MEEM 3700
Mechanical Vibrations

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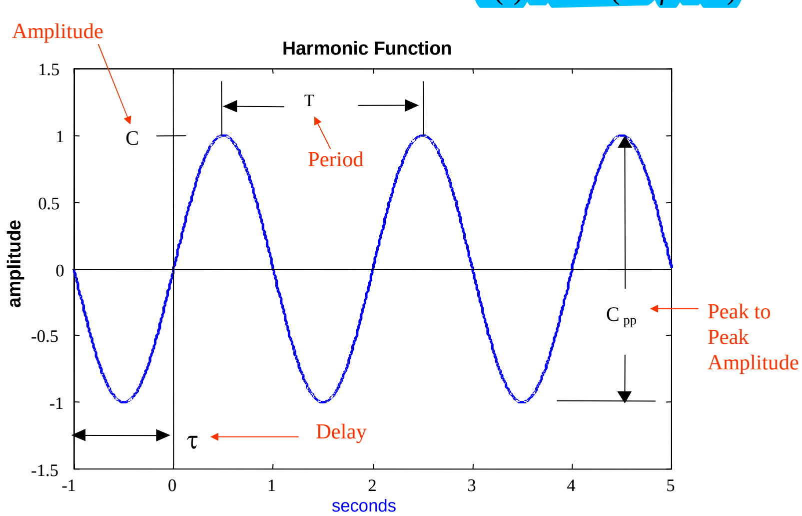
Harmonic Motion

$$x(t) = C\sin(2\pi ft + \Phi)$$



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Harmonic Motion: Basic Equations

$T = \text{period} = \frac{1}{f}(\text{sec})$ ⇒ $f = \frac{1}{T}\left(\frac{cycles}{\text{sec}}\right) \rightarrow (Hz)$ ⇒ $\omega = 2\pi f\left(\frac{rad}{\text{sec}}\right)$

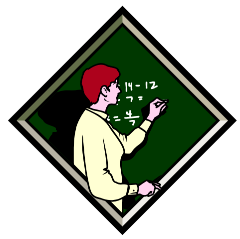
$C =$ amplitude

$C_{pp} =$ peak to peak amplitude

$$C_{\text{rms}} = \left[\frac{1}{T}\int_0^T C^2 \sin^2(2\pi ft + \Phi)dt\right]^{1/2} = \frac{\sqrt{2}}{2}C = 0.707C$$

rms=root mean square

$\Phi = \text{phase} = \omega\tau = 2\pi f\tau$



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Harmonic Motion: Basic Equations

$$x(t) = C\sin(\omega t + \phi)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$$

$$x(t) = C\sin(\phi)\cos\omega t + C(\phi)\sin\omega t$$

$A = C\sin\phi$

$B = C\cos\phi$

$$x(t) = A\cos\omega t + B\sin\omega t$$

Equivalent Forms

Harmonic Motion: Basic Equations

$$x(t) = A\cos\omega t + B\sin\omega t$$

$$A = C\sin\phi$$
$$B = C\cos\phi$$

$$\frac{A}{B} = \frac{C\sin\phi}{C\cos\phi}$$

$$\frac{A}{B} = \tan\phi$$

$$A^2 + B^2 = C^2\sin^2(\phi) + C^2\cos^2(\phi)$$

$$C^2\left(\sin^2(\phi) + \cos^2(\phi)\right) = A^2 + B^2$$

$$\phi = \tan^{-1}\left(\frac{A}{B}\right)$$

$$C = \sqrt{A^2 + B^2}$$

$$x(t) = C\sin(\omega t + \phi)$$

Harmonic Motion

$$x(t) = C\sin(2\pi ft + \Phi)$$
$$x(t) = C\sin(\omega t + \Phi)$$

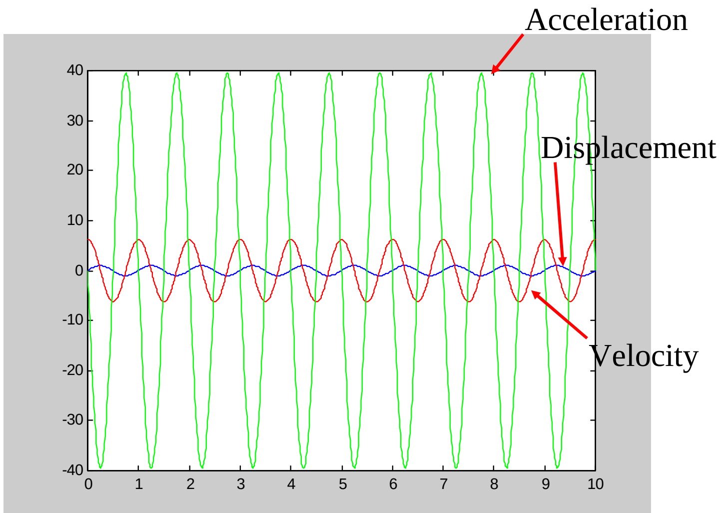
Displacement $(\omega = 2\pi f)$

$$\dot{x}(t) = \omega C\cos(\omega t + \Phi)$$ Velocity

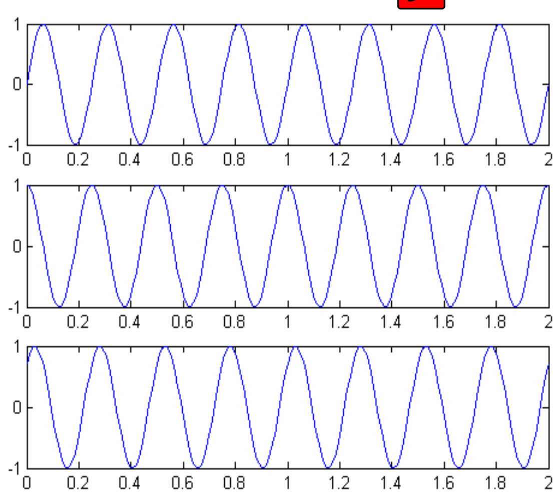
$$\ddot{x}(t) = -\omega^2 C\sin(\omega t + \Phi)$$ Acceleration

maximum velocity $= \omega C$ maximum acceleration $= \omega^2 C$

Harmonic Motion



Harmonic Motion



$$x(t) = C\sin(\omega t)$$

or

$$x(t) = C\cos\left(\omega t - \frac{\pi}{2}\right)$$

$$x(t) = C\cos(\omega t)$$

or

$$x(t) = C\sin\left(\omega t + \frac{\pi}{2}\right)$$

$$x(t) = C\sin\left(\omega t + \frac{\pi}{4}\right)$$

or

$$x(t) = C\cos\left(\omega t - \frac{\pi}{4}\right)$$

or

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

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Complex Number Representation of Harmonic Motion

Rotating Vector or Phasor

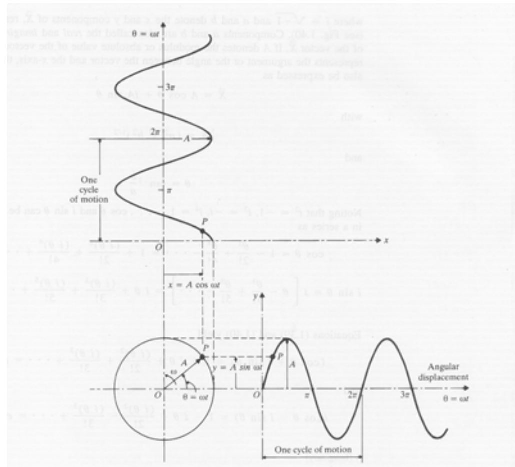

FIGURE 1.39 Harmonic motion as the projection of the end of a rotating vector.

Vertical Axis Projection:

$$x = C\cos\omega t$$

Horizontal Axis Projection:

$$y = C\sin\omega t$$

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Complex Number Representation of Harmonic Motion

Vectors in complex form: $\vec{X} = a + ib$

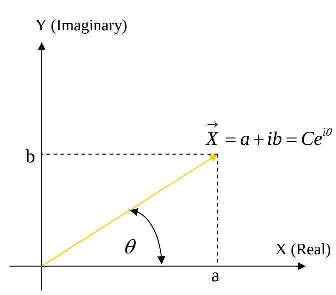
REAL IMAGINARY

Complex Trigonometric form:

$$\vec{X} = C\cos\theta + iC\sin\theta$$

Euler's Identity: $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$

Amplitude: $C = \sqrt{a^2 + b^2}$



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Complex Algebra

Complex number without vector notation: $z = a + ib$

Suppose

$$z_1 = a_1 + ib_1 = C_1 e^{i\theta_1}$$

$$z_2 = a_2 + ib_2 = C_2 e^{i\theta_2}$$

$$z_1 + z_2 = C_1 e^{i\theta_1} + C_2 e^{i\theta_2} = (a_1 + ib_1) + (a_2 + ib_2)$$
$$= (a_1 + a_2) + i(b_1 + b_2)$$

Similarly

$$z_1 - z_2 = C_1 e^{i\theta_1} - C_2 e^{i\theta_2} = (a_1 + ib_1) - (a_2 + ib_2)$$
$$= (a_1 - a_2) + i(b_1 - b_2)$$

$$C_j = \sqrt{a_j^2 + b_j^2};\ j = 1, 2 \qquad \theta_j = \tan^{-1}\left(\frac{b_j}{a_j}\right);\ j = 1, 2$$

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Complex Algebra

$$\frac{d\vec{X}}{dt} = \frac{d}{dt}\left(Ae^{i\omega t}\right) = i\omega Ae^{i\omega t} = i\omega \vec{X}$$ Velocity

$$\frac{d^2\vec{X}}{dt^2} = \frac{d}{dt}\left(i\omega Ae^{i\omega t}\right) = -\omega^2 Ae^{i\omega t} = -\omega^2 \vec{X}$$ Acceleration

Real

$$Displacement = \mathrm{Re}\left[Ce^{i\omega t}\right] = C\cos\omega t$$

$$Velocity = \mathrm{Re}\left[i\omega Ce^{i\omega t}\right] = -\omega C\sin\omega t$$
$$= \omega C\cos\left(\omega t + 90°\right)$$

$$Acceleration = \mathrm{Re}\left[-\omega^2 Ce^{i\omega t}\right] = -\omega^2 C\cos\omega t$$
$$= \omega^2 C\cos\left(\omega t + 180°\right)$$

Imaginary

$$Displacement = \mathrm{Im}\left[Ce^{i\omega t}\right] = C\sin\omega t$$

$$Velocity = \mathrm{Im}\left[i\omega Ce^{i\omega t}\right] = \omega C\sin\left(\omega t + 90°\right)$$

$$Acceleration = \mathrm{Im}\left[-\omega^2 Ce^{i\omega t}\right] = \omega^2 C\sin\left(\omega t + 180°\right)$$

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Differential Equations (Review)

$$a\ddot{x} + b\dot{x} + cx = F(t)$$

$x = x(t)$

$\dot{x} = \frac{dx}{dt}$

$\ddot{x} = \frac{d^2x}{d^2t}$

2nd order

linear

non-homogeneous

constant coefficients

Required solution

$x(t) = ?$

$x(t) = x_h(t) + x_p(t)$

MEEM 3700 13 of 20

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Differential Equations (Review)

Homogeneous Solution $x_h(t)$

$$a\ddot{x} + b\dot{x} + cx = 0$$

$$x_h(t) = ?$$

assume a solution....

$x(t) = C\sin(\omega t)$

or

$x(t) = Ce^{st}$

$\dot{x}(t) = sCe^{st}$

$\ddot{x}(t) = s^2Ce^{st}$

MEEM 3700 14 of 20

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Differential Equations (Review)

$$a\ddot{x} + b\dot{x} + cx = 0$$

substituting $x(t), \dot{x}(t), \ddot{x}(t)$

$$as^2Ce^{st} + bsCe^{st} + cCe^{st} = 0$$

rearranging

$$(as^2 + bs + c)Ce^{st} = 0$$

$$Ce^{st} = 0$$

$$as^2 + bs + c = 0$$

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Differential Equations (Review)

$$as^2 + bs + c = 0$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore there are two solutions

$$x_1(t) = C_1e^{s_1t} \quad \text{and} \quad x_2(t) = C_2e^{s_2t}$$

$$x(t) = C_1e^{s_1t} + C_2e^{s_2t}$$

Differential Equations (Review)

$$x_h(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

C_1 and C_2 are determined by the initial conditions $x(0), \dot{x}(0)$

$$x_h(0) = C_1 e^{s_1 0} + C_2 e^{s_2 0} = C_1 + C_2$$

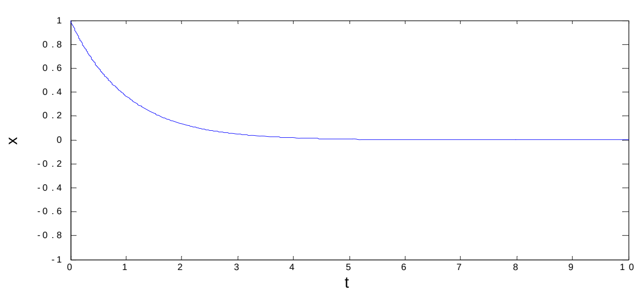
$$\dot{x}_h(0) = s_1 C_1 e^{s_1 0} + s_2 C_2 e^{s_2 0} = s_1 C_1 + s_2 C_2$$

s_1, s_2 can be real or complex valued

Differential Equations (Review)

$$x_h(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

s_1, s_2 are real valued



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Differential Equations (Review)

$$x_h(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

s_1, s_2 are complex valued

$s_1 = -\sigma + j\omega \qquad s_2 = -\sigma - j\omega$

$$x_h(t) = C_1 e^{(-\sigma + j\omega)t} + C_2 e^{(-\sigma - j\omega)t}$$

$$x_h(t) = e^{-\sigma t}\left(C_1 e^{(+j\omega)t} + C_2 e^{(-j\omega)t}\right)$$

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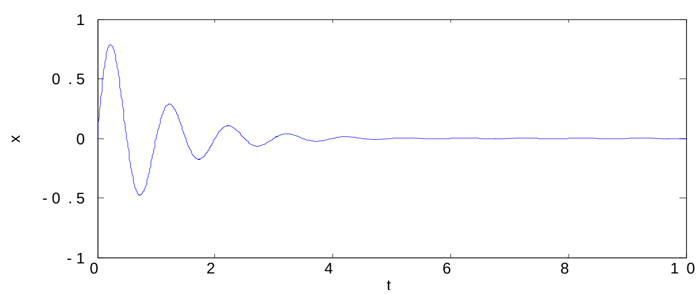
Differential Equations (Review)

Euler's Identity $\quad e^{\pm j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$$x_h(t) = e^{-\sigma t}\left(\left(C_1 + C_2\right)\cos(\omega t) + j\left(C_1 - C_2\right)\sin(\omega t)\right)$$



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