1

Single-degree-of-freedom Systems

1.1 INTRODUCTION

In this chapter the vibration of a single-degree-of-freedom system will be analyzed and reviewed. Analysis, measurement, design, and control of a single-degree-of-freedom system (often abbreviated SDOF) is discussed. The concepts developed in this chapter constitute an introductory review of vibrations and serve as an introduction for extending these concepts to more complex systems in later chapters. In addition, basic ideas relating to measurement and control of vibrations are introduced that will later be extended to multiple-degree-of-freedom systems and distributed-parameter systems. This chapter is intended to be a review of vibration basics and an introduction to a more formal and general analysis for more complicated models in the following chapters.

Vibration technology has grown and taken on a more interdisciplinary nature. This has been caused by more demanding performance criteria and design specifications for all types of machines and structures. Hence, in addition to the standard material usually found in introductory chapters of vibration and structural dynamics texts, several topics from control theory and vibration measurement theory are presented. This material is included not to train the reader in control methods (the interested student should study control and system theory texts) but rather to point out some useful connections between vibration and control as related disciplines. In addition, structural control has become an important discipline requiring the coalescence of vibration and control topics. A brief introduction to nonlinear SDOF systems and numerical simulation is also presented.

1.2 SPRING–MASS SYSTEM

Simple harmonic motion, or oscillation, is exhibited by structures that have elastic restoring forces. Such systems can be modeled, in some situations, by a spring–mass schematic, as illustrated in Figure 1.1. This constitutes the most basic vibration model of a structure and can be used successfully to describe a surprising number of devices, machines, and structures. The methods presented here for solving such a simple mathematical model may seem to be

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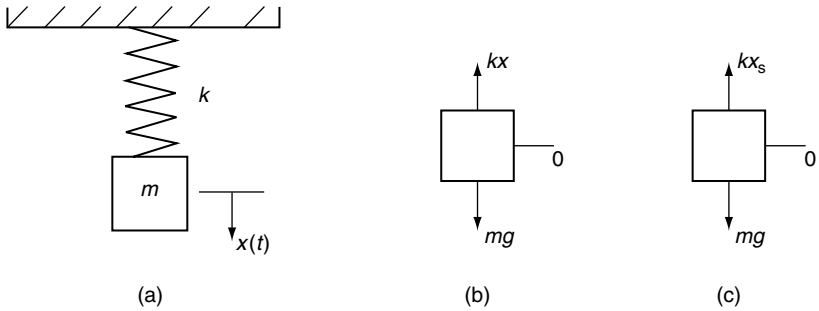

Figure 1.1 (a) Spring–mass schematic, (b) free body diagram, and (c) free body diagram of the static spring–mass system.

more sophisticated than the problem requires. However, the purpose of the analysis is to lay the groundwork for the analysis in the following chapters of more complex systems.

If $x = x(t)$ denotes the displacement (m) of the mass m (kg) from its equilibrium position as a function of time t (s), the equation of motion for this system becomes [upon summing forces in Figure 1.1(b)]

$$m\ddot{x} + k(x + x_s) - mg = 0$$

where k is the stiffness of the spring (N/m), x_s is the static deflection (m) of the spring under gravity load, g is the acceleration due to gravity (m/s^2), and the overdots denote differentiation with respect to time. (A discussion of dimensions appears in Appendix A, and it is assumed here that the reader understands the importance of using consistent units.) From summing forces in the free body diagram for the static deflection of the spring [Figure 1.1(c)], $mg = kx_s$ and the above equation of motion becomes

$$m\ddot{x}(t) + kx(t) = 0 \tag{1.1}$$

This last expression is the equation of motion of a single-degree-of-freedom system and is a linear, second-order, ordinary differential equation with constant coefficients.

Figure 1.2 indicates a simple experiment for determining the spring stiffness by adding known amounts of mass to a spring and measuring the resulting static deflection, x_s. The results of this static experiment can be plotted as force (mass times acceleration) versus x_s, the slope yielding the value of k for the linear portion of the plot. This is illustrated in Figure 1.3.

Once m and k are determined from static experiments, Equation (1.1) can be solved to yield the time history of the position of the mass m, given the initial position and velocity of the mass. The form of the solution of Equation (1.1) is found from substitution of an assumed periodic motion (from experience watching vibrating systems) of the form

$$x(t) = A\sin(\omega_n t + \phi) \tag{1.2}$$

where $\omega_n = \sqrt{k/m}$ is the *natural frequency* (rad/s). Here, the *amplitude*, A, and the *phase shift*, ϕ, are constants of integration determined by the initial conditions.

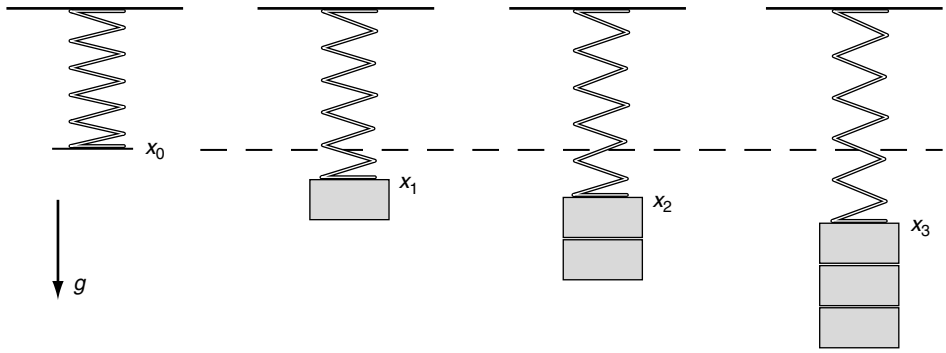

Figure 1.2 Measurement of the spring constant.

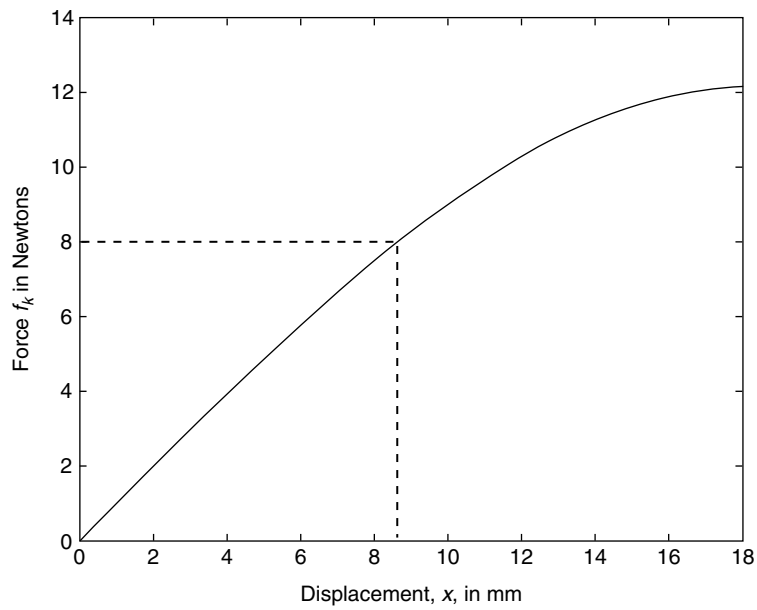

Figure 1.3 Determination of the spring constant.

The existence of a *unique* solution for Equation (1.1) with two specific initial conditions is well known and is given by, for instance, Boyce and DiPrima (2000). Hence, if a solution of the form of Equation (1.2) form is guessed and it works, then it is *the* solution. Fortunately, in this case the mathematics, physics, and observation all agree.

To proceed, if x_0 is the specified initial displacement from equilibrium of mass m, and v_0 is its specified initial velocity, simple substitution allows the constants A and ϕ to be evaluated. The unique solution is

$$x(t) = \sqrt{\frac{\omega_n^2 x_0^2 + v_0^2}{\omega_n^2}} \sin\left[\omega_n t + \tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)\right] \tag{1.3}$$

Alternatively, $x(t)$ can be written as

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t \tag{1.4}$$

by using a simple trigonometric identity.

A purely mathematical approach to the solution of Equation (1.1) is to assume a solution of the form $x(t) = A\,e^{\lambda t}$ and solve for λ, i.e.,

$$m\lambda^2 e^{\lambda t} + k e^{\lambda t} = 0$$

This implies that (because $e^{\lambda t} \neq 0$, and $A \neq 0$)

$$\lambda^2 + \left(\frac{k}{m}\right) = 0$$

or that

$$\lambda = \pm j \left(\frac{k}{m}\right)^{1/2} = \pm \omega_n j$$

where $j = (-1)^{1/2}$. Then the general solution becomes

$$x(t) = A_1\, e^{-\omega_n j t} + A_2\, e^{\omega_n j t} \tag{1.5}$$

where A_1 and A_2 are arbitrary complex conjugate constants of integration to be determined by the initial conditions. Use of Euler's formulae then yields Equations (1.2) and (1.4) (see, for instance, Inman, 2001). For more complicated systems, the exponential approach is often more appropriate than first guessing the form (sinusoid) of the solution from watching the motion.

Another mathematical comment is in order. Equation (1.1) and its solution are valid only as long as the spring is linear. If the spring is stretched too far, or too much force is applied to it, the curve in Figure 1.3 will no longer be linear. Then Equation (1.1) will be nonlinear (see Section 1.8). For now, it suffices to point out that initial conditions and springs should always be checked to make sure that they fall in the linear region if linear analysis methods are going to be used.

1.3 SPRING–MASS–DAMPER SYSTEM

Most systems will not oscillate indefinitely when disturbed, as indicated by the solution in Equation (1.3). Typically, the periodic motion dies down after some time. The easiest way to treat this mathematically is to introduce a velocity based force term, $c\dot{x}$, into Equation (1.1) and examine the equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1.6}$$

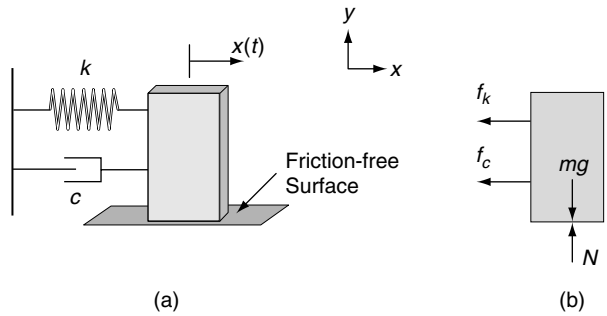

Figure 1.4 (a) Schematic of the spring–mass–damper system and (b) free body diagram of the system in part (a).

This also happens physically with the addition of a *dashpot* or *damper* to dissipate energy, as illustrated in Figure 1.4.

Equation (1.6) agrees with summing forces in Figure 1.4 if the dashpot exerts a dissipative force proportional to velocity on the mass m. Unfortunately, the constant of proportionality, c, cannot be measured by static methods as m and k are. In addition, many structures dissipate energy in forms not proportional to velocity. The constant of proportionality c is given in N s/m or kg/s in terms of fundamental units.

Again, the unique solution of Equation (1.6) can be found for specified initial conditions by assuming that $x(t)$ is of the form

$$x(t) = A\,e^{\lambda t}$$

and substituting this into Equation (1.6) to yield

$$A\left(\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m}\right)e^{\lambda t} = 0 \tag{1.7}$$

Since a trivial solution is not desired, $A \neq 0$, and since $e^{\lambda t}$ is never zero, Equation (1.7) yields

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0 \tag{1.8}$$

Equation (1.8) is called the *characteristic equation* of Equation (1.6). Using simple algebra, the two solutions for λ are

$$\lambda_{1,2} = -\frac{c}{2m} \pm \frac{1}{2}\sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}} \tag{1.9}$$

The quantity under the radical is called the *discriminant* and, together with the sign of m, c, and k, determines whether or not the roots are complex or real. Physically, m, c, and k are all positive in this case, so the value of the discriminant determines the nature of the roots of Equation (1.8).

It is convenient to define the dimensionless *damping ratio*, ζ, as

$$\zeta = \frac{c}{2\sqrt{km}}$$

In addition, let the *damped natural frequency*, ω_d, be defined (for $0 < \zeta < 1$) by

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{1.10}$$

Then, Equation (1.6) becomes

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0 \tag{1.11}$$

and Equation (1.9) becomes

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_d j, \qquad 0 < \zeta < 1 \tag{1.12}$$

Clearly, the value of the damping ratio, ζ, determines the nature of the solution of Equation (1.6). There are three cases of interest. The derivation of each case is left as a problem and can be found in almost any introductory text on vibrations (see, for instance, Meirovitch, 1986 or Inman, 2001).

Underdamped. This case occurs if the parameters of the system are such that

$$0 < \zeta < 1$$

so that the discriminant in Equation (1.12) is negative and the roots form a complex conjugate pair of values. The solution of Equation (1.11) then becomes

$$x(t) = e^{-\zeta\omega_n t} (A \cos \omega_d t + B \sin \omega_d t) \tag{1.13}$$

or

$$x(t) = Ce^{-\zeta\omega_n t} \sin (\omega_d t + \phi)$$

where A, B, C, and ϕ are constants determined by the specified initial velocity, v_0, and position, x_0:

$$A = x_0, \qquad C = \frac{\sqrt{(v_0 + \zeta\omega_n x_0)^2 + (x_0\omega_d)^2}}{\omega_d}$$

$$B = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}, \qquad \phi = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + \zeta\omega_n x_0}\right) \tag{1.14}$$

The underdamped response has the form given in Figure 1.5 and consists of a decaying oscillation of frequency ω_d.

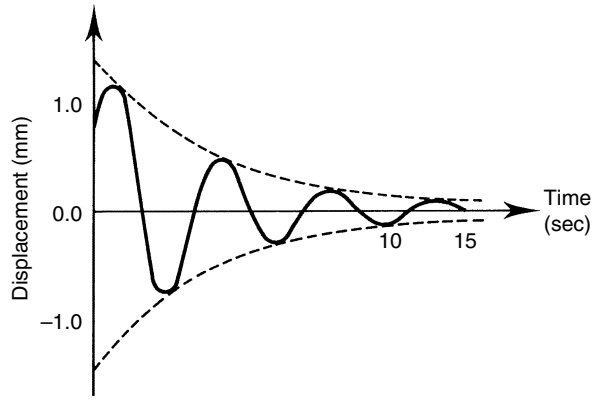

Figure 1.5 Response of an underdamped system.

Overdamped. This case occurs if the parameters of the system are such that

$$\zeta > 1$$

so that the discriminant in Equation (1.12) is positive and the roots are a pair of negative real numbers. The solution of Equation (1.11) then becomes

$$x(t) = A\,e^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + B\,e^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t} \tag{1.15}$$

where A and B are again constants determined by v_0 and x_0. They are

$$A = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$B = -\frac{v_0 + \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

The overdamped response has the form given in Figure 1.6. An overdamped system does not oscillate, but rather returns to its rest position exponentially.

Critically damped. This case occurs if the parameters of the system are such that

$$\zeta = 1$$

so that the discriminant in Equation (1.12) is zero and the roots are a pair of negative real repeated numbers. The solution of Equation (1.11) then becomes

$$x(t) = e^{-\omega_n t}\left[(v_0 + \omega_n x_0)t + x_0\right] \tag{1.16}$$

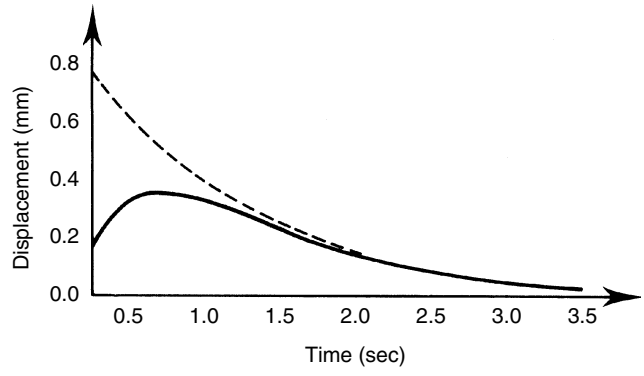

Figure 1.6 Response of an overdamped system.

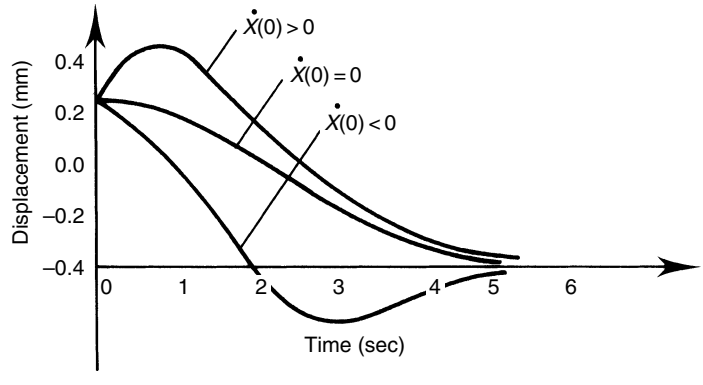

Figure 1.7 Response of a critically damped system.

The critically damped response is plotted in Figure 1.7 for various values of the initial conditions v_0 and x_0.

It should be noted that critically damped systems can be thought of in several ways. First, they represent systems with the minimum value of damping rate that yields a nonoscillating system (Problem 1.5). Critical damping can also be thought of as the case that separates nonoscillation from oscillation.

1.4 FORCED RESPONSE

The preceding analysis considers the vibration of a device or structure as a result of some initial disturbance (i.e., v_0 and x_0). In this section, the vibration of a spring–mass–damper system subjected to an external force is considered. In particular, the response to harmonic excitations, impulses, and step forcing functions is examined.

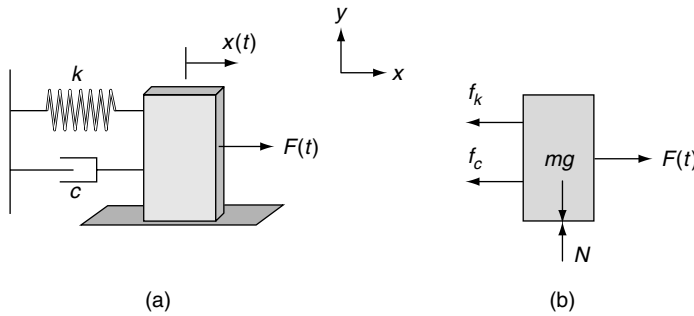

Figure 1.8 (a) Schematic of the forced spring–mass–damper system assuming no friction on the surface and (b) free body diagram of the system of part (a).

In many environments, rotating machinery, motors, and so on, cause periodic motions of structures to induce vibrations into other mechanical devices and structures nearby. It is common to approximate the driving forces, $F(t)$, as periodic of the form

$$F(t) = F_0 \sin \omega t$$

where F_0 represents the amplitude of the applied force and ω denotes the frequency of the applied force, or the driving frequency (rad/s). On summing the forces, the equation for the forced vibration of the system in Figure 1.8 becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \tag{1.17}$$

Recall from the discipline of differential equations (Boyce and DiPrima, 2000), that the solution of Equation (1.17) consists of the sum of the homogeneous solution in Equation (1.5) and a particular solution. These are usually referred to as the *transient response* and the *steady state response* respectively. Physically, there is motivation to assume that the steady state response will follow the forcing function. Hence, it is tempting to assume that the particular solution has the form

$$x_p(t) = X \sin(\omega t - \theta) \tag{1.18}$$

where X is the steady state amplitude and θ is the phase shift at steady state. Mathematically, the method is referred to as the method of undetermined coefficients. Substitution of Equation (1.18) into Equation (1.17) yields

$$X = \frac{F_0/k}{\sqrt{(1 - m\omega^2/k)^2 + (c\omega/k)^2}}$$

or

$$\frac{Xk}{F_0} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}} \tag{1.19}$$

and

$$\tan\theta = \frac{(c\omega/k)}{1 - m\omega^2/k} = \frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2} \tag{1.20}$$

where $\omega_n = \sqrt{k/m}$ as before. Since the system is linear, the sum of two solutions is a solution, and the total time response for the system of Figure 1.8 for the case $0 < \zeta < 1$ becomes

$$x(t) = e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t) + X\sin(\omega t - \theta) \tag{1.21}$$

Here, A and B are constants of integration determined by the initial conditions and the forcing function (and in general will be different from the values of A and B determined for the free response).

Examining Equation (1.21), two features are important and immediately obvious. First, as t becomes larger, the transient response (the first term) becomes very small, and hence the term steady state response is assigned to the particular solution (the second term). The second observation is that the coefficient of the steady state response, or particular solution, becomes large when the excitation frequency is close to the undamped natural frequency, i.e., $\omega \approx \omega_n$. This phenomenon is known as *resonance* and is extremely important in design, vibration analysis, and testing.

Example 1.4.1

Compute the response of the following system (assuming consistent units):

$$\ddot{x}(t) + 0.4\dot{x}(t) + 4x(t) = \frac{1}{\sqrt{2}}\sin 3t, \qquad x(0) = \frac{-3}{\sqrt{2}}, \qquad \dot{x}(0) = 0$$

First, solve for the particular solution by using the more convenient form of

$$x_p(t) = X_1\sin 3t + X_2\cos 3t$$

rather than the magnitude and phase form, where X_1 and X_2 are the constants to be determined. Differentiating x_p yields

$$\dot{x}_p(t) = 3X_1\cos 3t - 3X_2\sin 3t$$
$$\ddot{x}_p(t) = -9X_1\sin 3t - 9X_2\cos 3t$$

Substitution of x_p and its derivatives into the equation of motion and collecting like terms yield

$$\left(-9X_1 - 1.2X_2 + 4X_1 - \frac{1}{\sqrt{2}}\right)\sin 3t + (-9X_2 + 1.2X_1 + 4X_2)\cos 3t = 0$$

Since the sine and cosine are independent, the two coefficients in parentheses must vanish, resulting in two equations in the two unknowns X_1 and X_2. This solution yields

$$x_p(t) = -0.134\ \sin 3t - 0.032\ \cos 3t$$

Next, consider adding the free response to this. From the problem statement

$$\omega_n = 2\,\text{rad/s}, \qquad \zeta = \frac{0.4}{2\omega_n} = 0.1 < 1, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = 1.99\,\text{rad/s}$$

Thus, the system is underdamped, and the total solution is of the form

$$x(t) = e^{-\zeta\omega_n t}\,(A\sin\omega_d t + B\cos\omega_d t) + X_1\sin\omega t + X_2\cos\omega t$$

Applying the initial conditions requires the following derivative

$$\begin{aligned}\dot{x}(t) = {} & e^{-\zeta\omega_n t}\,(\omega_d A\cos\omega_d t - \omega_d B\sin\omega_d t) + \omega X_1\cos\omega t \\ & - \omega X_2\sin\omega t - \zeta\omega_n e^{-\zeta\omega_n t}\,(A\sin\omega_d t + B\cos\omega_d t)\end{aligned}$$

The initial conditions yield the constants A and B:

$$\begin{aligned}x(0) &= B + X_2 = \frac{-3}{\sqrt{2}} \Rightarrow B = -X_2 - \frac{3}{\sqrt{2}} = -2.089 \\ \dot{x}(0) &= \omega_d A + \omega X_1 - \zeta\omega_n B = 0 \Rightarrow A = \frac{1}{\omega_d}\,(\zeta\omega_n B - \omega X_1) = -0.008\end{aligned}$$

Thus the total solution is

$$x(t) = -e^{-0.2t}\,(0.008\sin 1.99t + 2.089\cos 1.99t) - 0.134\sin 3t - 0.032\cos 3t$$

Resonance is generally to be avoided in designing structures, since it means large-amplitude vibrations, which can cause fatigue failure, discomfort, loud noises, and so on. Occasionally, the effects of resonance are catastrophic. However, the concept of resonance is also very useful in testing structures. In fact, the process of modal testing (see Chapter 8) is based on resonance. Figure 1.9 illustrates how ω_n and ζ affect the amplitude at resonance. The dimensionless quantity Xk/F_0 is called the *magnification factor* and Figure 1.9 is called a *magnification curve* or *magnitude plot*. The maximum value at resonance, called the *peak resonance*, and denoted by M_p, can be shown (see, for instance, Inman, 2001) to be related to the damping ratio by

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \tag{1.22}$$

Also, Figure 1.9 can be used to define the *bandwidth* (BW) of the structure as the value of the driving frequency at which the magnitude drops below 70.7% of its zero frequency value (also said to be the 3 dB down point from the zero frequency point). The bandwidth can be calculated (Kuo and Golnaraghi, 2003, p. 359) in terms of the damping ratio by

$$\text{BW} = \omega_n\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \tag{1.23}$$

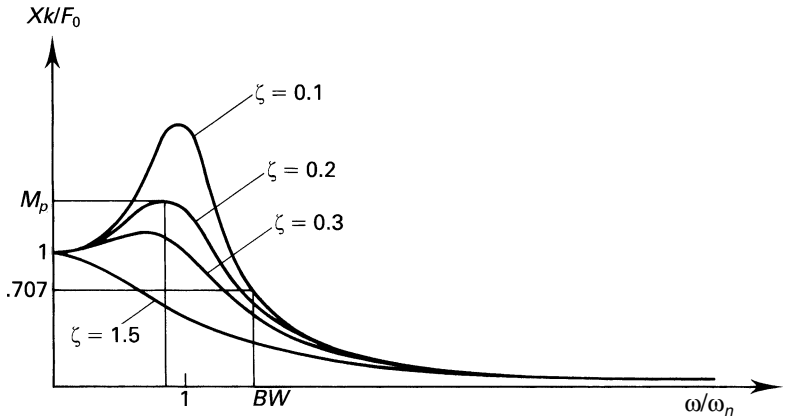

Figure 1.9 Magnification curves for a single-degree-of-freedom system ($r = \omega/\omega_n$).

Two other quantities are used in discussing the vibration of underdamped structures. They are the *loss factor* defined at resonance (only) to be

$$\eta = 2\zeta \tag{1.24}$$

and the *Q value*, or *resonance sharpness* factor, given by

$$Q = \frac{1}{2\zeta} = \frac{1}{\eta} \tag{1.25}$$

Another common situation focuses on the transient nature of the response, namely the response of (1.6) to an impulse, to a step function, or to initial conditions. Many mechanical systems are excited by loads, which act for a very brief time. Such situations are usually modeled by introducing a fictitious function called the *unit impulse function*, or the *Dirac delta function*. This delta function is defined by the two properties

$$\begin{aligned} &\delta(t-a) = 0, \qquad t \neq a \\ &\int_{-\infty}^{\infty} \delta(t-a)\,dt = 1 \end{aligned} \tag{1.26}$$

where a is the instant of time at which the impulse is applied. Strictly speaking, the quantity $\delta(t)$ is not a function; however, it is very useful in quantifying important physical phenomena of an impulse.

The response of the system of Figure 1.8 for the underdamped case (with $a = x_0 = v_0 = 0$) can be shown to be given by

$$x(t) = \begin{cases} 0, & t < a \\ \dfrac{e^{-\zeta\omega t}\sin\omega_d t}{m\omega_d}, & t > a \end{cases} \tag{1.27}$$

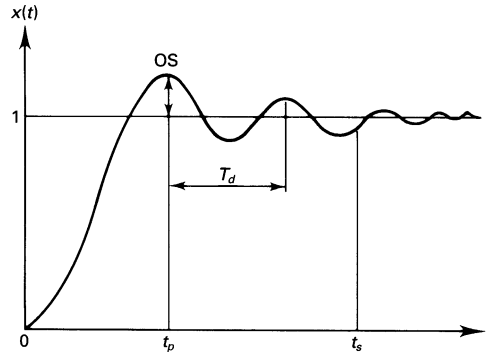

Figure 1.10 Step response of a single-degree-of-freedom system.

Note from Equation (1.13) that this corresponds to the transient response of the system to the initial conditions $x_0 = 0$ and $v_0 = 1/m$. Hence, the impulse response is equivalent to giving a system at rest an initial velocity of $(1/m)$. This makes the impulse response, $x(t)$, important in discussing the transient response of more complicated systems. The impulse is also very useful in making vibration measurements, as described in Chapter 8.

Often, design problems are stated in terms of certain specifications based on the response of the system to step function excitation. The response of the system in Figure 1.8 to a step function (of magnitude $m\omega_n^2$ for convenience), with initial conditions both set to zero, is calculated for underdamped systems from

$$m\ddot{x} + c\dot{x} + kx = m\omega_n^2\mu(t), \qquad \mu(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \tag{1.28}$$

to be

$$x(t) = 1 - \frac{e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{\sqrt{1-\zeta^2}} \tag{1.29}$$

where

$$\phi = \arctan\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right] \tag{1.30}$$

A sketch of the response is given in Figure 1.10, along with the labeling of several significant specifications for the case $m = 1$, $\omega_n = 2$, and $\zeta = 0.2$.

In some situations, the steady state response of a structure may be at an acceptable level, but the transient response may exceed acceptable limits. Hence, one important measure is the *overshoot*, labeled OS in Figure 1.10 and defined as the maximum value of the response minus the steady state value of the response. From Equation (1.29) it can be shown that

$$\mathrm{OS} = x_{\max}(t) - 1 = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \tag{1.31}$$

This occurs at the *peak time*, t_p, which can be shown to be

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{1.32}$$

In addition, the period of oscillation, T_d, is given by

$$T_d = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}} = 2t_p \tag{1.33}$$

Another useful quantity, which indicates the behavior of the transient response, is the *settling time*, t_s. This is the time it takes the response to get within ±5% of the steady state response and remain within ±5%. One approximation of t_s is given by (Kuo and Golnaraghi, 2003, p. 263):

$$t_s = \frac{3.2}{\omega_n\zeta} \tag{1.34}$$

The preceding definitions allow designers and vibration analysts to specify and classify precisely the nature of the transient response of an underdamped system. They also give some indication of how to adjust the physical parameters of the system so that the response has a desired shape.

The response of a system to an impulse may be used to determine the response of an underdamped system to any input $F(t)$ by defining the *impulse response function* as

$$h(t) = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\sin\omega_d t \tag{1.35}$$

Then the solution of

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

can be shown to be

$$x(t) = \int_0^t F(\tau)h(t-\tau)\,d\tau = \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t F(\tau)\,e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\,d\tau \tag{1.36}$$

for the case of zero initial conditions. This last expression gives an analytical representation for the response to any driving force that has an integral.

1.5 TRANSFER FUNCTIONS AND FREQUENCY METHODS

The preceding analysis of the response was carried out in the time domain. Current vibration measurement methodology (Ewins, 2000) as well as much control analysis (Kuo and Golnaraghi, 2003) often takes place in the frequency domain. Hence, it is worth the effort to reexamine these calculations using frequency domain methods (a phrase usually associated with linear control theory). The frequency domain approach arises naturally from mathematics (ordinary differential equations) via an alternative method of solving differential equations, such as Equations (1.17) and (1.28), using the Laplace transform (see, for instance, Boyce and DiPrima, 2000, Chapter 6).

Taking the Laplace transform of Equation (1.28), assuming both initial conditions to be zero, yields

$$X(s) = \left[\frac{1}{ms^2 + cs + k}\right]\mu(s) \tag{1.37}$$

where $X(s)$ denotes the Laplace transform of $x(t)$, and $\mu(s)$ is the Laplace transform of the right-hand side of Equation (1.28). If the same procedure is applied to Equation (1.17), the result is

$$X(s) = \left[\frac{1}{ms^2 + cs + k}\right] F_0(s) \tag{1.38}$$

where $F_0(s)$ denotes the Laplace transform of $F_0 \sin \omega t$. Note that

$$G(s) = \frac{X(s)}{\mu(s)} = \frac{X(s)}{F_0(s)} = \frac{1}{ms^2 + cs + k} \tag{1.39}$$

Thus, it appears that the quantity $G(s) = [1/(ms^2 + cs + k)]$, the ratio of the Laplace transform of the output (response) to the Laplace transform of the input (applied force) to the system, characterizes the system (structure) under consideration. This characterization is independent of the input or driving function. This ratio, $G(s)$, is defined as the *transfer function* of this system in control analysis (or of this structure in vibration analysis). The transfer function can be used to provide analysis of the vibrational properties of the structure as well as to provide a means of measuring the dynamic response of the structure.

In control theory, the transfer function of a system is defined in terms of an output to input ratio, but the use of a transfer function in structural dynamics and vibration testing implies certain physical properties, depending on whether position, velocity, or acceleration is considered as the response (output). It is quite common, for instance, to measure the response of a structure by using an accelerometer. The resultant transfer function is then $s^2X(s)/U(s)$, where $U(s)$ is the Laplace transform of the input and $s^2X(s)$ is the Laplace transform of the acceleration. This transfer function is called the *inertance* and its reciprocal is referred to as the *apparent mass*. Table 1.1 lists the nomenclature of various transfer functions. The physical basis for these names can be seen from their graphical representation.

The transfer function representation of a structure is very useful in control theory as well as in vibration testing. The variable s in the Laplace transform is a complex variable, which can be further denoted by

$$s = \sigma + j\omega_d$$

where the real numbers σ and ω_d denote the real and imaginary parts of s respectively. Thus, the various transfer functions are also complex valued.

In control theory, the values of s where the denominator of the transfer function $G(s)$ vanishes are called the *poles* of the transfer function. A plot of the poles of the compliance (also called receptance) transfer function for Equation (1.38) in the complex s plane is given

Table 1.1 Various transfer functions.

Response measurement	Transfer function	Inverse transfer function
Acceleration	Inertance	Apparent mass
Velocity	Mobility	Impedance
Displacement	Compliance	Dynamic stiffness

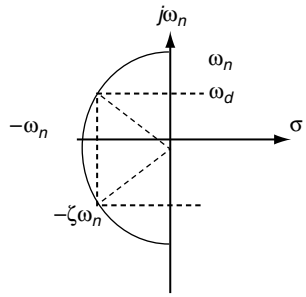

Figure 1.11 Complex s plane of the poles of Equation (1.39).

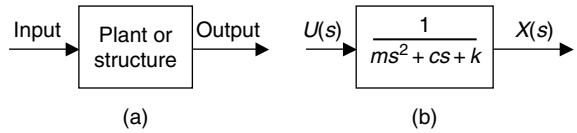

Figure 1.12 Block diagram representation of a single-degree-of-freedom system.

in Figure 1.11. The points on the semicircle occur where the denominator of the transfer function is zero. These values of $s(s=-\zeta\omega_n \pm \omega_d j)$ are exactly the roots of the characteristic equation for the structure. The values of the physical parameters m, c, and k determine the two quantities ζ and ω_n, which in turn determine the position of the poles in Figure 1.11.

Another graphical representation of a transfer function useful in control is the *block diagram* illustrated in Figure 1.12(a). This diagram is an icon for the definition of a transfer function. The control terminology for the physical device represented by the transfer function is the *plant*, whereas in vibration analysis the plant is usually referred to as the structure. The block diagram of Figure 1.12(b) is meant exactly to imply the formula

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + cs + k} \tag{1.40}$$

The response of Equation (1.38) to a sinusoidal input (forcing function) motivates a second description of the transfer function of a structure, called the *frequency response function* (often denoted by FRF). The frequency response function is defined as the transfer function evaluated at $s=j\omega$, i.e., $G(j\omega)$. The significance of the frequency response function follows from Equation (1.21), namely that the steady state response of a system driven sinusoidally is a sinusoid of the same frequency with different amplitude and phase. In fact, substitution of $j\omega$ into Equation (1.40) yields exactly Equations (1.19) and (1.20) from

$$\frac{X}{F_0} = |G(j\omega)| = \sqrt{x^2(\omega) + y^2(\omega)} \tag{1.41}$$

where $|G(j\omega)|$ indicates the magnitude of the complex frequency response function,

$$\phi = \tan^{-1} G(j\omega) = \tan^{-1}\left[\frac{y(\omega)}{x(\omega)}\right] \tag{1.42}$$

indicates the phase of the frequency response function, and

$$G(j\omega) = x(\omega) + y(\omega)j \tag{1.43}$$

This mathematically expresses two ways of representing a complex function, as the sum of its real part $[\mathrm{Re}\, G(j\omega) = x(\omega)]$ and its imaginary part $[\mathrm{Im}\, G(j\omega) = y(\omega)]$, or by its magnitude ($|G(j\omega)|$) and phase ($\phi$). In more physical terms, the frequency response function of a structure represents the magnitude and phase shift of its steady state response under sinusoidal excitation. While Equations (1.17), (1.21), (1.41), and (1.42) verify this for a single-degree-of-freedom viscously damped structure, it can be shown in general for any linear time-invariant plant (Melsa and Schultz, 1969, p. 187).

It should also be noted that the frequency response function of a linear system can be obtained from the transfer function of the system, and vice versa. Hence, the frequency response function uniquely determines the time response of the structure to any known input.

Graphical representations of the frequency response function form an extensive part of control analysis and also form the backbone of vibration measurement analysis. Next, three sets of frequency response function plots that are useful in testing vibrating structures are examined. The first set of plots consists simply of plotting the imaginary part of the frequency response function versus the driving frequency and the real part of the frequency response function versus the driving frequency. These are shown for a damped single-degree-of-freedom system in Figure 1.13 (the compliance frequency response function for $\zeta = 0.01$ and $\omega_n = 20\,\mathrm{rad/s}$).

The second representation consists of a single plot of the imaginary part of the frequency response function versus the real part of the frequency response function. This type of plot is called a *Nyquist plot* (also called an *Argand plane plot*) and is used for measuring the natural frequency and damping in testing methods and for stability analysis in control system design. The Nyquist plot of the mobility frequency response function of a structure modeled by Equation (1.37) is given in Figure 1.14.

The last plots considered for representing the frequency response function are called *Bode plots* and consist of a plot of the magnitude of the frequency response function versus the driving frequency and the phase of the frequency response function versus the driving frequency (a complex number requires *two* real numbers to describe it completely). Bode plots

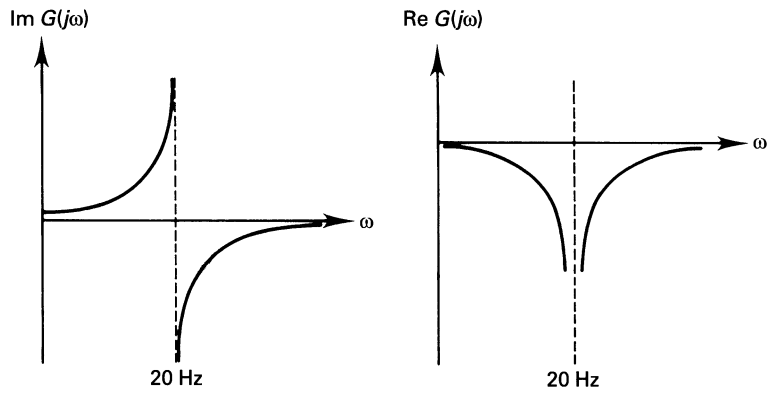

Figure 1.13 Plots of the real part and the imaginary part of the frequency response function.

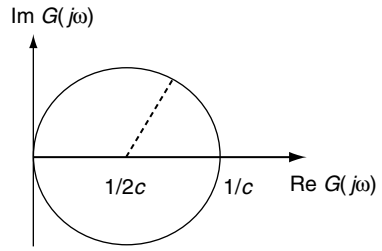

Figure 1.14 Nyquist plot for Equation 1.39.

have long been used in control system design and analysis as well as for determining the plant transfer function of a system. More recently, Bode plots have been used in analyzing vibration test results and in determining the physical parameters of the structure.

In order to represent the complete Bode plots in a reasonable space, $\log_{10}$ scales are often used to plot $|G(j\omega)|$. This has given rise to the use of the decibel and decades in discussing

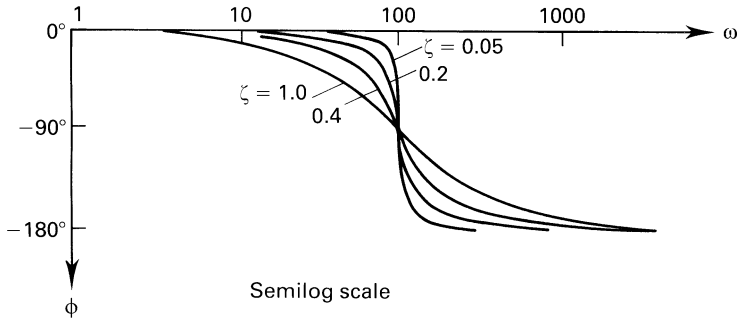

Figure 1.15 Bode phase plot for Equation (1.39).

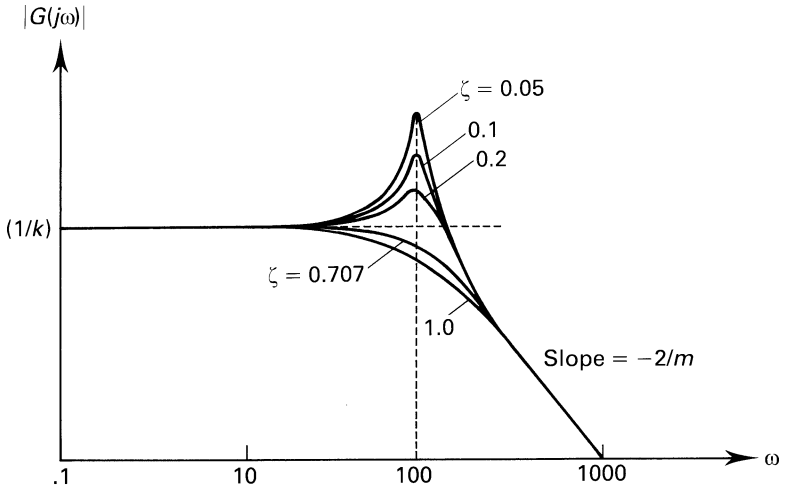

Figure 1.16 Bode magnitude plot for Equation (1.39).

the magnitude response in the frequency domain. The magnitude and phase plots (for the compliance transfer function) for the system of Equation (1.17) are shown in Figures 1.15 and 1.16 for different values of ζ. Note the phase change at resonance (90°), as this is important in interpreting measurement data.

1.6 MEASUREMENT AND TESTING

One can also use the quantities defined in the previous sections to measure the physical properties of a structure. As mentioned before, resonance can be used to determine the natural frequency of a system. Methods based on resonance are referred to as resonance testing (or modal analysis techniques) and are briefly introduced here and discussed in more detail in Chapter 8.

As mentioned earlier, the mass and stiffness of a structure can often be determined by making simple static measurements. However, damping rates require a dynamic measurement and hence are more difficult to determine. For underdamped systems one approach is to realize, from Figure 1.5, that the decay envelope is the function $e^{-\zeta\omega_n t}$. The points on the envelope illustrated in Figure 1.17 can be used to curve-fit the function e^{-at}, where a is the constant determined by the curve fit. The relation $a = \zeta\omega_n$ can next be used to calculate ζ and hence the damping rate c (assuming that m and k or ω_n are known).

A second approach is to use the concept of logarithmic decrement, denoted by δ and defined by

$$\delta = \ln \frac{x(t)}{x(t + T_d)} \tag{1.44}$$

where T_d is the period of oscillation. Using Equation (1.13) in the form

$$x(t) = Ae^{-\zeta\omega_n t} \sin(\omega_d t + \phi) \tag{1.45}$$

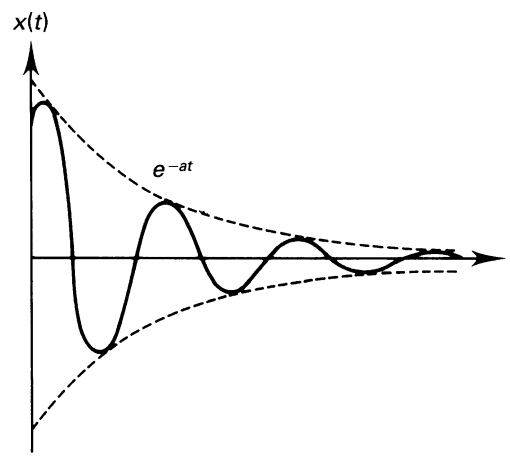

Figure 1.17 Free decay measurement method.

the logarithmic decrement δ becomes

$$\delta = \ln\left[\frac{e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)}{e^{-\zeta\omega_n(t+T_d)}\sin(\omega_d t + \omega_d T_d + \phi)}\right] = \ln e^{\zeta\omega_n T_d} = \zeta\omega_n T_d \tag{1.46}$$

where the sine functions cancel because $\omega_d T_d$ is a one-period shift by definition. Further evaluation of δ yields

$$\delta = \zeta\omega_n T_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{1.47}$$

Equation (1.47) can be manipulated to yield the damping ratio in terms of the decrement, i.e.,

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{1.48}$$

Hence, if the decrement is measured, Equation (1.48) yields the damping ratio.

The various plots of the previous section can also be used to measure ω_n, ζ, m, c, and k. For instance, the Bode diagram of Figure 1.16 can be used to determine the natural frequency, stiffness, and damping ratio. The stiffness is determined from the intercept of the frequency response function and the magnitude axis, since the value of the magnitude of the frequency response function for small ω is $\log(1/k)$. This can be seen by examining the function $\log_{10}|G(j\omega)|$ for small ω. Note that

$$\log|G(j\omega)| = \log\frac{1}{k} - \frac{1}{2}\log\left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right] = \log\left(\frac{1}{k}\right) \tag{1.49}$$

for very small values of ω. Also, note that $|G(j\omega)|$ evaluated at ω_n yields

$$k\,|G(j\omega_n)| = \frac{1}{2\zeta} \tag{1.50}$$

which provides a measure of the damping ratio from the magnitude plot of the frequency response function.

Note that Equations (1.48) and (1.22) appear to contradict each other, since

$$\frac{1}{2\zeta\sqrt{1-\zeta^2}} = k\,\max|G(j\omega)| = M_p \neq k\,|G(j\omega_n)| = \frac{1}{2\zeta}$$

except in the case of very small ζ (i.e., the difference between M_p and $|G(j\omega_n)|$ goes to zero as ζ goes to zero). This indicates a subtle difference between using the damping ratio obtained by taking resonance as the value of ω, where $|G(j\omega_n)|$ is a maximum, and using the point where $\omega = \omega_n$, the undamped natural frequency. This point is also illustrated by noting that the damped natural frequency [Equation (1.8)] is $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and the frequency at which $|G(j\omega_n)|$ is maximum is

$$\omega_p = \omega_n\sqrt{1-2\zeta^2} \tag{1.51}$$

Also note that Equation (1.51) is valid only if $0 < \zeta < 0.707$.

Finally, the mass can be related to the slope of the magnitude plot for the inertance transfer function, $G_1(s)$, by noting that

$$G_1(s) = \frac{s^2}{ms^2 + cs + k} \tag{1.52}$$

and for large ω (i.e., $\omega_n << \omega$), the value of $|G_1(j\omega)|$ is

$$|G_1(j\omega)| \approx (1/m) \tag{1.53}$$

Plots of these values are referred to as straight-line approximations to the actual magnitude plot (Bode, 1945).

The preceding formulae relating the physical properties of the structure to the magnitude Bode diagrams suggest an experimental way to determine the parameters of a structure: namely, if the structure can be driven by a sinusoid of varying frequency and if the magnitude and phase (needed to locate resonance) of the resulting response are measured, then the Bode plots and the preceding formulae can be used to obtain the desired physical parameters. This process is referred to as plant identification in the control's literature and can be extended to systems with more degrees of freedom (see, for instance, Melsa and Schultz, 1969, for a more complete account).

There are several other formulae for measuring the damping ratio and natural frequency from the results of such experiments (sine sweeps). For instance, if the Nyquist plot of the mobility transfer function is used, a circle of diameter $1/c$ results (see Figure 1.14). Another approach is to plot the magnitude of the frequency response function on a linear scale near the region of resonance, as shown in Figure 1.18. If the damping is small enough for the peak at resonance to be sharp, the damping ratio can be determined by measuring the frequencies at 0.707 at the maximum value (also called the 3 dB down point or half-power points), denoted by ω_1 and ω_2 respectively, and then using the formula (Ewins, 2000)

$$\zeta = \frac{1}{2}\left[\frac{\omega_2 - \omega_1}{\omega_d}\right] \tag{1.54}$$

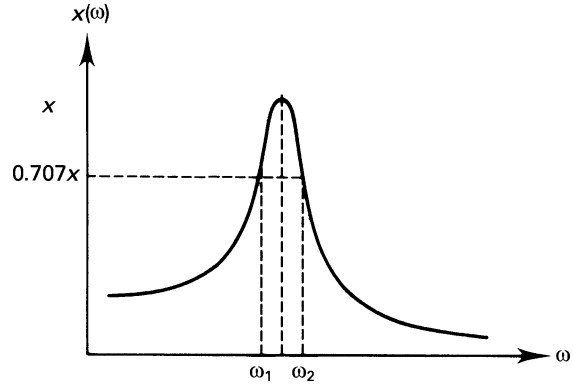

Figure 1.18 Quadrature peak picking method.

to compute the damping ratio. This method is referred to as *quadrature peak picking* and is illustrated in Figure 1.18.

1.7 STABILITY

In all the preceding analysis, the physical parameters m, c, and k are, of course, positive quantities. There are physical situations, however, in which expressions in the form of Equations (1.1) and (1.6) result but have one or more negative coefficients. Such systems are not well behaved and require some additional analysis.

Recalling that the solution to Equation (1.1) is of the form $A\sin(\omega t + \phi)$, where A is a constant, it is easy to see that the response, in this case $x(t)$, is bounded. That is to say,

$$|x(t)| < A \tag{1.55}$$

for all t, where A is some finite constant and $|x(t)|$ denotes the absolute value of $x(t)$. In this case, the system is well behaved or *stable* (called marginally stable in the control's literature). In addition, note that the roots (also called *characteristic values* or eigenvalues) of

$$\lambda^2 m + k = 0$$

are purely complex numbers $\pm j\omega_n$ as long as m and k are positive (or have the same sign). If k happens to be negative and m is positive, the solution becomes

$$x(t) = A\sinh\omega_n t + B\cosh\omega_n t \tag{1.56}$$

which increases without bound as t does. Such solutions are called *divergent* or *unstable*.

If the solution of the damped system of Equation (1.6) with positive coefficients is examined, it is clear that $x(t)$ approaches zero as t becomes large because of the exponential term. Such systems are considered to be *asymptotically stable* (called stable in the controls literature). Again, if one or two of the coefficients are negative, the motion grows without bound and becomes unstable as before. In this case, however, the motion may become unstable in one of two ways. Similar to overdamping and underdamping, the motion may grow without bound and not oscillate, or it may grow without bound and oscillate. The first case is referred to as *divergent instability* and the second case is known as *flutter instability*; together, they fall under the topic of self-excited vibrations.

Apparently, the sign of the coefficient determines the stability behavior of the system. This concept is pursued in Chapter 4, where these stability concepts are formally defined. Figures 1.19 through 1.22 illustrate each of these concepts.

These stability definitions can also be stated in terms of the roots of the characteristic equation [Equation (1.8)] or in terms of the poles of the transfer function of the system. In fact, referring to Figure 1.11, the system is stable if the poles of the structure lie along the imaginary axis (called the $j\omega$ axis), unstable if one or more poles are in the right half-plane, and asymptotically stable if all of the poles lie in the left half-plane. Flutter occurs when the poles are in the right half-plane and not on the real axis (complex conjugate pairs of roots

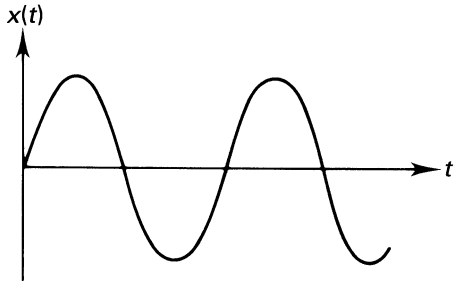

Figure 1.19 Response of a stable system.

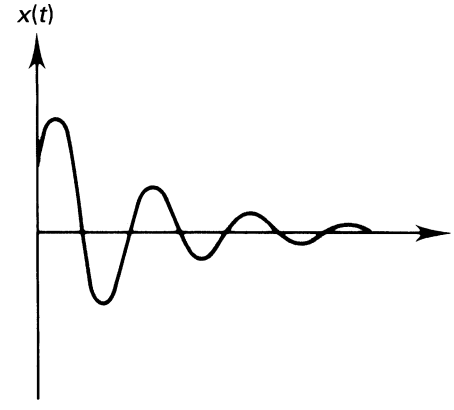

Figure 1.20 Response of an asympotically stable system.

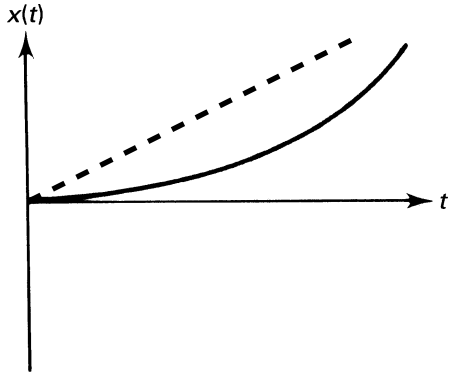

Figure 1.21 Response of a system with a divergent instability.

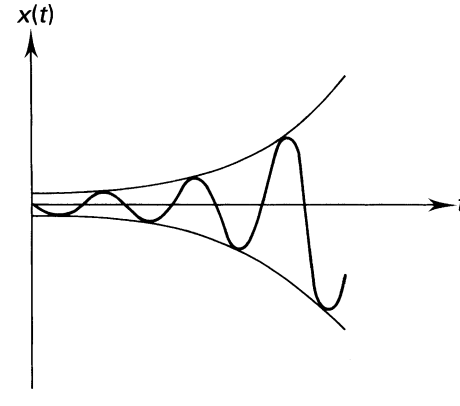

Figure 1.22 Response of a system with flutter instability.

with a positive real part), and divergence occurs when the poles are in the right half-plane along the real axis. In the simple single-degree-of-freedom case considered here, the pole positions are entirely determined by the signs of m, c, and k.

The preceding definitions and ideas about stability are stated for the free response of the system. These concepts of a well-behaved response can also be applied to the forced motion of a vibrating system. The stability of the forced response of a system can be defined by considering the nature of the applied force or input. The system is said to be *bounded-input, bounded-output stable* (or, simply, BIBO stable) if, for *any* bounded input (driving force), the output (response) is bounded for any arbitrary set of initial conditions. Such systems are manageable at resonance.

It can be seen immediately that Equation (1.17) with $c = 0$, the undamped system, is not BIBO stable, since, for $f(t) = \sin(\omega_n t)$, the response $x(t)$ goes to infinity (at resonance) whereas $f(t)$ is certainly bounded. However, the response of Equation (1.17) with $c > 0$ is

bounded whenever $f(t)$ is. In fact, the maximum value of $x(t)$ at resonance M_p is illustrated in Figure 1.9. Thus, the system of Equation (1.17) with damping is said to be BIBO stable.

The fact that the response of an undamped structure is bounded when $f(t)$ is an impulse or step function suggests another, weaker, definition for the stability of the forced response. A system is said to be *bounded*, or *Lagrange stable*, with respect to a given input if the response is bounded for any set of initial conditions. Structures described by Equation (1.1) are Lagrange stable with respect to many inputs. This definition is useful when $f(t)$ is known completely or known to fall in some specified class of functions.

Stability can also be thought of in terms of whether or not the energy of the system is increasing (unstable), constant (stable), or decreasing (asymptotically stable) rather than in terms of the explicit response. Lyapunov stability, defined in Chapter 4, extends this idea. Another important view of stability is based on how sensitive a motion is to small perturbations in the system parameters (m, c, and k) and/or small perturbations in initial conditions. Unfortunately, there does not appear to be a universal definition of stability that fits all situations. The concept of stability becomes further complicated for nonlinear systems. The definitions and concepts mentioned here are extended and clarified in Chapter 4.

1.8 DESIGN AND CONTROL OF VIBRATIONS

One can use the quantities defined in the previous sections to design structures and machines to have a desired transient and steady state response to some extent. For instance, it is a simple matter to choose m, c, and k so that the overshoot is a specified value. However, if one needs to specify the overshoot, the settling time, and the peak time, then there may not be a choice of m, c, and k that will satisfy all three criteria. Hence, the response cannot always be completely shaped, as the formulae in Section 1.4 may seem to indicate.

Another consideration in designing structures is that each of the physical parameters m, c, and k may already have design constraints that have to be satisfied. For instance, the material the structure is made of may fix the damping rate, c. Then, only the parameters m and k can be adjusted. In addition, the mass may have to be within 10% of a specified value, for instance, which further restricts the range of values of overshoot and settling time. The stiffness is often designed on the basis of the static deflection limitation.

For example, consider the system of Figure 1.10 and assume it is desired to choose values of m, c, and k so that ζ and ω_n specify a response with a settling time $t_s = 3.2$ units and a time to peak, t_p, of 1 unit. Then, Equations (1.32) and (1.34) imply that $\omega_n = 1/\zeta$ and that $\zeta = 1/\sqrt{1+\pi^2}$. This, unfortunately, also specifies the overshoot, since

$$\mathrm{OS} = \exp\left(\frac{-\zeta\pi}{\sqrt{(1-\zeta^2)}}\right)$$

Thus, all three performance criteria cannot be satisfied. This leads the designer to have to make compromises, to reconfigure the structure, or to add additional components.

Hence, in order to meet vibration criteria such as avoiding resonance, it may be necessary in many instances to alter the structure by adding vibration absorbers or isolators (Machinante,

1984, or Rivin, 2003). Another possibility is to use active vibration control and feedback methods. Both of these approaches are discussed in Chapters 6 and 7.

As just mentioned, the choice of the physical parameters m, c, and k determines the shape of the response of the system. In this sense, the choice of these parameters can be considered as the design of the structure. Passive control can also be considered as a redesign process of changing these parameters on an already existing structure to produce a more desirable response. For instance, some mass could be added to a given structure to lower its natural frequency. Although passive control or redesign is generally the most efficient way to control or shape the response of a structure, the constraints on m, c, and k are often such that the desired response cannot be obtained. Then the only alternative, short of starting over, is to try active control.

There are many different types of active control methods, and only a few will be considered to give the reader a feel for the connection between the vibration and control disciplines. As mentioned earlier, the comments made in this text on control should not be considered as a substitute for studying standard control or linear system texts. Output feedback control is briefly introduced here and discussed in more detail in Chapter 7.

First, a clarification of the difference between active and passive control is in order. Basically, an active control system uses some external adjustable or active (for example, electronic) device, called an actuator, to provide a means of shaping or controlling the response. Passive control, on the other hand, depends only on a fixed (passive) change in the physical parameters of the structure. Active control often depends on current measurements of the response of the system, and passive control does not. Active control requires an external energy source, and passive control typically does not.

Feedback control consists of measuring the output, or response, of the structure and using that measurement to determine the force to apply to the structure to obtain a desired response. The device used to measure the response (sensor), the device used to apply the force (actuator), and any electronics required to transfer the sensor signal into an actuator command (control law) make up the *control hardware*. This is illustrated by the block diagram in Figure 1.23. Systems with feedback are referred to as closed-loop systems, while control systems without feedback are called open-loop systems, as illustrated in Figures 1.23 and 1.24 respectively. A major difference between open-loop and closed-loop control is simply that closed-loop control depends on information about the response of the system, and open-loop control does not.

The rule that defines how the measurement from the sensor is used to command the actuator to effect the system is called the *control law*, denoted by $H(s)$ in Figure 1.23. Much

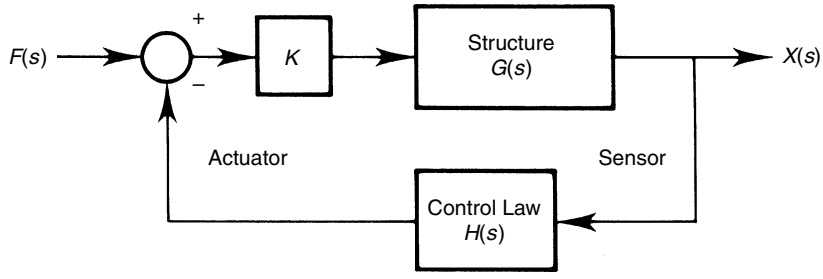

Figure 1.23 Closed-loop system.

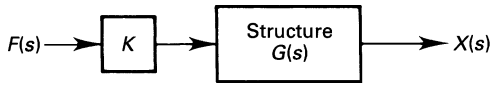

Figure 1.24 Open-loop system.

of control theory focuses on clever ways to choose the control law to achieve a desired response.

A simple open-loop control law is to multiply (or amplify) the response of the system by a constant. This is referred to as *constant gain control*. The magnitude of the frequency response function for the system in Figure 1.23 is multiplied by the constant K, called the *gain*. The frequency domain equivalent of Figure 1.23 is

$$\frac{X(s)}{F(s)} = KG(s) = \frac{K}{ms^2 + cs + k} \tag{1.57}$$

where the plant is taken to be a single-degree-of-freedom model of structure. In the time domain, this becomes

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = Kf(t) \tag{1.58}$$

The effect of this open-loop control is simply to multiply the steady state response by K and to increase the value of the peak response, M_p.

On the other hand, the closed-loop control, illustrated in Figure 1.23, has the equivalent frequency domain representation given by

$$\frac{X(s)}{F(s)} = \frac{KG(s)}{1 + KG(s)H(s)} \tag{1.59}$$

If the feedback control law is taken to be one that measures both the velocity and position, multiplies them by some constant gains g_1 and g_2 respectively, and adds the result, the control law $H(s)$ is given by

$$H(s) = g_1 s + g_2 \tag{1.60}$$

As the velocity and position are the *state variables* for this system, this control law is called *full state feedback*, or PD control (for position and derivative). In this case, Equation (1.56) becomes

$$\frac{X(s)}{F(s)} = \frac{K}{ms^2 + (Kg_1 + c)s + (Kg_2 + k)} \tag{1.61}$$

The time domain equivalent of this equation (obtained by using the inverse Laplace transform) is

$$m\ddot{x}(t) + (c + Kg_1)\dot{x}(t) + (k + Kg_2)x(t) = Kf(t) \tag{1.62}$$

By comparing Equations (1.58) and (1.62), the versatility of closed-loop control versus open-loop, or passive, control is evident. In many cases the choice of values of K, g_1, and

g_2 can be made electronically. By using a closed-loop control, the designer has the choice of three more parameters to adjust than are available in the passive case to meet the desired specifications.

On the negative side, closed-loop control can cause some difficulties. If not carefully designed, a feedback control system can cause an otherwise stable structure to have an unstable response. For instance, suppose the goal of the control law is to reduce the stiffness of the structure so that the natural frequency is lower. From examining Equation (1.62), this would require g_2 to be a negative number. Then, suppose that the value of k was overestimated and g_2 calculated accordingly. This could result in the coefficient of $x(t)$ becoming negative, causing instability. That is, the response of Equation (1.62) would be unstable if $(k + Kg_2) < 0$. This would amount to positive feedback and is not likely to arise by design on purpose, but it can happen if the original parameters are not well known. On physical grounds, instability is possible because the control system is adding energy to the structure. One of the major concerns in designing high-performance control systems is to maintain stability. This introduces another constraint on the choice of the control gains and is discussed in more detail in Chapter 7. Of course, closed-loop control is also expensive because of the sensor, actuator, and electronics required to make a closed-loop system. On the other hand, closed-loop control can always result in better performance provided the appropriate hardware is available.

Feedback control uses the measured response of the system to modify and add back into the input to provide an improved response. Another approach to improving the response consists of producing a second input to the system that effectively cancels the disturbance to the system. This approach, called *feedforward control*, uses knowledge of the response of a system at a point to design a control force that, when subtracted from the uncontrolled response, yields a new response with desired properties, usually a response of zero. Feedforward control is most commonly used for high-frequency applications and in acoustics (for noise cancellation) and is not considered here. An excellent treatment of feedforward controllers is given by Fuller, Elliot, and Nelson (1996).

1.9 NONLINEAR VIBRATIONS

The force versus displacement plot for a spring in Figure 1.3 curves off after the deflections and forces become large enough. Before enough force is applied to deform permanently or break the spring, the force deflection curve becomes nonlinear and curves away from a straight line, as indicated in Figure 1.25. Therefore, rather than the linear spring relationship $f_k = kx$, a model such as $f_k = \alpha x - \beta x^3$, called a *softening spring*, might better fit the curve. This nonlinear spring behavior greatly changes the physical nature of the vibratory response and complicates the mathematical description and analysis to the point where numerical integration usually has to be employed to obtain a solution. Stability analysis of nonlinear systems also becomes more complicated.

In Figure 1.25 the force–displacement curves for three springs are shown. Notice that the linear range for the two nonlinear springs is a good approximation until about 1.8 units of displacement or 2000 units of force. If the spring is to be used beyond that range, then the linear vibration analysis of the preceding sections no longer applies.

Consider, then, the equation of motion of a system with a nonlinear spring of the form

$$m\ddot{x}(t) + \alpha x(t) - \beta x^3(t) = 0 \tag{1.63}$$

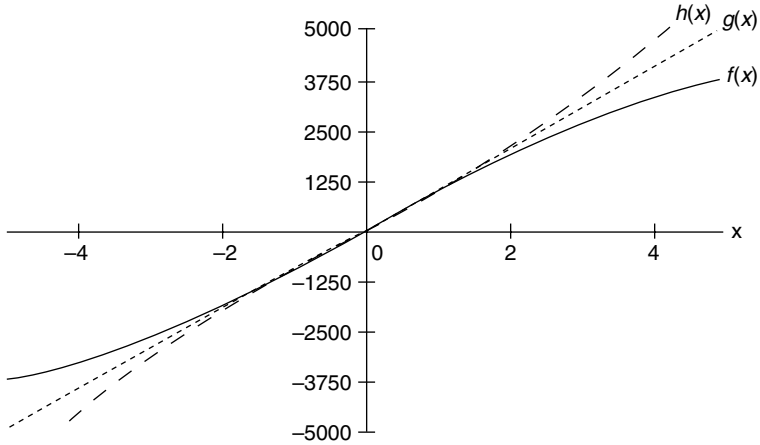

Figure 1.25 Force deflection curves for three different springs, indicating their linear range. The curve $g(x) = kx$ is a linear spring, the curve $f(x) = kx - bx^3$ is called a softening spring, and the curve $h(x) = kx + bx^3$ is called a hardening spring (here $k = 1000$, $b = 10$).

which is subject to two initial conditions. In the linear system there was only one equilibrium point to consider, $v(t) = x(t) = 0$. As will be shown in the following, the nonlinear system of Equation (1.63) has more than one equilibrium position. The equilibrium point of a system, or set of governing equations, may be defined best by first placing the equation of motion into state-space form.

A general single-degree-of-freedom system may be written as

$$\ddot{x}(t) + f(x(t), \dot{x}(t)) = 0 \tag{1.64}$$

where the function f can take on any form, linear or nonlinear. For example, for a linear spring–mass–damper system the function f is just $f(x, \dot{x}) = 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t)$, which is a linear function of the state variables of position and velocity. For a nonlinear system the function f will be some nonlinear function of the state variables. For instance, for the nonlinear spring of Equation (1.63), the function is $f(x, \dot{x}) = \alpha x - \beta x^3$.

The general state-space model of Equation (1.64) is written by defining the two state variables: the position $x_1 = x(t)$, and the velocity $x_2 = \dot{x}(t)$. Then, Equation (1.64) can be written as the first-order pair

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -f(x_1, x_2) \end{aligned} \tag{1.65}$$

This state-space form of the equation of motion is used for numerical integration, in control analysis, and for formally defining an equilibrium position. Define the state vector, $\mathbf{x}$, and a nonlinear vector function $\mathbf{F}$, as

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \text{and} \qquad \mathbf{F} = \begin{bmatrix} x_2(t) \\ -f(x_1, x_2) \end{bmatrix} \tag{1.66}$$

Then, Equations (1.65) may be written in the simple form of a vector equation

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{1.67}$$

An *equilibrium point* of this system, denoted by $\mathbf{x}_e$, is defined to be any value of the vector $\mathbf{x}$ for which $\mathbf{F}(\mathbf{x})$ is identically zero (called zero-phase velocity). Thus, the equilibrium point is any vector of constants, $\mathbf{x}_e$, that satisfies the relations

$$\mathbf{F}(\mathbf{x}_e) = \mathbf{0} \tag{1.68}$$

Placing the linear single-degree-of-freedom system into state-space form then yields

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -2\zeta\omega_n x_2 - \omega_n^2 x_1 \end{bmatrix} \tag{1.69}$$

The equilibrium of a linear system is thus the solution of the vector equality

$$\begin{bmatrix} x_2 \\ -2\zeta\omega_n x_2 - \omega_n^2 x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1.70}$$

which has the single solution $x_1 = x_2 = 0$. Thus, for any linear system the equilibrium point is a single point consisting of the origin. On the other hand, the equilibrium condition of the soft spring system of Equation (1.63) requires that

$$\begin{aligned} x_2 &= 0 \\ -\alpha x_1 + \beta x_1^3 &= 0 \end{aligned} \tag{1.71}$$

Solving for x_1 and x_2 yields the *three* equilibrium points

$$\mathbf{x}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -\sqrt{\dfrac{\alpha}{\beta}} \\ 0 \end{bmatrix} \tag{1.72}$$

In principle, the soft spring system of Equation (1.63) could oscillate around any of these equilibrium points, depending on the initial conditions. Each of these equilibrium points may also have a different stability property.

The existence of multiple equilibrium points also complicates the notion of stability introduced in Section 1.7. In particular, solutions near each equilibrium point could potentially have different stability behavior. Since the initial conditions may determine the equilibrium around which the solution centers, the behavior of a nonlinear system will depend on the initial conditions. In contrast, for a linear system with fixed parameters the solution form is the same regardless of the initial conditions. This represents another important difference to consider when working with nonlinear components.

1.10 COMPUTING AND SIMULATION IN MATLAB

Modern computer codes such as MATLAB make the visualization and computation of vibration problems available without much programming effort. Such codes can help enhance

understanding through plotting responses, can help find solutions to complex problems lacking closed-form solutions through numerical integration, and can often help with symbolic computations. Plotting certain parametric relations or plotting solutions can often aid in visualizing the nature of relationships or the effect of parameter changes on the response. Most of the plots used in this text are constructed from simple MATLAB commands, as the following examples illustrate. If you are familiar with MATLAB, you may wish to skip this section.

MATLAB is a high-level code, with many built-in commands for numerical integration (simulation), control design, performing matrix computations, symbolic manipulation, etc. MATLAB has two areas to enter information. The first is the command window, which is an active area where the entered command is compiled as it is entered. Using the command window is somewhat like a calculator. The second area is called an m-file, which is a series of commands that are saved and then called from the command window for execution. All of the plots in the figures in this chapter can be reproduced using these simple commands.

Example 1.10.1

Plot the free response of the underdamped system to the initial conditions $x_0 = 0.01\,\text{m}$, $v_0 = 0$ for values of $m = 100\,\text{kg}$, $c = 25\,\text{kg/s}$, and $k = 1000\,\text{N/m}$, using MATLAB and Equation (1.13).

To enter numbers in the command window, just type a symbol and use an equal sign after the blinking cursor. The following entries in the command window will produce the plot of Figure 1.26. Note that the prompt symbol `'>>'` is provided by MATLAB and the information following it is code typed in by

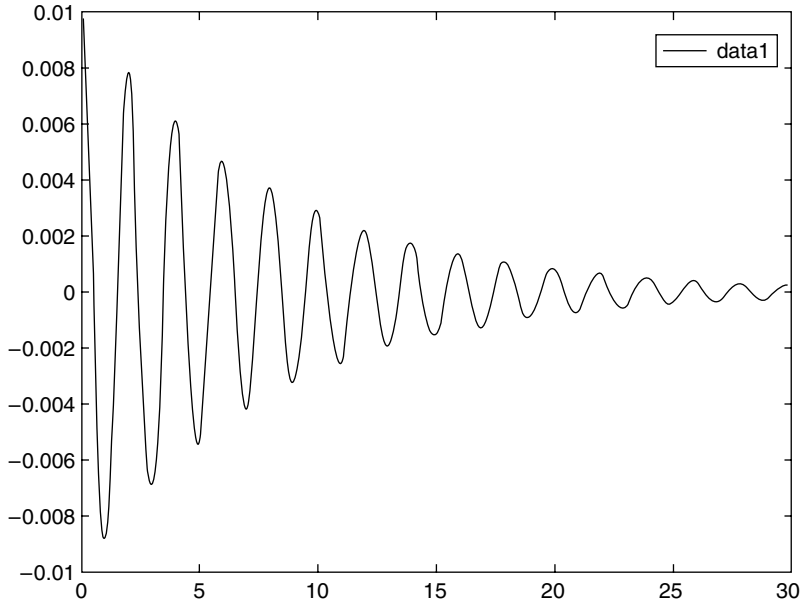

Figure 1.26 Response of an underdamped system ($m = 100\,\text{kg}$, $c = 25\,\text{kg/s}$, and $k = 1000\,\text{N/m}$) to the initial conditions $x_0 = 0.01\,\text{m}$, $v_0 = 0$, plotted using MATLAB.

the user. The symbol % is used to indicate comments, so that anything following this symbol is ignored by the code and is included to help explain the situation. A semicolon typed after a command suppresses the command from displaying the output. MATLAB uses matrices and vectors so that numbers can be entered and computed in arrays. Thus, there are two types of multiplication. The notation a*b is a vector operation demanding that the number of rows of a be equal to the number of columns of b. The product a.*b, on the other hand, multiplies each element of a by the corresponding element in b.

```
>> clear % used to make sure no previous values are stored
>> %assign the initial conditions, mass, damping and stiffness
>> x0=0.01;v0=0.0;m=100;c=25;k=1000;
>> %compute omega and zeta, display zeta to check if underdamped
>> wn=sqrt(k/m);z=c/(2*sqrt(k*m))
z =
0.0395
>> %compute the damped natural frequency
>> wd=wn*sqrt(1-z^2);
>> t=(0:0.01:15*(2*pi/wn));%set the values of time from 0 in
increments of 0.01 up to 15 periods
>> x=exp(-z*wn*t).*(x0*cos(wd*t)+((v0+z*wn*x0)/wd)*sin(wd*t));
% computes x(t)
>> plot(t,x)%generates a plot of x(t) vs t
```

The MATLAB code used in this example is not the most efficient way to plot the response and does not show the detail of labeling the axis, etc., but is given as a quick introduction.

The next example illustrates the use of m-files in a numerical simulation. Instead of plotting the closed-form solution given in Equation (1.13), the equation of motion can be numerically integrated using the ode command in MATLAB. The ode45 command uses a fifth-order Runge–Kutta, automated time step method for numerically integrating the equation of motion (see, for instance, Pratap, 2002).

In order to use numerical integration, the equations of motion must first be placed in first-order, or state-space, form, as done in Equation (1.69). This state-space form is used in MATLAB to enter the equations of motion.

Vectors are entered in MATLAB by using square brackets, spaces, and semicolons. Spaces are used to separate columns, and semicolons are used to separate rows, so that a row vector is entered by typing

```
>> u = [1  -1  2]
```

which returns the row

```
u =   1  -1  2
```

and a column vector is entered by typing

```
>> u = [1;  -1;  2]
```

which returns the column

```
u =
    1
    -1
    2
```

To create a list of formulae in an m-file, choose 'New' from the file menu and select 'm-file'. This will display a text editor window, in which you can enter commands. The following example illustrates the creation of an m-file and how to call it from the command window for numerical integration of the equation of motion given in example 1.10.1.

Example 1.10.2

Numerically integrate and plot the free response of the underdamped system to the initial conditions $x_0 = 0.01\,\text{m}$, $v_0 = 0$ for values of $m = 100\,\text{kg}$, $c = 25\,\text{kg/s}$, and $k = 1000\,\text{N/m}$, using MATLAB and equation (1.13).

First create an m-file containing the equation of motion to be integrated and save it. This is done by selecting 'New' and 'm-File' from the File menu in MATLAB, then typing

```
----------------------
Function xdot=f2(t,x)
c=25; k = 1000; m = 100;
% set up a column vector with the state equations
xdot=[x(2); -(c/m)*x(2)-(k/m)*x(1)];
----------------------
```

This file is now saved with the name `f2.m`. Note that the name of the file must agree with the name following the equal sign in the first line of the file. Now open the command window and enter the following:

```
>> ts=[0 30]; % this enters the initial and final time
>> x0 =[0.01 0]; % this enters the initial conditions
>> [t, x]=ode45('f2',ts,x0);
>> plot(t,x(:,1))
```

The third line of code calls the Runge–Kutta program ode45 and the state equations to be integrated contained in the file named `f2.m`. The last line plots the simulation of the first state variable $x_1(t)$ which is the displacement, denoted by `x(:,1)` in MATLAB. The plot is given in Figure 1.27.

Note that the plots of Figures 1.26 and 1.27 look the same. However, Figure 1.26 was obtained by simply plotting the analytical solution, whereas the plot of Figure 1.27 was obtained by numerically integrating the equation of motion. The numerical approach can be used successfully to obtain the solution of a nonlinear state equation, such as Equation (1.63), just as easily.

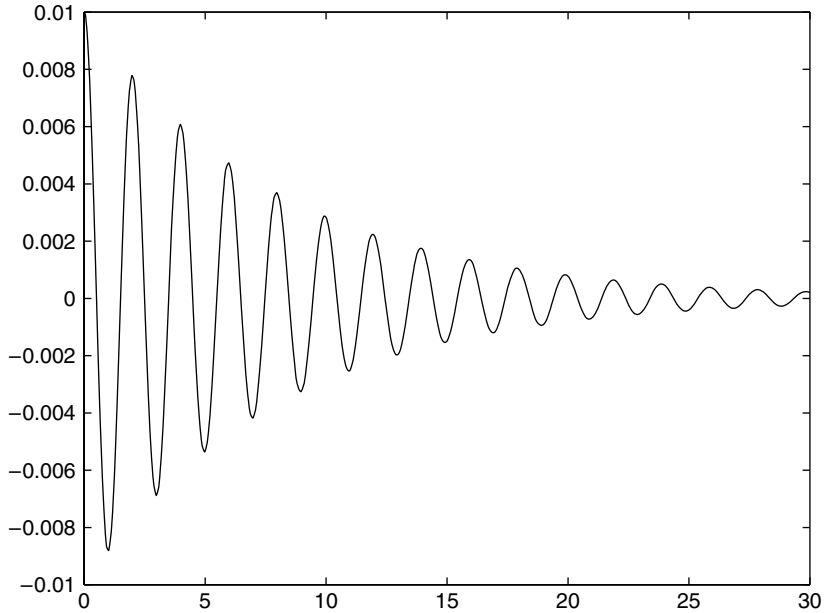

Figure 1.27 Plot of the numerical integration of the underdamped system of example 1.10.1 resulting from the MATLAB code given in example 1.10.2.

The forced response can also be computed using numerical simulation, and this is often more convenient than working through an analytical solution when the forcing functions are discontinuous or not made up of simple functions. Again, the equations of motion (this time with the forcing function) must be placed in state-space form. The equation of motion for a damped system with a general applied force is

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t)$$

In state-space form this expression becomes

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{1.73}$$

where $f(t) = F(t)/m$ and $F(t)$ is any function that can be integrated. The following example illustrates the procedure in MATLAB.

Example 1.10.3

Use MATLAB to compute and plot the response of the following system:

$$100\ddot{x}(t) + 10\dot{x}(t) + 500x(t) = 150\cos 5t, \qquad x_0 = 0.01, \qquad v_0 = 0.5$$

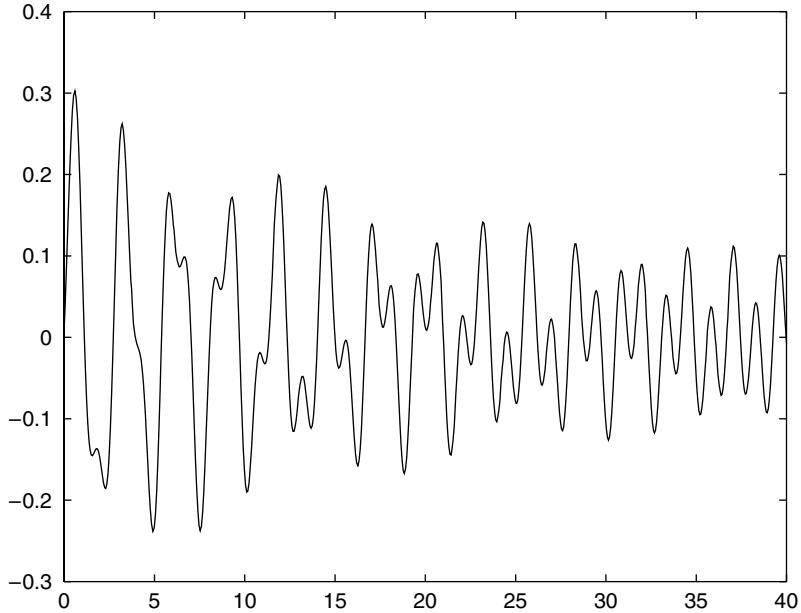

Figure 1.28 A plot of the numerical integration of the damped forced system resulting from the MATLAB code given in example 1.10.3.

The MATLAB code for computing these plots is given. First an m-file is created with the equation of motion given in first-order form:

```
---------------------------------------------
function v=f(t,x)
m=100; k=500; c=10; Fo=150; w=5;
v=[x(2); x(1)*-k/m+x(2)*-c/m + Fo/m*cos(w*t)];
---------------------------------------------
```

Then the following is typed in the command window:

```
>>clear all
>>xo=[0.01; 0.5]; %enters the initial conditions
>>ts=[0 40]; %enters the initial and final times
>>[t,x]=ode45('f',ts,xo); %calls the dynamics and integrates
>>plot(t,x(:,1)) %plots the result
```

This code produces the plot given in Figure 1.28. Note that the influence of the transient dynamics dies off owing to the damping after about 20 s.

Such numerical integration methods can also be used to simulate the nonlinear systems discussed in the previous section. Use of high-level codes in vibration analysis such as MATLAB

is now commonplace and has changed the way vibration quantities are computed. More detailed codes for vibration analysis can be found in Inman (2001). In addition there are many books written on using MATLAB (such as Pratap, 2002) as well as available online help.

CHAPTER NOTES

This chapter attempts to provide an introductory review of vibrations and to expand the discipline of vibration analysis and design by intertwining elementary vibration topics with the disciplines of design, control, stability, and testing. An early attempt to relate vibrations and control at an introductory level was made by Vernon (1967). More recent attempts have been made by Meirovitch (1985, 1990) and by Inman (1989) – the first edition of this text. Leipholz and Abdel-Rohman (1986) take a civil engineering approach to structural control. The latest attempts to combine vibration and control are by Preumont (2002) and Benaroya (2004) who also provides an excellent treatment of uncertainty in vibrations. The information contained in Sections 1.2 and 1.3, and in part of Section 1.4 can be found in every introductory text on vibrations, such as my own (Inman, 2001) and such as the standards by Thomson and Dahleh (1993), Rao (2004), and Meirovitch (1986). A complete summary of most vibration-related topics can be found in Braun, Ewins, and Rao (2002) and in Harris and Piersol (2002).

A good reference for vibration measurement is McConnell (1995). The reader is encouraged to consult a basic text on control such as the older text by Melsa and Schultz (1969), which contains some topics omitted from modern texts, or by Kuo and Golnaraghi (2003), which contains more modern topics integrated with MATLAB. These two texts also provide background to specifications and transfer functions given in Sections 1.4 and 1.5 as well as feedback control discussed in Section 1.8. A complete discussion of plant identification as presented in Section 1.6 can be found in Melsa and Schultz (1969). The excellent text by Fuller, Elliot, and Nelson (1996) examines the control of high-frequency vibration. Control is introduced here not as a discipline by itself but rather as a design technique for vibration engineers. A standard reference on stability is Hahn (1967), which provided the basic ideas for Section 1.7. The topic of flutter and self-excited vibrations is discussed in Den Hartog (1985). Nice introductions to nonlinear vibration can be found in Virgin (2000), in Worden and Tomlinson (2001), and in the standards by Nayfeh and Mook (1978) and Nayfeh and Balachandra (1995).

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PROBLEMS

1.1 Derive the solution of $m\ddot{x} + kx = 0$ and sketch your result (for at least two periods) for the case $x_0 = 1$, $v_0 = \sqrt{5}$, and $k/m = 4$.

1.2 Solve $m\ddot{x} - kx = 0$ for the case $x_0 = 1$, $v_0 = 0$, and $k/m = 4$, for $x(t)$ and sketch the solution.

1.3 Derive the solutions given in the text for $\zeta > 1$, $\zeta = 1$, and $0 < \zeta < 1$ with x_0 and v_0 as the initial conditions (i.e., derive Equations 1.14 through 1.16 and the corresponding constants).

1.4 Solve $\ddot{x} - \dot{x} + x = 0$ with $x_0 = 1$ and $v_0 = 0$ for $x(t)$ and sketch the solution.

1.5 Prove that $\zeta = 1$ corresponds to the smallest value of c such that no oscillation occurs. (*Hint*: Let $\lambda = -b$, b a positive real number, and differentiate the characteristic equation.)

1.6 Calculate t_p, OS, T_d, M_p, and BW for a system described by

$$2\ddot{x} + 0.8\dot{x} + 8x = f(t)$$

where $f(t)$ is either a unit step function or a sinusoidal, as required.

1.7 Derive an expression for the forced response of the undamped system

$$m\ddot{x}(t) + kx(t) = F_0 \sin \omega t, \qquad x(0) = x_0, \qquad \dot{x}(0) = v_0$$

to a sinusoidal input and nonzero initial conditions. Compare your result with Equation (1.21) with $\zeta = 0$.

1.8 Compute the total response to the system

$$4\ddot{x}(t) + 16x(t) = 8 \sin 3t, \qquad x_0 = 1\,\text{mm}, \qquad v_0 = 2\,\text{mm/s}$$

1.9 Calculate the maximum value of the peak response (magnification factor) for the system in Figure 1.18 with $\zeta = 1/\sqrt{2}$.

1.10 Derive Equation (1.22).

1.11 Calculate the impulse response function for a critically damped system.

1.12 Solve for the forced response of a single-degree-of-freedom system to a harmonic excitation with $\zeta = 1.1$ and $\omega_n^2 = 4$. Plot the magnitude of the steady state response versus the driving frequency. For what value of ω_n is the response a maximum (resonance)?

1.13 Calculate the compliance transfer function for the system described by the differential equation

$$a\dddddot{x} + b\dddot{x} + c\ddot{x} + d\dot{x} + ex = f(t)$$

where $f(t)$ is the input and $x(t)$ is a displacement. Also, calculate the frequency response function for this system.

1.14 Derive Equation (1.51).

1.15 Plot (using a computer) the unit step response of a single-degree-of-freedom system with $\omega_n^2 = 4, k = 1$ for several values of the damping ratio ($\zeta = 0.01, 0.1, 0.5$, and 1.0).

1.16 Let ω_p denote the frequency at which the peak response occurs [Equation (1.22)]. Plot ω_p/ω_n versus ζ and ω_d/ω_n versus ζ and comment on the difference as a function of ζ.

1.17 For the system of problem 1.6, construct the Bode plots for (a) the inertance transfer function, (b) the mobility transfer function, (c) the compliance transfer function, and (d) the Nyquist diagram for the compliance transfer function.

1.18 Discuss the stability of the following system: $2\ddot{x}(t) - 3\dot{x}(t) + 8x(t) = -3\dot{x}(t) + \sin 2t$.

1.19 Using the system of problem 1.6, refer to Equation (1.62) and choose the gains K, g_1, and g_2 so that the resulting closed-loop system has a 5% overshoot and a settling time of less than 10.

1.20 Calculate an allowable range of values for the gains K, g_1, and g_2 for the system of problem 1.6, such that the closed-loop system is stable and the formulae for overshoot and peak time of an underdamped system are valid.

1.21 Compute a feedback law with full state feedback [of the form given in Equation (1.62)] that stabilizes (makes asymptotically stable) the system $4\ddot{x}(t) + 16x(t) = 0$ and causes the closed-loop settling time to be 1 s.

1.22 Compute the equilibrium positions of the pendulum equation $m\ell^2\ddot{\theta}(t) + mg\ell \sin \theta(t) = 0$.

1.23 Compute the equilibrium points for the system defined by

$$\ddot{x} + \beta\dot{x} + x + x^2 = 0$$

1.24 The linearized version of the pendulum equation is given by

$$\ddot{\theta}(t) + \frac{g}{\ell}\theta(t) = 0$$

Use numerical integration to plot the solution of the nonlinear equation of problem 1.22 and this linearized version for the case where

$$g = 0.01\ell, \qquad \theta(0) = 0.1\,\text{rad}, \qquad \dot{\theta}(0) = 0.1\,\text{rad/s}$$

Compare your two simulations.