_2 \tag{5-5.5}$$

where $z = x +$ iy, $i = \sqrt{-1}$, is another analytic function, say $\psi(z)$. Thus,

$$\psi(z) = q_1 + iq_2 = \frac{1}{c}\int f(z)\,dz \tag{5-5.6}$$

is analytic, where c is as yet an arbitrary constant. The functions (q_1, q_2) are conjugate harmonic functions; that is, they satisfy Eqs. (5-5.3). We note by Eq. (5-5.6) that $\psi'(z) = (1/c)f(z)$, where the prime denotes differentiation with respect to z. Hence,

$$\frac{\partial q_1}{\partial x} + i\frac{\partial q_2}{\partial x} = \frac{\partial}{\partial x}\psi(z) = \frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x}$$

Because $\partial z/\partial x = 1$, we obtain from the above results and Eq. (5-5.5)

$$\frac{\partial q_1}{\partial x} + i\frac{\partial q_2}{\partial x} = \frac{1}{c}(Q_1 + iQ_2) \tag{5-5.7}$$

Equating real parts of Eq. (5-5.7), we obtain

$$\frac{\partial q_1}{\partial x} = \frac{1}{c}Q_1 \tag{5-5.8}$$

Because (q_1, q_2) satisfy Eqs. (5-5.3), we obtain from Eqs. (5-5.3) and (5-5.8)

$$\frac{\partial q_2}{\partial y} = \frac{1}{c}Q_1 \tag{5-5.9}$$

Accordingly, by Eqs. (5-5.8), (5-5.9), and (5-5.2), we find that p_0 defined by

$$p_0 = F - xq_1 - yq_2 \tag{5-5.10}$$

is harmonic, provided $c = 4$. Accordingly, the Airy stress function F may be written in the form

$$F = xq_1 + yq_2 + p_0 \tag{5-5.11}$$

where (q_1, q_2) are suitably chosen conjugate harmonic functions and p_0 is an arbitrary harmonic function. Alternatively, we may take F in the forms (provided $c = 4$)

$$F = 2xq_1 + p_1 \tag{5-5.12}$$

or

$$F = 2yq_2 + p_2 \tag{5-5.13}$$

where (p_1, p_2) are arbitrary harmonic functions.

5-6 Displacement Components for Plane Elasticity

Direct Integration Method. When the plane elasticity stress components $\sigma_x, \sigma_y, \tau_{xy}$ are known, the strain components $\epsilon_x, \epsilon_y, \epsilon_{xy}$ may be determined by Eqs. (5-3.6) for generalized plane stress or by Eqs. (5-3.2) for plane strain. Then integration of the strain–displacement relations [Eqs. (5-1.4) for plane strain or Eqs. (5-1.4) with ϵ_z given by Eqs. (5-3.4) for generalized plane stress] yields the (x, y) displacement components (u, v). The integration of the strain–displacement relations yields an arbitrary rigid-body displacement (see Section 2-15 in Chapter 2 and Examples 4-18.1 and 4-18.2 in Chapter 4). Accordingly, complete specification of the displacement (u, v) requires that the rigid-body displacement of the body be known. For example, in Example 4-18.1, it was specified that the point $x = y = z = 0$ be fixed and that the volumetric rotation for this point vanish. Consequently, the displacements and rotations of all other points and volume elements in the body were determined relative to the point and volume element at $x = y = z = 0$. Similarly, to fix the rigid-body displacement in the solution of the plane problem, we may specify the displacement of some point (say, x_0, y_0) and the rotation of a line element (say, a line element through point x_0, y_0).

Representation in Terms of Airy Stress Function. Alternatively, we may derive formulas for the plane displacement components (u, v) in terms of the Airy stress function. We carry out the calculation for the case of the plane stress. The results for plane strain may be obtained in a similar manner.

For plane stress relative to the (x, y) plane, the stress–strain relations are

$$
\begin{aligned}
\epsilon_x &= \frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y) \\
\epsilon_y &= \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_x) \\
\gamma_{xy} &= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{G}\tau_{xy}
\end{aligned}
\tag{5-6.1}
$$

where $(\epsilon_x, \epsilon_y, \gamma_{xy})$ are the strain components, $(\sigma_x, \sigma_y, \tau_{xy})$ are stress components, (u, v) denote the (x, y) displacement components, E denotes the modulus of elasticity, ν is Poisson's ratio, and $G = E/[2(1 + \nu)]$.

In terms of the Airy stress function F, the stress components are [Eqs. (5-4.9) with $V = 0$]

$$
\sigma_x = \frac{\partial^2 F}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{5-6.2}
$$

Equations (5-6.1) and (5-6.2) yield [with Eq. (5-5.2)]

$$
\begin{aligned}
E\frac{\partial u}{\partial x} &= -(1 + \nu)\frac{\partial^2 F}{\partial x^2} + Q_1 \\
E\frac{\partial v}{\partial y} &= -(1 + \nu)\frac{\partial^2 F}{\partial y^2} + Q_1
\end{aligned}
\tag{5-6.3}
$$

We replace Q_1 by $4(\partial q_1/\partial x)$ in the first of Eqs. (5-6.3) and by $4(\partial q_2/\partial y)$ in the second [see Eqs. (5-5.8) and (5-5.9)]. Thus, after dividing by $1+\nu$, we find

$$
\begin{aligned}
2G\frac{\partial u}{\partial x} &= -\frac{\partial^2 F}{\partial x^2} + \frac{4}{1+\nu}\frac{\partial q_1}{\partial x} \\
2G\frac{\partial v}{\partial y} &= -\frac{\partial^2 F}{\partial y^2} + \frac{4}{1+\nu}\frac{\partial q_2}{\partial y}
\end{aligned}
\tag{5-6.4}
$$

Integration of Eqs. (5-6.4) yields

$$
\begin{aligned}
2Gu &= -\frac{\partial F}{\partial x} + \frac{4}{1+\nu}q_1 + f_1(y) \\
2Gv &= -\frac{\partial F}{\partial y} + \frac{4}{1+\nu}q_2 + f_2(y)
\end{aligned}
\tag{5-6.5}
$$

where $f_1(y)$, $f_2(x)$ are arbitrary functions of integration.

To interpret (f_1, f_2) of Eqs. (5-6.5), we note that by the last of Eqs. (5-6.2) and Eqs. (5-6.5), with Eqs. (5-5.3)

$$
\tau_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\frac{\partial^2 F}{\partial x \partial y} + \frac{1}{2}\frac{df_1}{dy} + \frac{1}{2}\frac{df_2}{dx}
$$

Hence,

$$
\frac{df_1}{dy} + \frac{df_2}{dx} = 0 \tag{5-6.6}
$$

Integration of Eq. (5-6.6) yields

$$
f_1 = Ay + B \qquad f_2 = -Ax + C \tag{5-6.7}
$$

Hence, the functions (f_1, f_2) represent a rigid-body displacement (Section 2-15 in Chapter 2). Discarding them, we get

$$
\begin{aligned}
2Gu &= -\frac{\partial F}{\partial x} + \frac{4}{1+\nu}q_1 \\
2Gv &= -\frac{\partial F}{\partial y} + \frac{4}{1+\nu}q_2
\end{aligned}
\tag{5-6.8}
$$

Equations (5-6.8) determine displacement components (u, v) when the stress function F is known. The function Q_1 is determined by computing $\nabla^2 F$ [Eq. (5-5.2)]. Then the function Q_2 is determined by means of the Cauchy–Riemann equations [Eqs. (5-5.3)]. The functions (q_1, q_2) are then determined by integration of the function $f(z) = Q_1 + iQ_2$ [Eqs. (5-5.5) and (5-5.6)].

The method outlined above is useful for the determination of displacement components (u, v) for those cases in which direct integration of the strain–displacement relations fails (see Examples 4-18.1 and 4-18.2).

Example 5-6.1. Stress Function for the Flexural Wrinkling of a Sandwich Panel. Because of in-plane compressive forces $(\overline{F})$ in the compression facing of a sandwich panel (Fig. E5-6.1), flexural wrinkling (Chong and Hartsock, 1974), which is a localized instability, may occur prior to overall buckling. The compression facing can be treated approximately as a plate supported by the elastic core bounded by the tension facing. The core under plane strain conditions is governed by Eq. (5-4.12) in terms of the Airy stress function F:

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \tag{a}$$

Equation (a) may be satisfied by taking F in the form (Timoshenko and Goodier, 1970)

$$F = F(x, y) = f(y) \sin \alpha x \tag{b}$$

provided $f(y)$ satisfies the equation

$$\frac{\partial^4 f}{\partial y^4} - 2\alpha^2 \frac{\partial^2 f}{\partial y^2} + \alpha^4 f = 0 \tag{c}$$

The solution to Eq. (c) is

$$f(y) = C_1 \cosh \alpha y + C_2 \sinh \alpha y + C_3 y \cosh \alpha y + C_4 y \sinh \alpha y \tag{d}$$

To determine C_1, C_2, C_3, and C_4, the following four boundary conditions are used:

At $y = 0$: $\qquad \sigma_y = -q_m \sin \alpha x \qquad \epsilon_x = 0 \qquad$ (e)

At $y = D$: $\qquad \epsilon_x = 0 \qquad \dfrac{\partial v}{\partial x} = 0 \qquad$ (f)

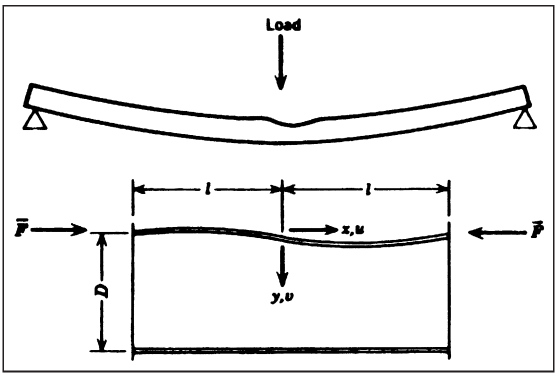


Figure E5-6.1

where q_m is the amplitude of the stress at the interface resulting from deformation of the compression facing. Expressing the stresses, hence the strain ϵ_x, in terms of the Airy stress function, we may employ Eqs. (e) and (f) to determine the constants C_1, C_2, C_3, and C_4. The resulting stress function F is

$$\begin{aligned} F(x, y) = \frac{q_m}{\alpha^2} \sin \alpha x \Bigg\{ &\cosh \alpha y - \frac{\alpha y}{2(1-\nu)} \sinh \alpha y \\ &- \left[(1+\nu)\beta^2 + \sinh^2 \beta(-6 + 8\nu + 6\nu^2 - 8\nu^3)\right] \\ &\times \frac{\alpha}{2(1-\nu)\Delta} \sinh \alpha y - \left[\alpha^2 \sinh^2 \beta (3 - \nu - 4\nu^2)\right] \frac{y \cosh \alpha y}{2(1-\nu)\Delta} \Bigg\} \end{aligned} \tag{g}$$

in which $\Delta = -\alpha[(1+\nu)\beta + (3 - \nu - 4\nu^2)\sinh\beta\cosh\beta]$, and $\beta = \alpha \mathrm{D}$.

Problem Set 5-6

1. The skewed plate of unit thickness is loaded by uniformly distributed stresses S_1 and S_2 applied perpendicularly to the sides of the plate (see Fig. P5-6.1).

(a) Determine all conditions of equilibrium for the plate in terms of S_1, S_2, a, b, and θ.

(b) For $\theta = 90°$, derive an expression for the elongation of the diagonal AC under the action of S_1 and S_2. Assume that the material is homogeneous, isotropic, and linearly elastic, and that the displacements are small.

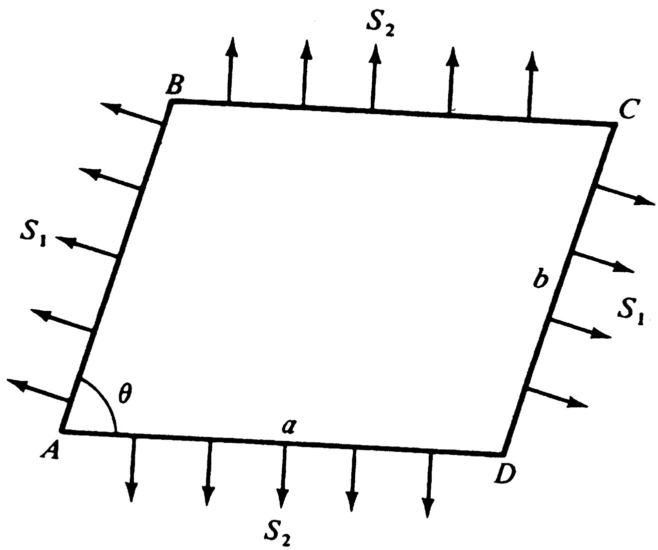


Figure P5-6.1

2. In Fig. P5-6.1, let S_1 and S_2 be applied so that they are directed parallel to the edges AB(DC) and AD(BC) of the skewed plate. Assuming that the plate is elastic, derive expressions for the principal stresses and the principal strains in terms of $S_1, S_2, a, b, \theta, E$, and ν, where E and ν denote Young's modulus and Poisson's ratio, respectively.

3. Let isotropic elastic material in the (x, y) plane be subjected to the stress components $\sigma_x = 0, \sigma_y = \sigma$, and $\tau_{xy} = \tau$. Let $u = v = \omega = 0$ for $x = y = 0$, where (u, v) denote (x, y) displacement components and ω denotes volumetric rotation.

(a) Show that the circle $x^2 + y^2 = a^2$ is deformed into an ellipse.

(b) For the case $\tau = 0$, show that the major and minor axes of the ellipse coincide with the (x, y) axes, and express their lengths in terms of a and the elastic properties of the material.

4. For the isotropic, homogeneous, and elastic cantilever beam shown in Fig. P5-6.4, the stresses are given by

$$\sigma_x = \frac{P}{I}(L - x)y \qquad \tau_{xy} = \frac{P}{2I}(y^2 - c^2) \qquad \sigma_y = 0$$

where P, I, L, and c are constants.

(a) Verify that these stresses satisfy equilibrium and compatibility conditions for plane stress.

(b) Determine the strains, hence the displacements u and v, as functions of x and y. The boundary conditions are for $x = y = 0, u = v = 0$, and an infinitesimal line segment originally in the y direction does not rotate.

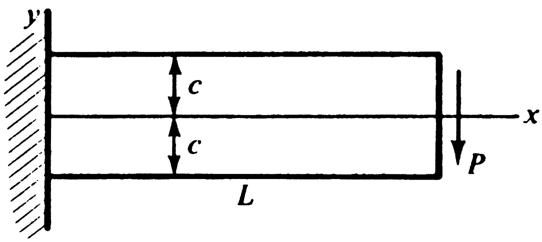


Figure P5-6.4

5. The rectangular plate shown in Fig. P5-6.5 is very thin in the z direction and has a length in the $\pm x$ directions that is very large compared to $2a$. The plate is made of a nonlinear elastic isotropic homogeneous material whose stress–strain relations are

$$\epsilon_x = A\sigma_x^3 - B\sigma_y^3 \qquad \epsilon_y = A\sigma_y^3 - B\sigma_x^3$$

where A and B are known constants. The plate is subjected to angular velocity ω about the x axis. The mass density of the plate is ρ. Assume that $\tau_{xy} = u = X = \partial/\partial x = 0$. Determine the stresses σ_x and σ_y and the displacement v as functions of y.

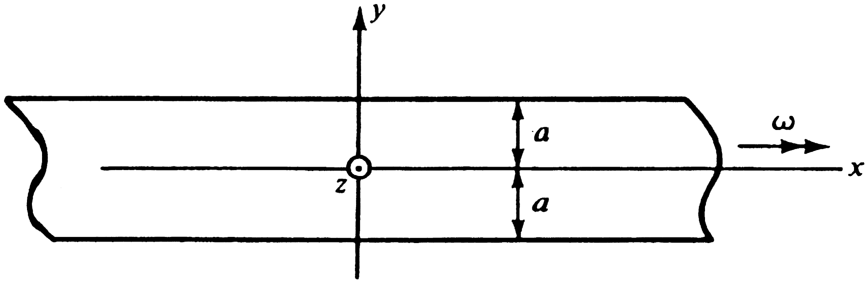


Figure P5-6.5

6. A narrow uniform bar of density ρ and length $2b$ is rotating with an angular velocity ω about an axis perpendicular to the bar through its center. Neglecting gravity effects and assuming linear elastic behavior, determine the increase in length of the bar resulting from the rotation.

7. Assume the plate of Problem 5 is linearly elastic. Determine the stress components σ_x, σ_y and the displacement component v as functions of y.

8. Consider the equations of linear elasticity of a homogeneous isotropic body. For example, the equations of motion are

$$(\lambda + G)\frac{\partial^2 u_\alpha}{\partial x_\alpha \partial x_\beta} + G\frac{\partial^2 u_\beta}{\partial x_\alpha \partial x_\alpha} = \rho\frac{\partial^2 u_\beta}{\partial t^2}$$

For the case of static equilibrium, assume that u_α is representable in the form

$$2Gu_\alpha = \frac{\partial \phi}{\partial x_\alpha}$$

where ϕ is a scalar function of rectangular Cartesian coordinates (x_1, x_2, x_3).

(a) Derive the defining equation for ϕ.

(b) In terms of ϕ, derive expressions for the volumetric strain (dilatation) e, the strain tensor (small displacement) $\epsilon_{\alpha\beta}$, and the stress tensor $\sigma_{\alpha\beta}$.

(c) Let $F = A(x^2 - y^2) + 2Bxy$ be an Airy stress function, where (A, B) are constants and (x, y) are plane rectangular Cartesian coordinates. Determine the problem solved by this function F for the plane rectangular region $-a \le x \le a$, $-b \le y \le b$.

9. The thin homogeneous plane strip of width $2h$ extends a great distance in the $\pm x$ direction (Fig. P5-6.9). The plate is rigidly restrained by the fixed walls at $y = \pm h$. The plate is loaded by gravity in the $-y$ direction. The density of the plate is ρ. Assume $\partial/\partial x = u = \tau_{xy} = X = 0$. The plate is made of a material whose stress–strain relations are

$$\epsilon_x = A\sigma_x^3 - B\sigma_y^3 \qquad \epsilon_y = A\sigma_y^3 - B\sigma_x^3 \qquad \gamma_{xy} = C\tau_{xy}$$

where A, B, and C are known constants. Determine formulas for σ_x, σ_y, and v as functions of y and the known constants A, B, C, ρ, g, and h.

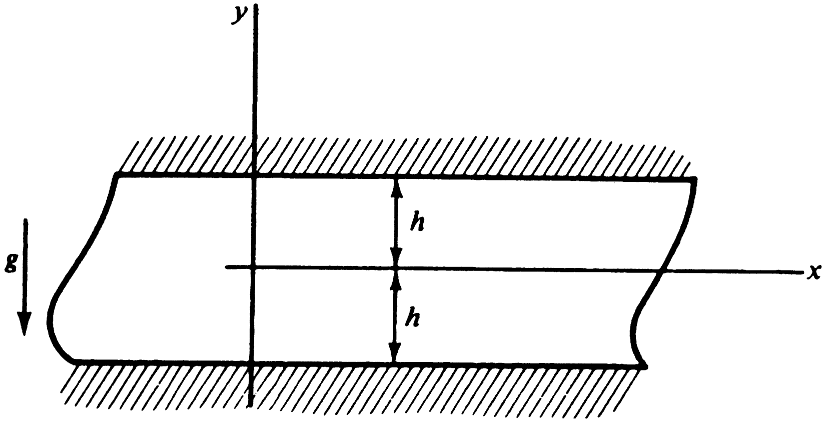


Figure P5-6.9

10. A flat strip is supported at one end ($x = 0$; Fig. P5-6.10) and hangs in a gravity field of acceleration g. The mass density of the strip is ρ. Let the thickness of the strip be 1 unit. Assume a state of plane stress relative to the (x, y) plane.

(a) Consider the equilibrium of the part of the bar from $x = x$ to $x = L$. Write expressions for the body forces X, Y and for the net force acting at section x.

(b) Assume the simplest possible stress distribution in the bar and derive an expression for the normal stress σ_x.

(c) By the semi-inverse method, determine whether equations of elasticity are satisfied by the results of parts (b) and (a).

(d) Derive explicit expressions for the (x, y) displacement components (u, v) in terms of properties of the bar and (x, y). Let $u = v = \partial u/\partial y = 0$ at $x = y = 0$.

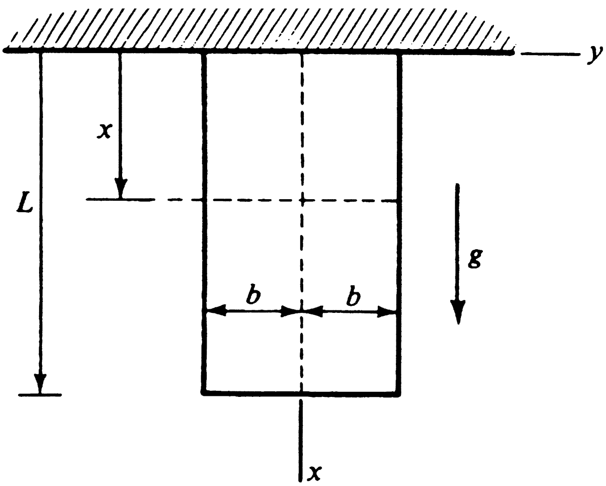


Figure P5-6.10

5-7 Polynomial Solutions of Two-Dimensional Problems in Rectangular Cartesian Coordinates

For plane elasticity with constant body forces or with body forces derivable from a potential function, the compatibility relations reduce to the following single equation in terms of a stress function F (for simply connected regions):

$$\nabla^2\nabla^2 F = 0 \tag{5-7.1}$$

where for plane rectangular Cartesian coordinates (x, y),

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5-7.2}$$

For zero body force, the stress components $\sigma_x, \sigma_y, \tau_{xy}$ are related to F by the equations

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \tag{5-7.3}$$

In the absence of body forces, Eqs. (5-7.3) automatically satisfy equilibrium [Eqs. (5-2.11)]. Accordingly, any solution to Eq. (5-7.1) represents the solution of a certain problem of plane elasticity. For example, any of the terms of Eq. (5-4.13) represents a solution to Eq. (5-7.1). Hence, Eq. (5-4.13) represents a set of solutions of the problem of plane elasticity.

If the stress function F is taken in the form of a polynomial in x and y, we note [see Eqs. (5-7.3)] that nontrivial (nonzero) stress components are obtained only for a polynomial of second degree or higher in x and y. Furthermore, Eq. (5-7.1) is satisfied identically by polynomials of third degree in x and y. For polynomials of degree higher than three, Eq. (5-7.1) requires the coefficients of all terms of degree higher than three to satisfy a set of $n - 3$ auxiliary conditions, where n is the degree of the polynomial.

For discontinuous loads on boundaries, the polynomial method has severe theoretical limitations, as discontinuous boundary conditions are not representable by polynomials. For continuously varying loads, however, the polynomial method seems to be unlimited theoretically, although in practice the computations may quickly become prohibitive if boundary conditions are to be precisely satisfied. Furthermore, because the computations soon become laborious in any case, the polynomial method requires a systematic approach. One such approach has been proposed by Neou (1957).

Method of Neou. The method proposed by C. Y. Neou (1957) systematically reduces the Airy stress function F expressed in a general doubly infinite power series to the desirable polynomial form for special cases. The method proceeds as follows: Let

$$F = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn} x^m y^n \tag{5-7.4}$$

where $m, n = 0, 1, 2, \ldots$, and A_{mn} are undetermined coefficients that may be arranged in the following rectangular array:

$$\begin{array}{cccccc} A_{00} & A_{01} & A_{02} & A_{03} & A_{04} & \cdots \\ A_{10} & A_{11} & A_{12} & A_{13} & A_{14} & \cdots \\ A_{20} & A_{21} & A_{22} & A_{23} & A_{24} & \cdots \\ A_{30} & A_{31} & A_{32} & A_{33} & A_{34} & \cdots \\ A_{40} & A_{41} & A_{42} & A_{43} & A_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \tag{5-7.5}$$

Substitution of Eq. (5-7.4) into Eqs. (5-7.3) yields

$$\sigma_x = \sum_{m=0}^{\infty}\sum_{n=2}^{\infty} n(n-1)A_{mn}x^m y^{n-2} \tag{5-7.6}$$

$$\sigma_y = \sum_{m=2}^{\infty}\sum_{n=0}^{\infty} m(m-1)A_{mn}x^{m-2} y^{n} \tag{5-7.7}$$

$$\tau_{xy} = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} mnA_{mn}x^{m-1} y^{n-1} \tag{5-7.8}$$

Because A_{00}, A_{01}, and A_{10} do not occur in Eqs. (5-7.6), (5-7.7), and (5-7.8), they may be omitted from Eq. (5-7.5).

Substitution of Eq. (5-7.4) into Eq. (5-7.1) yields

$$\begin{aligned} &\sum_{m=4}^{\infty}\sum_{n=0}^{\infty} m(m-1)(m-2)(m-3)x^{m-4}y^n A_{mn} \\ &\quad + 2\sum_{m=2}^{\infty}\sum_{n=2}^{\infty} m(m-1)n(n-1)x^{m-2}y^{n-2}A_{mn} \\ &\quad + \sum_{m=0}^{\infty}\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)x^m y^{n-4}A_{mn} = 0 \end{aligned} \tag{5-7.9}$$

Collecting similar powers of x and y and writing Eq. (5-7.9) under one summation sign, we obtain

$$\begin{aligned} \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \big[&(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)n(n-1)A_{mn} \\ &+ (n+2)(n+1)n(n-1)A_{m-2,n+2}\big]x^{m-2}y^{n-2} = 0 \end{aligned} \tag{5-7.10}$$

Because Eq. (5-7.10) must be satisfied for all values of x and y,

$$(m+2)(m+1)m(m-1)A_{m+2,n-2} + 2m(m-1)n(n-1)A_{mn} + (n+2)(n+1)n(n-1)A_{m-2,n+2} = 0 \tag{5-7.11}$$

Equation (5-7.11) establishes an interrelation among any three alternate coefficients in the diagonals of Eq. (5-7.5), running from lower left to upper right. For example, for $m = 2$ and $n = 2$, Eq. (5-7.11) yields

$$3A_{40} + A_{22} + 3A_{04} = 0$$

Similarly, other relations between the A_{mn} may be established by Eq. (5-7.11).

In the manner outlined above, the plane problem of elasticity with continuous boundary stress is reduced to the determination of A_{mn} [see Eqs. (5-7.4) and (5-7.5)] from the interdependence relations [Eq. (5-7.11)] and the prescribed boundary conditions.

Alternatively, the plane problem of elasticity may be solved by more general techniques, such as transform methods (Milne-Thompson, 1942; Stevenson, 1943; Green, 1945; Sneddon, 1995) or by methods of complex variables (Muskhelishvili, 1975).

Example 5-7.1. Stress Function Compatibility and Stresses. A prismatic cantilever beam has a length L, a rectangular cross section of unit thickness, and a depth $2c$. At its unsupported (free) end it is subjected to an axial tensile load P_1 applied at the centroid of the cross section and a vertical load P_2 parallel to the depth dimension $2c$. By the method of Neou (1957), an engineer develops the following formula for the corresponding Airy stress function:

$$F = \frac{1}{4c}\left(3P_2xy - \frac{P_2xy^3}{c^2} + P_1y^2\right) \tag{a}$$

where x, y are coordinates along the beam and along the depth direction, respectively, with origin at the centroid of the cross section of the free end. We wish to verify the correctness of Eq. (a).

To check the compatibility, we must ensure that F given by Eq. (a) satisfies Eq. (5-7.1). Substitution of Eq. (a) into Eq. (5-7.1) verifies the result $\nabla^2\nabla^2 F = 0$. Thus, F is a valid stress function. Next, we must examine the boundary conditions at the unsupported (free) end $x = 0$. By Eqs. (5-7.3) we find

$$\begin{aligned}
\sigma_x &= \frac{\partial^2 F}{\partial y^2} = \frac{P_1}{2c} - \frac{3P_2xy}{2c^3} \\
\sigma_y &= \frac{\partial^2 F}{\partial x^2} = 0 \\
\tau_{xy} &= -\frac{\partial^2 F}{\partial x \partial y} = -\frac{3P_2(c^2 - y^2)}{4c^3}
\end{aligned} \tag{b}$$

At the free end $x = 0$, the stress components must satisfy the conditions

$$\int_{-c}^{c} \sigma_x dy = P_1$$
$$\int_{-c}^{c} \tau_{xy} dy = -P_2 \tag{c}$$

Substitution of Eqs. (b) into Eq. (c) verifies that Eqs. (c) are satisfied. At the supported end of the beam, the support at $x = L$ must exert stress components σ_x, τ_{xy} on the beam, as given by Eq. (b), for the solution to be valid throughout the beam.

Problem Set 5-7

1. Determine the interrelations of A_{mn} [Eq. (5-7.11)] for $(m = 4, n = 2)$, $(m = 3, n = 3)$, and $(m = 2, n = 4)$.

2. By the method of Neou (1957), derive a polynomial in x and y for the Airy stress function F for the cantilever beam loaded as shown in Fig. P5-7.2 Hence, derive formulas for the stress components $\sigma_x, \sigma_y, \tau_{xy}$. What stress boundary conditions exist at $x = L$? Discuss the application of Saint-Venant's principle to this problem (see Section 4-15 in Chapter 4).

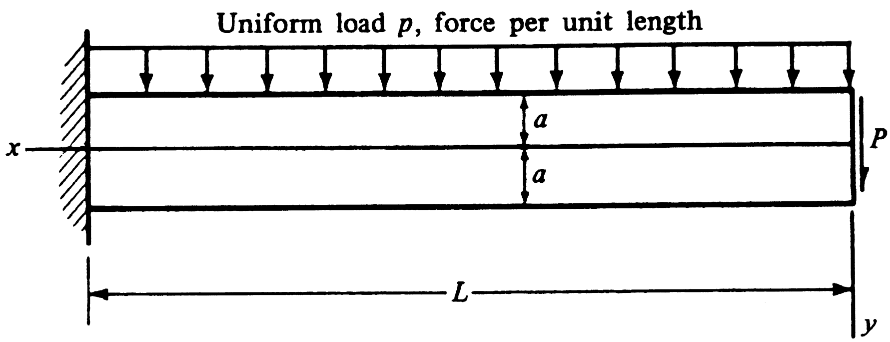


Figure P5-7.2

3. By the method of Neou (1957), derive a polynomial in x and y for the Airy stress function F for the beam loaded as shown in Fig. P5-7.3. Hence, derive formulas for the stress components $\sigma_x, \sigma_y, \sigma_{xy}$. Discuss the application of Saint-Venant's principle to this problem (see Section 4-15).

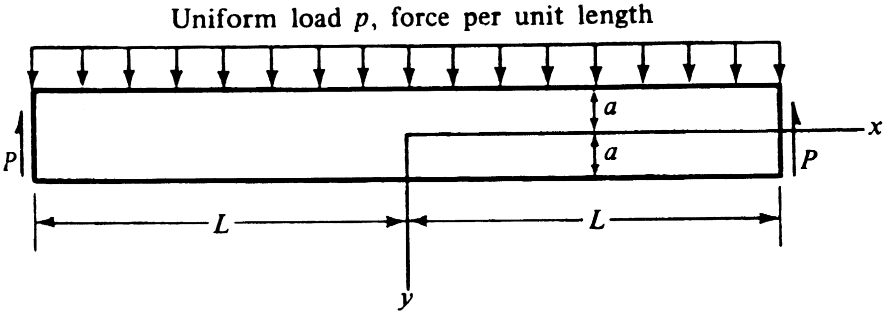


Figure P5-7.3

4. A cantilever beam is loaded as shown in Fig. P5-7.4.

(a) Derive expressions for the stresses in the beam using the stress function

$$\phi = C_1 xy + C_2 \frac{x^3}{6} + C_3 \frac{x^3 y}{6} + C_4 \frac{xy^3}{6} + C_5 \frac{x^3 y^3}{9} + C_6 \frac{xy^5}{20}$$

At the boundary $x = 0$ the solution is to satisfy the condition that the resultant force system vanishes (i.e., $F_x = F_y = M_z = 0$). What stress boundary conditions exist at $x = L$?

(b) Derive expressions for the displacement components u and v, assuming that the beam is in a state of plane stress and that it is fixed at the left end so that

$$u(L, 0) = v(L, 0) = 0 \qquad \frac{\partial u}{\partial y}(L, 0) = 0$$

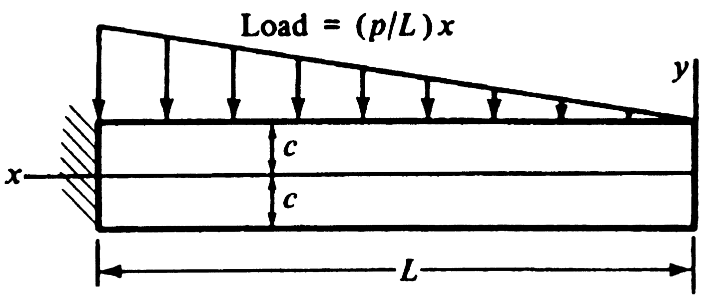


Figure P5-7.4

5. The Airy function $F = Ax^3 y$ generates a solution for a plane strain problem with zero body forces. Is this an exact three-dimensional solution? Explain. Determine the stresses and displacements by any valid procedure (Section 5-6).

6. A long prismatic dam is subjected to water pressure that increases linearly with depth. The dam has thickness $2b$ and height h (Fig. P5-7.6). Formulate the stress determination problem as a well-posed plane problem. State whether the problem is plain strain or generalized plane stress. Relax the boundary conditions at $x = 0$ and $x = h$ to require only restrictions on the resultant force system. Solve the problem using the stress function

$$F = A_1xy + A_2x^3 + A_3x^3y + A_4xy^3 + A_5(5x^3y^3 - 3xy^5)$$

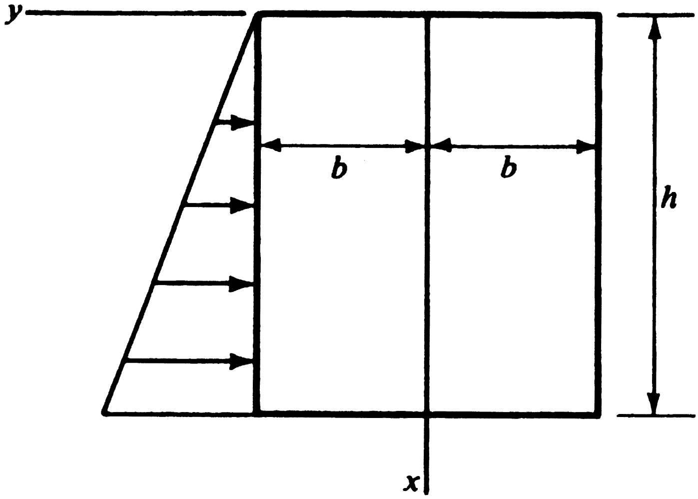


Figure P5-7.6

7. By the method of Neou (1957), the Airy stress function

$$F = \frac{p}{60a}\left(5\frac{L^2}{a^2} - 3\right)y^3 + \frac{p}{40a^3}y^5 - \frac{pa}{40L}xy + \frac{p}{20aL}xy^3 - \frac{p}{40a^3L}xy^5 - \frac{p}{4}x^2 + \frac{3p}{8a}x^2y - \frac{p}{8a^3}x^2y^3 + \frac{p}{12L}x^3 - \frac{p}{8aL}x^3y + \frac{p}{24a^3L}x^3y^3$$

is obtained for a rectangular beam supported by end shear load and subjected to a triangular load as shown in Fig. P5-7.7. Discuss the validity of the solution.

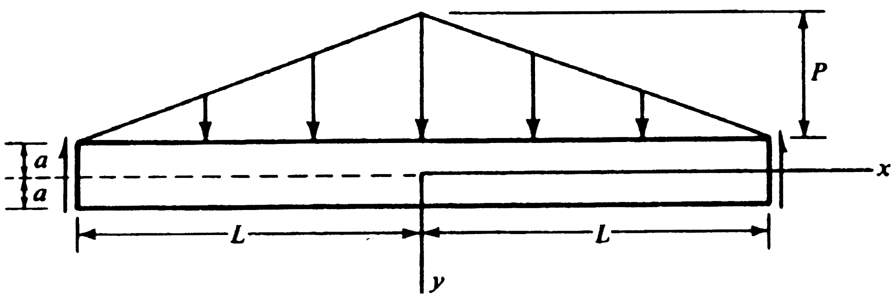


Figure P5-7.7

8. The cantilever beam shown in Fig. P5-7.8 is subjected to a distributed shear stress on the upper face. Assume the stress function for the problem to be of the form

$$F = C_1y^2 + C_2y^3 + C_3y^4 + C_4y^5 + C_5x^2 + C_6x^2y + C_7x^2y^2 + C_8x^2y^3$$

The boundary conditions are

At $y = -h$: $\tau_{xy} = \sigma_y = 0$

At $y = +h$: $\tau_{xy} = \dfrac{-\tau_0 x}{l}$ $\sigma_y = 0$

At the free end, the resultant forces and moment are zero. Determine the eight constants $C_1, C_2, \ldots, C_8$.

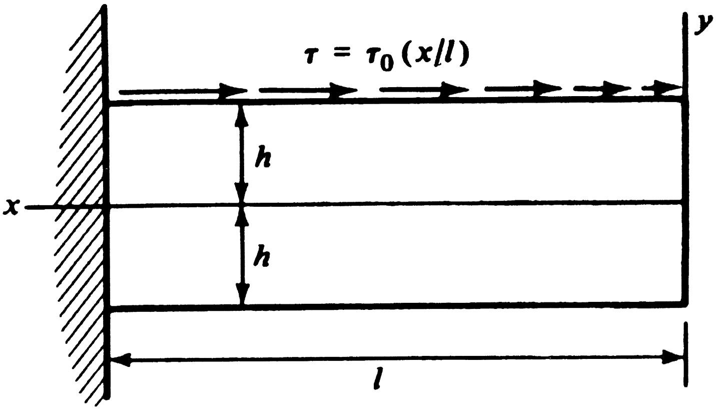


Figure P5-7.8

9. Consider the polynomial $F(x, y) = C_1x^5 + C_2x^4y + C_3x^3y^2 + C_4x^2y^3 + C_5xy^4 + C_6y^5$, where (x, y) are plane rectangular Cartesian coordinates and $C_1, C_2, \ldots, C_6$ are constants.

(a) Determine the conditions for which $F(x, y)$ is an Airy stress function (i.e., for which F is biharmonic).

(b) Derive formulas for the corresponding stress components. Are they compatible?

(c) Let $C_1 = C_3 = C_4 = C_6 = 1$. Specialize the stress formulas accordingly.

(d) Determine the boundary value stress problem for which $F(x, y)$ represents a solution for an isotropic homogeneous elastic medium in the region R bounded by $0 \le x \le 1, 0 \le y \le 1$; that is, determine the boundary stresses that act on the region R.

10. The Airy stress function,

$$F = Ax^2 + Bx^2y + Cy^3 + Dy^5 + Ex^2y^3 \tag{a}$$

where $A, B, \ldots, E$ are constants, can be used to get an approximate plane stress solution for a cantilever beam of unit width, length L, and depth $2c$, subject to a uniform pressure q (force/length) on its upper surface. The coordinates (x, y) have origin on the unsupported (free) end at the centroid of the end cross section, with x directed along the axis of the beam and y directed upward.

(a) Determine the requirements on $A, B, \ldots, E$ so that $F(x, y)$ is biharmonic.

(b) Determine the constants $A, B, \ldots, E$ so that the boundary conditions of the problem are satisfied (pressure q for $y = c$; zero net force and net moment on the free end $x = 0$).

5-8 Plane Elasticity in Terms of Displacement Components

In many problems it is convenient to seek solutions in terms of the displacement components. Accordingly, in this section we present equations of plane elasticity relative to the (x, y) plane in terms of (x, y) displacement components (u, v). We consider the case of plane stress, the results for plane strain being obtained in an analogous manner. We employ the approximations of small displacements.

In terms of (x, y) Cartesian coordinates, the strain components $\epsilon_x, \epsilon_y, \tau_{xy}$ in terms of (x, y) displacement components (u, v) are

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{5-8.1}$$

Hence, substitution of Eqs. (5-8.1) into Eqs. (5-3.10) yields the stress–displacement relations

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y} - (1+\nu)kT\right] \\ \sigma_y &= \frac{E}{1-\nu^2}\left[\nu\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - (1+\nu)kT\right] \\ \tau_{xy} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \end{aligned} \tag{5-8.2}$$

Equations (5-2.11) and (5-8.2) yield (in the absence of body forces and for variable modulus of elasticity E)

$$\begin{aligned} &u_{xx} + \frac{1}{2}(1-\nu)u_{yy} + \frac{1}{2}(1+\nu)v_{xy} + (u_x + \nu v_y)\frac{1}{E}\frac{\partial E}{\partial x} \\ &\quad + \frac{1}{2}(1-\nu)(u_y + v_x)\frac{1}{E}\frac{\partial E}{\partial y} = \frac{1+\nu}{E}\frac{\partial(EkT)}{\partial x} \\ &\frac{1}{2}(1+\nu)u_{xy} + \frac{1}{2}(1-\nu)v_{xx} + v_{yy} + \frac{1}{2}(1-\nu)(u_y + v_x)\frac{1}{E}\frac{\partial E}{\partial x} \\ &\quad + (v_y + \nu u_x)\frac{1}{E}\frac{\partial E}{\partial y} = \frac{1+\nu}{E}\frac{\partial(EkT)}{\partial y} \end{aligned} \tag{5-8.3}$$

where subscripts (x, y) on (u, v) denote partial derivatives. For $E =$ constant, $\partial E/\partial x = \partial E/\partial y = 0$.

Similarly, Eqs. (5-8.1), (5-3.8), and (5-2.11) yield for plane strain

$$\begin{aligned}
&(1-\nu)u_{xx} + \frac{1}{2}(1-2\nu)u_{yy} + \frac{1}{2}v_{xy} + [(1-\nu)u_x + \nu v_y]\frac{1}{E}\frac{\partial E}{\partial x} \\
&\quad + \frac{1}{2}(1-2\nu)(u_y + v_x)\frac{1}{E}\frac{\partial E}{\partial y} = \frac{1+\nu}{E}\frac{\partial(EkT)}{\partial x} \\
&\frac{1}{2}u_{xy} + \frac{1}{2}(1-2\nu)v_{xx} + (1-)v_{yy} + \frac{1}{2}(1-2\nu)(u_y + v_x)\frac{1}{E}\frac{\partial E}{\partial x} \\
&\quad + [\nu u_x + (1-\nu)v_y]\frac{1}{E}\frac{\partial E}{\partial y} = \frac{1+\nu}{E}\frac{\partial(EkT)}{\partial y}
\end{aligned} \tag{5-8.4}$$

The solution to Eqs. (5-8.3) or (5-8.4) subject to appropriate boundary conditions constitutes the solution of the plane problem of elasticity. Ordinarily, exact solutions to these equations are not readily achieved. Then we may resort to approximate numerical methods. For certain problems the concept of a displacement potential function may be useful (see Example 5-4.1).

Problem Set 5-8

1. Consider the small-displacement plane elasticity problem of plane stress relative to the (x, y) plane. Express the equilibrium equations in terms of (u, v), the (x, y) displacement components, including the effects of temperature $T(x, y)$, and letting the modulus of elasticity E be dependent on (x, y). Include body forces.

2. A state of plane strain relative to the (x, y) plane is defined by $u = u(x, y)$, $v = v(x, y)$, and $w = 0$. The strain energy density U_0 of a certain crystal undergoing plane strain is given by

$$U_0 = \tfrac{1}{2}\left(b_{11}\epsilon_x^2 + b_{22}\epsilon_y^2 + b_{33}\gamma_{xy}^2 + 2b_{12}\epsilon_x\epsilon_y + 2b_{13}\epsilon_x\gamma_{xy} + 2b_{23}\epsilon_y\gamma_{xy}\right)$$

where b_{ij}, $i, j = 1, 2, 3$ are elastic coefficients. For small-displacement theory, derive the differential equations of equilibrium in terms of (u, v) for plane strain of the crystal, including the effects of body forces.

5-9 Plane Elasticity Relative to Oblique Coordinate Axes

In certain classes of plane problems, it is convenient to employ elasticity equations relative to oblique coordinate axes. Accordingly, consider oblique coordinates (ξ, η) with axis ξ coincident with axis x of rectangular Cartesian axes (x, y) and axis η forming angle θ relative to axes ξ (Fig. 5-9.1). Hence, a typical point P in a region may be located by the coordinates (ξ, η) or (x, y), where

$$x = \xi + \eta\cos\theta \qquad y = \eta\sin\theta \tag{5-9.1}$$

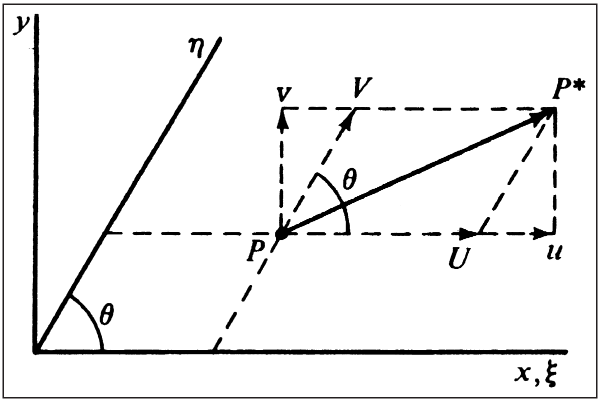


Figure 5-9.1

or

$$\xi = x - y \cot\theta \qquad \eta = y\csc\theta \tag{5-9.2}$$

Under a deformation, point P goes into point P^* under displacement components (u, v) relative to axes (x, y) or (U, V) relative to axes (ξ, η), where

$$u = U + V\cos\theta \qquad v = V\sin\theta \tag{5-9.3}$$

or

$$U = u - v\cot\theta \qquad V = v\csc\theta \tag{5-9.4}$$

We consider $u = u(x, y)$, $v = v(x, y)$ and $U = U(\xi, \eta)$, $V = V(\xi, \eta)$.

For small-displacement theory, we obtain from Eqs. (2B-13) in Chapter 2, discarding quadratic terms (and letting $x_1 = x$, $x_2 = y$, $y_1 = \xi$, $y_2 = \eta$, etc.) the strain components $(\epsilon_\xi, \epsilon_\eta, \gamma_{\xi\eta})$ relative to axes (ξ, η) as

$$\begin{aligned} \epsilon_\xi &= \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \xi}\cos\theta \\ \xi_\eta &= \frac{\partial V}{\partial \eta} + \frac{\partial U}{\partial \eta}\cos\theta \\ \gamma_{\xi\eta} &= \frac{\partial V}{\partial \xi} + \frac{\partial U}{\partial \eta} + \left(\frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \eta}\right)\cos\theta \end{aligned} \tag{5-9.5}$$

Also, by the chain rule of partial differentiation and Eqs. (5-9.3), we have for the strain components $(\epsilon_x, \epsilon_y, \gamma_{xy})$

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial U}{\partial \xi} + \frac{\partial V}{\partial \xi}\cos\theta \\ \epsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial V}{\partial \eta} - \frac{\partial V}{\partial \xi}\cos\theta \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\partial U}{\partial \eta} - \frac{\partial V}{\partial \xi}\cos 2\theta\right)\csc\theta + \left(\frac{\partial V}{\partial \eta} - \frac{\partial U}{\partial \xi}\right)\cot\theta \end{aligned} \tag{5-9.6}$$

For $\theta = \pi/2$, Eqs. (5-9.5) and (5-9.6) reduce to the usual results for orthogonal axes. By Eqs. (5-9.5) and (5-9.6), we obtain

$$\begin{aligned} \epsilon_x &= \epsilon_\xi \\ \epsilon_y &= \epsilon_\xi \cot^2\theta + \epsilon_\eta \csc^2\theta - \gamma_{\xi\eta} \cot\theta \csc\theta \\ \gamma_{xy} &= \gamma_{\xi\eta} \csc\theta - 2\epsilon_\xi \cot\theta \end{aligned} \tag{5-9.7}$$

We define stress components $(\sigma_\xi, \sigma_\eta, \tau_{\xi\eta}, \tau_{\eta\xi})$ relative to axes (ξ, η) by considering an element with sides coincident with (ξ, η) coordinate lines (Fig. 5-9.2; see also Problem 3-8.4 in Chapter 3). Hence, considering equilibrium of forces and moments as for the rectangular Cartesian element, we obtain the equilibrium equations

$$\begin{aligned} \frac{\partial \sigma_\xi}{\partial \xi} + \frac{\partial \tau_{\eta\xi}}{\partial \eta} + \left(\frac{\partial \tau_{\eta\xi}}{\partial \xi} + \frac{\partial \sigma_\eta}{\partial \eta} \right) \cos\theta &= 0 \\ \frac{\partial \tau_{\xi\eta}}{\partial \xi} + \frac{\partial \sigma_\eta}{\partial \eta} &= 0 \\ \tau_{\xi\eta} &= \tau_{\eta\xi} \end{aligned} \tag{5-9.8}$$

For $\theta = \pi/2$, Eqs. (5-9.8) reduce to the usual equation of equilibrium relative to orthogonal plane axes (x, y).

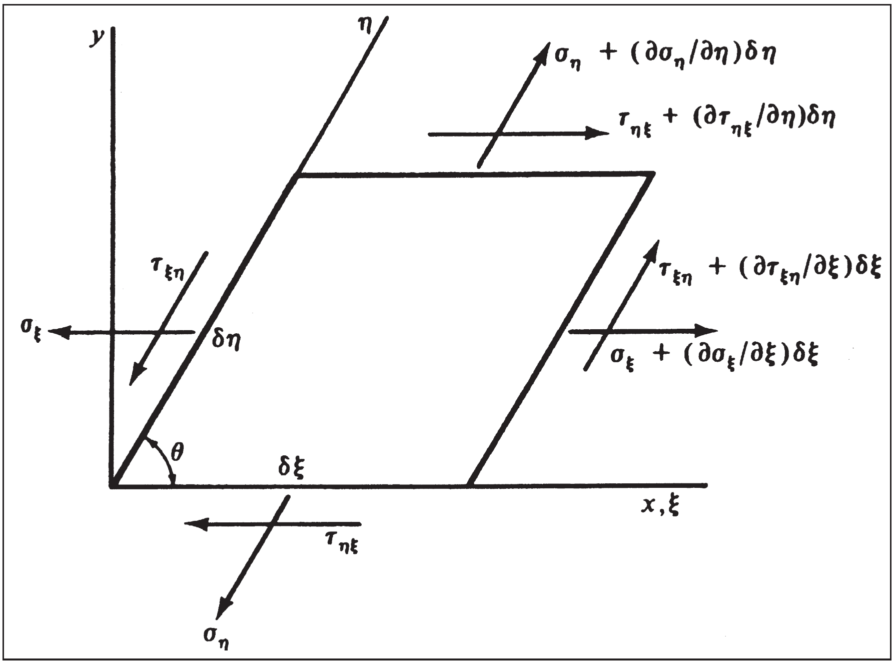


Figure 5-9.2

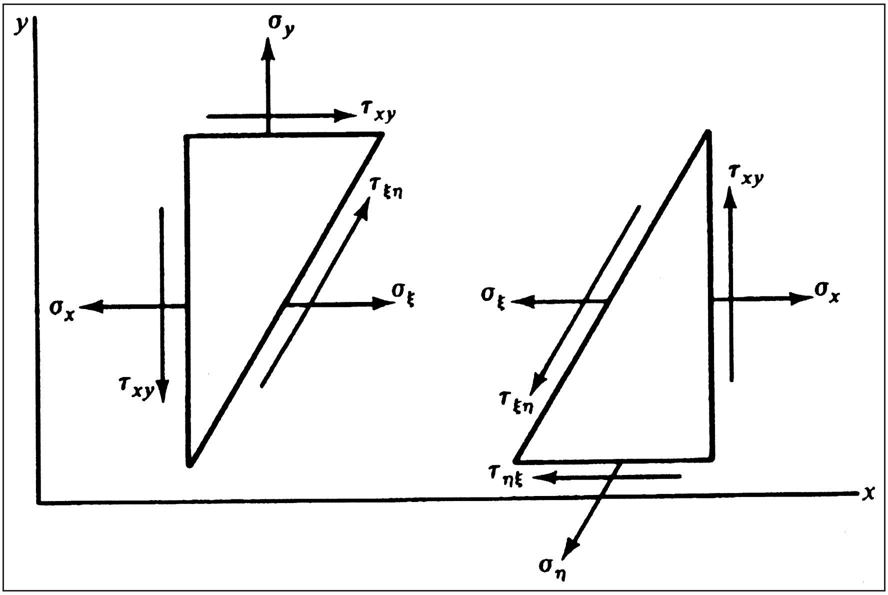


Figure 5-9.3

Relations between stress components $(\sigma_x, \sigma_y, \tau_{xy})$ defined relative to axes (x, y) and $(\sigma_\xi, \sigma_\eta, \tau_{\xi\eta}, \tau_{\eta\xi})$ defined relative to axes (ξ, η) may be derived by considering the equilibrium of appropriate elements. Accordingly, by the equilibrium conditions for the elements shown in Fig. 5-9.3, we obtain

$$\begin{aligned} \sigma_\xi &= \sigma_x \sin\theta - 2\tau_{xy}\cos\theta + \sigma_y \cos\theta\cot\theta \\ \sigma_\eta &= \sigma_y \csc\theta \\ \tau_{\xi\eta} &= \tau_{\eta\xi} = \tau_{xy} - \sigma_y \cot\theta \end{aligned} \tag{5-9.9}$$

Substitution of Eqs. (5-9.7) into Eqs. (5-3.8) yields for *plane strain*

$$\begin{aligned} \sigma_x &= K_1\big[(1-\nu+\nu\cot^2\theta)\epsilon_\xi + \nu\epsilon_\eta\csc^2\theta - \nu\gamma_{\xi\eta}\csc\theta\cot\theta - (1+\nu)kT\big] \\ \sigma_y &= K_1\big[(\nu+\cot^2\theta-\nu\cot^2\theta)\epsilon_\xi + (1-\nu)\epsilon_\eta\csc^2\theta \\ &\qquad - (1-\nu)\gamma_{\xi\eta}\csc\theta\cot\theta - (1+\nu)kT\big] \\ \tau_{xy} &= K_1\left[\frac{1-2\nu}{2}\gamma_{\xi\eta}\csc\theta - (1-2\nu)\epsilon_\xi\cot\theta\right] \end{aligned} \tag{5-9.10}$$

where

$$K_1 = \frac{E}{(1+\nu)(1-2\nu)} \tag{5-9.11}$$

Hence, substitution of Eqs. (5-9.10) into Eqs. (5-9.9) yields the stress–strain relations for plane strain states relative to coordinate axes (ξ, η). Thus, we find

$$\begin{aligned}
\sigma_\xi \sin^3\theta &= K_1\big[(1-\nu)\epsilon_\xi + (\cos^2\theta - \nu\cos 2\theta)\epsilon_\eta \\
&\qquad - (1-\nu)\gamma_{\xi\eta}\cos\theta - (1+\nu)kT\sin^2\theta\big] \\
\sigma_\eta \sin^3\theta &= K_1\big[(\cos^2\theta - \nu\cos 2\theta)\epsilon_\xi + (1-\nu)\epsilon_\eta \\
&\qquad - (1-\nu)\gamma_{\xi\eta}\cos\theta - (1+\nu)kT\sin^2\theta\big] \\
\tau_{\xi\eta} \sin^3\theta &= K_1\big[-(1-\nu)(\epsilon_\xi+\epsilon_\eta) + \tfrac{1}{2}(1-2\nu+\cos^2\theta)\gamma_{\xi\eta} \\
&\qquad + (1+\nu)kT\sin^2\theta\cos\theta\big]
\end{aligned} \tag{5-9.12}$$

Similarly, for plane stress [Eqs. (5-3.10)] we obtain

$$\begin{aligned}
\sigma_\xi \sin^3\theta &= K_2\big[\epsilon_\xi + (\cos^2\theta + \nu\sin^2\theta)\epsilon_\eta - \gamma_{\xi\eta}\cos\theta \\
&\qquad - (1+\nu)kT\sin^2\theta\big] \\
\sigma_\eta \sin^3\theta &= K_2\big[(\cos^2\theta + \nu\sin^2\theta)\epsilon_\xi + \epsilon_\eta - \gamma_{\xi\eta}\cos\theta \\
&\qquad - (1+\nu)kT\sin^2\theta\big] \\
\tau_{\xi\eta} \sin^3\theta &= K_2\big[-(\epsilon_\xi+\epsilon_\eta)\cos\theta + \tfrac{1}{2}(1+\cos^2\theta - \nu\sin^2\theta)\gamma_{\xi\eta} \\
&\qquad + (1+\nu)kT\sin^2\theta\cos\theta\big]
\end{aligned} \tag{5-9.13}$$

where

$$K_2 = \frac{E}{1-\nu^2} \tag{5-9.14}$$

The preceding equations find application in cases where orthogonal plane axes do not coincide with the boundary curves of the region, for example, in parallelogram regions such as swept-back airplane wings (Fig. 5-9.2). A general development for the theory of shells in nonorthogonal coordinates has been presented by Langhaar (1961).

APPENDIX 5A PLANE ELASTICITY WITH COUPLE STRESSES

5A-1 Introduction

The basic distinction between the classical theory of stress and the theory of stress including couple stresses lies in the nature of the assumed interaction of the material on two sides of a surface element. In the classical theory, it is assumed that the action of the material on one side of the surface upon the material on the other side of the surface is equipollent to a force (see Section 3-1 and Fig. 3-1.1 in Chapter 3). In couple stress theory, the interaction is assumed to be equipollent to a force and a couple (stress couple). Further refinement is also admitted in the nature of assumed body couples (analogous to body forces; see Section 3-8). The couple stresses are taken to be moments per unit area, and the body couples are moments per unit volume.

It has been noted that relatively few practical applications of couple stress (body couple) theories are known (Schijve, 1966; Ellis and Smith, 1967; Koiter, 1968). Nevertheless, the theory is less restrictive than the classical stress theory of Euler and Cauchy. Furthermore, applications of the simplest theory of elasticity, in which couple stresses are admitted, to problems in which the analogous classical solutions yield locally unbounded stresses or deformations indicate that the results (e.g., singularities) are changed, softened, or perhaps eliminated (Sternberg, 1968).

Accordingly, in this appendix we give a brief discussion of the linear couple stress theory for the equilibrium of homogeneous isotropic elastic solids under the conditions of *plane strain*. In particular, we follow the heuristic procedure employed by Mindlin (Mindlin, 1963; Weitsman, 1965; Kaloni and Ariman, 1967). Finally, although the whole of the classical theory of elasticity seems in agreement with the assumption that couple stresses vanish, a study of the couple stress theory may lead to a critical reexamination of the basic concepts and principles of the mechanics of continuum. In this last regard, one may read with profit the study by Toupin (1964).

5A-2 Equations of Equilibrium

For the plane problem relative to the (x, y) plane and in the absence of body forces and couples, the stress equations of equilibrium for a medium that can support couple stresses are, in (x, y) notation (see Fig. 5A-2.1; see also Appendix 3B in Chapter 3),

$$\begin{aligned} \sum F_x = 0 &: \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0 \\ \sum F_y = 0 &: \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \\ \sum M_0 = 0 &: \frac{\partial m_{xz}}{\partial x} + \frac{\partial m_{yz}}{\partial y} + \tau_{xy} - \tau_{yx} = 0 \end{aligned} \tag{5A-2.1}$$

Accordingly, for nonconstant couple stresses $(\partial m_{xz}/\partial x \neq 0, \partial m_{yz}/\partial y \neq 0)$, the shear stresses are not necessarily equal (i.e., $\tau_{xy} \neq \tau_{yx}$). Conversely, if (τ_{xy}, τ_{yx}) are equal to zero, the couple stresses (x_{xz}, m_{yz}) need not vanish. Equations (5A-2.1) are the Cosserat equations of equilibrium for plane problems with body forces and couples omitted (Cosserat and Cosserat, 1909).

5A-3 Deformation in Couple Stress Theory

We now treat the case of plane strain. As noted in Section 5-1, for plane strain relative to the (x, y) plane, the displacement components (u, v) are functions of (x, y) only and $w = 0$. Hence, for an isotropic elastic medium, the normal strains (ϵ_x, ϵ_y) are related to the normal stresses (σ_x, σ_y) by the first two of Eqs. (5-1.7),

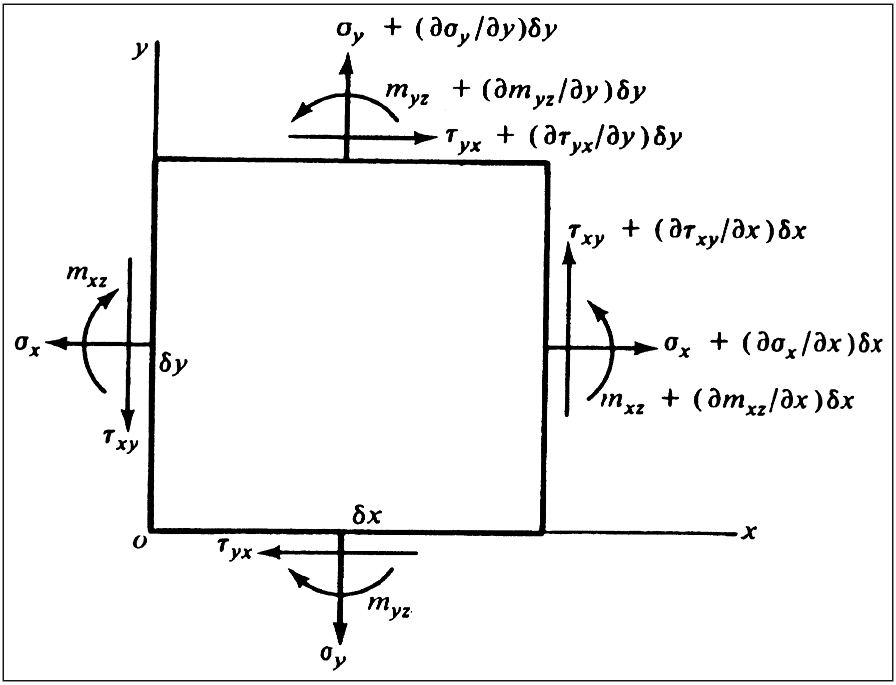


Figure 5A-2.1

and (ϵ_x, ϵ_y) are related to (u, v) by the first two of Eqs. (5-1.4). Furthermore, the shear strain γ_{xy} is related to (u, v) by the fourth of Eqs. (5-1.4). However, because in general $\tau_{xy} \neq \tau_{yx}$, the third of Eqs. (5-1.7) is no longer valid. Hence, following Mindlin (1963), we resolve τ_{xy} and τ_{yx} into a symmetric part τ_S and an antisymmetric part τ_A (see Section 1-25 in Chapter 1 and Fig. 5A-3.1):

$$\tau_S = \tfrac{1}{2}(\tau_{xy} + \tau_{yz}) \qquad \tau_A = \tfrac{1}{2}(\tau_{xy} - \tau_{yx}) \tag{5A-3.1}$$

Accordingly, by Fig. 5A-3.1 and Section 2-8 in Chapter 2, the symmetric part τ_S produces the shear strain

$$\gamma_{xy} = \frac{1}{G}\tau_S = \frac{1+\nu}{E}(\tau_{xy} + \tau_{yx}) \tag{5A-3.2}$$

where $G = E/[2(1+\nu)]$ is the modulus of shear. Similarly, the antisymmetric part τ_A produces a local rigid rotation (Fig. 5A-3.1 and Section 2-13):

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{5A-3.3}$$

Furthermore, the antisymmetric part τ_A is balanced by the couple stresses [Eq. (5A-2.1)].

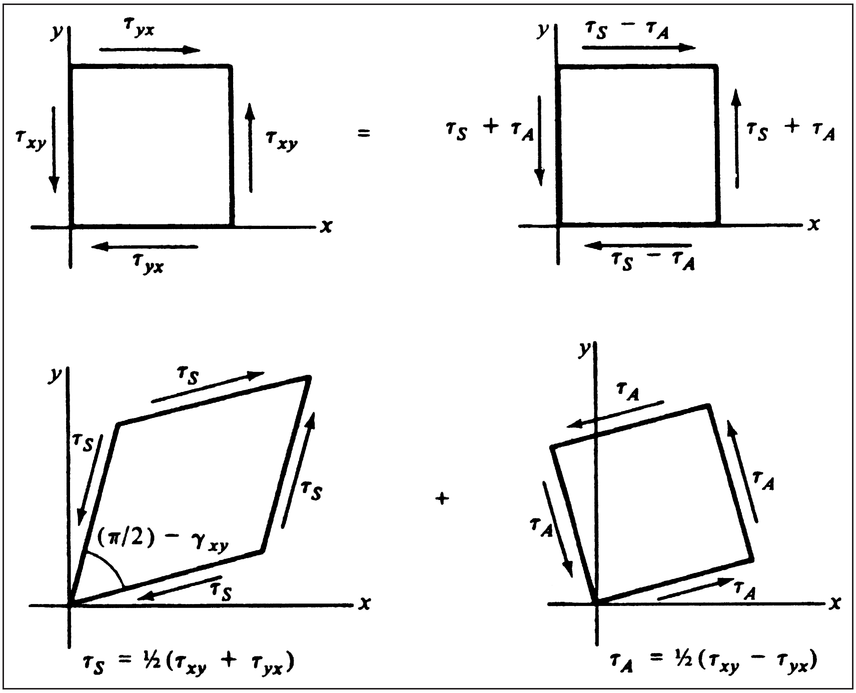


Figure 5A-3.1

Considering the effect of the couple stresses on the element $(\delta x, \delta y)$, Fig. 5A-3.2, we note that m_{xz}, m_{yz} produce curvatures κ_{xz} and κ_{yz} related to the rotation ω_z by the equations

$$R_{xz}\frac{\partial \omega_z}{\partial x}\delta x = \delta_x \qquad R_{yz}\frac{\partial \omega_z}{\partial y}\delta y = \delta y$$

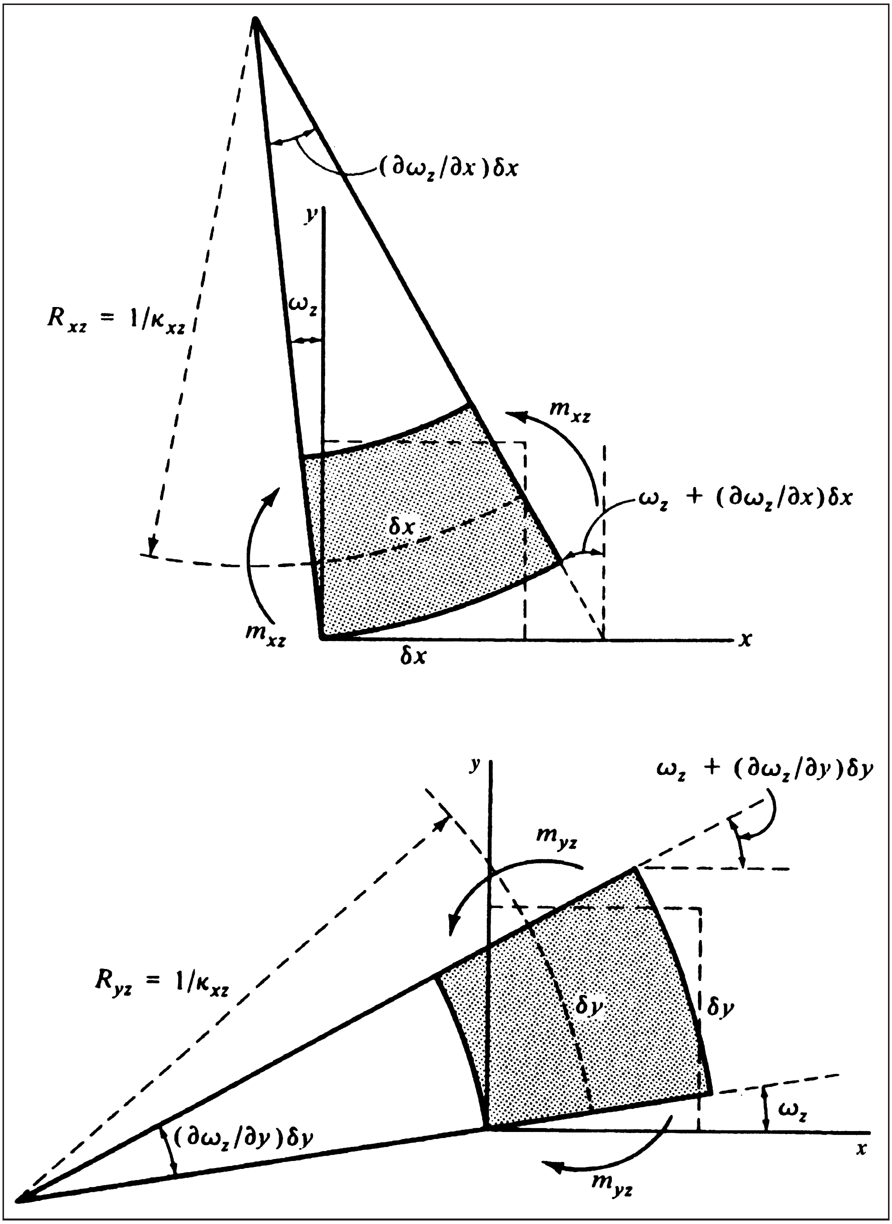
or

$$\kappa_{xz} = \frac{\partial \omega_z}{\partial x} \qquad \kappa_{yz} = \frac{\partial \omega_z}{\partial y} \tag{5A-3.4}$$

Analogous to the shearing strain γ_{xy} relation to the symmetric part τ_S of τ_{xy}, τ_{yz}, we assume that the curvatures $(\kappa_{xz}, \kappa_{yz})$ (deformations) are proportional to the stress couples (m_{xz}, m_{yz}) (forces):

$$\kappa_{xz} = \frac{1}{4B}m_{xz} \qquad \kappa_{yz} = \frac{1}{4B}m_{yz} \tag{5A-3.5}$$

where B [see Eq. (5A-3.2)] is a modulus of curvature or bending, and the factor 4 is taken for convenience in later calculations. We note that because the couple stresses have the dimensions of couple per unit area or force per unit length and curvature is the reciprocal of length, the modulus B has the dimensions of force.



(∂ωz/∂x)δx
y
Rxz = 1/κxz
ωz
mxz
ωz + (∂ωz/∂x)δx
δx
mxz
δx
x
y
ωz + (∂ωz/∂y)δy
myz
Ryz = 1/κxz
δy
δy
ωz
(∂ωz/∂y)δy
x
myz

Figure 5A-3.2

5A-4 Equations of Compatibility

Equations (5-1.4), (5A-3.3), and (5A-3.4) consist of five deformation quantities $(\epsilon_x, \epsilon_y, \gamma_{xy}, \kappa_{xz}, \kappa_{yz})$ expressed in terms of two displacement components. By elimination of the displacement components from Eqs. (5-1.4), we obtain the usual equations of strain compatibility [Eq. (5-3.1)].

Similarly, elimination of the rotation ω_z from Eqs. (5A-3.4) yields

$$\frac{\partial \kappa_{xz}}{\partial y} = \frac{\partial \kappa_{yz}}{\partial x} \tag{5A-4.1}$$

Now, by Eqs. (5-1.4) and (5A-3.3), we find

$$\begin{aligned} \frac{\partial \omega_z}{\partial x} &= \frac{1}{2}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y}\right) = \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y} \\ \frac{\partial \omega_z}{\partial y} &= \frac{1}{2}\left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}\right) = \frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y} \end{aligned}$$

Hence, by Eqs. (5A-3.4),

$$\begin{aligned} \kappa_{xz} &= \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y} \\ \kappa_{yz} &= \frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y} \end{aligned} \tag{5A-4.2}$$

Seemingly, we have obtained four compatibility relations [Eqs. (5-3.1), (5A-4.1), and (5A-4.2)]. However, we observe that Eqs. (5-4.2) imply Eq. (5A-4.1). Hence, we have the compatibility relations

$$\begin{aligned} \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \frac{\partial \kappa_{xz}}{\partial y} &= \frac{\partial \kappa_{yz}}{\partial x} \\ \kappa_{xz} &= \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial x} - \frac{\partial \epsilon_x}{\partial y} \\ \kappa_{yz} &= \frac{\partial \epsilon_y}{\partial x} - \frac{1}{2}\frac{\partial \gamma_{xy}}{\partial y} \end{aligned} \tag{5A-4.3}$$

where only three relations are independent, as the second equation is implied by the remaining three.

Finally, we note that the four compatibility relations may be written in terms of stress components $(\sigma_x, \sigma_y, \tau_{xy}, \tau_{yx})$ and couple stresses (m_{xz}, m_{yz}) by means of

Eqs. (5A-3.2), (5A-3.5), and the first two of Eqs. (5-1.7). Thus, we obtain

$$\begin{aligned}
\frac{\partial^2 \sigma_x}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu \nabla^2(\sigma_x + \sigma_y) &= \frac{\partial^2}{\partial x \partial y}(\tau_{xy} + \tau_{yx}) \\
\frac{\partial m_{xz}}{\partial y} &= \frac{\partial m_{yz}}{\partial x} \\
m_{xz} &= l^2 \frac{\partial}{\partial x}(\tau_{xy} + \tau_{yx}) - 2l^2 \frac{\partial}{\partial y}[\sigma_x - \nu(\sigma_x + \sigma_y)] \\
m_{yz} &= 2l^2 \frac{\partial}{\partial x}[\sigma_y - \nu(\sigma_x + \sigma_y)] - l^2 \frac{\partial}{\partial y}(\tau_{xy} + \tau_{yx})
\end{aligned} \tag{5A-4.4}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{5A-4.5}$$

and

$$l^2 = \frac{2(1+\nu)B}{E} = \frac{B}{G} \tag{5A-4.6}$$

where l^2 is the ratio of the material constants, B and G. By the last two of Eqs. (5A-4.4), we note that large stress gradients may lead to large values of the couple stresses (m_{xz}, m_{yz}) when $l^2 \neq 0$. If $l = 0$, the material has relatively no resistance to curvature effects $(B/G = 0)$, Eqs. (5A-3.5). Because the second of Eqs. (5A-4.4) is implied by the other three equations, only three of the four compatibility equations are independent.

5A-5 Stress Functions for Plane Problems with Couple Stresses

Equation (5A-2.1) may be solved by means of stress functions in a manner analogous to the solution of Eqs. (5-4.1) by means of the Airy stress function (Carlson, 1966).

According to the theory of total differentials (Section 1-19 in Chapter 1 and Section 5-4), the first of Eqs. (5A-2.1) is a necessary and sufficient condition for the existence of a function ϕ of (x, y) such that

$$\sigma_x = \frac{\partial \phi}{\partial y} \qquad \tau_{yx} = -\frac{\partial \phi}{\partial x} \tag{5A-5.1}$$

and the second of Eqs. (5A-2.1) yields in a similar manner

$$\sigma_y = \frac{\partial \theta}{\partial x} \qquad \tau_{xy} = -\frac{\partial \theta}{\partial y} \tag{5A-5.2}$$

where $\theta = \theta(x, y)$. Furthermore, the second of Eqs. (5A-4.4) admits a function $\psi = \psi(x, y)$ such that

$$m_{xz} = \frac{\partial \psi}{\partial x} \qquad m_{yz} = \frac{\partial \psi}{\partial y} \tag{5A-5.3}$$

Substitution of Eqs. (5A-5.1), (5A-5.2), and (5A-5.3) into the last of Eqs. (5A-2.1) yields

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x} + \phi\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y} - \theta\right) = 0 \tag{5A-5.4}$$

which in turn is a necessary and sufficient condition that the function $H = H(x, y)$ exists, such that

$$\frac{\partial \psi}{\partial x} + \phi = \frac{\partial H}{\partial y} \qquad \frac{\partial \psi}{\partial y} - \theta = -\frac{\partial H}{\partial x} \tag{5A-5.5}$$

or

$$\phi = \frac{\partial H}{\partial y} - \frac{\partial \psi}{\partial x} \qquad \theta = \frac{\partial H}{\partial x} + \frac{\partial \psi}{\partial y} \tag{5A-5.6}$$

Hence, substitution of Eqs. (5A-5.6) into Eqs. (5A-5.1) and (5A-5.2) yields expressions for $\sigma_x, \sigma_y, \tau_{yx}, \tau_{xy}$ in terms of ψ and H. Thus, we obtain the formulas

$$\begin{aligned}
\sigma_x &= \frac{\partial^2 H}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} \qquad \sigma_y = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \\
\tau_{xy} &= -\frac{\partial^2 H}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2} \qquad \tau_{yx} = -\frac{\partial^2 H}{\partial x \partial y} + \frac{\partial^2 \psi}{\partial x^2} \\
m_{xz} &= \frac{\partial \psi}{\partial x} \qquad m_{yz} = \frac{\partial \psi}{\partial y}
\end{aligned} \tag{5A-5.7}$$

where all components of stress and couple stress are expressed in terms of the two stress functions H and ψ. For $\psi = 0$, $m_{xz} = m_{yz} = 0$, and Eqs. (5A-5.7) reduce to the classical Airy stress function relations [Eqs. (5-4.9) with $V = 0$].

Differential Equations for H and ψ. The remaining equations [Eqs. (5A-4.4)] of compatibility define the functions H and ψ. Hence, substitution of the first four of Eqs. (5A-5.7) into the first of Eqs. (5A-4.4) yields

$$\nabla^2\nabla^2 H = \nabla^4 H = 0 \tag{5A-5.8}$$

Thus, H is the Airy stress function of classical stress theory [see (Eq. 5-4.12)]. Finally, substitution of Eqs. (5A-5.7) into the last two of Eqs. (5A-4.4) yields

$$\begin{aligned}
\frac{\partial}{\partial x}(\psi - l^2\nabla^2\psi) &= -2(1-\nu)l^2\frac{\partial}{\partial y}(\nabla^2 H) \\
\frac{\partial}{\partial y}(\psi - l^2\nabla^2\psi) &= -2(1-\nu)l^2\frac{\partial}{\partial x}(\nabla^2 H)
\end{aligned} \tag{5A-5.9}$$

Accordingly, the functions $\psi - l^2\nabla^2\psi$ and $2(1-\nu)l^2\nabla^2 H$ are conjugate harmonic functions; that is, they satisfy the Cauchy–Riemann equations [see Eqs. (5-5.3)].

By Eqs. (5A-5.9), we obtain, by differentiating the first of Eqs. (5A-5.9) by x and the second by y, and adding,

$$\nabla^2\psi - l^2\nabla^4\psi = 0 \tag{5A-5.10}$$

Similarly, differentiations with respect to y first and then x yield Eqs. (5A-5.8). Thus, the defining equations for H and ψ are Eqs. (5A-5.8) and (5A-5.10). The theory of plane strain with couple stresses is contained in Sections 5A-2 through 5A-5. The theory of plane stress may be derived in an analogous manner. In Appendix 6A, the plane strain theory is applied to the problem of a circular hole in a field of uniform tension as well as in a biaxial field of stress.

APPENDIX 5B PLANE THEORY OF ELASTICITY IN TERMS OF COMPLEX VARIABLES

The material treated in Sections 5-5 and 5-6 is essential for the topics discussed in this appendix.

5B-1 Airy Stress Function in Terms of Analytic Functions $\psi(z)$ and $\chi(z)$

It may be shown that the Airy (biharmonic) stress function $F(x, y)$ may be expressed in terms of two analytic functions of the complex variable $z = x+$ iy (Muskhelishvili, 1975). By this result, we transform the plane theory of elasticity into complex variable theory.

In Section 5-5 we introduced the analytic function $\psi(z) = q_1+$ iq$_2$ and noted that $F - xq_1 - yq_2$ is harmonic, where $i = \sqrt{-1}$, (q_1, q_2) are conjugate harmonic functions and F is the Airy (biharmonic) stress function. Hence, the Airy stress function may be written in the following forms [see Eqs. (5-5.11), (5-5.12), and (5-5.13)]:

$$\begin{aligned} F &= xq_1 + yq_2 + h_1 \\ F &= 2xq_1 + h_2 \\ F &= 2yq_2 + h_3 \end{aligned} \tag{5B-1.1}$$

where h_1, h_2, h_3 are arbitrary harmonic functions in the plane region D.

By the appropriate combination of two analytic functions defined in D, we now note that we may generate the Airy stress function in the form of the first of Eqs. (5B-1.1). To do this, we first introduce the analytic function

$$\chi(z) = p_1 + ip_2(z)$$

where (p_1, p_2) are conjugate harmomic functions. Next, we form the real part of $\overline{z}\psi(z) + \chi(z)$, where $\psi(z)$ is defined by Eq. (5-5.6) and $\overline{z} = x - iy$. Thus, we obtain

$$\begin{aligned} \text{Re}\,[\overline{z}\psi(z) + \chi(z)] &= \text{Re}[(x - iy)(q_1 + iq_2) + p_1 + ip_2] \\ &= xq_1 + yq_2 + p_1 \end{aligned} \tag{5B-1.2}$$

Accordingly, comparison of the first of Eqs. (5B-1.1) and Eq. (5B-1.2) yields

$$F = \text{Re}[\bar{z}\psi(z) + \chi(z)] \tag{5B-1.3}$$

Alternatively, Eq. (5B-1.3) may be written more symmetrically by employing the complex conjugations of ψ and χ and noting that the sum of a complex function and its conjugate yields a real function. Thus,

$$\bar{z}\psi(z) + z\overline{\psi(z)} + \chi(z) + \overline{\chi(z)} = 2(xq_1 + yq_2 + p_1)$$

Hence, we may write F in the form

$$2F = \bar{z}\psi(z) + z\overline{\psi(z)} + \chi(z) + \overline{\chi(z)} \tag{5B-1.4}$$

Equations (5B-1.3) and (5B-1.4) express the Airy stress function F in terms of the two analytic functions $\psi(z)$ and $\chi(z)$ and their complex conjugates. It is readily shown that Eq. (5B-1.4) satisfies the condition $\nabla^2\nabla^2 F = 0$.

5B-2 Displacement Components in Terms of Analytic Functions $\psi(z)$ and $\chi(z)$

For the case of plane stress, the (x, y) displacement components in terms of the Airy stress function F and the complex conjugate harmonic functions (q_1, q_2) are given by Eq. (5-6.8). By Eq. (5B-1.4), we obtain

$$\begin{aligned}\frac{\partial F}{\partial x} &= \tfrac{1}{2}[\psi(z) + \bar{z}\psi'(z) + \overline{\psi(z)} + z\overline{\psi'(z)} + \chi'(z) + \overline{\chi'(z)}]\\ \frac{\partial F}{\partial y} &= \tfrac{i}{2}[-\psi(z) + \bar{z}\psi'(z) + \overline{\psi(z)} - z\overline{\psi'(z)} + \chi'(z) - \overline{\chi'(z)}]\end{aligned} \tag{5B-2.1}$$

where primes denote differentiation with respect to z.

In developing the theory, it is expedient to express quantities in terms of $\partial F/\partial x + i(\partial F/\partial y)$. Hence, by Eq. (5B-1.2) we write

$$\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y} = \psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)} \tag{5B-2.2}$$

Consequently, multiplication of the second of Eqs. (5-6.8) by i and addition to the first of Eqs. (5-6.8) yields, with Eq. (5B-2.2),

$$2G(u + iv) = \kappa\psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)} \tag{5B-2.3}$$

where for plane stress (also generalized plane stress)

$$\kappa = \frac{3 - \nu}{1 + \nu} \qquad \text{(plane strain)} \tag{5B-2.4}$$

In an analogous manner, we also obtain Eq. (5B-2.3) for the case of plane strain, where for plane strain

$$\kappa = 3 - 4\nu \qquad \text{(plane stress)} \tag{5B-2.5}$$

One may transform the expression for plane strain into the equivalent expression for plane stress by the following substitutions:

$$\begin{aligned} \frac{1-\nu^2}{E} \quad &\text{(plane strain)} \to \frac{1}{E} \qquad \text{(plane stress)} \\ \nu \quad &\text{(plane strain)} \to \frac{\nu}{1+\nu} \qquad \text{(plane stress)} \end{aligned} \tag{5B-2.6}$$

Equation (5B-2.3) is the fundamental displacement relation in the complex variable theory of plane elasticity.

5B-3 Stress Components in Terms of $\psi(z)$ and $\chi(z)$

Consider a line element AB joining two points in a medium in the (x, y) plane, with positive direction from A to B. Axes n, t are normal and tangential, respectively, to AB at point P. They form a right-handed coordinate system as do (x, y) (Fig. 5B-3.1). Let the forces $\sigma_{nx}\, ds$, $\sigma_{ny}\, ds$ act on the infinitesimal element ds, with positive sense in the directions of positive (x, y), respectively. Hence, the stress components acting on an element of the medium with sides dx, dy, ds (Fig. 5B-3.2)

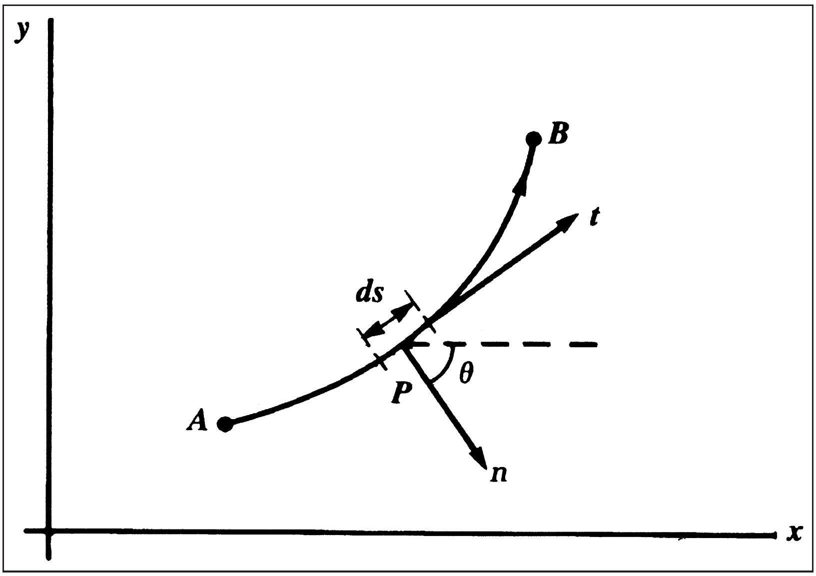


Figure 5B-3.1

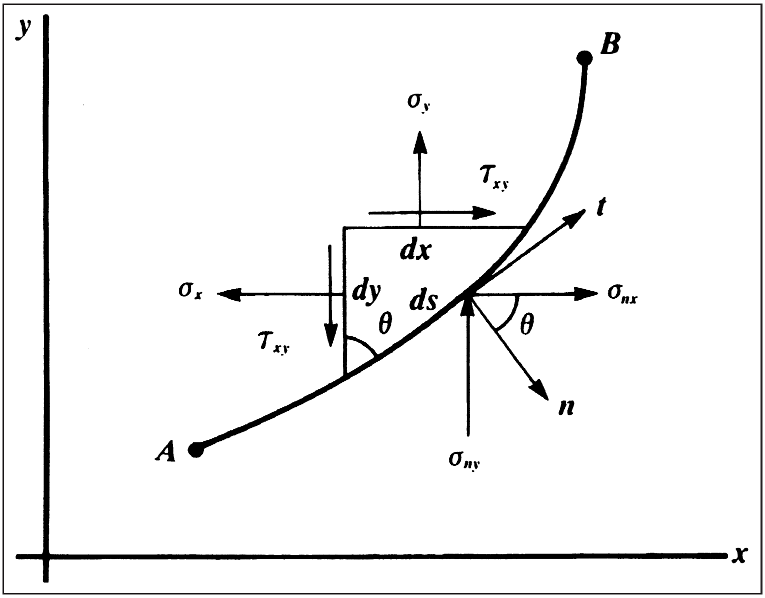


Figure 5B-3.2

are $\sigma_x, \sigma_y, \tau_{xy}, \sigma_{nx}, \sigma_{ny}$. For plane equilibrium of the element, we have (in the absence of body forces)

$$\sum F_x = \sigma_{nx}\,ds - \sigma_x\,dy + \tau_{xy}\,dx = 0$$
$$\sum F_y = \sigma_{ny}\,ds + \sigma_y\,dx + \tau_{xy}\,dy = 0$$

or

$$\begin{aligned}\sigma_{nx} &= \sigma_x \cos\theta - \tau_{xy}\sin\theta \\ \sigma_{ny} &= -\sigma_y \sin\theta + \tau_{xy}\cos\theta\end{aligned} \tag{a}$$

where

$$\cos\theta = \frac{dy}{ds} \qquad \sin\theta = \frac{dx}{ds} \tag{b}$$

Expressing σ_x, σ_y, and τ_{xy} in terms of the stress function F, we may write Eq. (a), with Eqs. (b), in the form

$$\sigma_{nx} = \frac{\partial^2 F}{\partial y^2}\frac{dy}{ds} + \frac{\partial^2 F}{\partial x \partial y}\frac{dx}{ds} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)\frac{dx}{ds} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)\frac{dy}{ds}$$

$$\sigma_{ny} = -\frac{\partial^2 F}{\partial x^2}\frac{dx}{ds} - \frac{\partial^2 F}{\partial x \partial y}\frac{dy}{ds} = -\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right)\frac{dx}{ds} - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)\frac{dy}{ds}$$

Accordingly, we may write by the chain rule of differentiation

$$\sigma_{nx} = \frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) \qquad \sigma_{ny} = -\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) \tag{5B-3.1}$$

Hence, multiplying the second of Eqs. (5B-3.1) by i and adding it to the first of Eqs. (5B-3.1), we obtain

$$\sigma_{nx} + i\sigma_{ny} = \frac{d}{ds}\left(\frac{\partial F}{\partial y} - i\frac{\partial F}{\partial x}\right) = i\frac{d}{ds}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right) \tag{5B-3.2}$$

or

$$(\sigma_{nx} + i\sigma_{ny})ds = -id\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right)$$

Substituting Eq. (5B-2.2) into Eq. (5B-3.2), we obtain

$$(\sigma_{nx} + i\sigma_{ny})ds = -id\left[\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)}\right] \tag{5B-3.3}$$

Now let ds have the direction of the y axis. Then $ds = dy, dz = i\,dy, d\overline{z} = -idy$, $\sigma_{nx} = \sigma_x$, and $\sigma_{ny} = \tau_{xy}$. Then, Eq. (5B-3.3) becomes

$$(\sigma_x + i\tau_{xy}) = \psi'(z) + \overline{\psi'(z)} - z\overline{\psi''(z)} - \overline{\chi''(z)} \tag{5B-3.4}$$

Similarly, let ds have the direction of the x axis. Then $ds = dx, dz = dx$, $d\overline{z} = dx, \sigma_{nx} = -\tau_{xy}$, and $\sigma_{ny} = -\sigma_y$, and Eq. (5B-3.3) becomes

$$(\sigma_y - i\tau_{xy}) = \psi'(z) + \overline{\psi'(z)} + z\overline{\psi''(z)} + \overline{\chi''(z)} \tag{5B-3.5}$$

Adding and subtracting Eqs. (5B-3.4) and (5B-3.5), we find

$$\begin{aligned} \nabla^2 F = \sigma_x + \sigma_y &= 2\left[\psi'(z) + \overline{\psi'(z)}\right] = 4\,\text{Re}\left[\psi'(z)\right] \\ \sigma_y - \sigma_x - 2i\tau_{xy} &= 2\left[z\overline{\psi''(z)} + \overline{\chi''(x)}\right] \end{aligned} \tag{5B-3.6}$$

or by complex conjugation we obtain from the second of Eqs. (5B-3.6)

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\overline{z}\psi''(z) + \chi''(z)] \tag{5B-3.7}$$

where $\psi(z)$ and $\chi(z)$ are analytic functions.

Accordingly, Eqs. (5B-2.3), (5B-3.6), and (5B-3.7) express the components (u, v) of the displacement vector and the components $(\sigma_x, \sigma_y, \tau_{xy})$ of the stress tensor in terms of analytic functions $\psi(z)$ and $\chi(z)$, inside region D occupied by the plane body under consideration.

5B-4 Expressions for Resultant Force and Resultant Moment

Let (F_x, F_y) be the resultant force that acts on an arc AB. Then, by Eqs. (5B-3.2) and (5B-3.3),

$$\begin{aligned} F_x + iF_y &= \int_A^B (\sigma_{nx} + i\sigma_{ny})ds \\ &= -i\int_A^B d\left[\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)}\right] \\ &= -i\left[\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)}\right]\Big|_A^B \end{aligned} \tag{5B-4.1}$$

Similarly, the moment M with respect to origin 0 of coordinate system (x, y) of the forces that act on AB is (Fig. 5B-3.2)

$$\begin{aligned} M &= \int_{AB} (x\sigma_{ny} - y\sigma_{nx})\, ds \\ &= -\int_{AB} \left[xd\left(\frac{\partial F}{\partial x}\right) + yd\left(\frac{\partial F}{\partial y}\right)\right] \\ &= -\left[x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}\right]_A^B + \int_{AB}\left[\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds}\right] ds \\ &= -\left[x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}\right]_A^B + \int_{AB}\frac{dF}{ds}ds \\ &= -\left[x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y}\right]_A^B + F|_A^B \end{aligned} \tag{5B-4.2}$$

Also,

$$x\frac{\partial F}{\partial x} + y\frac{\partial F}{\partial y} = \mathrm{Re}\left[z\left(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right)\right] \tag{5B-4.3}$$

Now, by Eq. (5B-2.1), we obtain

$$\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y} = \overline{\psi(z)} + \bar{z}\psi'(z) + \chi'(z) \tag{5B-4.4}$$

Accordingly, by Eqs. (5B-1.3), (5B-4.2), (5B-4.3), and (5B-4.4), the expression for M may be written

$$M = \mathrm{Re}[\chi(z) - z\chi'(z) - z\bar{z}\psi'(z)]_A^B \tag{5B-4.5}$$

Equations (5B-4.1) and (5B-4.5) represent boundary conditions for resultant force and moment in terms of the analytic functions $\psi(z)$ and $\chi(z)$.

Because we have assumed that region D is simply connected, the function $\psi(z)$ and $\chi(z)$ are single valued. Hence, if points A and B coincide (Fig. 5B-4.1), the curve AB is closed, and the values of ψ and χ are the same at points A and B.

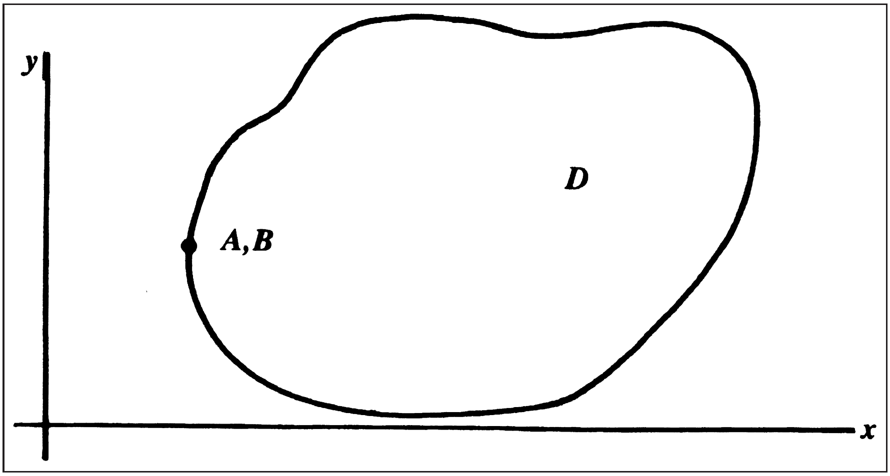


Figure 5B-4.1

Hence, if $A = B$, Eqs. (5B-4.1) and (5B-4.5) yield $F_x = F_y = M = 0$. Thus, for simply connected plane regions, the external forces acting on any part of the region contained inside a closed contour AB is statically equivalent to zero.

5B-5 Mathematical Form of Functions $\psi(z)$ and $\chi(z)$

In this section we consider the degree of arbitrariness of the functions ψ, χ in the cases when (a) the state of stress is given and (b) the displacement field is specified. It is convenient to treat these cases separately. Because $\chi(z)$ occurs in the stress and displacement relations only in the forms $\chi'(z)$ and $\chi''(z)$, it is expedient to define a function $\phi(z)$ such that

$$\chi'(z) = \phi(z) \tag{5B-5.1}$$

Case A. Stress State Given. By Eqs. (5B-5.1), (5B-3.6), and (5B-3.7),

$$\begin{aligned} \sigma_x + \sigma_y &= 2[\psi'(z) + \overline{\psi'(z)}] = 4\,\mathrm{Re}[\psi'(z)] \\ \sigma_y - \sigma_x - 2i\tau_{xy} &= 2[\bar{z}\psi''(z) + \phi'(z)] \end{aligned} \tag{5B-5.2}$$

To determine the nature of $\psi(z), \phi(z)$, we first note that for the simply connected region D, $\psi(z), \phi(z)$ may be specified to within certain arbitrary complex numbers without altering the stress distribution in region R. Thus, the stress quantities $\sigma_x + \sigma_y, \sigma_y - \sigma_x + 2i\tau_{xy}$ may be expressed in terms of either the functions (ψ, ϕ) or the functions (ψ_1, ϕ_1), where

$$\begin{aligned} \psi_1(z) &= \psi(z) + icz + a \\ \phi_1(z) &= \phi(z) + b \end{aligned} \tag{5B-5.3}$$

where (a, b) are complex numbers and c is a real constant. Equations (5B-5.3) follow directly from substitution of $\psi(z), \phi(z)$ and $\psi_1(z), \phi_1(z)$ into Eqs. (5B-5.2) and equating the quantities so obtained (i.e., requiring the same stresses for either set of functions). Integration then yields Eqs. (5B-5.3). In other words, if the state of stress in D is specified, the analytic functions ψ, ϕ are determined to within a linear function icz $+a$ and a complex constant b, respectively.

Case B. Displacement Specified. Let us specify the displacement components (u, v) in region D. By Eqs. (5B-5.1) and (5B-2.3), we find

$$
\begin{aligned}
2G(u+iv) &= \kappa\psi(z) - z\overline{\psi'(z)} - \overline{\phi(z)} \\
\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) &= \frac{(1+\nu)(1+\kappa)}{E}\operatorname{Im}\psi'(z)
\end{aligned}
\tag{5B-5.4}
$$

where ω is the volumetric rotation and $\operatorname{Im}\psi'(z)$ denotes the imaginary value of $\psi'(z)$, that is,

$$
\operatorname{Im}\psi'(z) = -\frac{i}{2}[\psi'(z) - \overline{\psi'(z)}] \tag{5B-5.5}
$$

The second of Eqs. (5B-5.4) follows from the fact that by the first of Eqs. (5B-5.4)

$$
\begin{aligned}
4Gu &= \kappa[\psi(z) + \overline{\psi(z)}] - z\overline{\psi'(z)} - \bar{z}\psi'(z) - \phi(z) - \overline{\phi(z)} \\
4Gv &= -i\kappa[\psi(z) - \overline{\psi(z)}] + i[z\overline{\psi'(z)} - \bar{z}\psi'(z) - \phi(z) - \overline{\phi(z)}]
\end{aligned}
\tag{5B-5.6}
$$

Because the stresses are determined uniquely, when the displacements are given, we conclude that the extent of the arbitrariness in the functions ψ, ϕ can be no greater than that exhibited by Eqs. (5B-5.3). Indeed, the requirement that the functions (ψ, ϕ) and (ψ_1, ϕ_1) yield the same displacements demands that

$$
c = 0 \qquad \kappa a = \bar{b} \tag{5B-5.7}
$$

This restriction is more severe than that of Eq. (5B-5.3). Thus, if the displacements (u, v) are prescribed in D, the function $\psi(z)$ is determined to within a complex constant a and the specification of a defines the constant b. Accordingly, the functions $\psi(z), \phi(z)$ are determined uniquely for a given state of stress, provided a, b, c are chosen so that for the plane region D, the displacement and rotation are specified to account for rigid-body motion. For example, we may specify the displacement and rotation for some point—say, z_0—in R. Then, for example, the conditions

$$
\psi(z_0) = 0 \qquad \operatorname{Im}\psi'(z_0) = 0 \qquad \phi(z_0) = 0 \tag{5B-5.8}
$$

are sufficient to determine the values of a, b, c. If the displacements are specified $c = 0$, and we may choose a so that $\psi(z_0) = 0$. Then, by Eq. (5B-5.7), b is defined.

Form of Functions $\psi(z)$ and $\phi(z)$. By the theory of analytic functions, we know that in a simply connected region D, the analytic functions $\psi(z), \phi(z)$ are single valued and may be represented in the power series (Carrier et al., 2005; Brown and Churchill, 2008) over R:

$$\psi(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \phi(z) = \sum_{n=0}^{\infty} b_n z^n \tag{5B-5.9}$$

If the region D is multiply connected, the functions $\psi(z), \phi(z)$ may be multivalued; that is, they may undergo finite incremental changes in traversing a closed contour defining the interior of D (Brown and Churchill, 2008). Consider for simplicity the doubly connected region R (Fig. 5B-5.1). In circumscribing the boundary C_1, let the functions $\psi(z)$ and $\phi(z)$ receive the increments

$$\Delta\psi(z) = i2\pi\alpha \qquad \Delta\phi(z) = i2\pi\beta \tag{5B-5.10}$$

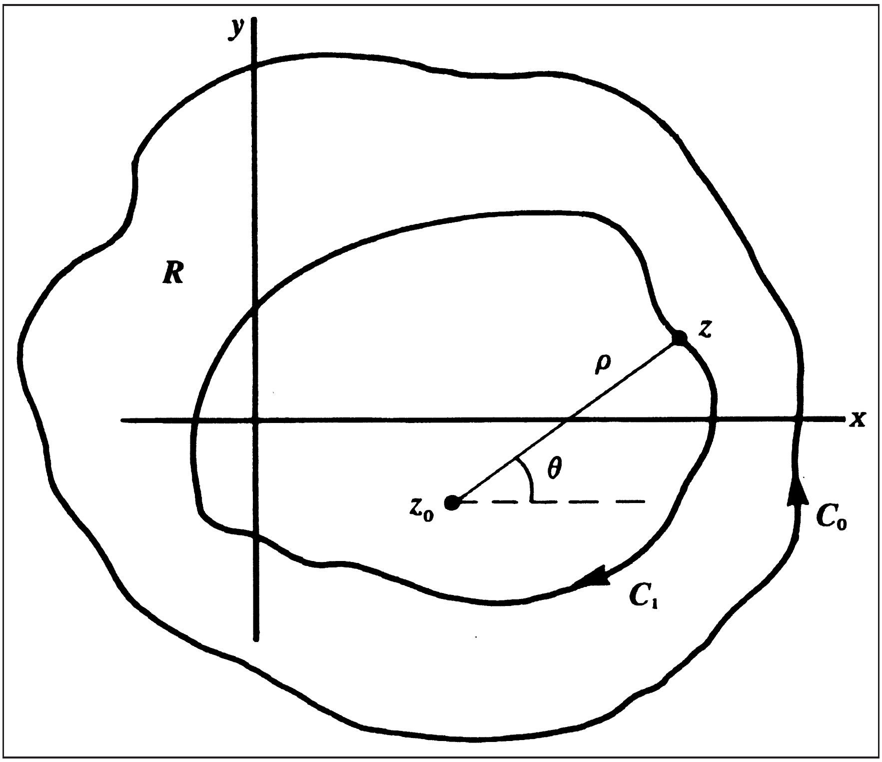


Figure 5B-5.1

where in general (α, β) are complex constants. This type of behavior is exhibited by the function $\log(z - z_0)$, where z_0 is a point inside the contour C_1. Because $z - z_0 = \rho e^{i\theta}$, we have, upon circumscribing C_1,

$$\begin{aligned}\Delta\left[c\log(z - z_0)\right] &= \left[c\log(\rho e^{i\theta})\right]\Big|_{\rho,0}^{\rho,2\pi} \\ &= c\left[\log\rho + \log e^{i\theta}\right]\Big|_{\rho,0}^{\rho,2\pi} \\ &= c\left[\log\rho + i\theta\right]\Big|_{\rho,0}^{\rho,2\pi} = i2\pi c\end{aligned} \tag{5B-5.11}$$

Hence, we may write

$$\begin{aligned}\psi_0(z) &= \psi(z) - \alpha\log(z - z_0) \\ \phi_0(z) &= \phi(z) - \beta\log(z - z_0)\end{aligned} \tag{5B-5.12}$$

where $\psi_0(z), \phi_0(z)$ are analytic within the doubly connected region R, as within R, $\psi_0(z)$ and $\phi_0(z)$ are finite, differentiable, and single valued.

Consequently, we may represent $\psi(z)$ and $\phi(z)$ in the form

$$\begin{aligned}\psi(z) &= \psi_0(z) + \alpha\log(z - z_0) \\ \phi(z) &= \phi_0(z) + \beta\log(z - z_0)\end{aligned} \tag{5B-5.13}$$

The requirement that the displacement $u + iv$ be single valued demands that a relation between the constants α and β exist. Thus, substitution of Eqs. (5B-5.13) into the first of Eqs. (5B-5.4) has the result, with the requirement of single-valued displacements, that the term $\kappa\alpha\log(z - z_0) - \overline{\beta\log(z - z_0)}$ vanishes in circumscribing C_1. Hence, noting that $\Delta[\alpha\log(z - z_0)] = 2\pi i\alpha$ and $\Delta\overline{[\beta\log(z - z_0)]} = -2\pi i\overline{\beta}$, we obtain the relation

$$\kappa\alpha + \overline{\beta} = 0 \tag{5B-5.14}$$

Hence, Eqs. (5B-5.13) become

$$\begin{aligned}\psi(z) &= \psi_0(z) + \alpha\log(z - z_0) \\ \phi(z) &= \phi_0(z) - \kappa\overline{\alpha}\log(z - z_0)\end{aligned} \tag{5B-5.15}$$

Finally, we note that $\psi_0(x), \phi_0(z)$ may be represented by the Laurent series (Brown and Churchill, 2008)

$$\begin{aligned}\psi_0(z) &= \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n \\ \phi_0(z) &= \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n\end{aligned} \tag{5B-5.16}$$

as they are analytic in R. Generalization of these results for n-connected regions is given by Muskhelishvili (1975, Chapter 5).

Transformation under Translation and Rotation of Rectilinear Coordinate Axes. For a given state of stress in a plane region, translation of the origin of rectilinear coordinate axes requires that $\psi(z)$ remain invariant, whereas $\chi(z)$ must be modified to maintain the stress state. For a rotation of axes (x, y) into axes (x_1, y_1) through angle α, the functions (ψ, χ) are given by

$$\psi = \psi_1(\zeta)e^{i\alpha} \qquad \chi = \chi_1(\zeta) \tag{5B-5.17}$$

where $\zeta = ze^{-i\alpha}$ and (ψ_1, χ_1) are functions relative to axes (x_1, y_1), which play the same role as (ψ, χ) relative to axes (x, y) (Muskhelishvili, 1975, p. 137).

5B-6 Plane Elasticity Boundary Value Problems in Complex Form

As with the three-dimensional theory, we may state the following plane boundary value problems of elasticity (in the absence of body forces):

1. Determine the states of stress and displacement in region R for given stresses applied to the boundary B of region R.
2. Determine the states of stress and displacement in region R for given displacement of the boundary B of region R.

The uniqueness of solutions of the above problems may be shown (Section 4-16 in Chapter 4) for bounded displacement field and for stress fields that vanish at infinity (Muskhelishvili, 1975).

For the first problem, the plane theory of elasticity is characterized by the equation (in absence of body forces) (see Sections 5-1 and 5-2):

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0 \end{aligned}\right\} \quad \text{over } R \tag{5B-6.1}$$

$$\nabla^2(\sigma_x + \sigma_y) = 0\} \quad \text{over } R \tag{5B-6.2}$$

$$\left.\begin{aligned} \sigma_{nx} = \sigma_x l + \tau_{xy} m \\ \sigma_{ny} = \tau_{xy} l + \sigma_y m \end{aligned}\right\} \quad \text{on } B \tag{5B-6.3}$$

In terms of the Airy stress function F, we may write these equations as follows (Section 5-4):

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} \qquad \tau_{xy} = \frac{\partial^2 F}{\partial x \partial y} \tag{5B-6.1$'$}$$

$$\nabla^2\nabla^2 F = 0 \tag{5B-6.2$'$}$$

$$\begin{aligned}\sigma_{nx} &= \frac{\partial^2 F}{\partial y^2}\ell - \frac{\partial^2 F}{\partial x \partial y}m \\ \sigma_{ny} &= \frac{\partial^2 F}{\partial x \partial y}\ell + \frac{\partial^2 F}{\partial x^2}m\end{aligned} \tag{5B-6.3$'$}$$

Noting the relations (Fig. 5B-6.1)

$$\ell = \cos\theta = \frac{dy}{ds} \qquad m = \sin\theta = -\frac{dx}{ds} \tag{5B-6.4}$$

we obtain by Eqs. (5B-6.3′) and (5B-6.4) and the chain rule of differentiation [see Eqs. (5B-3.1)]

$$\sigma_{nx} = \frac{d}{ds}\left(\frac{\partial F}{\partial y}\right) \qquad \sigma_{ny} = -\frac{d}{ds}\left(\frac{\partial F}{\partial x}\right) \tag{5B-6.5}$$

Integration of Eqs. (5B-6.5) yields

$$\begin{aligned}\frac{\partial F}{\partial x} &= -\int_B \sigma_{ny}\, ds = f_1(s) + C_1 \\ \frac{\partial F}{\partial y} &= \int_B \sigma_{nx}\, ds = f_s(s) + C_2\end{aligned} \tag{5B-6.6}$$

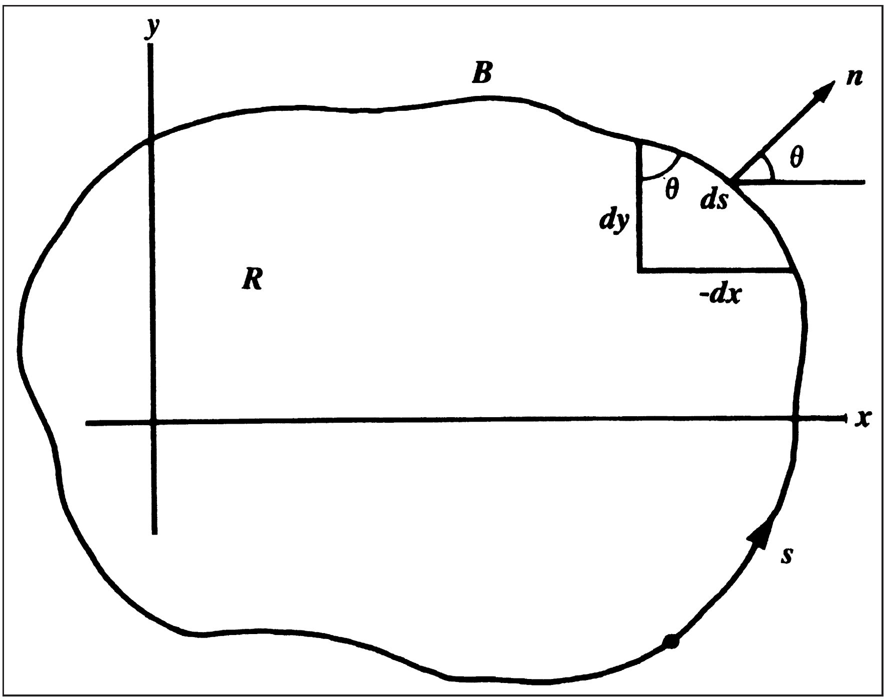


Figure 5B-6.1

where $f_1(s)$, $f_2(s)$ are functions of s on boundary B and (C_1, C_2) are arbitrary constants. Thus, Eqs. (5B-6.6) define the derivatives of F to within arbitrary constants.

Because Eqs. (5B-6.6) are equivalent to Eqs. (5B-6.3), the first fundamental problem of the plane theory of elasticity may be written in the form

$$\begin{aligned} &\nabla^2\nabla^2 F = \nabla^4 F = 0 \qquad \text{over } R \\ &\left.\begin{aligned} \frac{\partial F}{\partial x} &= f_1(s) + C_1 \\ \frac{\partial F}{\partial y} &= f_2(s) + C_2 \end{aligned}\right\} \qquad \text{on } C \end{aligned} \tag{5B-6.7}$$

where f_1, f_2 are prescribed functions of s. By Eqs. (5B-2.2), we may write the last two of Eqs. (5B-6.7) in terms of ψ, χ. Thus,

$$\psi(z) + z\overline{\psi'(z)} + \overline{\chi'(z)} = f_1(s) + if_2(s) + \text{constant} \qquad \text{on } B \tag{5B-6.8}$$

We recall that the first of Eqs. (5B-6.7) is satisfied identically by Eq. (5B-1.3) [or Eq. (5B-1.4)].

For the second fundamental problem, we require that $u = g_1(s)$, $v = g_2(s)$ on B, where (g_1, g_2) are prescribed functions. Hence, for this problem we replace Eq. (5B-6.8) by the boundary condition [see Eq. (5B-2.3)]

$$\kappa\psi(z) - z\overline{\psi'(z)} - \overline{\chi'(z)} = 2G(g_1 + ig_2) \qquad \text{on } B \tag{5B-6.9}$$

We have noted the nature of the arbitrariness of functions ψ, χ in Section 5B-5. To within this degree of arbitrariness for the simply connected region, the functions ψ and χ are determined completely by Eqs. (5B-6.8) and (5B-6.9) for the first and second fundamental problems. For details of the mixed fundamental problem (Section 4-15 in Chapter 4), refer to the literature (Muskhelishvili, 1975). For the simply connected region, Eqs. (5B-6.8) [or Eqs. (5B-6.9)] in conjunction with Eqs. (5B-5.9) [or Eqs. (5B-5.13) and (5B-5.16) for the doubly connected region] serve to define $\psi(z)$ and $\phi(z)$, that is, to define the coefficients a_n, b_n [recall $\phi = \psi'(z)$, Eq. (5B-5.1)].

5B-7 Note on Conformal Transformation

Let z and ζ be two complex variables related by the equation

$$z = w(\zeta) \tag{5B-7.1}$$

where $w(\zeta)$ is an analytic function in some domain D in the w plane. Hence, Eq. (5B-7.1) relates every point ζ in the w plane to some definite point in the z plane; that is, Eq. (5B-7.1) defines a one-to-one correspondence between the points

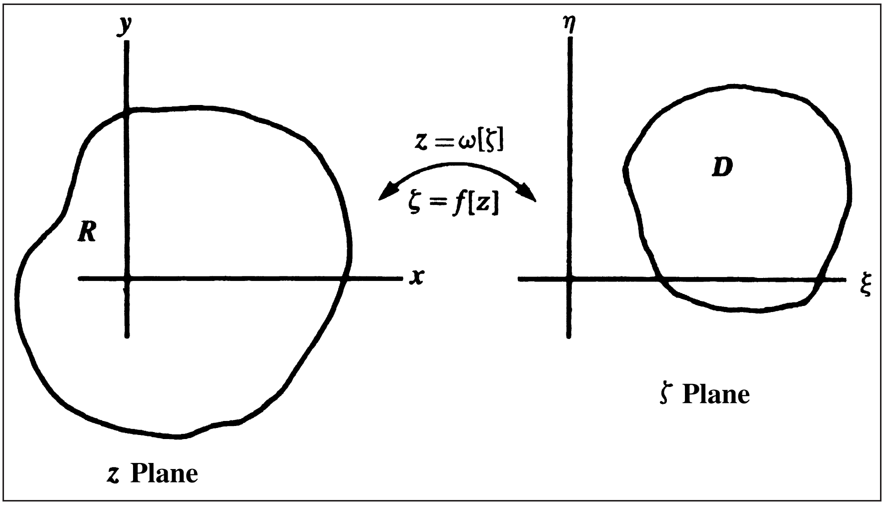


Figure 5B-7.1

in the w plane and the points in the z plane. Also, Eq. (5B-7.1) may be inverted to yield

$$\zeta = f(z) \tag{5B-7.2}$$

Because the points in the z plane cover some region R in the z plane (Fig. 5B-7.1), we say that Eq. (5B-7.1) represents an invertible single-valued "conformal mapping" of region R into the region D (or conversely). The mapping is called conformal because of the following property, which relations of the type of Eq. (5B-7.1) possess where $w(\zeta)$ is analytic: If in D two-line elements emanate from some point ζ and subtend angle θ, then the corresponding elements in R form the same angle, with the sense of θ maintained. The following discussion depends heavily on topics treated in Brown and Churchill (2008).

Many of the solutions of plane problems of elasticity by the method of complex variables rely heavily on the theorems relative to the unit circle. Thus, fundamental to these solutions is the conformal mapping of a region R in the z plane into a unit circle in the w plane. In particular, two cases are distinguished: (1) the transformation of a simply connected region R interior to a contour C, and (2) the transformation of the region R^* exterior to a contour C (Fig. 5B-7.2).

By the theory of conformal mapping, the transformation (mapping)

$$z = \sum_{k=0}^{\infty} c_k \zeta^k = w(\zeta) \tag{5B-7.3}$$

transforms the interior region R (Fig. 5B-7.2) bounded by the simple contour C (i.e., a contour that consists of one closed curve that does not intersect itself) into

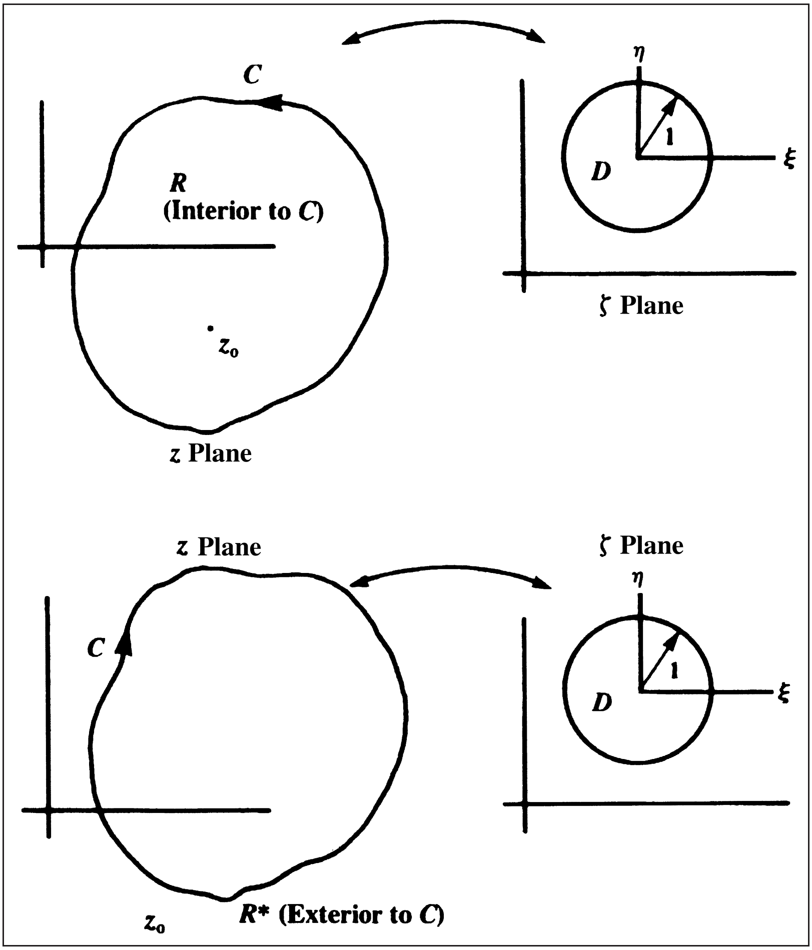


Figure 5B-7.2

the unit circle. The arbitrary point z_0 can be transformed into an arbitrarily chosen point in the unit circle (say, $\xi = \eta = 0$).

For the region R^* outside contour C, the mapping

$$
\begin{aligned}
z = w(\zeta) &= \frac{C_{-1}}{\zeta} + \text{ananalytic fuction} \\
&= \frac{C_{-1}}{\zeta} + \sum_{k=0}^{\infty} c_k \zeta^k
\end{aligned}
\tag{5B-7.4}
$$

transforms region R^* (Fig. 5B-7.2) exterior to C into the unit circle.

In Eq. (5B-7.3), $w'(\zeta)$, where prime denotes derivative with respect to ζ, has no zero within the unit circle (as it is conformal); hence, $\overline{w'(\zeta)}$ has no zero in D.

Equations (5B-7.3) and (5B-7.4) contain an infinite number of terms in general. However, in practice often only a finite number of terms are used. Hence, instead of transforming the actual region R (or R^*) into the unit circle, an approximation R_a of R is employed. If an exact transformation $w(\zeta)$ is unknown, the coefficients c_n are sometimes determined by methods of the approximate theory of conformal transformations.

Example 5B-7.1. The mapping

$$z = w(\zeta) = -a \int_1^\zeta (1 - p^3)^{2/3} \frac{dp}{p^2} + \text{constant} \tag{a}$$

where a is a real constant, transforms an equilateral triangle [Fig. (E5B-7.1)] in the z plane into a unit circle in the ζ plane. Noting by the binomial expansion that $(1 - p^3)^{2/3} = 1 - \frac{2}{3}p^3 + \frac{1}{9}p^6 - \frac{4}{27}p^9 + \cdots$, and choosing the constant in Eq. (a) properly, we find

$$z = w(\zeta) = -a\left(\frac{1}{\zeta} + \frac{1}{3}\zeta^2 + \frac{1}{45}\zeta^5 + \cdots\right) \tag{b}$$

For the boundary of the unit circle, $\zeta = 1e^i$. Thus, for the contour of region R, Eq. (b) yields

$$z = -a(e^{-i\theta} + \tfrac{1}{3}e^{2i\theta} + \tfrac{1}{45}e^{5i\theta} + \cdots) \tag{c}$$

Approximations R_a to the equilateral triangle (region R) may be obtained by taking 2, 3, 4, ... terms in Eq. (c). With three terms, a fairly good approximation to the equilateral triangle is obtained.

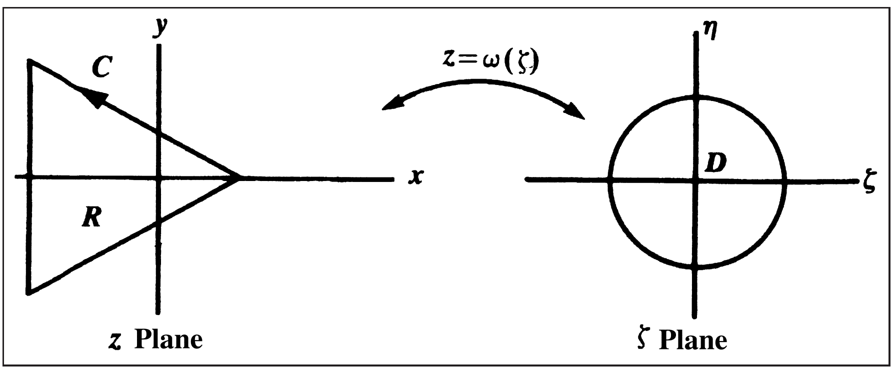


Figure E5B-7.1

Curvilinear Coordinates in the Plane. Because much of the complex variable method relates to the conformal mapping of a given region R in the z plane into a region D (unit circle) in the ζ plane, it is natural to introduce polar coordinates (r, θ) in the ζ plane (see Chapter 6). Then, $\zeta = \xi + i\eta$, where $\xi = r\cos\theta$, $\eta = r\sin\theta$ may be written as $\zeta = re^{i\theta}$. Hence,

$$z = x + iy = w(\zeta) = w(re^{i\theta}) \tag{5B-7.5}$$

Accordingly, the circles $r =$ constant and the radii $\theta =$ constant in the ζ plane are transformed into orthogonal curvilinear coordinate lines (a, b) in the z plane by Eq. (5B-7.5) (see Section 1-20, Chapter 1), as $z = w(\zeta)$ is a conformal transformation (Fig. 5B-7.3). The tangents to the coordinate lines are denoted by the symbols A, B and form a base for the axes of the curvilinear coordinate system at point z_0. Because the transformation is conformal, the axes (A, B) are right handed (conform to axes x, y) as a conformal transformation preserves the orientation of directions. The axis A forms the angle α with respect to the x direction.

In the sequel we require expressions for the transformations of displacement components (u, v), which are vectors. Accordingly, consider a vector $\mathbf{V}$ in the z plane at the point $z = w(\mathrm{re}^{i\theta})$. By Fig. 5B-7.3 we find

$$V_x + iV_y = (V_A + iV_B)e^{i\alpha} \tag{5B-7.6}$$

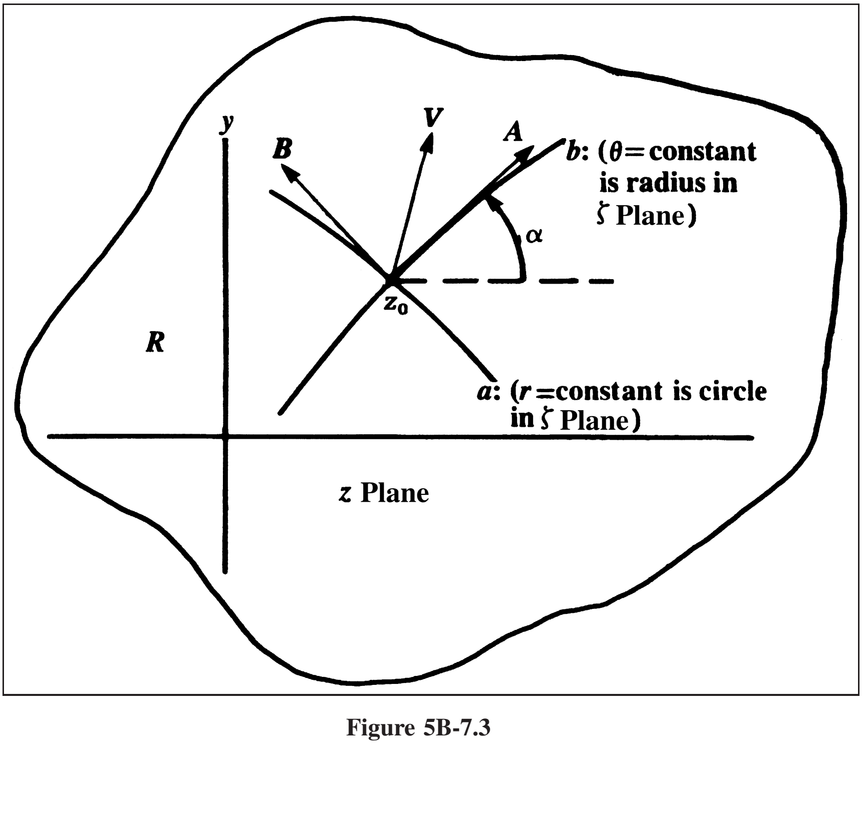


Figure 5B-7.3

Equation (5B-7.6) relates the components (V_x, V_y) relative to the (x, y) axes to the components (V_A, V_B) relative to curvilinear coordinates (a, b). To express $e^{i\alpha}$ in terms of the transformation $z = w(\zeta)$, we note that if we consider a displacement dz of the point z in the direction of the tangent A, the corresponding point ζ (in D) will undergo displacement dζ in the radial direction ($\theta =$ constant). Thus,

$$dz = e^{i\alpha}|dz| \qquad d\zeta = e^{i\theta}|d\zeta|$$

and, with Eq. (5B-7.5),

$$\begin{aligned} e^{i\alpha} &= \frac{dz}{|dz|} = \frac{w'(\zeta)d\zeta}{|w'(\zeta)||d\zeta|} = e^{i\theta}\frac{w'(\zeta)}{|w'(\zeta)|} \\ &= \frac{\zeta}{r}\frac{w'(\zeta)}{|w'(\zeta)|} \end{aligned} \tag{5B-7.7}$$

Equations (5B-7.6) and (5B-7.7) yield

$$V_x + iV_y = (V_A + iV_B)\frac{\zeta}{r}\frac{w'(\zeta)}{|w'(\zeta)|}$$

or

$$V_A + iV_B = e^{-i\alpha}(V_x + iV_y)\frac{\overline{\xi}}{r}\frac{\overline{w'(\zeta)}}{|w'(\zeta)|} \tag{5B-7.8}$$

where $\overline{\xi} = \text{re}^{-i\theta}$ and $\overline{w'(\zeta)}$ are complex conjugates of ζ and $w'(\zeta)$.

Problem Set 5B-7

1. Let $z = c\cosh\zeta$, where $z = x + iy, \zeta = \xi + i\eta$. Derive the equations that define the coordinate lines in the z plane that correspond to the coordinate lines $\xi = \xi_0 =$ constant, $\eta = \eta_0 =$ constant in the ζ plane. Show that the coordinate lines form an orthogonal system.

2. Let $z = ia\coth(\zeta/2)$, where $z = x + iy$, $\zeta = \xi + i\eta$. Repeat Problem 1.

5B-8 Plane Elasticity Formulas in Terms of Curvilinear Coordinates

To transform the stress components and the displacement components to curvilinear coordinates (a, b) we must transform $\psi(z)$, $\chi(z)$ into functions of ζ, that is, into functions $\psi(\zeta)$, $\chi(\zeta)$, where $z = w(\zeta)$.

Stress Components. Let $\sigma_a, \sigma_b, \tau_{ab}$ be defined as follows (Fig. 5B-8.1):

$\sigma_a =$ normal stress component on curve $a =$ constant

$\sigma_b =$ normal stresscomponent on curve $b =$ constant

$\tau_{ab} = \tau_{ba} =$ shear component on both curves

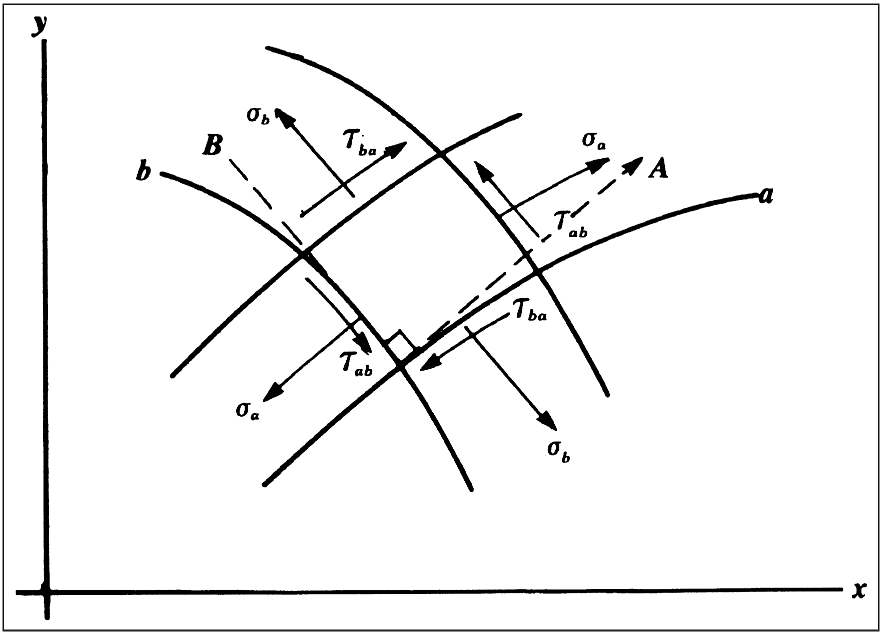


Figure 5B-8.1

By plane transformation laws of stress (Section 3-7 in Chapter 3), we obtain

$$\begin{aligned} \sigma_a &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \tau_{ab} &= -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{aligned} \tag{5B-8.1}$$

Letting $\alpha \to \alpha + \pi/2$, we obtain from the first of Eq. (5B-8.1)

$$\sigma_b = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\alpha - \tau_{xy}\sin 2\alpha \tag{5B-8.2}$$

Hence, by Eqs. (5B-8.1) and (5B-8.2), we find

$$\sigma_a + \sigma_b = \sigma_x + \sigma_y \tag{5B-8.3}$$

$$\sigma_b - \sigma_a + 2i\tau_{ab} = e^{2i\alpha}(\sigma_y - \sigma_x + 2i\tau_{xy}) \tag{5B-8.4}$$

To obtain an expression for the term $e^{2i\alpha}$, we note by Eq. (5B-7.7) that

$$\begin{aligned} e^{2i\alpha} &= \frac{\zeta^2}{r^2}\frac{[w'(\zeta)]^2}{|w'(\zeta)|^2} = \frac{\zeta^2}{r^2}\frac{[w'(\zeta)]^2}{w'(\zeta)\overline{w'(\zeta)}} \\ &= \frac{\zeta^2}{r^2}\frac{w'(\zeta)}{\overline{w'(\zeta)}} \end{aligned} \tag{5B-8.5}$$

Thus, Eqs. (5B-8.4) and (5B-8.5) yield

$$\sigma_b - \sigma_a + 2i\tau_{ab} = \frac{\zeta^2}{r^2}\frac{w'(\zeta)}{\overline{w'(\zeta)}}(\sigma_y - \sigma_x + 2i\tau_{xy})$$
$$\sigma_a + \sigma_b = \sigma_x + \sigma_y \tag{5B-8.6}$$

To express σ_a, σ_b in terms of ζ, we note that by Eqs. (5B-3.6) and (5B-7.5)

$$\sigma_x + \sigma_y = 2[\psi'(z) + \overline{\psi'(z)}] = 2\left[\frac{\psi_1'(\zeta)}{w'(\zeta)} + \overline{\left(\frac{\psi_1'(\zeta)}{w'(\zeta)}\right)}\right] \tag{5B-8.7}$$

where $\psi_1(\zeta) = \psi(z)$. In a similar manner, we may express σ_b—$\sigma_a + 2i\tau_{ab}$ in terms of ζ.

Displacement Components. Let (u, v) denote the (x, y) components of displacement (Fig. 5B-8.2). Let (u_a, u_b) denote the (a, b) components of displacement. Then, by vector projections, we find

$$(u_a + u_b) = e^{-i\alpha}(u + iv) \tag{5B-8.8}$$

where $(u + iv)$ is expressed in terms of ψ and χ by Eq. (5B-2.3), which in turn may be expressed in terms of ζ.

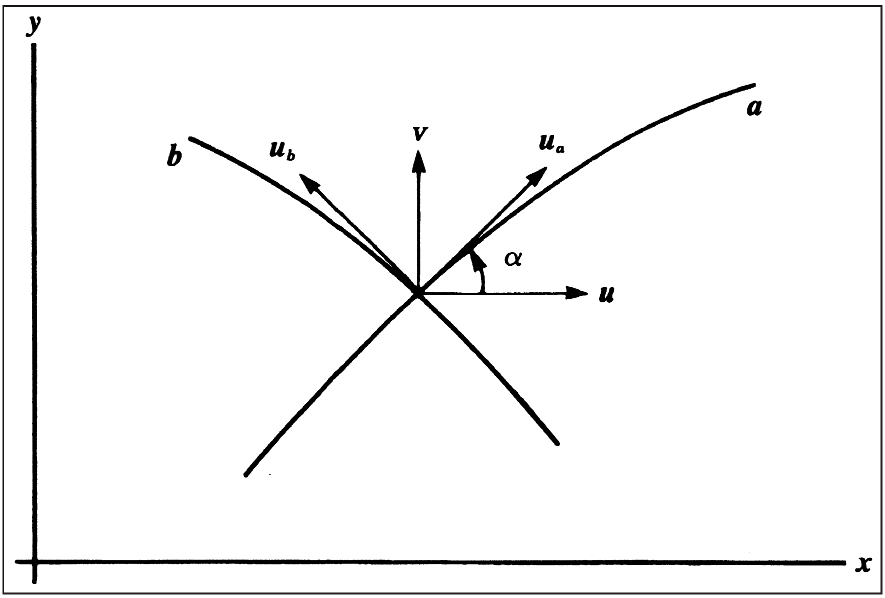


Figure 5B-8.2

Equations (5B-8.6) and (5B-8.8) express the stress components and the displacement components of plane elasticity in curvilinear plane coordinates in the z plane (polar coordinates r, θ in the ζ plane).

5B-9 Complex Variable Solution for Plane Region Bounded by Circle in the z Plane

In this section we demonstrate the complex variable method for the case of a simply connected circular region R in the z plane, with prescribed boundary stresses on the circle C (Fig. 5B-9.1). The case of prescribed displacement on C may be treated in an analogous manner. Although the example is elementary, the essential features of the complex variable method are illustrated. (The more complicated problem of the plane region with circular hole is treated in Section 6-10 of Chapter 6.)

Solution Relative to z. We take axes (x, y) with origin at the center of the circle C. We consider the components of the boundary stress $(\sigma_{nr}, \sigma_{n\alpha})$ on C to be known, continuous, and single-valued functions of α on C. Accordingly, by Eqs. (5B-3.3) and (5B-6.8), we have (with constant = 0)

$$f_1(s) + if_2(s) = i \int_0^s (\sigma_{nx} + i\sigma_{ny})\, ds = ia \int_0^\alpha (\sigma_{nx} + i\sigma_{ny})\, d\alpha \tag{5B-9.1}$$

Overall equilibrium of region R requires

$$\sum F_x = a \int_0^{2\pi} \sigma_{nx}\, d\alpha = a \int_0^{2\pi} (\sigma_{nr} \cos\alpha - \sigma_{n\alpha} \sin\alpha)\, d\alpha = 0 \tag{5B-9.2}$$

$$\sum F_y = a \int_0^{2\pi} \sigma_{ny}\, d\alpha = a \int_0^{2\pi} (\sigma_{nr} \sin\alpha + \sigma_{n\alpha} \cos\alpha)\, d\alpha = 0 \tag{5B-9.3}$$

$$\sum M_0 = a \int_0^{2\pi} \sigma_{n\alpha}\, d\alpha = 0 \tag{5B-9.4}$$

Hence, $(\sigma_{nr}, \sigma_{n\alpha})$ are periodic in α (with period 2π). Assuming $(\sigma_{nr}, \sigma_{n\alpha})$ are continuous, single-valued functions of α (Dirichlet conditions; Carrier et al., 2005), we may represent them (and hence σ_{nx}, σ_{ny}) in the form of convergent Fourier series. Thus, we may express $f_1 +$ if$_2$ in the known form

$$f_1 + if_2 = \sum_{m=-\infty}^{m=\infty} A_m e^{im\alpha} \tag{5B-9.5}$$

where by Fourier series theory

$$A_m = \frac{1}{2\pi} \int_0^{2\pi} (f_1 + if_2) e^{-im\alpha}\, d\alpha \tag{5B-9.6}$$

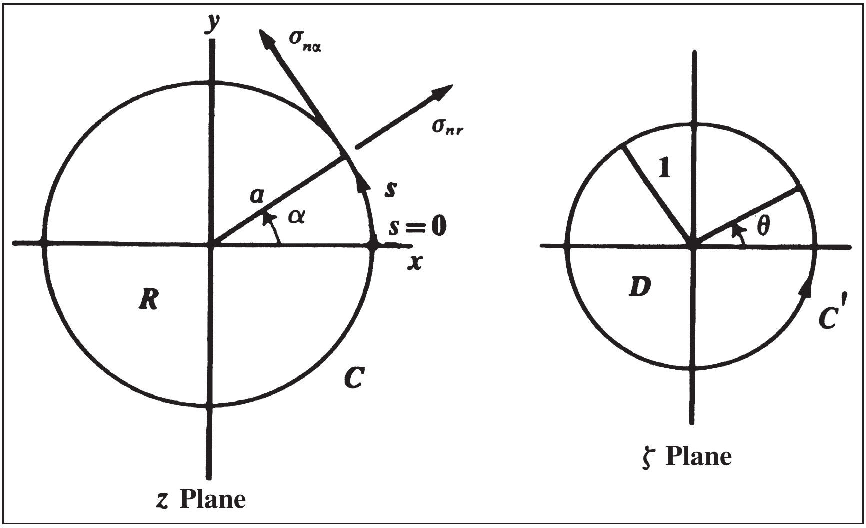


Figure 5B-9.1

By Section 5B-5 [Eq. (5B-5.9)], we have in R, $|z| < a$ for analytic functions ψ, χ'

$$\psi(z) = \sum_{n=1}^{\infty} a_n z^n$$
$$\chi'(z) = \phi(z) = \sum_{n=0}^{\infty} b_n z^n \tag{5B-9.7}$$

where we have taken $\psi(0) = 0$. Assuming that the series of Eqs. (5B-9.7) converge in R and on C, we have by Eq. (5B-6.8) and (5B-9.7)

$$\sum_{n=1}^{\infty} a_n a^n e^{in\alpha} + \overline{a}_1 a e^{i\alpha} + \sum_{n=0}^{\infty} (n+2)\overline{a}_{n+2} a^{n+2} e^{-in\alpha} + \sum_{n=0}^{\infty} \overline{b}_n a^n e^{-in\alpha}$$
$$= \sum_{m=-\infty}^{\infty} A_m e^{im\alpha} \tag{5B-9.8}$$

where we note the formulas

$$z = ae^{i\alpha} \qquad \overline{z} = ae^{-i\alpha}$$
$$\overline{\psi'(z)} = \sum_{n=1}^{\infty} n\overline{a}_n \overline{z}^{n-1} \qquad \overline{\phi(z)} = \overline{\chi'(z)} = \sum_{n=0}^{\infty} \overline{b}_n z^{-n}$$
$$z\sum_{n=1}^{\infty} n\overline{a}_n \overline{z}^{n-1} = \sum_{n=1}^{\infty} n\overline{a}_n a^n e^{-(n-2)i\alpha}$$
$$= \overline{a}_1 a e^{i\alpha} + \sum_{n=0}^{\infty} (n+2)\overline{a}_{n+2} a^{n+2} e^{-in\alpha} \tag{5B-9.9}$$

Comparing like powers of e in Eq. (5B-9.8), we obtain

$$\begin{aligned} e^{i\theta}:& \qquad a(a_1+\overline{a}_1)=A_1=\text{real number} \qquad (n=1)\\ e^{in\theta}:& \qquad a_n=A_n \qquad (n>1)\\ e^{-in\theta}:& \qquad (n+2)\overline{a}_{n+2}a^{n+2}+a^n\overline{b}_n=-A_{-n} \qquad (n>0)\end{aligned} \tag{5B-9.10}$$

Equations (5B-9.10) define all the coefficients a_n, b_n except a_1. Only the real value of a_1 is defined by the first of Eqs. (5B-9.10), as $a_1+\overline{a}_1=\text{Re}(a_1)$. However, this condition is sufficient because the imaginary part of $\psi'(z)$ may be chosen arbitrarily for $z=0$. For example, we may take Im $\psi'(0)=0$ [Eq. (5B-5.8)]. Furthermore, the constant A_1 has the physical significance that it is the average (mean) value of radial load acting on the boundary C or R. This result follows from Eqs. (5B-9.6) and (5B-9.1). Thus, by Eq. (5B-9.6),

$$\begin{aligned} 2\pi A_1 &= \int_0^{2\pi}(f_1+if_2)e^{-i\alpha}\,d\alpha\\ &= \int_0^{2\pi}(f_1\cos\alpha+f_2\sin\alpha)\,d\alpha + i\int_0^{2\pi}(f_2\cos\alpha-f_1\sin\alpha)\,d\alpha\end{aligned}$$

However, we note that by Eqs. (5B-9.2) to (5B-9.4),

$$\begin{aligned} \sum M_0 &= a\int_0^{2\pi}\sigma_{n\alpha}d\alpha = a\int_0^{2\pi}(\sigma_{ny}\cos\alpha-\sigma_{nx}\sin\alpha)\,d\alpha\\ &= -\int_0^{2\pi}(\cos\alpha df_1+\sin\alpha df_2)\\ &= -[f_1\cos\alpha+f_2\sin\alpha]_0^{2\pi}+\int_0^{2\pi}(-f_1\sin\alpha+f_2\cos\alpha)\,d\alpha\\ &= \int_0^{2\pi}(-f_1\sin\alpha+f_2\cos\alpha)\,d\alpha=0\end{aligned}$$

and in a similar manner

$$\begin{aligned} \sum F_r &= a\int_0^{2\pi}\sigma_{nr}\,d\alpha = a\int_0^{2\pi}(\sigma_{nx}\cos\alpha+\sigma_{ny}\sin\alpha)\,d\alpha\\ &= \int_0^{2\pi}(f_1\cos\alpha+f_2\sin\alpha)\,d\alpha\end{aligned}$$

Hence,

$$A_1=\frac{a}{2\pi}\int_0^{2\pi}\sigma_{nr}\,d\alpha \tag{5B-9.11}$$

and A_1 equals the mean value of the radial load.

Solution Relative to ζ. Alternatively, the solution may be derived in the ζ plane (Fig. 5B-9.1). For example, the region R in the z plane may be transformed into the unit circle D in the ζ plane by the mapping

$$z = w(\zeta) = a\zeta \tag{5B-9.12}$$

Hence, on the boundary C',

$$\zeta = e^{i\theta} = \gamma$$
$$\frac{w(\zeta)}{\overline{w'(\zeta)}} = \frac{a\zeta}{a} = \zeta (= \gamma \text{ on } C') \tag{5B-9.13}$$

To write boundary conditions on C', we require that $\psi(z)$ and $\chi'(z)$ be transformed into functions of ζ. For this purpose, we remark that with the notation

$$\psi_1(\zeta) = \psi(z) = \psi[w(\zeta)]$$
$$\phi_1(\zeta) = \phi(\zeta) = \phi[w(\zeta)]$$

we have

$$\psi'(z) = \frac{d\psi}{dz} = \frac{d\psi_1(\zeta)}{d\zeta}\frac{d\zeta}{dz} = \frac{\psi_1'}{w'(\zeta)}$$

Consequently, the boundary conditions [Eq. (5B-6.8)] in terms of $\zeta (= \gamma$ on $C')$ become, with $\phi(z) = \chi'(z)$,

$$\psi_1(\gamma) + \frac{w(\gamma)}{\overline{w'(\gamma)}}\overline{\psi_1'(\gamma)} + \overline{\phi_1(\gamma)} = f_1 + if_2 \tag{5B-9.14}$$

or with Eqs. (5B-9.13)

$$\psi_1(\gamma) + \gamma\overline{\psi_1'(\gamma)} + \overline{\phi_1(\gamma)} = f_1 + if_2 \tag{5B-9.15}$$

With

$$\psi_1(\zeta) = \sum_{n=0}^{\infty} a_n \zeta^n \qquad \phi_1(\zeta) = \sum b_n \zeta^n \tag{5B-9.16}$$

the analysis proceeds as in the z plane [following Eqs. (5B-9.7)]. Then substitution of $\psi(z), \phi(z)$ [or $\psi_1(\zeta), \phi_1(\zeta)$] into expressions for $\sigma_x + \sigma_y$, $\sigma_y - \sigma_x + 2i\tau_{xy}$, $2G(u + iv)$ yields (x, y) components in the z plane (in the ζ plane), provided ψ, ϕ are absolutely and uniformly convergent on the boundary circle $|z| = a$. If first derivatives of $\sigma_{nr}, \sigma_{n\theta}$ (or σ_{nx}, σ_{ny}) satisfy Dirichlet conditions (Brown and Churchill, 2008), this requirement is satisfied.

Problem Set 5B

1. Consider the problem of small deflections, plane thermoelasticity for which

$$\epsilon_x = \gamma_{xz} = \gamma_{yz} = 0$$

(a) Derive an expression for σ_z in terms of stress components σ_x and σ_y, material properties k (thermal coefficients of linear expansion) and E (modulus of elasticity), and temperature change T measured from an arbitrary zero.

(b) Assume the additional conditions that stress components $\sigma_x = \sigma_y = \tau_{xy} = 0$. Hence, derive expressions for the strain components ϵ_x, ϵ_y, and γ_{xy}.

(c) Show that under the combined conditions of parts (a) and (b), the compatibility conditions reduce to $\nabla^2 T = 0$ for constant E and k.

(d) Using the results of part (b), show that the rotation of a volume element in the xy plane is

$$\omega_z = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

Hence, show that

$$\frac{\partial \epsilon'}{\partial x} = \frac{\partial \omega_x}{\partial y} \qquad \frac{\partial \epsilon'}{\partial y} = -\frac{\partial \omega_x}{\partial x}$$

where $\epsilon' = (1+\nu)kT$. That is, show that ϵ' and ω_z satisfy the Cauchy–Riemann equations. (Consequently, the theory associated with the Cauchy–Riemann equations may be applied to ϵ' and ω_z.)

2. Let z denote the complex variable $z = x + iy$, where (x, y) denote plane rectangular Cartesian coordinates. Let $\overline{z} = x - iy$ denote the complex conjugate of z.

(a) Show that the equilibrium equations of plane elasticity in the absence of body forces may be transformed into the result ($i^2 = -1$)

$$\frac{\partial}{\partial z}(\sigma_x - \sigma_y + 2i\tau_{xy}) + \frac{\partial}{\partial \overline{z}}(\sigma_x + \sigma_y) = 0$$

(b) Let the displacement s be given by $s = u + iv$ where (u, v) denotes (x, y) displacement components. Express $\partial s/\partial \overline{z}$ in terms i and derivatives of u, v relative to (x, y).

(c) Noting that the equilibrium equation of part (a) is a necessary and sufficient condition that there exists a function $F(z, \overline{z})$ such that

$$\frac{\partial F}{\partial z} = \sigma_x + \sigma_y \qquad \frac{\partial F}{\partial \overline{z}} = \sigma_y - \sigma_x - 2i\tau_{xy}$$

and expressing $(\sigma_x, \sigma_y, \tau_{xy})$ in terms of (u, v), show for plane strain, employing the results of part (b), that

$$4Gs = -F(z, \overline{z}) + f(z)$$

where $2G(1+\nu) = E$, $E =$ Young's modulus, and $\nu =$ Poisson's ratio.

(d) Compute the derivative $\partial s/\partial z$ in terms of i and derivatives of (u, v) with respect to (x, y). Hence, show that

$$\sigma_x + \sigma_y - f'(z) = -4G\frac{\partial s}{\partial z} = \frac{\partial F}{\partial z} - f'(z)$$
$$4(\lambda + G)\frac{\partial s}{\partial z} = \sigma_x + \sigma_y + 2i(\lambda + G)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

where

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

3. Show that the equation $\sigma_x + \sigma_y = 4\text{Re}[\psi'(z)]$ may be written in the form

$$\sigma_x + \sigma_y + \frac{4iE\omega}{(1+\nu)(1+\kappa)} = 4\psi'(z)$$

where ω is the volumetric rotation.

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CHAPTER 6

PLANE ELASTICITY IN POLAR COORDINATES

The use of polar coordinates is advantageous in problems involving boundaries formed by circular arcs or radially straight lines. Furthermore, certain problems of symmetry lend themselves well to polar coordinates. Accordingly, in this chapter we express the basic plane elasticity equations in polar coordinates.

6-1 Equilibrium Equations in Polar Coordinates

Consider an element of volume bounded by the polar coordinate lines (r, θ) and $(r + dr, \theta + d\theta)$ (Fig. 6-1.1). Let the thickness h of the element [dimension perpendicular to the (x, y) plane] be a function of (r, θ). Let the element be subjected to stress as shown (R and Θ denote body forces per unit volume in the radial and tangential directions, respectively). Because $d\theta$ is an infinitesimal angle, summations of forces in the radial and tangential directions yield for equilibrium, assuming that the thickness is sufficiently small compared to the in-plane dimensions so that variations of radial and tangential stresses over the thickness can be neglected:

$$\begin{aligned} \frac{\partial(h\sigma_r)}{\partial r} + \frac{1}{r}\frac{\partial(h\tau_{r\theta})}{\partial\theta} + \frac{h(\sigma_r - \sigma_\theta)}{r} + hR = 0 \\ \frac{\partial(h\tau_{r\theta})}{\partial r} + \frac{1}{r}\frac{\partial(h\sigma_\theta)}{\partial\theta} + \frac{2(h\tau_{r\theta})}{r} + h\Theta = 0 \end{aligned} \tag{6-1.1}$$

Equations (6-1.1) are the equilibrium equations for plane elasticity in polar coordinates. They are equivalent to Eqs. (5-2.11) in Chapter 5. Alternatively, Eqs. (6-1.1) may be derived by mathematically transforming Eqs. (5-2.11) from (x, y) coordinates to (r, θ) coordinates by tensor theory (see also Appendix 3A in Chapter 3). For $h =$ constant h may be canceled from Eqs. (6-1.1).

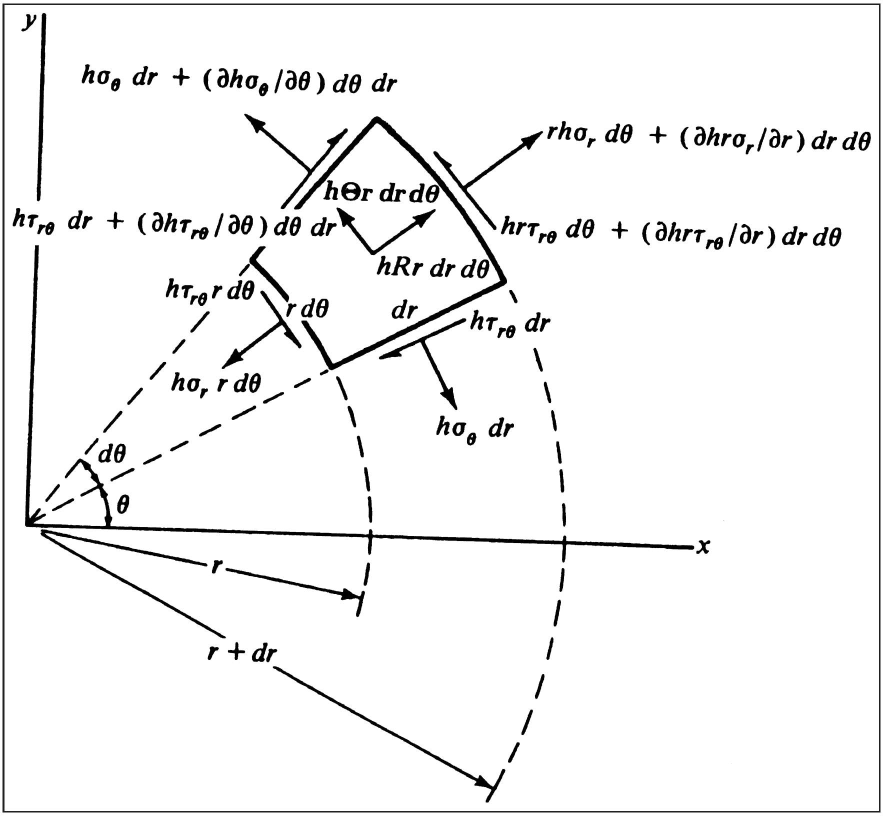


Figure 6-1.1

6-2 Stress Components in Terms of Airy Stress Function $F = F(r, \theta)$

To derive stress components in terms of the Airy stress function F, where F is considered to be a function of polar coordinates (r, θ), we may transform Eqs. (5-4.3) (for constant thickness and in the absence of body forces) to polar coordinates as follows. By Fig. (6-1.1), we obtain the following relations between (x, y) and (r, θ):

$$r^2 = x^2 + y^2$$
$$x = r\cos\theta \quad y = r\sin\theta \tag{6-2.1}$$
$$\tan\theta = \frac{y}{x}$$

Consider first the transformation of σ_x. By Eqs. (5-4.9), we note that we require $\partial^2 F/\partial y^2$ in terms of (r, θ). By the chain rule of partial differentiation and Eq. (6-2.1), we have

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial F}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial F}{\partial \theta}\cos\theta$$

Similarly,

$$\begin{aligned}\frac{\partial^2 F}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial F}{\partial y}\\ &= \frac{\partial^2 F}{\partial r^2}\sin^2\theta + \frac{2}{r}\frac{\partial^2 F}{\partial r\,\partial\theta}\sin\theta\cos\theta - \frac{2}{r^2}\frac{\partial F}{\partial\theta}\sin\theta\cos\theta\\ &\quad + \frac{1}{r}\frac{\partial F}{\partial r}\cos^2\theta + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\cos^2\theta\end{aligned}$$

Now, noting that as $\theta \to 0$, $\sigma_x \to \sigma_r$, $\cos\theta \to 1$, $\sin\theta \to 0$, we obtain

$$\sigma_r = \left.\frac{\partial^2 F}{\partial y^2}\right|_{\theta\to 0} = \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}$$

Also, noting that as $\theta \to \pi/2$, $\sigma_y \to \sigma_\theta$, $\cos\theta \to 0$, $\sin\theta \to 1$, we find

$$\sigma_\theta = \left.\frac{\partial^2 F}{\partial y^2}\right|_{\theta\to\pi/2} = \frac{\partial^2 F}{\partial r^2}$$

In a similar manner, we may evaluate $\partial^2 F/\partial x\partial y$. Then, noting that as $\theta \to 0$, $\tau_{xy} \to \tau_{r\theta}$, we find

$$\tau_{r\theta} = -\left.\frac{\partial^2 F}{\partial x\,\partial y}\right|_{\theta\to 0} = -\frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial\theta} + \frac{1}{r^2}\frac{\partial F}{\partial\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)$$

Accordingly, the stress components are given in terms of the Airy stress function $F(r, \theta)$ by the relations

$$\begin{aligned}\sigma_r &= \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial\theta^2}\\ \sigma_\theta &= \frac{\partial^2 F}{\partial r^2}\\ \tau_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial\theta}\right)\end{aligned} \tag{6-2.2}$$

More generally, the preceding transformations may be carried out with respect to orthogonal curvilinear coordinates (Section 1-22). For variable thickness $h = h(r, \theta)$, we replace $\sigma_r, \sigma_\theta, \tau_{r\theta}$ in Eq. (6-2.2) by $h\sigma_r, h\sigma_\theta, h\tau_{r\theta}$ [see Eqs. (6-1.1)]. Also, for certain cases, body forces may be introduced simply (see Section 6-6).

6-3 Strain–Displacement Relations in Polar Coordinates

Consider a point P in a medium that undergoes a deformation (Fig. 6-3.1). Under the deformation, the point P moves to P^*. With respect to rectangular Cartesian coordinates (x, y), the displacement components of point P are (u, v); with respect

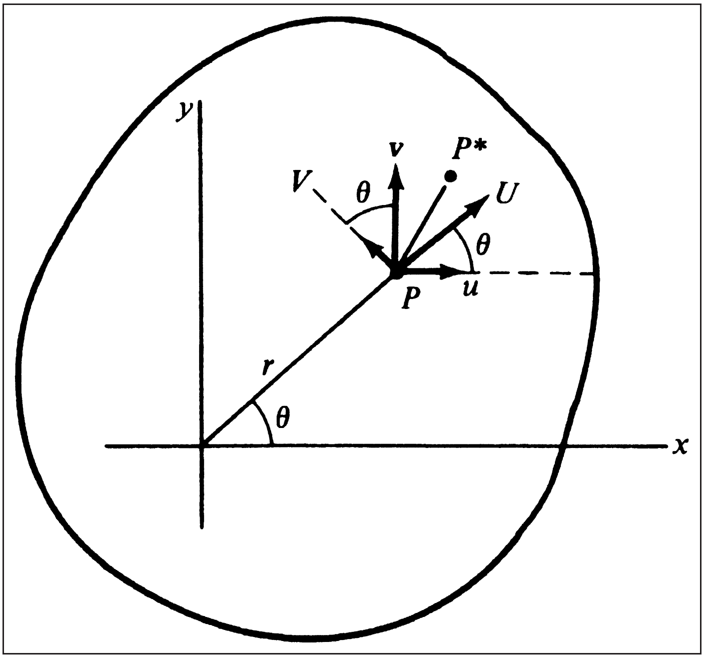


Figure 6-3.1

to polar coordinates, the displacement components are (U, V). Accordingly, by Fig. 6-3.1,

$$\begin{aligned} u &= U\cos\theta - V\sin\theta \\ v &= U\sin\theta + V\cos\theta \end{aligned} \tag{6-3.1}$$

Substitution of Eqs. (6-3.1) into Eqs. (2-15.14) yields $\epsilon_x, \epsilon_y, \gamma_{xy}$ in terms of U, V, and θ. For example, consider ϵ_x. By Eqs. (2-15.14) and the chain rule for partial differentiation, we obtain

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} \tag{6-3.2}$$

where, by Eqs. (6-3.1) and (6-2.1),

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial U}{\partial \theta}\cos\theta - U\sin\theta - \frac{\partial V}{\partial \theta}\sin\theta - V\cos\theta \\ \frac{\partial u}{\partial r} &= \frac{\partial U}{\partial r}\cos\theta - \frac{\partial V}{\partial r}\sin\theta \\ \frac{\partial \theta}{\partial x} &= -\frac{\sin\theta}{r} \\ \frac{\partial r}{\partial x} &= \cos\theta \end{aligned} \tag{6-3.3}$$

Accordingly, by Eqs. (6-3.2) and (6-3.3), we obtain

$$\epsilon_x = \left(-\frac{\partial U}{\partial\theta}\cos\theta + U\sin\theta + \frac{\partial V}{\partial\theta}\sin\theta + V\cos\theta\right)\frac{\sin\theta}{r} + \left(\frac{\partial U}{\partial r}\cos\theta - \frac{\partial V}{\partial r}\sin\theta\right)\cos\theta$$

Noting that $\epsilon_x \to \epsilon_r$, $\sin\theta \to 0$, and $\cos\theta \to 1$ as $\theta \to 0$, we obtain

$$\epsilon_r = \epsilon_x|_{\theta\to 0} = \frac{\partial U}{\partial r}$$

Analogously, $\epsilon_x \to \epsilon_\theta$, $\sin\theta \to 1$, and $\cos\theta \to 0$ as $\theta \to \pi/2$. Hence,

$$\epsilon_\theta = \epsilon_x|_{\theta\to\pi/2} = \frac{1}{r}\frac{\partial V}{\partial\theta} + \frac{U}{r}$$

Finally, in a similar manner, we may express γ_{xy} as a function of U, V, and θ, and noting that $\gamma_{xy} \to \gamma_{r\theta}$ as $\theta \to 0$, we obtain

$$\gamma_{r\theta} = \gamma_{xy}|_{\theta\to 0} = \frac{\partial V}{\partial r} - \frac{V}{r} + \frac{1}{r}\frac{\partial U}{\partial\theta}$$

Accordingly, the strain components $\epsilon_r, \epsilon_\theta, \gamma_{r\theta}$ with respect to polar coordinates (r, θ) are

$$\begin{aligned} \epsilon_r &= \frac{\partial U}{\partial r} \\ \epsilon_\theta &= \frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial\theta} \\ \gamma_{r\theta} &= \frac{1}{r}\frac{\partial U}{\partial\theta} + \frac{\partial V}{\partial r} - \frac{V}{r} \end{aligned} \tag{6-3.4}$$

where $U = U(r, \theta)$, $V = V(r, \theta)$ are the radial and tangential displacement components (Fig. 6-3.1).

Alternatively, Eqs. (6-3.4) may be derived by the method of Section 2-6 in Chapter 2 (see also Appendix 2B).

Problem. Derive the last of Eqs. (6-3.4).

With the understanding that (u, v) denote radial and tangential components of displacement relative to (r, θ) coordinates, we may write

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial\theta} \\ \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \frac{1}{r}\frac{\partial u}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) \end{aligned} \tag{6-3.5}$$

The strain–compatibility relations in polar coordinates may be obtained either by elimination of (u, v) from Eqs. (6-3.5) or by transformation of Eqs. (5-3.1) in Chapter 5 into polar coordinates. Thus, for plane deformations we obtain the compatibility relation

$$\frac{\partial}{\partial r}\left(r\frac{\partial \gamma_{r\theta}}{\partial \theta} - r^2\frac{\partial \epsilon_\theta}{\partial r}\right) + r\frac{\partial \epsilon_r}{\partial r} - \frac{\partial^2 \epsilon_r}{\partial \theta^2} = 0 \tag{6-3.6}$$

For the special case of rotationally symmetric problems where all quantities are functions of radial coordinate r only, by Eqs. (6-3.5), we obtain the compatibility relations

$$\begin{aligned} \epsilon_r &= \frac{d}{dr}(r\epsilon_\theta) \\ \gamma_{r\theta} &= r\frac{d}{dr}\left(\frac{v}{r}\right) \end{aligned} \tag{6-3.7}$$

Problem Set 6-3

1. Consider two orthogonal line elements, ds_1 and ds_2, one radial and one tangential in a plane R (Fig. P6-3.1). Consider the following separate deformations: (a) all points in the body (region) undergo a radial displacement; $U = U_1(r, \theta)$, $V = V_1 = 0$, where (U, V) denote radial and tangential components of displacement; (b) all points undergo a displacement such that $U = U_2 = 0$, $V = V_2(r, \theta)$. Derive expressions for the strain components $\epsilon_r, \epsilon_\theta, \gamma_{r\theta}$ corresponding to the deformations (a) and (b). Superimpose the results of deformations (a) and (b) to arrive at Eqs. (6-3.4).

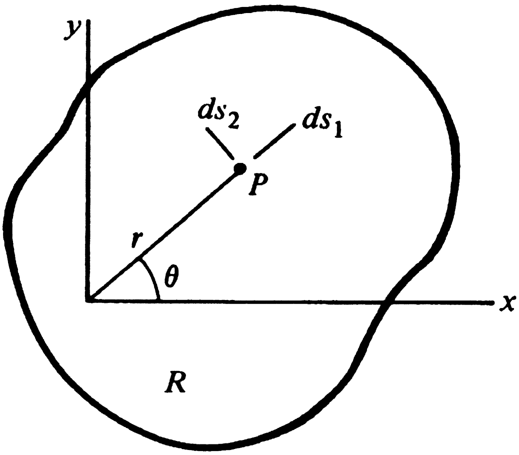


Figure P6-3.1

2. The line t is tangent to the centerline of a circular arc ring AB at point P (see Fig. P6-3.2). When the ring is loaded, point P undergoes radial and tangential displacement components (w, u). Derive an expression for $\tan(\phi^* - \phi)$, the tangent of the angle through which line t rotates. Linearize this formula for small rotations, that is, for $\tan(\phi^* - \phi) \approx \phi^* - \phi$. Recall that $\tan(\phi^* - \phi) \approx (\tan\phi^* - \tan\phi)/(1 + \tan\phi^* \tan\phi)$. Note that $u = u(\theta)$, $w = w(\theta)$. *Express the results in terms of* a, w, u *and derivatives of* w *and* u.

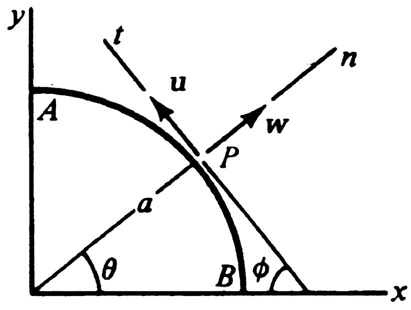


Figure P6-3.2

6-4 Stress–Strain–Temperature Relations

Equations (5-2.12) and (5-2.13) in Chapter 5 remain valid for any orthogonal plane coordinates, except that the derivatives $\partial/\partial x$, $\partial/\partial y$ must be transformed appropriately. Accordingly, relative to polar coordinates (r, θ), we have the stress–strain relations

$$\begin{aligned} \sigma_r &= \lambda e + 2G\epsilon_r \\ \sigma_\theta &= \lambda e + 2G\epsilon_\theta \\ \tau_{r\theta} &= G\gamma_{r\theta} \\ e &= \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \end{aligned} \tag{6-4.1}$$

where (u, v) are displacement components relative to polar coordinates (r, θ); see Fig. 6-3.1 (where U, V are used).

Accordingly, for plane strain we have the stress–strain–temperature relations [Eqs. (5-3.8)]

$$\begin{aligned} \sigma_r &= \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_r + \nu\epsilon_\theta - (1+\nu)kT] \\ \sigma_\theta &= \frac{E}{(1+\nu)(1-2\nu)}[\nu\epsilon_r + (1-\nu)\epsilon_\theta - (1+\nu)kT] \\ \tau_{r\theta} &= \frac{E}{2(1+\nu)}\gamma_{r\theta} \\ \sigma_z &= \frac{E}{(1+\nu)(1-2\nu)}[\nu(\epsilon_r + \epsilon_\theta) - (1+\nu)kT] \\ e &= \epsilon_r + \epsilon_\theta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \\ \epsilon_z &= \gamma_{rz} = \gamma_{\theta z} = \tau_{rz} = \tau_{\theta z} = 0 \end{aligned} \tag{6-4.2}$$

and for the compatibility relations in terms of stress components [Eq. (5-3.9)]

$$\nabla^2(\sigma_r + \sigma_\theta) + \frac{E}{1-\nu}\nabla^2(kT) + \frac{1}{1-\nu}\left(\frac{\partial B_r}{\partial r} + \frac{1}{r}B_r + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta}\right) = 0 \tag{6-4.3}$$

where (B_r, B_θ) denote body forces relative to (r, θ) coordinates, T denotes temperature, and k is the coefficient of linear thermal expansion. For plane stress, we have the stress–strain–temperature relations [Eq. (5-3.10)]

$$\begin{aligned}
\sigma_r &= \frac{E}{1-\nu^2}[\epsilon_r + \nu\epsilon_\theta - (1+\nu)kT] \\
\sigma_\theta &= \frac{E}{1-\nu^2}[\nu\epsilon_r + \epsilon_\theta - (1+\nu)kT] \\
\tau_{r\theta} &= \frac{E}{2(1+\nu)}\gamma_{r\theta} \\
\epsilon_z &= -\frac{1}{1-\nu}[\nu(\epsilon_r + \epsilon_\theta) - (1+\nu)kT] \\
e = \epsilon_r + \epsilon_\theta + \epsilon_z &= \frac{1}{1-\nu}[(1-2\nu)(\epsilon_r + \epsilon_\theta) + (1+\nu)kT] \\
\sigma_z = \tau_{rz} = \tau_{\theta z} &= \gamma_{rz} = \gamma_{\theta z} = 0
\end{aligned} \tag{6-4.4}$$

and the compatibility relations [Eq. (5-3.11)]

$$\nabla^2(\sigma_r + \sigma_\theta) + E\nabla^2(kT) + (1+\nu)\left(\frac{\partial B_r}{\partial r} + \frac{1}{r}B_r + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta}\right) = 0 \tag{6-4.5}$$

Problem Set 6-4

1. **(a)** For the case of plane stress relative to the (x, y) plane, write the integral V of the strain energy density U in terms of rectangular Cartesian coordinates (x, y). Neglect temperature effects.

(b) Express the integral V in terms of polar coordinates (r, θ).

(c) Derive expressions for the stress components relative to polar coordinates (Section 4-3 in Chapter 4).

2. A circular ring, with rectangular cross section, has a unit thickness perpendicular to its plane. Its inner boundary $(r = a)$ is fixed. Its outer boundary $(r = b)$ is subjected to a uniform shearing stress S directed in the counterclockwise sense.

(a) In terms of polar coordinates (r, θ), with origin at the center of the ring, and polar coordinate stress components, write the integral V for the strain energy density U of the ring (plane stress case).

It may be shown that the stress solution for this problem is given by $\sigma_r = \sigma_\theta = 0$, $\tau_{r\theta} = Sb^2/r^2$.

(b) Evaluate the integral V of the strain energy density U.

(c) By equating V to the work done during loading (the shear stress at $r = b$ is increased from zero to S), compute the rotation of the ring at $r = b$.

3. In Problem 2, determine the tangential (θ) displacement v as a function of r, where $v = 0$ at $r = a$, and $\tau_{r\theta} = S$ at $r = b$.

4. In Problem 2, assume $\sigma_r = \sigma_\theta = u = 0$, where u is the radial displacement. Show that $\tau_{r\theta} = Sb^2/r^2$. (Assume $v = 0$ at $r = a$ and $\tau_{r\theta} = S$ at $r = b$.)

6-5 Compatibility Equation for Plane Elasticity in Terms of Polar Coordinates

Expressing the second derivative of F with respect to x in terms of polar coordinates and adding it to the second derivative of F with respect to y derived in Section 6-2, we obtain

$$\sigma_x + \sigma_y = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \tag{6-5.1}$$

Also, by Eqs. (6-2.2) we note that

$$\sigma_r + \sigma_\theta = \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} \tag{6-5.2}$$

Accordingly, by Eqs. (6-5.1), (6-5.2), and (5-7.1), we obtain the compatibility relation (for constant body forces, or body forces derivable from a potential function) in terms of polar coordinates (r, θ):

$$\nabla^2\nabla^2 F = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}\right) = 0 \tag{6-5.3}$$

Accordingly, in polar coordinates [see Section 1-22 and Eq. (1-22.13) in Chapter 1]

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{6-5.4}$$

A solution of the compatibility equation $\nabla^2\nabla^2 F = 0$ in polar coordinates was derived by Michell (1899) for a certain class of plane problems. A modified form of the solution given by Michell[1] is

$$\begin{aligned} F = {} & A_0 \log r + B_0 r^2 + C_0 r^2 \log r + D_0 r^2 \theta + A_0' \theta \\ & + \frac{A_1}{2} r\theta \sin\theta + (B_1 r^3 + A_1' r^{-1} + B_1' r \log r)\cos\theta \\ & - \frac{C_1}{2} r\theta\cos\theta + (D_1 r^3 + C_1' r^{-1} + D_1' r \log r)\sin\theta \\ & + \sum_{n=2}^{\infty} (A_n r^n + B_n r^{n+2} + A_n' r^{-n} + B_n' r^{-n+2})\cos n\theta \\ & + \sum_{n=2}^{\infty} (C_n r^n + D_n r^{n+2} + C_n' r^{-n} + D_n' r^{-n+2})\sin n\theta \end{aligned} \tag{6-5.5}$$

[1]The term $D_0 r^2 \theta$ was not given by Michell (1899). Also, Michell included the terms $r\cos\theta$, $r\sin\theta$, which are not included here. However, these terms yield zero stress components. See Timoshenko and Goodier (1970, Chapter 4). See also Timpe (1905, 1923).

Problem Set 6-5

1. Consider a ring loaded as shown in Fig. P6-5.1. Show that the function

$$\phi = \left(Ar^2 + Br^4 + \frac{C}{r^2} + D \right) \cos 2\theta + Fr^2 + H \log r$$

satisfies $\nabla^2\nabla^2\phi = 0$. Determine the constants A, B, C, D, F, H to satisfy the stress boundary conditions. Hence, derive formulas for $\sigma_r, \sigma_\theta, \tau_{r\theta}$.

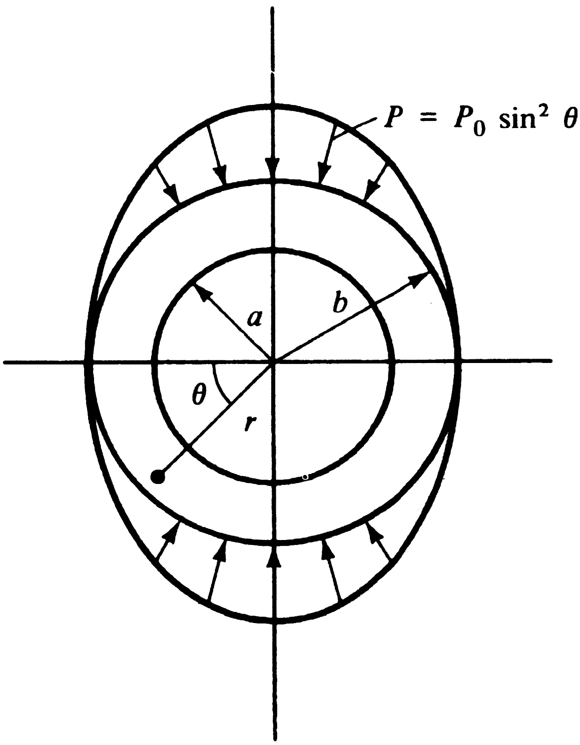


Figure P6-5.1

2. Derive the equation of compatibility for plane problems in polar coordinates in terms of the strain components [see Eq. (6-3.6)].

3. The stress function $F = (E\delta/4\pi)r \log r \sin\theta$ has been proposed as a possible solution for a circular ring with a radial slit (Fig. P6-5.3), where δ is the radial displacement at the slit.

(a) Write down a complete set of boundary conditions in terms of stress components and displacement components.

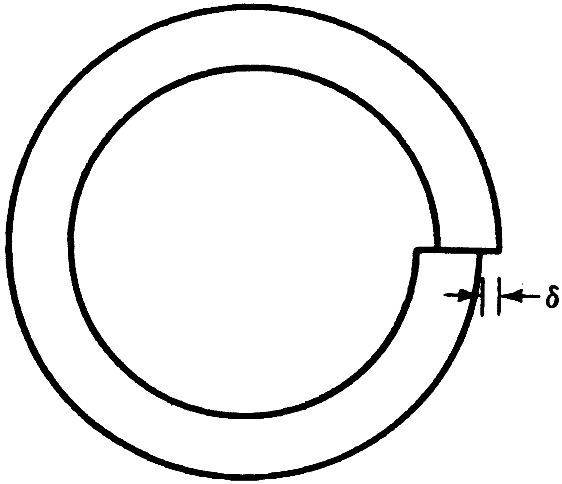


Figure P6-5.3

(b) Outline a procedure to determine a stress function that satisfies *all* boundary conditions.

4. For a problem of plane stress,

$$Eu = (1 - \nu)(\log r)\cos\theta - 2\cos\theta + 2\theta\sin\theta$$

$$Ev = (1 - \nu)(1 - \log r)\sin\theta + 2\theta\cos\theta$$

where (u, v) are displacement components in polar coordinates (r, θ), E is the modulus of elasticity, and ν is Poisson's ratio. There is no body force.

(a) Is this a possible displacement vector if the origin is included in the body? Explain.

(b) Is this a possible displacement vector for a closed ring with center at the origin? Explain.

(c) Does the corresponding Airy stress function satisfy the compatibility condition $\nabla^2\nabla^2 F = 0$? Explain.

(d) Show that for this problem the stress components σ_r and σ_θ are equal.

5. In addition to the terms given in Eq. (6-5.5) (obtained by the method of separation of variables), the terms

$$F_1 = A\theta r^2 \log r \qquad F_2 = B\theta \log r$$

$$F_3 = C\theta r \cos\theta \log r \qquad F_4 = D\theta r \sin\theta \log r$$

are also solutions to the biharmonic equation of plane elasticity, in the absence of body forces and thermal effects. Discuss the application of these terms to regions R_1, R_2, R_3, with polar coordinate systems shown in Fig. P6-5.5.

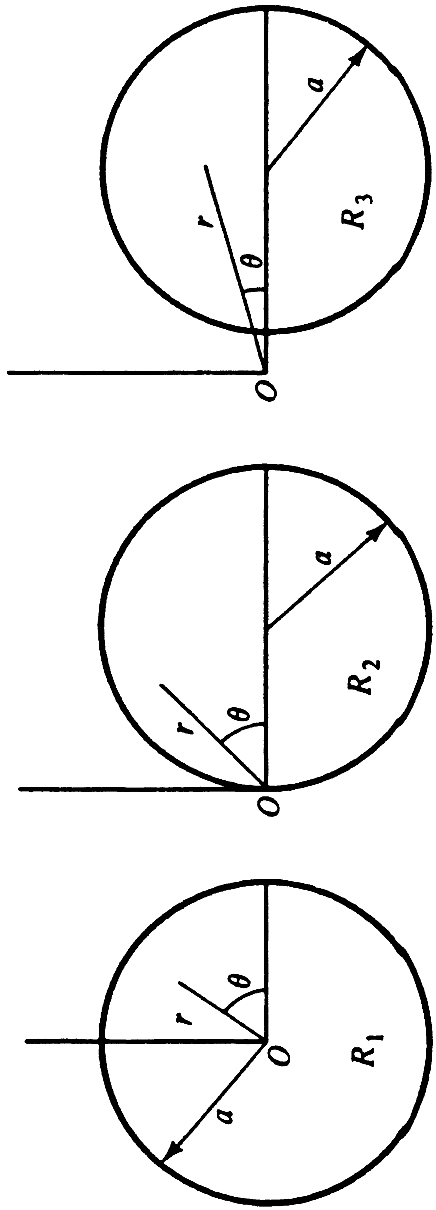


Figure P6-5.5

6-6 Axially Symmetric Problems

For axially symmetric problems, $F = F(r)$. Then the equilibrium equations [see Eqs. (6-1.1)] reduce to (for h = constant)

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) + R = 0 \qquad \Theta = 0 \tag{6-6.1}$$

Accordingly, for axially symmetric problems of equilibrium the tangential body force Θ is zero, and the two stress components $(\sigma_r, \sigma_\theta)$ and the radial body force R are functions of r only. Furthermore, the shearing stress $\tau_{r\theta}$ [see Eqs. (6-2.2)] is zero.

The compatibility relation simplifies to

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr}\right) = 0 \tag{6-6.2}$$

Equation (6-4.2) may be written in the form

$$\frac{1}{r}\frac{d}{dr}\left\{r\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right)\right]\right\} = 0 \tag{6-6.3}$$

In this latter form, the Airy stress function F may be determined by direct integration. Accordingly, for problems of axial symmetry, integration of Eq. (6-6.3) yields the Airy stress function in the form

$$F = A\log r + Br^2\log r + Cr^2 + D \tag{6-6.4}$$

where A, B, and C are arbitrary constants of integration, which are determined by boundary conditions. The constant D does not enter into the formulas for the stress components, as they depend on derivatives of F. Thus, by Eqs. (6-2.2) and (6-6.4), we obtain

$$\begin{aligned}\sigma_r &= \frac{1}{r}\frac{dF}{dr} = \frac{A}{r^2} + B(1 + 2\log r) + 2C \\ \sigma_\theta &= \frac{d^2F}{dr^2} = -\frac{A}{r^2} + B(3 + 2\log r) + 2C\end{aligned} \tag{6-6.5}$$

For a doubly connected region bounded by contours L_1 and L_2 and with the origin of coordinates (r, θ) inside the inner contour (Fig. 6-6.1), the requirement that the displacement be single valued dictates that $B = 0$. (See Example 6-6.2; see also remarks at the end of Section 5-4 in Chapter 5.)

Inclusion of Body Forces. A direct and elementary treatment of the most generally rotationally symmetric plane state of stress for linear isotropic elastic materials under arbitrary body forces has been given by Stern (1965). The main results follow.

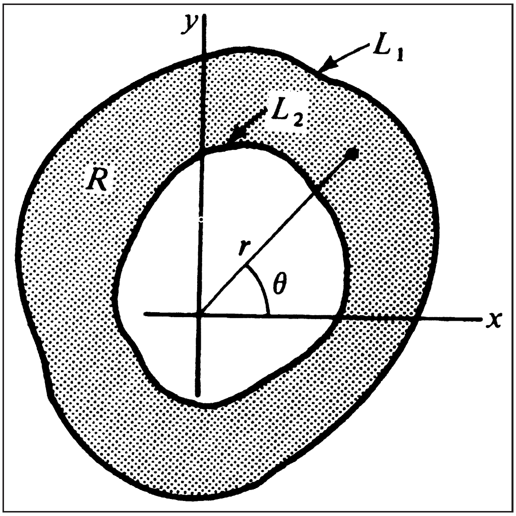


Figure 6-6.1

For the most general rotationally symmetric plane problem, relative to polar coordinates (r, θ) we assume that the stress components are independent of θ. Thus, Eqs. (6-1.1), with h = constant, yield

$$\begin{aligned} \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \\ \frac{d\tau_{r\theta}}{dr} + \frac{2\tau_{r\theta}}{r} + \Theta = 0 \end{aligned} \tag{6-6.6}$$

Recalling Eq. (5-3.12) and expressing ∇^2 and $\partial/\partial x$, $\partial/\partial y$ in terms of (r, θ), in the absence of temperature effects, we obtain for the equation of compatibility

$$\frac{d^2}{dr^2}(\sigma_r + \sigma_\theta) + \frac{1}{r}\frac{d}{dr}(\sigma_r + \sigma_\theta) = -K_2\left(\frac{dR}{dr} + \frac{R}{r}\right) \tag{6-6.7}$$

where for plane strain $K_2 = 1/(1-\nu)$ and for plane stress $K_2 = 1 + \nu$. For purposes of integration, it is convenient to rewrite Eqs. (6-6.6) and (6-6.7) in the forms

$$\frac{1}{r^2}\frac{d}{dr}(r^2\sigma_r) = \frac{1}{r}(\sigma_r + \sigma_\theta) - R \tag{6-6.8}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\tau_{r\theta}) = -\Theta \tag{6-6.9}$$

$$\frac{1}{r}\frac{d}{dr}\left\{r\left[\frac{d}{dr}(\sigma_r + \sigma_\theta) + K_2 R\right]\right\} = 0 \tag{6-6.10}$$

where by the assumption of independency of θ, R and Θ must be independent of θ.

Integration of Eq. (6-6.10) yields

$$\sigma_r + \sigma_\theta = A \log r + B - K_2 H(r) \tag{6-6.11}$$

where A and B are constants to be defined by the boundary conditions, and

$$H(r) = \int_{r_0}^{r} R(\xi)\, d\xi \tag{6-6.12}$$

where r_0 is some fixed (arbitrary) value of r. Hence, by Eqs. (6-6.8) and (6-6.11), we find, after integration, that

$$\sigma_r = \frac{C}{r^2} + \frac{A}{4}[2\log(r) - 1] + \frac{B}{2} - \frac{K_2}{2}H(r) + \frac{K_2 - 2}{2}I(r) \tag{6-6.13}$$

where C is a constant of integration and

$$I(r) = \frac{1}{r^2}\int_{r_0}^{r} \xi^2 R(\xi)\, d\xi \tag{6-6.14}$$

By Eqs. (6-6.11) and (6-6.13), we obtain

$$\sigma_\theta = -\frac{C}{r^2} + \frac{A}{4}[2\log(r) + 1] + \frac{B}{2} - \frac{K_2}{2}H(r) + \frac{2 - K_2}{2}I(r) \tag{6-6.15}$$

Finally, integration of Eq. (6-6.9) yields

$$\tau_{r\theta} = \frac{D}{r^2} - J(r) \tag{6-6.16}$$

where D is a constant and

$$J(r) = \frac{1}{r^2}\int_{r_0}^{r} \xi^2 \Theta(\xi)\, d\xi \tag{6-6.17}$$

The displacement components are obtained by integrating the strain–displacement relations. However, the displacement need not be rotationally symmetric. Let u and v denote the radial and transverse components of displacement. Then the strain–displacement relations in conjunction with Hooke's law give for plane stress [see Eqs. (5-3.10) with $kT = 0$]

$$\epsilon_r = \frac{\partial u}{\partial r} = \frac{1}{E}\sigma_r - \frac{\nu}{E}\sigma_\theta \tag{6-6.18}$$

$$\epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{E}\sigma_\theta - \frac{\nu}{E}\sigma_r \tag{6-6.19}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) = \frac{2(1+\nu)}{E}\tau_{r\theta} \tag{6-6.20}$$

With the aid of Eqs. (6-6.13) and (6-6.15) and integration by parts, Eq. (6-6.18) yields

$$u(r,\theta) = -\frac{1+\nu}{E}\frac{C}{r} + \frac{1-\nu}{2E}Ar\log r - \frac{3-\nu}{4E}Ar + \frac{1-\nu}{2E}Br - \frac{1-\nu^2}{2E}r[H(r) - I(r)] + f(\theta)$$

where $f(\theta)$ is an undetermined function of θ only. Putting this result in Eq. (6-6.20) and noting Eq. (6-6.16), we can write

$$v(r,\theta) = -\frac{1+\nu}{E}\frac{D}{r} - \frac{1+\nu}{E}r[G(r) - J(r)] + \frac{df}{d\theta} + rg(\theta)$$

where $g(\theta)$ is another undetermined function and

$$G(r) = \int_{r_0}^{r} \Theta(\xi)\,d\xi$$

As a consequence of Eq. (6-6.19), however, we conclude that

$$\frac{d^2 f}{d\theta^2} + f = 0 \qquad \frac{dg}{d\theta} = \frac{A}{E}$$

so that finally we find

$$u(r,\theta) = -\frac{1+\nu}{E}\frac{C}{r} + \frac{1-\nu}{2E}Ar\log r - \frac{3-\nu}{4E}Ar + \frac{1-\nu}{2E}Br - \frac{1-\nu^2}{2E}r[H(r) - I(r)] + M\cos\theta + N\sin\theta$$

$$v(r,\theta) = -\frac{1+\nu}{E}\frac{D}{r} + \frac{A}{E}r\theta - \frac{1+\nu}{E}r[G(r) - J(r)] - M\sin\theta + N\cos\theta + Lr$$

where the constants M and N represent the Cartesian components of a rigid-body translation and L is a rigid-body rotation angle.

In certain cases, restrictions may be imposed on the constants. For example, we should note that if the origin is contained in the body, then the constants A, C, and D must necessarily vanish. The constant A must also vanish whenever the origin can be encircled by a contour entirely in the body, even though the origin itself is not; this guarantees single-valued displacements. Finally, if any portion of the body extends indefinitely, the constant A must vanish for stresses to remain bounded.

The accelerating disk affords a rather simple application of the preceding results. Consider a circular disk of radius b clamped to a rotating shaft on a concentric circular portion of the disk of radius a, $0 < a < b$. We suppose that at some particular instant the shaft is rotating with angular velocity ω and angular acceleration α. In a quasi-static analysis the problem may be rephrased as a circular ring clamped along the inner boundary $r = a$ and free of traction on the outer boundary $r = b$, and further subjected to the body-force densities

$$R = \rho r\omega^2 \qquad \Theta = -\rho r\alpha$$

where ρ is the mass density of the disk, assumed uniform throughout. Integrating from the inner boundary ($r_0 = a$), we obtain

$$\begin{aligned} H(r) &= \tfrac{1}{2}\rho\omega^2(r^2 - a^2) \\ I(r) &= \frac{\rho\omega^2}{4r^2}(r^4 - a^4) \\ G(r) &= -\tfrac{1}{2}\rho\alpha(r^2 - a^2) \\ J(r) &= -\frac{\rho\alpha}{4r^2}(r^4 - a^4) \end{aligned}$$

Because the ring is complete, $A = 0$. Furthermore, on the outer boundary $\sigma_r = \tau_{r\theta} = 0$. Hence,

$$\begin{aligned} \frac{C}{b^2} + \frac{B}{2} - \frac{\rho\omega^2}{8b^2}(b^2 - a^2)[(3+\nu)b^2 + (1-\nu)a^2] &= 0 \\ \frac{D}{b^2} + \frac{\rho\alpha}{4b^2}(b^4 - a^4) &= 0 \end{aligned}$$

At $r = a$, $u = v = 0$, so that $M = N = 0$ and

$$\begin{aligned} -\frac{1+\nu}{E}\frac{C}{a} + \frac{1-\nu}{E}\frac{aB}{2} &= 0 \\ -\frac{1+\nu}{E}\frac{D}{a} + aL &= 0 \end{aligned}$$

Thus, we find

$$\begin{aligned} \frac{1}{2}B &= \frac{(1+\nu)\rho\omega^2}{8}K \\ C &= \frac{(1-\nu)\rho\omega^2 a^2}{8}K \\ D &= -\frac{\rho\alpha}{4}(b^4 - a^4) \\ L &= -\frac{(1+\nu)\rho\alpha}{4Ea^2}(b^4 - a^4) \end{aligned}$$

where

$$K = (b^2 - a^2)\frac{(3+\nu)b^2 + (1-\nu)a^2}{(1+\nu)b^2 + (1-\nu)a^2}$$

Then the stresses are given by

$$\sigma_r = \frac{\rho\omega^2}{8}\left\{\left(1+\frac{a^2}{r^2}\right)K - \left(1-\frac{a^2}{r^2}\right)[(3+\nu)r^2 + (1-\nu)a_2 - \nu K]\right\}$$

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left\{\left(1+\frac{a^2}{r^2}\right)\nu K - \left(1-\frac{a^2}{r^2}\right)[(1+3\nu)r^2 - (1-\nu)a^2 - K]\right\}$$

$$\tau_{r\theta} = -\frac{q\alpha}{4r^2}(b^4 - r^4)$$

while the displacement components are simply

$$u = \frac{\rho\omega^2(1-\nu^2)}{8E} r \left(1-\frac{a^2}{r^2}\right)[K - (r^2 - a^2)]$$

$$v = -\frac{\rho\alpha(1+\nu)}{4E} r \left(1-\frac{a^2}{r^2}\right)\left(\frac{b^4}{a^2} - r^2\right)$$

Example 6-6.1. Let $A = B = 0$ in Eq. (6-6.5). Then Eq. (6-6.5) yields

$$\sigma_r = \sigma_\theta = 2C \tag{a}$$

Equation (a) represents the case of constant stress throughout the plane [see Fig. E6-6.1].

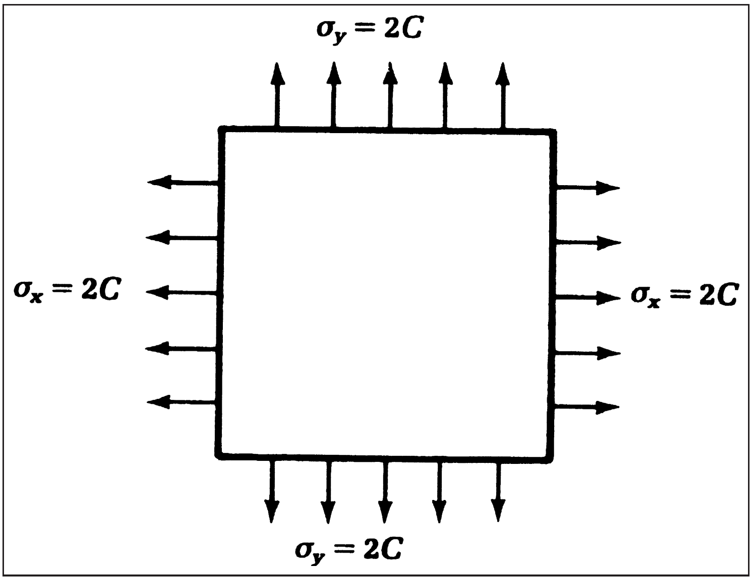


Figure E6-6.1

Example 6-6.2. Let $B = 0$ in Eqs. (6-6.5). Then

$$\sigma_r = \frac{A}{r^2} + 2C \qquad \sigma_\theta = -\frac{A}{r^2} + 2C \tag{b}$$

Equation (b) may be used to represent the stress in a thick-walled cylinder with inner radius a and outer radius b and with internal pressure p_i and external pressure p_0 (Fig. E6-6.2). Then the boundary conditions are

$$\begin{aligned} \sigma_r &= -p_0 \quad \text{for} \quad r = b \\ \sigma_r &= -p_i \quad \text{for} \quad r = a \end{aligned} \tag{c}$$

Substitution of Eqs. (c) into Eqs. (b) yields

$$\begin{aligned} A &= \frac{a^2 b^2 (p_0 - p_i)}{b^2 - a^2} \\ 2C &= \frac{p_i a^2 - p_0 b^2}{b^2 - a^2} \end{aligned} \tag{d}$$

To investigate the variation of $(\sigma_r, \sigma_\theta)$ through the wall of the cylinder, consider the case $p_i = p$, $p_0 = 0$. Then Eqs. (b) and (d) yield

$$\begin{aligned} \sigma_r &= -\frac{a^2 b^2 p}{(b^2 - a^2) r^2} + \frac{a^2 p}{b^2 - a^2} \\ \sigma_\theta &= \frac{a^2 b^2 p}{(b^2 - a^2) r^2} + \frac{a^2 p}{b^2 - a^2} \end{aligned} \tag{e}$$

The change of $(\sigma_r, \sigma_\theta)$ with radial distance r is pictured in Fig. E6-6.3.

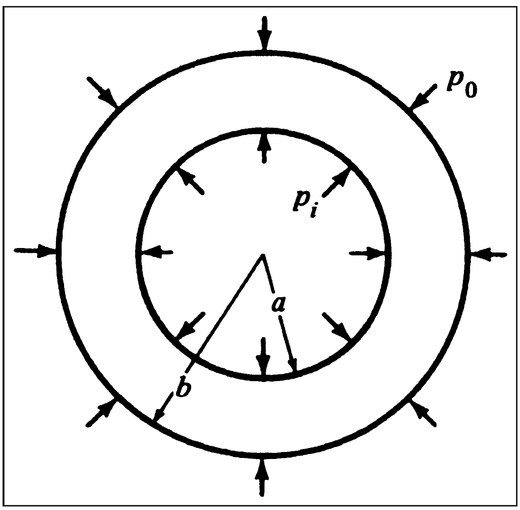


Figure E6-6.2

Example 6-6.3. Plane Strain Axisymmetrical Deformation of a Circular Cylinder. A thick-wall cylindrical pressure vessel with circular cross section undergoes linearly elastic deformation when subjected to a uniform external pressure acting on its outer lateral surface $r = b$. Its inner lateral surface at radius $r = a$ is constrained by a rigid cylindrical core so that its radial displacement $u = 0$ at $r = a$ (similar to Fig. P6-6.2, with $u = 0$). We wish to determine the stress components $(\sigma_r, \sigma_\theta, \sigma_z)$, where (r, θ) are polar coordinates in the cross section and z is the coordinate along the axis of the cylinder. The origin of coordinates (r, θ, z) is located at the center $(r = 0)$ of one of its end cross sections (where $z = 0$). We assume that the cylinder is free to expand laterally except at $r = a$ but is constrained axially so that a condition of plane strain relative to the (r, θ) plane exists.

Because the cylinder is loaded axisymmetrically, the theory of this Section applies. Thus, the stress components are independent of θ, and the tangential displacement component v (Fig. 6-1.1) is zero. Also, $\tau_{r\theta} = 0$ by Eqs. (6-6.2) and (6-6.4).

In the absence of body forces and temperature field, Eqs. (6-6.13) and (6-6.14) yield

$$\begin{aligned} \sigma_r &= \frac{C}{r^2} + \frac{A}{4}[2\log(r) - 1] + \frac{B}{2} \\ \sigma_\theta &= -\frac{C}{r^2} + \frac{A}{4}[2\log(r) + 1] + \frac{B}{2} \end{aligned} \tag{a}$$

Because the (r, θ) origin can be encircled by a contour entirely in the body even though the origin itself is not in the body (i.e., it is located at $r = 0$), the constant $A = 0$. [See the discussion following Eq. (6-6.20).]

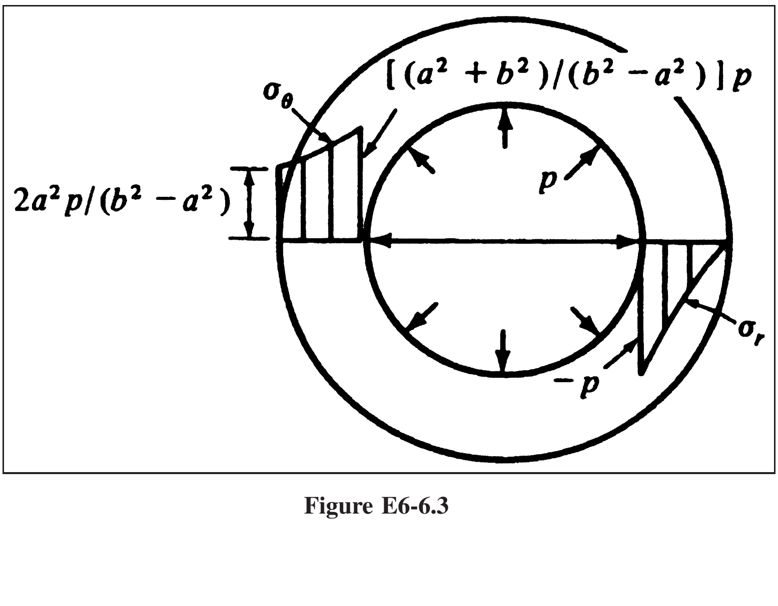


Figure E6-6.3

The strain–displacement relations for a linearly elastic isotropic medium for plane strain are [Eqs. (6-4.2) with $kT = 0$]

$$\begin{aligned} \epsilon_r &= \frac{du}{dr} = \frac{1-\nu^2}{E}\left(\sigma_r - \frac{\nu\sigma_\theta}{1-\nu}\right) \\ \epsilon_\theta &= \frac{u}{r} = \frac{1-\nu^2}{E}\left(\sigma_\theta - \frac{\nu\sigma_r}{1-\nu}\right) \end{aligned} \tag{b}$$

The second of Eqs. (b) yields, with Eqs. (a) and $A = 0$,

$$u = \frac{(1+\nu)r}{E}\left[-\frac{C}{r^2} + \frac{B(1-2\nu)}{2}\right] \tag{c}$$

The boundary condition $u = 0$ for $r = a$ yields with Eq. (c)

$$C = \frac{B(1-2\nu)a^2}{2} \tag{d}$$

The boundary condition $\sigma_r = -p$ for $r = b$ yields with Eqs. (a) and (d)

$$B = -\frac{2pb^2}{a^2(1-2\nu)+b^2} \qquad C = -\frac{p(1-2\nu)a^2b^2}{a^2(1-2\nu)+b^2} \tag{e}$$

Equations (a) and (e) yield

$$\begin{aligned} \sigma_r &= -\frac{pb^2}{a^2(1-2\nu)+b^2}\left[1 + (1-2\nu)\frac{a^2}{r^2}\right] \\ \sigma_\theta &= -\frac{pb^2}{a^2(1-2\nu)+b^2}\left[1 - (1-2\nu)\frac{a^2}{r^2}\right] \end{aligned} \tag{f}$$

Therefore, because for plane strain $\sigma_z = \nu(\sigma_r + \sigma_\theta)$, we obtain by Eq. (f)

$$\sigma_z = -\frac{2\nu pb^2}{a^2(1-2\nu)+b^2} = \text{constant} \tag{g}$$

Equations (c) and (e) yield

$$u = -\frac{(1+\nu)(1-2\nu)pb^2r}{E[a^2(1-2\nu)+b^2]}\left[1 - \frac{a^2}{r^2}\right] \tag{h}$$

Atheromatous Plaque on Artery Wall. An atheroma, commonly referred to as atheromatous plaque, is an accumulation and swelling in artery walls made up of cells or cell debris, which contains lipids (cholesterol and fatty acids), calcium, and fibrous connective tissue. Plaques are found in arteries, not veins, unless surgically moved to function as arteries as in bypass surgery, of most humans older than 10. The process of atheroma development is called atherogenesis, an unhealthy condition that may lead to heart attack.

The endothelium (the cell monolayer on the inside of the vessel) and the covering tissue, termed fibrous cap, separate atheroma from the blood in the lumen (artery opening). If a rupture occurs of the endothelium and fibrous cap, then a platelet and clotting response over the rupture rapidly develops and results in a shower of debris. Platelet and clot accumulation over the rupture may produce narrowing/closure of the lumen, and tissue damage may occur due to either closure of the lumen and loss of blood flow beyond the ruptured atheroma and/or by occlusion of smaller downstream vessels by debris (Waller et al., 1992).

Of course, the atheroma process and the mechanism of the rupture of the endothelium and fibrous cap are very complex. To understand the effect of the plaque on the stress distribution in the artery, consider an axis-symmetric problem of a hollow cylinder made of two kinds of materials, the first one for the plaque $r \in [a, b]$ and the second one for the health artery $r \in [b, c]$ with $a < b < c$. Let the hollow cylinder subject to an internal pressure p and the material constants be (λ_1, G_1) and (λ_2, G_2) for the first and the second material, respectively. Due to axis symmetry the displacement vector of every point in the cylinder is radial, one may write in cylindrical coordinates

$$\mathbf{u}^i = [u_r^i, u_\theta^i, u_z^i] = [U_i(r), 0, 0] \qquad i = 1, 2 \tag{6-6.21}$$

From the strain–displacement relations [Eqs. (2A-2.7)], one obtains

$$\begin{aligned} \epsilon_{rr}^i &= \frac{dU_i}{dr} \qquad \epsilon_{\theta\theta}^i = \frac{U_i}{r} \qquad \epsilon_{zz}^i = 0 \\ \gamma_{r\theta}^i &= 0 \qquad \gamma_{rz}^i = 0 \qquad \gamma_{\theta z}^i = 0 \end{aligned} \tag{6-6.22}$$

The general stress–strain relation for linear isotropic elastic solid in cylindrical coordinates can be expressed as (Sadd, 2009)

$$\begin{aligned} \sigma_{rr} &= \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2G\epsilon_{rr} \\ \sigma_{\theta\theta} &= \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2G\epsilon_{\theta\theta} \\ \sigma_{zz} &= \lambda(\epsilon_{rr} + \epsilon_{\theta\theta} + \epsilon_{zz}) + 2G\epsilon_{zz} \\ \sigma_{r\theta} &= G\gamma_{r\theta} \\ \sigma_{rz} &= G\gamma_{rz} \\ \sigma_{\theta z} &= G\gamma_{\theta z} \end{aligned} \tag{6-6.23}$$

Substituting Eqs. (6-6.22) into Eqs. (6-6.23) yields

$$\begin{aligned} \sigma_{rr}^i &= \lambda_i \left(\frac{dU_i}{dr} + \frac{U_i}{r} \right) + 2G_i \frac{dU_i}{dr} \\ \sigma_{\theta\theta}^i &= \lambda_i \left(\frac{dU_i}{dr} + \frac{U_i}{r} \right) + 2G_i \frac{U_i}{r} \\ \sigma_{zz}^i &= \sigma_{r\theta}^i = \sigma_{rz}^i = \sigma_{\theta z}^i = 0 \end{aligned} \tag{6-6.24}$$

Substituting Eqs. (6-6.24) into Eqs. (3A-2.3) and assuming that there is no body force, the only nontrivial equilibrium equation is obtained as

$$\frac{d^2U_i}{dr^2} + \frac{1}{r}\frac{dU_i}{dr} - \frac{U_i}{r^2} = 0 \tag{6-6.25}$$

The solutions for Eq. (6-6.25) are

$$U_1 = Ar + Br^{-1} \qquad U_2 = Cr + Dr^{-1} \tag{6-6.26}$$

and the stresses are

$$\begin{aligned} \sigma_{rr}^1 &= 2(\lambda_1 + G_1)A - 2G_1 r^{-2} B \\ \sigma_{\theta\theta}^1 &= 2(\lambda_1 + G_1)A + 2G_1 r^{-2} B \\ \sigma_{rr}^2 &= 2(\lambda_2 + G_2)C - 2G_2 r^{-2} D \\ \sigma_{\theta\theta}^2 &= 2(\lambda_2 + G_2)C + 2G_2 r^{-2} D \end{aligned} \tag{6-6.27}$$

The boundary conditions are (1) the internal pressure acting on the wall of plaque $(r = a)$ is p; (2) the outer pressure at $r = c$ is zero; (3) the normal stress is continuous at the interface between the plaque and the healthy artery $(r = b)$; and (4) the displacement is also continuous at the interface $(r = b)$. One may express the four boundary conditions as

$$\begin{aligned} \sigma_{rr}^1(r = a) &= -p \\ \sigma_{rr}^2(r = c) &= 0 \\ \sigma_{rr}^1(r = b) &= \sigma_{rr}^2(r = b) \\ U_1(r = b) &= U_2(r = b) \end{aligned} \tag{6-6.28}$$

which imply

$$\begin{bmatrix} \lambda_1 + G_1 & -G_1 a^{-2} & 0 & 0 \\ 0 & 0 & \lambda_2 + G_2 & -G_2 c^{-2} \\ \lambda_1 + G_1 & -G_1 b^{-2} & -\lambda_2 - G_2 & G_2 b^{-2} \\ 1 & b^{-2} & -1 & b^{-2} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -\frac{p}{2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{6-6.29}$$

The coefficients A, B, C, and D are solved as

$$\begin{aligned} A &= -\frac{\alpha}{\lambda_1 + 2G_1} P \\ B &= \frac{\beta}{G_1}\left(1 - a^2 \frac{\alpha}{\beta} \frac{\lambda_1 + G_1}{\lambda_1 + 2G_1}\right) P \\ C &= P \\ D &= c^2 \frac{\lambda_2 + G_2}{G_2} P \end{aligned} \tag{6-6.30}$$

where

$$\alpha \equiv -(\lambda_2 + G_1 + G_2) + (\lambda_2 + G_2)\frac{c^2}{b^2}\frac{G_2 - G_1}{G_2}$$
$$\beta \equiv (\lambda_2 + G_2)(c^2 - b^2) - \alpha(b^2 - a^2)\frac{\lambda_1 + G_1}{\lambda_1 + 2G_1} \tag{6-6.31}$$
$$P \equiv \frac{pa^2}{2\beta}$$

To verify the solutions, consider the following special cases:

1. If the material properties of the plaque and the healthy artery are the same, that is,

$$\lambda_1 = \lambda_2 = \lambda \qquad \mu_1 = \mu_2 = \mu \tag{a}$$

then

$$\alpha = -(\lambda + 2G) \qquad \beta = (\lambda + G)(c^2 - a^2) \qquad P = \frac{pa^2}{2(\lambda + G)(c^2 - a^2)} \tag{b}$$

$$A = C = P \qquad B = D = \frac{\lambda + G}{G}c^2 P \tag{c}$$

$$\sigma_{rr} = \frac{pa^2(1 - c^2 r^{-2})}{c^2 - a^2} \qquad \sigma_{\theta\theta} = \frac{pa^2(1 + c^2 r^{-2})}{c^2 - a^2} \tag{d}$$

which are exactly the same solutions for an artery without plaque [cf. Eq. (e) in Example 6-6.2].

2. If the thickness of the plaque is very thin, that is,

$$a = b \tag{e}$$

then

$$\beta = (\lambda_2 + G_2)(c^2 - a^2) \qquad P = \frac{pa^2}{2(\lambda_2 + G_2)(c^2 - a^2)} \tag{f}$$

$$C = P \qquad D = c^2\frac{\lambda_2 + G_2}{G_2}P \tag{g}$$

$$\sigma^2_{rr} = \frac{pa^2(1 - c^2 r^{-2})}{c^2 - a^2} \qquad \sigma^2_{\theta\theta} = \frac{pa^2(1 + c^2 r^{-2})}{c^2 - a^2} \tag{h}$$

which are again the same solutions for an artery without plaque.

Significance of Active Stress. What is the *fundamental difference between living biological tissues and lifeless materials?* As we have discussed in Chapter 4, all lifeless materials are passive; but the muscle, when it is activated, may exert active tensile stress even in the state of contraction. Now we are going to demonstrate the effect of active stress in living biological tissue through an example.

Recall the full constitutive relation of the compressible muscle (Humphrey, 2002)

$$\sigma_{ij} = \sigma^p_{ij} + Am_i m_j \tag{6-6.32}$$

where $\boldsymbol{\sigma}$ is the total Cauchy stress (active plus passive); $\boldsymbol{\sigma}^p$ is the passive contribution to the stress; $A > 0$ is the muscle tension in the direction $\mathbf{m}$, which is a unit vector in the direction of a muscle fiber in a deformed state (Eulerian discription). For an axially symmetric and plane strain problem, let the direction of muscle tension $\mathbf{m}$ coincide with the circumferential direction, namely, the θ direction. The displacement vector has only one component

$$\mathbf{u} = [u_r, u_\theta, u_z] = [U(r), 0, 0]$$

From the strain–displacement relations, Eqs. (2A-2.7), one may obtain

$$\epsilon_{rr} = \frac{dU}{dr} \triangleq U' \qquad \epsilon_{\theta\theta} = r^{-1}U \qquad \epsilon_{zz} = \gamma_{r\theta} = \gamma_{rz} = \gamma_{\theta z} = 0 \tag{6-6.33}$$

From the general stress–strain relation for linear isotropic elastic solid in cylindrical coordinates, the Cauchy stress tensor for muscle can be obtained as [cf. Eq. (6-6.32)]

$$\begin{aligned} \sigma_{rr} &\triangleq \sigma_r = (\lambda + 2G)U' + \lambda r^{-1}U \\ \sigma_{\theta\theta} &\triangleq \sigma_\theta = (\lambda + 2G)r^{-1}U + \lambda U' + A \\ \sigma_{zz} &\triangleq \sigma_\theta = \lambda(U' + r^{-1}U) \end{aligned} \tag{6-6.34}$$

The governing equation, Eq. (6-6.1), now leads to

$$\frac{d\sigma_r}{dr} + \frac{1}{r}(\sigma_r - \sigma_\theta) = (\lambda + 2G)(U'' + r^{-1}U' - r^{-2}U) - r^{-1}A = 0 \tag{6-6.35}$$

The solution of this differential equation has two parts: the homogeneous solution and the particular solution, which is obtained as

$$U = Cr + Dr^{-1} + \alpha r \ln r \tag{6-6.36}$$

where the active stress ratio is defined as

$$\alpha \equiv \frac{A}{2(\lambda + 2G)} \tag{6-6.37}$$

The stresses are obtained as

$$\begin{aligned} \sigma_r &= (\lambda + 2G)U' + \lambda r^{-1}U \\ &= 2(\lambda + G)C - 2GDr^{-2} + \alpha[2(\lambda + G)\ln r + \lambda + 2G] \\ &\equiv 2(\lambda + G)C - 2GDr^{-2} + \alpha F(r) \\ \sigma_\theta &= (\lambda + 2G)r^{-1}U + \lambda U' + A \\ &= 2(\lambda + G)C + 2GDr^{-2} + \alpha[2(\lambda + G)\ln r + 3\lambda + 4G] \\ \sigma_z &= \lambda(r^{-1}U + U') \\ &= 2\lambda C + 2\alpha[\lambda \ln r + \lambda] \end{aligned} \tag{6-6.38}$$

The boundary conditions $\sigma_r(r = a) = -p$ and $\sigma_r(r = b) = 0$ lead to

$$\begin{bmatrix} 2(\lambda + G) & -2Ga^{-2} \\ 2(\lambda + G) & -2Gb^{-2} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} -p - \alpha F(a) \\ -\alpha F(b) \end{bmatrix} \tag{6-6.39}$$

Then C and D are determined to be

$$\begin{aligned} C &= \frac{a^2 b^2}{2(\lambda + G)(b^2 - a^2)} \left\{ pb^{-2} - \alpha[-b^{-2}F(a) + a^{-2}F(b)] \right\} \\ D &= \frac{a^2 b^2}{2G(b^2 - a^2)} \{p - \alpha[F(b) - F(a)]\} \end{aligned} \tag{6-6.40}$$

For illustrative purpose, let the dimensionless parameters used in this example be

$$\begin{aligned} &\text{Poisson's ratio:} && \nu = 0.3 \\ &\text{Outer radius/inner radius:} && \frac{b}{a} = 1.5 \\ &\text{Active stress ratio:} && \frac{A}{E} = 0.4 \\ &\text{Pressure/Young's modulus:} && \frac{p}{E} = 0.1 \end{aligned} \tag{6-6.41}$$

For $\nu = 0.3$, the two Lamé constants are calculated to be

$$\lambda = \frac{3}{5.2}E \qquad G = \frac{1}{2.6}E \tag{6-6.42}$$

Define the normalized stresses, displacement, and position as

$$\begin{aligned} \overline{\sigma}_{rr} &\equiv \frac{\sigma_{rr}}{p} \\ \overline{\sigma}_{\theta\theta} &\equiv \frac{\sigma_{\theta\theta}}{p} \\ \overline{\sigma}_{zz} &\equiv \frac{\sigma_{zz}}{p} \\ \overline{U} &\equiv \frac{U}{a} \\ \overline{r} &\equiv \frac{r - a}{b - a} \end{aligned} \tag{6-6.43}$$

The normalized stresses and displacement as functions of normalized position with and without active stress are plotted in Figs. 6-6.2 to 6-6.5. The curves in solid lines are for the case with active stress, that is, when the muscle is activated, the curves in broken lines are for the case without active stress, that is, the muscle has not been activated. From Fig. 6-6.2, it is seen that at $r = a$, that is, $\overline{r} = 0$, the radial stress is equal to $-p$, that is, $\overline{\sigma}_{11} = -1$; and at $r = b, \overline{\sigma}_{11} = 0$. This means in both cases the boundary conditions are satisfied and the only difference

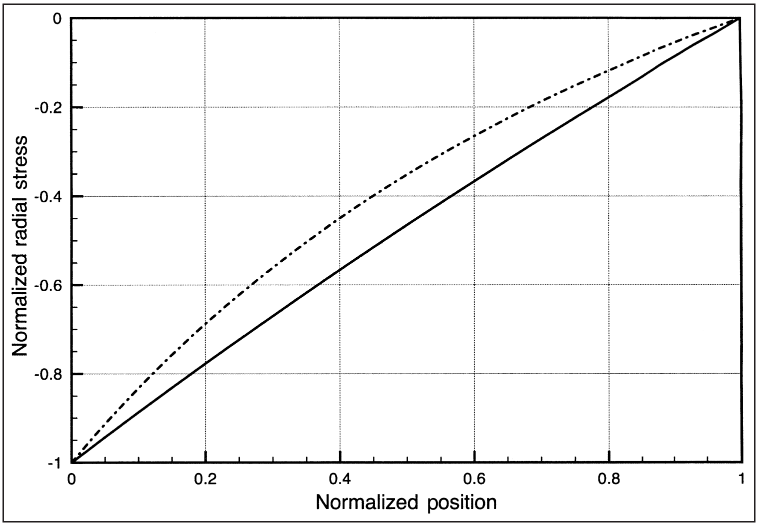


Figure 6-6.2

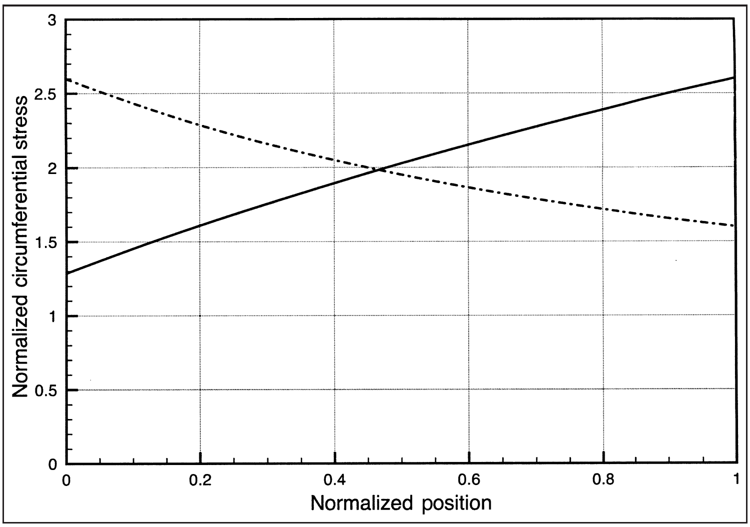


Figure 6-6.3

is quantitative. From Fig. 6-6.3, the circumferential stress $\overline{\sigma}_{22}$ in case of no active stress is monotonically decreasing with the radius r, which is a well-known solution for the problem of cylindrical tube subjected to inner pressure. However, in case of muscle being activated, the circumferential stress is monotonically increasing with the radius. From Fig. 6-6.4, the longitudinal stress in case of no active stress is a constant tensile stress, which means that a tensile stress in the z direction is needed

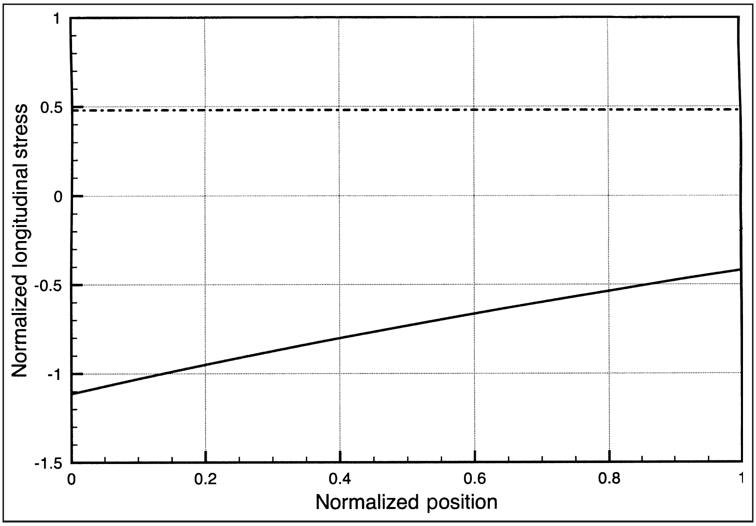


Figure 6-6.4

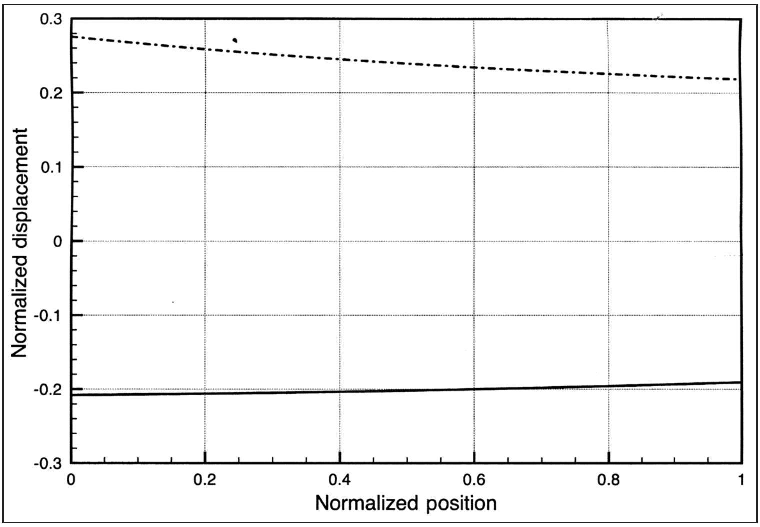


Figure 6-6.5

to maintain the plane strain condition. On the contrary, in the case with active stress, σ_{33} is a varying compressive stress. Actually this phenomenon can be seen even more vividly in Fig. 6-6.5, from which it is seen that the radial displacement $u_r = U(r)$ is positive in case of no active stress. Of course, it is positive simply because the inner pressure pushes everything outward. However, when the muscle is activated, the active stress, although a tensile stress, tends to squeeze the muscle fiber. That is why the radial displacement is smaller than the corresponding one in the case of no active stress and even becomes negative. Once again this example demonstrates the effect of active stress—the hallmark of living biological tissue.

Problem Set 6-6

1. Derive expressions for the radial and tangential components of displacement for the problem of Example 6–6.2.

2. A thin circular disk is given, which has outer radius b and inner radius a. The hole is expanded and a smooth, rigid plug of radius $a + \epsilon$ is inserted. Determine the stresses in the disk for this problem of generalized plane stress (Fig. P6-6.2).

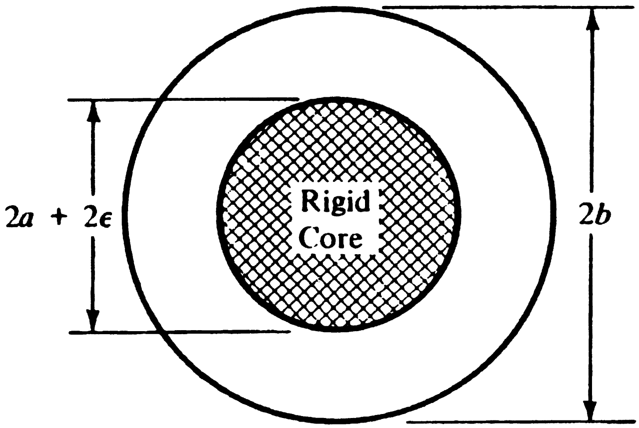


Figure P6-6.2

3. A cylinder is cast of thermoplastic material in a steel mold (Fig. P6-6.3). The material solidifies at 210°F. It is then cooled to room temperature, during which process the material "shrinks" (by thermal contraction) around the steel core. Estimate the maximum normal stress in the cylinder. The steel core has a 2-in. radius, and the plastic cylinder an original radius of 4 in. The coefficient of linear expansion is $k = 0.0002$ in./in./°F. $E = 10^5$ psi, $\nu = 0.5$.

4. Noting that the radial body force for a solid constant-thickness (thin) rotating disk is $R = \rho\omega^2 r$, where ρ is the mass density and ω is the angular frequency, show that a solution of the elasticity problem is given by $r\sigma_r = F$, $\sigma_\theta = (dF/dr) + \rho\omega^2 r^2$, where F satisfies the equation

$$r^2 \frac{d^2F}{dr^2} + r\frac{dF}{dr} - F = -(3+\nu)\rho\omega^2 r^3 \tag{a}$$

Hence, show that the solution for F is

$$F = Ar + \frac{B}{r} - \frac{(3+\nu)\rho\omega^2 r^3}{8} \tag{b}$$

Derive expressions for the constants A and B for the solid disk (Fig. P6-6.4).

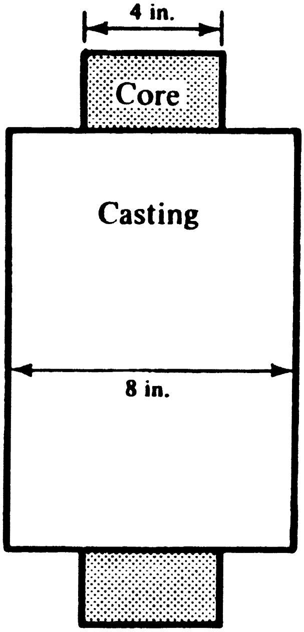


Figure P6-6.3

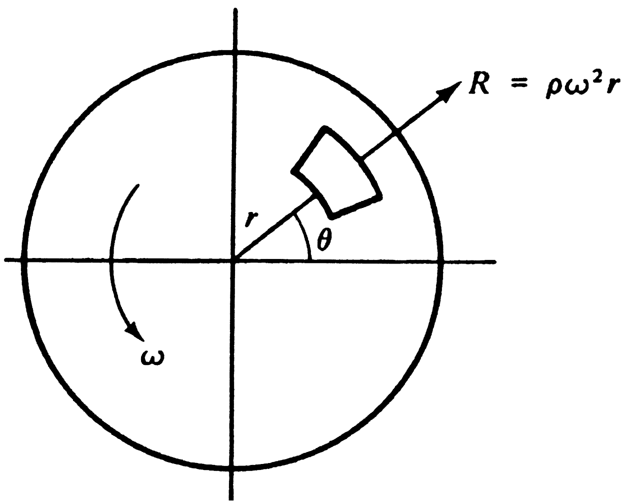


Figure P6-6.4

5. A steel disk with a hole 2 in. in diameter is shrunk on a shaft 2.003 in. in diameter. The disk has a constant thickness, and its outside diameter is 20 in. Assuming that the shaft is rigid, calculate the angular velocity at which the disk will become loose on the shaft (see Problem 4).

6. Consider the Airy stress function $F = Ar^2 \log r$, where (r, θ) are polar coordinates.

(a) Compute the associated stress components $(\sigma_r, \sigma_\theta, \tau_{r\theta})$.

(b) Is the above Airy stress function a possible solution to a boundary value problem of a complete ring $(a \le r \le b, 0 \le \theta \le 2\pi)$? Explain.

(c) Is the above Airy stress function a possible solution to a boundary value problem of a disk $(0 \le r \le b, 0 \le \theta \le 2\pi)$? Explain.

(d) Is the above Airy stress function a possible solution to a boundary value problem of an incomplete ring $(a \le r \le b, 0 \le \theta \le \theta_0 < 2\pi)$? Explain.

7. Plain strain axisymmetric deformation of a circular cylinder: The cylinder has an inner radius a and an outer radius b. The inside of the cylinder is restrained such that the radial displacement is zero at $r = a$. The outside is subjected to a pressure p. Determine the stress σ_r, σ_θ, and σ_z as functions of r.

8. A thin circular disk has outer radius b and inner radius a (similar to Fig. P6-6.2). The radial displacement u at $r = a$ is zero. The outer boundary $(r = b)$ is subjected to pressure p and is otherwise unconstrained. Determine the stress components $(\sigma_r, \sigma_\theta)$ as functions of polar coordinate r. *Hint*: See Eqs. (6-6.4) and (6-6.5), and note that the boundary conditions must be satisfied. The problem is one of plane stress in the plane of the disk.

9. A thin circular annulus (inner radius $= a$; outer radius $= b$) is subjected to a temperature distribution T defined by the relation $kT = A(r^2 - a^2)$, where k and A are known constants. Derive expressions for the polar coordinate stress components $(\sigma_r, \sigma_\theta, \tau_{r\theta})$. *Hint*: The compatibility equation for the axisymmetric plane stress problem is $d(r\epsilon_\theta)/dr = \epsilon_r$.

10. Let the disk of Problem 8 be subjected to pressure p at its inner surface $(r = a)$, which is now unconstrained. Let the outer radius be fixed so that the radial displacement $u = 0$. Determine $(\sigma_r, \sigma_\theta)$ as functions of r.

11. In addition to the constraints and load of Problem 8, let the disk be subjected to the temperature distribution T defined by $kT = A(r^2 - a^2)$, where k and A are known constants (see Problem 9). Determine the polar stress components $(\sigma_r, \sigma_\theta)$ as functions of r.

12. Determine and plot the displacement U and stresses $\sigma_{rr}, \sigma_{\theta\theta}$, and σ_{zz} as functions of r in the artery wall with atheromatous plaque for the following two cases:

(a) $a = 1.0,\ b = 1.1,\ c = 1.2,\ \lambda_1 = \lambda_2 = G_1 = G_2 = p = 1$

(b) $a = 1.0,\ b = 1.1,\ c = 1.2,\ \lambda_2 = G_2 = p = 1,\ \lambda_1 = G_1 = 2$

13. Let the muscle tension A in Eq. (6-6.34) be replaced by 0.2 Er/a. Find the significance of active stress by determining and plotting the displacement U and stresses $\sigma_{rr}, \sigma_{\theta\theta}$, and σ_{zz} as functions of r. Use the following parameters: $p/E = 0.1$, $\nu = 0.3$, $b/a = 1.5$.

6-7 Plane Elasticity Equations in Terms of Displacement Components

In this section we develop the plane stress equilibrium equations for an isotropic homogeneous elastic material in the absence of temperature effects. In Section 6-9 we consider the plane stress problem of a variable-thickness disk of nonhomogeneous anisotropic material.

For plane stress, the stress–strain equilibrium equations in polar coordinates (r, θ), in the absence of temperature effects, are [see Eqs. (6-4.4)]

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) \\ \sigma_\theta &= \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r) \\ \tau_{r\theta} &= G\gamma_{r\theta} = \frac{E}{2(1+\nu)}\gamma_{r\theta} \end{aligned} \tag{6-7.1}$$

where $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ and $(\epsilon_r, \epsilon_\theta, \gamma_{r\theta})$ denote polar coordinate components of stress and strain, respectively, and where ν denotes Poisson's ratio, E Young's modulus, and G the shear modulus. The strain–displacement relations in plane polar coordinates are [Eq. (6-3.5)]

$$\begin{aligned} \epsilon_r &= u_r \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}v_\theta \\ \gamma_{r\theta} &= \frac{1}{r}u_\theta + v_r - \frac{v}{r} \end{aligned} \tag{6-7.2}$$

where (u, v) denote (r, θ) components of displacement, and where (r, θ) subscripts on (u, v) denote differentiation relative to (r, θ). Substitution of Eqs. (6-7.2) into Eqs. (6-7.1) yields

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}\left(u_r + \nu\frac{u}{r} + \nu\frac{v_\theta}{r}\right) \\ \sigma_\theta &= \frac{E}{1-\nu^2}\left(\frac{u}{r} + \frac{v_\theta}{r} + \nu u_r\right) \\ \tau_{r\theta} &= \frac{E}{2(1+\nu)}\left(\frac{u_\theta}{r} + v_r - \frac{v}{r}\right) \end{aligned} \tag{6-7.3}$$

The equilibrium equations are, with $h =$ constant [see Eqs. (6-1.1)],

$$\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + B_r &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{2\tau_{r\theta}}{r} + B_\theta &= 0 \end{aligned} \tag{6-7.4}$$

Substitution of Eqs. (6-7.3) into Eqs. (6-7.4) yields

$$\begin{aligned} &\frac{E}{1-\nu^2}\left[u_{rr} + \frac{u_r}{r} - \frac{u}{r^2} + \frac{(1+\nu)v_{r\theta}}{2r} - \frac{(3-\nu)v_\theta}{2r^2} + \frac{(1-\nu)u_{\theta\theta}}{2r^2}\right] + B_r = 0 \\ &\frac{E}{1-\nu^2}\left[\frac{(1+\nu)u_{r\theta}}{2r} + \frac{(3-\nu)u_\theta}{2r^2} + \frac{(1-\nu)v_{rr}}{2}\right. \\ &\qquad \left. + \frac{(1-\nu)v_r}{2r} - \frac{(1-\nu)v}{2r^2} + \frac{v_{\theta\theta}}{r^2}\right] + B_\theta = 0 \end{aligned} \tag{6-7.5}$$

In Eqs. (6-7.4) and (6-7.5) we have denoted body forces in the (r, θ) directions by (B_r, B_θ), respectively. Equations (6-7.5) are the equilibrium equations for plane stress problems in terms of displacement components (u, v) relative to polar coordinates (r, θ). They form the basis for study of plane stress boundary value problems in polar coordinates. For the classical axisymmetric problem, $u = u(r)$, $v = 0$. Then Eqs. (6-7.5) reduce to the single equation

$$\frac{E}{1-\nu^2}\left(\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2}\right) + B_r = 0 \tag{6-7.6}$$

For $B_r = 0$, Eq. (6-7.6) may be written

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ru)\right] = 0$$

and direct integration yields the solution

$$u = C_1 r + \frac{C_2}{r} \tag{6-7.7}$$

where the constants C_1, C_2 are determined by boundary conditions. For example, by Eqs. (6-7.3) and (6-7.7), we obtain, because $v = 0$,

$$\begin{aligned} \sigma_r &= \frac{EC_1}{1-\nu} - \frac{EC_2}{1+\nu}\frac{1}{r^2} \\ \sigma_\theta &= \frac{EC_1}{1-\nu} + \frac{EC_2}{1+\nu}\frac{1}{r^2} \end{aligned} \tag{6-7.8}$$

With the boundary conditions $\sigma_r = -p_0$ for $r = b$, $\sigma_r = -p_i$ for $r = a$, we obtain (see Example 6-6.2)

$$\begin{aligned} C_1 &= \frac{1-\nu}{E}\left(\frac{p_i a^2 - p_0 b^2}{b^2 - a^2}\right) \\ C_2 &= \frac{1+\nu}{E}\left[\frac{a^2 b^2 (p_i - p_0)}{b^2 - a^2}\right] \end{aligned} \tag{6-7.9}$$

Example 6-7.1. Stresses in a Rotating Disk Subjected to a Temperature Gradient. A thin solid disk of radius a rotates about an axis through its center $r = 0$ with a constant angular velocity ω. It is also subjected to a temperature field T defined by the relation $T = T_0 r/a$, where T_0 is a constant. We wish to determine the stresses in the disk and the increase of its diameter resulting from these effects.

The radial body force is $R = \rho r \omega^2$, and because $\omega =$ constant ($\alpha = 0$), the tangential body force $\Theta = 0$ (see Section 6-6). Hence by Eqs. (6-6.8) and (6-6.9), we have

$$\frac{1}{r^2}\frac{d}{dr}(r^2\sigma_r) = \frac{1}{r}(\sigma_r + \sigma_\theta) - \rho r \omega^2 \tag{a}$$

$$\frac{1}{r^2}\frac{d}{dr}(r^2\tau_{r\theta}) = 0 \tag{b}$$

By Eq. (6-4.5), for plane stress we have (with $B_r = R = \rho r \omega^2$ and $B_\theta = \Theta = 0$)

$$\frac{1}{r}\frac{d}{dr}\left\{r\left[\frac{d}{dr}(\sigma_r + \sigma_\theta + EkT) + (1+\nu)\rho r \omega^2\right]\right\} = 0 \tag{c}$$

Solving Eqs. (a) and (c) for $(\sigma_r, \sigma_\theta)$, with $T = T_0 r/a$, we find by the procedure used to obtain Eqs. (6-6.13) and (6-6.15), because $A = 0$,

$$\sigma_r = \frac{C}{r^2} + \frac{B}{2} - \frac{3+\nu}{8}\rho r^2 \omega^2 - \frac{Ek}{3}T_0\frac{r}{a} \tag{d}$$

$$\sigma_\theta = -\frac{C}{r^2} + \frac{B}{2} - \frac{1+3\nu}{8}\rho r^2 \omega^2 - \frac{2}{3}EkT_0\frac{r}{a} \tag{e}$$

Integration of Eq. (b) yields

$$\tau_{r\theta} = \frac{D}{r^2} \tag{f}$$

where D is a constant.

The boundary conditions at $r = a$ are $\sigma_r = 0$ and $\tau_{r\theta} = 0$. With these conditions, Eqs. (d) and (f) yield

$$\frac{C}{a^2} + \frac{B}{2} = \frac{3+\nu}{8}\rho a^2 \omega^2 + \frac{Ek}{3}T_0 \tag{g}$$

$$D = 0 \tag{h}$$

At $r = 0$, $u = 0$. Hence, we must obtain an expression for u in terms of $(\sigma_r, \sigma_\theta)$ or constants C and B.

For plane stress, the strain–displacement relations are by the first two of Eqs. (6-4.4)

$$\epsilon_r = \frac{du}{dr} = \frac{1}{E}(\sigma_r - \nu\sigma_\theta) + kT \tag{i}$$

$$\epsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) + kT \tag{j}$$

Equation (j) yields

$$u = \frac{1}{E}(r\sigma_\theta - \nu r\sigma_r) + kTr \tag{k}$$

For $r = 0$, Eqs. (d), (e), and (k) yield

$$u = 0 = \frac{1}{E}\left(-\frac{C}{0} - \nu\frac{C}{0}\right) \tag{l}$$

Consequently, in order for u to be zero, $C = 0$. Hence Eq. (g) yields

$$B = \frac{3+\nu}{4}\rho a^2 \omega^2 + \frac{2EkT_0}{3} \tag{m}$$

and Eqs. (d), (e), and (m) give

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(a^2 - r^2) + \frac{EkT_0}{3}\left(1 - \frac{r}{a}\right) \tag{n}$$

$$\sigma_\theta = \frac{\rho\omega^2}{8}[(3+\nu)a^2 - (1+3\nu)r^2] + \frac{EkT_0}{3}\left(1 - \frac{2r}{a}\right) \tag{o}$$

By Eqs. (k), (n), and (o), the general expression for u is

$$u = \frac{(1-\nu)\rho\omega^2 r}{8E}[(3+\nu)a^2 - (1+\nu)r^2] + \frac{kT_0 r}{3}\left(1 + \frac{r}{a}\right) - \frac{\nu kT_0 r}{3}\left(1 - \frac{r}{a}\right) \tag{p}$$

Thus, for $r = a$,

$$u = \frac{1-\nu}{4}\rho a^3\omega^2 + \frac{2}{3}kT_0 a$$

and the increase in the diameter of the disk is

$$\Delta d = 2u = \frac{1-\nu}{2}\rho a^3\,\omega^2 + \frac{4}{3}kT_0 a \tag{q}$$

6-8 Plane Theory of Thermoelasticity

The plane theory of thermoelasticity is based on assumptions equivalent to those of plane elasticity theory. Consequently, plane thermoelasticity consists of two cases: plane strain and plane stress (or, more generally, *generalized plane stress*).

Plane Strain. We recall that a body is in a state of plane strain parallel to the (x, y) plane if the z displacement component w is constant and if (u, v), the (x, y) components of displacement, are functions of (x, y) only. Consequently, the strain–displacement relations in (x, y) coordinates reduce to

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \epsilon_z = 0 \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{xz} = \gamma_{yz} = 0 \end{aligned} \tag{6-8.1}$$

Substituting Eqs. (6-8.1) into the equations of thermoelasticity (see Section 4-12 in Chapter 4), we obtain relations for the plane strain theory of thermoelasticity.

In cylindrical coordinates (r, θ, z) the plane strain condition is expressed by the relations

$$u = u(r, \theta) \qquad v = v(r, \theta) \qquad w = \text{const.}$$

Hence, the strain–displacement relations in cylindrical coordinates are [see Eqs. (2A-2.7) in Chapter and (6-7.2)]

$$\begin{aligned} \epsilon_r &= \frac{\partial u}{\partial r} \qquad \epsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \epsilon_z = 0 \\ \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \qquad \gamma_{rz} = \gamma_{\theta z} = 0 \end{aligned} \tag{6-8.2}$$

The stress–strain–temperature relations in cylindrical coordinates are [see Eqs. (4-11.6) and (6-4.2)]

$$\begin{aligned} \epsilon_r &= E^{-1}[\sigma_r - \nu(\sigma_\theta + \sigma_z)] + kT & \gamma_{r\theta} &= G^{-1}\tau_{r\theta} \\ \epsilon_\theta &= E^{-1}[\sigma_\theta - \nu(\sigma_r + \sigma_z)] + kT & \gamma_{rz} &= \gamma_{\theta z} = 0 \\ \epsilon_z &= E^{-1}[\sigma_z - \nu(\sigma_r + \sigma_\theta)] + kT \end{aligned} \tag{6-8.3}$$

For plane strain $\epsilon_z = 0$; hence, the last of Eqs. (6-8.3) yields

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) - EkT \tag{6-8.4}$$

Substitution of Eq. (6-8.4) into Eqs. (6-8.3) yields the stress–strain–temperature relations for plane strain:

$$\begin{aligned} \epsilon_r &= E^{-1}[(1-\nu^2)\sigma_r - \nu(1+\nu)\sigma_\theta] + (1+\nu)kT \\ \epsilon_\theta &= E^{-1}[(1-\nu^2)\sigma_\theta - \nu(1+\nu)\sigma_r] + (1+\nu)kT \\ \gamma_{r\theta} &= G^{-1}\tau_\theta \end{aligned} \tag{6-8.5}$$

For axisymmetric problems $v = 0$ and $\partial/\partial\theta = 0$, and Eqs. (6-8.2) are modified accordingly. Consequently, u and T are functions of r only.

For axially symmetric plane strain in the absence of body forces, the equilibrium equations reduce to the single equation [see Eqs. (3A-2.7) and (2A-2.7) and let $\partial/\partial\theta = 0$, $v = 0$]

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{6-8.6}$$

Substituting Eqs. (6-8.2) into Eqs. (6-8.5), solving Eqs. (6-8.5) for $(\sigma_r, \sigma_\theta)$, and substituting the resulting equations into the equilibrium equation [Eq. (6-8.6)], we obtain

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = \frac{1+\nu}{1-\nu}\frac{d(kT)}{dr}$$

Rewriting this equation, we obtain

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru)}{dr}\right] = \frac{1+\nu}{1-\nu}\frac{d(kT)}{dr} \tag{6-8.7}$$

Integration of Eq. (6-8.7) yields

$$u = \left(\frac{1+\nu}{1-\nu}\right)\frac{1}{r}\int_a^r \rho kT\,d\rho + Ar + \frac{B}{r} \tag{6-8.8}$$

Equations (6-8.7) and (6-8.8) and corresponding modifications of the equations of Section 4-12 in Chapter 4 summarize the plane strain theory of thermoelasticity.

Plane Stress. A body is in a state of plane stress in the (x, y) plane if $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. Substitution of these conditions into the general thermoelasticity theory of Section 4-12 in Chapter 4 yields the corresponding equations of plane stress thermoelasticity.

In cylindrical coordinates, the stress–strain–temperature relations for plane stress are [see Eqs. (6-8.3)]

$$\begin{aligned} \epsilon_r &= E^{-1}(\sigma_r - \nu\sigma_\theta) + kT \\ \epsilon_\theta &= E^{-1}(\sigma_\theta - \nu\sigma_r) + kT \\ \epsilon_z &= -\frac{\nu}{E}(\sigma_r + \sigma_\theta) + kT \end{aligned} \tag{6-8.9}$$

Inverting the first two of Eq. (6-8.9), we obtain

$$\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}(\epsilon_r + \nu\epsilon_\theta) - \frac{EkT}{1-\nu} \\ \sigma_\theta &= \frac{E}{1-\nu^2}(\epsilon_\theta + \nu\epsilon_r) - \frac{EkT}{1-\nu} \end{aligned} \tag{6-8.10}$$

Substitution of Eqs. (6-8.10) into the last of Eqs. (6-8.9) yields

$$\epsilon_z = \frac{-\nu}{1-\nu}(\epsilon_r + \epsilon_\theta) + \frac{1+\nu}{1-\nu}kT \tag{6-8.11}$$

Equation (6-8.6) is the equilibrium condition for axially symmetric plane stress thermoelasticity, as $\sigma_z = \tau_{rz} = \tau_{\tau z} = 0$. Also, because $v = 0$, and $\partial/\partial\theta = 0$ for axial symmetry, the strain–displacement relations [Eqs. (6-8.2)] reduce to

$$\epsilon_r = \frac{du}{dr} \qquad \epsilon_\theta = \frac{u}{r} \tag{6-8.12}$$

where u is the displacement in the r direction.

Substitution of Eqs. (6-8.12) and (6-8.10) into Eqs. (6-8.6) yields

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = (1+\nu)\frac{d(kT)}{dr} \tag{6-8.13}$$

Integration of Eq. (6-8.13) yields

$$u = (1+\nu)\frac{1}{r}\int_a^r kT\rho d\rho + Ar + \frac{B}{r} \tag{6-8.14}$$

Equations to (6-8.9) to (6-8.14) and corresponding modifications of the equations of Section 4-12 summarize the theory of plane stress thermoelasticity. Plane stress thermoelasticity problems of radial heating of a thin circular disk and axial heating of beams and strips are important in practice.

Problem Set 6-8

1. **(a)** Show that $u = \sum_{n=0}^{\infty} a_n \cos n\theta$, $v = \sum_{n=0}^{\infty} b_n \sin n\theta$, where (u, v) denote polar coordinates (r, θ) components of displacement and the coefficients a_n, b_n are functions of r only, is a possible solution of the plane stress equations of equilibrium expressed in terms of displacement components (u, v).

(b) Derive the differential equations that define the coefficients a_n, b_n.

2. Assume that the Airy stress function F is of the form $F = f(\theta)$, where $f(\theta)$ is a function of θ, the polar coordinate angle of polar coordinates (r, θ).

(a) Derive the explicit form for $f(\theta)$ for the case of the plane stress problem of a ring, in the region $a \leq r \leq b$, under uniform shearing stresses applied at the inner $(r = a)$ and outer $(r = b)$ surfaces of the ring. Neglect body forces and inertia forces.

(b) Derive explicit expressions for displacement components (u, v) relative to polar coordinates (r, θ), respectively, expressing the results in terms of the applied stresses and the radii a and b.

3. A thin circular disk of radius a is subjected to a temperature distribution

$$T = T_0 \left(1 - \frac{r}{a}\right)$$

where T_0 is a known constant. The compatibility equation for axisymmetric polar coordinate problems with thermal effects is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dF}{dr}\right) = -EkT + C$$

where k is the coefficient of thermal expansion and C is an unknown constant of integration (to be determined by boundary conditions). Determine the change in diameter of the disk due to the applied temperature.

4. Derive the compatibility equation given in Problem 3.

5. A long mine tunnel of radius a is cut in deep rock. Before the tunnel is cut, the rock is subjected to uniform pressure p. Considering the rock to be an infinite, homogeneous elastic medium with elastic constants E and ν, determine the inward radial displacement at the surface of the tunnel due to the excavation.

6. A circular annular disk rotates with constant angular velocity ω about the axis O, perpendicular to the plane of the disk (Fig. P6-8.6). The inner radius of the disk is located at $r = a$, the outer radius at $r = b$. The inside radius is restrained to prevent radial displacement. Assume that a state of plane stress relative to the plane of the disk exists.

(a) Derive the equations of motion of the disk in terms of displacement components relative to polar coordinates (r, θ).

(b) Integrate the equations to determine the radial displacement u.

(c) Determine the constants of integrations.

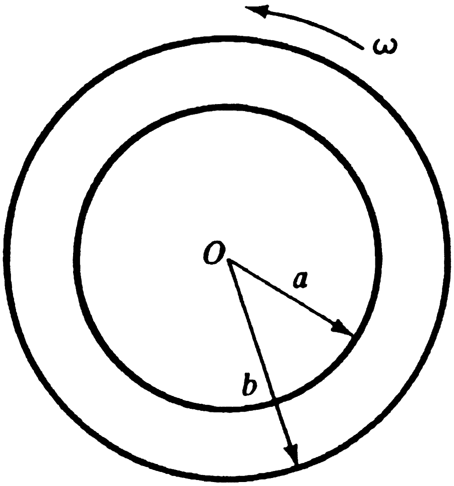


Figure P6-8.6

7. Modify the equations of Section 4-12 in Chapter 4 for plane strain. Repeat for plane stress.

8. Let $T = T(r, \theta)$ for a plane thermoelasticity problem in polar coordinates (r, θ). Determine an explicit expression for $T(r, \theta)$ for the steady-state case in absence of heat source, expressing $T(r, \theta)$ in the form $T = T_1(r) + T_2(r, \theta)$, where $T_1(r)$ is the part of $T(r, \theta)$ dependent upon r alone. That is, show that

$$T_1(r) = A_0 + B_0 \log r$$

$$T_2(r, \theta) = \sum_{n=1}^{\infty} [(A_n r^n + B_n r^{-n}) \cos n\theta + (C_n r^n + D_n r^{-n}) \sin n\theta]$$

where A_n, B_n, C_n, D_n are constants.

9. In Problem 8 set all constants except B_1 and D_1 equal to zero. For the resulting temperature field, determine the stress produced in a hollow circular cylinder defined by cylindrical coordinates (r, θ, z) the z axis coinciding with the longitudinal axis of the cylinder. Assume that axial displacement of the cylinder is prevented.

10. A nuclear fuel element in the form of a solid right-circular cylinder is free to expand laterally but not axially. It is subjected to a radiation heat source in the form of the Gaussian distribution

$$Q = Ae^{-\alpha^2 r^2}$$

where α^2 is a constant and r is the radial coordinate. Generally, $\alpha^2 \ll 1$. Compute the temperature distribution T. What reasonable approximation may be used for T? Determine the stress distribution in the cylinder. What practical restriction must be imposed on A?

11. A solid plane circular disk of radius a is subjected to the temperature distribution T given by $kT = A + Br\cos\theta + Cr\sin\theta$, where A, B, C are constants and (r, θ) are polar coordinates with origin at the center of the disk. The disk is not restrained at its boundary $r = a$.

(a) Show that the solution of the plane stress problem of the disk is $\sigma_r = \sigma_\theta = \tau_{r\theta} = 0$.

(b) Derive explicit expressions for the radial and tangential displacement components (u, v), respectively.

(c) Write the boundary conditions that determine the arbitrary constants of integration of part (b).

6-9 Disk of Variable Thickness and Nonhomogeneous Anisotropic Material

In this section we treat the variable-thickness elastic disk made of nonhomogeneous anisotropic material relative to polar coordinates (r, θ). We assume that the stress components and body forces are functions of radial distance r from the center of the disk.

The equilibrium equations are [Eqs. (6-1.1)]

$$\begin{aligned} \frac{d}{dr}(h\sigma_r) + \frac{h}{r}(\sigma_r - \sigma_\theta) + hB_r &= 0 \\ \frac{d}{dr}(h\tau_{r\theta}) + \frac{2h}{r}\tau_{r\theta} + hB_\theta &= 0 \end{aligned} \tag{6-9.1}$$

where $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ denote stress components relative (r, θ) coordinates, $h = h(r)$ denotes the disk thickness, and (B_r, B_θ) denote the body forces per unit volume in the (r, θ) directions, respectively.

For the material being considered, the stress–strain-temperature relations are

$$\begin{aligned} \sigma_r &= C_{11}\epsilon_r + C_{12}\epsilon_\theta + C_{13}\gamma_{r\theta} - C_1 T \\ \sigma_\theta &= C_{12}\epsilon_r + C_{22}\epsilon_\theta + C_{23}\gamma_{r\theta} - C_2 T \\ \tau_{r\theta} &= C_{13}\epsilon_r + C_{23}\epsilon_\theta + C_{33}\gamma_{r\theta} - C_3 T \end{aligned} \tag{6-9.2}$$

where $C_{ij} = C_{ji} = C_{ij}(r)$ are elastic constants, $C_i = C_i(r)$ are thermoelastic constants, $T = T(r)$ denotes temperature, and $(\epsilon_r, \epsilon_\theta, \gamma_{r\theta})$ are strain components.

Inverting Eqs. (6-9.2), we obtain

$$\begin{aligned} \epsilon_r &= S_{11}\sigma_r + S_{12}\sigma_\theta + S_{13}\tau_{r\theta} + k_1 T \\ \epsilon_\theta &= S_{12}\sigma_r + S_{22}\sigma_\theta + S_{23}\tau_{r\theta} + k_2 T \\ \gamma_{r\theta} &= S_{13}\sigma_r + S_{23}\sigma_\theta + S_{33}\tau_{r\theta} + k_3 T \end{aligned} \tag{6-9.3}$$

where (k_1, k_2, k_3) are linear thermal expansion coefficients related to C_i and C_{ij} by the relations

$$\begin{aligned}
Ck_1 &= C_1 \begin{vmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{vmatrix} - C_2 \begin{vmatrix} C_{12} & C_{13} \\ C_{23} & C_{33} \end{vmatrix} + C_3 \begin{vmatrix} C_{12} & C_{13} \\ C_{22} & C_{23} \end{vmatrix} \\
Ck_2 &= -C_1 \begin{vmatrix} C_{12} & C_{23} \\ C_{13} & C_{33} \end{vmatrix} + C_2 \begin{vmatrix} C_{11} & C_{13} \\ C_{13} & C_{33} \end{vmatrix} - C_3 \begin{vmatrix} C_{11} & C_{13} \\ C_{12} & C_{23} \end{vmatrix} \\
Ck_3 &= C_1 \begin{vmatrix} C_{12} & C_{22} \\ C_{13} & C_{23} \end{vmatrix} - C_2 \begin{vmatrix} C_{11} & C_{12} \\ C_{13} & C_{23} \end{vmatrix} + C_3 \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix}
\end{aligned} \tag{6-9.4}$$

and

$$\begin{aligned}
S_{11} &= \frac{1}{C}\begin{vmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{vmatrix} \qquad S_{12} = S_{21} = -\frac{1}{C}\begin{vmatrix} C_{12} & C_{13} \\ C_{23} & C_{33} \end{vmatrix} \\
S_{13} &= S_{31} = \frac{1}{C}\begin{vmatrix} C_{12} & C_{13} \\ C_{22} & C_{23} \end{vmatrix} \qquad S_{22} = \frac{1}{C}\begin{vmatrix} C_{11} & C_{13} \\ C_{13} & C_{33} \end{vmatrix} \\
S_{23} &= S_{32} = -\frac{1}{C}\begin{vmatrix} C_{11} & C_{13} \\ C_{12} & C_{23} \end{vmatrix} \qquad S_{33} = \frac{1}{C}\begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix} \\
C &= \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix}
\end{aligned} \tag{6-9.5}$$

For the type of problem considered here, $u = u(r)$ and $v = v(r)$. Then Eqs. (6-3.5) reduce to

$$\epsilon_r = u' \qquad \epsilon_\theta = \frac{u}{r} \qquad \gamma_{r\theta} = v' - \frac{v}{r} \tag{6-9.6}$$

where primes denote derivatives with respect to r.

Equations (6-9.1), (6-9.2), and (6-9.6) yield

$$\begin{aligned}
\sigma_r &= C_{11}u' + C_{12}\frac{u}{r} + C_{13}\left(v' - \frac{v}{r}\right) - C_1 T \\
\sigma_\theta &= C_{12}u' + C_{22}\frac{u}{r} + C_{23}\left(v' - \frac{v}{r}\right) - C_2 T \\
\tau_{r\theta} &= C_{13}u' + C_{23}\frac{u}{r} + C_{33}\left(v' - \frac{v}{r}\right) - C_3 T
\end{aligned} \tag{6-9.7}$$

and

$$\begin{aligned}
u'' + R_1 u' + R_2 u + R_3 v'' + R_4 v' + R_5 v &= R_6 \\
v'' + P_1 v' + P_2 v + P_3 u'' + P_4 u' + P_5 u &= P_6
\end{aligned} \tag{6-9.8}$$

where

$$\begin{aligned}
rR_1 &= 1 + \frac{r}{\overline{C}_{11}}\overline{C}'_{11} \\
r^2R_2 &= \frac{r}{\overline{C}_{11}}\overline{C}'_{12} - \frac{\overline{C}_{22}}{\overline{C}_{11}} \\
R_3 &= \frac{\overline{C}_{13}}{\overline{C}_{11}} \\
rR_4 &= \frac{r}{\overline{C}_{11}}\overline{C}'_{13} - \frac{\overline{C}_{23}}{\overline{C}_{11}} \\
rR_5 &= -R_4 \\
R_6 &= \frac{1}{\overline{C}_{11}}\left[-\overline{B}_r + \overline{C}_1 T' + \left(\overline{C}'_1 + \frac{\overline{C}_1}{r} - \frac{\overline{C}_2}{r}\right)T\right] \\
rP_1 &= 1 + \frac{r}{\overline{C}_{33}}\overline{C}'_{33} \\
rP_2 &= -P_1 \\
P_3 &= \frac{\overline{C}_{13}}{\overline{C}_{33}} \\
rP_4 &= \frac{2\overline{C}_{13} + \overline{C}_{23}}{\overline{C}_{33}} + \frac{r}{\overline{C}_3}\overline{C}'_{13} \\
r^2P_5 &= \frac{\overline{C}_{23}}{\overline{C}_{33}} + \frac{r}{\overline{C}_{33}}\overline{C}'_{23} \\
P_6 &= \frac{1}{\overline{C}_{33}}\left[-\overline{B}_\theta + \left(\frac{2\overline{C}_3 + r\overline{C}'_3}{r}\right)T + \overline{C}_3 T'\right]
\end{aligned} \tag{6-9.9}$$

where

$$\overline{C}_{ij} = hC_{ij} \qquad \overline{C}_i = hC_i \qquad \overline{B}_r = hB_r \qquad \overline{B}_\theta = jB_\theta \tag{6-9.10}$$

If the disk rotates with angular velocity ω and angular acceleration α,

$$B_r = \rho\omega^2 \qquad B_\theta = -\rho\alpha r \tag{6-9.11}$$

where ρ denotes mass per unit volume.

Boundary Conditions. We consider two cases.

Case 1.

$$\begin{aligned}
&\text{For } r = a: \qquad \sigma_r = \sigma_a \qquad \tau_{r\theta} = \tau_a \\
&\text{For } r = b: \qquad \sigma_r = \sigma_b \qquad \tau_{r\theta} = \tau_b
\end{aligned} \tag{6-9.12}$$

where $\sigma_a, \sigma_b, \tau_a, \tau_b$ are prescribed constants.

Case 2.

$$\begin{aligned}
&\text{For } r = a: \qquad u = u_a \qquad v = v_a \\
&\text{For } r = b: \qquad \sigma_r = \sigma_b \qquad \tau_{r\theta} = \tau_b
\end{aligned} \tag{6-9.13}$$

where $\sigma_b, \tau_b, u_a, v_a$ are prescribed constants.

Substitution of Eqs. (6-9.7) into Eqs. (6-9.12) and (6-9.13) yields the boundary conditions in terms of (u, v) in the following form:

Case 1. For $r = a$,

$$\begin{aligned} C_{11}u' + C_{12}\frac{u}{a} + C_{13}\left(v' - \frac{v}{a}\right) - C_1 T_a = \sigma_a \\ C_{13}u' + C_{23}\frac{u}{a} + C_{33}\left(v' - \frac{v}{a}\right) - C_3 T_a = \tau_a \end{aligned} \tag{6-9.14}$$

where $T_a = T$ evaluated at $r = a$. For $r = b$,

$$\begin{aligned} C_{11}u' + C_{12}\frac{u}{b} + C_{13}\left(v' - \frac{v}{b}\right) - C_1 T_b = \sigma_b \\ C_{13}u' + C_{23}\frac{u}{b} + C_{33}\left(v' - \frac{v}{b}\right) - C_3 T_b = \tau_b \end{aligned} \tag{6-9.15}$$

where $T_b = T$ evaluated at $r = b$.

Case 2. For $r = a$.

$$u = u_a \qquad v = v_a \tag{6-9.16}$$

For $r = b$,

$$\begin{aligned} C_{11}u' + C_{12}\frac{u}{b} + C_{13}\left(v' - \frac{v}{b}\right) - C_1 T_b = \sigma_b \\ C_{13}u' + C_{23}\frac{u}{b} + C_{33}\left(v' - \frac{v}{b}\right) - C_3 T_b = \tau_b \end{aligned} \tag{6-9.17}$$

Equations (6-9.8) with appropriate boundary conditions [Eqs. (6-9.14) and (6-9.15) or Eqs. (6-9.16) and (6-9.17)] define the described disk problem for plane stress with nonhomogeneous anisotropic material. If $C_{13} = C_{23} = 0$ and $v = 0$, the above theory reduces to the axisymmetric plane stress problem of the orthotropic disk. If $C_{13} = C_{23} = 0$ and $v \neq 0$, the above theory uncouples into two problems, one that defines u and the other that defines v. The defining equations for the u problem are the first of Eqs. (6-9.8) with $R_3 = R_4 = R_5 = 0$ and the boundary conditions $\sigma_r = \sigma_a$ for $r = a$, $\sigma_r = \sigma_b$ for $r = b$ [Case 1, Eq. (6-9.12)], or $u = u_a$ for $r = a$, $\sigma_r = \sigma_b$ for $r = b$ [Case 2, Eq. (6-9.13)]. The defining equations for v are the second of Eqs. (6-9.8) with $P_3 = P_4 = P_5 = 0$ and the boundary conditions $\tau_{r\theta} = \tau_a$ for $r = a$, $\tau_{r\theta} = \tau_b$ for $r = b$ [Case 1, Eq. (6-9.12)], or $v = v_a$ for $r = a$, $\tau_{r\theta} = \tau_b$ for $r = b$ [Case 2, Eq. (6-9.13)]. The finite difference method may be used to solve the boundary value problem described above.

Problem Set 6-9

1. An annular plane region R defined by $a \leq r \leq b$ is subjected to uniform pressure p_a at $r = a$ and p_b at $r = b$. The stress–strain relations of the material relative to polar coordinates (r, θ) are

$$\begin{aligned} \sigma_r &= C_{11}\epsilon_r + C_{12}\epsilon_\theta \\ \sigma_\theta &= C_{12}\epsilon_r + C_{22}\epsilon_\theta \\ \tau_{r\theta} &= C_{33}\gamma_{r\theta} \end{aligned}$$

where C_{ij} are constant elastic coefficients.

(a) Considering the physical nature of the problem, express the equilibrium equations in terms of (u, v), the (r, θ) displacement components.

(b) Show that the radial displacement component u is of the form

$$u = Ar^{-n} + Br^{n}$$

where n is an explicit function of the elastic constants C_{ij} and (A, B) are constants.

(c) Write the conditions that define the constants A and B.

6-10 Stress Concentration Problem of Circular Hole in Plate

The general solution for the Airy stress function [Eq. (6-5.5)] includes a number of special cases of importance (Section 6-11). In this section we single out the particularly important problem of a plane rectangular region with interior circular hole and subjected to uniformly distributed edge stresses (Kirsch, 1898). As a special case of this problem, we treat in detail the case of uniformly distributed normal stress along two opposite edges (Fig. 6-10.1). More generally, normal stress and shear stress may be distributed uniformly along all edges. We assume that the hole is sufficiently small compared to typical overall dimensions of the region so that there exists regions far removed from the hole in which the stresses are essentially unaffected by the hole. Hence, for a circle $r = b$ scribed in the region $(b \gg a)$, the stress distribution is obtained by considering the equilibrium state of an element (Fig. 6-10.2). Thus, we find

$$\begin{aligned} N &= \sigma \cos^2\theta = \frac{\sigma}{2}(1 + \cos 2\theta) \\ S &= -\sigma \sin\theta \cos\theta = -\frac{\sigma}{2}\sin 2\theta \end{aligned} \tag{6-10.1}$$

For simplicity, we may consider the stress components [Eq. (6-10.1)] as the sum of two stress states (Fig. 6-10.3)

$$\begin{aligned} N &= N_1 + N_2 \\ S &= S_1 + S_2 \end{aligned} \tag{6-10.2}$$

where

$$N_1 = \frac{\sigma}{2} \qquad S_1 = 0 \tag{6-10.3}$$

and

$$N_1 = \frac{\sigma}{2}\cos 2\theta \qquad S_2 = -\frac{\sigma}{2}\sin 2\theta \tag{6-10.4}$$

The stress distribution for state 1 is described by the results of Example (6-6.2), with $p_i = 0$, $p_0 = -\sigma/2$, and $a \ll b$. The stress distribution of state 2 may be described by the Airy stress function

$$F(r, \theta) = f(r)\cos 2\theta \tag{6-10.5}$$

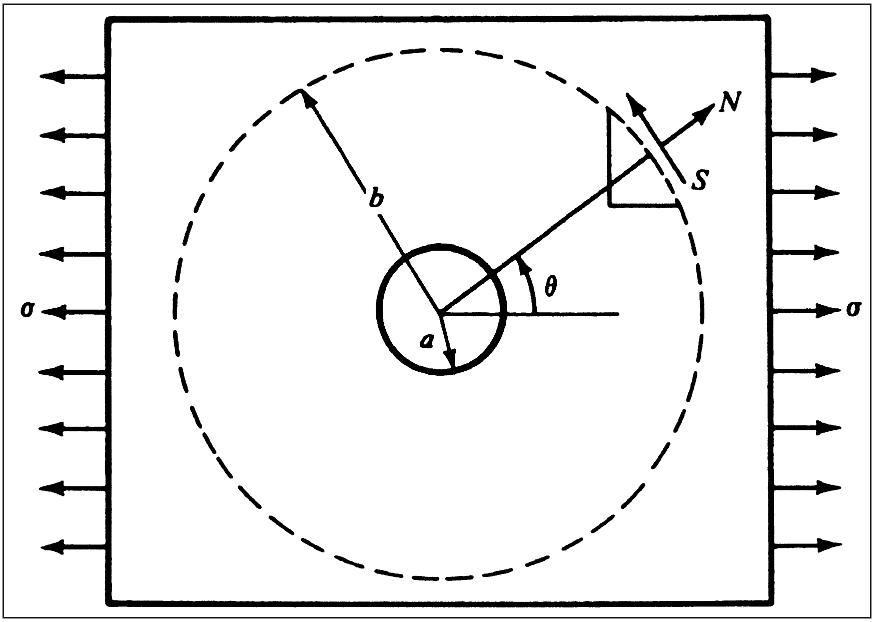


Figure 6-10.1

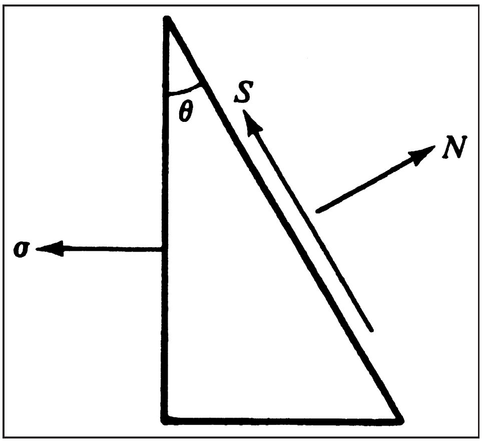


Figure 6-10.2

Substitution of Eq. (6-10.5) into the compatibility equation $\nabla^2\nabla^2 F = 0$ yields the equation for $f(r)$:

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{4f}{r^2}\right) = 0$$

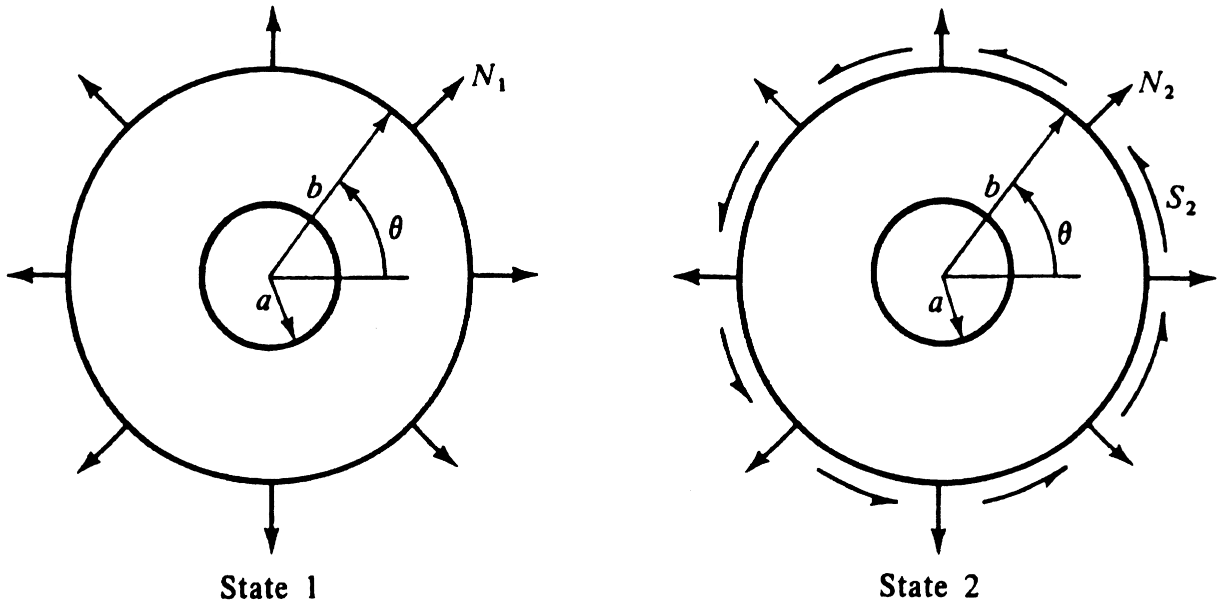


Figure 6-10.3

the solution of which is

$$f(r) = Ar^2 + Br^4 + C\frac{1}{r^2} + D \tag{6-10.6}$$

Hence,

$$F(r, \theta) = \left(Ar^2 + Br^4 + C\frac{1}{r^2} + D\right)\cos 2\theta \tag{6-10.7}$$

Equation (6-10.7) corresponds to the term in the first summation of Eq. (6-5.5), with $n = 2$, where A, B, C, D are constants to be determined by the boundary conditions for state 2 (Fig. 6-10.3).

The stress components for state 2 are, by Eqs. (6-10.7) and (6-2.2),

$$\begin{aligned}\sigma_r &= -\left(2A + \frac{6C}{r^4} + \frac{4D}{r^2}\right)\cos 2\theta \\ \sigma_\theta &= \left(2A + 12Br^2 + \frac{6C}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} &= \left(2A + 6Br^2 - \frac{6C}{r^4} - \frac{2D}{r^2}\right)\sin 2\theta\end{aligned} \tag{6-10.8}$$

Accordingly, with Eqs. (6-10.8) and the boundary conditions

$$\begin{aligned}&\sigma_r = \tau_{r\theta} = 0 \qquad \text{for} \qquad r = a \\ &\sigma_r = \tfrac{1}{2}\sigma\cos 2\theta \qquad \tau_{r\theta} = -\tfrac{1}{2}\sigma\sin 2\theta \qquad \text{for} \qquad r = b\end{aligned} \tag{6-10.9}$$

the values of A, B, C, D are

$$A = -\frac{\sigma}{4} \qquad B = 0 \qquad C = -\frac{a^4}{4}\sigma \qquad D = \frac{a^2}{2}\sigma \tag{6-10.10}$$

Then superposition of Eqs. (6-10.8) and Eqs. (b) of Example 6–6.2 [with Eqs. (d) under the conditions that $p_i = 0$, $p_0 = -(\sigma/2)$ and $b \gg a$] yields the stress state in the plane region with small circular hole (Fig. 6-10.1):

$$\begin{aligned}\sigma_r &= \frac{\sigma}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right)\cos 2\theta \\ \sigma_\theta &= \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2}\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \\ \tau_{r\theta} &= -\frac{\sigma}{2}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta\end{aligned} \tag{6-10.11}$$

We note that as $r \to b$ ($\gg a$), the stress state given by Eqs. (6-10.11) satisfies the conditions for $r = b$ [Eqs. (6-10.1)]. Also for $r = a$,

$$\sigma_r = \tau_{r\theta} = 0 \qquad \sigma_\theta = \sigma(1 - 2\cos 2\theta)$$

For $\theta = \pi/2, 3\pi/2$, σ_θ attains its maximum value of $(\sigma_\theta)_{\max} = 3\sigma$. [In general, $(\sigma_\theta)_{\max} = k\sigma$, where k is called the *stress concentration factor*.] For $\theta = 0, \pi$, σ_θ

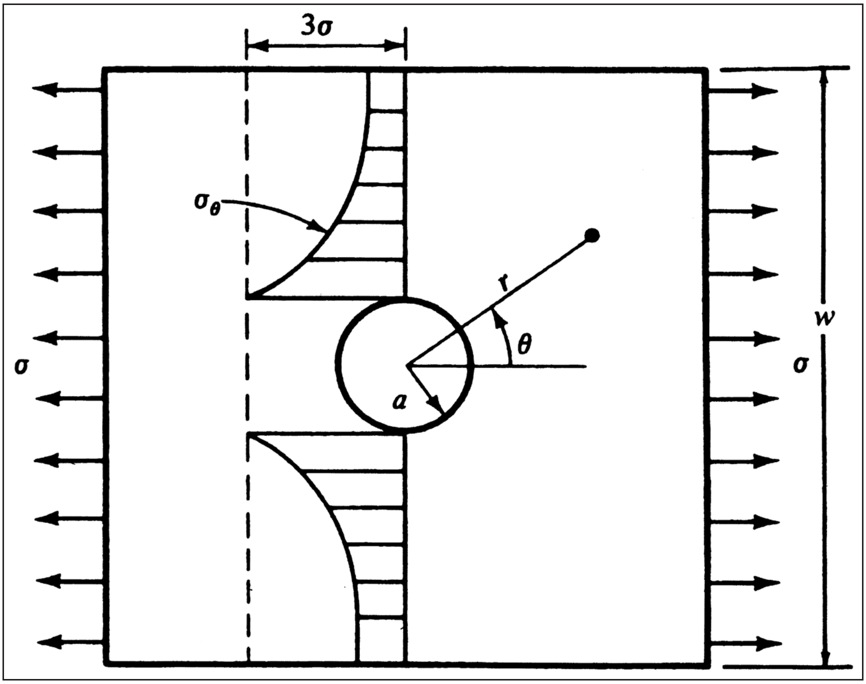


Figure 6-10.4

attains a compressive value of $-\sigma$. Thus, σ_θ attains a maximum tensile value of three times the uniformly distributed stress σ, at the hole $r = a$, for $\theta = \pi/2, 3\pi/2$ (Fig. 6-10.4).

Because for $\theta = \pi/2, 3\pi/2$, $\sigma_\theta = (\sigma/2)(2 + a^2/r^2 + 3a^4/r^4)$, $\sigma_\theta \to \sigma$ rapidly as r increases. Hence, the effect of the hole is of local character, the hole producing a *stress concentration* effect that increases the maximum stress several fold in the vicinity of the hole over the nominal stress value σ.

By superposition, we may also show $(\sigma_\theta)_{max} = 2\sigma$ everywhere at the boundary of the hole, when uniform tensile stress σ is applied along all straight edges of the plate. Furthermore, if a uniform compressive stress of magnitude σ is applied to two opposite edges (say, the horizontal edges in Fig. 6-10.1) and a uniform tensile stress σ is applied simultaneously to the other edges (Fig. 6-10.1), at the hole, then

$$\sigma_\theta = 4\sigma \quad \text{for} \quad \theta = \pi/2, 3\pi/2$$

$$\sigma_\theta = -4\sigma \quad \text{for} \quad \theta = 0, \pi$$

The large stress concentration effect that occurs at small holes in structural elements is of considerable importance to the designer. Much effort is expended to determine these effects and to design elements that minimize such effects (Savin, 1961).[2]

[2]Savin's book is devoted entirely to methods of calculating stress concentration factors around holes.

The displacement components in the region may be determined by the method noted in Section 6-6, that is, by direct integration of the strain–displacement relations [Eqs. (6-3.5)].

Plane Strain under General Loading. More generally, the stress concentration problem of a circular hole in a plate subject to boundary stresses $\sigma_x, \sigma_y, \tau_{xy}$ may be solved by the method of superposition. For example, consider (x, y) axes with origin at the center of the hole, with the x axis in the horizontal direction and the y axis in the vertical direction (Figs. 6-10.1 and 6-10.4). On distant boundary planes perpendicular to the x axis, stresses σ_x, τ_{xy} act, and on distant boundary planes perpendicular to the y axis, stresses σ_y, τ_{xy} act. The stress components $\sigma_x, \sigma_y, \tau_{xy}$ are assumed to act in the positive sense (see Fig. 3-2.2 in Chapter 3).

The stress components $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ (see Fig. 6.1.1) at a point (r, θ), as in Fig. 6-10.4, are

$$
\begin{aligned}
\sigma_r &= \left(\frac{\sigma_x + \sigma_y}{2}\right)\left(1 - \frac{a^2}{r^2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\left(1 + \frac{3a^4}{r^4} - 4\frac{a^2}{r^2}\right)\cos 2\theta \\
&\quad + \tau_{xy}\left(1 + \frac{3a^4}{r^4} - 4\frac{a^2}{r^2}\right)\sin 2\theta \\
\sigma_\theta &= \left(\frac{\sigma_x + \sigma_y}{2}\right)\left(1 + \frac{a^2}{r^2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\left(1 + \frac{3a^4}{r^4}\right)\cos 2\theta \\
&\quad - \tau_{xy}\left(1 + \frac{3a^4}{r^4}\right)\sin 2\theta \\
\tau_{r\theta} &= -\left(\frac{\sigma_x - \sigma_y}{2}\right)\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\sin 2\theta + \tau_{xy}\left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right)\cos 2\theta
\end{aligned}
\tag{6-10.12}
$$

For plane strain, $\epsilon_z = 0$. Hence,

$$\sigma_z = \nu(\sigma_r + \sigma_\theta) \tag{6-10.13}$$

The first two of Eqs. (6-10.12) and Eq. (6-10.13) yield

$$\sigma_z = \nu\left[\sigma_x + \sigma_y - 2(\sigma_x - \sigma_y)\frac{a^2}{r^2}\cos 2\theta - \frac{4a^2}{r^2}\tau_{xy}\sin 2\theta\right] \tag{6-10.14}$$

For $r = a$, Eqs. (6-10.12) and (6-10.14) yield

$$
\begin{aligned}
&\sigma_r = 0 \qquad \tau_{r\theta} = 0 \\
&\sigma_\theta = \sigma_x + \sigma_y - 2(\sigma_x - \sigma_y)\cos 2\theta - 4\tau_{xy}\sin 2\theta \\
&\sigma_z = \nu\sigma_\theta
\end{aligned}
\tag{6-10.15}
$$

The maximum, minimum values of σ_θ, hence σ_z, are given by the condition

$$\tan 2\theta = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2} \tag{6-10.16}$$

For example, for $\tau_{xy} = 0$, Eq. (6-10.16) yields $\tan 2\theta = 0$, or $\theta = 0$ (or π), $\pi/2$ (or $3\pi/2$). Thus, by Eq. (6-10.15) we obtain

$$\sigma_\theta = -\sigma_x + 3\sigma_y \quad \text{for} \quad \theta = 0, \pi \qquad r = a \tag{6-10.17}$$

and

$$\sigma_\theta = 3\sigma_x - \sigma_y \quad \text{for} \quad \theta = \pi/2, 3\pi/2 \qquad r = a \tag{6-10.18}$$

For the cases ($\sigma_x = \sigma, \sigma_y = 0$), ($\sigma_x = \sigma_y = \sigma$) and ($\sigma_x = -\sigma_y = \sigma$), Eqs. (6-10.17) and (6-10.18) yield the results obtained in the discussion following Eqs. (6-10.12). For $\sigma_x = \sigma_y = 0$, $\sigma_\theta = -4\tau_{xy} \sin 2\theta$. Hence,

$$\begin{aligned} (\sigma_\theta)_{\max} &= 4\tau_{xy} \quad \text{for} \quad \theta = 3\pi/4 \quad r = a \\ (\sigma_\theta)_{\min} &= -4\tau_{xy} \quad \text{for} \quad \theta = \pi/4 \quad r = a \end{aligned} \tag{6-10.19}$$

Applications of Eqs. (6-10.12) to rock mechanics problems have been given by Leeman and Hayes (1966) and to deep mine shaft problems by Chan and Beus (1985).

Large Holes. Equations (6-10.11) and (6-10.12) are applicable for the condition $a \ll b$; that is, for small circular holes relative to the loaded regions (Fig. 6-10.1). For a large hole (the radius of the hole being large compared to the smallest dimension of the region, which is the lateral width w in Fig. 6-10.4), these equations and the associated concentration factors are no longer valid. Chong and Pinter (1984) employed finite elements to investigate the effect of the ratio a/w (hole radius/width of strip) on the stress concentration factor k for the loading shown in Fig. 6-10.4. They found that in the range of a/w from 0.3 to 0.9, k varies from 3.44 to 19.50, respectively. As the ratio a/w approaches 0.99, k increases to 163. For small values of a/w (< 0.1), k becomes essentially constant and equal to approximately 3. An extensive literature survey, including experimental results, is also presented in Chong and Pinter (1984).

Problem Set 6-10

1. A very large plate has a small circular hole in it. At a long distance from the hole, $\sigma_x = 20$ kips/in.^{2}, $\sigma_y = 30$ kips/in.^{2}, $\tau_{xy} = 0$. Calculate the maximum tensile stress in the plate adjacent to the hole.

2. Consider the Airy stress function $F = f(r)\cos 2\theta$, where (r, θ) are polar coordinates and $f(r)$ is a function of r only.
   (a) Derive the differential equation that defines $f(r)$.
   (b) Show that $f(r) = C_1 r^2 + C_2 R^4 + C_3(1/r^2) + C_4$ is the solution of the differential equation of part (a).
   (c) Consider the polar coordinate region bounded by the θ coordinate lines $r = a$, $r = b$, $a < b$. Determine the equations that define C_1, C_2, C_3, C_4, supposing that $\sigma_r = \tau_{r\theta} = 0$ for $r = a$, and $\sigma_r = \sigma \cos 2\theta$, $\tau_{r\theta} = -\sigma \sin 2\theta$ for $r = b$, where σ is a known constant.

6-11 Examples

A large number of special cases of the general solution of Eq. (6-5.5) find important practical applications in practice. Rather than discuss these cases in detail, we merely note briefly some important specializations of Eq. (6-5.5) and the corresponding applications.

Example 6-11.1. Pure Bending of Curved Bars. The Airy stress function [see Eq. (6-6.4)]

$$F = A \log r + Br^2 \log r + Cr^2 + D \tag{E6-11.1}$$

may be used to study the problem of pure bending of curved bars (Fig. E6-11.1). The corresponding stress components are given by Eqs. (6-6.5). The constants A, B, and C are determined from the boundary conditions (the beam has unit thickness)

$$\begin{aligned} \sigma_r &= 0 \qquad & r &= a, b \\ \int_a^b \sigma_\theta dr &= 0 \qquad & \int_a^b \sigma_\theta r dr &= -M \\ \tau_{r\theta} &= 0 \qquad & &\text{on all boundaries} \end{aligned} \tag{E6-11.2}$$

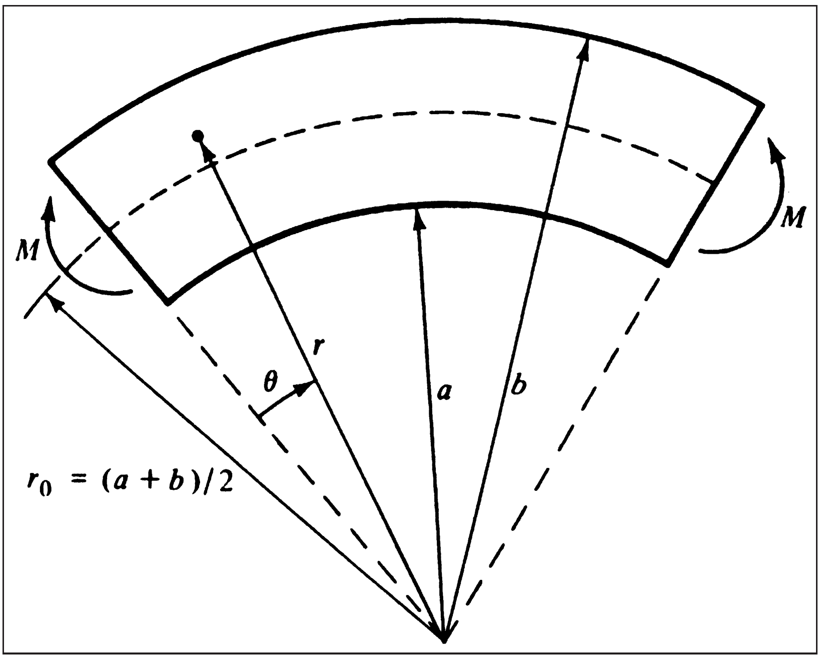


Figure E6-11.1

Hence, the stress components are then defined by Eqs. (6-6.5) with

$$\begin{aligned}
A &= -\frac{4M}{N}a^2b^2\log\frac{b}{a} \\
B &= -\frac{2M}{N}(b^2-a^2) \\
C &= \frac{M}{N}[b^2-a^2+2(b^2\log b - a^2\log a)] \\
N &= (b^2-a^2)^2 - 4a^2b^2\left(\log\frac{b}{a}\right)^2
\end{aligned} \tag{E6-11.3}$$

The strain components may be obtained for either plane strain or plane stress conditions. The displacement components may be obtained then by direct integration of the strain–displacement relations [Eqs. (6-3.5)].

Example 6-11.2. Circular Cantilever Beam. The Airy stress function

$$F(r,\theta) = f(r)\sin\theta \tag{E6-11.4}$$

may be used to study the problem of the circular cantilever beam subject to end shear (Fig. E6-11.2). With the compatibility condition $\nabla^2\nabla^2 F = 0$ and

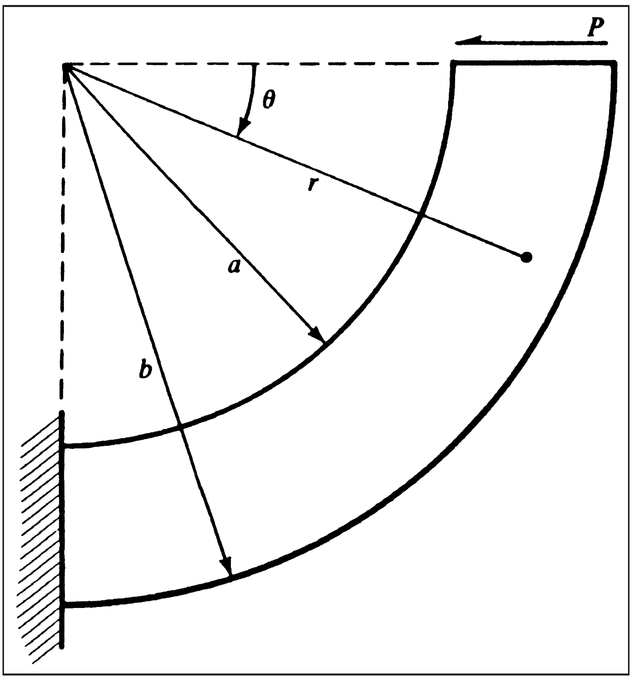


Figure E6-11.2

Eq. (E6-11.4), we find

$$f(r) = Ar^3 + \frac{B}{r} + Cr + Dr \log r \tag{E6-11.5}$$

Hence, by Eqs. (6-2.2), (E6-11.4), and (E6-11.5), the stress components are

$$\begin{aligned} \sigma_r &= \left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta \\ \sigma_\theta &= \left(6Ar + \frac{2B}{r^3} + \frac{D}{r}\right)\sin\theta \\ \tau_{r\theta} &= -\left(2Ar - \frac{2B}{r^3} + \frac{D}{r}\right)\cos\theta \end{aligned} \tag{E6-11.6}$$

With the boundary conditions

$$\sigma_r = \tau_{r\theta} = 0 \quad \text{at} \quad r = a, b$$

and the end shear condition

$$\int_a^b \tau_{r\theta}\, dr = P$$

Eqs. (E6-11.6) yield

$$\begin{aligned} &A = \frac{P}{2N} \qquad B = -\frac{Pa^2b^2}{2N} \qquad D = -\frac{P}{N}(a^2 + b^2) \\ &N = a^2 - b^2 + (a^2 + b^2)\log\frac{b}{a} \end{aligned} \tag{E6-11.7}$$

Again the strain components and the displacement components may be obtained by the equations of Section 6-4, and by integration of the strain–displacement relations [Eqs. (6-3.5)].

It may also be shown that the Airy stress function

$$F = f(r)\cos\theta \tag{E6-11.8}$$

yields a solution to the circular cantilever beam subjected to end tension T and end moment M (Fig. E6-11.3). Then, by appropriate superposition of the results obtained with Eqs. (E6-11.1), (E6-11.4), and (E6-11.8), solutions of the problems illustrated in Figs. E6-11.4 and E6-11.5 may be obtained.

Example 6-11.3. Normal Point Load on Edge of Half-Plane. The problem of the point load P on the half-plane boundary (Fig. E6-11.6) may be analyzed by means of the stress function (under the condition that stresses vanish as $r \to \infty$):

$$F(r, \theta) = -\frac{P}{\pi} r\theta \sin\theta \tag{E6-11.9}$$

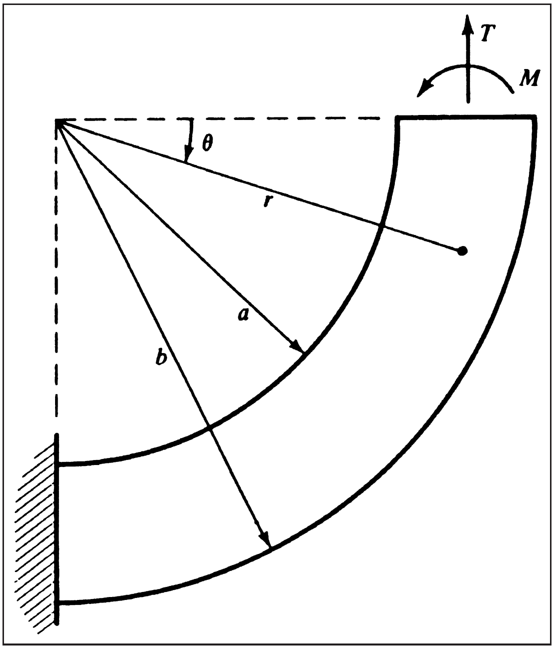


Figure E6-11.3

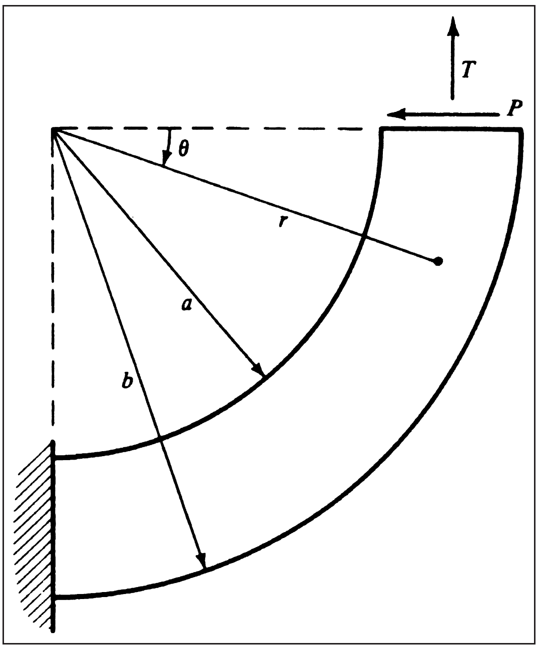


Figure E6-11.4

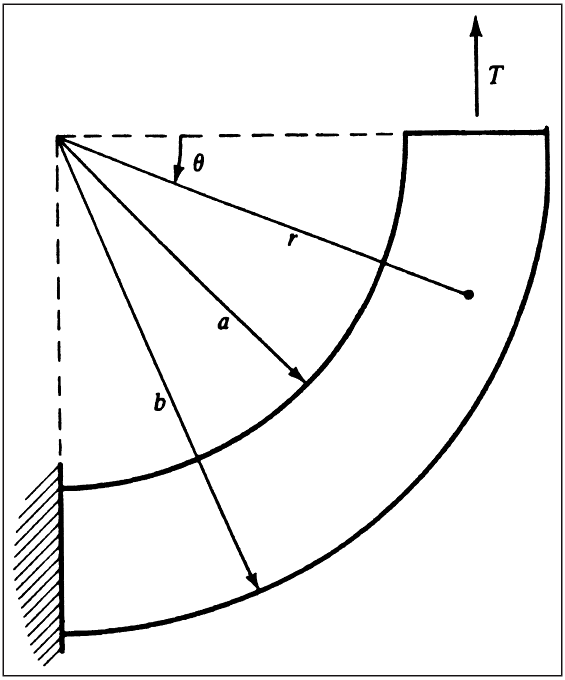


Figure E6-11.5

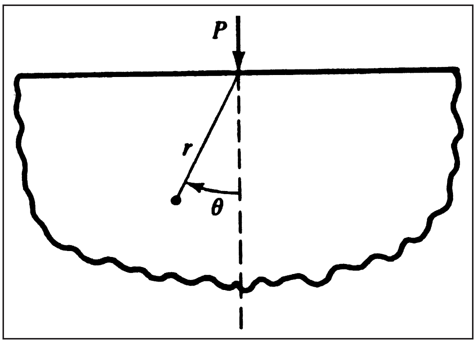


Figure E6-11.6

The derivation of the stress components is left as an exercise. Note that the point $r = 0$ is a singular point (yields an infinite stress). This result may be used to obtain the stress distribution in the half-plane under the action of several point forces (see Problem Set 6-11).

Example 6-11.4. Plane Wedge under Load at Tip. The wedge problem under point load at the tip may be studied by the stress function (Fig. E6-11.7)

$$F = Ar\theta \sin\theta \tag{E6-11.10}$$

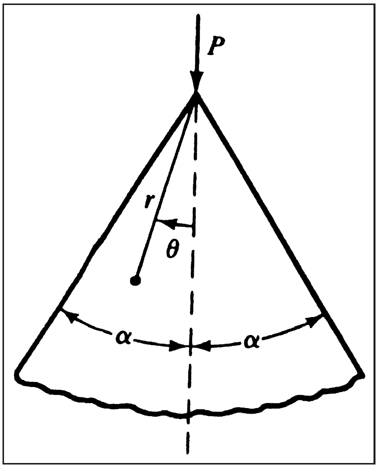


Figure E6-11.7

where A is a constant determined by the boundary condition of equilibrium for a tip element. The stresses vanish as $r \to \infty$ (compare Example 6-11.3). Certain paradoxes of the wedge problem have been treated in the literature (Sternberg and Koiter, 1958; Ting, 1984a, 1984b).

Problem Set 6-11

1. Derive the strain components and the displacement components of Example 6-11.1. Assume a state of plane stress in the (r, θ) plane.

2. Repeat Problem 1 for Example 6-11.2. Assume appropriate constraints at the wall (support).

3. Repeat Problem 1 for Example 6-11.3. Discuss the behavior at $r = 0$.

4. Repeat Problem 1 for Example 6-11.4. Discuss the cases

$$\alpha = \pi/2 \qquad \pi/2 < \alpha < \pi$$

5. A circular cantilever beam is loaded in pure bending (Fig. P6-11.5). Determine the displacement of the end (point A). For $r = (a + b)/2$, $\theta = 0$, let the radial and tangential displacement components vanish, $u = v = 0$, and $\partial u/\partial \theta = 0$.

6. A thick rectangular plate is rolled into a cylindrical shape (Fig. P6-11.6). Residual stresses resulting from the rolling process are removed by annealing. After annealing, the end planes 1 and 2 are a small angle α apart. The end planes are then brought together by applying a moment M to each plane, and the faces are welded together. Then uniform internal pressure p_i and external pressure p_0 are applied to the lateral surfaces of the cylinder. Derive expressions for the radial and tangential stress components σ_r and σ_θ.

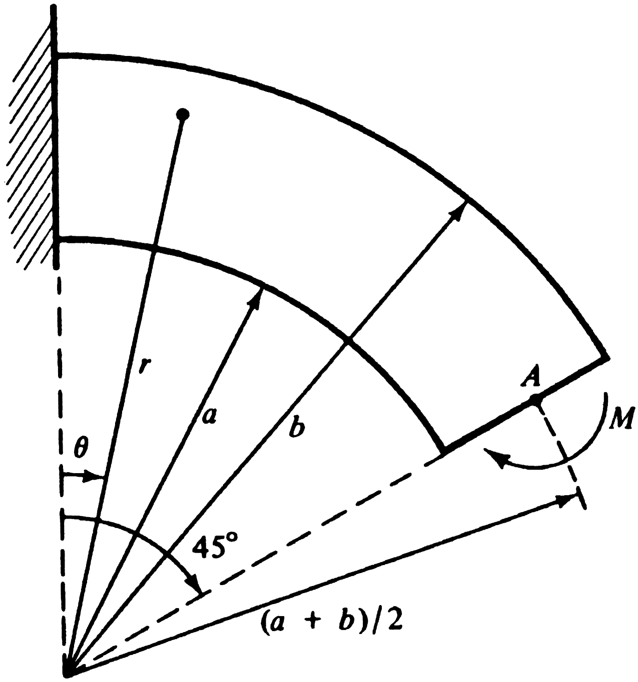


Figure P6-11.5

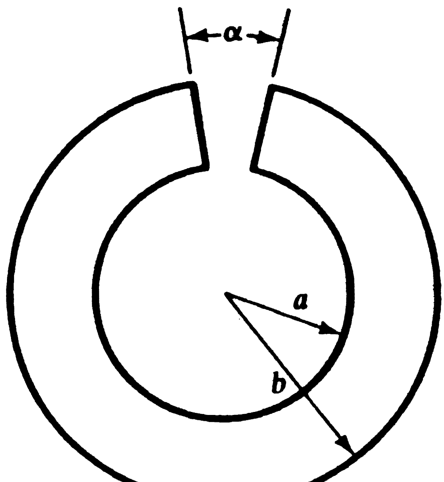


Figure P6-11.6

7. The stress function $F = Ar\theta \sin\theta$ yields the solution to the problem of a semi-infinite plate loaded by a concentrated force perpendicular to its straight-line boundary, where $0 < r$, $-\pi/2 \le \theta \le \pi/2$. Derive a formula for the maximum shearing stress at a point in the plate some distance from the load. Derive an equation for curves along which the maximum shearing stress is a constant, and trace several of these curves on a sketch of the plate. Derive expressions for radial and tangential components of displacement.

8. A semi-infinite plate is loaded normally to its free boundary by a concentrated force P (Fig. P6-11.8). Assume that $\sigma_\theta = \tau_{r\theta} = 0$. Hence, show that $r\sigma_r = f(\theta)$, where $f(\theta)$ is a function of θ alone. Derive the formula for $f(\theta)$. Hence, express σ_r as a known function of r, θ, and the load P. Derive expressions for radial and tangential components of displacement.

9. The stress function for a single concentrated force P acting perpendicular to the straight boundary of a semi-infinite plate is

$$F = -\frac{P}{\pi} r\theta \sin\theta$$

By the method of superposition, derive expressions for the principal stresses and the maximum shear at point A for the semi-infinite plate loaded as shown in Fig. P6-11.9, for the cases $Q = P$ and $Q = 2P$.

10. Two forces P are applied a distance $2b$ apart perpendicularly to the edge of a semi-infinite plate (Fig. P6-11.10).

(a) Determine the principal stresses at a point D at a depth d below the surface in the line of symmetry.

(b) The two forces P are replaced by a single force $2P$ applied in the line of symmetry. Determine the depth c below which the minimum principal stress at D is changed by less than 4%.

11. Consider a plane disk subjected to diametrically directed forces P, as shown in Fig. P6-11.11a. (See Appendix 6B for a more advanced discussion of this problem.)

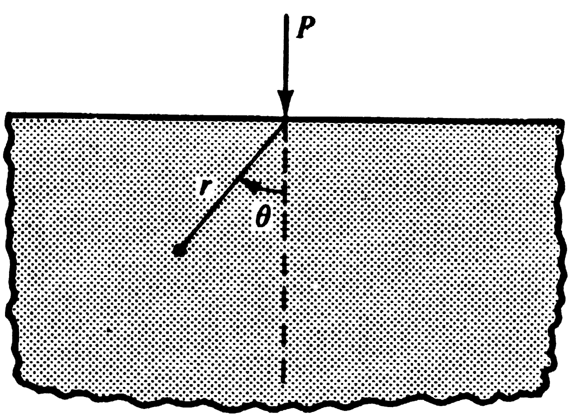


Figure P6-11.8

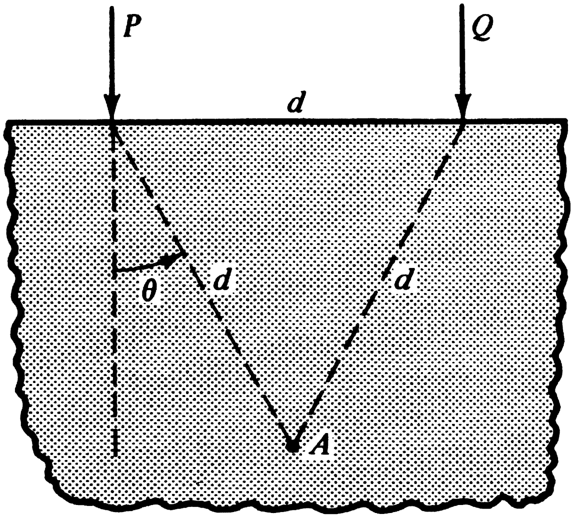


Figure P6-11.9

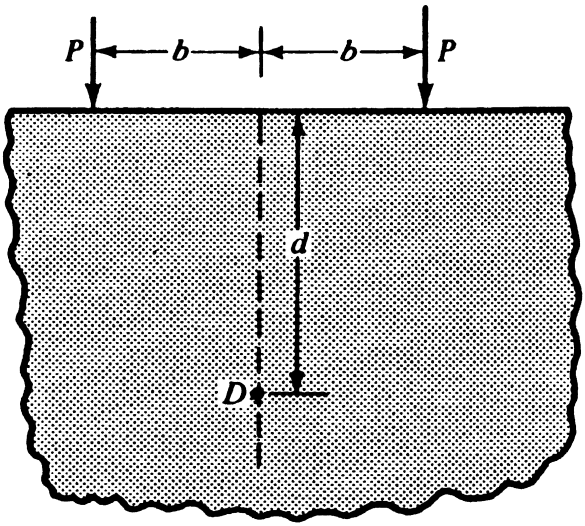


Figure P6-11.10

(a) By considering the solution of the half-plane subjected to point load P acting normal to the straight-line boundary, show by superposition of two appropriate half-plane problems that we may obtain a solution to the disk problem for boundary stresses as shown in Fig. P6-11.11b.

(b) Then, select a state of stress that when superposed upon that of Fig. P6-11.11b is a solution to the problem of Fig. P6-11.11a.

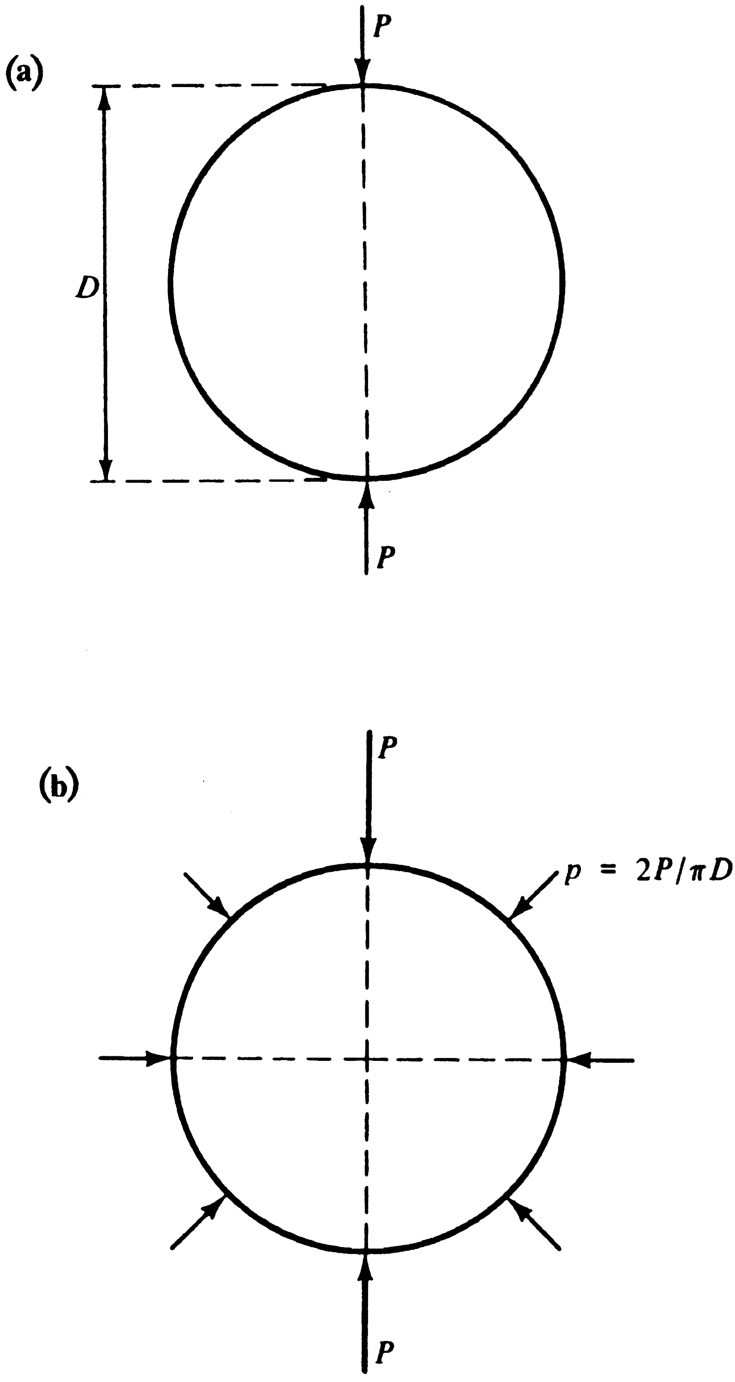


Figure P6-11.11

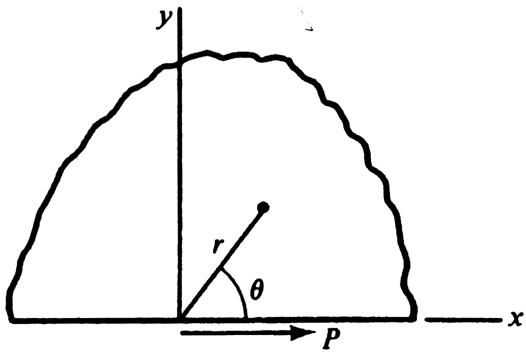


Figure P6-11.12

12. A tangential concentrated force P is applied to the upper half-plane ($y \geq 0$) at the origin (Fig. P6-11.12). Formulate the problem in terms of the Airy function. Determine the stress components. (*Hint*: See Problems 7 and 8.)

13. The semi-infinite plate is loaded uniformly along the straight-line boundary $\theta = \pi$ (Fig. P6-11.13). Show that the stress components may be derived from the stress function $F = Cr^2(\theta - \sin\theta\cos\theta)$. Evaluate the stress component for $\theta = \pi/2$; for $\theta = 0$. Discuss any discrepancies in these components. Derive expressions for radial and tangential components of displacement.

14. For a state of plane stress expressed in polar coordinates, assume that all stress components except σ_r are zero.

(a) In the absence of body forces and acceleration, show that $r\sigma_r = f(\theta)$, where $f(\theta)$ is an arbitrary function of θ.

(b) Derive a general formula for $f(\theta)$.

(c) Apply the results of parts (a) and (b) to the problem of a cantilever wedge loaded in its plane by a concentrated force P applied at its tip (Fig. P6-11.14); that is, express σ_r as a completely determined function of r and θ. Discuss the boundary conditions at the support.

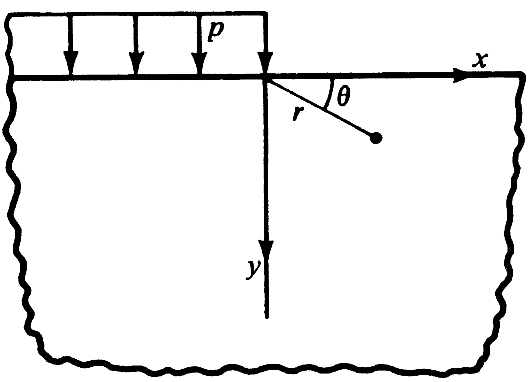


Figure P6-11.13

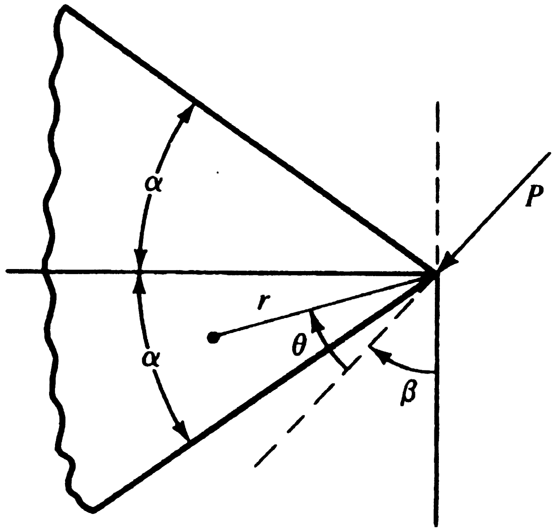


Figure P6-11.14

15. A thin plate in the shape of a wedge is subjected to uniform pressure p acting along its side $\theta = -\alpha$ and a uniform pressure q acting along its side $\theta = \alpha$ $(0 < \alpha < \pi/2)$. Because $\sigma_\theta = -p, -q$ for $\theta = -\alpha, +\alpha$, $\partial^2 F/\partial r^2$ must be independent of the radial coordinate r, where (r, θ) denote polar coordinates and F is the Airy stress function. The tip of the wedge is located at $r = \theta = 0$. Hence, the Airy stress function F may at most be proportional to r^2. Accordingly, by the general solution of $\nabla^2\nabla^2 F = 0$ [see Eq. (6-5.5)], we take

$$F = (A + B\theta + C\cos 2\theta + D\sin 2\theta)r^2$$

(a) Derive the conditions that define the constants A, B, C, D.

(b) For the case $p = 0$, $\alpha = \pi/2$, show that the solution yields the case of a semi-infinite plate subjected to uniform pressure on one-half of its boundary.

16. (a) In the absence of body forces and temperature field, show that $F = Cr^2(2\theta - \sin 2\theta)$, $C > 0$ a constant, is an Airy stress function.

(b) For a plane wedge (Fig. P6-11.16), employ F of part (a) to determine possible boundary conditions for the surfaces $\theta = \pm\alpha$, where 2α is the wedge angle.

(c) In terms of the constant C and polar coordinates (r, θ), determine the explicit formulas for the stress components σ_x, τ_{xy} on the vertical section ab.

17. A plane wedge (tapered beam with thickness of 1 unit) is loaded at its tip by a force P (Fig. P6-11.17). In terms of polar coordinates (r, θ) the stress components are $\sigma_r = -(kP\cos\theta)/r$, $\sigma_\theta = \tau_{r\theta} = 0$, where $k = 2/(2\alpha - \sin 2\alpha)$ and α is the half-angle of the wedge. In terms of k, P, x, y, derive expressions for the stress components $\sigma_x, \sigma_y, \tau_{xy}$ relative to the rectangular Cartesian axes (x, y). Evaluate the maximum shearing stress at the point $x = 1$, $y = -1$. (*Hint*: Consider the equilibrium of appropriate elements or parts of the wedge.)

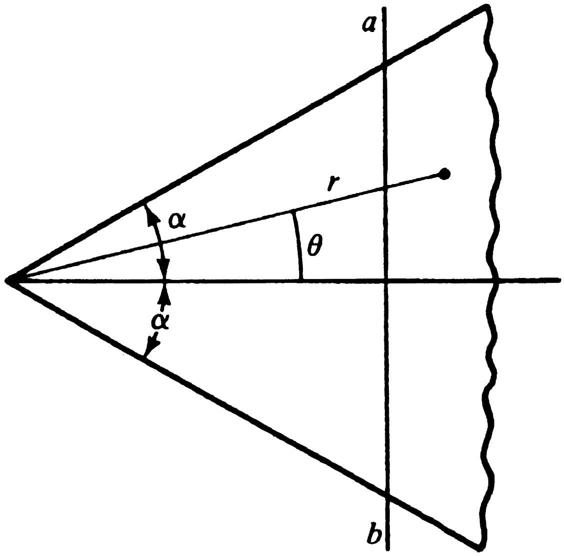


Figure P6-11.16

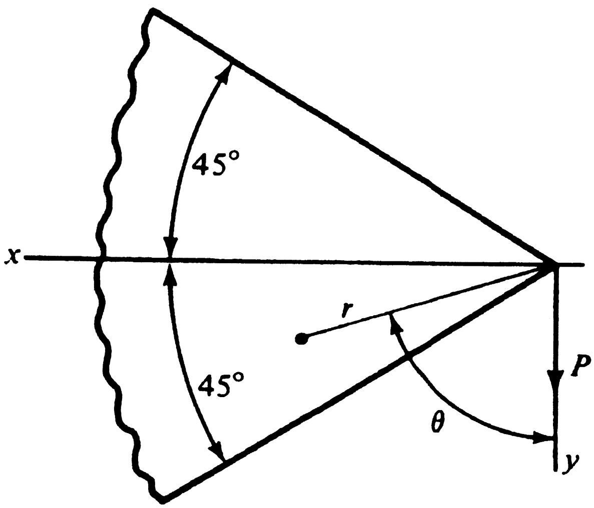


Figure P6-11.17

18. Determine the value of the constant C in the stress function

$$F = C[r^2(\alpha - \theta) + r^2 \sin\theta \cos\theta - r^2 \cos^2\theta \tan\alpha]$$

required to satisfy the conditions on the upper and lower edges of the triangular plate shown in Fig. P6-11.18. Evaluate σ_x and τ_{xy} for a vertical section mn.

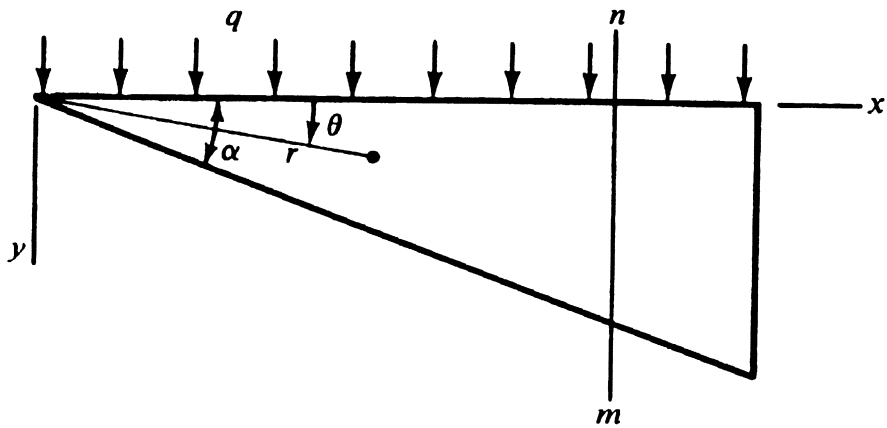


Figure P6-11.18

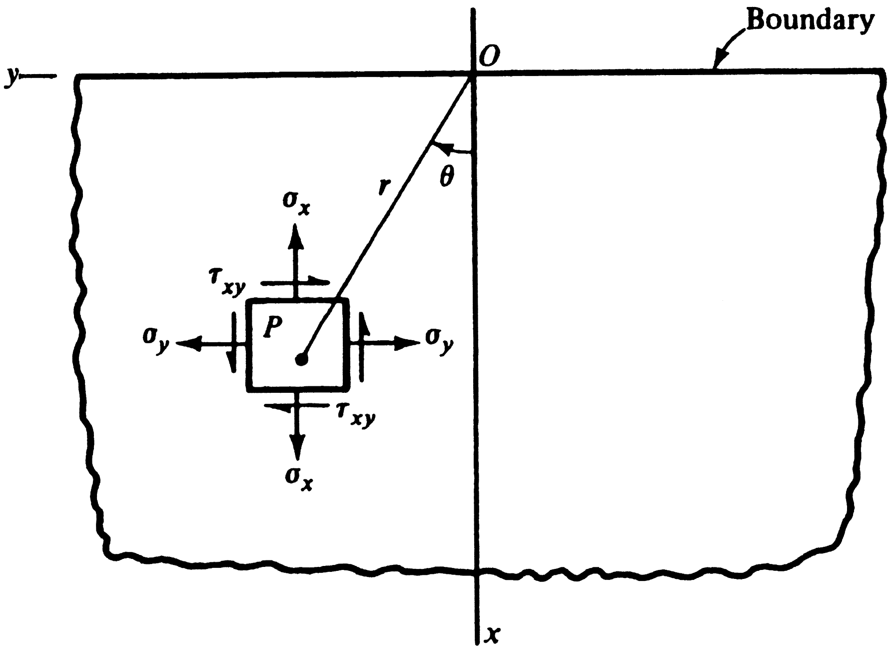


Figure P6-11.19

19. A stress function used in solving the problem of vertical loading of a straight boundary of a semi-infinite plane region is $F = Ar^2\theta$. Consider a point P: (r, θ) (see Fig. P6-11.19). Transform F into a function of the rectangular coordinates (x, y). Hence, derive expressions for the stress components $\sigma_x, \sigma_y, \tau_{xy}$ that act at point P. Examine the boundary conditions for $\theta = \pm\pi/2$.

APPENDIX 6A STRESS–COUPLE THEORY OF STRESS CONCENTRATION RESULTING FROM CIRCULAR HOLE IN PLATE

The theory of plane elasticity with couple stresses is treated in Appendix 5A. In the present appendix we give the solution to the plane elasticity theory with couple stresses for the circular hole in a plane region under uniform tension σ (Fig. 6-10.1). The governing equations for the function H and ψ are Eqs. (5A-5.8) and (5A-5.10). The solutions for H and ψ in polar coordinates are (see Mindlin, 1963; Weitsman, 1965; Kaloni and Ariman, 1967, Chapter 5 References)

$$\begin{aligned} H &= \frac{\sigma}{4} r^2 (1 - \cos 2\theta) + A \log r + \left(\frac{B}{r^2} + C \right) \cos 2\theta \\ \psi &= \left[\frac{D}{r^2} + E K_2 \left(\frac{r}{l} \right) \right] \sin 2\theta \end{aligned} \tag{6A-1}$$

where A, B, C, D, E are constants and $K_2(r/l)$ is the modified Bessel function of the second kind and second order. Equations (6A-1) may be shown to satisfy Eqs. (5A-5.8), (5A-5.9), and (5A-5.10).

In terms of polar coordinates (r, θ), the stress components and couples are $\sigma_r, \sigma_\theta, \tau_{r\theta}, \tau_{\theta r}, m_{rz}, m_{\theta z}$ (Fig. 6A-1). By equilibrium of triangular elements 1 and 2 (Figs. 6A-1 and 6A-2), we obtain

$$\begin{aligned} \sigma_r &= \sigma_x \cos^2 \theta + \sigma_y \sin^2 \theta + (\tau_{xy} + \tau_{yx}) \sin \theta \cos \theta \\ \sigma_\theta &= \sigma_x \sin^2 \theta + \sigma_y \cos^2 \theta - (\tau_{xy} + \tau_{yx}) \sin \theta \cos \theta \\ \tau_{r\theta} &= (\sigma_y - \sigma_x) \sin \theta \cos \theta + \tau_{xy} \cos^2 \theta - \tau_{yx} \sin^2 \theta \\ \tau_{\theta r} &= (\sigma_y - \sigma_x) \sin \theta \cos \theta - \tau_{xy} \sin^2 \theta + \tau_{yx} \cos^2 \theta \\ m_{rz} &= m_{xz} \cos \theta + m_{yz} \sin \theta \\ m_{\theta z} &= -m_{xz} \sin \theta + m_{yz} \cos \theta \end{aligned} \tag{6A-2}$$

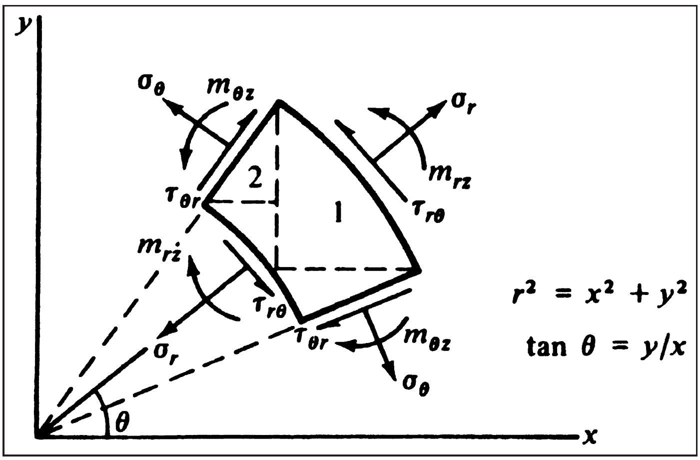


Figure 6A-1

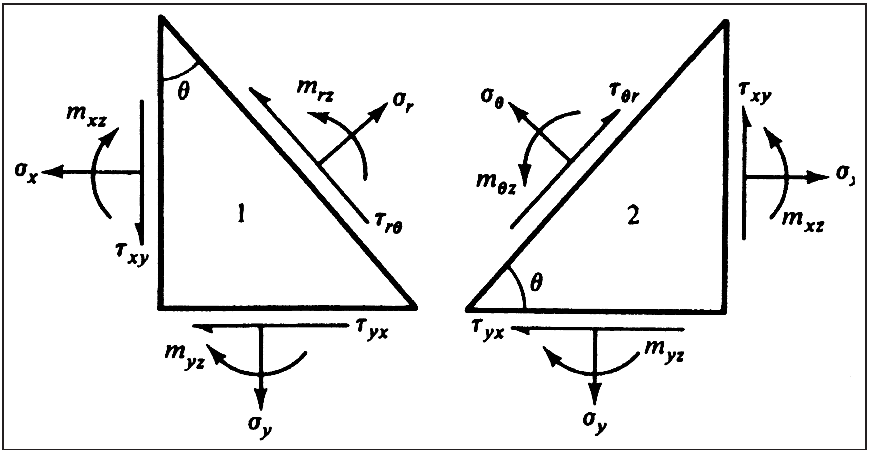


Figure 6A-2

Accordingly, by Eqs. (5A-5.7) and (6A-2) and the relations

$$\begin{aligned}\frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\end{aligned} \tag{6A-3}$$

we find in terms of H and ψ

$$\begin{aligned}\sigma_r &= \frac{1}{r}\frac{\partial H}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H}{\partial\theta^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\\ \sigma_\theta &= \frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\\ \tau_{r\theta} &= -\frac{1}{r}\frac{\partial^2 H}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial H}{\partial\theta} - \frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}\\ \tau_{\theta r} &= -\frac{1}{r}\frac{\partial^2 H}{\partial r\partial\theta} + \frac{1}{r^2}\frac{\partial H}{\partial\theta} + \frac{\partial^2\psi}{\partial r^2}\\ m_{rz} &= \frac{\partial\psi}{\partial r}\\ m_{\theta z} &= \frac{1}{r}\frac{\partial\psi}{\partial\theta}\end{aligned} \tag{6A-4}$$

Substitution of Eqs. (6A-1) into Eq. (6A-4) yields, with the observation that

$$\begin{aligned}K_2\left(\frac{r}{l}\right) &= \frac{2l}{r}K_1\left(\frac{r}{l}\right) + K_0\left(\frac{r}{l}\right)\\ \frac{\partial K_0(r/l)}{\partial r} &= -\frac{1}{l}K_1\left(\frac{r}{l}\right)\\ \frac{\partial K_1(r/l)}{\partial r} &= -\frac{1}{r}K_1\left(\frac{r}{l}\right) - \frac{1}{l}K_0\left(\frac{r}{l}\right)\end{aligned} \tag{6A-5}$$

where K_0 and K_1 are the modified Bessel functions of the second kind of orders zero and one (Irving and Mullineaux, 1966),

$$
\begin{aligned}
\sigma_r &= \frac{\sigma}{2}(1+\cos 2\theta) + \frac{A}{r^2} - \left(\frac{6B}{r^4} + \frac{4C}{r^2} - \frac{6D}{r^4}\right)\cos 2\theta \\
&\quad + \frac{2E}{Ir}\left[\frac{3l}{r}K_0\left(\frac{r}{l}\right) + \left(1 + \frac{6l^2}{r^2}\right)K_1\left(\frac{r}{l}\right)\right]\cos 2\theta \\
\sigma_\theta &= \frac{\sigma}{2}(1-\cos 2\theta) - \frac{A}{r^2} + \left(\frac{6B}{r^4} - \frac{6D}{r^4}\right)\cos 2\theta \\
&\quad - \frac{2E}{lr}\left[\frac{3l}{r}K_0\left(\frac{r}{l}\right) + \left(1 + \frac{6l^2}{r^2}\right)K_1\left(\frac{r}{l}\right)\right]\cos 2\theta \\
\tau_{r\theta} &= -\left(\frac{\sigma}{2} + \frac{6B}{r^4} + \frac{2C}{r^2} - \frac{6D}{r^4}\right)\sin 2\theta \\
&\quad + \frac{E}{lr}\left[\frac{6l}{r}K_0\left(\frac{r}{l}\right) + \left(1 + \frac{12l^2}{r^2}\right)K_1\left(\frac{r}{l}\right)\right]\sin 2\theta \\
\tau_{\theta r} &= -\left(\frac{\sigma}{2} + \frac{6B}{r^4} + \frac{2C}{r^2} - \frac{6D}{r^4}\right)\sin 2\theta \\
&\quad + \frac{E}{l^2}\left[\left(1 + \frac{6l^2}{r^2}\right)K_0\left(\frac{r}{l}\right) + \left(\frac{3l}{r} + \frac{12l^3}{r^3}\right)K_1\left(\frac{r}{l}\right)\right]\sin 2\theta \\
m_{rz} &= -\frac{2D}{r^3}\sin 2\theta - \frac{E}{l}\left[\frac{2l}{r}K_0\left(\frac{r}{l}\right) + \left(1 + \frac{4l^2}{r^2}\right)K_1\left(\frac{r}{l}\right)\right]\sin 2\theta \\
m_{\theta z} &= \left\{\frac{2D}{r^3} + \frac{2E}{r}\left[K_0\left(\frac{r}{l}\right) + \frac{2l}{r}K_1\left(\frac{r}{l}\right)\right]\right\}\cos 2\theta
\end{aligned}
\tag{6A-6}
$$

The constants A, B, C, D, E are determined by the boundary conditions

$$\sigma_r = \tau_{r\theta} = m_{rz} = 0 \qquad r = a \tag{6A-7}$$

and the condition

$$D = 8(1-\nu)l^2C$$

where ν is Poisson's ratio, which is required for the satisfaction of Eq. (5A-5.9) in polar coordinates. Thus, we find

$$
\begin{aligned}
A &= -\frac{\sigma a^2}{2} \qquad B = -\frac{\sigma a^4(1-F)}{4(1+F)} \\
C &= \frac{\sigma a^2}{2(1+F)} \qquad D = \frac{4(1-\nu)a^2l^2\sigma}{1+F} \\
E &= -\frac{palF}{(1+F)K_1(a/l)} \\
F &= \frac{8(1-\nu)}{4 + \dfrac{a^2}{l^2} + \dfrac{2a}{l}\dfrac{K_0(a/l)}{K_1(a/l)}}
\end{aligned}
\tag{6A-8}
$$

The terms containing σ correspond to the stress distribution due to simple tension (Section 6-10). The terms in r diminish as r increases. Hence, the stress state at points far from the hole is due to simple tension σ.

If the couple stresses are ignored, $\ell = 0$. Then noting that

$$\lim_{a/l \to \infty} \frac{K_0(a/l)}{K_1(a/l)} = 1$$

we see that the stress components of Eq. (6A-6) reduce to those of Eq. (6-10.11).

With $l \neq 0$, the stress σ_θ at the hole ($r = a$) for $\theta = \pi/2, 3\pi/2$, is (Fig. 6-10.1)

$$\sigma_\theta = \sigma \frac{3 + F}{1 + F} \tag{6A-9}$$

Accordingly, if stress couples are maintained in the theory, the stress concentration factor depends both on Poisson's ratio ν and the ratio of the radius a of the hole and the material constant l [Eq. (5A-4.6)]. If couple stresses are discarded, $F = 0$, and the stress concentration factor is 3 [Eq. (6A-9)] as usual (Section 6-10). As a/l decreases, so does the stress concentration factor. With $a/l = 3$ and $\nu = 0(0.5)$, the stress concentration factor is 2.4(2.6).

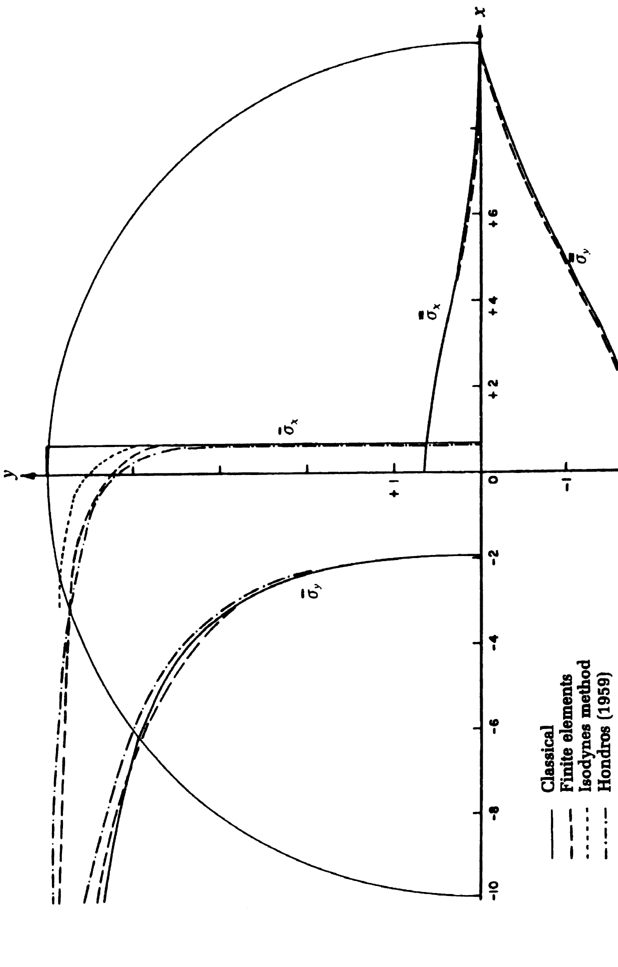
Although the above theory implies that the ratio a/l influences the stress concentration factor, experiments indicate that in order to do so the material constant l must be of order of the grain size (Ellis and Smith, 1967).

Indeed, on the basis of these experiments, it may be concluded that the reduction (from 3) in stress concentration factors that is experimentally observed for small-radius notches and holes cannot be accounted for by the above simple couple stress theory. The requirement that l must be about the order of magnitude of the grain size or smaller implies that theoretical foundations of the simple, isotropic, homogeneous continuum must be extended to examine the problem in finer detail (Ellis and Smith, 1967).

APPENDIX 6B STRESS DISTRIBUTION OF A DIAMETRICALLY COMPRESSED PLANE DISK

An experimental test of a plane disk subjected to diametrically directed forces P (Fig. P6-11.11a) is known as the split cylinder test (so called because the disk or cylinder tends to split along the line of action of forces P) or the Brazilian test. The split cylinder test is an extremely useful method for determining the tensile strengths of brittle materials that have much higher compressive strengths than tensile strengths (Chong and Kuruppu, 1984). Typically, tensile failure will occur along the loaded diameter, splitting the cylinder (or disk) into two halves

The classical theory (Timoshenko and Goodier, 1970) assumes that the line load is applied over an infinitesimally small width. If we assume a simple radial stress distributions for each force P and superimpose boundary stresses (similar to Problem 6-11.11), we find that the horizontal tensile stress (Fig. 6B-1) along



y
x
+1
0
-1
-2
-10
-8
-6
-4
-2
+2
+4
+6
Classical
Finite elements
Isodynes method
Hondros (1959)

Figure 6B-1

the diameter is constant (Timoshenko and Goodier, 1970, and Problem 6-11.11) and equal to (for unit thickness)

$$\sigma_x = \frac{2P}{\pi D} \tag{6B-1}$$

This stress distribution violates equilibrium (Fairhurst, 1964) because if half of the disk (say, left of the loaded diameter) is taken as a free body, $\Sigma F_x \neq 0$. To overcome these difficulties, Hondros (1959) developed a modified theory assuming negligible body forces and a finite width of loading applied radially. Numerical results based on a series solution agree closely with his experimental results monitored by strain gages. Stresses along the loaded (vertical) diameter are given by

$$\sigma_x = \frac{2P}{\pi ab}\left[\frac{(1 - r^2/R^2)\sin 2\alpha}{(1 - 2r^2/R^2\cos 2\alpha + r^4/R^4)} - \tan^{-1}\left(\frac{1 + r^2/R^2}{1 - r^2/R^2}\tan\alpha\right)\right] \tag{6B-2}$$

$$\sigma_y = \frac{2P}{\pi ab}\left[\frac{(1 - r^2/R^2)\sin 2\alpha}{(1 - 2r^2/R^2\cos 2\alpha + r^4/R^4)} + \tan^{-1}\left(\frac{1 + r^2/R^2}{1 - r^2/R^2}\tan\alpha\right)\right]$$

$$\tau_{xy} = 0 \tag{6B-3}$$

where r = radial distance from the origin; R = radius of the disk; a = width of the applied load; and $2\alpha = a/R$. At any other point on the disk, the stresses are given in a series form. For long cylinders (plane strain case) and thin disks (plane strain case), the stress expressions given remain unchanged. However, the stress–strain relationships are different.

The finite element method can be used to model the split cylinder test (Chong et al., 1982). As a result of symmetry, only one-quarter of the disk needs to be considered. In the Chong et al. (1982) study, a total of 250 two-dimensional elements with 146 nodes were used. Each node had two degrees of freedom. The nodal stresses were computed using consistent stress distributions. The load of $P/2$ was assumed to act at the apex node.

The stress distributions from the above theories, experiments, and the finite element method along the vertical diameter $(\overline{\sigma}_x, \overline{\sigma}_y)$ and the horizontal diameter $(\overline{\overline{\sigma}}_x, \overline{\overline{\sigma}}_y)$ are presented in Fig. 6B-1. These stresses have been normalized (divided) by the quantity $\sigma_0 = P/(bd)$ for comparison with other references. Four different methods are compared in the figure: (a) classical theory of Timoshenko and Goodier (1970), (b) finite element analysis (Chong et al., 1982), (c) isodynes method (Pindera et al., 1978), and (d) Hondros' theory (1959) with bearing width a equal to one-sixth of the disk radius. Methods (a) and (b) are plotted for all four curves. For simplicity, methods (c) and (d) are shown only if they deviate from the classical theory.

It can be seen that the classical theory agrees well with all methods except for the tensile stress across the loaded diameter $\overline{\sigma}_x$. For $\overline{\sigma}_x$, methods (b), (c), and (d) show good agreement, indicating a very high compressive stress close to the load. This represents the reversal of stresses necessary for equilibrium and balance of internal horizontal forces. Both methods (b) and (d) indicate zero stress at 0.85 of the disk radius measuring from the center, whereas method (c) measures zero stress at 0.90 of the disk radius.

Physically, the region under the load experiences very high uniform compressive pressures in $\overline{\sigma}_x$ and $\overline{\sigma}_y$ [as indicated by Eqs. (6B-2) and (6B-3); this also can be seen from the finite element analysis]. Apparently this region wedges its way into the disk, causing an ultimate tensile failure in the brittle materials. This wedging action can be seen in the displacement contours based on finite element analysis.

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CHAPTER 7

PRISMATIC BAR SUBJECTED TO END LOAD

In this chapter we consider the formulation of the classical problem of cylindrical elastic bars subjected to forces acting on the end planes of the bar. After developing the general theory, we examine bars of certain typical cross sections by elementary means. First, we consider the classical problem of torsion of prismatic bars after Saint-Venant. Next, we treat briefly the problem of bending of prismatic bars. The latter theory is again attributed principally to Saint-Venant.

7-1 General Problem of Three-Dimensional Elastic Bars Subjected to Transverse End Loads

Consider a cylindrical bar made of linearly elastic, homogeneous, isotropic material. Let the bar occupy the region bounded by a cylindrical lateral surface S and by two end planes distance L apart and perpendicular to the surface S (Fig. 7-1.1). The lateral surface of the bar is free of external load. The end planes of the bar are subjected to forces that satisfy equilibrium conditions of the bar as a whole. If the body forces are zero, the following sets of equations apply:

(a) Equilibrium equations:

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} &= 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} &= 0 \end{aligned} \tag{7-1.1}$$

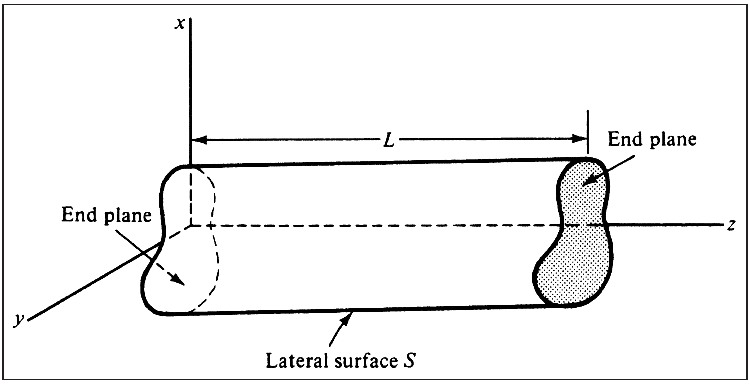

Figure 7-1.1

(b) Stress–strain relations:

$$\begin{array}{lll} \sigma_x = \lambda e + 2G\epsilon_x & \sigma_y = \lambda e + 2G\epsilon_y & \sigma_z = \lambda e + 2G\epsilon_z \\ \tau_{xy} = G\gamma_{xy} & \tau_{xz} = G\gamma_{xz} & \tau_{yz} = G\gamma_{yz} \end{array} \tag{7-1.2}$$

or, alternatively,

$$\begin{aligned} \epsilon_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \\ \epsilon_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] \\ \epsilon_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \end{aligned} \tag{7-1.3}$$

(c) Boundary conditions:
On lateral surfaces (direction cosines $l, m, n = l, m, 0$):

$$\begin{aligned} \sigma_{Px} &= l\sigma_x + m\tau_{xy} = 0 \\ \sigma_{Py} &= l\tau_{xy} + m\sigma_y = 0 \\ \sigma_{Pz} &= l\tau_{xz} + m\tau_{yz} = 0 \end{aligned} \tag{7-1.4a}$$

On ends ($z = 0$, $z = L$; direction cosines $l, m, n = 0, 0, \mp 1$):

$$\tau_{xz}, \tau_{yz} \qquad \text{prescribed functions} \tag{7-1.4b}$$

such that

$$\sum F_x = P_x \qquad \sum F_y = P_y \qquad \sum M_z = M$$

where P_x, P_y denote (x, y) components of the resultant force and M denotes the moment of the resultant couple. The problem of solving the equations formulated in the above generality poses considerable mathematical difficulties, particularly if the solution sought is to permit reasonably simple calculations. Fortunately, in a large number of practical cases, it is unnecessary to consider the problem in such general terms. Even though in practice we rarely know the true distribution of forces that act in the end planes of the bar, we often know a force system that is approximately statically equivalent to the actual force system. Accordingly, if we are considering a member with cross-sectional dimensions that are small compared to the length of the member, it may be adequate merely to ensure that the solution yields resultant forces and resultant moments that are approximately equal to actual values at the ends of the bar. For example, by Saint-Venant's principle, the stress distribution in regions sufficiently far removed from the end planes will be little affected by different distribution of forces over the end planes, provided the resultant force and moment for all distributions considered are the same (Chapter 4, Section 4-15).

Finally, the stress component σ_{ij} must satisfy the Beltrami–Mitchell compatibility equations (in the absence of body forces and for uniform temperature distribution)

$$\nabla^2\sigma_{ij} + \frac{1}{1+\nu}\frac{\partial^2 I_1}{\partial x_i \partial x_j} = 0 \qquad i, j = 1, 2, 3 \tag{7-1.5}$$

where

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_x + \sigma_y + \sigma_z \tag{7-1.6}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{7-1.7}$$

In the following discussion, we consider first the problem of twisting (torsion) of the bar by couples whose planes lie in the end planes of the bar. Then we treat the problem of bending of the bar by transverse end forces. The problems of bars subjected to axial forces at the ends and to couples whose planes are perpendicular to the end planes of the bar are left as exercises (see Review Problems R-1 and R-2, which appear before Appendix 7A at the end of this chapter).

7-2 Torsion of Prismatic Bars. Saint-Venant's Solution. Warping Function

In Chapter 4, Section 4-19, we treated the problem of torsion of a bar with simply connected circular cross section by the semi-inverse method. By taking displacement components in the form

$$u = -\beta yz \qquad v = \beta xz \qquad w = 0 \tag{7-2.1}$$

where (x, y, z) denote rectangular Cartesian coordinates and β denotes the angle of twist per unit length of the bar, we were able to satisfy the equations of elasticity exactly, provided the end shears were applied in a particular manner (Section 4-19).

However, if we proceed to apply Eqs. (7-2.1) to the torsion problem of a bar with simply connected non-circular cross section, we find that in general it is not possible to satisfy the boundary conditions on the lateral surface [see Eqs. (7-14.4)]. Accordingly, Eqs. (7-2.1) do not represent the solution to the torsion problem of bars with non-circular cross section. Hence, we are faced with the choice of either modifying Eqs. (6-2.1) or abandoning the semi-inverse method with regard to displacement components. For example, one may attempt to add more generality to Eqs. (7-2.1) (after Saint-Venant) or one may attempt to reformulate the problem in terms of stress components (after Prandtl). Initially, in this section, we modify Eqs. (7-2.1). In Section 7-3 we return to the formulation of the problem in terms of stress components.

The concept of allowing a section distance z from the end $z = 0$ to rotate as a rigid body about the axis of twist (the z axis, Fig. 7-2.1) is analytically attractive. Accordingly, we retain the same form for (u, v) [see Eq. (7-2.1) and Section 4-19]; however, we relax the condition $w = 0$.

Because the end forces tend to twist the bar about the z axis, physically it seems reasonable that extension of the bar along its axis is of secondary importance. Hence, the dependency of w, the displacement component in the z direction, upon z appears to be of secondary importance. Physically, the dependency of w upon coordinates (x, y) is difficult to guess. Accordingly, we do not attempt to specify an explicit relation between w and (x, y): rather, we arbitrarily take (after Saint-Venant) w in the form $w = \beta\psi(x, y)$, where $\psi(x, y)$ is an arbitrary function of (x, y). Because $\psi(x, y)$ is a measure of how much a point in the plane $z =$ constant displaces in the z direction, it is called the *warping function*. Thus, for the small-displacement torsion problem of a bar with non-circular cross section, we take the displacement vector (u, v, w) in the form

$$u = -\beta z y \qquad v = \beta z x \qquad w = \beta\psi(x, y) \tag{7-2.2}$$

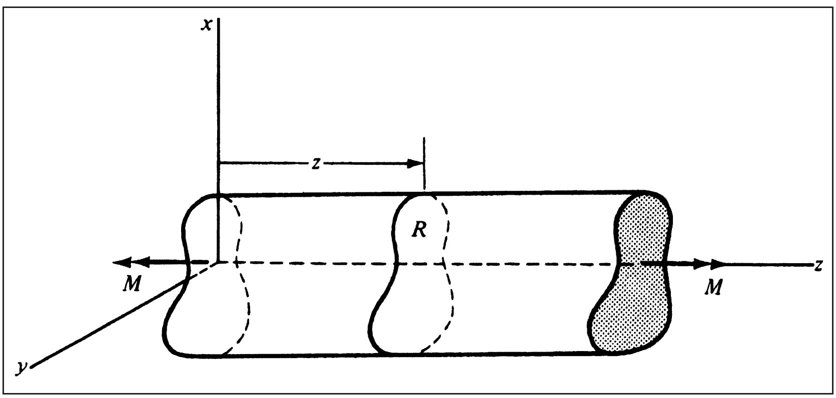

Figure 7-2.1

We now proceed to determine whether the equations of elasticity may be satisfied by this assumption. In other words, we seek to determine the function $\psi(x, y)$ such that the equations of elasticity are satisfied.

For small-displacement theory, Eqs. (2-15.14) and (7-2.2) yield

$$
\begin{aligned}
&\epsilon_x = \epsilon_y = \epsilon_z = \gamma_{xy} = 0 \\
&\gamma_{xz} = \beta\left(\frac{\partial\psi}{\partial x} - y\right) \qquad \gamma_{yz} = \beta\left(\frac{\partial\psi}{\partial y} + x\right)
\end{aligned}
\tag{7-2.3}
$$

Substitution of Eqs. (7-2.3) into Eqs. (7-1.2) yields the stress components

$$
\begin{aligned}
&\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \\
&\tau_{xz} = G\beta\left(\frac{\partial\psi}{\partial x} - y\right) \qquad \tau_{yz} = G\beta\left(\frac{\partial\psi}{\partial y} + x\right)
\end{aligned}
\tag{7-2.4}
$$

Now substitution of Eqs. (7-2.4) into Eqs. (7-1.1) yields for equilibrium

$$
\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \nabla^2\psi = 0 \tag{7-2.5}
$$

where now

$$
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
$$

Accordingly, the assumption of displacement components in the form of Eqs. (7-2.2) yields the requirement that $\nabla^2\psi = 0$, that is, that ψ be harmonic over the region R of the cross section of the bar (Fig. 7-2.1). Because we have assumed displacement components (u, v, w), compatibility conditions are automatically satisfied (Chapter 2, Section 2-16). Consequently, we have satisfied the equations of elasticity, provided that we can find a harmonic function (warping function) ψ that by Eqs. (7-2.4) yields stress components that satisfy the boundary conditions [Eqs. (7-14.4)].

Substituting Eqs. (7-2.4) into the boundary conditions for the lateral surface, we see that the first two of Eqs. (7-1.4a) are satisfied identically. The third equation yields

$$
\left(\frac{\partial\psi}{\partial x} - y\right) l + \left(\frac{\partial\psi}{\partial y} + x\right) m = 0 \tag{7-2.6}
$$

where (l, m) denote the components of the unit normal vector to the lateral surface S bounding the simply connected region R (Fig. 7-2.2). By Fig. 7-2.2, we find

$$
\begin{aligned}
l &= \cos\phi = \frac{dy}{ds} \\
m &= \sin\phi = -\frac{dx}{ds}
\end{aligned}
\tag{7-2.7}
$$

Substitution of Eq. (7-2.7) into Eq. (7-2.6) yields

$$\frac{\partial \psi}{\partial x}\frac{dy}{ds} - \frac{\partial \psi}{\partial y}\frac{dx}{ds} = x\frac{dx}{ds} + y\frac{dy}{ds} = \frac{1}{2}\frac{d}{ds}(x^2 + y^2) \tag{7-2.8}$$

Furthermore, by Fig. 7-2.2, we have

$$\frac{dy}{ds} = \frac{dx}{dn} \qquad \frac{dx}{ds} = -\frac{dy}{dn} \tag{7-2.9}$$

Consequently, Eqs. (7-2.8) and (7-2.9) yield

$$\frac{\partial \psi}{dn} = \frac{\partial \psi}{\partial x}\frac{dx}{dn} + \frac{\partial \psi}{\partial y}\frac{dy}{dn} = \frac{1}{2}\frac{d}{ds}(x^2 + y^2) \tag{7-2.10}$$

For a circular cross section of radius $a, x^2 + y^2 = a^2 =$ constant. Then Eq. (7-2.10) yields $d\psi/dn = 0$ on S, or $\psi =$ constant on S. This result agrees with that obtained in Section 4-19.

In general, we note that if the cross section is noncircular Eqs. (7-2.6), (7-2.7), and (7-2.9) yield the result

$$\frac{\partial \psi}{\partial n} = yl - xm = f(s) \tag{7-2.11}$$

where $f(s)$ denotes a function of the parameter s on the bounding curve S (Fig. 7-2.2).

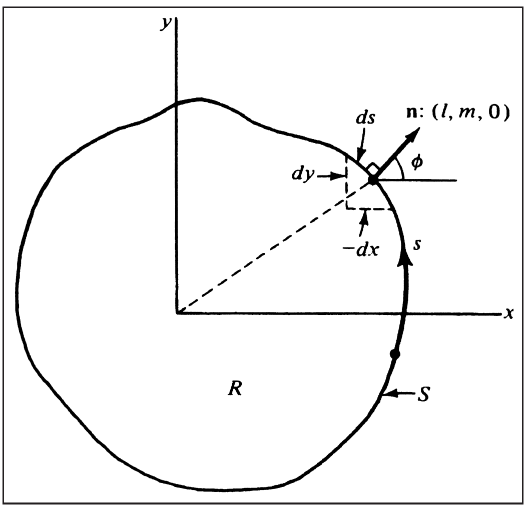

Figure 7-2.2

Finally, it may be shown (see Problem 7-2.1 at the end of this section) that

$$\begin{aligned} \sum F_x &= \int_A \tau_{xz}\, dA = 0 \\ \sum F_y &= \int_A \tau_{yz}\, dA = 0 \\ \sum M_z &= \int_A (x\tau_{yz} - y\tau_{xz})\, dA = M \end{aligned} \tag{7-2.12}$$

Accordingly, we have obtained a solution of the torsion problem of a bar with simply connected cross section, provided $\psi(x, y)$ satisfies the equations

$$\begin{aligned} \nabla^2\psi &= 0 \qquad \text{in } R \\ \frac{d\psi}{dn} &= yl - xm = f(s) \qquad \text{on } S \end{aligned} \tag{7-2.13}$$

Equations (7-2.13) define a well-known, extensively studied problem of potential theory (Kellogg, 2008): The Neumann boundary value problem.[1] In other words, the torsion problem expressed in terms of the warping function $\psi(x, y)$ may be stated as follows:

Determine a function $\psi(x, y)$ that is harmonic ($\nabla^2\psi = 0$) in R, such that it is regular in R and continuous in $R + S$, and such that its normal derivative takes on prescribed values $f(s)$ on S.

Alternatively, Eqs. (7-2.13) may be reformulated by utilizing the complex conjugate of $\psi(x, y)$, that is, by utilizing the function $\chi(x, y)$ related to $\psi(x, y)$ by the Cauchy–Riemann equation (Brown and Churchill, 2008)[2]:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\chi}{\partial y} \qquad \frac{\partial\psi}{\partial y} = -\frac{\partial\chi}{\partial x} \tag{7-2.14}$$

Differentiating the first of Eqs. (7-2.14) by y, the second by x, and subtracting, we obtain $\nabla^2\chi = 0$. Substitution of Eqs. (7-2.14) and (7-2.9) into the second of Eqs. (7-2.13) yields

$$\frac{\partial\chi}{ds} = yl - xm = \frac{1}{2}\frac{d}{ds}(x^2 + y^2) \quad \text{or} \quad \chi = \tfrac{1}{2}(x^2 + y^2) + \text{const.}$$

Accordingly, in terms of the complex conjugate χ of ψ, Eqs. (7-2.13) may be written

$$\begin{aligned} \nabla^2\chi &= 0 \qquad \text{in } R \\ \chi &= \tfrac{1}{2}(x^2 + y^2) = g(s) \qquad \text{on } S \end{aligned} \tag{7-2.15}$$

[1]A solution ψ to the Neumann problem exists, provided that the integral of the normal derivative of the function ψ, calculated over the entire boundary S, vanishes. Then the solution ψ is determined to within an arbitrary constant. For the torsion problem [Eqs. (7-2.13)], the solution ψ exists (see Problem 7-2.1).

[2]See also Eqs. (5-5.3) in Chapter 5.

where the constant in the second equation has been set equal to zero, as it does not affect the state or stress or displacement [see Eqs. (7-2.3), (7-2.4), and (7-2.14)].

In terms of χ, the strain and stress components are, by Eqs. (7-2.3), (7-2.4), and (7-2.14),

$$\gamma_{xz} = \beta\left(\frac{\partial\chi}{\partial y} - y\right) \qquad \gamma_{yz} = -\beta\left(\frac{\partial\chi}{\partial x} - x\right) \tag{7-2.16}$$

and

$$\tau_{xz} = G\beta\left(\frac{\partial\chi}{\partial y} - y\right) \qquad \tau_{yz} = -G\beta\left(\frac{\partial\chi}{\partial x} - x\right) \tag{7-2.17}$$

The boundary value problem defined by Eqs. (7-2.15), that of seeking a harmonic function χ in region R, whose values are prescribed on the boundary S of R, is known as the Dirichlet problem. The Dirichlet problem has been studied extensively (Kellogg, 2008; Courant and Hilbert, 1996).

Problem Set 7-2

1. Verify the first two of Eqs. (7-2.12). Verify that a solution ψ to the Neumann problem exists for the torsion of a bar [see Eqs. (7-2.13)].

7-3 Prandtl Torsion Function

In the preceding section we formulated the torsion problem of the bar with simply connected cross section in terms of two associated boundary value problems [see Eqs. (7-2.13) and (7-2.15)]. In this section we consider an alternative approach originally formulated by Prandtl (1903).[3] Prandtl employed the semi-inverse procedure as follows.

Because in the classical torsion problem the lateral surface and the end planes of the bar are free from normal tractions, one might initially guess that the normal tractions are zero throughout the bar. Furthermore, because the end faces are subjected to shear stress components that produce a couple **M**, one might initially assume as a first guess that the shear component not associated with the couple **M** also vanishes. Then one has (with respect to x, y, z axes designated in Fig. 7-2.1)

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \tag{7-3.1}$$

Next, because the left and right end planes are loaded identically, it appears reasonable that the remaining two components of stress (τ_{xz}, τ_{yz}) are approximately independent of the axial coordinate z. Accordingly, assuming that τ_{xz}, τ_{yz} are functions of (x, y) only and substituting Eqs. (7-3.1) into Eqs. (7-1.1), we find

$$\frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} = 0 \tag{7-3.2}$$

[3] As we will see, the results obtained by Prandtl are related simply to those obtained by Saint-Venant.

Equation (7-3.2) represents the necessary and sufficient condition that there exist a function $\phi(x, y)$ such that (see Chapter 1, Section 1-19)

$$\tau_{xz} = \frac{\partial \phi}{\partial y} \qquad \tau_{yz} = -\frac{\partial \phi}{\partial x} \tag{7-3.3}$$

where here the function ϕ is called the *Prandtl torsion function*.

Equation (7-3.3) automatically satisfies the equation of equilibrium [Eq. (7-3.2)]. Substitution of Eqs. (7-3.1) and (7-3.3) into Eqs. (7-1.5) yields

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = c = \text{constant} \tag{7-3.4}$$

Accordingly, compatibility is satisfied provided $\nabla^2 \phi = c$. The constant c may be shown to have a physical significance in that it is related to the angle of twist. Before verifying this statement, we consider the boundary conditions on the lateral surface and on the end planes [Eqs. (7-1.4)]. The first two of Eqs. (7-1.4a) are satisfied automatically; the last of Eqs. (7-1.4a), with Eqs. (7-2.7) and (7-3.3), yields (see Fig. 7-2.2)

$$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} = 0 \qquad \text{on } S$$

or

$$\phi = K = \text{constant on } S \tag{7-3.5}$$

where K denotes an arbitrary constant. For the simply connected cross section we may set $K = 0$ (see Section 7-6).

Finally, substitution of Eqs. (7-2.7) and (7-3.3) into Eqs. (7-1.4b) yields the following integrations over the end planes:

$$\begin{aligned}
\sum F_x &= \iint \sigma_{Px}\, dx\, dy = \iint \tau_{xz}\, dx\, dy \\
&= \int dx \int \frac{\partial \phi}{\partial y} \partial y = \int \phi \Big|_{y_1}^{y_2} dx \\
\sum Fy &= \iint \sigma_{Py}\, dx\, dy = \iint \tau_{yz}\, dx\, dy \\
&= -\int dy \int \frac{\partial \phi}{\partial x} dx = -\int \phi \Big|_{x_1}^{x_2} dy \\
\sum M_z = M &= \iint (x\tau_{yz} - y\tau_{xz})\, dx\, dy \\
&= -\iint \left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \right) dx\, dy \\
&= -\int x\phi \Big|_{x_1}^{x_2} dy - \int y\phi \Big|_{y_1}^{y_2} dx + 2\iint \phi\, dx\, dy
\end{aligned}$$

Because ϕ = constant on the lateral surface [we take $K = 0$ for the simply connected region; see Eq. (7-3.5)] and x_1, x_2, y_1, y_2 denote points on the lateral surface, it follows that

$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum M_z = M = 2\iint \phi\, dx\, dy \tag{7-3.6}$$

By the above discussion, we see that the torsion problem for a simply connected cross section R is solved precisely, provided we obtain a function ϕ such that

$$\begin{aligned} \nabla^2\phi = c = \text{const.} \qquad &\text{in } R \\ \phi = 0 \qquad &\text{on } S \end{aligned} \tag{7-3.7}$$

and provided the shears τ_{xz}, τ_{yz} are distributed over the end planes in accordance with Eq. (7-3.3). The twisting moment M is then defined by Eq. (7-3.6). The constant c may be related to the angle of twist per unit length of the bar, as we now proceed to show.

Displacement Components. Substitution of Eqs. (7-3.1) and (7-3.3) into the stress–strain relations [Eqs. (7-1.3)] yields with Eqs. (2-15.14)

$$\begin{aligned} &\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial w}{\partial z} = 0 \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \\ &\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{1}{G}\tau_{yz} \end{aligned} \tag{7-3.8}$$

Integration of Eqs. (7-3.8) yields

$$u = -Az(y - b) \qquad v = Az(x - a) \tag{7-3.9}$$

where A is a constant of integration and where $x = a$, $y = b$ defines the *center of twist*, that is, the z axis about which each cross section rotates as a rigid body (see Section 4-19; there, $a = y = 0$ denotes the axis of twist).

Substitution of Eqs. (7-3.9) into the last two of Eqs. (7-3.8) yields

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{1}{G}\tau_{xz} + A(y - b) \\ \frac{\partial w}{\partial y} &= \frac{1}{G}\tau_{yz} - A(x - a) \end{aligned} \tag{7-3.10}$$

Integration of Eqs. (7-3.10) yields

$$w = w_0 - A(xb - ya) \tag{7-3.11}$$

where $w_0 = w_0(x, y)$ represents the warping of the cross section. The terms involving the constants (a, b) in Eqs. (7-3.9) and (7-3.11) represent a rigid-body displacement relative to the center of twist.

To determine the angle of twist per unit length of the bar, we recall that the rotation ω_z of a volume element relative to the z axis is [see Eqs. (2-13.2)]

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{7-3.12}$$

Substitution of Eqs. (7-3.9) into Eqs. (7-3.12) yields $\omega_z = Az$. Hence, the angle of twist β per unit length of the bar is

$$\beta = \frac{\partial \omega_z}{\partial z} = A \tag{7-3.13}$$

Therefore, the constant of integration A in Eqs. (7-3.9) is identical to the angle of twist per unit length of the bar. Furthermore, by the last two of Eqs. (7-3.8), we note that by differentiating γ_{xz} by y and γ_{yz} by x and subtracting, we obtain

$$2\beta = 2\frac{\partial \omega_z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{G}\left(\frac{\partial \tau_{yz}}{\partial x} - \frac{\partial \tau_{xz}}{\partial y}\right) \tag{7-3.14}$$

Hence, substitution of Eq. (7-3.3) into Eq. (7-3.14) yields [with Eq. (7-3.7)]

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = c = -2G\beta \tag{7-3.15}$$

Accordingly, in terms of the Prandtl stress function ϕ, the torsion problem of a bar with simply connected cross section R bounded by S is defined by

$$\begin{aligned} \nabla^2 \phi &= -2G\beta \qquad \text{in } R \\ \phi &= 0 \qquad \text{on } S \end{aligned} \tag{7-3.16}$$

For the case where $a = b = 0$, the warping displacement $w_0(x, y)$ is related to the warping function $\psi(x, y)$ by the equation [see Eqs. (7-2.2) and (7-3.11)]

$$w_0 = \beta \psi(x, y) \tag{7-3.17}$$

Furthermore, the Prandtl stress function $\phi(x, y)$ is related to the warping function $\psi(x, y)$ by the equation [see Eqs. (7-2.4) and (7-3.3)]

$$\frac{\partial \phi}{\partial y} = G\beta\left(\frac{\partial \psi}{\partial x} - y\right) \qquad \frac{\partial \phi}{\partial x} = -G\beta\left(\frac{\partial \psi}{\partial y} + x\right) \tag{7-3.18}$$

and to the complex conjugate χ of ψ by the relations [see Eqs. (7-2.14), (7-3.3), and (7-3.18)]

$$\frac{\partial \phi}{\partial y} = G\beta\left(\frac{\partial \chi}{\partial y} - y\right) \qquad \frac{\partial \phi}{\partial x} = G\beta\left(\frac{\partial \chi}{\partial x} - x\right) \tag{7-3.19}$$

Integration of these relations yields

$$\phi = G\beta[\chi - \tfrac{1}{2}(x^2 + y^2) + b] \qquad (7\text{-}3.20)$$

where b denotes a constant. Thus, the Prandtl stress function ϕ may be simply related to the Saint-Venant warping function ψ [Eqs. (7-3.18)] or to the conjugate harmonic function χ of ψ [Eq. (7-3.20)].

Problem Set 7-3

1. Show that cylinders with circular cross sections are the only bodies whose lateral surface can be free from external load when the stress components are characterized by

$$\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \qquad \tau_{xz} = -G\beta y \qquad \tau_{yz} = G\beta x$$

7-4 A Method of Solution of the Torsion Problem: Elliptic Cross Section

A direct approach to the solution of the torsion problem is difficult in most practical cases. However, in terms of Prandtl's stress function ϕ, the following indirect approach is sometimes useful, although it is not generally applicable.

Because $\phi = 0$ on the lateral boundary [Eq. (7-3.16)], we may seek stress functions ϕ_i such that $\phi_i = 0$ on the lateral boundary of the shaft, leaving sufficient arbitrariness in ϕ so that the equation $\nabla^2\phi = -2G\beta$ may be satisfied over the region R occupied by the cross section. For a certain class of cross sections with boundaries simply expressible in the form $f(x, y) = 0$, this procedure is sometimes fruitful.

Example 7-4.1. Bar with Elliptical Cross Section. The equation of the bounding curve C of a bar with elliptical cross section is (Fig. E7-4.1)

$$f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 \qquad (\text{E7-4.1})$$

Hence, if we assume a stress function ϕ in the form

$$\phi = A\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \qquad (\text{E7-4.2})$$

where A is a constant, the boundary condition $\phi = 0$ on C is automatically satisfied. To yield a solution to the torsion problem, the function ϕ must be chosen so that both of Eqs. (7-3.16) are satisfied. By Eq. (E7-4.2) we find that

$$\nabla^2\phi = 2A\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

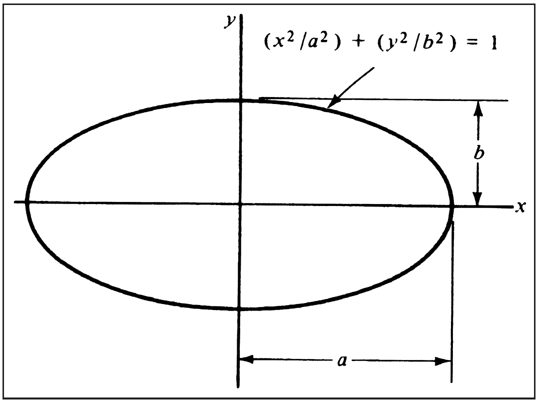

Figure E7-4.1

Hence, in order that ϕ satisfy Eq. (7-3.16) we must have

$$A = -\frac{a^2 b^2 G\beta}{a^2 + b^2} \tag{E7-4.3}$$

Accordingly, if A is given by Eq. (E7-4.3), Eq. (E7-4.2) yields the solution of the torsion of a bar with elliptic cross section. With ϕ so determined, the theory of Section 7-3 yields the stress components (τ_{xz}, τ_{yz}) and the moment M in terms of the dimensions a, b of the cross section, the shear modulus G, and the angle of twist β per unit length of the bar.

Moment–Angle of Twist Relation. The moment–stress function relation [Eq. (7-3.6)], with Eqs. (E7-4.2) and (E7-4.3), now yields

$$M = -\frac{2G\beta a^2 b^2}{a^2 + b^2}\left[\frac{1}{a^2}\iint x^2\,dx\,dy + \frac{1}{b^2}\iint y^2\,dx\,dy - \iint dx\,dy\right] \tag{7-4.4}$$

Now, for the ellipse,

$$\begin{aligned}\iint x^2\,dx\,dy &= I_y = \frac{\pi a^3 b}{4}\\ \iint y^2\,dx\,dy &= I_x = \frac{\pi a b^3}{4}\\ \iint dx\,dy &= \pi ab\end{aligned} \tag{7-4.5}$$

where (I_x, I_y) denote the moment of inertia of the cross-sectional area with respect to the (x, y) axes, respectively. Consequently, Eqs. (7-4.4) and (7-4.5) yield

$$M = \frac{\pi G\beta a^3 b^3}{a^2 + b^2} = C\beta \tag{7-4.6}$$

where

$$C = \frac{\pi a^3 b^3 G}{a^2 + b^2} \tag{7-4.7}$$

is called the *torsional rigidity* of the bar. Equation (7-4.6) relates the twisting moment M to the angle of twist β, the constant of proportionality being C, the torsional rigidity.

Also, by Eqs. (7-4.3) and (7-4.6), we find

$$A = -\frac{M}{\pi ab} \tag{7-4.8}$$

Therefore, we may write ϕ in the form

$$\phi = -\frac{M}{\pi ab}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \tag{7-4.9}$$

Stress Components. By Eqs. (7-3.3) and (7-4.9), we obtain

$$\begin{aligned} \tau_{xz} &= \frac{\partial \phi}{\partial y} = -\frac{2M}{\pi ab^3}y \\ \tau_{yz} &= -\frac{\partial \phi}{\partial x} = \frac{2M}{\pi a^3 b}x \end{aligned} \tag{7-4.10}$$

Hence, (τ_{xz}, τ_{yz}) vary linearly over the cross section with respect to (y, x), respectively. To determine the direction of the shearing stress vector $\boldsymbol{\tau} = \mathbf{i}\tau_{xz} + \mathbf{j}\tau_{yz}$ on the boundary of the shaft, we note that the tangent of the angle between the vector $\boldsymbol{\tau}$ and the positive x axis is given by [Eq. (7-4.10)]

$$\frac{\tau_{yz}}{\tau_{xz}} = -\frac{b^2 x}{a^2 y} \tag{7-4.11}$$

However, by the equation of the bounding curve C of the cross section [Eq. (E7-4.1)], we see that the angle formed by the tangent to C and the positive x axis is

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y} \tag{7-4.12}$$

Equations (7-4.11) and (7-4.12) show that the shearing stress vector $\boldsymbol{\tau}$ is tangent to the boundary C of the cross section. For $x = a$, $y = 0$, $\boldsymbol{\tau} = \mathbf{j}\tau_{yz}$; hence, $\boldsymbol{\tau}$ is directed perpendicular to the x axis. For $x = 0$, $y = b$, $\boldsymbol{\tau} = \mathbf{i}\tau_{xz}$; then $\boldsymbol{\tau}$ is directed perpendicular to the y axis (see Fig. 7-4.2). Also, the magnitude of $\boldsymbol{\tau}$ is

$$\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2} = \frac{2M}{\pi ab}\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4}} \tag{7-4.13}$$

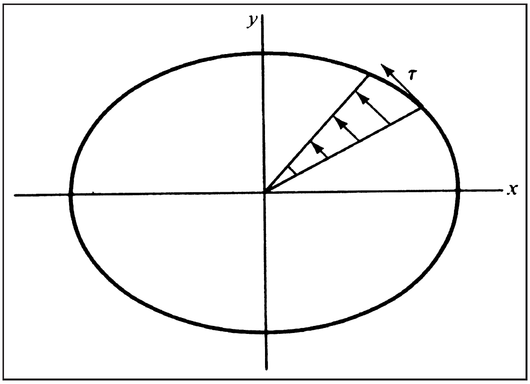

Figure 7-4.2

Determining the maximum value of τ from Eq. (7-4.13), we find

$$\tau_{\max} = \frac{2M}{\pi ab^2} \qquad y = b \qquad x = 0 \tag{7-4.14}$$

For a circular shaft $a = b = r$; then $\tau_{\max} = 2M/\pi r^3$, everywhere on the boundary C.

Displacement Components. With β determined as a function of M and C [Eq. (7-4.6)], the displacement components (u, v) are known for all points in any cross section for a given moment and a given bar. They are $u = -\beta yz$, $v = \beta xz$ [Eq. (7-3.9), with $a = b = 0$]. To compute the displacement component w, we must compute $\psi(x, y)$, the warping function [Eqs. (7-2.2) or (7-3.17)], from its relation to the stress function $\phi(x, y)$ [Eq. (7-3.18)].

By Eqs. (7-3.18) and (7-4.9), we obtain

$$\begin{aligned} \frac{\partial \psi}{\partial x} &= \frac{1}{G\beta}\frac{\partial \phi}{\partial y} + y = \left(1 - \frac{2M}{\pi ab^3 G\beta}\right) y \\ \frac{\partial \psi}{\partial y} &= -\frac{1}{G\beta}\frac{\partial \phi}{\partial x} - x = \left(\frac{2M}{\pi a^3 bG\beta} - 1\right) x \end{aligned} \tag{7-4.15}$$

Integration of Eqs. (7-4.15) yields

$$\psi = \frac{b^2 - a^2}{a^2 + b^2} xy + \text{const.} \tag{7-4.16}$$

If we set $w = 0$ for $x = y = 0$, the constant in Eq. (7-4.16) is zero. Consequently,

$$w = \beta\psi = \frac{\beta(b^2 - a^2)}{a^2 + b^2} xy$$

or

$$w = -Kxy \tag{7-4.17}$$

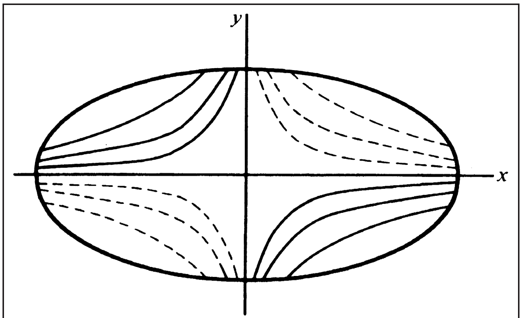

Figure 7-4.3

where

$$K = \frac{\beta(a^2 - b^2)}{a^2 + b^2} = \frac{M(a^2 - b^2)}{\pi a^3 b^3 G} \tag{7-4.18}$$

Equation (7-4.17) is the equation of a hyperbola. Accordingly, the contour map of w over the cross section of the bar is represented by a family of hyperbolas (Fig. 7-4.3), with the (x, y) axes representing lines of zero displacement.

Because K is a positive constant, w is positive (in the direction of the positive z axis) in the second and fourth quadrants and negative in the first and third quadrants of the (x, y) plane.

Problem Set 7-4

1. Derive Eq. (7-4.14).

2. Apply the method outlined in Section 7-4 to the bar with circular cross section.

7-5 Remarks on Solutions of the Laplace Equation, $\nabla^2 F = 0$

In the theory of complex variables (Brown and Churchill, 2008) it is shown that the real and imaginary parts of an analytic function F of the complex variable $z = x + iy$ satisfy the Laplace equation $\nabla^2 F = 0$; that is, the real and imaginary parts of an analytic function are harmonic functions. Accordingly, by considering the real and the imaginary parts of analytic functions F_n, one may proceed, inversely so to speak, to determine the equations of the boundaries of simply connected cross sections for which the real and imaginary parts of F_n represent solutions of the torsion problem. For example, we have previously noted that $f(z) = \psi + i\chi$ is an analytic function where χ is the conjugate harmonic of the warping function ψ, and that the torsion problem may be represented either in terms of ψ or χ (Section 7-3).

One of the simplest sets of analytic functions of the complex variable $z = x + iy$ is the set $F_n = z^n = (x + iy)^n$. By letting $n = \pm 1, \pm 2, \pm 3, \ldots$, solutions of the torsion problem may be developed in the form of polynomials. For example, for $n = 2$, we obtain the solutions $x^2 - y^2$ and $2xy$. For $n = 3$, we find $x^3 - 3xy^2$ and $3x^2y - y^3$. For $n = 4$, we have $x^4 - 6x^2y^2 + y^4$ and $4x^3y - 4xy^3$, and so on. Sums and differences of these polynomial solutions may also be employed, as the sums and the differences of harmonic functions yield other harmonic functions. A systematic application of this technique to the torsion problem has been employed by Weber and Günther (1958). Here we merely present a classical example of the method. Other examples are considered in the problems.

Example 7-5.1. Equilateral Triangle. Consider the harmonic polynomial $\phi_1 = A(x^3 - 3xy^2)$ (obtained from z^n, with $n = 3$), where A is a constant. Because ϕ_1 is harmonic, by setting $\chi = \phi_1$, we may write Prandtl's stress function ϕ in the form [see Eq. (7-3.20)]

$$\phi = -G\beta\left[\frac{x^2 + y^2}{2} - \frac{x^3 - 3xy^2}{2a} - b\right] \tag{E7-5.1}$$

where a and b denote constants. If we assign the value $2a^2/27$ to the constant b, we may factor Eq. (E7-5.1) into the form

$$\phi = \frac{G\beta}{2a}\left(x - \sqrt{3}y - \frac{2a}{3}\right)\left(x + \sqrt{3}y - \frac{2a}{3}\right)\left(x + \frac{a}{3}\right) \tag{E7-5.2}$$

Accordingly, for $b = 2a^2/27$, the condition that ϕ vanish on the lateral boundary of a bar in torsion [Eqs. (7-3.16)] is satisfied identically by the three conditions

$$\begin{aligned} x - \sqrt{3}y - \frac{2a}{3} &= 0 \\ x + \sqrt{3}y - \frac{2a}{3} &= 0 \\ x + \frac{a}{3} &= 0 \end{aligned} \tag{E7-5.3}$$

Equations (E7-5.3) represent the equations of three straight lines in the (x, y) plane that form an equilateral triangle (Fig. E7-5.1). The region bounded by the three straight lines may be considered as the cross section of a bar in torsion.

Shear–Stress Components. By Eqs. (7-3.3) and (E7-5.1), we find that the shear–stress components are

$$\begin{aligned} \tau_{xz} &= -\frac{3G\beta y}{a}\left(x + \frac{a}{3}\right) \\ \tau_{yz} &= G\beta\left(x - \frac{3x^2}{2a} + \frac{3y^2}{2a}\right) \end{aligned} \tag{E7-5.4}$$

Equations (E7-5.4) show that $\tau_{xz} = 0$ for $y = 0$ and for $x = -a/3$, and that τ_{yz} is parabolically distributed along the y axis $(x = 0)$.

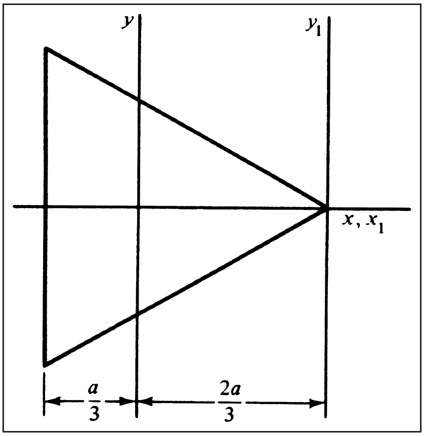

Figure E7-5.1

Warping of Cross Section. Letting $\chi = (x^3 - 3xy^2)/2a$ and integrating Eqs. (7-2.14), we obtain the warping function

$$\psi = \frac{y}{2a}(y^2 - 3x^2) + C_0 \tag{E7-5.5}$$

where C_0 is a constant. If we set $w = 0$ for $x = y = 0$, then Eq. (E7-5.5) and the last of Eqs. (7-2.2) yield

$$w = \frac{\beta y}{2a}(y^2 - 3x^2) \tag{E7-5.6}$$

By Eq. (E7-5.6), we note that $w = 0$ for $y = 0$ and $y = \pm\sqrt{3}x$. In general, the w contour lines for which $w =$ constant are described by the equation

$$x^2 = \frac{y^2}{3} + \frac{K}{y} \tag{E7-5.7}$$

where $K =$ constant. If $K > 0$, $x \to \infty$ as $y \to 0$ and as $y \to \infty$. These conditions facilitate the visualization of the contour map for w (Problem 7-5.1), where positive w is taken in the direction of positive z where (x, y, z) are for a right-handed coordinate system. The sign of w changes upon crossing the lines $y = 0$ and $y = \pm\sqrt{3}x$. Consequently, the cross section warps into alternate convex $(+w)$ and concave $(-w)$ regions.

Problem Set 7-5

1. Sketch the contour map for the warping of the triangular cross section under torsion [see Eq. (E7-5.7) and Fig. E7-5.1].

2. Derive Eqs. (E7-5.2), (E7-5.4), and (E7-5.5).

3. Considering terms obtained from the analytic function $(x + iy)^4$, we can express a Prandtl stress function in the form

$$\phi = -G\beta\left[\frac{x^2 + y^2}{2} - \frac{a(x^4 - 6x^2y^2 + y^4)}{2} + \frac{a - 1}{2}\right]$$

Set $a = 0.2$; plot the cross section of the bar for which ϕ solves the torsion problem. Calculate the stress at the boundary point for which the radius vector forms an angle of $\theta = 45^\circ$ with the positive x axis. Use $G = 12 \times 10^6$ psi, $\beta = 0.001$ rad/in. Compare the result to that of a circle with radius equal to the radius vector of the plotted cross section at $\theta = 45^\circ$. Repeat for $a = 0.5$. (In his investigations, Saint-Venant found that the torsional rigidity of a given cross section may be approximated by replacing the given cross section with an elliptical cross section with the same area and the same polar moment of inertia.) Is the *circular* approximation noted above a good approximation?

4. Choosing axes (x_1, y_1) at the tip of the equilateral triangular cross section (Fig. E7-5.1), by means of Eqs. (7-3.6) and (E7-5.2) show that

$$M = \frac{G\beta a^4}{15\sqrt{3}}$$

5. C. Weber proposed the following elementary method of examining the effects of a circular groove or slot in a circular bar [for other kinds of groove and bar combinations, see Weber and Günther (1958)]: Considering a pair of harmonic functions x and $x/(x^2 + y^2)$ obtained from z^n with $n = \pm 1$, Weber transformed the functions into polar coordinates (r, θ). Thus, $x = r\cos\theta$ and $x/(x^2 + y^2) = (\cos\theta)/r$. Hence, he took [see Eq. (7-3.20)] a Prandtl stress function in the form

$$\phi = \frac{G\beta}{2}\left[b^2 - r^2 + 2a(r^2 - b^2)\frac{\cos\theta}{r}\right] \tag{a}$$

where β is taken to denote the angle of twist per unit length. Setting $\phi = 0$ on the boundary, Weber obtained the equation of the boundary of the cross section as

$$(r^2 - b^2)\left(1 - \frac{2a}{r}\cos\theta\right) = 0 \tag{b}$$

Equation (b) is satisfied identically by the conditions

$$\begin{aligned} r^2 - b^2 &= 0 \\ r - 2a\cos\theta &= 0 \end{aligned} \tag{c}$$

Equations (c) may be considered to represent the cross section R of a circular shaft with a circular groove (Fig. P7-5.5). Hence, with Eq. (a), the stress components τ_{xz}, τ_{yz} may be computed by Eqs. (7-3.3). Derive the formulas for τ_{xz}, τ_{yz}.

6. Using the results derived in Problem 5, derive formulas for the stress components τ_{xz}, τ_{yz} on the boundary of the shaft and on the boundary of the groove. Compute the maximum value of stress on the boundary of the shaft and then on the groove.

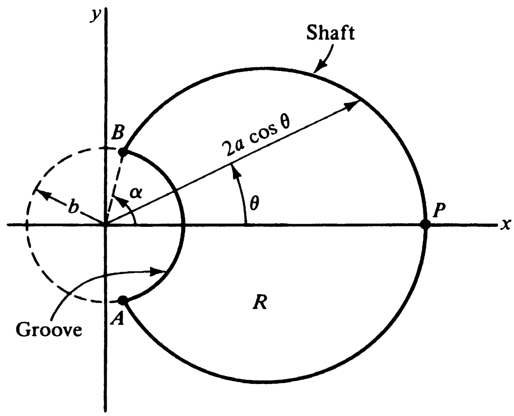

Figure P7-5.5

7. Compute $\tau_{\max}$ in terms of M and a for $\alpha = 60^\circ$, $\alpha = 45^\circ$, and $\alpha = 30^\circ$ (Fig. P7-5.5). Compute τ at the point P for these cases. Verify that $\tau_{xz} = \tau_{yz} = 0$ for corners A and B.

8. For the cross section given in Problem 5, derive the formula for the torsional rigidity of the member.

9. Consider the torsion of a shaft with circular cross section that varies along the axis of the shaft. Let (r, θ, z) be cylindrical coordinates such that (r, θ) lies in the plane of the cross section and z lies along the axis of the shaft. Thus, the radius of the circular cross section varies with z. As in the torsion of a bar with constant circular cross section, assume that $u = w = 0$, where u, w denote displacement components in the (r, z) directions, respectively. Because of the symmetry of the circular cross section, the displacement component v in the θ direction is independent of polar coordinate θ. The dependence of v on r and z is difficult to guess. Hence, take $v = v(r, z)$.

(a) Determine the corresponding strain components of the shaft.

(b) For a linearly elastic, isotropic material, determine the corresponding stress components of the shaft.

(c) Express the equilibrium equations in terms of v.

(d) Show that there exists a torsion function $F(r, z)$ such that F satisfies the equations of equilibrium, provided

$$\frac{\partial F}{\partial r} = r^3 \frac{\partial}{\partial z}\left(\frac{v}{r}\right) \qquad \frac{\partial F}{\partial z} = -r^3 \frac{\partial}{\partial r}\left(\frac{v}{r}\right)$$

(e) Show that the defining equation for $F(r, z)$ is

$$\frac{\partial^2 F}{\partial r^2} - \frac{3}{r}\frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} = 0$$

(f) Determine the boundary conditions that F must satisfy. Hence, define the mathematical problem that determines F. *Hint:* Consider a section of the shaft in the r, z plane and write the boundary conditions for the lateral surface of the shaft.

7-6 Torsion of Bars with Tubular Cavities

Consider a bar with cross section R, where R is the multiply connected region interior to C_0 and exterior to the longitudinal tubular cavities $C_2, C_2, \ldots, C_n$ (Fig. 7-6.1). As in the torsion problem of the simply connected cross section, the displacement components are taken in the form

$$\begin{aligned} u &= -\beta z y \\ v &= \beta z x \\ w &= \beta \psi(x, y) \end{aligned} \tag{7-6.1}$$

where β and ψ are a constant and a function of (x, y), respectively, which are to be determined.

The shearing–stress components in region R are given by the relations [see Eqs. (7-2.4)]

$$\tau_{xz} = \beta G\left(\frac{\partial \psi}{\partial x} - y\right) \qquad \tau_{yz} = \beta G\left(\frac{\partial \psi}{\partial y} + x\right) \tag{7-6.2}$$

Because the boundaries $C_0, C_1, C_2, \ldots, C_n$ are free from external loads, the boundary conditions are

$$l\tau_{xz} + m\tau_{yz} = 0 \quad \text{on } C_i \qquad i = 0, 1, \ldots, n \tag{7-6.3}$$

In terms of ψ, the boundary conditions may be written in the form

$$\frac{\partial \psi}{dn} = ly - mx \quad \text{on } C_i \qquad i = 0, 1, 2, \ldots, n \tag{7-6.4}$$

Introducing the stress function ϕ, defined by Eqs. (7-3.3), we may write the boundary conditions in terms of the stress function ϕ in the form

$$l\frac{\partial \phi}{\partial y} - m\frac{\partial \phi}{\partial x} = \frac{d\phi}{ds} = 0 \quad \text{on } C_i \qquad i = 0, 1, \ldots, n \tag{7-6.5}$$

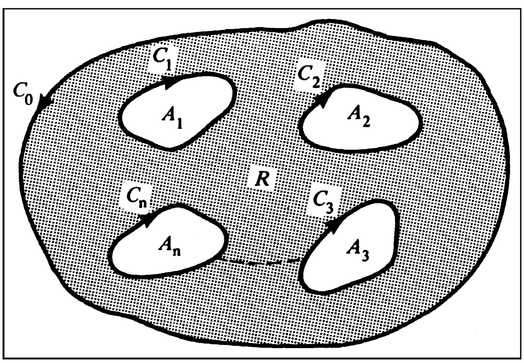

Figure 7-6.1

or

$$\phi = K_i \quad \text{on } C_i \qquad i = 0, 1, 2, \ldots, n \tag{7-6.6}$$

where the K_i are constants.

In general, the function ϕ may be multiple valued. However, the function ψ is determined by the boundary condition, Eq. (7-6.4), to within an arbitrary constant, and it follows by Eqs. (7-6.2) and (7-3.3) that the function ϕ is determined to within an arbitrary constant. Consequently, the stress function ϕ defined by Eqs. (7-3.3) must satisfy the conditions of Eq. (7-6.6), where the value of only one of the constants K_i may be assigned arbitrarily. If the region R is simply connected (i.e., if there are no tubular cavities), $i = 0$, and $\phi = K_0$ on C_0. The constant K_0 may then be assigned an arbitrary value—for example, zero.

The remaining n constants must be chosen so that the displacement component w [and hence ψ, see Eq. (7-6.1)] is a single-valued function, the constants K_i being related to the function ψ through Eqs. (7-3.18) and (7-6.6) or to the complex conjugate χ of ψ through Eqs. (7-3.20) and (7-6.6). For example, the values of K_i may be established so that the solution of the Dirichlet problem [with $b = 0$ in Eqs. (7-3.20)]

$$\begin{aligned} \nabla^2 \chi &= 0 \quad \text{over } R \\ \chi &= \tfrac{1}{2}(x^2 + y^2) + \overline{K}_i \qquad i = 1, \ldots, n \quad \text{on } C_i \\ G\beta \overline{K}_i &= K_i \end{aligned}$$

satisfies the conditions for the existence of a single-valued function in a multiply connected region.[4]

Substituting Eqs. (7-3.3) into Eqs. (7-6.2), differentiating the first of Eqs. (7-6.2) by y and the second by x, and subtracting the resulting equations, we obtain the condition

$$\nabla^2 \phi = -2G\beta \qquad \text{in region } R \tag{7-6.7}$$

The twisting moment M that results from the shearing forces that act on the end plane of the bar is

$$M = \iint\limits_{\text{over } R} (x\tau_{yz} - y\tau_{xz})\, dx\, dy \tag{7-6.8}$$

Substituting Eqs. (7-3.3) into Eqs. (7-6.8), we obtain

$$M = -\iint\limits_{\text{over } R} \left(x\frac{\partial \phi}{\partial x} + y\frac{\partial \phi}{\partial y} \right) dx\, dy \tag{7-6.9}$$

[4]See Eqs (7-2.15) and the discussion at the end of Section 5-4 in Chapter 5, particularly Eqs. (5-4.24) and (5-4.25). Here, $m = n$ and $G = \chi$.

Equation (7-6.9) may be written in the form

$$M = \iint\limits_{\text{over } R} 2\phi\, dx\, dy - \iint\limits_{\text{over } R} \left[\frac{\partial (x\phi)}{\partial x} + \frac{\partial (y\phi)}{\partial y} \right] dx\, dy \tag{7-6.10}$$

Transforming the second integral of Eq. (7-6.10) by Green's theorem for the plane (Section 1-16), we may write Eq. (7-6.10) in the form

$$M = 2 \iint\limits_{\text{over } R} \phi\, dx\, dy = \sum_{i=0}^{n} \oint_{C_i} \phi (x\, dy - y\, dx) \tag{7-6.11}$$

Because we may assign the value of one of the $K_i's$ in Eq. (7-6.6) arbitrarily, let K_0 on the boundary C_0 be zero; that is, let $\phi = 0$ on C_0. Then, substitution of Eq. (7-6.6) into Eq. (7-6.11) yields

$$M = 2 \iint\limits_{\text{over } R} \phi\, dx\, dy + \sum_{i=1}^{n} K_i \oint_{C_i} (y\, dx - x\, dy)$$

Noting that

$$\oint_{C_i} (y\, dx - x\, dy) = 2 \iint\limits_{\text{over } A_i} dx\, dy = 2A_i$$

where A_i is the area bounded by the curve C_i, we obtain

$$M = 2 \iint\limits_{\text{over } R} \phi\, dx\, dy + 2 \sum_{i=1}^{n} K_i A_i \tag{7-6.12}$$

Equation (7-6.12) is the moment–stress function relation for the torsion problem of bars with multiply connected cross sections. Alternatively, by means of Eqs. (7-3.20) and (7-6.12), M may be expressed in terms of the function χ.

Problem Set 7-6

1. For the hollow circular shaft of inner radius a and outer radius b, by the above theory, evaluate M using the stress function $\phi = A(r^2 - b^2)$.

7-7 Transfer of Axis of Twist

In the previous analysis of the torsion problem, we assumed that any cross section of the beam was subjected to an infinitesimal rotation θ about a z axis. No assumption was made as to the location of the z axis relative to the cross section. In calculations, it may be convenient to choose a particular z axis. Hence, let us consider an axis z_1 that is parallel to the axis z, but that intersects the (x, y) plane at point (a, b). With respect to the z_1 axis, the displacement components are

$$u_1 = -\beta z(y - b) \qquad v_1 = \beta z(x - a) \qquad w_1 = \beta \psi_1(x, y) \tag{7-7.1}$$

where ψ_1, not necessarily identical to ψ, is the warping function with respect to the z_1 axis (see also Review Problem R-4, later in this chapter before Appendix 7A).

In terms of the stress function ψ_1, the stress components are

$$\begin{aligned} \tau_{xz} &= G\beta\left(\frac{\partial \psi_1}{\partial x} - y + b\right) \\ \tau_{yz} &= G\beta\left(\frac{\partial \psi_1}{\partial y} + x - a\right) \\ \sigma_x &= \sigma_y = \sigma_z = \tau_{xy} = 0 \end{aligned} \tag{7-7.2}$$

Substitution of these stress components into the equilibrium equations [Eqs. (7-1.1)] yields the result

$$\nabla^2 \psi_1 = \frac{\partial^2 \psi_1}{\partial x^2} + \frac{\partial^2 \psi_1}{\partial y^2} = 0 \tag{7-7.3}$$

Also, the boundary conditions [Eqs. (7-1.4)] reduce to the condition

$$\frac{d}{dn}(\psi_1 + bx - ay) = ly - mx \tag{7-7.4}$$

Now the function $\psi_1 + bx - ay$ is harmonic, and it satisfies the same boundary conditions as the warping function ψ. Hence, by the uniqueness (Courant and Hilbert, 1996) of the solution of the problem of Neumann, ψ and $\psi_1 + bx - ay$ can differ only by a constant; that is, $\psi_1 = \psi - bx + ay + c$, where c is a constant. Consequently, the displacement components measured with respect to axis z_1 are given by the formulas

$$\begin{aligned} u_1 &= -\beta zy + \beta zb \\ v_1 &= \beta zx - \beta za \\ w_1 &= \beta\psi + \beta ya - \beta xb + \beta c \end{aligned} \tag{7-7.5}$$

These components differ by a rigid-body displacement from those with respect to the z axis [Eqs. (7-2.2)]. Consequently, the stress components are identical with those with respect to the z axis. Thus, the choice of the origin of coordinates is immaterial in the torsion problem of the bar with regard to the stress components.

7-8 Shearing–Stress Component in Any Direction

Directional Derivative. Let $P(x, y)$ be any point on a curve in the (x, y) plane. Let the scalar function $\phi(x, y)$ be defined on C with its partial derivatives $\partial\phi/\partial x$

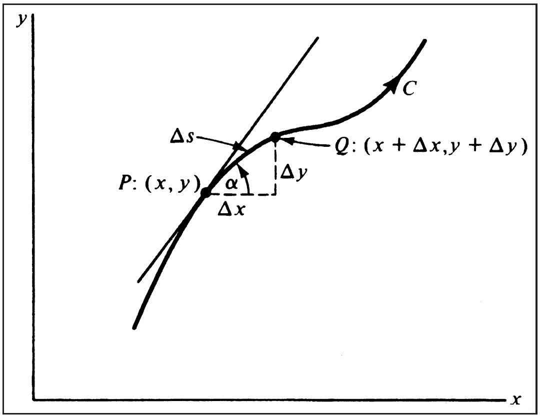

Figure 7-8.1

and $\partial\phi/\partial y$; for example, ϕ may be the stress function in torsion. Let $Q : (x + \Delta x, y + \Delta y)$ be a point on C in the neighborhood of P (see Fig. 7-8.1). Let Δs be the length of arc PQ and $\Delta\phi$ be the change in ϕ due to increments Δx and Δy. Then, the derivative

$$\frac{d\phi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s}$$

determines the rate of change of ϕ along the curve C at the point P: (x, y). Now the total differential of ϕ is

$$d\phi = \frac{\partial\phi}{\partial x}\, dx + \frac{\partial\phi}{\partial y}\, dy$$

and

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds}$$

Also,

$$\frac{dx}{ds} = \lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s} = \cos\alpha$$
$$\frac{dy}{ds} = \lim_{\Delta s \to 0} \frac{\Delta y}{\Delta s} = \sin\alpha$$

Hence, $d\phi/ds = (\partial\phi/\partial x)\cos\alpha + (\partial\phi/\partial y)\sin\alpha$. By this equation, it is apparent that $d\phi/ds$ depends on the direction of s. For this reason, $d\phi/ds$ is called the *directional derivative*. It represents the rate of change of ϕ in the direction of the tangent to the particular curve chosen for point $P : (x, y)$. For example, if $\alpha = 0$,

$$\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}$$

is the rate of change of ϕ in the direction of the x axis.

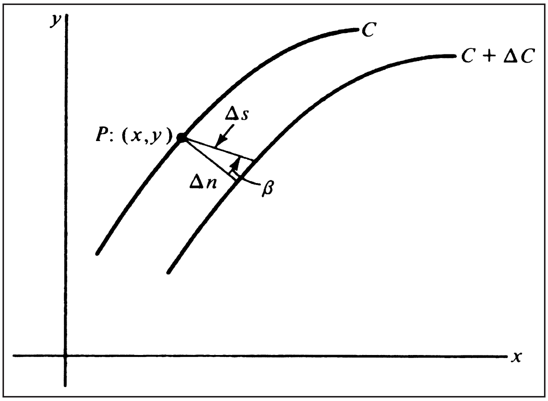

Figure 7-8.2

Maximum Value of the Directional Derivative: Gradient. Consider two neighboring curves in the (x, y) plane; say, C and $C + \Delta C$ (Fig. 7-8.2). Let the respective values of ϕ on these curves be ϕ and $\phi + \Delta\phi$. Then $\Delta\phi/\Delta s$ is the average rate of change of ϕ with respect to the distance Δs measured from curve C to the curve $C + \Delta C$. Now consider the ratio $\Delta n/\Delta s$, where Δn denotes the distance from C to $C + \Delta C$ measured along the normal to C at point P: $((x, y)$. The limiting value of this ratio is $\cos\beta$; that is,

$$\frac{dn}{ds} = \lim_{\Delta C \to 0} \frac{\Delta n}{\Delta s} = \cos\beta$$

Hence,

$$\frac{d\phi}{ds} = \frac{d\phi}{dn}\frac{dn}{ds} = \frac{d\phi}{dn}\cos\beta$$

Therefore, $d\phi/dn$, that is, the derivative of ϕ in the direction normal to C, is the maximum value that $d\phi/ds$ may take in any direction. Hence, $(d\phi/ds)_{\max} = |d\phi/dn|$. The vector in the direction of the normal, of magnitude $|d\phi/dn|$, is called the gradient of ϕ; that is, $(\phi_x, \phi_y) = \text{gradient } \phi = \text{grad } \phi$, where (x, y) subscripts on ϕ denote partial derivatives. Consequently, the maximum value of $d\phi/ds$ is equal to the magnitude of the gradient of ϕ, $|\text{grad } \phi|$.

Stress Component–Directional Derivative. Consider an arbitrary point P: (x, y) in the cross section of a bar in torsion (Fig. 7-8.3). The stress component τ_θ in the direction θ is

$$\tau_\theta = \tau_{xz}\cos\theta + \tau_{yz}\sin\theta$$

In terms of the stress function ϕ, by Eqs. (7-3.3),

$$\tau_{xz} = \frac{\partial\phi}{\partial y} \qquad \tau_{yz} = -\frac{\partial\phi}{\partial x}$$

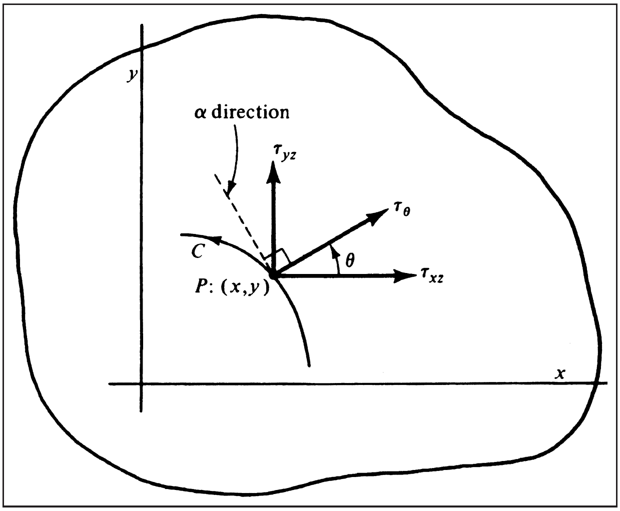

Figure 7-8.3

Therefore,

$$\begin{aligned}\tau_\theta &= \frac{\partial\phi}{\partial y}\cos\theta - \frac{\partial\phi}{\partial x}\sin\theta \\ &= \phi_y\cos\theta - \phi_x\sin\theta\end{aligned}$$

Now set $\alpha = \theta + \pi/2$. Then

$$\begin{aligned}\tau_\theta &= \phi_y\cos\left(\alpha - \frac{\pi}{2}\right) - \phi_x\sin\left(\alpha - \frac{\pi}{2}\right) \\ &= \phi_x\cos\alpha + \phi_y\sin\alpha = \frac{d\phi}{ds}\end{aligned}$$

Consequently, τ_θ is equal to the directional derivative of ϕ in a direction leading θ by 90°. Note that if the direction α corresponds to a direction for which $\phi =$ constant, $d\phi/ds = 0$. Hence, the shearing–stress perpendicular to the line $\phi =$ constant is zero. Therefore, lines $\phi =$ constant are shearing–stress trajectories, and the stress vector on lines $\phi =$ constant has magnitude

$$|\tau_\theta| = (\phi_x^2 + \phi_y^2)^{1/2} = \left(\frac{d\phi}{ds}\right)_{\max} = |\text{grad } \phi|$$

The stress vector is tangent to lines $\phi =$ constant.

In polar coordinates (r, β) (see Fig. 7-8.4),

$$\tau_r = \frac{1}{r}\frac{\partial\phi}{\partial\beta} \qquad \tau_\beta = -\frac{\partial\phi}{\partial r} \tag{7-8.1}$$

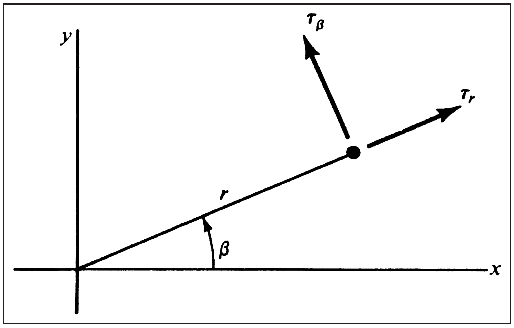

Figure 7-8.4

For example, in terms of polar coordinates (r, β), the Prandtl stress function of a circular shaft with circular groove is [see Eq. (a), Problem 7-5.5]

$$\phi = \frac{G\theta}{2}\left[b^2 - r^2 + 2a(r^2 - b^2)\frac{\cos\beta}{r}\right] \tag{7-8.2}$$

where here θ denotes the unit angle of twist. Consequently, Eqs. (7-8.1) and (7-8.2) yield

$$\begin{aligned} \tau_r &= \tau_{rz} = \frac{1}{r}\frac{\partial\phi}{\partial B} = -\frac{G\theta a}{r^2}(r^2 - b^2)\sin\beta \\ \tau_\beta &= \tau_{\beta z} = -\frac{\partial\phi}{\partial r} = G\theta\left[r - \frac{a}{r^2}(r^2 + b^2)\cos\beta\right] \end{aligned} \tag{7-8.3}$$

Thus, for $\beta = 0$ and $r = b$ (Fig. P7-5.5 and Problem 7-5.5) we have

$$\tau_{rz} = 0 \qquad \tau_{\beta z} = -G\theta(2a - b)$$

and for $\beta = 0$ and $r = 2a$ (point P in Fig. P7-5.5) we obtain

$$\tau_{rz} = 0 \qquad \tau_{\beta z} = \frac{G\theta}{4a}(4a^2 - b^2)$$

Problem Set 7-8

1. Plot out several shearing–stress trajectories for the cross section shown in Fig. P7-5.5.

7-9 Solution of Torsion Problem by the Prandtl Membrane Analogy

In this section we consider an analogy method proposed by Prandtl (1903)[5] that leads itself to obtaining approximate solutions to the torsion problem. Although

[5]See Prandtl (1903, p. 758). Another analogy method, a hydrodynamic analogy, has been proposed by Pestel (1955a, 1955b); see also Grossmann (1957). We discuss only the analogy proposed by Prandtl.

this method is of historical interest, it is rarely used today to obtain quantitative results, and it is treated here primarily from the heuristic viewpoint.

The analogy is based upon the equivalence of the torsion equation (7-3.15)

$$\nabla^2 \phi = -2G\beta \tag{7-9.1}$$

and the membrane equation

$$\nabla^2 z = -\frac{q}{S} \tag{7-9.2}$$

where z denotes the lateral displacement of a membrane subjected to a lateral pressure q in terms of force per unit area and an initial (large) tension S (Fig. 7-9.1) in terms of force per unit length.

For example, consider an element $ABCD$ of dimensions dx, dy of a membrane (Fig. 7-9.1). The net vertical force due to the tension S acting along edge AD is (assuming small displacements so that $\sin\alpha \approx \tan\alpha$)

$$-S\,dy \sin\alpha \approx -S\,dy \tan\alpha = -S\,dy \frac{\partial z}{\partial x}$$

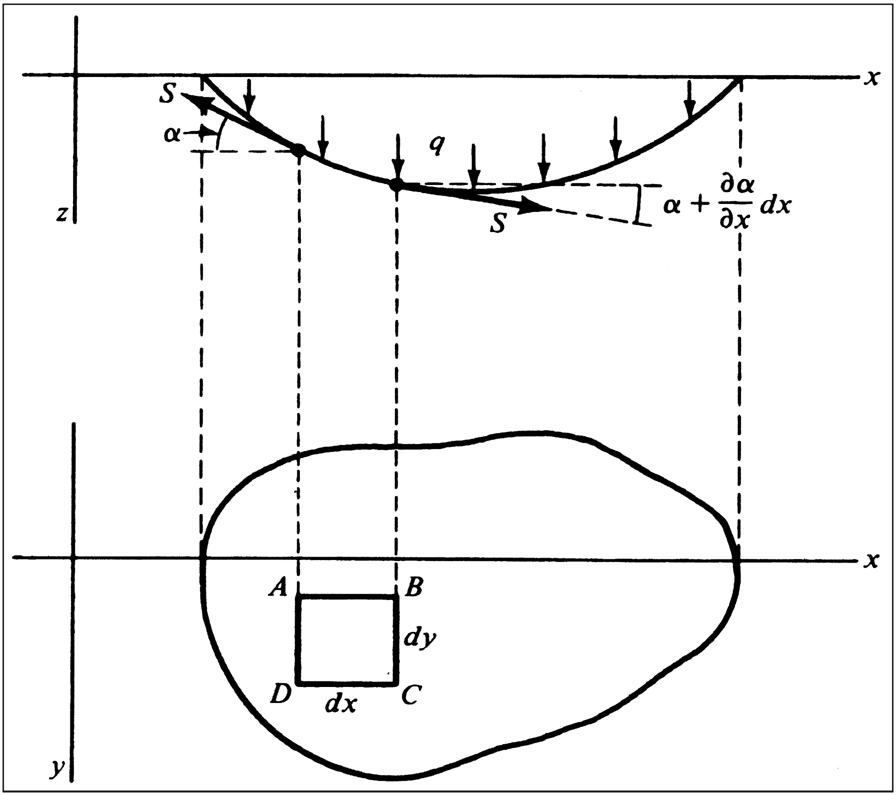

Figure 7-9.1

and similarly the net vertical force due to the tension S (assumed to remain constant for sufficiently small values of q) acting along edge BC is

$$S\,dy \tan\left(\alpha + \frac{\partial \alpha}{\partial x}\,dx\right) = S\,dy \frac{\partial}{\partial x}\left(z + \frac{\partial z}{\partial x}\,dx\right)$$

Similarly, for edges AB and DC we obtain

$$-S\,dx\frac{\partial z}{\partial y} \qquad S\,dx\frac{\partial}{\partial y}\left(z + \frac{\partial z}{\partial y}\,dy\right)$$

Consequently, summation of force in the vertical direction yields for equilibrium of the membrane element $dx\,dy$:

$$S\frac{\partial^2 z}{\partial x^2}\,dx\,dy + S\frac{\partial^2 z}{\partial y^2}\,dx\,dy + q\,dx\,dy = 0$$

or

$$\nabla^2 z = -\frac{q}{S}$$

Prandtl showed that the shearing–stress components in a straight elastic bar in torsion may be related to the slopes of a membrane (soap film) extended over a hole in a flat plate and subjected to a small pressure q, the hole having the shape of the cross section of the bar and the membrane being attached to the boundary of the hole.

By comparison of Eqs. (7-9.1) and (7-9.2), we arrive at the following analogous quantities:

$$z = c\phi \qquad \frac{q}{S} = c2G\beta \tag{7-9.3}$$

where c is a constant of proportionality. Hence,

$$\frac{z}{q/S} = \frac{\phi}{2G\beta} \qquad \phi = \frac{2G\beta S}{q}z \tag{7-9.4}$$

Accordingly, the membrane displacement z is proportional to the Prandtl stress function ϕ, and because the shearing–stress components τ_{xz}, τ_{yz} are equal to the appropriate derivatives of ϕ with respect to x and y [see Eqs. (7-3.3)], it follows that the stress components are proportional to the derivatives of the membrane displacement z with respect to the coordinates (x, y) in the flat plate to which the membrane is attached (Fig. 7-9.1). In other words, the stress components at a point (x, y) of the bar are proportional to the slopes of the membrane at the corresponding point (x, y) of the membrane. Consequently, the distribution of shear–stress components in the cross section of the bar is easily visualized by forming a mental image of the slope of the corresponding membrane. Furthermore, for simply connected cross sections, because z is proportional to ϕ, by Eqs. (7-3.6) and (7-9.4) we note that the twisting moment M is proportional to the volume enclosed by the membrane and the (x, y) plane (Fig. 7-9.1).

For the multiply connected cross section, additional conditions arise. For example, consider the cross section shown in Fig. 7-6.1. For this cross section, Eq. (7-6.12) shows that the twisting moment M is proportional to the integral of ϕ over R plus twice the sum of the products of area of the holes and the corresponding constant values of ϕ on the boundaries of the holes. With regard to the membrane analogy, one must then consider a membrane stretched over region R in such a manner that the membrane has a constant value on a boundary of a hole. Such an effect may be obtained if one stretches a membrane over a flat plate P_0 with a cutout corresponding to region R and with flat plates $P_1, P_2, \ldots, P_n$ placed over the holes $A_1, A_2, \ldots, A_n$, the plates $P_1, P_2, \ldots, P_n$ having appropriate heights $z_1, z_2, \ldots, z_n$ with respect to the holes $A_1, A_2, \ldots, A_n$. For example, for a cross section with a single tubular hole, the equivalent membrane is shown in Fig. 7-9.2. This simple idea can be extended to n holes. On the basis of the directional derivative concept [see Section 7-8 and particularly Eqs. (7-8.1)] and the membrane analogy, we see that for a curve C on the membrane defined by $z = \text{constant}$ (i.e., for $\phi = \text{constant}$) the shear–stress resultant τ is everywhere tangent to the curve (Fig. 7-9.3), where by Eq. (7-8.1),

$$\tau = -\frac{\partial \phi}{\partial n} = -\frac{d\phi}{dn} \quad \text{on } C \tag{7-9.5}$$

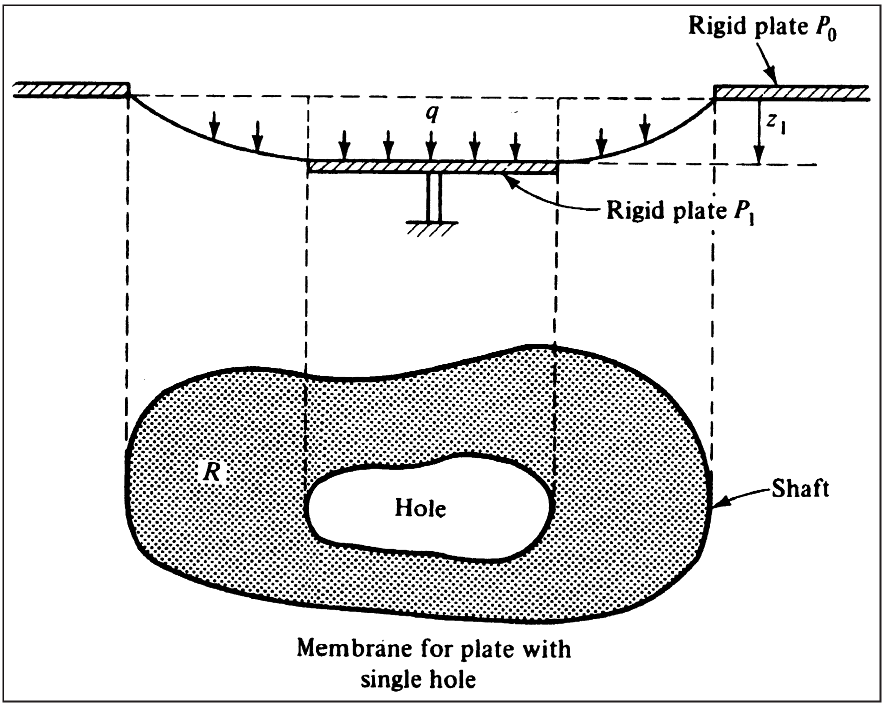

Figure 7-9.2

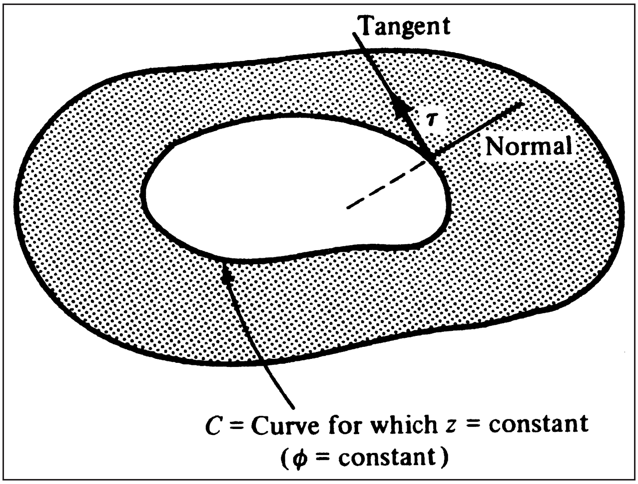

Figure 7-9.3

Considering the equilibrium of the part of the membrane enclosed by C, we find

$$qA = \int S \sin\theta \, ds \tag{7-9.6}$$

where A denotes the plane area bounded by C (Fig. 7-9.4).

By Fig. 7-9.4 and Eqs. (7-9.4) and (7-9.5), we have

$$\sin\theta = -\frac{\partial z}{\partial n} = -\frac{d\phi}{dn}\frac{q}{2G\beta S} = \frac{\tau q}{2G\beta S} \tag{7-9.7}$$

Hence, Eqs. (7-9.6) and (7-9.7) yield

$$\int_C \tau \, ds = 2G\beta A \tag{7-9.8}$$

Accordingly, for multiply connected regions Eq. (7-9.8) becomes (see Figs. 7-6.1 and 7-9.2)

$$\int_{C_i} \tau \, ds = 2G\beta A_i \tag{7-9.9}$$

where C_i denotes the boundary of the plane area A_i.

Several cross sections and their associated membranes are shown schematically in Fig. 7-9.5.

Some useful conclusions may be drawn from consideration of Fig. 7-9.5. For example, noting that by Eqs. (7-4.6) and (7-6.12)

$$M = 2\iint\limits_R \phi \, dx \, dy + 2\sum_{i=1}^{k} K_i A_i = C\beta \tag{7-9.10}$$

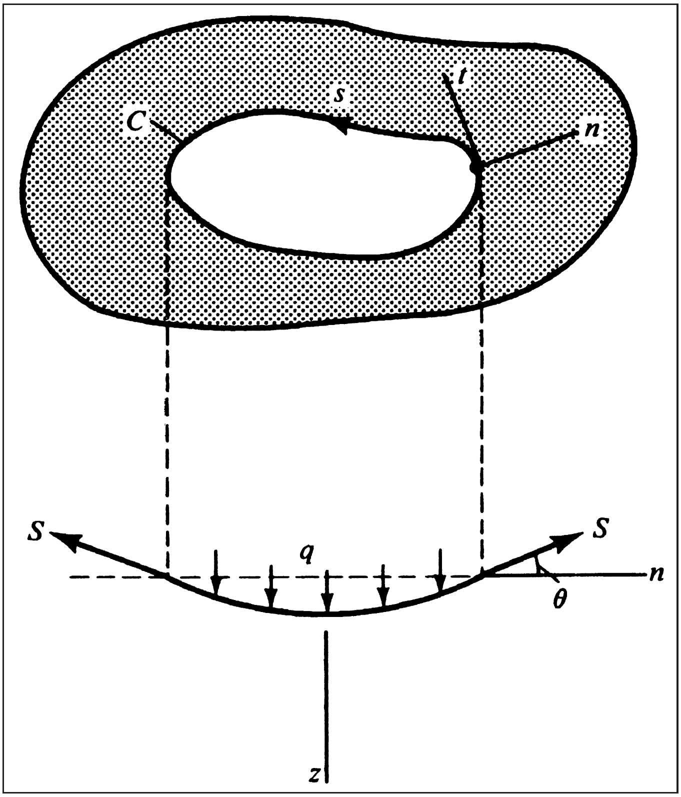

Figure 7-9.4

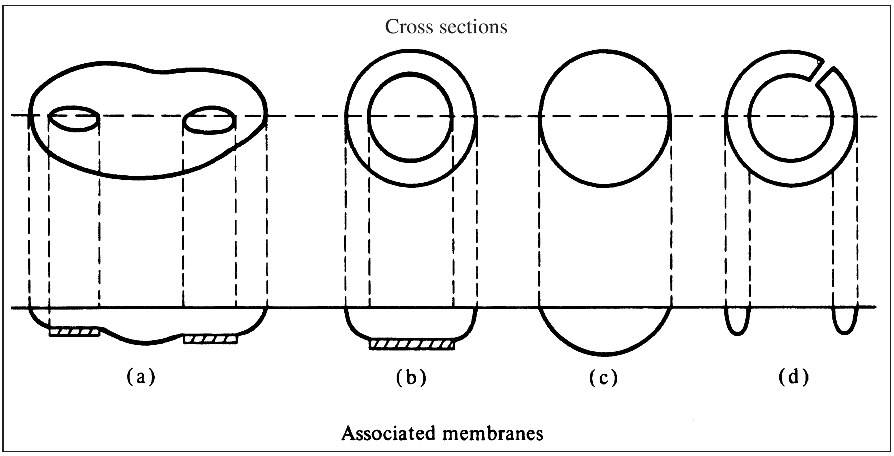

Figure 7-9.5

it appears from Fig. 7-9.5 that for a bar with circular cross section and a given angle of twist (i.e., for a given pressure q and tension S for the associated membrane), the required moment M is not changed as greatly by cutting a concentric circular hole in the shaft as it is cutting a concentric circular hole *and* slit in the shaft (Figs. 7-9.5b, c, and d). Calculations bear out this observation.

Certain kinds of approximations may also be suggested by examination of the membrane. For example, if the wall thickness of a circular tube is small (Fig. 7-9.5b), then by Eq. (7-9.10) we have, with $k = 1$,

$$M = 2 \iint_R \phi \, dx \, dy + 2K_1 A_1 \approx 2K_1 A_1 \tag{7-9.11}$$

where K_1 is the value of ϕ on the boundary of the hole and A_1 is the area of the hole. Other approximations of this type are often employed in practice (Weber and Günther, 1958).

Example 7-9.1. Narrow Rectangular Cross Section. Consider a bar subjected to torsion. Let the cross section of the bar be a solid rectangle with width $2a$ and depth $2b$, where $b \gg a$ (Fig. E7-9.1). The associated membrane is shown in Fig. E7-9.2.

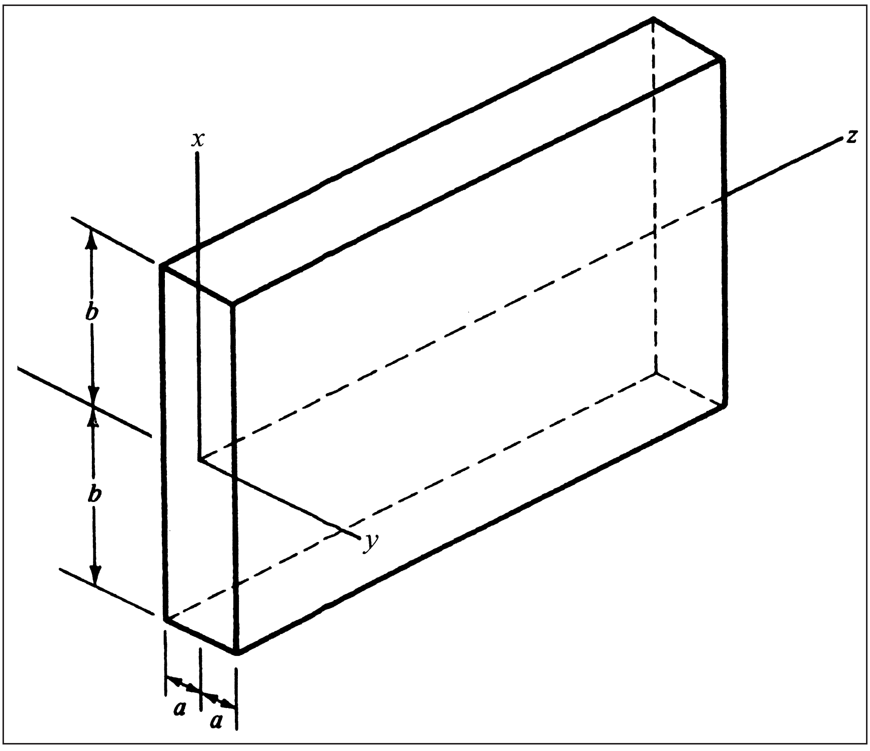

Figure E7-9.1

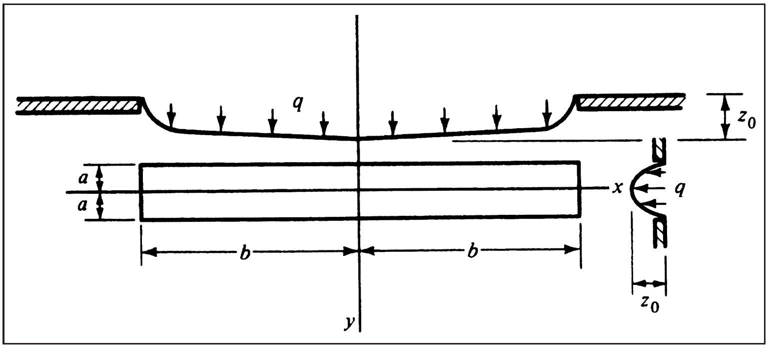

Figure E7-9.2

Except for the region near $x = \pm b$, the membrane deflection is approximately independent of x. For a given x, the deflection with respect to y is assumed to be parabolic. Then

$$z = z_0 \left[1 - \left(\frac{y}{a} \right)^2 \right] \tag{a}$$

Hence,

$$\nabla^2 z = -\frac{2z_0}{a^2} \tag{b}$$

By Eqs. (b), (7-9.2), and (7-9.3), we may write $\nabla^2 z = -2z_0/a^2 = -2cG\beta$ or

$$\phi = G\beta a^2 \left[1 - \left(\frac{y}{a} \right)^2 \right] \tag{c}$$

Consequently, Eqs. (7-3.3) yield

$$\tau_{xz} = \frac{\partial \phi}{\partial y} = -2G\beta y \qquad \tau_{yz} = 0 \tag{d}$$

and the last of Eqs. (7-3.6) yields

$$M = 2 \int_{-b}^{b} \int_{-a}^{a} \phi \, dx \, dy = \frac{16}{3} G\beta a^3 b \tag{e}$$

By Eqs. (d), we note that the maximum value of $|\tau_{xz}|$ is $\tau_{\max} = 2G\beta a$ for $y = \pm a$.

In summary, we note that the solution is approximate, and in particular the boundary conditions for $x = \pm b$ are not satisfied. See also Timoshenko (1983) for the case of a narrow trapezoid.

Problem Set 7-9

1. A torsion bar has a cross section in the shape of an isosceles triangle of height h and base $2b$, with $h \gg b$. Let (x, y) axes be defined such that the origin is at the center of the base, with the x axis in the height direction. Define the torsion function to be $\phi = Gb^2\beta[1 - (y/b)^2]$, based upon the membrane of the cross section.

(a) Derive expressions for the corresponding stress components.

(b) Determine the formula for the torsional rigidity in terms of G, b, h.

(c) Examine the boundary conditions and discuss them.

7-10 Solution by Method of Series. Rectangular Section

In Example 7-9.1 the torsion problem of a bar with narrow rectangular cross section was approximated by noting the deflection of the corresponding membrane. In this section we again consider the rectangular section $-a \leq x \leq a, -b \leq y \leq b$, but we discard the restriction $a \ll b$ (Fig. 7-10.1).

By visualizing the membrane corresponding to the cross section of Fig. 7-10.1, we note that the torsion stress function ϕ must be even in x and y. Also, we recall

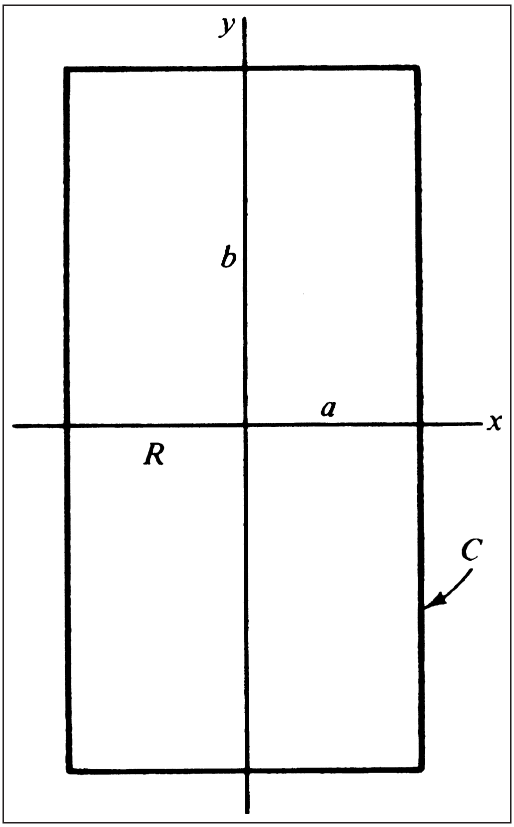

Figure 7-10.1

that in terms of ϕ the torsion problem is defined by the equations

$$\begin{aligned} \nabla^2\phi &= -2G\beta \quad \text{over } R \\ \phi &= 0 \quad \text{on } C \end{aligned} \tag{7-10.1}$$

By Example 7-9.1, we have seen that $G\beta(a^2 - x^2)$ is a particular integral of the first of Eqs. (7-10.1). Accordingly, we take the stress function ϕ in the form [see also Eq. (7-3.20)]

$$\phi = G\beta(a^2 - x^2) + V(x, y) \tag{7-10.2}$$

where $V(x, y)$ is an even function of (x, y). Substitution of Eq. (7-10.2) into Eqs. (7-10.1) yields

$$\begin{aligned} \nabla^2 V &= 0 \quad \text{over } R \\ V &= 0 \quad \text{for } x = \pm a \\ V &= G\beta(x^2 - a^2) \quad \text{for } y = \pm b \end{aligned} \tag{7-10.3}$$

Equations (7-10.3) represent a special case of the Dirichlet problem (Section 7-2).

We seek solutions of Eqs. (7-10.3) by the method of separation of variables. Thus, we take

$$V = f(x)g(y) \tag{7-10.4}$$

where $f(x)$ and $g(y)$ are functions of x and y, respectively. The first of Eqs. (7-10.3) and (7-10.4) yield

$$\nabla^2 V = gf'' + g''f = 0 \tag{7-10.5}$$

where primes denote derivatives with respect to x or y. In order that Eq. (7-10.5) be satisfied, we must have

$$\frac{f''}{f} = -\frac{g''}{g} = -\lambda^2 \tag{7-10.6}$$

where λ^2 is a positive constant. Hence,

$$f'' + \lambda^2 f = 0 \qquad g'' - \lambda^2 g = 0 \tag{7-10.7}$$

The solutions of Eqs. (7-10.7) are

$$\begin{aligned} f &= A\cos\lambda x + B\sin\lambda x \\ g &= C\cosh\lambda y + D\sinh\lambda y \end{aligned} \tag{7-10.8}$$

Because V must be even in x and y, it follows that $B = D = 0$. Consequently, the function V takes on the form [Eq. (7-10.4)]

$$V = A\cos\lambda x \,\cosh\lambda y \tag{7-10.9}$$

where A denotes an arbitrary constant.

To satisfy the second of Eqs. (7-10.3), Eq. (7-10.9) yields the result

$$\lambda = \frac{n\pi}{2a} \qquad n = 1, 3, 5, \ldots \tag{7-10.10}$$

To satisfy the last of Eqs. (7-10.3) we employ the method of superposition ($\nabla^2 V = 0$ is a linear, homogeneous partial differential equation), and we write

$$V = \sum_{n=1,3,5,\ldots}^{\infty} A_n \cos\frac{n\pi x}{2a}\cosh\frac{n\pi y}{2a} \tag{7-10.11}$$

Equation (7-10.11) satisfies $\nabla^2 V = 0$ in R, provided the series converges and is termwise differentiable (Brown and Churchill, 2007). Equation (7-10.11) automatically satisfies the boundary condition for $x = \pm a$. The boundary condition for $y = \pm b$ yields the condition [Eqs. (7-10.3)]

$$\sum_{n=1,3,5,\ldots}^{\infty} C_n \cos\frac{n\pi x}{2a} = G\beta(x^2 - a^2) = h(x) \tag{7-10.12}$$

where

$$C_n = A_n \cosh\frac{n\pi b}{2a} \tag{7-10.13}$$

By the theory of Fourier series, we multiply both sides of Eq. (7-10.12) by $\cos(n\pi x/2a)$ and integrate between the limits $-a$ and $+a$ to obtain the coefficients C_n as follows:

$$C_n = \frac{1}{a}\int_{-a}^{a} h(x)\cos\frac{n\pi x}{2a}\,dx \tag{7-10.14}$$

Because $h(x)\cos(n\pi x/2a) = G\beta(x^2 - a^2)\cos(n\pi x/2a)$ is symmetrical about $x = 0$, we may write

$$C_n = \frac{2G\beta}{a}\int_0^a (x^2 - a^2)\cos\frac{n\pi x}{2a}\,dx$$

or

$$C_n = \frac{2G\beta}{a}\int_0^a x^2\cos\frac{n\pi x}{2a}\,dx - 2G\beta a\int_0^a \cos\frac{n\pi x}{2a}\,dx$$

Integration yields [see Pierce and Foster (1956), Formula 350, or Ryzhik et al. (1994)]:

$$C_n = \frac{-32G\beta a^2(-1)^{(n-1)/2}}{n^3\pi^3} \tag{7-10.15}$$

Hence, Eqs. (7-10.11), (7-10.13), and (7-10.15) yield

$$A_n = -\frac{32G\beta a^2(-1)^{(n-1)/2}}{n^3\pi\cosh\dfrac{n\pi b}{2a}} \tag{7-10.16}$$

and

$$\phi = G\beta(a^2 - x^2) - \frac{32G\beta a^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}\cos\dfrac{n\pi x}{2a}\cosh\dfrac{n\pi y}{2a}}{n^3\cosh\dfrac{n\pi b}{2a}} \tag{7-10.17}$$

Note that as $\cosh x = 1 + x^2/2! + x^4/4! + \cdots$, the series in Eq. (7-10.17) goes to zero if $b/a \to \infty$ (i.e., if the section is very narrow $b \gg a$). Then Eq. (7-10.17) reduces to

$$\phi \approx G\beta(a^2 - x^2) \tag{7-10.18}$$

This result verifies the assumption employed in Example 7-9.1 for the slender rectangular cross section.

By Eqs. (7-3.3) and (7-10.17), we obtain

$$\begin{aligned}
\tau_{xz} &= \frac{\partial\phi}{\partial y} = -\frac{16G\beta a}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}\cos\dfrac{n\pi x}{2a}\sinh\dfrac{n\pi y}{2a}}{n^3\cosh\dfrac{n\pi b}{2a}}\\
\tau_{yz} &= -\frac{\partial\phi}{\partial x} = 2G\beta x - \frac{16G\beta a}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}\sin\dfrac{n\pi x}{2a}\cosh\dfrac{n\pi y}{2a}}{n^3\cosh\dfrac{n\pi b}{2a}}
\end{aligned} \tag{7-10.19}$$

By Eqs. (7-3.6) and (7-10.17), the twisting moment is

$$M = 2\int_{-b}^{b}\int_{-a}^{a}\phi\,dx\,dy = C\beta = GJ\beta \tag{7-10.20}$$

where J, a factor dependent on geometry of the cross section, is

$$\begin{aligned}
J &= 2\int_{-b}^{b}\int_{-a}^{a}(a^2 - x^2)dxdy\\
&\quad - \frac{64a^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}\frac{(-1)^{(n-1)/2}}{n^3\cosh\dfrac{n\pi b}{2a}}\int_{-b}^{b}\int_{-a}^{a}\left(\cos\frac{n\pi x}{2a}\cosh\frac{n\pi y}{2a}\right)dxdy
\end{aligned}$$

Integration yields [see Pierce and Foster (1956), Formula 489, or Ryzhik et al. (1994)]:

$$J = \frac{(2a)^3(2b)}{3}\left[1 - \frac{192}{\pi^5}\left(\frac{a}{b}\right)\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}\right] \tag{7-10.21}$$

The factor outside the brackets on the right side of Eq. (7-10.21) is an approximation for a thin rectangular cross section because the series goes to zero as b/a becomes large.

In general, Eq. (7-10.21) may be written in the form

$$J = k_1(2a)^3(2b) \tag{7-10.22}$$

where

$$k_1 = \frac{1}{3}\left[1 - \frac{192}{\pi^5}\left(\frac{a}{b}\right)\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}\right] \tag{7-10.23}$$

Equation (7-10.20) may then be written in the form

$$M = G\beta k_1(2a)^3(2b) \tag{7-10.24}$$

Values of k_1 for various ratios of b/a are given by Timoshenko and Goodier (1970).

Problem Set 7-10

1. Verify Eq. (7-10.21).

2. With $b > a$, show that the maximum shear for the rectangular cross section (Fig. 7-10.1) occurs at $x = a$, $y = 0$. Hence, show that

$$\tau_{\max} = 2G\beta ak$$

where

$$k = 1 - \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2\cosh(n\pi b/2a)}$$

3. Derive the warping function for the rectangular cross section. Consider the case $a = b$, and sketch in contour lines.

4. Calculate τ_{xz}, τ_{yz} at the indicated points in the cross section (Fig. P7-10.4). Calculate J [Eq. (7-10.21)].

5. Consider a shaft with a sector cross section with angle α and radius a (Fig. P7-10.5). Let (r, β) denote polar coordinates. Let the torsion stress function ϕ be given by

$\phi = V - G\theta r^2/2$, where here θ denotes the unit angle of twist. By the method employed in Section 7-10, show that

$$\phi = \frac{G\theta}{2}\left[-r^2\left(1-\frac{\cos 2\beta}{\cos\alpha}\right)+\frac{16a^2\alpha^2}{\pi^3}\sum_{n=1,3,5,\ldots}^{\infty}(-1)^{(n+1)/2}\left(\frac{r}{a}\right)^{n\pi/\alpha}\right.$$

$$\left.\times\frac{\cos(n\pi\beta/\alpha)}{n\left(n+\dfrac{2\alpha}{\pi}\right)\left(n-\dfrac{2\alpha}{\pi}\right)}\right]$$

6. Consider the torsion problem of a shaft whose cross section is shown in Fig. P7-10.6. Assume a stress function of the form $\phi = V - \frac{1}{2}G\theta r^2$, where V is a function of r alone, G denotes the shear modulus, θ denotes the angle of twist per unit length of the shaft, and r is the radial polar coordinate. For $h/a \ll 1$, derive an expression for V in terms of a, h, and r. Hence, derive an expression for the shearing stress τ. Discuss the validity of the solution in the vicinity of $\beta = \pi/2$.

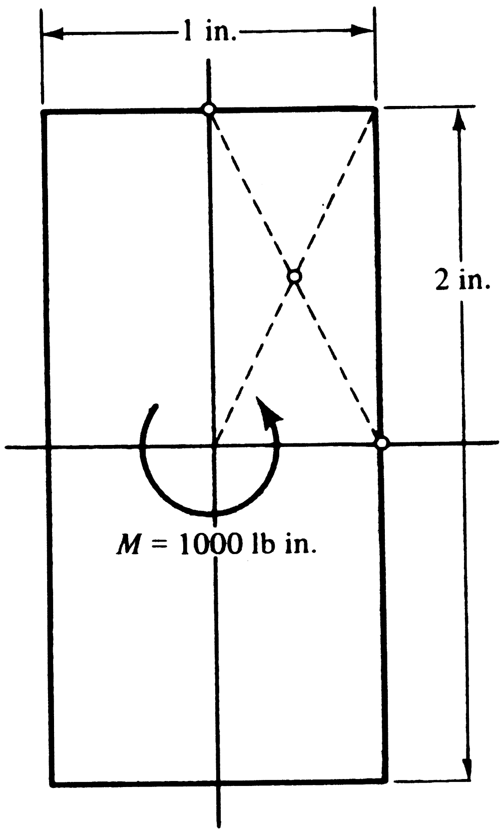

Figure P7-10.4

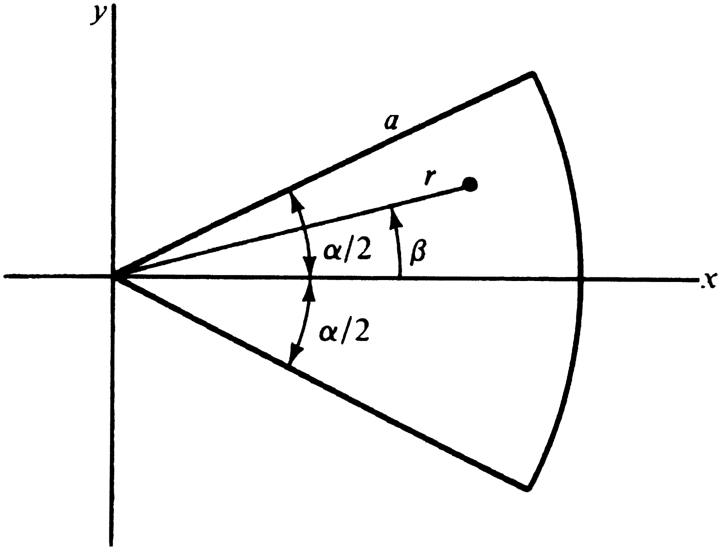

Figure P7-10.5

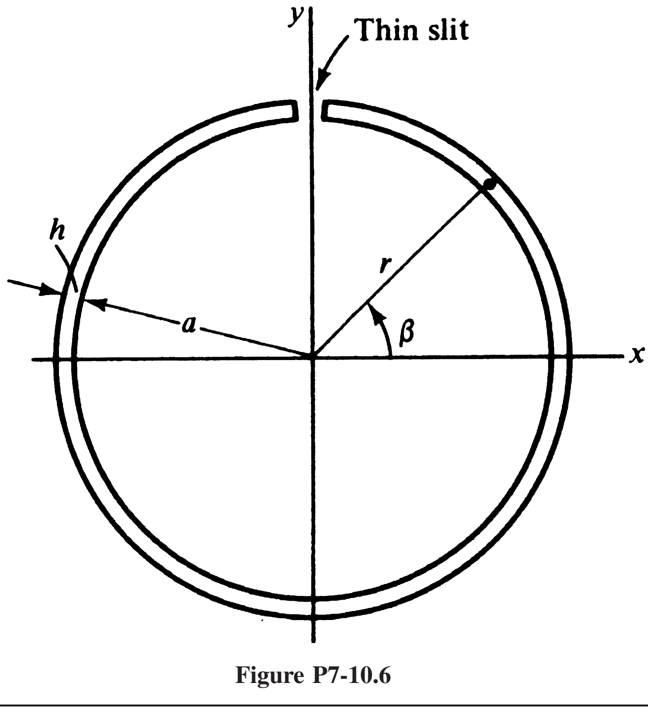

Figure P7-10.6

7-11 Bending of a Bar Subjected to Transverse End Force

Consider a prismatic elastic bar fixed[6] at the end $z = 0$ and subjected to a lateral force P at the end $z = L$ (Fig. 7-11.1). The cross section of the bar is contained in region R bounded by the surface S. We restrict discussion to the case of simply connected regions R (see Sections 7-2 and 7-6).

We let the origin of axes (x, y, z) be located arbitrarily in the cross section at $z = 0$. Furthermore, we take the x axis coincident with the line of action of force P. Then summation of forces on the end face $z = L$ yields

$$P_x = \iint \tau_{zx}\, dx\, dy = P \qquad P_y = P_z = M_x = M_y = M_z = 0 \tag{7-11.1}$$

Accordingly, overall equilibrium of any portion of the bar (say, between the sections $z = z$, $z = L$; Figs. 7-11.1 and 7-11.2) requires that

$$\begin{aligned} &\iint \tau_{zx} dx dy = P \qquad \iint \sigma_z x dx dy = -P(L - z) \\ &\iint \tau_{zx} dx dy = \iint \sigma_z dx dy = \iint y \sigma_z dx dy \\ &\qquad = \iint (x\tau_{zy} - y\tau_{zx}) dx dy = 0 \end{aligned} \tag{7-11.2}$$

It follows from the first two of Eqs. (7-11.2) that τ_{zx} and σ_z are not zero. Also, in general, τ_{yz} is not zero by the last of Eqs. (7-11.2).

Following the semi-inverse method of Saint-Venant, we seek solutions such that σ_z, τ_{zx}, τ_{zy} are the only nonvanishing stress components; that is, we assume that

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \tag{7-11.3}$$

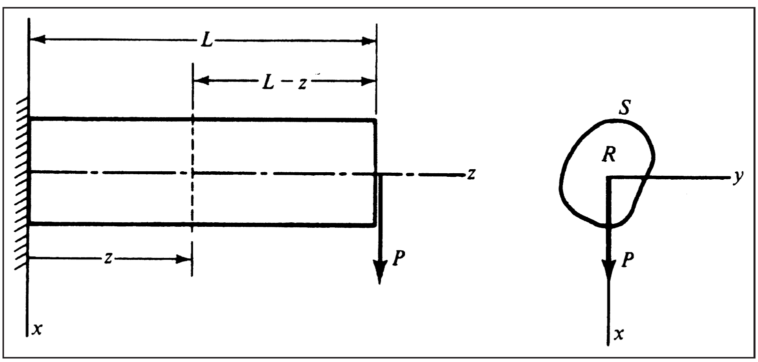

Figure 7-11.1

[6]For example, the conditions at $z = 0$ may be taken such that the displacement components $u = v = w = 0$ at $x = y = z = 0$, and the rotation $\omega = 0$ at $x = y = z = 0$.

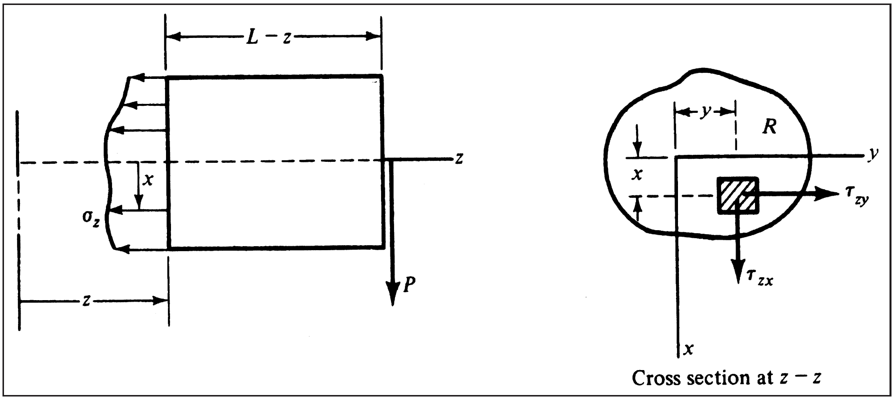

Figure 7-11.2

Furthermore, we take the simplest linear dependence on (x, y) for the component σ_z; that is, we assume that σ_z is proportional to $Ax + By + C$, where A, B, C are constants. More explicitly, on the basis of the second of Eqs. (7-11.2) we assume that

$$\sigma_z = P(Ax + By + C)(L - z) \tag{7-11.4}$$

Substitution of Eq. (7-11.4) into Eqs. (7-11.2) yields the result

$$\begin{aligned} AI_{yy} + BI_{xy} + CS_y &= -1 \\ AI_{xy} + BI_{xx} + CS_x &= 0 \\ AS_y + BS_x + CS_0 &= 0 \end{aligned} \tag{7-11.5}$$

where (I_{xx}, I_{yy}, I_{xy}) and (S_x, S_y) are the moments of inertia and the first moments, respectively, of the area of the cross section of the bar relative to axes (x, y), and S_0 is the area of the cross section of the bar.

Equations (7-11.5) are three linear algebraic equations in the unknowns A, B, C. Solving Eqs. (7-11.5), we obtain

$$\begin{aligned} A &= -\frac{I_{xx}S_0 - S_x^2}{\Delta} = \frac{S_x^2 - I_{xx}S_0}{\Delta} \\ B &= \frac{I_{xy}S_0 - S_xS_y}{\Delta} \\ C &= \frac{I_{xx}S_y - I_{xy}S_x}{\Delta} = -A\overline{x} - B\overline{y} \end{aligned} \tag{7-11.6}$$

where

$$\Delta = \begin{vmatrix} I_{yy} & I_{xy} & S_y \\ I_{xy} & I_{xx} & S_x \\ S_y & S_x & S_0 \end{vmatrix} \tag{7-11.7}$$

and $\overline{x}, \overline{y}$, denote the coordinates of the center of gravity of the area of the cross section.

In the absence of body forces ($X = Y = Z = 0$). Equations (7-1.1), (7-11.3), and (7-11.4) yield

$$\frac{\partial \tau_{xz}}{\partial z} = \frac{\partial \tau_{yz}}{\partial z} = 0$$
$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = P(Ax + By + C) \tag{7-11.8}$$

It follows by the first two of Eqs. (7-11.8) that τ_{xz}, τ_{yz} are independent of z. Furthermore, the last of Eqs. (7-11.8) may be written in the form

$$\frac{\partial}{\partial x}\left[\tau_{xz} - \frac{P}{2}(Ax^2 + Cx)\right] + \frac{\partial}{\partial y}\left[\tau_{yz} - \frac{P}{2}(By^2 + Cy)\right] = 0 \tag{7-11.9}$$

By the theory of Section 1-19, Eq. (7-11.9) represents necessary and sufficient conditions that a function F exist such that

$$\tau_{xz} - \frac{P}{2}(Ax^2 + Cx) = \frac{P}{2}\frac{\partial F}{\partial y}$$
$$\tau_{yz} - \frac{P}{2}(By^2 + Cy) = -\frac{P}{2}\frac{\partial F}{\partial x}$$

or

$$\tau_{xz} = \frac{P}{2}\left[\frac{\partial F}{\partial y} + Ax^2 + Cx\right]$$
$$\tau_{yz} = \frac{P}{2}\left[-\frac{\partial F}{\partial x} + By^2 + Cy\right] \tag{7-11.10}$$

Hence, if τ_{xz} and τ_{yz} are expressed in the form of Eqs. (7-11.10), the equations of equilibrium are satisfied. Furthermore, as τ_{xz} and τ_{yz} are independent of z, it follows that $F = F(x, y)$. The governing equations for F are the compatibility equations [Eqs. (7-1.5)] and the boundary conditions [Eqs. (7-1.4)]. Substitution of Eqs. (7-11.3), (7-11.4), and (7-11.10) into Eqs. (7-1.5) yields

$$\frac{\partial}{\partial y}(\nabla^2 F) = -\frac{2\nu A}{1+\nu}$$
$$\frac{\partial}{\partial x}(\nabla^2 F) = \frac{2\nu B}{1+\nu}$$

Integration yields

$$\nabla^2 F = \frac{2\nu}{1+\nu}(Bx - Ay) - 2C_0 \qquad \text{over } R \tag{7-11.11}$$

where C_0 is a constant of integration that may be interpreted physically [see Section 7-12; see also Eq. (7-11.29)].

The boundary conditions [Eqs. (7-1.4a)] reduce to $l\tau_{xz} + m\tau_{yz} = 0$ or

$$\tau_{xz}\frac{dy}{ds} - \tau_{yz}\frac{dx}{ds} = 0 \tag{7-11.12}$$

where [see Eq. (7-2.7) and Fig. 7-2.2]

$$l = \frac{dy}{ds} \qquad m = -\frac{dx}{ds} \tag{7-11.13}$$

Substitution of Eqs. (7-11.10) into Eq. (7-11.12) yields

$$\frac{\partial F}{\partial s} = (By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds} \qquad \text{on } S \tag{7-11.14}$$

Equation (7-11.11), which holds over region R, and Eq. (7-11.14), which holds on the lateral surface S, are the defining equations for F.

The above results may be simplified somewhat by noting the nature of Eqs. (7-11.11) and (7-11.14), and representing F in terms of two new functions. Thus, we set

$$F = \Gamma + C_0\phi \tag{7-11.15}$$

Then Eqs. (7-11.11) and (7-11.14) yield

$$\left.\begin{aligned} \nabla^2\phi &= -2 \\ \nabla^2\Gamma &= \frac{2\nu}{1+\nu}(Bx - Ay) \end{aligned}\right\} \qquad \text{over } R \tag{7-11.16}$$

and

$$\left.\begin{aligned} &\frac{\partial\phi}{\partial s} = 0 \quad \text{or} \quad \phi = 0 \qquad \text{(Section 7-6)} \\ &\frac{\partial\Gamma}{\partial s} = (By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds} \end{aligned}\right\} \qquad \text{on } S \tag{7-11.17}$$

By Eqs. (7-11.16) and (7-11.17), we see from the theory of Section 7-3 that ϕ (except for the constant factor $G\beta$) is the Prandtl stress function. Accordingly, the problem of the bending of the cantilever bar subjected to transverse end load may be expressed in terms of the Prandtl stress function of torsion and an auxiliary function Γ, which must satisfy the last of Eqs. (7-11.16) and (7-11.17). The function Γ is called the *flexural function* or the *bending function*.

If (x, y) are axes of symmetry with origin at the centroid of the section (then x, y are called principal axes of the cross section),

$$B = C = 0 \qquad A = -\frac{1}{I_{yy}} = -\frac{1}{I} \tag{7-11.18}$$

where I denotes I_{yy}. Then, analogous to the principal axes theories of stress ($\sigma_{ij} = 0, i \neq j$) and strain ($\epsilon_{ij} = 0, i \neq j$), axes (x, y) are called principal axes of inertia ($I_{xy} = 0$). With x a principal axis of inertia of the cross section, the equations for the flexural function [Eqs. (7-11.16) and (7-11.17)] reduce to

$$\begin{aligned} \nabla^2\Gamma &= \frac{2\nu}{1+\nu}\frac{y}{I} \qquad \text{over } R \\ \frac{\partial\Gamma}{\partial s} &= \frac{x^2}{I}\frac{dy}{ds} \qquad \text{on } S \end{aligned} \tag{7-11.19}$$

For a certain class of problems it is convenient to redefine Γ in terms of two functions, as follows:

$$\Gamma = \frac{1}{P}[\Psi(x, y) + h(y)] \tag{7-11.20}$$

where Ψ is a function of both x and y, and h is a function of y only.[7] Then Eqs. (7-11.19) become

$$\begin{aligned} \nabla^2\Psi &= \frac{2\nu}{1+\nu}\frac{P}{I}y - \frac{df}{dy} \qquad \text{over } R \\ \frac{\partial\Psi}{\partial s} &= \left(\frac{Px^2}{I} - f\right)\frac{dy}{ds} \qquad \text{on } S \end{aligned} \tag{7-11.21}$$

where $f = dh/dy = f(y)$. The objective of the substitution of Eq. (7-11.20) is to arrive at simpler boundary conditions. For example, if we can choose f such that

$$\left(\frac{Px^2}{I} - f\right)\frac{dy}{ds} = 0 \qquad \text{on } S \tag{7-11.22}$$

then

$$\frac{\partial\Psi}{\partial s} = 0 \qquad \text{on } S \tag{7-11.23}$$

and as R is a simply connected region, it follows that we may take (see Section 7-6)

$$\Psi = 0 \qquad \text{on } S \tag{7-11.24}$$

We will employ this technique below to obtain the solution of the flexure problem for the rectangular and the elliptic cross sections.

Alternatively, we may seek solutions of Eq. (7-11.11) by taking a particular integral in the form of a polynomial in x and y. For example, we may express F in the form

$$F = h(x, y) + \frac{\nu}{3(1+\nu)}(Bx^3 - Ay^3) - \frac{1}{2}C_0(x^2 + y^2) \tag{7-11.25}$$

[7]This substitution was employed by Timoshenko (1913) to solve the problem of flexure of certain kinds of cross sections (Sections 7-14 and 7-15).

where $\nabla^2 h = 0$; that is, h is a harmonic function. Then the problem of bending of a bar by transverse end force transforms into seeking a function such that [see Eqs. (7-11.10), (7-11.11), and (7-11.14)]

$$\nabla^2 h = 0 \qquad \text{over region } R$$

$$\frac{\partial h}{\partial s} = (By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds} + \left(\frac{\nu}{1+\nu}Bx^2 - C_0 x\right)\frac{dx}{ds} + \left(\frac{\nu}{1+\nu}Ay^2 + C_0 y\right)\frac{dy}{ds} \qquad \text{on } S \tag{7-11.26}$$

where the stress components are given by

$$\begin{aligned}\tau_{xz} &= \frac{P}{2}\left[\frac{\partial h}{\partial y} + A\left(x^2 - \frac{\nu y^2}{1+\nu}\right) + Cx - C_0 y\right]\\ \tau_{yz} &= \frac{P}{2}\left[-\frac{\partial h}{\partial x} + B\left(y^2 - \frac{\nu x^2}{1+\nu}\right) + Cy + C_0 x\right]\end{aligned} \tag{7-11.27}$$

For principal axes of the cross section, $B = C = 0$, $A = -1/I$, and Eqs. (7-11.26) and (7-11.27) are simplified accordingly.

Determination of the Constant of Integration, C_0. The above formulation of the flexural problem of the bar (cantilever beam) subjected to end force P is complete except for the determination of the integration constant C_0 [Eq. (7-11.11)]. We find that if we substitute Eqs. (7-11.10) into Eqs. (7-11.2), all the equations are satisfied identically with the exception of the last equation, that is,

$$M_z = \iint (x\tau_{yz} - y\tau_{xz})\, dx\, dy = 0 \tag{7-11.28}$$

The constant C_0 must be chosen to satisfy Eq. (7-11.28). Accordingly, if we employ the definitions of Eqs. (7-11.10) and (7-11.15), we obtain, after some calculations,

$$\begin{aligned}C_0 \iint \phi\, dx\, dy = &- \iint \Gamma\, dx\, dy - \frac{1}{2}\iint (By - Ax)xy\, dx\, dy\\ &- \oint \left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dx}{ds}\right] R_s\, ds\end{aligned} \tag{7-11.29}$$

where the double integrals are evaluated over R, the line integral is taken over S, and

$$R_s = \frac{1}{2}\int_0^s (x\, dy - y\, dx) \tag{7-11.30}$$

For principal axes of the cross section, $B = C = 0$, $A = -1/I$, and Eq. (7-11.29) is simplified accordingly. With Eq. (7-11.29), the formulation of the problem of bending of a bar subjected to transverse end load is complete.

In general, $C_0 \neq 0$. Hence, there is twisting of the bar (torsion) when a transverse end load is applied arbitrarily. It is for this reason that the Prandtl torsion function [Eqs. (7-11.15) through (7-11.17)] enters into the bending problem of bars.

The constant C_0 may be related to the average rotation of a cross section of the bar with respect to the axis z. For example, for the state of stress defined above, we obtain by Eqs. (7-1.3)

$$\begin{aligned} \epsilon_x &= \epsilon_y = -\frac{\nu\sigma_z}{E} = -\frac{\nu P}{E}(Ax + By + C)(L - z) \\ \epsilon_z &= \frac{P}{E}(Ax + By + C)(L - z) \\ \gamma_{xy} &= 0 \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz} \end{aligned} \tag{7-11.31}$$

where

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} & \epsilon_y &= \frac{\partial v}{\partial y} & \epsilon_z &= \frac{\partial w}{\partial z} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \end{aligned}$$

and where E is the modulus of elasticity and ν is Poisson's ratio of the material. With Eqs. (7-11.31), the three strain compatibility equations of the type (see Chapter 2, Section 2-16)

$$\frac{\partial^2\epsilon_x}{\partial y^2} + \frac{\partial^2\epsilon_y}{\partial x^2} = \frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

are satisfied identically. Also, the equation

$$2\frac{\partial^2\epsilon_z}{\partial x\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{xz}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right)$$

is satisfied. The remaining two equations of compatibility simplify to

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{xz}}{\partial y}\right) &= -\frac{2\nu PB}{E} \\ \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{xz}}{\partial y}\right) &= \frac{2\nu PA}{E} \end{aligned} \tag{7-11.32}$$

Integration of Eqs. (7-11.32) leads to

$$\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{xz}}{\partial y} = \frac{2\nu P}{E}(-Bx + Ay) + 2K \tag{7-11.33}$$

where K is a constant of integration.

Recalling the definition of γ_{yz}, γ_{xz}, and ω_z in terms of (u, v, w) [see Eqs. (2-15.14) and (2-5.3)], we note that Eq. (7-11.33) may be written as

$$\frac{\partial \omega_z}{\partial z} = \frac{\nu P}{E}(-Bx + Ay) + K$$

The term ω_z is the angle of rotation of an element of volume in the rod about the z axis. The term $\partial \omega_z / \partial z$ is thus the twist of fibers in the rod parallel to the z axis. Integration of the twist over the cross section R of the bar yields the result

$$\frac{\partial \overline{\omega}_z}{\partial z} = \frac{\nu P}{E}(-B\overline{x} + A\overline{y}) + K \tag{7-11.34}$$

where

$$\begin{aligned} \overline{\omega}_z &= \frac{1}{S_0} \iint \omega_z \, dx \, dy \\ \overline{x} &= \frac{1}{S_0} \iint x \, dx \, dy \\ \overline{y} &= \frac{1}{S_0} \iint y \, dx \, dy \end{aligned} \tag{7-11.35}$$

denote, respectively, the average value of the angle of rotation ω_z, the x value of the centroid of the cross section, and the y value of the centroid, and S_0 denotes the area of the cross section. Accordingly, by Eq. (7-11.34), the integration constant K may be related to the average angle of rotation of a cross section about the z axis. Furthermore, if the x axis is an axis of symmetry, $B = \overline{y} = 0$. Then $\partial \overline{\omega}_z / \partial z = K$. However, because x is an axis of symmetry (a principal axis passing through the centroid of the cross section), $\overline{\omega}_z = 0$. Hence, when x is a principal axis passing through the centroid of the cross section, $K = 0$.

Alternatively, the compatibility condition, Eq. (7-11.33), may be expressed in terms of τ_{xz} and τ_{yz} by means of the last two of Eqs. (7-11.31). Then, by Eqs. (7-11.10), the compatibility relation may be formulated in terms of the function F. This latter expression, with Eq. (7-11.11), yields the result

$$C_0 = \frac{E}{(1+\nu)P} K \tag{7-11.36}$$

Accordingly, the above remarks made with regard to K hold also for the constant C_0. For example, the constant C_0 defined by Eqs. (7-11.29) vanishes when x is a principal axis. In general, C_0 is related to the mean rotation $\overline{\omega}_z$ by Eqs. (7-11.34) and (7-11.36). That is,

$$\frac{\partial \overline{\omega}_z}{\partial z} = \frac{(1+\nu)P}{E} \left[\frac{\nu}{1+\nu}(-B\overline{x} + A\overline{y}) + C_0 \right] \tag{7-11.37}$$

Remark on Solution of $\nabla^2\chi = F(x, y)$. The basic equation of the theories of torsion and of bending of bars is of the form

$$\nabla^2\chi = F(x, y) \tag{7-11.38}$$

where χ must satisfy certain requirements on the lateral surface of the bar [see, e.g., Eqs. (7-2.5), (7-2.10), (7-2.13), (7-2.15), (7-3.10), (7-11.11), (7-11.14), (7-11.16), (7-11.17), (7-11.21), and (7-11.26)]. In general, Eq. (7-11.38) is a linear, nonhomogeneous, partial differential equation of second order. Because it is linear, it may be transformed into an equivalent homogeneous equation. The following basic theorem holds for the equivalent homogeneous case ($\nabla^2\chi = 0$) (Brown and Churchill, 2007).

Theorem 7-11.1. *If $\chi_1, \chi_2, \ldots, \chi_n$ are n solutions of a homogeneous linear partial differential equation, then $C_1\chi_1 + C_2\chi_2 + \cdots + C_n\chi_n$ is also a solution, where $C_1, C_2, \ldots, C_n$ are arbitrary constants.*

Any function of x and y that satisfies Eq. (7-11.38) identically is called a *particular* integral. There are in general an infinite number of particular solutions to Eq. (7-11.38). Because of the linear character of Eq. (7-11.38), the sum of a complementary function ($\chi_1, \chi_2, \ldots, \chi_n$) and *any* particular integral will satisfy Eq. (7-11.38). In the torsion and bending problems of bars, the solution of Eq. (7-11.38) must also satisfy the boundary conditions. In general, the boundary conditions are extremely complex. Particularly, we have seen that in general the bending problem of a bar entails both bending and twisting [see Eqs. (7-11.11), (7-11.15), (7-11.16), and (7-11.17)]. In Section 7-13 we will examine explicitly the conditions under which a bar loaded by a transverse end force will bend without twisting of its end section about the z axis. By application of the conditions for which twisting of the end section is eliminated, we obtain some simplification of the boundary conditions.

Problem Set 7-11

1. Derive Eqs. (7-11.6).
2. Derive Eq. (7-11.29). Simplify the results for principal axes of the cross section.
3. Verify that all but the last of Eqs. (7-11.2) are satisfied by Eq. (7-11.10).

7-12 Displacement of a Cantilever Beam Subjected to Transverse End Force

In this section we derive formulas for the (x, y, z) displacement components (u, v, w) for the stress components defined in Section 7-11. Hence, our task is to integrate Eqs. (7-11.31).

By the third of Eqs. (7-11.31) we have

$$\frac{\partial w}{\partial z} = \frac{P}{E}(Ax + By + C)(L - z)$$

Integration yields

$$w = \frac{PL}{E}(Ax + By + C)z - \frac{P}{2E}(Ax + By + C)z^2 + f(x, y) \tag{7-12.1}$$

where $f(x, y)$ denotes a function of (x, y) only.

To obtain expressions for the displacement components (u, v), we consider simultaneously certain of Eqs. (7-11.31) and Eq. (7-12.1). In the development of these expressions it is convenient to employ the following transformations. As noted by Eqs. (7-11.25) and (7-11.26), the bending problem of the bar may be defined in terms of a harmonic function h. Now we introduce a function $g(x, y)$, the conjugate harmonic of $h(x, y)$, defined by the relations

$$\frac{\partial g}{\partial x} = \frac{\partial h}{\partial y} \qquad \frac{\partial g}{\partial y} = -\frac{\partial h}{\partial x} \qquad \nabla^2 g = 0 \tag{7-12.2}$$

Then, with Eqs. (7-11.27) and (7-12.2) and the last of Eqs. (7-11.31), we obtain

$$\begin{aligned} \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{(1+\nu)P}{E}\left[\frac{\partial g}{\partial x} + A\left(x^2 - \frac{\nu y^2}{1+\nu}\right) + Cx - C_0 y\right] \\ \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{(1+\nu)P}{E}\left[\frac{\partial g}{\partial y} + B\left(y^2 - \frac{\nu x^2}{1+\nu}\right) + Cy + C_0 x\right] \end{aligned} \tag{7-12.3}$$

By the first of Eqs. (7-11.31) and (7-12.3) and Eq. (7-12.1), we find

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{\nu P}{E}(Ax + By + C)(L - z) \\ \frac{\partial u}{\partial z} &= \frac{(1+\nu)P}{E}\left[\frac{\partial g}{\partial x} + A\left(x^2 - \frac{\nu y^2}{1+\nu}\right) + Cx - C_0 y\right] \\ &\quad - \frac{PAL}{E}z + \frac{PA}{2E}z^2 - \frac{\partial f}{\partial x} \end{aligned} \tag{7-12.4}$$

Equations (7-12.4) are compatible, provided that

$$\frac{\partial^2(g - \overline{f})}{\partial x^2} = -\frac{(2+\nu)A}{1+\nu}x + \frac{\nu B}{1+\nu}y - \frac{C}{1+\nu} \tag{7-12.5}$$

where

$$\overline{f} = \frac{Ef}{(1+\nu)P} \tag{7-12.6}$$

Similarly, we find

$$\frac{\partial v}{\partial y} = -\frac{\nu P}{E}(Ax + By + C)(L - z)$$

$$\frac{\partial v}{\partial z} = \frac{(1+\nu)P}{E}\left[\frac{\partial g}{\partial y} + B\left(y^2 - \frac{\nu x^2}{1+\nu}\right) + Cy + C_0 x\right] - \frac{PBL}{E}z + \frac{PB}{2E}z^2 - \frac{\partial f}{\partial y}$$

and

$$\frac{\partial^2 (g - \overline{f})}{\partial y^2} = \frac{\nu A}{1+\nu}x - \frac{(2+\nu)B}{1+\nu}y - \frac{C}{1+\nu} \tag{7-12.7}$$

Finally, differentiation of the equation

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

with respect to z yields

$$\frac{\partial^2 (g - \overline{f})}{\partial x \partial y} = \frac{\nu}{1+\nu}(Bx + Ay) \tag{7-12.8}$$

Equations (7-12.5), (7-12.7), and (7-12.8) require that [with Eq. (7-12.6)]

$$\begin{aligned} f = {} & \frac{(1+\nu)P}{E}g + \frac{PA}{2E}\left(\frac{2+\nu}{3}x^3 - \nu xy^2\right) \\ & + \frac{PB}{2E}\left(-\nu x^2 y + \frac{2+\nu}{3}y^3\right) + \frac{PC}{2E}(x^2 + y^2) - \beta x + \alpha y + \gamma_0 \end{aligned} \tag{7-12.9}$$

where α, β, γ_0 are constants.

With Eqs. (7-12.1) and (7-12.9), the displacement component w is now determined in terms of the harmonic function g. Next, we substitute the expression for f into the equations for $\partial u/\partial z$ and $\partial v/\partial z$ to obtain [with Eq. (7-11.36)]

$$\begin{aligned} \frac{\partial u}{\partial z} &= -Ky - \frac{P}{E}\left\{A\left[Lz - \frac{z^2}{2} - \frac{\nu}{2}(x^2 - y^2)\right] - \nu Bxy - \nu Cx\right\} + \beta \\ \frac{\partial v}{\partial z} &= Kx - \frac{P}{E}\left\{-\nu Axy + B\left[Lz - \frac{z^2}{2} + \frac{\nu}{2}(x^2 - y^2)\right] - \nu Cy\right\} - \alpha \end{aligned} \tag{7-12.10}$$

From the equations for $\partial u/\partial x$ and $\partial u/\partial z$, we determine u in the form

$$\begin{aligned} u = -Kyz - \frac{P}{E}\bigg\{ & A\left[\frac{Lz^2}{2} - \frac{z^3}{6} + \frac{\nu}{2}(L - z)x^2 + \frac{\nu}{2}y^2 z\right] \\ & + \nu B(L - z)xy + \nu C(L - z)x\bigg\} + \beta z + F_1(y) \end{aligned}$$

where $F_1(y)$ is an unknown function of y. Similarly, we find

$$v = Kxz - \frac{P}{E}\left\{\nu A(L-z)xy + B\left[\frac{Lz^2}{2} - \frac{z^3}{6} + \frac{\nu}{2}(L-z)y^2 + \frac{\nu}{2}x^2 z\right] + \nu C(L-z)y\right\} - \alpha z + F_2(x)$$

where $F_2(x)$ is an unknown function of x.

The functions $F_1(y)$ and $F_2(x)$ are determined by the condition

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Hence,

$$F_1(y) = \frac{\nu PAL}{2E}y^2 - \gamma y + \alpha_0$$
$$F_2(x) = \frac{\nu PBL}{2E}x^2 + \gamma x + \beta_0$$

where $\alpha_0, \beta_0, \gamma$ are constants.

In summary, by the analysis above we have determined the displacement components (u, v, w) in the form

$$\begin{aligned}
u &= -Kyz - \frac{P}{E}\left\{A\left[\frac{Lz^2}{2} - \frac{z^3}{6} + \frac{\nu}{2}(L-z)(x^2-y^2)\right] + \nu B(L-z)xy + \nu C(L-z)x\right\} - \gamma y + \beta z + \alpha_0 \\
v &= Kxz - \frac{P}{E}\left\{\nu A(L-z)xy + B\left[\frac{Lz^2}{2} - \frac{z^3}{6} - \frac{\nu}{2}(L-z)(x^2-y^2)\right] + \nu C(L-z)y\right\} + \gamma x - \alpha z + \beta_0 \\
w &= \overline{g} + \frac{P}{E}\left\{A\left[x\left(Lz - \frac{z^2}{2}\right) + \frac{2+\nu}{6}x^3 - \frac{\nu}{2}xy^2\right] + B\left[y\left(Lz - \frac{z^2}{2}\right) - \frac{\nu}{2}x^2 y + \frac{2+\nu}{6}y^3\right] + C\left[Lz + \frac{1}{2}(x^2+y^2-z^2)\right]\right\} \\
&\quad - \beta x + \alpha y + \gamma_0
\end{aligned} \tag{7-12.11}$$

where $\alpha_0, \beta_0, \gamma_0, \alpha, \beta, \gamma$ are constants and

$$\overline{g} = \frac{(1+\nu)Pg}{E} \tag{7-12.12}$$

In Eq. (7-12.11), the terms in $\alpha_0, \beta_0, \gamma_0, \alpha, \beta, \gamma$ represent a rigid-body displacement (see Chapter 2, Section 2-15 and Problem 2-15.1). To evaluate the rigid-body displacement, we may require that the displacement (u, v, w) and the rotation $(\omega_x, \omega_y, \omega_z)$ be prescribed at a point (x, y, z).

Problem Set 7-12

1. Discuss conditions that may be employed to evaluate $\alpha_0, \beta_0, \gamma_0, \alpha, \beta, \gamma$ of Eqs. (7-12.11).

7-13 Center of Shear

The condition for which there occurs no twisting of the end section of a bar loaded by transverse end force is defined from Eq. (7-11.37) by settling the twist $\partial(\overline{\omega}_z)/\partial z = 0$. Thus, we obtain

$$C_0 = \frac{\nu}{1+\nu}(B\overline{x} - A\overline{y}) \tag{7-13.1}$$

as the necessary and sufficient condition that the twist vanish. In general, if C_0 is defined by Eq. (7-13.1), the moment M_z does not vanish. For example, in general,

$$M_z = \iint (x\tau_{yz} - y\tau_{xz})\,dx\,dy \tag{7-13.2}$$

Accordingly, with Eqs. (7-11.10), (7-11.15), and (7-13.1), Eq. (7-13.2) yields

$$\begin{aligned} M_z = P\Bigg\{&\frac{\nu}{1+\nu}(B\overline{x} - A\overline{y})\iint \phi\,dx\,dy + \iint \Gamma\,dx\,dy \\ &+ \frac{1}{2}\iint (By - Ax)xy\,dx\,dy \\ &+ \oint \left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right] R_s ds\Bigg\} \end{aligned} \tag{7-13.3}$$

Thus, Eq. (7-13.3) defines the moment that must be applied to the end of the bar, together with a force P directed along the x axis, to give zero average twist of the end. By elementary statics and Saint-Venant's principle, we replace the moment M_z and the force P acting along the x axis by a force P_i, parallel to P and equal in magnitude to P, but located at a distance y_i from the x axis, where

$$\begin{aligned} y_i = -\frac{M_z}{P} = \frac{\nu}{1+\nu}&(-B\overline{x} + A\overline{y})\iint \phi\,dx\,dy - \iint \Gamma\,dx\,dy \\ -\frac{1}{2}\iint (By - Ax)xy\,dx\,dy &- \oint \left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right] R_s ds \end{aligned} \tag{7-13.4}$$

The above theory defines bending of a bar with zero average rotation when a force P is applied parallel to the x axis at a distance y_i from the x axis. Similarly, if a force P is applied parallel to the y axis, it must be located at a distance x_i from the y axis for bending of the rod with zero average rotation of the end, where, as with the computation for y_i [Eq. (7-13.4)], we find

$$x_i = \frac{\nu}{1+\nu}(b\overline{x} - a\overline{y}) \iint \phi \, dx\, dy + \iint \gamma \, dx\, dy + \frac{1}{2}\iint (by - ax)xy\, dx\, dy$$
$$+ \oint \left[(by^2 + cy)\frac{dx}{ds} - (ax^2 + dx)\frac{dy}{ds} \right] R_s ds \tag{7-13.5}$$

where over the cross section R

$$\nabla^2 \gamma = \frac{2\nu}{1+\nu}(bx - ay) \tag{7-13.6}$$

and on the boundary S

$$\gamma = \oint \left[(by^2 + cy)\frac{dx}{ds} - (ax^2 + cx)\frac{dy}{ds} \right] ds \tag{7-13.7}$$

and where

$$a = \frac{I_{xy}S_0 - S_x S_y}{\Delta} \qquad b = \frac{S_y^2 - S_0 I_{xy}}{\Delta} \qquad c = \frac{I_{yy}S_x - I_{xy}S_y}{\Delta} \tag{7-13.8}$$

where Δ is defined by Eq. (7-11.7).

The intersection of the lines $x = x_i$, $y = y_i$ locates a point in the (x, y) plane. This point is called the *shear center* because if a transverse force is applied at (x_i, y_i), it produces zero average twist at the end of the rod.

It may be shown that the location of the shear center may be determined provided the solution of the torsion problem is known; that is, in general, it is not necessary to know the solution to the bending problem to compute (x_i, y_i) (see Problems 7-13.1 to 7-13.4). In the strength of materials definition of shear center, Poisson's ratio is usually discarded.

Problem Set 7-13

1. With the fact that (Green's theorem)

$$\iint (F\nabla^2 G - G\nabla^2 F)\, dx\, dy = \oint \left(F\frac{\partial F}{\partial n} - G\frac{\partial F}{\partial n} \right) ds \tag{a}$$

where F and G are functions of (x, y), let $F = \phi$, $G = \Gamma$, take into consideration Eqs. (7-11.16) and (7-11.17), and show that

$$\frac{2\nu}{1+\nu}\iint \phi(Bx - Ay)\, dx\, dy + 2\iint \Gamma \, dx\, dy = -\oint \Gamma \frac{\partial \phi}{\partial n} ds \tag{b}$$

2. Noting by Eqs. (7-2.4) and (7-3.3) that

$$\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x} - y \qquad \frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y} - x \tag{a}$$

where the factor $G\beta$ has been absorbed in ϕ, show that [with Eqs. (7-2.7), (7-2.9), and (7-11.30)]

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x}\frac{dx}{dn} + \frac{\partial \phi}{\partial y}\frac{dy}{dn} = -\frac{\partial \psi}{\partial s} - 2\frac{dR_s}{ds} \tag{b}$$

Hence, show that

$$\begin{aligned} \oint \Gamma \frac{\partial \phi}{\partial n}\, ds &= -\oint \Gamma\left(\frac{\partial \psi}{\partial s} + 2\frac{dR_s}{ds}\right) ds \\ &= \oint (\psi + 2R_s)\frac{\partial \Gamma}{\partial s}\, ds \\ &= \oint (\psi + 2R_s)\left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right] ds \end{aligned} \tag{c}$$

3. With Eqs. (7-2.7) and Eq. (a) of Problem 1, show that

$$\begin{aligned} I &= \oint \psi\left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right] ds \\ &= -\iint \left\{\frac{\partial}{\partial y}[(By^2 + Cy)\psi] + \frac{\partial}{\partial x}[(Ax^2 + Cx)\psi]\right\} dx\, dy \\ &= -2\iint (Ax + By + C)\psi\, dx\, dy \\ &\quad - \iint \left[(By^2 + Cy)\frac{\partial \psi}{\partial y} + (Ax^2 + Cx)\frac{\partial \psi}{\partial x}\right] dx\, dy \end{aligned}$$

Hence, with Eq. (a) of Problem 2, Eq. (a) of Problem 1, and the fact that $\phi = 0$ on S for a simply connected region R [see Eq. (7-11.17)], show that

$$I = -2\iint (Ax + By + C)\psi\, dx\, dy + \iint (By - Ax)xy\, dx\, dy$$

4. With the results of Problems 1, 2, and 3, show that

$$\begin{aligned} \iint \Gamma\, dx\, dy = &-\frac{\nu}{1+\nu}\iint \phi(Bx - Ay)\, dx\, dy \\ &- \oint \left[(By^2 + Cy)\frac{dx}{ds} - (Ax^2 + Cx)\frac{dy}{ds}\right] R_s ds \\ &+ \iint (Ax + By + C)\psi\, dx\, dy - \frac{1}{2}\iint (By - Ax)xy\, dx\, dy \end{aligned}$$

Hence, show that [see Eqs. (7-13.4) and (7-13.5)]

$$\begin{aligned} y_i &= -\iint (Ax + By + C)\psi\, dx\, dy + \frac{\nu}{1+\nu}\iint [B(x - \overline{x}) - A(y - \overline{y})]\phi\, dx\, dy \\ x_i &= \iint (ax + by + c)\psi\, dx\, dy + \frac{\nu}{1+\nu}\iint [b(x - \overline{x}) - a(y - \overline{y})]\phi\, dx\, dy \end{aligned} \tag{7-13.9}$$

Equation (7-13.9) shows that if the solution to the torsion problem for region R is known—that is, if either ϕ or ψ is known [see Eq. (a) of Problem 2]—the coordinates (x_i, y_i) of the shear center may be calculated. In other words, (x_i, y_i) may be determined even though the solution of the bending problem ($F = \Gamma + C_0\phi$) is not known.

5. Show that when the cross section of a bar has one axis of symmetry, the shear center will lie on this axis. Show that when the cross section of a bar has two axes of symmetry, the shear center coincides with the intersection of these two axes.

6. Show by calculations and examples that the shear center of a cross section of a bar does not necessarily lie in the region R occupied by the cross section.

7-14 Bending of a Bar with Elliptic Cross Section

In this section we consider a technique introduced by Timoshenko (1921, 1913, 1983) for solving the bending problem of bars for certain types of cross section. The motivation of the method lies in seeking to represent the boundary conditions [taken in the form of the second of Eqs.(7-11.21)] in the simplest possible form. For example, for a simply connected cross section we may choose $f(y)$ to make the right side of the second of Eqs. (7-11.21) equal to zero. Then, $\partial\Psi/\partial s = 0$ on S. Because the cross section is simply connected, it follows that Ψ may be taken equal to zero on S (see Section 7-6). We illustrate the method for a bar with elliptic cross section.

For an elliptic cross section, the lateral surface of the cross section is defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7-14.1}$$

where (a, b) denotes the major and minor semiaxes of the ellipse. Accordingly, the right side of the second of Eqs. (7-11.21) vanishes identically, provided we set

$$f(y) = -\frac{Pa^2}{Ib^2}(y^2 - b^2) \tag{7-14.2}$$

Substitution of Eq. (7-14.2) into the first of Eqs. (7-11.21) yields

$$\nabla^2\Psi = \frac{2Py}{I}\left(\frac{a^2}{b^2} + \frac{\nu}{1+\nu}\right) \tag{7-14.3}$$

The boundary condition $\Psi = 0$ on S [see Eq. (7-11.24)] will be satisfied if we take Ψ in the form

$$\Psi(x, y) = D\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)y \tag{7-14.4}$$

where $D =$ constant. Substitution of Eq. (7-14.4) into Eq. (7-14.3) yields

$$D = \frac{P}{I}\left(\frac{a^2b^2}{3a^2+b^2}\right)\left(\frac{a^2}{b^2}+\frac{\nu}{1+\nu}\right) \tag{7-14.5}$$

With the cross section defined by Eq. (7-14.1), the axes (x, y) are axes of symmetry. Hence, with the resultant force P directed along the x axis, $C_0 = 0$ (see Section 7-11). Then Eqs. (7-11.10), (7-11.15), (7-11.20), and (7-11.21) yield the following expressions for the stress components τ_{xz}, τ_{yz}:

$$\begin{aligned} \tau_{xz} &= \frac{1}{2}\left[\frac{\partial \Psi}{\partial y} + f - \frac{Px^2}{I}\right] \\ \tau_{yz} &= -\frac{1}{2}\frac{\partial \Psi}{\partial x} \end{aligned} \tag{7-14.6}$$

Substitution of Eqs. (7-14.2), (7-14.4), and (7-14.5) into Eqs. (7-14.6) yields

$$\begin{aligned} \tau_{xz} &= \frac{Pa^2}{2I}\left[\frac{(1+\nu)a^2+\nu b^2}{(1+\nu)(3a^2+b^2)}\left(\frac{x^2}{a^2}+\frac{3y^2}{b^2}-1\right) - \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\right] \\ \tau_{yz} &= -\frac{(1+\nu)a^2+\nu b^2}{(1+\nu)(3a^2+b^2)}\frac{Pxy}{I} \end{aligned} \tag{7-14.7}$$

The normal stress component σ_z for this case ($A = -1/I$, $B = C = 0$) is found from Eq. (7-11.4) to be

$$\sigma_z = -\frac{Px}{I}(L - z) \tag{7-14.8}$$

Equation (7-14.8) agrees precisely with elementary beam theory. However, the shearing–stress components differ from results predicted by elementary beam theory. Elementary beam theory predicts that τ_{yz} vanishes everywhere and that τ_{xz} is a function of x only.

If $b \ll a$, Eqs. (7-14.7) may be approximated by the equations

$$\tau_{xz} = \frac{P}{3I}(a^2 - x^2) \qquad \tau_{yz} = -\frac{Pxy}{3I} \tag{7-14.9}$$

Then, τ_{xz} agrees with the stress component computed by elementary theory. However, again τ_{yz} is in disagreement with elementary theory although it is very small (because y is small; it is at most equal to b). The maximum value of τ_{xz} predicted by Eq. (7-14.9) is (for $x = 0$)

$$(\tau_{xz})_{\max} = \frac{Pa^2}{3I} = \frac{4P}{3A} \tag{7-14.10}$$

where $I = Aa^2/4$, where $A =$ cross-sectional area of the ellipse.

By Eqs. (7-14.7), the maximum value of τ_{xz} is (for $x = y = 0$)

$$(\tau_{xz})_{\max} = \frac{Pa^2}{2I}\left[1 - \frac{(1+\nu)a^2 + \nu b^2}{(1+\nu)(3a^2+b^2)}\right] \tag{7-14.11}$$

Again, for $b \ll a$, Eq. (7-14.11) yields the result given by Eq. (7-14.10).

Bar with Circular Cross Section. If in the above analysis we let $a = b$, the cross section of the bar becomes circular. Thus, for the circular bar we obtain from Eqs. (7-14.7)

$$\begin{aligned} \tau_{xz} &= \frac{P}{2I}\left[\frac{1+2\nu}{4(1+\nu)}(x^2+3y^2-a^2) - (x^2+y^2-a^2)\right] \\ \tau_{xz} &= -\frac{1+2\nu}{4(1+\nu)}\frac{Pxy}{I} \end{aligned} \tag{7-14.12}$$

Hence

$$(\tau_{xz})_{\max} = \frac{3+2\nu}{8(1+\nu)}\frac{Pa^2}{I} \tag{7-14.13}$$

7-15 Bending of a Bar with Rectangular Cross Section

Consider a cantilever beam with rectangular cross section R and with lateral surface S. Let end load P be applied to the end of the bar (beam) and directed along the vertical centroidal axis (x axis, Fig. 7-15.1). The cross section is defined by the equation

$$(x^2 - a^2)(y^2 - b^2) = 0 \tag{7-15.1}$$

By the theory of Section 7-11, the beam undergoes bending with no twisting of the end plane ($A = -1/I$, $B = C = C_0 = 0$).

Because the net load P is equivalent to shear–stress components τ_{xz}, τ_{yz} distributed over the end of the bar, we may employ the semi-inverse method by assuming simple distributions for τ_{xz}, τ_{yz} and then attempt to satisfy the elasticity equations. For example, as $\sum F_x = P$, $\sum F_y = 0$, it appears reasonable to assume τ_{yz} to be odd in y and τ_{xz} to be even in x and y (see Fig. 7-15.1). Furthermore, we employ the technique demonstrated in Section 7-14 for the elliptic cross section. Hence, taking $f = Pa^2/I$ and noting the nature of the dependency of (τ_{xz}, τ_{yz}) on x and y, we find by Eqs. (7-14.6) that Ψ is even in x and odd in y. Also, by choosing $f = Pa^2/I$ and noting that $dy/ds = 0$ for $y = \pm b$, by Eqs. (7-11.21), we obtain

$$\begin{aligned} \nabla^2\Psi &= \frac{2\nu}{1+\nu}\frac{Py}{I} \qquad \text{over } R \\ \Psi &= 0 \qquad \text{on } S \end{aligned} \tag{7-15.2}$$

By inspection, a particular solution of the first of Eqs. (7-15.2) is

$$\Psi_1 = Ay^3 + By \tag{7-15.3}$$

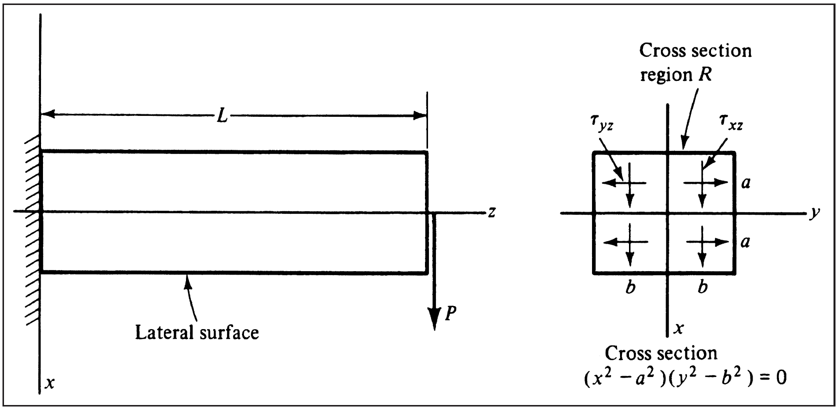

Figure 7-15.1

Substitution of Eq. (7-15.3) into Eq. (7-15.2) yields

$$A = \frac{\nu P}{3(1+\nu)I} \tag{7-15.4}$$

with B arbitrary.

By the discussion at the end of Section 7-11, we choose Ψ in the form

$$\Psi = \Omega + \frac{\nu P}{3(1+\nu)I} y^3 + By \tag{7-15.5}$$

where by Eqs. (7-15.2) and (7-15.5)

$$\begin{aligned} \nabla^2 \Omega &= 0 \qquad \text{on } R \\ \Omega &= -\frac{\nu P}{3(1+\nu)I} y^3 - By \qquad \text{on } S \end{aligned} \tag{7-15.6}$$

Let us choose[8] B so that $\Omega = 0$ for $y = \pm b$. Then, Eq. (7-15.6) yields

$$B = -\frac{\nu P b^2}{3(1+\nu)I} \tag{7-15.7}$$

Thus, by Eqs. (7-15.5) and (7-15.6), we arrive at the stress function

$$\Psi = \Omega + \frac{\nu P}{3(1+\nu)I}(y^3 - b^2 y) \tag{7-15.8}$$

where

$$\nabla^2 \Omega = 0 \qquad \text{on } R \tag{7-15.9}$$

[8]Note that we could assume a particular solution of Eq. (7-15-2) in the form $Ay^3 + B$. Then we could choose B so that $\Omega = 0$ for $y = b$, but Ω would not be zero on the line $y =—b$. Hence, our choice of Ψ_1 [Eq. (7-15-3)] leads to a simpler boundary condition for Ω.

and

$$\begin{aligned} \Omega &= 0 \qquad \text{for } y = \pm b \\ \Omega &= \frac{\nu P}{3(1+\nu)I}(b^2 y - y^3) \qquad \text{for } x = \pm a \end{aligned} \tag{7-15.10}$$

Because Ψ is even in x and odd in y, Ω is even in x and odd in y.

Consider solutions of Eq. (7-15.9) of the form

$$\Omega = f(x)g(y) \tag{7-15.11}$$

where $f(x), g(y)$ are functions of x and y, respectively. Substitution of Eq. (7-15.11) into Eq. (7-15.9) with the requirement that Ω be even in x and odd in y yields solutions of the form

$$\Omega = A \cosh kx \sin ky \tag{7-15.12}$$

where A and k are constants.

Substitution of Eq. (7-15.12) into the first of Eqs. (7-15.10) yields $A \cosh kx \sin kb = 0$, or $k = n\pi/b, n = 1, 2, 3, \ldots$. Hence, superposition of solutions of the type given by Eq. (7-15.12) yields

$$\Omega = \sum_{n=1}^{\infty} A_n \cosh \frac{n\pi x}{b} \sin \frac{n\pi y}{b} \tag{7-15.13}$$

Let $A_n \cosh(n\pi a/b) = a_n$. Then, by Eq. (7-15.13) and the second of Eqs. (7-15.10), we must require that

$$\sum_{n=1}^{\infty} a_n \sin \frac{n\pi y}{b} = \frac{\nu P}{3(1+\nu)I}(b^2 y - y^3) \tag{7-15.14}$$

Multiplying Eq. (7-15.14) by $\sin(m\pi y/b)$ and integrating from $-b$ to b, we obtain

$$\sum_{n=1}^{\infty} a_n \int_{-b}^{b} \sin \frac{n\pi y}{b} \sin \frac{m\pi y}{b}\, dy = \frac{\nu P}{(1+\nu)I} \int_{-b}^{b} (b^2 y - y^3) \sin \frac{m\pi y}{b}\, dy \tag{7-15.15}$$

Observing that

$$\int_{-b}^{b} \sin \frac{m\pi y}{b} \sin \frac{n\pi y}{b}\, dy = \begin{cases} 0, & m \neq n \\ b, & m = n \end{cases}$$

$$\int_{-b}^{b} y \sin \frac{m\pi y}{b}\, dy = -\frac{2(-1)^m b^2}{m\pi}.$$

$$\int_{-b}^{b} y^3 \sin \frac{m\pi y}{b}\, dy = -\frac{2(-1)^m b^4}{m^3\pi^3}(m^2\pi^2 - 6)$$

we obtain after integration of Eq. (7-15.15)

$$a_n = -\frac{4\nu P b^3}{(1+\nu)I}\frac{(-1)^n}{n^3\pi^3}$$

Hence, the constant A_n in Eq. (7-15.13) is determined, and the stress function Ψ is given by the formula

$$\Psi = \frac{\nu P}{3(1+\nu)I}\left[y^3 - b^2 y - \frac{12b^3}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\frac{\cosh\dfrac{n\pi x}{b}\sin\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\right] \tag{7-15.16}$$

Then, because $f = Pa^2/I$, substitution of Eq. (7-15.16) into Eqs. (7-14.6) yields

$$\begin{aligned}\tau_{xz} &= \frac{P}{2I}(a^2 - x^2) + \frac{\nu P}{6(1+\nu)I}\left[3y^2 - b^2 - \frac{12b^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cosh\dfrac{n\pi x}{b}\cos\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\right]\\ \tau_{xz} &= \frac{2\nu P b^2}{\pi^2(1+\nu)I}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\sinh\dfrac{n\pi x}{b}\sin\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\end{aligned} \tag{7-15.17}$$

Equations (7-15.17) express the solution to the bending of a cantilever beam with rectangular cross section and with load P directed along the vertical centroidal axis in the end plane $z = L$.

Examination of τ_{xz}. On the horizontal line $x = 0$, Eqs. (7-15.17) yield

$$\begin{aligned}\tau_{xz} &= \frac{Pa^2}{2I}\left\{1 + \frac{\nu}{1+\nu}\left[\frac{y^2}{a^2} - \frac{b^2}{3a^2} - \frac{4b^2}{\pi^2 a^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cos\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\right]\right\}\\ \tau_{yz} &= 0\end{aligned} \tag{7-15.18}$$

Elementary theory of beams yields the result (for $x = 0$) $\tau_{xz} = Pa^2/2I$. Hence, the quantity in braces in Eq. (7-15.18) represents a correction factor K to elementary beam theory; that is, the result of elementary beam theory must be multiplied by the factor

$$K = 1 + \frac{\nu}{1+\nu}\left[\frac{y^2}{a^2} - \frac{b^2}{3a^2} - \frac{4b^2}{\pi^2 a^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\frac{\cos\dfrac{n\pi y}{b}}{\cosh\dfrac{n\pi a}{b}}\right] \tag{7-15.19}$$

If $\nu = 0$, the correction factor is 1. Also, if $b \ll a$ the correction factor is approximately 1. That is, elementary theory is approximately correct (at $x = 0$) for beams of narrow cross section (Fig. 7-15.1, with $b \ll a$). The fact that the correction factor approaches 1 as b/a approaches zero is apparent from Eq. (7-15.19), as $\cos(n\pi y/b)$ is never larger than 1 and $\cosh(n\pi a/b)$ becomes very large as $b/a \to 0$.

It may also be shown that $K \to 1$ as b/a becomes very large. For example, consider the point $x = y = 0$. Then, Eq. (7-15.19) yields

$$K = 1 - \frac{\nu}{1+\nu}\left[\frac{b^2}{3a^2} + \frac{4b^2}{\pi^2 a^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\operatorname{sech}\frac{n\pi a}{b}\right]$$

Note that as $b/a \to \infty$, $\operatorname{sech}(n\pi a/b) \to 1$. To evaluate the series $\sum_{n=1}^{\infty}(-1)^n/n^2$, we first observe that by Fourier series we may express θ^2 in series form as

$$\theta^2 = \frac{C^2}{3} - \frac{4C^2}{\pi^2}\left(\cos\frac{\pi\theta}{C} - \frac{1}{2^2}\cos\frac{2\pi\theta}{C} + \frac{1}{3^2}\cos\frac{3\pi\theta}{C} - \frac{1}{4^2}\cos\frac{4\pi\theta}{C} + \cdots\right)$$

where C is a constant. Letting $\theta = 0$ and $C = \frac{1}{2}$, we obtain

$$0 = \frac{1}{12} - \frac{1}{\pi^2}\left(1 - \frac{1}{2^2} + \frac{3}{3^2} - \frac{1}{4^4} + \cdots\right)$$

or

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

Accordingly, for $b/a \to \infty$, $\tau_{xz} \to Pa^2/2I$ at $x = y = 0$. That is, the elementary theory of beam also gives the correct result for a very wide beam (Fig. 7-15.1 for $b \gg a$).

Problem Set 7-15

1. Let $b/a = 6$ and $\nu = 0.3$. For the horizontal line $x = 0$, evaluate the correction factor K [Eq. (7-15.19)] for $y/b = 0$, 0.2, 0.4, 0.6, 0.8, and 1.0.

Review Problems

R-1. Let the resultant vector of the forces acting on the end $z = L$ of a bar be directed along the z axis. Let the resultant moment be zero. Consider the simplest stress distribution that is statically equivalent to the resultant vector and the resultant moment. Hence, by the semi-inverse method, solve the problem of the cylindrical bar subjected to a longitudinal end force.

R-2. Let the forces that act on the end of a rod at $z = L$ be statically equivalent to a couple of moment $M_y = M$, where M_y denotes the moment relative to the y axis in the end plane at $z = L$. Compute a statically equivalent system for the end plane $z = 0$.

Assume the simplest stress distribution that is statically equivalent to M_y. Hence, solve the problem of bending of a bar subjected to end couple M. Express the stress components, the strain components, and the displacement components in terms of M and material and geometrical properties of the bar.

R-3. Figure R7-3 represents the cross section of a cantilever beam subjected to transverse end load P directed along the x axis. Derive a formula for $f(y)$ to make Ψ vanish on the lateral boundary [see Eq. (7-11.22)]. Discuss the application of the method demonstrated in Sections 7-14 and 7-15 for this problem.

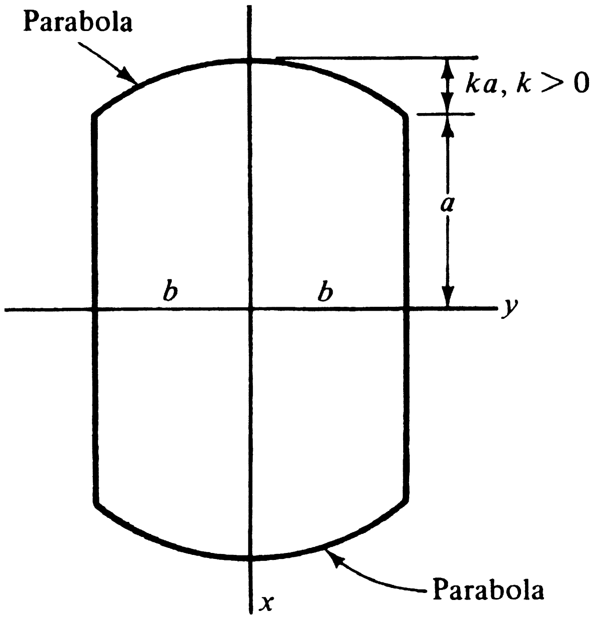

Figure R7-3

R-4.[9] In Section 7-7 it was assumed that the angle of twist $\theta = \beta z$ was the same for both axes z and z_1. Then it was shown that the stresses, hence the moment M with respect to axis z_1, are identical to the stresses and the moment with respect to axis z. Alternatively, we may assume that the twisting moment M for axis z is the same as for axis z_1. Then it may be shown by the equations of elasticity that the twist θ_1 relative to axis z_1 is equal to the twist θ relative to axis z. Verify this statement.

APPENDIX 7A ANALYSIS OF TAPERED BEAMS

Chapter 7 is devoted to prismatic beams. However, for certain structural applications, tapered beams, which have variable moments of inertia to counteract different acting moments, are more efficient than prismatic beams.

[9]This problem was suggested by Professor James G. Goree, Clemson University, Clemson, South Carolina.

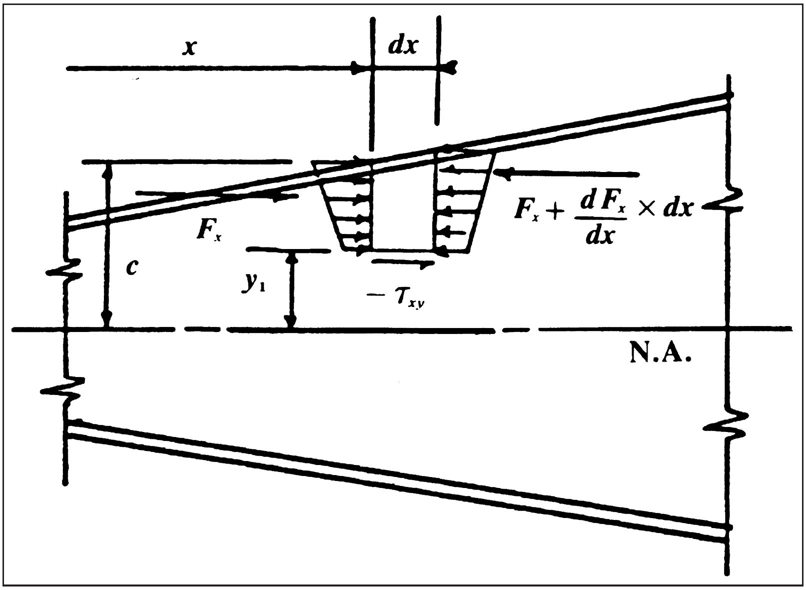

Figure 7A-1 Stresses in tapered beam.

As a result of their structural efficiency and suitability for fabrication, web-tapered beams (Fig. 7A-1) are becoming popular in various types of construction. The flexural and torsional behavior of tapered beams has been studied extensively by Lee and his associates (1967, 1972) as well as by Davis et al. (1973). The stability aspects have also been investigated by Kitipornchai and Trahair (1972). As for the shear stresses, Chong et al. (1976) have showed, using principles of mechanics, that the sloping flanges possess vertical components of forces that can either increase or decrease the web shear, depending on the direction of taper and the direction of acting shear.

Assumptions of small deflection theory were used. Referring to the stress block in Fig. 7A-1 and summing up forces in the horizontal direction, Chong et al. (1976) found

$$-\tau_{xy} t\, dx = \frac{\partial F_x}{\partial x}\, dx \tag{7A-1}$$

But

$$F_x = \int_{y_1}^{c} \sigma_x\, dA \tag{7A-2}$$

and

$$\sigma_x = \frac{M_x y}{I_x} \tag{7A-3}$$

and therefore

$$F_x = \frac{M_x}{I_x} \int_{y_1}^{c} y\, dA \tag{7A-4}$$

Let

$$Q_{xy} = \int_{y_1}^{c} y\,dA \tag{7A-5}$$

Then

$$F_x = \frac{M_x Q_{xy}}{I_x} \tag{7A-6}$$

Substituting Eq. (7A-6) into Eq. (7A-1), Chong et al. (1976) found

$$\tau_{xy} = -\frac{1}{t}\frac{\partial}{\partial x}\left(\frac{M_x Q_{xy}}{I_x}\right) \tag{7A-7}$$

Expansion of Eq. (7A-7) yields

$$\tau_{xy} = -\frac{1}{t}\left(\frac{V_x Q_{xy}}{I_x} + \frac{M_x}{I_x}\frac{\partial Q_{xy}}{\partial x} - \frac{M_x Q_{xy}}{I_x^2}\frac{\partial I_x}{\partial x}\right) \tag{7A-8}$$

in which t = thickness of web; F_x = internal force in the x direction; σ_x = normal stress in the x direction; M_x = moment at x; V_x = shear at x; I_x = moment of inertia at x; dA = differential cross-sectional area; and τ_{xy} = shear stress at x.

The first term of Eq. (7A-8) corresponds to the shear stress in beams of constant cross section. The additional terms account for the taper. Equation (7A-7) can be applied to the classical wedge cantilever (Shepherd, 1935), as shown in Fig. 7A-2. The classical solution is

$$\tau_{xy} = -\frac{Py^2}{I_x}\left[\left(\frac{\tan\alpha}{\alpha}\right)^3 \sin^4\theta\right] \tag{7A-9}$$

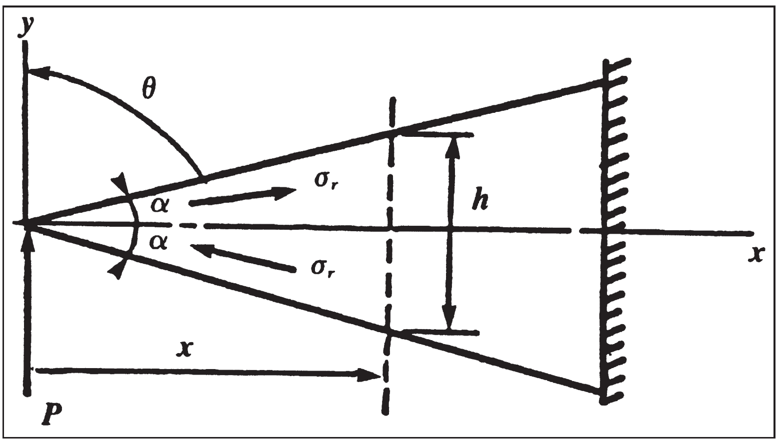

Figure 7A-2 Wedge cantilever loaded at tip.

For small tapers

$$\left[\left(\frac{\tan\alpha}{\alpha}\right)^3 \sin^4\theta\right] \to 1 \tag{7A-10}$$

For $\alpha = 10^\circ$, the maximum error amounts to 3% if the bracketed term is set equal to unity. Thus, for regular small tapers

$$\tau_{xy} = -\frac{Py^2}{I_x} \tag{7A-11}$$

Using Eq. (7A-7), Chong and co-workers (1976) obtained

$$M_x = Px \tag{7A-12}$$

$$Q_{xy} = \frac{1}{2}\left[\left(\frac{h}{2}\right)^2 - y^2\right]t \qquad Q_{xy} = \frac{b}{2}(x^2\tan^2\alpha - y^2) \tag{7A-13}$$

in which b = uniform thickness of the wedge. Substitution of Eqs. (7A-2) and (7A-13) into Eq. (7A-7) yields $\tau_{xy} = -Py^2/I_x$, which is identical to Eq. (7A-11).

For shear stresses in tapered beams loaded away from the tip (Fig. 7A-3)

$$I_x = \frac{bh^3}{12} \tag{7A-14}$$

$$Q_{xy} = \frac{b}{2}\left(\frac{h^2}{4} - y^2\right) \tag{7A-15}$$

Substitution of Eqs. (7A-14) and (7A-15) into (7A-7) gives

$$\tau_{xy} = -\frac{3M_x}{bh^2}\frac{dh}{dx} + \frac{6}{b}\left(\frac{h^2}{4} - y^2\right)\frac{d}{dx}\left(\frac{M}{h^3}\right) \tag{7A-16}$$

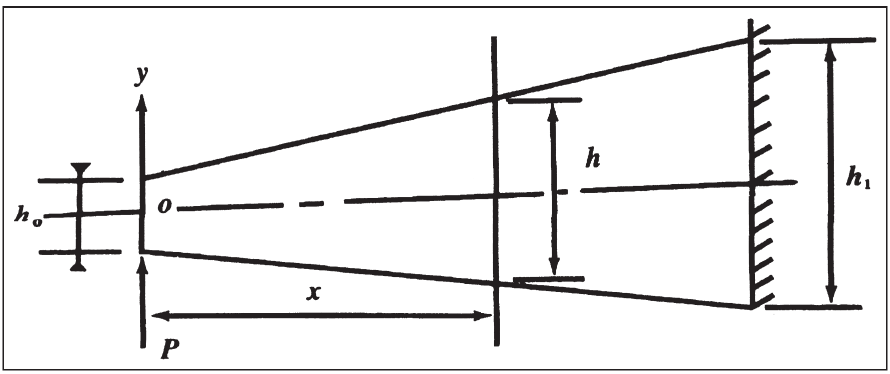

Figure 7A-3 Tapered cantilever beam.

The shear distribution of web-tapered beams was investigated using a theory that assumes a radial flexural stress pattern. Finite element analysis and the classical wedge theory were used to check the accuracy of the theory. These independent methods agreed well with each other (Chong et al., 1976). The presented theory is applicable to wide-flange or box-tapered Hookean beams. Conventionally, the shear–stress distribution is assumed to be uniform with the external shear carried solely by the web. By the proposed theory, significant shears are carried by the flanges, which can be deducted or added to the total web shear.

On the basis of the proposed theory, a simplified analysis procedure was described. We may simply calculate the vertical components of the flange flexural load and subtract or add them from the total vertical shear. The resulting shear was assumed to be carried by the web as a relatively uniform stress distribution (Chong et al., 1976).

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CHAPTER 8

GENERAL SOLUTIONS OF ELASTICITY

In this chapter we present particular forms of general solutions of the three-dimensional equations of elasticity. In essence, these solutions are contained in the Galerkin–Papkovich vector.

8-1 Introduction

Galerkin (1930) represented strain components by three functions X, Y, Z, in terms of which he defined stress components. Later he expressed the corresponding displacement components and showed a number of applications. Papkovich (1932a, 1932b) noted that the Galerkin strain functions X, Y, Z could be considered components of a vector $\mathbf{F}$; that is,

$$\mathbf{F} = \mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z \tag{8-1.1}$$

The vector $\mathbf{F}$ is called the Galerkin–Papkovich vector.

Papkovich showed that the vector $\mathbf{F}$ is related to the displacement vector

$$\boldsymbol{\rho} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w \tag{8-1.2}$$

where (u, v, w) denote (x, y, z) displacement components, the relation being

$$2G\boldsymbol{\rho} = [2(1-\nu)\nabla^2 - \nabla\ \text{div}]\,\mathbf{F}, \tag{8-1.3}$$

provided that

$$\nabla^4\mathbf{F} = -\frac{\mathbf{B}}{1-\nu} \tag{8-1.4}$$

where

$$\mathbf{B} = \mathbf{i}B_x + \mathbf{j}B_y + \mathbf{k}B_z \tag{8-1.5}$$

denotes the body force vector (B_x, B_y, B_z) and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Through the Helmholtz transformation, Mindlin (1936) also showed that there exists a correlation [Eq. (8-1.3)] between the displacement vector $\boldsymbol{\rho}$ (sometimes referred to as the Galerkin vector) and the Galerkin–Papkovich vector. An alternative procedure has been given by Westergaard (1952).

Problem Set 8-1

1. Verify Eqs. (8-1.3), (8-1.4), and (8-1.5).

8-2 Equilibrium Equations

In this section we express the equilibrium equations in terms of the displacement vector $\boldsymbol{\rho}$. Consideration of the equilibrium of a tetrahedron shows that the stress vector $\boldsymbol{\sigma}_p$ associated with the inclined plane of the tetrahedron is given by [see Eq. (3-3.7) in Chapter 3]

$$\boldsymbol{\sigma}_p = \ell\boldsymbol{\sigma}_x + m\boldsymbol{\sigma}_y + n\boldsymbol{\sigma}_z \tag{8-2.1}$$

where (ℓ, m, n) are direction cosines of the normal to the tetrahedron oblique plane and where $(\tau_{ij} = \tau_{ji})$:

$$\begin{aligned} \boldsymbol{\sigma}_x &= \mathbf{i}\sigma_x + \mathbf{j}\tau_{xy} + \mathbf{k}\tau_{xz} \\ \boldsymbol{\sigma}_y &= \mathbf{i}\tau_{yx} + \mathbf{j}\sigma_y + \mathbf{k}\tau_{yz} \\ \boldsymbol{\sigma}_z &= \mathbf{i}\tau_{zx} + \mathbf{j}\tau_{zy} + \mathbf{k}\sigma_z \end{aligned} \tag{8-2.2}$$

where σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, and τ_{yz} are the components of the stress tensor.

Because

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma}_x &= \frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} \\ \operatorname{div}\boldsymbol{\sigma}_y &= \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} \\ \operatorname{div}\boldsymbol{\sigma}_z &= \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} \end{aligned} \tag{8-2.3}$$

the general equilibrium equations may be written

$$\text{div}\,\boldsymbol{\sigma}_x + B_x = 0 \qquad \text{div}\ \boldsymbol{\sigma}_y + B_y = 0 \qquad \text{div}\ \boldsymbol{\sigma}_z + B_z = 0 \tag{8-2.4}$$

The stress–strain relations of linear elastic materials may be written

$$\begin{aligned}
\epsilon_x &= \frac{1+v}{E}\sigma_x - \frac{v}{E}I_1 = \frac{\partial u}{\partial x}\\
\epsilon_y &= \frac{1+v}{E}\sigma_y - \frac{v}{E}I_1 = \frac{\partial v}{\partial y}\\
\epsilon_z &= \frac{1+v}{E}\sigma_z - \frac{v}{E}I_1 = \frac{\partial w}{\partial x}\\
\gamma_{xy}&\frac{1}{G}\tau_{xy} \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}
\end{aligned} \tag{8-2.5}$$

where

$$I_1 = \sigma_x + \sigma_y + \sigma_z \tag{8-2.6}$$

Also, by Eqs. (8-1.2), (8-2.5), and (8-2.6), we may write $J_1 = \epsilon_x + \epsilon_y + \epsilon_z$ in the form

$$J_1 = \text{ div } \boldsymbol{\rho} = \frac{1-2\nu}{E}I_1 \tag{8-2.7}$$

Inverting Eqs. (8-2.5), we have

$$\begin{aligned}
\sigma_x &= 2G\left(\epsilon_x + \frac{\nu}{1-2\nu}\text{ div }\boldsymbol{\rho}\right) = 2G\left(\frac{\partial u}{\partial x} + \frac{\nu}{1-2\nu}\text{ div }\boldsymbol{\rho}\right)\\
\sigma_y &= 2G\left(\frac{\partial v}{\partial y} + \frac{\nu}{1-2\nu}\text{ div }\boldsymbol{\rho}\right)\\
\sigma_z &= 2G\left(\frac{\partial w}{\partial z} + \frac{\nu}{1-2\nu}\text{ div }\boldsymbol{\rho}\right)\\
\tau_{xy} &= G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \qquad \tau_{yz} = G\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right),
\end{aligned} \tag{8-2.8}$$

where

$$G = \frac{E}{2(1+\nu)}$$

Substitution of Eqs. (8-2.8) into Eqs. (8-2.4) yields

$$G\left(\nabla^2 u + \frac{1}{1-2\nu}\frac{\partial}{\partial x}\ \text{div}\ \boldsymbol{\rho}\right) + B_x = 0$$

$$G\left(\nabla^2 v + \frac{1}{1-2\nu}\frac{\partial}{\partial y}\ \text{div}\ \boldsymbol{\rho}\right) + B_y = 0$$

$$G\left(\nabla^2 w + \frac{1}{1-2\nu}\frac{\partial}{\partial z}\ \text{div}\ \boldsymbol{\rho}\right) + B_z = 0 \qquad (8\text{-}2.9)$$

or, with Eqs. (8-1.2) and (8-1.5), addition of Eqs. (8-2.9) yields

$$G\left(\nabla^2 + \frac{1}{1-2\nu}\boldsymbol{\nabla}\ \text{div}\right)\boldsymbol{\rho} + \mathbf{B} = 0 \qquad (8\text{-}2.10)$$

Equation (8-2.10) represents the equilibrium equation in terms of the displacement vector $\boldsymbol{\rho}$ and body force vector $\mathbf{B}$.

Problem Set 8-2

1. Derive Eqs. (8-2.9).

8-3 The Helmholtz Transformation

The Helmholtz transformation illustrates the condition that an arbitrary displacement vector may be decomposed into a dilatation and a rotation. According to Helmholtz's theorem, the displacement vector may be resolved into a lamellar part and a solenoidal part. Thus, we write

$$\boldsymbol{\rho} = \boldsymbol{\nabla}\phi + \text{curl}\,\mathbf{S} \qquad (8\text{-}3.1)$$

where $\boldsymbol{\nabla}\phi$, the lamellar part (curl $\boldsymbol{\nabla}\phi = 0$), may be shown to represent the dilatation, and the solenoidal part curl $\mathbf{S}$ (div $\mathbf{S} = 0$) may be shown to represent the rotation. The function ϕ is a scalar potential (or strain potential), whereas the function $\mathbf{S}$ is a vector potential function such that div $\mathbf{S} = \boldsymbol{\nabla}\cdot\mathbf{S} = 0$. Because

$$\boldsymbol{\rho} = \mathbf{i}u + \mathbf{j}v + \mathbf{k}w$$

$$\boldsymbol{\nabla}\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

and

$$\text{curl } \mathbf{S} = \nabla \times \mathbf{S} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ S_x & S_y & S_z \end{vmatrix}$$
$$= \left(\frac{\partial S_z}{\partial y} - \frac{\partial S_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial S_x}{\partial z} - \frac{\partial S_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial S_y}{\partial x} - \frac{\partial S_x}{\partial y}\right)\mathbf{k} \qquad (8\text{-}3.2)$$

we have

$$\begin{aligned} u &= \frac{\partial \phi}{\partial x} + \frac{\partial S_z}{\partial y} - \frac{\partial S_y}{\partial z} \\ v &= -\frac{\partial S_z}{\partial x} + \frac{\partial \phi}{\partial y} + \frac{\partial S_x}{\partial z} \\ w &= \frac{\partial S_y}{\partial x} - \frac{\partial S_x}{\partial y} + \frac{\partial \phi}{\partial z} \end{aligned} \qquad (8\text{-}3.3)$$

Equations (8-3.3) yield the dilatation

$$J_1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla^2 \phi \qquad (8\text{-}3.4)$$

and the rotation vector

$$\omega_x = -\tfrac{1}{2}\nabla^2 S_x \qquad \omega_y = -\tfrac{1}{2}\nabla^2 S_y \qquad \omega_z = -\tfrac{1}{2}\nabla^2 S_z \qquad (8\text{-}3.5)$$
$$\boldsymbol{\omega} = -\tfrac{1}{2}\nabla^2 \mathbf{S} = i\omega_x + \mathbf{j}\omega_y + \mathbf{k}\omega_z$$

Hence, the dilatation is expressed in terms of ϕ, and the rotation is expressed in terms of **S**.

Problem Set 8-3

1. Derive Eqs. (8-3.3).

2. Derive Eqs. (8-3.4) and (8-3.5).

8-4 The Galerkin (Papkovich) Vector

Substitution of Eq. (8-3.1) into Eq. (8-2.10) yields

$$\nabla^2[\alpha \nabla \phi + [\text{curl } \mathbf{S}] = -\frac{\mathbf{B}}{G} \tag{8-4.1}$$

where we have utilized the conditions

$$\text{div curl } \mathbf{S} = 0$$

$$\text{div } \nabla \phi = \nabla^2 \phi$$

$$\nabla^2 \nabla \phi = \nabla \nabla^2 \phi \tag{8-4.2}$$

and where

$$\alpha = \frac{2(1-\nu)}{1-2\nu} = \frac{\lambda + 2G}{G}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{8-4.3}$$

Equation (8-4.1) represents the equilibrium equation in terms of the functions ϕ and **S**.

Because the curl of a vector and the divergence of a vector are independent quantities, we take **S** and ϕ in the form (where **W** is an arbitrary vector)

$$\mathbf{S} = -\text{curl } \mathbf{W} \qquad \phi = \frac{1}{\alpha} \text{div } \mathbf{W} \tag{8-4.4}$$

Substitution of Eq. (8-4.4) into Eq. (8-3.1) yields

$$\boldsymbol{\rho} = \frac{1}{\alpha} \nabla \text{ div } \mathbf{W} - \text{curl curl } \mathbf{W} \tag{8-4.5}$$

Noting the relation

$$\text{curl curl} = \nabla \text{ div} - \nabla^2$$

we may write Eq. (8-4.5) in the form

$$\boldsymbol{\rho} = \nabla^2 \mathbf{W} - \frac{1}{2(1-\nu)} \nabla \text{ div } \mathbf{W} \tag{8-4.6}$$

Substitution of Eqs. (8-4.4) into Eq. (8-4.1) yields the equilibrium equation in the form

$$\nabla^2(\nabla \text{ div } \mathbf{W} - \text{curl curl } \mathbf{W}) = -\frac{\mathbf{B}}{G}$$

Again noting the relation curl curl $= \nabla$ div $- \nabla^2$, we find

$$\nabla^2\nabla^2\mathbf{W} = \nabla^4\mathbf{W} = -\frac{\mathbf{B}}{G} \tag{8-4.7}$$

Analogously, substitution of Eq. (8-1.3) into (8-2.10) yields

$$\nabla^2\nabla^2\mathbf{F} = -\frac{\mathbf{B}}{1-\nu} \tag{8-4.8}$$

Comparison of Eqs. (8-4.7) and (8-4.8) [or alternatively Eqs. (8-1.3) and (8-4.6)] yield the following relation between the vector **W** and the vector function **F**:

$$\mathbf{W} = \frac{1-\nu}{G}\mathbf{F} \tag{8-4.9}$$

Problem Set 8-4

1. Derive Eq. (8-4.1).

2. Derive Eq. (8-4.5).

3. Derive Eq. (8-4.8).

8-5 Stress in Terms of the Galerkin Vector F

Equation (8-1.3) defines the displacement components (u, v, w) in terms of $\mathbf{F} = (X, Y, Z)$ by the relations

$$\begin{aligned}
2Gu &= 2(1-\nu)\nabla^2 X - \frac{\partial}{\partial x}\,\mathrm{div}\,\mathbf{F} \\
2Gv &= 2(1-\nu)\nabla^2 Y - \frac{\partial}{\partial y}\,\mathrm{div}\,\mathbf{F} \\
2Gw &= 2(1-\nu)\nabla^2 Z - \frac{\partial}{\partial z}\,\mathrm{div}\,\mathbf{F}
\end{aligned} \tag{8-5.1}$$

Equation (8-1.3) also yields

$$2G\ \mathrm{div}\ \boldsymbol{\rho} = (1-2\nu)\nabla^2\mathrm{div}\,\mathbf{F} \tag{8-5.2}$$

Now, Eqs. (8-2.2), (8-2.8), (8-1.3), (8-5.2), and (8-5.1) yield

$$\begin{aligned}
\boldsymbol{\sigma}_x &= (1-\nu)\left(\boldsymbol{\nabla}\nabla^2 X + \frac{\partial}{\partial x}\nabla^2\mathbf{F}\right) + \left(\mathbf{i}\nu\nabla^2 - \frac{\partial}{\partial x}\boldsymbol{\nabla}\right)\mathrm{div}\,\mathbf{F} \\
\boldsymbol{\sigma}_y &= (1-\nu)\left(\boldsymbol{\nabla}\nabla^2 Y + \frac{\partial}{\partial y}\nabla^2\mathbf{F}\right) + \left(\mathbf{j}\nu\nabla^2 - \frac{\partial}{\partial y}\boldsymbol{\nabla}\right)\mathrm{div}\,\mathbf{F} \\
\boldsymbol{\sigma}_z &= (1-\nu)\left(\boldsymbol{\nabla}\nabla^2 Z + \frac{\partial}{\partial z}\nabla^2\mathbf{F}\right) + \left(\mathbf{k}\nu\nabla^2 - \frac{\partial}{\partial z}\boldsymbol{\nabla}\right)\mathrm{div}\,\mathbf{F}
\end{aligned} \tag{8-5.3}$$

Equations (8-5.3) satisfy Eqs. (8-2.4) (in the absence of body force **B**), provided that Eq. (8-4.8) is satisfied. Substitution of Eqs. (8-5.3) into Eqs. (8-2.2) yields

$$\begin{aligned}
\sigma_x &= 2(1-\nu)\frac{\partial}{\partial x}\nabla^2 X + \left(\nu\nabla^2 - \frac{\partial^2}{\partial x^2}\right)\operatorname{div}\mathbf{F} \\
\sigma_y &= 2(1-\nu)\frac{\partial}{\partial y}\nabla^2 Y + \left(\nu\nabla^2 - \frac{\partial^2}{\partial y^2}\right)\operatorname{div}\mathbf{F} \\
\sigma_z &= 2(1-\nu)\frac{\partial}{\partial z}\nabla^2 Z + \left(\nu\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\operatorname{div}\mathbf{F}
\end{aligned} \tag{8-5.4}$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = (1+\nu)\nabla^2 \operatorname{div}\mathbf{F} \tag{8-5.5}$$

$$\begin{aligned}
\tau_{xy} &= (1-\nu)\left(\frac{\partial}{\partial y}\nabla^2 X + \frac{\partial}{\partial x}\nabla^2 Y\right) - \frac{\partial^2}{\partial x\,\partial y}\operatorname{div}\mathbf{F} \\
\tau_{yz} &= (1-\nu)\left(\frac{\partial}{\partial z}\nabla^2 Y + \frac{\partial}{\partial y}\nabla^2 Z\right) - \frac{\partial^2}{\partial y\,\partial z}\operatorname{div}\mathbf{F} \\
\tau_{zx} &= (1-\nu)\left(\frac{\partial}{\partial x}\nabla^2 Z + \frac{\partial}{\partial z}\nabla^2 X\right) - \frac{\partial^2}{\partial x\,\partial z}\operatorname{div}\mathbf{F}
\end{aligned} \tag{8-5.6}$$

The equations in this section may also be represented in terms of the vector function $\mathbf{W} = \mathbf{i}W_x + \mathbf{j}W_y + \mathbf{k}W_z$.

Problem Set 8-5

1. Derive Eqs. (8-5.3).
2. Derive Eqs. (8-5.4), (8-5.5), and (8-5.6).
3. Express the equations in Section 8-5 in terms of the vector function $\mathbf{W} = \mathbf{i}W_x + \mathbf{j}W_y + \mathbf{k}W_z$.

8-6 The Galerkin Vector: A Solution of the Equilibrium Equations of Elasticity

To show that the Galerkin vector is a general solution of the equilibrium equations of elasticity, it is necessary and sufficient that for any displacement vector $\boldsymbol{\rho}$ there exists an infinite number of vector functions **F** that satisfy Eqs. (8-1.3) and (8-1.4). The criterion of the generality of a given form of solution lies in the possibility of determining the arbitrary functions that occur in the solution so that boundary conditions are fulfilled. To show that an infinite number of vector functions **F** exist, we proceed as follows.

Let

$$\psi = \operatorname{div} \mathbf{F} \tag{8-6.1}$$

Then Eqs. (8-1.3) and (8-5.2) may be written

$$2G\boldsymbol{\rho} = 2(1-\nu)\nabla^2\mathbf{F} - \nabla\psi \tag{8-6.2}$$

and

$$2G\operatorname{div}\ \boldsymbol{\rho} = (1-2\nu)\nabla^2\psi \tag{8-6.3}$$

Next, choose a function ψ_1 that satisfies Eq. (8-6.3). Assume that ψ_1 exists, then, an infinite number of such functions exist, as any harmonic function may be added to ψ_1 without affecting the validity of Eq. (8-6.3). Now substitute ψ_1 into Eq. (8-6.2). Let $F = F_1$ denote a vector that satisfies Eq. (8-6.2). Assume $\mathbf{F}_1$ exists; then, an infinite number of such functions exist, as any harmonic vector function may be added to $\mathbf{F}_1$ without violating Eq. (8-6.2).

To demonstrate this last statement, select a Galerkin vector in the form

$$\mathbf{F}_2 = \mathbf{F}_1 + \mathbf{k}\chi \tag{8-6.4}$$

The vector $\mathbf{F}_2$ is required to satisfy Eq. (8-1.3). Also, by Eqs. (8-6.2), (8-6.3), and (8-6.4), we have [also noting Eq. (8-6.1)]

$$2G\boldsymbol{\rho} = 2(1-\nu)\nabla^2\mathbf{F}_1 - \nabla\psi_1 \tag{8-6.5}$$

$$2G\operatorname{div}\boldsymbol{\rho} = (1-2\nu)\nabla^2\psi_1 \tag{8-6.6}$$

provided that

$$\begin{aligned} \nabla^2\chi &= 0 \\ \frac{\partial\chi}{\partial z} &= \psi_1 - \operatorname{div}\mathbf{F}_1 \end{aligned} \tag{8-6.7}$$

A function χ that satisfies Eq. (8-6.7) can be determined only if $(\psi_1 - \operatorname{div}\mathbf{F})$ is harmonic. For example, let $(\psi_1 - \operatorname{div}\mathbf{F}_1) = \theta$. Then $\partial\chi/\partial z = \theta$, and $\nabla^2\chi = 0$ yields

$$\nabla^2\frac{\partial\chi}{\partial z} = \frac{\partial}{\partial z}\nabla^2\chi = \nabla^2\theta = 0$$

Hence $\theta = (\psi_1 - \operatorname{div}\mathbf{F}_1)$ must be harmonic. Alternatively, we have by Eq. (8-6.5)

$$2G\operatorname{div}\boldsymbol{\rho} = 2(1-\nu)\nabla^2\operatorname{div}\mathbf{F}_1 - \nabla^2\psi_1 \tag{8-6.8}$$

Subtraction of Eq. (8-6.8) from Eq. (8-6.6) yields

$$\nabla^2(\psi_1 - \operatorname{div}\mathbf{F}_1) = 0$$

Hence, $\psi_1 - \operatorname{div}\mathbf{F}_1$ is harmonic. The harmonic function χ may also be added to Eq. (8-6.4) in the forms $\boldsymbol{i}\chi$ and $\boldsymbol{j}\chi$. That is, the vector

$$\boldsymbol{\chi} = \mathbf{i}\chi + \mathbf{j}\chi + \mathbf{k}\chi \tag{8-6.9}$$

may be added to Eq. (8-6.4) without altering the above argument. The proof that states the generality of the Galerkin vector is complete.

Problem Set 8-6

1. Derive Eq. (8-6.5).

2. Derive Eq. (8-6.8).

3. Generalize the results of Section 8-6 for the vector $\boldsymbol{\chi}$ defined by Eq. (8-6.9).

8-7 The Galerkin Vector kZ and Love's Strain Function for Solids of Revolution

Let

$$\mathbf{F} = \mathbf{k}Z \tag{8-7.1}$$

where Z may be considered a general function of the rectangular space coordinates (x, y, z), or alternatively of cylindrical coordinates (r, θ, z), and so on.

For rectangular Cartesian coordinates (x, y, z)

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{8-7.2}$$

For cylindrical coordinates (r, θ, z)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \tag{8-7.3}$$

Substitution of Eq. (8-7.1) into Eq. (8-4.8) yields

$$\nabla^4 Z = -\frac{B_z}{1-\nu} \tag{8-7.4}$$

If the body force B_z is zero, Z is biharmonic; that is, $\nabla^4 Z = 0$.

By Eqs. (8-1.3), (8-5.1), (8-5.5), (8-7.1), and (8-5.4), we have for either rectangular or cylindrical coordinates

$$2G\boldsymbol{\rho} = \mathbf{k}[2(1-\nu)\nabla^2 Z] - \boldsymbol{\nabla}\frac{\partial Z}{\partial z} \tag{8-7.5}$$

$$2Gw = \left[2(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right] Z \tag{8-7.6}$$

$$I_1 = (1+\nu)\frac{\partial}{\partial z}(\nabla^2 Z) \tag{8-7.7}$$

$$\text{div } \mathbf{F} = \frac{\partial Z}{\partial z} \tag{8-7.8}$$

$$\sigma_z = \frac{\partial}{\partial z}\left\{\left[(2-\nu)\boldsymbol{\nabla}^2 - \frac{\partial^2}{\partial z^2}\right] Z\right\} \tag{8-7.9}$$

For rectangular Cartesian coordinates. Eqs. (8-5.1), (8-5.4), and (8-5.6) yield

$$2Gu = -\frac{\partial^2 Z}{\partial x\,\partial z} \qquad 2Gv = -\frac{\partial^2 Z}{\partial y\,\partial z} \tag{8-7.10}$$

$$\sigma_x = \frac{\partial}{\partial z}\left[\left(\nu\boldsymbol{\nabla}^2 - \frac{\partial^2}{\partial x^2}\right) Z\right] \qquad \sigma_y = \frac{\partial}{\partial z}\left[\left(\nu\boldsymbol{\nabla}^2 - \frac{\partial^2}{\partial y^2}\right) Z\right] \tag{8-7.11}$$

$$\tau_{xy} = -\frac{\partial^3 Z}{\partial x \partial y \partial z} \tag{8-7.12}$$

$$\tau_{yz} = \frac{\partial}{\partial y}\left\{\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right] Z\right\} = \frac{\partial}{\partial y}\left[\left(-\nu\boldsymbol{\nabla}^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) Z\right] \tag{8-7.13}$$

$$\tau_{xz} = \frac{\partial}{\partial x}\left\{\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right] Z\right\} = \frac{\partial}{\partial x}\left[\left(-\nu\boldsymbol{\nabla}^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) Z\right] \tag{8-7.14}$$

Noting the following relations between rectangular and cylindrical coordinates,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r} \qquad \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial r^2} \qquad \frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial \theta} \qquad \frac{\partial^2}{\partial y^2} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z} \qquad \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial z^2} \qquad \frac{\partial^2}{\partial x\,\partial y} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) = \frac{\partial^2}{\partial r \partial \theta}\left[\frac{1}{r}(\,)\right]$$

$$\frac{\partial^2}{\partial y \partial z} = \frac{1}{r}\frac{\partial^2}{\partial \theta\,\partial z} \qquad \frac{\partial^2}{\partial z \partial x} = \frac{\partial^2}{\partial z \partial r}$$

we may write, for cylindrical coordinates, formulas that correspond to Eqs. (8-7.10) through (8-7.14). Thus,

$$2Gu_r = -\frac{\partial^2 Z}{\partial r\, \partial z} \qquad 2Gv_\theta = -\frac{1}{r}\frac{\partial^2 Z}{\partial \theta \partial z} \tag{8-7.15}$$

$$\sigma_r = \frac{\partial}{\partial z}\left[\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right) Z\right] \tag{8-7.16}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) Z\right] \tag{8-7.17}$$

$$\tau_{r\theta} = -\frac{\partial^3}{\partial r \partial \theta \partial z}\left(\frac{Z}{r}\right) \tag{8-7.18}$$

$$\tau_{\theta z} = \frac{1}{r}\frac{\partial}{\partial \theta}\left\{\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right] Z\right\} \tag{8-7.19}$$

$$\tau_{zr} = \frac{\partial}{\partial r}\left\{\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right] Z\right\} \tag{8-7.20}$$

If $Z = Z(r, z)$, it is identical to the strain function introduced in 1960 by A. E. H. Love (2009) for solids of revolution loaded symmetrically about the axis of revolution z. For axisymmetrical cases,

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{8-7.21}$$

and dependency on θ vanishes. Hence, $v_\theta, \tau_{r\theta}, \tau_{\theta z}$ vanish identically in Eqs. (8-7.15), (8-7.18), and (8-7.19), while Eqs. (8-7.16), (8-7.17), and (8-7.20) are simplified accordingly. Hence, in terms of cylindrical coordinates (r, θ, z), for axially symmetric situations, $Z = Z(r, z)$, and the stress components become

$$\sigma_r = \frac{\partial}{\partial z}\left[\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right) Z\right] \qquad \sigma_\theta = \frac{\partial}{\partial z}\left[\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right) Z\right]$$

$$\sigma_z = \frac{\partial}{\partial z}\left\{\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right] Z\right\}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left\{\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right] Z\right\} \tag{8-7.22}$$

Problem Set 8-7

1. Verify Eq. (8-7.4).

2. Derive Eqs. (8-7.5) through (8-7.9).

8-8 Kelvin's Problem: Single Force Applied in the Interior of an Infinitely Extended Solid

Let the z axis be taken positive vertically downward. Let a point force $2\mathbf{k}P$ be applied at the origin of coordinates r, z and be directed along z. Because the problem is axially symmetrical, Love's strain function $Z(r, z)$ hence Galerkins' vector $\mathbf{k}Z(r, z)$, may be applied to the problem

Consider

$$R^2 = r^2 + z^2 \tag{8-8.1}$$

The function R is biharmonic; that is,

$$\nabla^2\nabla^2 R = 0 \tag{8-8.2}$$

Accordingly, we take

$$Z = BR \tag{8-8.3}$$

where B is a constant because $\nabla^2\nabla^2 Z = 0$, provided the body force $B_z = 0$. By Eq. (8-8.3), we have

$$\begin{aligned}
\frac{\partial Z}{\partial r} &= \frac{Br}{R} \qquad \frac{\partial^2 Z}{\partial r^2} = B\left(\frac{1}{R} - \frac{r^2}{R^3}\right) = \frac{Bz^2}{R^3} \\
\frac{\partial Z}{\partial z} &= \frac{Bz}{R} \qquad \frac{\partial^2 Z}{\partial z^2} = B\left(\frac{1}{R} - \frac{z^2}{R^3}\right) = \frac{Br^2}{R^3} \\
\nabla^2 Z &= \frac{2B}{R}
\end{aligned} \tag{8-8.4}$$

By Eqs. (8-7.6), (8-7.7), (8-7.15), (8-7.22), and (8-8.4), we obtain

$$\begin{aligned}
2Gu_r &= -\frac{\partial^2 Z}{\partial r\,\partial z} = \frac{Brz}{R^3} \\
2Gw &= \left[(1-2\nu)\nabla^2 + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right] Z \\
&= B\left[\frac{2(1-2\nu)}{R} + \frac{1}{R} + \frac{z^2}{R^3}\right]
\end{aligned} \tag{8-8.5}$$

and

$$\begin{aligned}
I_1 &= (1+\nu)\frac{\partial \nabla^2 Z}{\partial z} = -\frac{2(1+\nu)Bz}{R^3} \\
\sigma_r &= \frac{\partial}{\partial z}\left[\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right) Z\right] = B\left[\frac{(1-2\nu)z}{R^3} - \frac{3r^2 z}{R^5}\right]
\end{aligned}$$

$$\sigma_\theta = \frac{\partial}{\partial z}\left[\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)Z\right] = \frac{(1-2\nu)Bz}{R^3}$$

$$\sigma_z = \frac{\partial}{\partial z}\left\{\left[(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]Z\right\} = -B\left[\frac{(1-2\nu)z}{R^3} + \frac{3z^3}{R^5}\right]$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left\{\left[(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right]Z\right\} = -B\left[\frac{(1-2\nu)r}{R^3} + \frac{3rz^2}{R^5}\right] \qquad (8\text{-}8.6)$$

We note that if $\nu = \frac{1}{2}$, the stresses are zero for $z = 0$ except for the point $r = 0$. (See Problem 8-8.3.)

The constant B is determined by considering the forces that act on a spherical cavity surrounding the origin of coordinates (the point of application of the point force). Alternatively, we may consider the total vertical load that acts on a plane $z =$ constant. For $z > 0$, this total load should be $-P$. For $z < 0$, it is P. For constant z, Eq. (8-8.1) yields

$$r\,dr = R\,dR \qquad (8\text{-}8.7)$$

Hence, for $z > 0$,

$$-P = \int_0^\infty 2\pi r\,dr(\sigma_z) = \int_z^\infty 2\pi R\,dR(\sigma_z)$$

or

$$P = 2\pi B\left[(1-2\nu)z\int_z^\infty \frac{R\,dR}{R^3} + 3z^3\int_z^\infty \frac{RdR}{R^5}\right.$$
$$= 4\pi(1-\nu)B$$

Consequently,

$$B = \frac{P}{4\pi(1-\nu)} \qquad (8\text{-}8.8)$$

By Eqs. (8-8.6) we may show that the shearing stress τ_{rz} on a cylindrical surface $r =$ constant between any two horizontal planes $z = \pm$ constant converges to zero as $r \to \infty$. Hence, the total applied load between the two planes is $2\mathbf{k}P$. Furthermore, it may be noted that $\sigma_z = 0$ for the plane $z = 0$ except for the point $r = 0$.

Accordingly, Kelvin's problem is solved by Eqs. (8-8.5), (8-8.6), and (8-8.8). The solution to Kelvin's problem may be employed to build solutions to other problems of axial symmetry (Timoshenko and Goodier, 1970; Luré, 1964).

Problem Set 8-8

1. Derive Eqs. (8-8.5).

2. Derive Eqs. (8-8.6).

3. Show that for $\nu = \frac{1}{2}$, with the force $\mathbf{k}P$ acting at the origin of the coordinates of one-half of the infinite space (solid) that was considered in the Kelvin problem, Eqs. (8-8.5) and (8-8.6) give the solution of the problem of a point load on the half-space with $\nu = \frac{1}{2}$ (Boussinesq's problem). [See Westergaard (1952, p. 136), and Boussinesq (1885).] For $\nu \neq \frac{1}{2}$, see the section that follows.

8-9 The Twinned Gradient and Its Application to Determine the Effects of a Change of Poisson's Ratio

Westergaard (1940) has devised a procedure suitable for certain problems of elasticity that have a simple solution for a particular value of Poisson's ratio, say, ν_0. For example, he showed that Boussinesq's problem of normal force (Boussinesq, 1885) and Cerruti's problem of a tangential force (Cerruti, 1882) acting on the plane surface of a semi-infinite solid may be solved when Poisson's ratio is $\nu_0 = \frac{1}{2}$ by the results of Kelvin's problem (Section 8-8 and Problem 8-8.3) of a force at a point in the interior of an infinite solid. In the case where Poisson's ratio $\nu_0 = \frac{1}{2}$, the solution of Kelvin's problem can be represented in terms of one principal stress at each point, acting along a radial line from the point of application of the force. The other principal stresses are zero, and one-half of the total force may be assigned to one-half of the infinite solid. For other values of Poisson's ratio $\nu \neq \frac{1}{2}$ (based on the positive definiteness of the strain energy, $\nu < \frac{1}{2}$; see Chapter 4, Section 4-6), certain terms must be added to the formulas for the displacements and stresses. The previous derivations of the Boussinesq solution and particularly for the Cerruti solution are somewhat lengthy. Westergaard overcame this difficulty by the application of a simple analytical device, which he called the *twinned gradient*. The displacement to be added to the solution for $\nu_0 = \frac{1}{2}$ to obtain the solution for the case $\nu \neq \frac{1}{2}$ is obtained as the gradient of a potential except that one of the components is replaced by its so-called *twin*, an identical component but in the reverse direction (opposite sign). The procedure is outlined below.

The Twinned Gradient Method. Assume that a set of displacements (denoted by primes)

$$\boldsymbol{\rho}' = \mathbf{i}u' + \mathbf{j}v' + \mathbf{k}w' \tag{8-9.1}$$

and stresses σ' and τ' have been found for a particular value ν_0 of Poisson's ratio and for body forces per unit volume $\mathbf{B}'$. Then by Eqs. (8-2.10) and (8-9.1),

$$G\left(\nabla^2 + \frac{1}{1-2\nu_0}\nabla\text{div}\right)\boldsymbol{\rho}' + \mathbf{B}' = 0 \tag{8-9.2}$$

The corresponding stresses are defined by the equations [Eqs. (8-2.8)]

$$\sigma_x' = 2G\left(\frac{\partial u'}{\partial x} + \frac{\nu_0}{1-2\nu_0}\text{div}\boldsymbol{\rho}'\right), \ldots, \ldots$$

$$\tau'_{xy} = G\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right), \ldots, \ldots$$

$$I'_1 = \sigma'_x + \sigma'_y + \sigma'_z = \frac{2(1+\nu_0)G}{1-2\nu_0}\operatorname{div}\boldsymbol{\rho}' \tag{8-9.3}$$

Assume now that Poisson's ratio is changed from ν_0 to ν, while the shear modulus of elasticity G remains unchanged. Also assume for the time being that the displacements $\boldsymbol{\rho}'$ are held unchanged. Then according to Eqs. (8-2.8), the shearing stresses will not be changed, but each normal stress σ' will receive the increment

$$\begin{aligned}\overline{\sigma} &= 2G\left(\frac{\nu}{1-2\nu} - \frac{\nu_0}{1-2\nu_0}\right)\operatorname{div}\boldsymbol{\rho}' \\ &= \frac{2(\nu-\nu_0)G}{(1-2\nu)(1-2\nu_0)}\operatorname{div}\boldsymbol{\rho}' = \frac{(\nu-\nu_0)I'_1}{(1+\nu_0)(1-2\nu)}\end{aligned} \tag{8-9.4}$$

Because in this correction the displacements $\boldsymbol{\rho}'$ remained unchanged, further corrections must be provided. The correction is completed by adding additional displacements $\boldsymbol{\rho}''$ to $\boldsymbol{\rho}'$, thereby obtaining the total displacements

$$\boldsymbol{\rho} = \boldsymbol{\rho}' + \boldsymbol{\rho}'' \tag{8-9.5}$$

and the final stresses

$$\sigma = \sigma' + \sigma'' \qquad \tau = \tau' + \tau'' \tag{8-9.6}$$

which require the final body forces

$$\mathbf{B} = \mathbf{B}' + \mathbf{B}'' \tag{8-9.7}$$

Westergaard (1940) investigated the possibility of expressing the correction $\boldsymbol{\rho}''$ by means of a *twinned gradient*. He gave the operator

$$\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} - \boldsymbol{k}\frac{\partial}{\partial z} = \nabla - 2\mathbf{k}\frac{\partial}{\partial z} \tag{8-9.8}$$

this name because the part $\mathbf{k}(\partial/\partial z)$ in ∇ is replaced by its "twin" $-\mathbf{k}(\partial/\partial z)$. He then expressed the correction $\boldsymbol{\rho}''$ in terms of the negative of the twinned gradient operating on a scalar function Φ as

$$2G\boldsymbol{\rho}'' = -\mathbf{i}\frac{\partial\Phi}{\partial x} - \mathbf{j}\frac{\partial\Phi}{\partial y} + \mathbf{k}\frac{\partial\Phi}{\partial z} = -\nabla\Phi + 2\mathbf{k}\frac{\partial\Phi}{\partial z} \tag{8-9.9}$$

By Eqs. (8-9.9) and (8-2.8), we find

$$\tau''_{xy} = -\frac{\partial^2\Phi}{\partial x\,\partial y} \qquad \tau''_{xz} = \tau''_{yz} = 0 \tag{8-9.10}$$

and

$$2G\text{div}\,\boldsymbol{\rho}'' = -\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = -\nabla^2\Phi + 2\frac{\partial^2\Phi}{\partial z^2} \tag{8-9.11}$$

Then, one may obtain the total additional normal stresses σ'' that must be added to σ' to obtain the final stresses by means of Eq. (8-2.8). For example, for σ_z'' we have (because the normal stresses σ' were increased previously by $\overline{\sigma}$)

$$\sigma_z'' = \overline{\sigma} + 2G\left(\frac{\partial w''}{\partial z} + \frac{\nu}{1-2\nu}\text{div}\,\boldsymbol{\rho}''\right) \tag{8-9.12}$$

By Eq. (8-9.1) with double primes, Eqs. (8-9.4) and (8-9.9), we have alternatively for Eq. (8-9.12)

$$\sigma_z'' = \frac{(\nu-\nu_0)I_1''}{(1+\nu_0)(1-2\nu)} + \frac{1}{1-2\nu}\left(\frac{\partial^2\Phi}{\partial z^2} - \nu\nabla^2\Phi\right) \tag{8-9.13}$$

Westergaard (1940) arbitrarily took $\sigma_z'' = 0$ with the objective of selecting Φ accordingly and computing the remaining stresses and body forces so that equilibrium is maintained. Hence, by Eq. (8-9.13), Φ is taken so that

$$\frac{\partial^2\Phi}{\partial z^2} - \nu\nabla^2\Phi = \frac{(\nu_0-\nu)I_1''}{1+\nu_0} \tag{8-9.14}$$

Because $\sigma_z'' = 0, \sigma_x''$ and σ_y'' may be computed in terms of Φ by means of Eqs. (8-9.3) using double primes. Thus, with Eqs. (8-9.1) and (8-9.9)

$$\sigma_x'' = \sigma_x'' - \sigma_z'' = 2G\left(\frac{\partial u''}{\partial x} - \frac{\partial w''}{\partial z}\right) = -\frac{\partial^2\Phi}{\partial x^2} - \frac{\partial^2\Phi}{\partial z^2} \tag{8-9.15}$$

or

$$\sigma_x'' = \frac{\partial^2\Phi}{\partial y^2} - \nabla^2\Phi$$

and similarly for σ_y''. Thus, the complete set of stresses produced by the changes is

$$\sigma_x'' = \frac{\partial^2\Phi}{\partial y^2} - \nabla^2\Phi \qquad \sigma_y'' = \frac{\partial^2\Phi}{\partial x^2} - \nabla^2\Phi \qquad \tau_{xy}'' = -\frac{\partial^2\Phi}{\partial x\partial y}$$
$$\sigma_z'' = \tau_{xz}'' = \tau_{yz}'' = 0 \tag{8-9.16}$$

The body forces required to maintain the stresses of Eq. (8-9.16) in a state of equilibrium are determined by the equations

$$\frac{\partial\sigma_x''}{\partial x} + \frac{\partial\tau_{xy}''}{\partial y} + B_x'' = 0 \qquad \frac{\partial\tau_{xy}''}{\partial x} + \frac{\partial\sigma_y''}{\partial y} + B_y'' = 0 \tag{8-9.17}$$

Hence

$$B''_x = \frac{\partial(\nabla^2\Phi)}{\partial x} \qquad B''_y = \frac{\partial(\nabla^2\Phi)}{\partial y} \tag{8-9.18}$$

or

$$\mathbf{B}'' = \mathbf{i}\frac{\partial}{\partial x}(\nabla^2\Phi) + \mathbf{j}\frac{\partial}{\partial y}(\nabla^2\Phi)$$

In many important problems, the original and the final body forces are zero. Then, $\mathbf{B}' = \mathbf{B}'' = \mathbf{B} = 0$, and Eq. (8-9.18) is satisfied if $\nabla^2\Phi = 0$, that is, if Φ is a harmonic function. In addition, Eq. (8-9.14) reduces to the condition

$$\frac{\partial^2\Phi}{\partial z^2} = \frac{(\nu_0 - \nu)I'_1}{1 + \nu_0} \tag{8-9.19}$$

It is possible to satisfy the equation $\nabla^2\Phi = 0$ and Eq. (8-9.19) at the same time, as $\nabla^2 I'_1 = 0$; that is, I'_1 is a harmonic function. Also, because $\nabla^2\Phi = 0$, Φ becomes an Airy stress function of (x, y); see Eqs. (8-9.17) and Chapter 5, Section 5-4.

Some of the equations developed above become indefinite when $\nu_0 = \frac{1}{2}$. However, considerations of convergence show that the remaining equations remain applicable for this limiting case. This conclusion is also reached by a direct study of the limiting case of $\nu_0 = \frac{1}{2}$.

8-10 Solutions of the Boussinesq and Cerruti Problems by the Twinned Gradient Method

The Boussinesq Problem. The Boussinesq problem is that of determining the displacements and the stresses in a half-space (semi-infinite solid) occupying the region $z > 0$ and subjected to a load $\mathbf{k}P$ acting at the origin of coordinates (x, y, z), where (x, y) are rectangular coordinates in the bounding plane of the half-space, and coordinate z, perpendicular to the bounding plane, may be visualized as being directed vertically, positive downward.

In Kelvin's problem (Section 8-8) a solid is considered that extends infinitely in all directions. At the origin of coordinates a concentrated force $2P$ is applied in the direction of z, which is assumed to be directed vertically, positive downward. In the Boussinesq problem the semi-infinite solid occupying the region $z \geq 0$ is considered, with concentrated normal force P acting at the origin of coordinates (x, y, z) in the direction of positive z.

The Cerruti Problem. In the Cerruti problem the semi-infinite solid $z \geq 0$ is acted on by a force P applied at the origin of coordinates (x, y, z) but directed in the positive x direction; that is, force P lies in the bounding (x, y) plane of the region. In the following we develop the solutions for the Boussinesq and the Cerruti problems by the method of twinned gradient.

Solutions for $\nu = \frac{1}{2}$. When Poisson's ratio $\nu = \frac{1}{2}$, the Kelvin, Boussinesq, and Cerruti solutions can all be expressed by the same formulas. In considering these solutions, it is appropriate to use cylindrical coordinates (r, θ, z) and the polar coordinate R, in addition to axes (x, y, z), where r and R are defined by

$$R^2 = r^2 + z^2 = x^2 + y^2 + z^2 \tag{8-10.1}$$

For $\nu = \frac{1}{2}$, Eqs. (8-8.5) yield

$$u'_r = \frac{P}{4\pi G}\frac{rz}{R^3} \qquad w' = \frac{P(R^2 + z^2)}{4\pi G R^3} \tag{8-10.2}$$

In terms of (x, y, z) coordinates, using Eq. (8-9.1) we may write [Eqs. (8-7.10) with $Z = Br$; Eq. (8-8.3)] with $\mathbf{r} = \mathbf{i}x + \mathbf{j}y$

$$\boldsymbol{\rho}' = \frac{P}{4\pi G R^3}[\mathbf{i}xz + \mathbf{j}yz + \mathbf{k}(R^2 + z^2)] \tag{8-10.3}$$

The principal stress in the direction of the radius vector R is equal to I'_1, as the other two principal stresses are zero. Thus, by the expression for I_1 given in Eq. (8-8.6), we have

$$\sigma'_R = I'_1 = -\frac{3P}{2\pi}\frac{z}{R^3} \tag{8-10.4}$$

With $Z = BR$, the corresponding components of stress in cylindrical coordinates (r, θ, z) and rectangular coordinates (x, y, z) are, from Eqs. (8-8.6) and Eqs. (8-7.9) through (8-7.14), respectively,

$$\begin{aligned} &\sigma'_r = -\frac{3P}{2\pi}\frac{r^2 z}{R^3} \qquad \sigma'_\theta = 0 \qquad \sigma'_z = -\frac{3P}{2\pi}\frac{z^3}{R^5} \\ &\tau'_{r\theta} = \tau'_{\theta z} = 0 \qquad \tau'_{rz} = -\frac{3P}{2\pi}\frac{rz^2}{R^5} \end{aligned} \tag{8-10.5}$$

and

$$\begin{aligned} &\sigma'_x = -\frac{3P}{2\pi}\frac{x^2 z}{R^5} \qquad \sigma'_y = -\frac{3P}{2\pi}\frac{y^2 z}{R^5} \qquad \sigma'_z = -\frac{3P}{2\pi}\frac{z^3}{R^5} \\ &\tau'_{xy} = -\frac{3P}{2\pi}\frac{xyz}{R^5} \qquad \tau'_{yz} = -\frac{3P}{2\pi}\frac{yz^2}{R^5} \qquad \tau'_{xz} = -\frac{3P}{2\pi}\frac{xz^2}{R^5} \end{aligned} \tag{8-10.6}$$

Boussinesq's Problem for Any Value of Poisson's Ratio. With $\nu_0 = \frac{1}{2}$ and with I'_1 given by Eq. (8-10.4), Eq. (8-9.19) yields

$$\begin{aligned}\frac{\partial^2\Phi}{\partial z^2} &= \frac{1-2\nu}{3}I_1' = -\frac{(1-2\nu)P}{2\pi}\frac{z}{R^3} \\ \frac{\partial\Phi}{\partial z} &= \frac{(1-2\nu)P}{2\pi R} \qquad \Phi = \frac{(1-2\nu)P}{2\pi}\log(R+z)\end{aligned} \tag{8-10.7}$$

which shows that Φ is a harmonic function. The displacements to be added to Eqs. (8-10.2) and (8-10.3) are, by Eq. (8-9.9),

$$\begin{aligned}u_r'' &= -\frac{1}{2G}\frac{\partial\Phi}{\partial r} = -\frac{(1-2\nu)P}{4\pi G}\frac{r}{R(R+z)} \\ w'' &= \frac{1}{2G}\frac{\partial\Phi}{\partial z} = \frac{(1-2\nu)P}{4\pi GR}\end{aligned} \tag{8-10.8}$$

The two stress components that must be added to those in Eqs. (8-10.5) are, by Eqs. (8-9.16), with $\nabla^2\Phi = 0$,

$$\begin{aligned}\sigma_r'' &= \frac{1}{r}\frac{\partial\Phi}{\partial r} = \frac{(1-2\nu)P}{2\pi}\frac{1}{R(R+z)} \\ \sigma_\theta'' &= \frac{\partial^2\Phi}{\partial r^2} = -\frac{1}{r}\frac{\partial\Phi}{\partial r} - \frac{\partial^2\Phi}{\partial z^2} = \frac{(1-2\nu)P}{2\pi}\left[-\frac{1}{R(R+z)} + \frac{z}{R^3}\right]\end{aligned} \tag{8-10.9}$$

Cerruti's Problem for Any Value of Poisson's Ratio. In treating the Cerruti problem, it is expedient to change the directions so that the force P that acts at the origin of coordinates will act in the direction of the positive x axis on the semi-infinite solid $z \geq 0$. Then, Eqs. (8-10.1) to (8-10.6) must be interpreted under the cyclic changes $(x, y, z) \rightarrow (y, z, x)$ and $(\mathbf{i}, \mathbf{j}, \mathbf{k}) \rightarrow (\mathbf{j}, \mathbf{k}, \mathbf{i})$. Then, by Eqs. (8-9.14) with $\nabla^2\Phi = 0$ and $\nu_0 = \frac{1}{2}$,

$$\frac{\partial^2\Phi}{\partial z^2} = \frac{1-2\nu}{3}I_1' = -\frac{(1-2\nu)P}{2\pi}\frac{x}{R^3} \tag{8-10.10}$$

Integrations yield

$$\frac{\partial\Phi}{\partial z} = \frac{(1-2\nu)P}{2\pi}\frac{x}{R(R+z)} \qquad \Phi = -\frac{(1-2\nu)P}{2\pi}\frac{x}{R+z} \tag{8-10.11}$$

By substituting the function Φ into Eqs. (8-9.9) and (8-9.16) and adding the values to those derived from Eqs. (8-10.3) and (8-10.6), we obtain the following formulas, which are the solution of the Cerruti problem:

$$u = \frac{P}{4\pi GR}\left[1 + \frac{x^2}{R^2} + (1-2\nu)\left(\frac{R}{R+z} - \frac{x^2}{(R+z)^2}\right)\right]$$

$$v = \frac{P}{4\pi GR}\left[\frac{xy}{R^2} - \frac{(1-2\nu)xy}{(R+z)^2}\right]$$

$$w = \frac{P}{4\pi GR}\left[\frac{xz}{R^2} + \frac{(1-2\nu)x}{R+z}\right]$$

$$\sigma_x = \frac{Px}{2\pi R^3}\left[-\frac{3x^2}{R^2} + \frac{1-2\nu}{(R+z)^2}\left(R^2 - y^2 - \frac{2Ry^2}{R+z}\right)\right]$$

$$\sigma_y = \frac{Px}{2\pi R^3}\left[-\frac{3y^2}{R^2} + \frac{1-2\nu}{(R+z)^2}\left(3R^2 - x^2 - \frac{2Rx^2}{R+z}\right)\right]$$

$$\sigma_z = -\frac{3Pxz^2}{2\pi R^5} \qquad I_1 = -\frac{(1+\nu)Px}{\pi R^3}$$

$$\tau_{xy} = \frac{Py}{2\pi R^3}\left[-\frac{3x^2}{R^2} + \frac{1-2\nu}{(R+z)^2}\left(-R^2 + x^2 + \frac{2Rx^2}{R+z}\right)\right]$$

$$\tau_{yz} = -\frac{3Pxyz}{2\pi R^5} \qquad \tau_{xz} = -\frac{3Px^2z}{2\pi R^5} \tag{8-10.12}$$

Normal Line Load on a Semi-Infinite Orthotropic Solid. The case of a normal load distributed uniformly along a strip of width 2ϵ on the bounding plane of a semi-infinite solid with orthotropic material properties has been presented by Lekhnitskii (1981). As in the case of an isotropic semi-infinite solid, the stress distribution is radially directed and is symmetrical with respect to the centerline of the strip. However, depending upon the relative magnitude of the elastic coefficients, the trajectories of constant radial stress may vary considerably from the circle trajectories in the case of the isotropic solid.

Problem Set 8-10

1. Verify the results given by Eqs. (8-10.5) and (8-10.6).

2. Verify the results given by Eqs. (8-10.8) and (8-10.9).

3. Derive the expression for Φ given by Eqs. (8-10.7).

4. Derive the formula for Φ given by Eq. (8-10.11).

5. Derive the solution for the Cerruti problem [Eqs. (8-10.12)].

8-11 Additional Remarks on Three-Dimensional Stress Functions

Four types of stress functions are known for solving elasticity problems. They are (1) the components of the displacement vector, (2) the components of the Galerkin vector, (3) the Maxwell stress functions, and (4) the Morera stress functions

(Langhaar and Stippes, 1954). For problems with stress-type boundary conditions, the Maxwell stress functions are in many respects the simplest to use, but they lack the simple transformation properties of vectors. It was shown by Weber (1948) that the Maxwell and Morera functions supplement each other, and that together they are the components of a second-order symmetric Cartesian tensor.

Langhaar and Stippes (1954) developed the compatibility equations for an isotropic Hookean body subjected to boundary stresses and temperature gradients in terms of the Maxwell stress functions, and they presented the general solution for steady temperature fields. They also showed that when the complementary energy of a homogeneous body with arbitrary elastic properties is expressed in terms of the components of the Maxwell–Morera tensor, the Euler equations for the integral of the complementary energy density are the complete set of compatibility equations in terms of the stress components. In addition, they generalized the Maxwell–Morera tensor so that it represents the general solution of the equilibrium equations in any curvilinear coordinate system. As an application, they gave the general solution of the equilibrium equations in cylindrical coordinates.

A comprehensive discussion of stress functions has been presented in a review article by Sternberg (1960), which gives an extensive list of references dating to 1958. In a highly mathematical treatment, Marsden and Hughes (1994), using modern differential geometry and functional analysis, discuss parts of the mathematical foundations of three-dimensional elasticity. As stated in their preface, their work "is intended for mathematicians, engineers, and physicists who wish to see this classical subject in a modern setting and to see examples of what newer mathematical tools have to contribute." Several hundred references are listed.

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