13

Approximations of Distributed-parameter Models

13.1 INTRODUCTION

This chapter is devoted to examining approximations of distributed-parameter systems with lumped-parameter models. Since the solutions of distributed-parameter systems are often given in terms of an infinite series, and since only a few configurations have closed-form solutions, there is a need to cast distributed-parameter systems into finite-dimensional systems that can easily be solved numerically. In addition, control and design are well developed for lumped-parameter systems, providing further motivation to approximate distributed systems with the more easily manipulated lumped systems. From the experimentalist point of view, most common measurement methods only 'see' a finite (dimensional) number of points.

In this chapter, several common methods of approximating distributed-mass structures by lumped-mass models are presented. Most of these methods eliminate the spatial dependence in the solution technique by discretizing the spatial variable in some way, effectively approximating an eigenfunction with an eigenvector. This chapter ends with a discussion of the effects of active control of distributed-mass structures and the accuracy of the approximation.

13.2 MODAL TRUNCATION

Since the solution of the vibration problem given by

$$w_{tt}(x,t) + L_1 w_t(x,t) + L_2 w(x,t) = f(x,t), \qquad x \in \Omega \tag{13.1}$$

plus appropriate boundary conditions is of the form

$$w(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x) \tag{13.2}$$

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which converges uniformly, it is possible to approximate the solution by

$$w_N(x,t) = \sum_{n=1}^{N} a_n(t)\phi_n(x) \tag{13.3}$$

where N is finite. This finite sum approximation ignores the sum given by

$$w_R(x,t) = \sum_{n=N+1}^{\infty} a_n(t)\phi_n(x) \tag{13.4}$$

called the *residual*. The modes in this sum are called the *truncated modes*, i.e., the functions $\phi_n(x)$ for values of the index $n = N+1 \to \infty$. The assumption is that the residual solution is small, i.e., that $\| w_R(x,t) \| < \varepsilon$. This assumption is often satisfied by physical structures, giving rise to the statement that structures behave like low-pass filters.

Substitution of Equation (13.3) into Equation (13.1) yields

$$\sum_{n=1}^{N} [\ddot{a}_n(t)\phi_n(x) + \dot{a}_n L_1 \phi_n(x) + a_n(t) L_2 \phi_n(x)] = \sum_{n=1}^{N} b_n(t)\phi_n(x) \tag{13.5}$$

where $f(x,t)$ has also been expanded in terms of the functions $\phi_n(x)$ with coefficients $b_n(t)$. Premultiplying Equation (13.5) by $\phi_m(x)$ and integration over Ω yields two possibilities. Note that the sum is now finite, so that convergence is not a problem. First, if $L_1 L_2 = L_2 L_1$ on the appropriate domain, then Equation (13.5) becomes N decoupled ordinary differential equations of the form

$$\ddot{a}_n(t) + \lambda_n^{(1)} \dot{a}_n(t) + \lambda_n^{(2)} a_n(t) = b_n(t) \tag{13.6}$$

In matrix form this becomes

$$I\ddot{\mathbf{a}} + \Lambda_D \dot{\mathbf{a}} + \Lambda_K \mathbf{a} = \mathbf{f} \tag{13.7}$$

which can then be analyzed by the methods of Chapter 5. Here, Λ_D and Λ_K are diagonal matrices and $\mathbf{a}$ and $\mathbf{f}$ are N vectors of obvious definition.

If the commutivity condition does not hold, then Equation (13.6) becomes

$$I\ddot{\mathbf{a}} + D\dot{\mathbf{a}} + \Lambda_K \mathbf{a} = \mathbf{f} \tag{13.8}$$

where the elements of D are

$$d_{ij} = \int_{\Omega} \phi_i L_1 \phi_j \, d\Omega \tag{13.9}$$

and the functions $\phi_j(x)$ are the eigenfunctions of L_2. The boundary conditions are incorporated in the matrices D, Λ_D, and Λ_K automatically by virtue of the integration. The initial conditions on $\mathbf{a}(t)$ are defined by

$$a_i(0) = \int_{\Omega} w(x,0)\phi_i(x)\, d\Omega \qquad \text{and} \qquad \dot{a}_i(0) = \int_{\Omega} w_t(x,0)\phi_i(x)\, d\Omega \tag{13.10}$$

In both cases it is required that $w_R(x, t)$ be as small as possible, i.e., that the higher modes do not contribute much to the solution. In practice this is often so. For instance, $N = 3$ is often adequate to describe the longitudinal vibration of a simple cantilevered beam (recall example 12.3.1). Equations (13.6) and (13.8) are finite-dimensional approximations of Equations (13.1) derived by truncating the higher modes of the response of the structure (i.e., setting $w_R = 0$) and as such are referred to as a *truncated modal model*.

13.3 RAYLEIGH–RITZ–GALERKIN APPROXIMATIONS

The Rayleigh quotient was introduced in Section 11.7 as a means of approximating the natural frequencies of a conservative system. Ritz used this concept to calculate an approximate solution for the eigenfunctions (mode shapes) in terms of an assumed series of trial functions. This approach is similar to modal truncation but, rather than using the exact mode shapes as the expanding basis, any complete set of basis functions that satisfy the boundary conditions is used. In other words, the Rayleigh–Ritz (as it is usually called) approximation does not require any knowledge of the eigenfunctions. Furthermore, the Rayleigh quotient can be written in terms of energy, rather than in terms of the eigenvalue problem, reducing the number of derivatives and boundary conditions that need to be satisfied by the choice of 'trial' functions.

Trial functions are functions that (a) satisfy the boundary conditions or at least some of them, (b) are orthogonal to each other, and (c) have enough derivatives to be fit into the equation of motion. Trial functions are further divided up into those that satisfy all of the boundary conditions (called comparison functions) and those that satisfy only the geometric boundary conditions (called admissible functions). In forming the sum of Equation (13.5) there are three classifications of functions that can be used:

1. *Eigenfunctions*. These satisfy the equation of motion plus all the boundary conditions.
2. *Comparison functions*. These are orthogonal and satisfy all the boundary conditions (but not the equation of motion).
3. *Admissible functions*. These are orthogonal and satisfy only the geometric boundary conditions (i.e. things like displacements and slopes).

Boundary conditions are classified as either (a) natural boundary conditions (those that involve force and moment balances) or (b) geometric boundary conditions (those that satisfy displacement and slope conditions at the boundary).

Using trial functions eliminates the need to know the eigenfunctions of the structure before approximating the system. Let $\{\theta_n(x)\}$ be a linearly independent set of basis functions that are complete in a subspace of $D(L_2)$ and satisfy the appropriate boundary conditions. The Nth approximate solution of Equation (13.1) is then given by the expression

$$w_N(x, t) = \sum_{n=1}^{N} a_n(t)\theta_n(x) \tag{13.11}$$

Likewise, $f(x, t)$ is approximated by

$$f_n(x, t) = \sum_{n=1}^{N} b_n(t)\theta_n(x) \tag{13.12}$$

Substitution of Equations (13.9) and (13.10) for $w(x, t)$ and $f(x, t)$ in (13.1), respectively, yields

$$\sum_{n=1}^{N} [\ddot{a}(t)\theta_n + \dot{a}_n(t)L_1\theta_n + a_n(t)L_2\theta_n] = \sum_{n=1}^{N} b_n(t)\theta_n(x) \tag{13.13}$$

Premultiplying Equation (13.13) by $\theta_m(x)$ and integrating (thus using the boundary conditions) yields the finite-dimensional approximation

$$M\ddot{\mathbf{x}} + D\dot{\mathbf{x}} + K\mathbf{x} = \mathbf{f}(t) \tag{13.14}$$

where the matrices M, D, and K and the vector $\mathbf{f}$ are defined by

$$m_{ij} = \int \theta_i \theta_j \, d\Omega \tag{13.15}$$

$$d_{ij} = \int \theta_i L_1 \theta_j \, d\Omega \tag{13.16}$$

$$k_{ij} = \int \theta_i L_2 \theta_j \, d\Omega \tag{13.17}$$

$$f_i = \int f(x, t)\theta_i \, d\Omega \tag{13.18}$$

Unlike the coefficient matrices of the modal truncation scheme of Equations (13.7) and (13.8), the matrices M, D, and K in this case are not necessarily diagonal. Note, however, that they are symmetric as long as the operators L_1 and L_2 are self-adjoint. The order, N, of the finite-dimensional approximation [Equation (13.11)] is chosen so that $w_n(x, t)$ is as small as possible for the purpose at hand. Note that the difference between the functions $\phi_i(x)$ in Section 13.2 and the $\theta_n(x)$ in this section is that the $\phi_i(x)$ are eigenfunctions of the stiffness operator. In this section the trial functions $\theta_n(x)$ are chosen in a somewhat arbitrary fashion. Hence, this method is also called the *assumed mode method*. The bottom line with approximation methods is that, the closer the trial function to the exact eigenfunction (mode shape), the better is the estimate. If, in fact, an exact set of mode shapes is used, and the damping is proportional, the approximation will be exact.

Starting with the Rayleigh quotient, the Ritz method minimizes the quotient over the coefficients of expansion for $\phi(x)$ and provides an approximation of the system natural frequencies and mode shapes for undamped systems. Let the approximate spatial dependence have the form

$$\phi(x) = \sum_{i=1}^{N} c_i \phi_i(x) \tag{13.19}$$

where the $\phi_i(x)$ are the trial functions and the constants c_i are to be determined. Recall the statement of the operator eigenvalue problem resulting from separation of variables, as given in Equation (10.7). Rewriting Equation (10.7) with the mass density placed on the right-hand side yields

$$L\phi(x) = \lambda\rho\phi(x) \tag{13.20}$$

subject to the appropriate boundary conditions. Multiplying Equation (13.20) by $\phi(x)$ and integrating yields the Rayleigh quotient

$$\lambda = \frac{\int_0^\ell \phi(x) L\phi(x)\, dx}{\int_0^\ell \rho\phi(x)\phi(x)\, dx} = \frac{N}{D} \tag{13.21}$$

where

$$N = \int_0^\ell \phi(x) L\phi(x)\, dx \qquad \text{and} \qquad D = \int_0^\ell \rho\phi(x)\phi(x)\, dx$$

The Ritz approximation process is to substitute Equation (13.19) into Equation (13.21) and compute the coefficients c_i that minimize the Rayleigh quotient given by Equation (13.21).

Differentiating Equation (13.21) with respect to the coefficients c_i yields

$$\frac{\partial\lambda}{\partial c_i} = \frac{D\left(\dfrac{\partial N}{\partial c_i}\right) - N\left(\dfrac{\partial D}{\partial c_i}\right)}{D^2} = 0 \Rightarrow \frac{\partial N}{\partial c_i} - \lambda\frac{\partial D}{\partial c_i} = 0 \tag{13.22}$$

since D is never zero. Equation (13.22) computes the values of the expansion coefficients that minimize the Rayleigh quotient and hence allow the approximation of the eigenvalues. Next, consider writing N and D in terms of the constants c_i using Equation (13.19). With a little manipulation, it can be shown (see problem 13.11) that Equation (13.22) is the generalized eigenvalue–eigenvector problem

$$K\mathbf{c} = \lambda M\mathbf{c} \tag{13.23}$$

where the column vector $\mathbf{c}$ consists of the expansion coefficients c_i . The elements of the 'mass' and 'stiffness' matrix are given by

$$k_{ij} = \int_0^\ell \phi_i(x) L\phi_j(x)\, dx \qquad \text{and} \qquad m_{ij} = \int_0^\ell \rho\phi_i(x)\phi_j(x)\, dx \tag{13.24}$$

The solution of the generalized eigenvalue problem (13.23) yields an approximation of the eigenvalues λ_n and hence the natural frequencies. The eigenvectors $\mathbf{c}$ approximate the eigenfunctions of the system and hence the mode shapes. The number of approximated frequencies and mode shapes is N, the number of trial functions used in Equation (13.19).

The power of this approach is in finding approximate solutions when Equation (13.20) cannot be solved analytically, such as for odd boundary conditions and/or for spatially varying coefficients such as $\rho(x)$ and $EI(x)$. Note that, if exact eigenfunctions are used, exact frequencies result. Also note that the boundary conditions come into play when evaluating the integrals in Equation (13.24).

13.4 FINITE ELEMENT METHOD

Probably the most popular method of representing distributed-mass structures is the *finite element method* (FEM). This section presents a very brief introduction to the topic. A classic

reference for FEM is Hughes (2000). The method divides the structure of interest into subsections of finite size, called *finite elements*. These elements are connected to adjacent elements at various points on their boundaries, called *nodes*. Once this procedure is finished, the distributed-mass structure is represented by a finite number of nodes and elements referred to as a finite element *grid*, or *mesh*.

The displacement of each element is approximated by some function of the spatial variables between nodes. The next step in the finite element analysis (often abbreviated FEA) is to calculate the energy in each element as a function of the displacement. The total energy of the structure is then expressed as the sum of the energy in each element. External forces are included by using the principle of virtual work to derive forces per element. Lagrange's equations (see, for instance, Meirovitch, 2001) are then applied to the total energy of the structure, which yields the approximate equations of motion. These equations are finite-dimensional. This procedure is illustrated in the following example.

Example 13.4.1

This example considers the longitudinal vibration of a bar of length ℓ and derives a finite element stiffness matrix of the bar. The bar of Figure 13.1 is configured as one finite element with a node at each end. The axial stiffness is regarded as time independent throughout the element, so that the displacement must satisfy

$$EA\frac{d^2u(x)}{dx^2} = 0, \qquad x \in (0, \ell) \tag{13.25}$$

Integrating this expression yields

$$u(x) = c_1 x + c_2, \qquad x \in (0, \ell) \tag{13.26}$$

where c_1 and c_2 are constants of integration. At each node, the value of u is allowed to be a time-dependent coordinate denoted by $u_1(t)$, as labeled in the figure. Using these as boundary conditions, the constants c_1 and c_2 are evaluated to be

$$c_2 = u(t) \tag{13.27}$$

$$c_1 = \frac{u_2(t) - u_1(t)}{\ell} \tag{13.28}$$

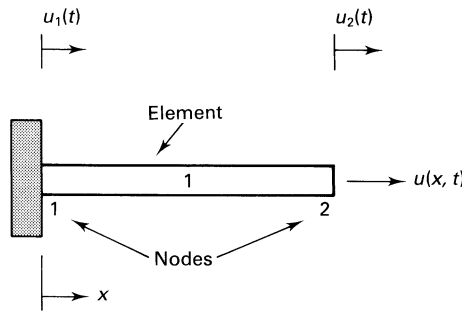

Figure 13.1 Two-node, one-element model of a cantilevered beam.

so that $u(x, t)$ is approximated by

$$u(x, t) = \left(1 - \frac{x}{\ell}\right) u_1(t) + \frac{x}{\ell} u_2(t) \tag{13.29}$$

Next, the nodal forces f_1 and f_2 are related to the displacement $u(x)$ by

$$EAu'(0) = -f_1, \qquad EAu'(\ell) = f_2 \tag{13.30}$$

or

$$EA\frac{u_2 - u_1}{\ell} = -f_1, \qquad EA\frac{u_2 - u_1}{\ell} = f_2 \tag{13.31}$$

where the prime indicates differentiation with respect to x. This last expression can be written in the matrix form

$$K\mathbf{u} = \mathbf{f} \tag{13.32}$$

where $\mathbf{u}(t) = [u_1(t) \quad u_2(t)]^T$, $\mathbf{f} = [f_1(t) \quad f_2(t)]^T$ and

$$K = \frac{EA}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{13.33}$$

Here, the vector $\mathbf{u}(t)$ is called the *nodal displacement vector*, the vector $\mathbf{f}(t)$ is called the *nodal force vector*, and the matrix K is the *element stiffness matrix*.

In example 13.4.1, note that the displacement in the element is written in the form

$$u(x, t) = a_1(x)u_1(t) + a_2(x)u_2(t) = \mathbf{a}^T(x)\mathbf{u}(t) \tag{13.34}$$

where $\mathbf{a}(x) = [a_1(x) \;\; a_2(x)]^T$. The functions $u_1(t)$ and $u_2(t)$ are the time-dependent nodal displacements, and in example 13.4.1 they approximate $u(0, t)$ and $u(\ell, t)$ respectively. The functions $a_1(x)$ and $a_2(x)$ are called *shape functions*, or *interpolation functions*. In the example, $a_1(x) = (1 - x/\ell)$ and $a_2(x) = (x/\ell)$. However, the shape functions are not unique in general. They are referred to as interpolation functions because they allow the displacement to be specified, or interpolated, at points along the structure that lie between nodes. As will be illustrated in the following, the solution of the dynamic finite element equations yields only the nodal displacements $u_1(t)$ and $u_2(t)$.

Next, a dynamic model is needed. A mass matrix is required that is consistent with the preceding stiffness matrix for the bar element. The mass matrix can be determined from an expression for the kinetic energy of the element, denoted by $T(t)$ and defined by

$$T(t) = \frac{1}{2}\int_0^\ell \rho(x)u_t(x, t)^2\, dx \tag{13.35}$$

Substitution of $u_t(x, t)$ from Equation (13.32) yields

$$T(t) = \frac{1}{2}\int_0^\ell \rho(x)\dot{\mathbf{u}}^T(t)\mathbf{a}(x)\mathbf{a}^T(x)\dot{\mathbf{u}}^T(t)\, dx \tag{13.36}$$

or

$$T(t) = \frac{1}{2}\dot{\mathbf{u}}^T(t)\left[\int_o^\ell \rho(x)\mathbf{a}(x)\mathbf{a}^T(x)\,dx\right]\dot{\mathbf{u}}(t) \tag{13.37}$$

The expression in brackets is clearly a matrix that is defined as the element mass matrix, denoted by M. Examination of Equation (13.37) indicates that the mass matrix is given by

$$M = \int_0^\ell \rho(x)\mathbf{a}(x)\mathbf{a}^T(x)\,dx \tag{13.38}$$

Since the mass matrix is calculated by using the same shape functions as the stiffness matrix, the resulting mass matrix is called a *consistent mass matrix*. An alternative means of constructing the mass matrix is just to lump the mass of the structure at the various nodes. If this is done, the result is called an *inconsistent mass matrix*.

Note that the stiffness matrix of Equation (13.36) can also be represented in terms of the *shape* functions $\mathbf{a}(t)$. Examination of the potential energy in the system yields (for the bar of example 13.4.1)

$$K = \int_0^\ell EA(x)\mathbf{a}(x)\mathbf{a}^T(x)\,dx \tag{13.39}$$

With K defined by Equation (13.31), the potential energy per element, denoted by $V(t)$, is given by

$$V(t) = \frac{1}{2}\mathbf{u}^T(t)K\mathbf{u}(t) \tag{13.40}$$

Example 13.4.2

Calculate the consistent mass matrix for the bar element of example 13.4.1. Substituting the shape functions of Equation (13.29) into equation (13.38) yields

$$M = \rho\int_0^\ell \left(\begin{bmatrix} \frac{1-x}{\ell} \\ \frac{x}{\ell} \end{bmatrix}\begin{bmatrix} \frac{1-x}{\ell} & \frac{x}{\ell} \end{bmatrix}\right) dx = \frac{\rho\ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{13.41}$$

These definitions of the finite element mass and stiffness matrix can be assembled by using the potential and kinetic energies along with Lagrange's equations to formulate the approximate equations of a distributed parameter structure.

Recall that Lagrange's equations (see, for instance, Thomson, 1988) simply state that the equations of motion of an n-degree-of-freedom structure with coordinates u_i can be calculated from the energy of the structure by

$$\left[\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{u}_i}\right)\right] - \frac{\partial T}{\partial u_i} + \frac{\partial V}{\partial u_i} = f_i \tag{13.42}$$

where the f_i denote external forces.

With Lagrange's equations, the equations of motion of a structure modeled by one or more finite elements can be derived. This is done by first modeling the structure of interest as several finite elements (like the bar of examples 13.4.1 and 13.4.2). Next, the total energy of each element is added to produce the total energy of the distributed structure. Then Lagrange's equations are applied to produce the dynamic equations for the structure. The procedure is best illustrated by the following example.

Example 13.4.3

Again, consider the bar element of examples 13.4.1 and 13.4.2 and use these to model the vibration of a cantilevered bar. In this example, the clamped free bar will be modeled by three (an arbitrary choice) finite elements – and hence four nodes – as depicted in Figure 13.2. Note that, because of the clamped boundary condition, $u_1(t)=0$. Taking this into consideration, the total potential energy, denoted by $V_T(t)$, is the sum of the potential energy in each element:

$$V_T(t)=\sum_{i=1}^{3} V_i(t) \tag{13.43}$$

With $\ell/3$ substituted for ℓ in Equation (13.33) and the appropriate displacement vector $\mathbf{u}$, $V_T(t)$ becomes

$$\begin{aligned} V_T(t)=&\frac{3EA}{2\ell}\begin{bmatrix}0\\u_2\end{bmatrix}^T\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}0\\u_3\end{bmatrix}+\frac{3EA}{2\ell}\begin{bmatrix}u_2\\u_3\end{bmatrix}^T\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}u_2\\u_3\end{bmatrix}\\ &+\frac{3EA}{2\ell}\begin{bmatrix}u_3\\u_4\end{bmatrix}^T\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}u_3\\u_4\end{bmatrix}\end{aligned} \tag{13.44}$$

Calculating the derivatives of V with respect to u_i yields

$$\begin{bmatrix}\dfrac{\partial V_T}{\partial u_2}\\ \dfrac{\partial V_T}{\partial u_3}\\ \dfrac{\partial V_T}{\partial u_4}\end{bmatrix}=\frac{3EA}{\ell}\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}\begin{bmatrix}u_2\\u_3\\u_4\end{bmatrix} \tag{13.45}$$

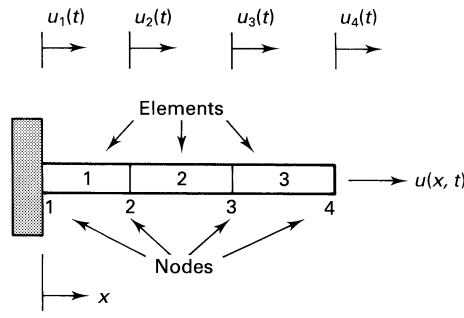

Figure 13.2 Four-node, three-element model of a cantilevered beam.

where the coefficient of the displacement vector $\mathbf{u} = [u_2\ u_3\ u_4]^T$ is the global stiffness matrix, K, for the entire structure based on a three-element finite approximation.

Calculation of the total kinetic energy $T(t)$ yields

$$T = \frac{1}{2}\frac{\rho A \ell}{18}\left\{\begin{bmatrix} 0 \\ \dot{u}_2 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ \dot{u}_2 \end{bmatrix} + \begin{bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} + \begin{bmatrix} \dot{u}_3 \\ \dot{u}_4 \end{bmatrix}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{u}_3 \\ \dot{u}_4 \end{bmatrix}\right\} \tag{13.46}$$

Calculation of the various derivatives of T required for Lagrange's equations yields

$$\begin{bmatrix} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}_2}\right) \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}_3}\right) \\ \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{u}_4}\right) \end{bmatrix} = \frac{\rho A \ell}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\ddot{\mathbf{u}} \tag{13.47}$$

where the coefficient of $\ddot{\mathbf{u}}$ is the consistent mass matrix of the three-element finite element approximation.

Substitution of Equations (13.45) and (13.47) into Lagrange's equation [Equation (13.42)] yields the three-degree-of-freedom model of the undamped bar as

$$M\ddot{\mathbf{u}} + K\mathbf{u} = \mathbf{0} \tag{13.48}$$

This last expression can be solved for the vibration response of the undamped bar at the nodal point. The response between nodes can be interpolated by using the shape functions (or interpolation functions), i.e., $u(x, t) = \mathbf{a}^T\mathbf{u}$.

These procedures can be generalized to any type of distributed-mass structure or combination of structures. The matrices M and K that result are similar to those that result from the Rayleigh–Ritz method. In fact, the finite element method can be thought of as a piecewise version of the Rayleigh–Ritz method. For an accurate representation of a response, 10–20 elements per wavelength of the highest frequency of interest must be used.

13.5 SUBSTRUCTURE ANALYSIS

A distributed-mass structure often yields a large-order finite element model with hundreds or even thousands of nodes. This is especially true of large, complicated, and/or very flexible structures. Substructure analysis is a method of predicting the dynamic behavior of such a complicated large-order system by first dividing the model up into several parts, called substructures, and analyzing these smaller parts first. The dynamic solution of each substructure is then combined to produce the response of the entire structure.

Let the n-dimensional vector $\mathbf{x}$ denote the coordinates of a large finite element model. First, divide the structure up into parts according to the modal coordinates via the following scheme:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{13.49}$$

Here, $\mathbf{x}_1$ represents those nodes associated with the first substructure, and $\mathbf{x}_2$ represents the nodes associated with the second substructure. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be further partitioned into those coordinates that are unique to substructure 1 and those that are common to $\mathbf{x}_1$ and $\mathbf{x}_2$. Divide $\mathbf{x}_1$ into internal coordinates $\mathbf{x}_{1i}$ and common coordinates $\mathbf{x}_c$, i.e., $\mathbf{x}_1^T = \left[\mathbf{x}_{1i}^T \;\; \mathbf{x}_c^T\right]^T$. Likewise, the nodal coordinates for the second substructure, $\mathbf{x}_2$, are partitioned as

$$\mathbf{x}_2 = \begin{bmatrix} \mathbf{x}_{2i} \\ \mathbf{x}_c \end{bmatrix} \tag{13.50}$$

The subset of nodes $\mathbf{x}_c$ is the same in both $\mathbf{x}_1$ and $\mathbf{x}_2$.

Next, partition the mass and stiffness matrices for each of the two (could be $N < n$) parts according to internal ($\mathbf{x}_{2i}$) and external ($\mathbf{x}_c$) coordinates. Let T_1 and V_1 denote the kinetic energy and potential energy, respectively, in substructure 1. These energies are

$$T_1 = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{x}}_{1i} \\ \dot{\mathbf{x}}_c \end{bmatrix}^T \begin{bmatrix} M_{ii}(1) & M_{ic}(1) \\ M_{ci}(1) & M_{cc}(1) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_{1i} \\ \dot{\mathbf{x}}_c \end{bmatrix} \tag{13.51}$$

$$V_1 = \frac{1}{2}\begin{bmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_c \end{bmatrix}^T \begin{bmatrix} K_{ii}(1) & K_{ic}(1) \\ K_{ci}(1) & K_{cc}(1) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_c \end{bmatrix} \tag{13.52}$$

Likewise, the energy in substructure 2 is

$$T_2 = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{x}}_{2i} \\ \dot{\mathbf{x}}_c \end{bmatrix}^T \begin{bmatrix} M_{ii}(2) & M_{ic}(2) \\ M_{ci}(2) & M_{cc}(2) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_{2i} \\ \dot{\mathbf{x}}_c \end{bmatrix} \tag{13.53}$$

$$V_2 = \frac{1}{2}\begin{bmatrix} \mathbf{x}_{2i} \\ \mathbf{x}_c \end{bmatrix}^T \begin{bmatrix} K_{ii}(2) & K_{ic}(2) \\ K_{ci}(2) & K_{cc}(2) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{2i} \\ \mathbf{x}_c \end{bmatrix} \tag{13.54}$$

Next, the modes $\mathbf{u}_i$ of each substructure are calculated by assuming that the common coordinates (also called connecting nodes) are free and not really constrained by the rest of the structure, i.e., that the coordinates satisfy the equation of motion

$$\begin{bmatrix} M_{ii}(j) & M_{ic}(j) \\ M_{ci}(j) & M_{cc}(j) \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{x}}_{2i} \\ \ddot{\mathbf{x}}_c \end{bmatrix} + \begin{bmatrix} K_{ii}(j) & K_{ic}(j) \\ K_{ci}(j) & K_{cc}(j) \end{bmatrix} \begin{bmatrix} \mathbf{x}_{2i} \\ \mathbf{x}_c \end{bmatrix} = \mathbf{0} \tag{13.55}$$

for each substructure ($j = 1, 2$). Equation (13.55) is obtained by using the energy expressions of Equations (13.51) through (13.54) substituted into Lagrange's equations. Each of the dynamic substructure equations (13.55) is next solved for the system eigenvalues and eigenvectors. Let $[\phi_{1i}^T(n) \;\; \phi_c^T(n)]^T$ denote the nth eigenvector of substructure 1. The modal matrix of substructure 1, denoted by $\phi(1)$, is the square matrix defined by

$$\phi(1) = \begin{bmatrix} \phi_{1i}(1) & \phi_{1i}(2) \cdots \phi_{1i}(n) \\ \phi_c(1) & \phi_c(2) \cdots \phi_c(n) \end{bmatrix} = \begin{bmatrix} \phi_i(1) \\ \phi_c \end{bmatrix}$$

where $\phi_i(1)$ and ϕ_c are rectangular matrix partitions of $\phi(1)$. These partitions are used to define a new coordinate $\mathbf{q}(1)$ by

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{x}_{1i} \\ \mathbf{x}_c \end{bmatrix} = \begin{bmatrix} \phi_i(1) \\ \phi_c \end{bmatrix} \mathbf{q}(1) \tag{13.56}$$

This yields

$$\mathbf{x}_{1i} = \phi_i(1)\mathbf{q}(1) \tag{13.57}$$

where it should be noted that $\phi_i(1)$, a rectangular matrix, relates the internal coordinates of substructure 1, $\mathbf{x}_{1i}$, to the new coordinate $\mathbf{q}(1)$ yet to be determined. This procedure can be repeated using the information from the second substructure to determine $\phi_i(2)$ and to define $\mathbf{q}(2)$. These quantities are related by

$$\mathbf{x}_{2i} = \phi_i(2)\mathbf{q}(2) \tag{13.58}$$

The substitution of Equation (13.57) into the expressions for the energy [Equations (13.51) and (13.52)] yields

$$T(1) = \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}(1) \\ \dot{\mathbf{x}}_c \end{bmatrix}^T \begin{bmatrix} \phi_i^T(1)M_{ii}(1)\phi_i(1) & \phi_i^T(1)M_{ic}(1) \\ M_{ci}(1)\phi_i(1) & M_{cc}(1) \end{bmatrix} \begin{bmatrix} \dot{\mathbf{q}}(1) \\ \dot{\mathbf{x}}_c \end{bmatrix} \tag{13.59}$$

$$V(1) = \frac{1}{2}\begin{bmatrix} \mathbf{q}(1) \\ \mathbf{x}_c \end{bmatrix}^T \begin{bmatrix} \phi_i^T(1)K_{ii}(1)\phi_i(1) & \phi_i^T(1)K_{ic}(1) \\ K_{ci}(1)\phi_i(1) & M_{cc}(1) \end{bmatrix} \begin{bmatrix} \mathbf{q}(1) \\ \mathbf{x}_c \end{bmatrix} \tag{13.60}$$

Similar expressions are obtained for the energies of the second substructure, $T(2)$ and $V(2)$, by substitution of Equation (13.58) into Equations (13.53) and (13.54). The total energy in the complete structure is now considered to be defined by $[T(1) + T(2)]$ and $[V(1) + V(2)]$.

These energy expressions are substituted into Lagrange's equations to produce the equations of motion in terms of substructure quantities. Lagrange's equations for the system are

$$\begin{bmatrix} \phi_i^T(1)M_{ii}(1)\phi_i(1) & 0 & \phi_i^T(1)M_{ic}(1) \\ 0 & \phi_i(2)M_{ii}(2)\phi_i(2) & \phi_i^T(2)M_{ic}(2) \\ M_{ci}(1)\phi_i(1) & M_{ci}(2)\phi_i(2) & M_{cc}(1)+M_{cc}(2) \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}(1) \\ \ddot{\mathbf{q}}(2) \\ \ddot{\mathbf{x}}_c \end{bmatrix} + \begin{bmatrix} \phi_i^T(1)K_{ii}(1)\phi_i(1) & 0 & \phi_i^T(1)K_{ic}(1) \\ 0 & \phi_i(2)K_{ii}(2)\phi_i(2) & \phi_i^T(2)K_{ic}(2) \\ K_{ci}(1)\phi_i(1) & K_{ci}(2)\phi_i(2) & K_{cc}(1)+K_{cc}(2) \end{bmatrix} \begin{bmatrix} \mathbf{q}(1) \\ \mathbf{q}(2) \\ \mathbf{x}_c \end{bmatrix} = \mathbf{0} \tag{13.61}$$

This last expression constitutes a substructure representation of the original structure. The solution of Equation (13.61) is determined by any of the methods discussed in Chapter 3. The matrix coefficients are determined by analyzing each substructure independently. Equations (13.57) and (13.50) are used to recover the solution in physical coordinates from the solution of the substructure equations given by (13.61). Each of the quantities in Equation (13.61) are determined by solving the two substructures separately. Each of these is of an order less than the original system. Equation (13.61) is also of an order less than the original structure. In fact, it is of order n minus the order of $\mathbf{x}_c$. Hence, the response of the entire structure $\mathbf{x}_n$ can be obtained by analyzing several systems of smaller order.

13.6 TRUNCATION IN THE PRESENCE OF CONTROL

The majority of practical control schemes are implemented by actuators and sensors that are fixed at various points throughout the structure and hence behave fundamentally as lumped-mass elements rather than as distributed-mass elements. In addition, most control algorithms are based on finite-dimensional lumped-mass models of small order. Thus, it is quite natural to use a 'truncated' model or other finite-dimensional approximation of distributed-mass structures when designing control systems for them.

This section examines the problem of controlling the vibrations of a distributed-mass structure by using a finite number of lumped-mass actuators and sensors acting at various points on the structure. The approach discussed here is first to cast the structure into an infinite-dimensional matrix equation that is transformed and then truncated. A combination of modal methods and impedance methods is used to solve a simple structural control problem. The goal of the section is to present a simple, representative method of reducing vibration levels in flexible mechanical structures.

Consider a distributed-mass structure described by a partial differential equation of the form

$$Ly(x,t) = f(x,t), \qquad x \in \Omega \tag{13.62}$$

and associated boundary and initial conditions. Here, the functions $y(x,t)$ and $f(x,t)$ are in $\mathcal{L}_2^R(\Omega)$, $y(x,t)$ being the system output, and $f(x,t)$ the system input. This model is an abbreviated formulation of the structures presented in Chapter 9. In terms of the notation of Chapter 9, the operator L is of the form

$$L = \frac{\partial^2}{\partial t^2}(\cdot) + L_1 \frac{\partial}{\partial t}(\cdot) + L_2(\cdot) \tag{13.63}$$

where the output equation is just $y(x,t) = w(x,t)$. If the operator L had an easily calculated inverse, the solution would be given by $y(x,t) = L^{-1}f(x,t)$. To that end, consider taking the Laplace or Fourier transform on the temporal variable of Equation (13.62). This yields

$$Ly(x,s) = f(x,s) \tag{13.64}$$

plus boundary conditions. For ease of notation, no special distinction will be made between y in the time domain and y in the s domain (i.e., between y and its transform), as the remainder of the section deals only with the transformed system.

The control problem of interest here is one that could be implemented by sensors and actuators acting at discrete points along the structure, which may have dynamics of their own. Suppose that the structure is measured at m points along its length, labeled by x_i'. Let $y(s)$ denote an $m \times 1$ column vector with the ith component defined as $y(x_i', s)$, i.e., the time-transformed output (displacement) measured at point x_i'. In addition, r actuators are used to apply time-dependent forces $u_i(t)$, or transformed forces $u_i(s)$, at the r points x_i''. The control action, denoted by $f_c(x,s)$, can be written as

$$f_c(x,s) = \sum_{i=1}^{r} \delta(x - x_i'')u_i(s) = \mathbf{r}^T(x)\mathbf{u}(s) \tag{13.65}$$

Here, $\mathbf{r}(x)$ is an $r \times 1$ vector with ith component $\delta(x - x_i'')$, the Dirac delta function, and $\mathbf{u}(s)$ is an $r \times 1$ vector with ith component $u_i(s)$.

Negative feedback is used, so that the total force applied to the structure is given by

$$f(x, s) = f_{\text{ext}}(x, s) - f_c(x, s) \tag{13.66}$$

where $f_{\text{ext}}(x, s)$ represents an externally applied disturbance force and $f_c(x, s)$ represents the control forces. With the actuator just described, Equation (13.66) becomes

$$f(x, s) = f_{\text{ext}}(x, s) - \mathbf{r}^T \mathbf{u}(s) \tag{13.67}$$

To complete the feedback loop, $\mathbf{u}(s)$ must depend on the output, or measured response, of the structure. Let $H(s)$ be an $r \times m$ transfer matrix defining the dependence of the control action on the output via the expression

$$\mathbf{u}(s) = H(s)\mathbf{y}(s) \tag{13.68}$$

so that Equation (13.67) becomes

$$f(x, s) = f_{\text{ext}} - \mathbf{r}^T(x)H(s)\mathbf{y}(s) \tag{13.69}$$

This last expression represents output feedback control. An alternative here would be to use state feedback control, as discussed in Chapter 7.

Next, consider casting the problem into modal coordinates. Let $\{\Psi_i(x)\}$ be a set of basis functions in $\mathcal{L}_2^{\text{C}}(\Omega)$ and consider $\mathcal{L}\Psi_i(x)$, also in $L_2^{\text{C}}(\Omega)$. Then

$$L\Psi_i = \sum_{j=1}^{\infty} \lambda_{ij}(s)\Psi_j(x) \tag{13.70}$$

where $\lambda_{ij}(s)$ is an expansion coefficient. Note that, if $\lambda_{ij}(s) = 0$ for $i \neq j$, then Equation (13.70) becomes

$$L\Psi_i(x) = \lambda_i(s)\Psi_i(x) \tag{13.71}$$

so that the expansion coefficient, $\lambda_{ii}(s) = \lambda_i(s)$, is an eigenvalue of the operator L with eigenfunction $\Psi_i(x)$. The $\lambda_{ii}(s)$ are also called modal *impedances* (see Section 10.4 for conditions under which this is true).

Example 13.6.1

For a pinned–pinned uniform beam in transverse vibration of length ℓ with no damping

$$\Psi_n(x) = \frac{\sqrt{2}}{\ell}\sin(k_n x) \tag{13.72}$$

$$\lambda_n(s) = \left[k_n^4 + \frac{s^2}{c^2 k^2}\right] EI \tag{13.73}$$

where

$$k_n = \frac{n\pi}{\ell}, \qquad c^2 = \frac{E}{\rho}, \qquad k^2 = \frac{I}{A}$$

Expanding the functions $y(x, s)$ and $f(x, s)$ in terms of this same set of basis functions, $\{\Psi_i(x)\}$, yields

$$y(x, s) = \sum_{i=1}^{\infty} d_i(s)\Psi_i(s) \tag{13.74}$$

and

$$f(x, s) = \sum_{i=1}^{\infty} c_i(s)\Psi_i(x) \tag{13.75}$$

The expansion coefficients $d_i(s)$ are called the modal response coefficients and the coefficients $c_i(s)$ are called the modal input coefficients.

Next, compute (assuming proper convergence, i.e., that Ψ_i is an eigenfunction of L)

$$Ly = \sum_{i=1}^{\infty} d_i(s)L\Psi_i(x) = f(x, s) \tag{13.76}$$

or

$$\sum_{i=1}^{\infty} \lambda_i(s)d_i(s)\Psi_i(x) = \sum_{i=1}^{\infty} c_i(s)\Psi_i(x) \tag{13.77}$$

Note that for Equation (13.77) it is assumed that the $\Psi_i(x)$ are, in fact, the eigenfunctions of L.

Using the orthogonality of the $\{\Psi_i(x)\}$, Equation (13.77) implies that

$$\lambda_i(s)d_i(s) = c_i(s) \tag{13.78}$$

for each index i, so that

$$\lambda_i(s) = \frac{c_i(s)}{d_i(s)} \tag{13.79}$$

This gives rise to the interpretation of $\lambda_i(s)$ as a 'modal impedance'. Note also that, as before,

$$d_i(s) = \int_{\Omega} y(x, s)\Psi_i(x)\, d\Omega \tag{13.80}$$

and

$$c_i(s) = \int_{\Omega} f(x, s)\Psi_i(x)\, d\Omega \tag{13.81}$$

If L is not self-adjoint and/or the functions $\Psi_i(x)$ are not the normal modes of the system, then this procedure can be completed using the orthogonality of the complex set of basis functions. In this coupled case, substitution of Equation (13.70) into Equation (13.76) yields

$$\sum_{i=1}^{\infty} d_i(s)\left[\sum_{j=1}^{\infty} \lambda_{ij}(s)\Psi_j(x)\right] = \sum_{j=1}^{\infty} c_j(s)\Psi_j(x) \tag{13.82}$$

Multiplying Equation (13.82) by $\Psi_k^*(x)$, the conjugate of $\Psi_k(x)$, and integrating over Ω yields

$$\sum_{i=1}^{\infty} d_i(s)\lambda_{ik}(s) = c_k(s) \tag{13.83}$$

where the summation over the index j has been eliminated by the assumed orthogonality of the set $\{\Psi_k(x)\}$. Equation (13.83) constitutes the equivalent version of Equation (13.78) for the case in which the system does not possess classical normal modes.

Next, consider applying linear feedback control in a modal coordinate system defined by the set $\{\Psi_k(x)\}$. Equation (13.78) can be written as a single infinite-dimensional matrix equation of the form

$$\Lambda \mathbf{d} = \mathbf{c} \tag{13.84}$$

where Λ is the $\infty \times \infty$ modal impedance matrix with the ijth element defined by $\lambda_{ij}(s)$ and $\mathbf{c}$ and $\mathbf{d}$ are $\infty \times 1$ column matrices defined by $c_i(s)$ and $d_i(s)$ respectively. Defining $\mathbf{\Psi}(x)$ as the $\infty \times 1$ column matrix of eigenfunctions $\mathbf{\Psi}_i(x)$, the other relevant terms can be written as

$$f_{\text{ext}} = \sum_{i=1}^{\infty} e_i(s)\Psi_i(x) = \mathbf{e}^T(s)\mathbf{\Psi}(x) \tag{13.85}$$

$$f(x, s) = \sum_{i=1}^{\infty} c_i(s)\Psi_i(x) = \mathbf{c}^T(s)\mathbf{\Psi}(x) \tag{13.86}$$

and

$$y(x, s) = \sum_{i=1}^{\infty} d_i(s)\Psi_i(x) = \mathbf{d}^T(s)\mathbf{\Psi}(x) \tag{13.87}$$

where the various column vectors have the obvious definitions. For instance, the vector $\mathbf{e}(s)$ is the vector of expansion coefficients for the external disturbance force f_{ext} with components $e_i(s)$, and so on.

A measurement matrix, denoted by M, can be defined by

$$M_{ij} = \Psi_j(x_i') \tag{13.88}$$

which is an $m \times \infty$ matrix and relates $\mathbf{d}(s)$ directly to $\mathbf{y}(s)$ by

$$\mathbf{y}(s) = M\mathbf{d}(s) \tag{13.89}$$

Likewise, an $r \times \infty$ modal coefficient matrix, denoted by R, can be defined by

$$R_{ij} = \int_{\Omega} r_i(x)\Psi_j(x)\ d\Omega \tag{13.90}$$

which relates $\mathbf{r}(x)$ to $\Psi(x)$ by

$$\mathbf{r}(x) = R\boldsymbol{\Psi}(x) \tag{13.91}$$

Using the orthogonality of $\Psi_i(x)$, the inner product

$$\int_{\Omega} \boldsymbol{\Psi}(x)\boldsymbol{\Psi}^T(x)\ d\Omega = I_{\infty} \tag{13.92}$$

where I_{∞} denotes the $\infty \times \infty$ identity matrix with elements $\int_{\Omega} \Psi_i(x)\Psi_{\varphi}(x)\ d\Omega = \delta_{ij}$. Note that, if $\boldsymbol{\Psi}(x)$ is complex, then the transpose should be interpreted as the conjugate transpose. Multiplying Equation (13.91) by $\boldsymbol{\Psi}^T$ from the right and integrating over Ω yields

$$R = \int_{\Omega} \mathbf{r}(x)\boldsymbol{\Psi}^T(x)\ d\Omega \tag{13.93}$$

This last expression provides a more useful definition of the modal coefficient matrix R.

A relationship between R, $\mathbf{c}$, and $\mathbf{e}$ can be found by substituting Equations (13.85) and (13.86) into Equation (13.67). This yields

$$\mathbf{c}^T\boldsymbol{\Psi} = \mathbf{e}^T\boldsymbol{\Psi} - \mathbf{r}^T\mathbf{u} \tag{13.94}$$

Since $\mathbf{r}^T\mathbf{u}$ is a scalar, this can also be written as

$$\mathbf{c}^T(s)\boldsymbol{\Psi}(x) = \mathbf{e}^T(s)\boldsymbol{\Psi}(x) - \mathbf{u}^T(s)\mathbf{r}(x) \tag{13.95}$$

Multiplication from the right by $\Psi^T(x)$ and integrating over Ω yields

$$\mathbf{c}^T(s)\int_{\Omega} \boldsymbol{\Psi}(x)\boldsymbol{\Psi}^T(x)\ d\Omega = \mathbf{e}^T(s)\int_{\Omega} \boldsymbol{\Psi}(x)\boldsymbol{\Psi}^T(x)\ d\Omega - \mathbf{u}^T(s)\int_{\Omega} \mathbf{r}(x)\boldsymbol{\Psi}^T(x)\ d\Omega \tag{13.96}$$

Using Equations (13.92) and (13.93) then yields

$$\mathbf{c}^T(s) = \mathbf{e}^T(s) - \mathbf{u}^T(s)R \tag{13.97}$$

or

$$\mathbf{c}(s) = \mathbf{e}(s) - R^T\mathbf{u}(s) \tag{13.98}$$

Equation (13.84) now becomes

$$\Lambda\mathbf{d}(s) = \mathbf{e}(s) - R^T\mathbf{u}(s) \tag{13.99}$$

or upon substitution of Equation (13.68) for $\mathbf{u}(s)$

$$\Lambda\mathbf{d}(s) = \mathbf{e}(s) - R^T H(s)\mathbf{y}(s) \tag{13.100}$$

Using Equation (13.89), the last term in Equation (13.100) can be placed in terms of $\mathbf{d}(s)$ to yield

$$\Lambda\mathbf{d}(\mathbf{s}) = \mathbf{e}(\mathbf{s}) - R^T H(\mathbf{s}) M \mathbf{d}(s) \tag{13.101}$$

Assuming that Λ^{-1} exists, this last expression can be manipulated to yield

$$\left[I_\infty + \Lambda^{-\mathbf{1}} Q(s)\right] \mathbf{d}(s) = \Lambda^{-1}\mathbf{e}(s) \tag{13.102}$$

where $Q(s) = R^T H(s) M$. Equation (13.102) represents the closed-loop configuration for the output feedback control of a distributed-parameter structure in terms of infinite-dimensional matrices.

If the infinite matrix inverse $\Lambda^{-\mathbf{1}}$ exists, if the inverse of the impedance matrix $[I_\infty + \Lambda^{-1}Q]$ can be calculated, and if the functions $\{\Psi_i(x)\}$ are known, Equation (13.102) along with Equation (13.74) yields the response $\mathbf{d}(s)$ in terms of the input, $\mathbf{e}(s)$. Several common examples, such as uniform beams and plates of simple geometry, satisfy these assumptions. Unfortunately, in many practical cases these assumptions are not satisfied, and the matrix Λ must be truncated in some fashion. Even in cases where $\Lambda^{-\mathbf{1}}$ can be calculated, the control $Q(s)$ may be such that $[I_\infty + \Lambda^{-\mathbf{1}}Q]$ is difficult to calculate. In cases where truncation of the model is required, Equation (13.102) provides a convenient formula for studying the effects of truncation in the presence of control.

As was true for the procedure of Section 7.8, the truncation method presented here is based on partitioning the various infinite-dimensional matrices of Equation (13.102). Let $\Lambda_{n\infty}^{-\mathbf{1}}$ denote the matrix formed from the matrix $\Lambda^{-\mathbf{1}}$ by partitioning off the first n rows and all the columns. Using this notation, the matrix $\Lambda^{-\mathbf{1}}$ is partitioned as

$$\Lambda^{-1} = \begin{bmatrix} \Lambda_{nm}^{-1} & \Lambda_{n\infty}^{-1} \\ \Lambda_{\infty n}^{-1} & \Lambda_{\infty\infty}^{-1} \end{bmatrix} \tag{13.103}$$

In a similar fashion, the matrices M, R, and Q are partitioned as

$$M = [M_{mn}\, M_{m\infty}] \tag{13.104}$$

$$R^T = \begin{bmatrix} R_{nr}^T \\ R_{\infty r}^T \end{bmatrix} \tag{13.105}$$

and

$$Q = \begin{bmatrix} Q_{nn} & Q_{n\infty} \\ Q_{\infty n} & Q_{\infty\infty} \end{bmatrix} \tag{13.106}$$

The submatrices of Q can all be written in terms of R, M, and H as

$$Q_{nn} = R_{nr}^T H M_{mn} \tag{13.107}$$

$$Q_{n\infty} = R_{nr}^T H M_{m\infty} \tag{13.108}$$

$$Q_{\infty n} = R_{\infty r}^T H M_{mn} \tag{13.109}$$

and

$$Q_{\infty\infty} = R_{\infty r}^T H M_{m\infty} \tag{13.110}$$

Substitution of these partitioned matrices into Equation (13.102) yields the partitioned system

$$\begin{bmatrix} I_n + \Lambda_{mn}^{-1} Q_{nn} + \Lambda_{n\infty}^{-1} Q_{\infty n} & \Lambda_{nn}^{-1} Q_{n\infty} + \Lambda_{n\infty}^{-1} Q_{\infty\infty} \\ \Lambda_{\infty n}^{-1} Q_{nn} + \Lambda_{\infty\infty}^{-1} Q_{\infty n} & I_\infty + \Lambda_{\infty n}^{-1} Q_{n\infty} + \Lambda_{\infty\infty}^{-1} Q_{\infty\infty} \end{bmatrix} \begin{bmatrix} \mathbf{d}_n \\ \mathbf{d}_\infty \end{bmatrix} = \begin{bmatrix} \Lambda_{nn}^{-1} & \Lambda_{n\infty}^{-1} \\ \Lambda_{\infty n}^{-1} & \Lambda_{\infty\infty}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{e}_n \\ \mathbf{e}_\infty \end{bmatrix} \tag{13.111}$$

Here, the response vector $\mathbf{d}$ and the input vector $\mathbf{e}$ have also been partitioned, dividing these infinite-dimensional vectors into an $n \times 1$ finite-dimensional part and an $\infty \times 1$ infinite-dimensional part.

The various partitions of Equation (13.103) can be used to interpret the effects of truncating the modal description of a structure at n modes in the presence of a control law. Structures are generally thought of as low-pass filters in the sense that $\lambda_n^{-1} \to 0$ as $n \to \infty$. Thus, for structures it is reasonable to assume that the matrix $\Lambda_{\infty\infty}^{-1}$ is zero for some value of n.

Sensors often behave like low-pass filters as well, so that it is also reasonable to assume that $M_{n\infty}$ is the zero matrix. This, in turn, causes $Q_{n\infty} = Q_{\infty\infty} = 0$. If the actuators are slow enough, it can also be argued that $R_{\infty r}^T = 0$, which causes $Q_{\infty n} = Q_{\infty\infty} = 0$. With these three assumptions, the system of Equation (13.111) is reduced to

$$\begin{bmatrix} I_n + \Lambda_{nn}^{-1} Q_{nn} & 0 \\ \Lambda_{\infty n}^{-1} Q_{nn} & I_\infty \end{bmatrix} \begin{bmatrix} \mathbf{d}_n \\ \mathbf{d}_\infty \end{bmatrix} = \begin{bmatrix} \Lambda_{nn}^{-1} & \Lambda_{n\infty}^{-1} \\ \Lambda_{\infty n}^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_n \\ \mathbf{e}_\infty \end{bmatrix} \tag{13.112}$$

This can be written as the two coupled vector equations

$$(I_n + \Lambda_{nn}^{-1} Q_{nn})\mathbf{d}_n = \Lambda_{nn}^{-1}\mathbf{e}_n + \Lambda_{n\infty}\mathbf{e}_\infty \tag{13.113}$$

and

$$\Lambda_{\infty n}^{-1} Q_{nn}\mathbf{d}_n + \mathbf{d}_\infty = \Lambda_{\infty n}^{-1}\mathbf{e}_n \tag{13.114}$$

These last two equations provide a simple explanation of some of the problems encountered in the control of distributed-mass structures using truncated models.

First, consider the case with $\mathbf{e}_\infty = \mathbf{0}$. This corresponds to a 'band-limited' input. That is, the external disturbance provides energy only to the first n modes. In this case, Equation (13.105) becomes

$$\mathbf{d}_n = (I_n + \Lambda_{nn}^{-1} Q_{nn})^{-1} \Lambda_{nn}^{-1}\mathbf{e}_n \tag{13.115}$$

Equation (13.115) can now be used to solve the control problem, i.e., to calculate Q_{nn} such that the response $\mathbf{d}_n$ has a desired form. In fact, Equation (13.115) is equivalent to first approximating a distributed-parameter system by a finite-dimensional system and then designing a finite-dimensional control system for it. However, this is slightly misleading, as can be seen by considering Equation (13.114).

Rearrangement of Equation (13.114) yields

$$\mathbf{d}_\infty = \Lambda_{\infty n}^{-1}\mathbf{e}_n - \Lambda_{\infty n}^{-1} Q_{nn}\mathbf{d}_n \tag{13.116}$$

This states that, unless the dynamics of the structure decouple (i.e., $\Lambda_{\infty n}^{-1} = 0$), or unless it can be argued that $\Lambda_{\infty n}^{-1}$ is small, the higher, uncontrolled modes of the response $\mathbf{d}_\infty$ will be excited by the control action, Q_{nn}. Such unwanted excitation is called *control spillover*.

In the case where $\Lambda_{\infty n}^{-1}$ is close to zero, Equation (13.115) provides a good approximation to the control problem for distributed-mass structures. In fact, the requirement that $\mathbf{e}_\infty = \mathbf{0}$ provides a criterion for determining the proper order, n, to be chosen for the approximation for a given disturbance. The value of n is chosen so that $\mathbf{e}_\infty$ is approximately zero.

Next, consider the case where the sensors are not low-pass filters, i.e., $M_{m\infty} \neq 0$, so that $Q_{n\infty} \neq 0$. In this case the first partition of Equation (13.111) yields

$$(I_n + \Lambda_{nn}^{-1} Q_{nn})\mathbf{d}_n + \Lambda_{nn}^{-1} Q_{n\infty}\mathbf{d}_\infty = \Lambda_{nn}^{-1}\mathbf{e}_n \tag{13.117}$$

The equation describing $\mathbf{d}_n$ is recoupled to the truncated dynamics constrained in the vector $\mathbf{d}_\infty$. If the term $Q_{n\infty}$ is erroneously neglected and Equation (13.115) is used to design the control system, then the resulting solution $\mathbf{d}_n$ will be in error, and the resulting calculation of R_{nr}^T and H will be in error. The response will suffer from what is often referred to as *observation spillover*, meaning that the sensors have caused a coupling of the truncated system with the neglected modes, producing error in the closed-loop response.

A similar problem arises if the actuators are 'fast', i.e., if $R_{\infty r}^T \neq 0$. In this case, Equations (13.113) and (13.114) become

$$(I_n + \Lambda_{mn}^{-1} Q_{nn} + \Lambda_{n\infty}^{-1} Q_{\infty n})\mathbf{d}_n = \Lambda_{nn}^{-1}\mathbf{e}_n \tag{13.118}$$

and

$$(\Lambda_{nn}^{-1} Q_{nn} + \Lambda_{n\infty}^{-1} Q_{\infty n})\mathbf{d}_\infty = \Lambda_{\infty n}^{-1}\mathbf{e}_n \tag{13.119}$$

Again, the introduction of the term $Q_{\infty n}$, associated with high-speed actuator excitation, couples the equation for the solution $\mathbf{d}_n$ and the truncated, or residual, solution $\mathbf{d}_\infty$. Thus, if $R_{\infty n}^T$ is not actually zero and Equation (13.115) is used to compute the control law, error will result. The interpretation here is that $R_{\infty n}^T$ excites the neglected modes, $\mathbf{d}_\infty$, and hence causes energy to appear in the neglected part of the model. This again causes control spillover.

13.7 IMPEDANCE METHOD OF TRUNCATION AND CONTROL

The modal description of a structure presented in the previous section lends itself to an interpretation of potential problems encountered when using point actuators and sensors in designing a control system for a distributed-mass structure. In this section an alternative approach is presented that uses the modal equation [Equation (13.102)] but, rather than truncating the response, uses an impedance method to calculate the closed-loop response vector $\mathbf{d}$.

The sensor–actuator admittance (inverse of impedance) matrix $Y(s)$ is defined by

$$Y(s) = M\Lambda^{-1}(s)R^T \tag{13.120}$$

and is related to the dynamic stiffness matrix. Note that $Y(s)$ is a finite-dimensional matrix, the elements of which are infinite sums.

Consider again the infinite matrix description of the structural control problem, as formulated in Equation (13.102). This expression can be written as

$$I\mathbf{d} + \Lambda^{-1}R^T HM\mathbf{d} = \Lambda^{-1}\mathbf{e} \tag{13.121}$$

From Equation (13.89), the vector $M\mathbf{d}$ can be replaced with $\mathbf{y}$ to yield

$$I\mathbf{d} + \Lambda^{-1}R^T H\mathbf{y} = \Lambda^{-1}\mathbf{e} \tag{13.122}$$

Multiplying this expression by M yields

$$\mathbf{y} + M\Lambda^{-1}R^T H\mathbf{y} = M\Lambda^{-1}\mathbf{e} \tag{13.123}$$

$$\mathbf{y} + Y(s)H(s)\mathbf{y} = M\Lambda^{-1}\mathbf{e} \tag{13.124}$$

This can be written as

$$[I_m + Y(s)H(s)]\mathbf{y} = M\Lambda^{-1}\mathbf{e} \tag{13.125}$$

where I_m is the $m \times m$ identity matrix. Thus, the coefficient of $\mathbf{y}$ is a finite-dimensional matrix. Assuming that the coefficient of $\mathbf{y}$ has an inverse, Equation (13.125) can be written as

$$\mathbf{y} = [I_m + Y(s)H(s)]^{-1}M\Lambda^{-1}\mathbf{e} \tag{13.126}$$

This expression can be substituted into Equation (13.122) to yield

$$\mathbf{d} = \{I_\infty - \Lambda^{-1}R^T H[I_m + Y(s)H(s)]^{-1}M\}\Lambda^{-1}\mathbf{e} \tag{13.127}$$

which expresses the system response, $\mathbf{d}$, in terms of the disturbance input, $\mathbf{e}$. Equation (13.127) represents the impedance method of dealing with truncation in the control of distributed-mass structures, as developed by Berkman and Karnopp (1969). The open-loop system, as represented by Λ, still needs to be truncated using the low-pass filter argument of the previous section, i.e., $\Lambda^{-1} \cong \Lambda_n^{-1}$. However, the feedback control portion is now finite-dimensional and of low order (i.e., equal to the number of measurement points or sensors). Hence, truncation or partitioning of an inverse matrix is required in order to compute the control. Instead, the elements of the matrix $[I_m + Y(s)H(s)]$, which are all infinite sums, can be first calculated and/or truncated. Then, the exact inverse can be calculated. Thus, the value of the truncation index for the structure and that for the control can be separately chosen. This approach also allows for the inclusion of actuator dynamics due to the presence of the matrix $H(s)$. The following example serves to clarify this method.

Example 13.7.1

Consider the transverse vibrations of a pined–pinned beam with an actuator providing dynamics (or an impedance), $Z_1(s)$, acting at point x_1. Also, consider a single displacement-measuring sensor,

located at x_1, so that $y(x)$ is the scalar quantity $w(x_1, s)$, i.e., so that the index $m = 1$. Let $\Psi_i(x)$ denote the modes (or eigenfunctions) of the pinned–pinned beam without the actuator attached. From Equation (13.88), the matrix M becomes the $1 \times \infty$ vector

$$M = [\Psi_1(x_1)\ \Psi_2(x_2)\ \cdots]\ \boldsymbol{\Psi}(x_1)$$

Likewise, from Equation (13.82) the matrix R becomes the $1 \times \infty$ vector

$$R = [\Psi_1(x_1)\ \Psi_2(x_1) \cdots]\boldsymbol{\Psi}(x_1)$$

The matrix $H(s)$ in this case is just the scalar element $H(s) = Z_1(s)$.

The sensor actuator admittance matrix becomes the scalar element

$$Y(s) = M\Lambda^{-1}R^T = \sum_{i=1}^{\infty} \lambda_I^{-1}(s)\Psi_i^2(x_I)$$

where the $\lambda_i(s)$ are the open-loop system eigenvalues, since $\Lambda(s)$ is diagonal in this example (i.e., $\Lambda(s)$ consists simply of the eigenvalues of the structure).

The response can now be computed from Equation (13.119) to be

$$\mathbf{d}(s) = \left[\Lambda^{-1}(s) - \frac{[\Lambda^{-1}\boldsymbol{\Psi}^T(x_1)][\Lambda^{-1}\boldsymbol{\Psi}^T(x)]^T}{1 + \sum_{i=1}^{\infty} \lambda_i^{-1}(s)\Psi_i^2(x_1)} \right] \mathbf{e}$$

It is important to note that the response (or, more exactly, the Laplace transform of the response) is calculated here by truncating (approximating) the structural dynamics $\Lambda^{-1}(s)$ and $\Lambda^{-1}\boldsymbol{\Psi}(x_1)$ independently of the control. The actuator representation, $Z_1(s)$, is not truncated at all in this example. This is in contrast to the completely modal approach of the previous section.

As the example illustrates, the modal impedance inversion technique described in this section reduces the problem of truncation in the presence of control from one of approximating an infinite-order matrix with a finite-order matrix to that of approximating infinite sums with partial finite summations.

CHAPTER NOTES

This chapter introduces some methods of approximating distributed-mass models of structures with lumped-mass models more suitable for digital computing. Section 13.2 introduced the obvious and popular method of modal truncation.

Modal methods are quite common and can be found in most texts. See, for instance, Meirovitch (1980, 2001), for a more complete discussion. The Ritz–Galerkin method of Section 13.3 is again very common and is found in most vibration texts at almost every level. The common name of Raleigh–Ritz has always been surrounded with a bit of controversy over who actually first penned the method, and this is nicely settled in Leissa (2005). The finite element method briefly discussed in Section 13.4 is currently the most often used

and written about method. An excellent short introduction to finite element methods can be found in Meirovitch (1986). A more comprehensive approach can be found in the excellent book by Shames and Dym (1985) or the more advanced treatment by Hughes (2000).

Meirovitch's (1980) book contains a complete treatment of the substructure methods discussed in Section 13.5, as does the paper by Hale and Meirovitch (1980). The paper by Craig (1987) reviews the related topic of component mode methods. Sections 13.6 and 13.7, dealing with the topic of truncation and control system design, are taken directly from the paper by Berkman and Karnopp (1969), which was one of the first papers written in this area. Many other approaches to this same problem can be found in the literature. The survey paper by Balas (1982) provides a useful introduction to the topic.

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PROBLEMS

13.1 Estimate the amount of energy neglected in a three-mode approximation of a fixed–fixed beam of length ℓ in the longitudinal vibration.

13.2 Calculate a three-mode approximation of a clamped square plate using modal truncation.

13.3 Use trigonometric functions and perform a Ritz–Galerkin approximation for a transversely vibrating beam (undamped) that is clamped at one end and attached to a spring with constant k and mass m at the other end. Use three terms. Calculate the natural frequencies and compare them with those obtained by the method of Section 12.4.

13.4 Compare the finite element model of example 13.4.1 with a three-mode Ritz–Galerkin model of the same structure. How do the eigenvalues compare with those of the distributed-parameter model?

13.5 Show that the matrices M and K defined by the finite element method of Section 13.4 are both symmetric (in general).

13.6 Derive the finite element matrix for a transversely vibrating beam modeled with three elements.

13.7 Consider a three-degree-of-freedom system with mass matrix $M = I$ and stiffness matrix

$$K = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 1.5 & -.5 \\ 0 & -.5 & .5 \end{bmatrix}$$

that corresponds to three masses connected in series by three springs. Define two substructures by letting substructure 1 be the first two masses and substructure 2 be the remaining mass. Calculate the coefficient matrices of Equation (13.61).

13.8 Calculate $\lambda(s)$ for a cantilevered beam in transverse vibration. Use this information to calculate the matrix Λ^{-1}.

13.9 For problem (13.8), suppose that a disturbance force of $\sin 2t$ is applied to the structure at the midpoint and calculate the value of the index n such that $\mathbf{e}_\infty$ is negligible. For simplicity, set each of the physical parameter values to unity.

13.10 Recalculate the equations of example 13.7.1 using two elements as part of the control, i.e. $z_1(s)$ and $z_2(s)$, acting at points x_1 and x_2, respectively.

13.11 Derive Equations (13.23) and (13.24) for the case $N = 2$, by substituting the sum of Equation (3.19) into the Rayleigh quotient and taking the indicated derivatives. (Hint: Use the fact that L is self-adjoint and write the two derivative equations as one equation in matrix form.)