3

Matrices and the Free Response

3.1 INTRODUCTION

As illustrated in Chapter 1, the nature of the free response of a single-degree-of-freedom system is determined by the roots of the characteristic equation [Equation (1.8)]. In addition, the exact solution is calculated using these roots. A similar situation exists for the multiple-degree-of-freedom systems described in the previous chapter. Motivated by the single-degree-of-freedom system, this chapter examines the problem of characteristic roots for systems in matrix notation and extends many of the ideas discussed in Chapter 1 to the multiple-degree-of-freedom systems described in Chapter 2. The mathematical tools needed to extend the ideas of Chapter 1 are those of linear algebra, which are introduced here in an informal way, as needed.

Chapter 2 illustrated that many types of mechanical system can be characterized by vector differential equations with matrix coefficients. Just as the nature of the scalar coefficients in the single-degree-of-freedom case determines the form of the response, the nature of the matrix coefficients determines the form of the response of multiple-degree-of-freedom systems.

In fact, if we attempt to follow the method of solving single-degree-of-freedom vibration problems in solving multiple-degree-of-freedom systems, we are led immediately to a standard matrix problem called the algebraic eigenvalue problem. This chapter introduces the matrix eigenvalue problem and applies it to the multiple-degree-of-freedom vibration problems introduced in Chapter 2. The eigenvalues and eigenvectors can be used to determine the time response to initial conditions by the process called modal analysis which is introduced here. The use of high-level codes such as MATLAB is introduced to compute mode shapes and natural frequencies. The chapter concludes with simulation of the time response to initial condition disturbances, using numerical integration as an alternative to modal analysis.

Vibration with Control D. J. Inman
© 2006 John Wiley & Sons, Ltd. ISBN: 0-470-01051-7

3.2 EIGENVALUES AND EIGENVECTORS

This section introduces topics from linear algebra and the matrix eigenvalue problem needed to study the vibrations of multiple-degree-of-freedom systems. Consider first the simple conservative vibration problem of Equation (2.11), repeated here:

$$M\ddot{\mathbf{x}} + K\mathbf{x} = \mathbf{0}$$

for the free response case where $\mathbf{F} = \mathbf{0}$. Since M is assumed to be positive definite, it has an inverse. Premultiplying the equation of motion by the matrix M^{-1} yields the following equation for the free response:

$$\ddot{\mathbf{q}} + M^{-1}K\mathbf{q} = \mathbf{0}$$

Following the mathematical approach of Section 1.2 and the physical notion that the solution should oscillate suggests that a solution may exist of the form of nonzero constant $\mathbf{u}$, in this case a vector, times the exponential $e^{\mu jt}$, i.e., $\mathbf{q}(t) = \mathbf{u}\, e^{\mu jt}$. Substitution of this expression into the preceding equation yields

$$-\mu^2\mathbf{u} + A\mathbf{u} = \mathbf{0}, \qquad \mathbf{u} \neq \mathbf{0}$$

where $A = M^{-1}K$. Rearrangement of this expression yields the equation

$$A\mathbf{u} = \lambda\mathbf{u}, \qquad \mathbf{u} \neq \mathbf{0}$$

where $\lambda = \mu^2$ and $\mathbf{u}$ cannot be zero. This expression is exactly a statement of the matrix eigenvalue problem. As in the case of the single-degree-of-freedom system, the constants $\lambda = \mu^2$ characterize the natural frequencies of the system. With this as a motivation, the matrix eigenvalue problem is described in detail in this section and applied to the linear vibration problem in Section 3.3. Computational considerations are discussed in Section 3.8.

Square matrices can be characterized by their eigenvalues and eigenvectors, defined in this section. Let A denote an $n \times n$ square matrix. The scalar λ is defined as an *eigenvalue* of matrix A with corresponding *eigenvector* $\mathbf{x}$, which must be nonzero, if λ and $\mathbf{x}$ satisfy the equation

$$A\mathbf{x} = \lambda\mathbf{x}, \qquad \mathbf{x} \neq \mathbf{0} \tag{3.1}$$

Geometrically, this means that the action of matrix A on vector $\mathbf{x}$ just changes the length of vector $\mathbf{x}$ and does not change its direction or orientation in space. Physically, the eigenvalue λ will yield information about the natural frequencies of the system described by matrix A. It should be noted that, if $\mathbf{x}$ is an eigenvector of A, then so is the vector $\alpha\mathbf{x}$, where α is any scalar. Thus, the magnitude of an eigenvector is arbitrary.

A rearrangement of Equation (3.1) yields

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3.2}$$

where I is the $n \times n$ identity matrix. Since $\mathbf{x}$ cannot be zero, by the definition of an eigenvector, the inverse of the matrix $(A - \lambda I)$ must not exist. That is, there cannot exist a matrix

$(A - \lambda I)^{-1}$ such that, $(A - \lambda I)^{-1}(A - \lambda I) = I$. Otherwise, premultiplying Equation (3.2) by this inverse would mean that the only solution to Equation (3.2) is $\mathbf{x} = \mathbf{0}$, violating the definition of an eigenvector. Matrices that do not have inverses are said to be *singular*, and those that do have an inverse are called *nonsingular*.

Whether or not a matrix is singular can also be determined by examining the *determinant* of the matrix. The determinant of an $n \times n$ matrix A is defined and denoted by

$$\det A = |A| = \sum_{s=1}^{n} a_{rs}|A_{rs}| \tag{3.3}$$

for any fixed r, where a_{rs} is the element of A at the intersection of the rth row and sth column of A and $|A_{rs}|$ is the determinant of the matrix formed from A by striking out the rth row and sth column multiplied by $(-1)^{r+s}$. An illustration of this for $n = 2$ is given in Section 2.3. The value of the determinant of a matrix is a unique scalar. In addition, it is a simple matter to show that

$$|A| = |A^T| \tag{3.4}$$

$$|AB| = |A||B| \tag{3.5}$$

Whether or not the determinant of a matrix is zero is very significant and useful. The following five statements are entirely equivalent:

1. A is nonsingular.
2. A^{-1} exists.
3. $\det A \neq 0$.
4. The only solution of the equation $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$.
5. Zero is not an eigenvalue of A.

Note that, if $\det(A) = 0$, then A^{-1} does not exist, A is singular, and $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution; i.e., zero is an eigenvalue of A.

Example 3.2.1

The determinant of matrix A is calculated from Equation (3.3) (r is chosen as the fixed value 1) as

$$\det A = \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 1 \\ 2 & 5 & 3 \end{bmatrix} = 1[(1)(3) - (1)(5)] - 3[(0)(3) - (1)(2)] - 2[(0)(5) - (1)(2)] = 8$$

Applying the concept of the determinant of a matrix to the eigenvalue problem stated in Equation (3.2) indicates that, if λ is to be an eigenvalue of matrix A, then λ must satisfy the equation

$$\det(A - \lambda I) = 0 \tag{3.6}$$

This expression results in a polynomial in λ, which is called the *characteristic equation* of matrix A.

Since A is an $n \times n$ matrix, Equation (3.6) will have n roots (or A will have n eigenvalues), which are denoted by λ_i. Then, Equation (3.6) can be rewritten as

$$\det(A - \lambda I) = \prod_{i=1}^{n} (\lambda - \lambda_i) = 0 \tag{3.7}$$

If λ_i happens to be a root that is repeated m_i times, then this becomes

$$\det(A - \lambda I) = \prod_{i=1}^{k} (\lambda - \lambda_i)^{m_i}, \qquad \text{where } \sum_{i=1}^{k} m_i = n \tag{3.8}$$

Also, note from examination of Equation (3.2) that any given eigenvalue may have many eigenvectors associated with it. For instance, if $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ, and α is any scalar, $\alpha\mathbf{x}$ is also an eigenvector of A with corresponding eigenvalue λ. Eigenvectors have several other interesting properties, many of which are useful in calculating the free response of a vibrating system.

The first property has to do with the concept of linear independence. A set of vectors, denoted by $\{\mathbf{e}_i\}_{i=1}^{n} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, is said to be *linearly independent*, or just *independent*, if

$$\alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \cdots + \alpha_n \mathbf{e}_n = \mathbf{0} \tag{3.9}$$

implies that each of the scalars α_i is zero. If this is not the case, i.e., if there exists one or more nonzero scalars α_i satisfying Equation (3.9), then the set of vectors $\{\mathbf{x}_i\}$ is said to be *linearly dependent*. The set of all linear combinations of all n-dimensional real vectors is called the *span* of the set of all n-dimensional real vectors. A set of n linearly independent vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is said to form a *basis* for the span of vectors of dimension n. This means that, if $\mathbf{x}$ is any vector of dimension n, then there exists a unique representation of vector $\mathbf{x}$ in terms of the basis vectors $\mathbf{e}_i$, given by

$$\mathbf{x} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + \cdots + a_n \mathbf{e}_n \tag{3.10}$$

The coefficients a_i are sometimes called the *coordinates* of vector $\mathbf{x}$ in the basis $\{\mathbf{e}_i\}_{i=1}^{n}$. One familiar basis is the basis consisting of unit vectors $\left(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\right)$ of a rectangular coordinate system, which forms a basis for the set of all three-dimensional real vectors.

Another important use of the idea of linear independence is contained in the concept of the *rank* of a matrix. The rank of a matrix is defined as the number of independent rows (or columns) of the matrix when the rows (columns) are treated like vectors. This property is used in stability analysis in Chapter 4, and in control in Chapter 7. Note that a square $n \times n$ matrix is nonsingular if and only if its rank is n (i.e., if and only if it has *full rank*).

If the scalar product, or dot product, of two vectors is zero, i.e., if $\mathbf{x}_i^T \mathbf{x}_j = 0$, then the two vectors are said to be *orthogonal*. If $\mathbf{x}_i^T \mathbf{x}_i = 1$, the vector $\mathbf{x}_i$ is called a *unit vector*. If a set of unit vectors is also orthogonal, i.e., if

$$\mathbf{x}_i^T \mathbf{x}_j = \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

they are said to be an *orthonormal* set. Here, δ_{ij} is the Kronecker delta. Again, the familiar unit vectors of rectangular coordinate systems are an orthonormal set of vectors. Also, as discussed later, the eigenvectors of a symmetric matrix can be used to form an orthonormal set. This property is used in this chapter and again in Chapter 5 to solve various vibration problems.

Another important property of eigenvectors is as follows. If A is a square matrix and if the eigenvalues of A are distinct, then the eigenvectors associated with those eigenvalues are independent. If A is also symmetric, then an independent set of eigenvectors exist even if the eigenvalues are repeated. Furthermore, if zero is not an eigenvalue of A and A has eigenvalues λ_i with corresponding eigenvectors $\mathbf{x}_i$, then the eigenvectors of A^{-1} are also $\mathbf{x}_i$ and the eigenvalues are λ_i^{-1}. Thus, A and A^{-1} have related eigenvalue problems. Yet another useful result for the eigenvalues, λ_i, of matrix A is that the eigenvalues of $(A \pm \beta I)$ are just $\lambda_i \pm \beta$, where β is any scalar (called a shift).

Matrix A is *similar* to matrix B if there exists a nonsingular matrix P such that

$$A = PBP^{-1} \tag{3.11}$$

In this case, P is referred to as a *similarity transformation* (matrix) and may be used to change vibration problems from one coordinate system, which may be complicated, to another coordinate system that has a simple or canonical form.

The reason that similarity transformations are of interest is that, if two matrices are similar, they will have the same eigenvalues. Another way to state this is that similarity transformations preserve eigenvalues, or that eigenvalues are *invariant* under similarity transformations. Some square matrices are similar to diagonal matrices. Diagonal matrices consist of all zero elements except for those on the diagonal, making them easy to manipulate. The algebra of diagonal matrices is much like that of scalar algebra. This class of matrices is examined in detail next.

If matrix A is similar to a diagonal matrix, denoted by Λ, then A can be written as

$$A = P\Lambda P^{-1} \tag{3.12}$$

Postmultiplying this expression by P yields

$$AP = P\Lambda \tag{3.13}$$

Now, let the vectors p_i, $i = 1, 2, \ldots, n$, be the columns of matrix P, i.e.,

$$P = [\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3 \cdots \mathbf{p}_n] \tag{3.14}$$

Note that no $\mathbf{p}_i$ can be a zero vector since P is nonsingular. If λ_{ii} denotes the ith diagonal element of diagonal matrix Λ, then Equation (3.13) can be rewritten as the n separate equations

$$A\mathbf{p}_i = \lambda_{ii}\mathbf{p}_i, \qquad i = 1, 2, \ldots, n \tag{3.15}$$

Equations (3.15) state that $\mathbf{p}_i$ is the ith eigenvector of matrix A and that λ_{ii} is the associated eigenvalue, λ_i. The preceding observation can be summarized as follows:

1. If A is similar to a diagonal matrix, the diagonal elements of that matrix are the eigenvalues of A (i.e., $\lambda_i = \lambda_{ii}$).
2. A is similar to a diagonal matrix if and only if A has a set of n linearly independent eigenvectors.
3. If A has distinct eigenvalues, then it is similar to a diagonal matrix.

As an important note for vibration analysis: if A is a real symmetric matrix, then there exists a matrix P such that Equation (3.12) holds.

If the eigenvectors of A are linearly independent, they can be used to form an orthonormal set. Let $\mathbf{s}_i$ denote the orthonormal eigenvectors of A so that $\mathbf{s}_i^T\mathbf{s}_j = \delta_{ij}$, the Kronecker delta. Forming a matrix out of this set of normalized eigenvectors then yields

$$S = [\mathbf{s}_1 \quad \mathbf{s}_2 \quad \mathbf{s}_3 \quad \ldots \quad \mathbf{s}_n] \tag{3.16}$$

Here, note that expanding the matrix product S^TS yields

$$S^TS = I \tag{3.17}$$

where I is the $n \times n$ identity matrix, because of the orthonormality of the rows and columns of S. Equation (3.17) implies immediately that $S^T = S^{-1}$. Such real-valued matrices are called *orthogonal matrices*, and Equation (3.12) can be written as

$$A = S\Lambda S^T \tag{3.18}$$

In this case, A is said to be *orthogonally similar* to Λ. (If S is complex valued, then $S^*S = I$, where the asterisk indicates the complex conjugate transpose of S, and S is called a Hermitian matrix.) Orthonormal sets are used to compute the time response of vibrating systems from the eigenvalues and eigenvectors.

Often it is convenient in vibration analysis to modify the concept of orthogonally similar matrices by introducing the concept of a weighting matrix. To this end, the eigenvectors of a matrix K can be normalized with respect to a second positive definite matrix, which in this case is chosen to be the matrix M. That is, the magnitude of the eigenvectors of K, $\mathbf{x}_i$, are chosen such that

$$\mathbf{x}_i^T M \mathbf{x}_j = \delta_{ij} \tag{3.19}$$

In this case the weighted transformation, denoted by S_m, has the following properties:

$$S_m^T M S_m = I \tag{3.20}$$

$$S_m^T K S_m = \mathrm{diag}[\omega_i^2] \tag{3.21}$$

where ω_i^2 denote the eigenvalues of matrix K. This is not to be confused with the diagonal matrix S^TKS, where S is made up of the (not weighted) eigenvectors of matrix K.

3.3 NATURAL FREQUENCIES AND MODE SHAPES

As mentioned previously, the concept of the eigenvalue of a matrix is closely related to the concept of natural frequency of vibration in mechanical structures, just as the roots of the characteristic equation and natural frequency of a single-degree-of-freedom system are related. To make the connection formally, consider again the undamped nongyroscopic conservative system described by

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{0} \tag{3.22}$$

subject to initial conditions $\mathbf{q}_0$ and $\dot{\mathbf{q}}_0$. Here, the matrices M and K are assumed to be symmetric and positive definite.

In an attempt to solve Equation (3.22), a procedure similar to the method used to solve a single-degree-of-freedom system is employed by assuming a solution of the form

$$\mathbf{q}(t) = \mathbf{u}\, e^{\mu j t} \tag{3.23}$$

Here, $\mathbf{u}$ is a nonzero, unkown vector of constants, μ is a scalar value to be determined, $j = \sqrt{-1}$, and t is, of course, the time. Substitution of Equation (3.23) into Equation (3.22) yields

$$(-M\mu^2 + K)\mathbf{u}\, e^{\mu j t} = \mathbf{0} \tag{3.24}$$

This is identical to the procedure used in Section 1.2 for single-degree-of-freedom systems. Since $e^{\mu j t}$ is never zero for any value of μ or t, Equation (3.24) holds if and only if

$$(-M\mu^2 + K)\mathbf{u} = \mathbf{0} \tag{3.25}$$

This is starting to look very much like the eigenvalue problem posed in Equation (3.2). To make the analogy more complete, let $\mu^2 = \lambda$, so that Equation (3.25) becomes

$$(K - \lambda M)\mathbf{u} = \mathbf{0} \tag{3.26}$$

Since it is desired to calculate nonzero solutions of Equation (3.22), the vector $\mathbf{u}$ should be nonzero. This corresponds very well to the definition of an eigenvector, i.e., that it be nonzero. Eigenvalue problems stated in terms of two matrices of the form $A\mathbf{x} = \lambda B\mathbf{x}$, $\mathbf{x} \neq \mathbf{0}$, are called *generalized eigenvalue problems*. Now recall, that a nonzero solution $\mathbf{u}$ of Equation (3.26) exists if and only if the matrix $(K - \lambda M)$ is singular or if and only if

$$\det(K - \lambda M) = \mathbf{0} \tag{3.27}$$

Next, note that, since M is positive definite, it must have an inverse. To see this, note that, if M^{-1}does not exist, then there is a nonzero vector $\mathbf{x}$ such that

$$M\mathbf{x} = \mathbf{0}, \qquad \mathbf{x} \neq \mathbf{0} \tag{3.28}$$

Premultiplying by $\mathbf{x}^T$ results in

$$\mathbf{x}^T M\mathbf{x} = \mathbf{0}, \qquad \mathbf{x} \neq \mathbf{0} \tag{3.29}$$

which clearly contradicts the fact that M is positive definite (recall the end of Section 2.1).

Since M^{-1} exists, $\det(M^{-1}) \neq 0$ and we can multiply Equation (3.27) by $\det(M^{-1})$ [invoking Equation (3.5)] to obtain

$$\det(M^{-1}K - \lambda I) = 0 \tag{3.30}$$

which is of the same form as Equation (3.6) used to define eigenvalues and yields a polynomial in λ of order n. As will be illustrated, each root of Equation (3.30), or eigenvalue of the matrix $M^{-1}K$, is the square of one of the natural frequencies of Equation (3.22).

There are several alternative ways to relate the eigenvalue problem of Equation (3.1) to the natural frequency problem of Equation (3.22). For instance, since M is positive definite, it has a positive definite square root. That is, there exists a unique positive definite matrix $M^{1/2}$ such that $M^{1/2}M^{1/2} = M$. The eigenvalues of $M^{1/2}$ are $\beta_i^{1/2}$, where β_i are the eigenvalues of M. Both M and its matrix square root have the same eigenvectors. Furthermore, if P is the matrix of eigenvectors of M, then

$$M^{1/2} = P\Lambda_M^{1/2}P^{-1} \tag{3.31}$$

where $\Lambda_M^{1/2}$ is a diagonal matrix with diagonal elements $\beta_i^{1/2}$. Many times in modeling systems, M is already diagonal, in which case the matrix square root is calculated by taking the square root of each of the diagonal elements. Systems with a non-diagonal mass matrix are called *dynamically coupled* systems. The existence of this matrix square root provides an important alternative relationship between matrix eigenvalues and vibrational natural frequencies and allows a direct analogy with the single-degree-of-freedom case. Matrix factorizations, such as the square root, lead to more computationally efficient algorithms (see Section 3.8).

Since $M^{1/2}$ is positive definite, it has an inverse $M^{-1/2}$, and pre- and postmultiplying Equation (3.27) by $\det(M^{-1/2})$ and factoring out -1 yields

$$\det(\lambda I - M^{-1/2}KM^{-1/2}) \quad = \quad 0 \tag{3.32}$$

Equation (3.32) is an alternative way of expressing the eigenvalue problem. The difference between Equations (3.32) and (3.30) is that the matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ is symmetric and positive definite, whereas $M^{-1}K$ is not necessarily symmetric. Matrix symmetry provides both a theoretical and computational advantage. Specifically, a symmetric matrix is similar to a diagonal matrix consisting of its eigenvalues along the diagonal, and the eigenvectors of a symmetric matrix are linearly independent and orthogonal. The corresponding differential equation then becomes

$$I\ddot{\mathbf{r}}(t) + M^{-1/2}KM^{-1/2}\mathbf{r}(t) = \mathbf{0} \tag{3.33}$$

where $\mathbf{q}(t) = M^{-1/2}\mathbf{r}(t)$ has been substituted into Equation (3.22) and the result premultiplied by $M^{-1/2}$.

As expected and shown later, the numbers λ_i are directly related to the natural frequencies of vibration of the system described by Equation (3.22): $\omega_i^2 = \mu_i^2 = \lambda_i$. It is expected, as in the case of single-degree-of-freedom systems with no damping, that the natural frequencies will be such that the motion oscillates without decay. Mathematically, this result follows from realizing that the matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ is symmetric and positive definite, ensuring the nature of the natural frequencies and eigenvectors.

To see that a real, symmetric, positive definite matrix such as $\tilde{K} = M^{-1/2}KM^{-1/2}$ has positive real eigenvalues (and, hence, real eigenvectors) requires some simple manipulation of the definitions of these properties. First, note that, if $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ, then

$$A\mathbf{x} = \lambda\mathbf{x} \tag{3.34}$$

Assuming that λ and $\mathbf{x}$ are complex and taking the conjugate transpose of this expression yields (because A is symmetric)

$$\mathbf{x}^* A = \mathbf{x}^* \lambda^* \tag{3.35}$$

Premultiplying Equation (3.34) by $\mathbf{x}^*$, postmultiplying Equation (3.35) by $\mathbf{x}$, and subtracting the two yields

$$0 = \mathbf{x}^* A\mathbf{x} - \mathbf{x}^* A\mathbf{x} = (\lambda - \lambda^*)\mathbf{x}^*\mathbf{x}$$

or, since $\mathbf{x} \neq \mathbf{0}$, that $\lambda = \lambda^*$. Hence, λ must be real valued.

Next, consider that A can be written as $A = S\Lambda S^T$. Therefore, for any and all arbitrary vectors $\mathbf{x}$,

$$\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T S\Lambda S^T\mathbf{x} = \mathbf{y}^T \Lambda\mathbf{y}$$

where $\mathbf{y} = S^T\mathbf{x}$ is also free to take on any real value. This can be expressed as

$$\mathbf{y}^T \Lambda\mathbf{y} = \sum_{i=1}^{n} \lambda_i y_i^2 > 0$$

since A is positive definite. If the vectors $\mathbf{y}_1 = [1\ 0\ 0\ \cdots\ 0]^T$, $\mathbf{y}_2 = [0\ 1\ 0\ \cdots\ 0]^T$, $\ldots$, $\mathbf{y}_n = [0\ 0\ 0\ \cdots\ 1]^T$ are, in turn, substituted into this last inequality, the result is $\lambda_i > 0$, for each of the n values of index i. Hence, a positive definite symmetric matrix has positive real eigenvalues (the converse is also true).

Applying this fact to Equation (3.32) indicates that each eigenvalue of the mass normalized stiffness matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ is a positive real number. From Equation (3.25) we see that the natural frequencies of Equation (3.22) are $\mu = \omega$, where $\omega^2 = \lambda$, a positive real number. Hence, the coefficient of t in Equation (3.23) has the form $\omega = \pm\sqrt{\lambda}j$, just as in the single-degree-of-freedom case. The square roots of λ_i are the natural frequencies of the system, i.e., $\omega_i = \sqrt{\lambda_i}$, where i ranges from 1 to n, n being the number of degrees of freedom. That is, there is one natural frequency for each degree of freedom.

The concept of a positive definite matrix can also be related to conditions on the elements of the matrix in a useful manner. Namely, it can be shown that a symmetric matrix A is positive definite if and only if the leading principal minors of A are positive. That is, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{mn} \end{bmatrix}$$

then A is positive definite if and only if

$$a_{11} > 0$$

$$\det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > 0$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} > 0$$

$$\vdots$$

$$\det A > 0$$

This condition provides a connection between the condition that a matrix be positive definite and the physical parameters of the system. For example, the stiffness matrix of example 2.4.1 will be positive definite if and only if $k_1 + k_2 > m\Omega^2$ and $2k_2 > m\Omega^2$, by the preceding principle minor condition. That is, for $A = K$ in example 2.4.1, the first two conditions yield the two inequalities in k_i, m, and Ω. This provides physical insight as it indicates that stability may be lost if the system spins faster (Ω) than the stiffness can handle. These inequalities are very useful in vibration design and in stability analysis.

Another interesting fact about symmetric matrices is that their eigenvectors form a complete set, or a basis. Recall that a set of real vectors $\{\mathbf{u}_i\}$ of dimension n is a *basis* for the set of all real n-dimensional vectors if and only if they are linearly independent and every other real vector of dimension n can be written as a linear combination of $\mathbf{u}_i$. Thus, the solution $\mathbf{q}(t)$ can be expanded in terms of these eigenvectors. The set of eigenvectors of the matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ forms a linearly independent set such that any vector of dimension n can be written as a linear combination of these vectors. In particular, the solution of the vibration problem can be expanded in terms of this basis.

Combining the preceding matrix results leads to the following solution for the response $\mathbf{r}(t)$. There are n solutions of Equation (3.33) of the form

$$\mathbf{r}_k(t) = \mathbf{u}_k\, e^{\mu_k jt} \tag{3.36}$$

As just shown, under the assumption that $M^{-1/2}KM^{-1/2}$ is positive definite, the numbers μ_k must all be of the form

$$\mu_k = \pm\sqrt{\lambda_k} \tag{3.37}$$

where λ_k are the positive eigenvalues of the matrix $M^{-1/2}KM^{-1/2}$. Combining Equation (3.36) and (3.37) it can be seen that each $\mathbf{r}_k(t)$ must have the form

$$\mathbf{r}_k(t) = \left(a_k\, e^{-\sqrt{\lambda_k}jt} + b_k\, e^{+\sqrt{\lambda_k}jt}\right)\mathbf{u}_k \tag{3.38}$$

where a_k and b_k are arbitrary constants. Since $\mathbf{u}_k$ are the eigenvectors of a symmetric matrix, they form a basis, so the n-dimensional vector $\mathbf{r}(t)$ can be expressed as a linear combination of these. That is,

$$\mathbf{r}(t) = \sum_{k=1}^{n} \left(a_k \, e^{-\sqrt{\lambda_k} jt} + b_k \, e^{\sqrt{\lambda_k} jt} \right) \mathbf{u}_k \tag{3.39}$$

where the arbitrary constants a_k and b_k can be determined from the initial conditions $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$. This amounts to solving the $2n$ algebraic equations given by

$$\begin{aligned} \mathbf{r}(0) &= \sum_{k=1}^{n} (a_k + b_k)\mathbf{u}_k \\ \dot{\mathbf{r}}(0) &= j \sum_{k=1}^{n} \sqrt{\lambda_k}(b_k - a_k)\mathbf{u}_k \end{aligned} \tag{3.40}$$

for the $2n$ constants a_k and b_k, $k = 1, \ldots, n$.

Since the symmetric properties of the matrix $M^{-1/2}KM^{-1/2}$ were used to develop the solution given by Equation (3.39), note that the solution expressed in Equation (3.39) is the solution of a slightly different problem to the solution $\mathbf{q}(t)$ of Equation (3.22). The two are related by the transformation

$$\mathbf{q}(t) = M^{-1/2}\mathbf{r}(t) \tag{3.41}$$

which also specifies how the initial conditions in the original coordinates are to be transformed.

Equation (3.39) can be manipulated, using Euler's formulae for trigonometric functions, to become

$$\mathbf{r}(t) = \sum_{k=1}^{n} c_k \sin(\omega_k t + \phi_k) \, \mathbf{u}_k \tag{3.42}$$

where c_k and ϕ_k are constants determined by the initial conditions. This form clearly indicates the oscillatory nature of the system and defines the concept of natural frequency. Here, $\omega_k = +\sqrt{\lambda_k}$ denotes the undamped natural frequencies. Note that the frequencies are always positive because the Euler formula transformation from $e^{\pm\sqrt{\lambda_k} t}$ to $\sin \omega_k t$ effectively uses the $\pm$ sign in defining oscillation at the (positive) frequency ω_k. This expression extends the undamped single-degree-of-freedom result to undamped multiple-degree-of-freedom systems.

To evaluate the constants c_k and ϕ_k, the orthonormality of vectors $\mathbf{u}_k$ is again used. Applying the initial conditions to Equation (3.42) yields

$$\mathbf{r}(0) = \sum_{k=1}^{n} c_k \sin(\phi_k)\mathbf{u}_k \tag{3.43}$$

and

$$\dot{\mathbf{r}}(0) = \sum_{k=1}^{n} c_k \omega_k \cos(\phi_k)\mathbf{u}_k \tag{3.44}$$

Equation (3.41) is used to yield $\mathbf{r}(0) = M^{1/2}\mathbf{q}(0)$ and $\dot{\mathbf{r}}(0) = M^{1/2}\dot{\mathbf{q}}(0)$ from the given initial conditions $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$. Premultiplying Equation (3.43) by $\mathbf{u}_i^T$ yields

$$\mathbf{u}_i^T\mathbf{r}(0) = \sum_{k=1}^{n} c_k \sin(\phi_k)\mathbf{u}_i^T\mathbf{u}_k$$

Invoking the orthonormality for vectors $\mathbf{u}_i$ yields

$$c_i \sin\phi_i = \mathbf{u}_i^T\mathbf{r}(0) \tag{3.45}$$

Likewise, Equation (3.44) yields

$$c_i \cos\phi_i = \frac{\mathbf{u}_i^T\dot{\mathbf{r}}(0)}{\omega_i} \tag{3.46}$$

Combining Equations (3.45) and (3.46) and renaming the index yields

$$\phi_i = \tan^{-1}\left\{\frac{\omega_i\mathbf{u}_i^T\mathbf{r}(0)}{\mathbf{u}_i^T\dot{\mathbf{r}}(0)}\right\}$$

and

$$c_i = \frac{\mathbf{u}_k^T\mathbf{r}(0)}{\sin\phi_i}$$

Note that, if the initial position $\mathbf{r}(0)$ is zero, then Equation (3.45) would imply that $\phi_i = 0$ for each i, then Equation (3.46) is used to compute the coefficients c_i. Once the constants c_i and ϕ_i are determined, then the index is changed to k to fit into the sum of Equation (3.42) which is written in terms of c_k and ϕ_k.

Next, consider the eigenvectors $\mathbf{u}_k$ to see how they represent the physical motion of the system. Suppose the initial conditions $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$ are chosen in such a way that $c_k = 0$ for $k = 2, 3, \cdots, n$, $c_1 = 1$, and $\phi_k = 0$ for all k. Then, the expansion (3.42) reduces to one simple term, namely

$$\mathbf{r}(t) = \sin(\omega_1 t)\mathbf{u}_1 \tag{3.47}$$

This implies that every mass is vibrating with frequency ω_1 or is stationary and that the relative amplitude of vibration of each of the masses is the value of the corresponding element of $\mathbf{u}_1$. Thus, the size and sign of each element of the eigenvector indicates the positions of each mass from its equilibrium position, i.e., the 'shape' of the vibration at any instant of time. Transforming this vector back into the physical coordinate system via $\mathbf{v}_1 = M^{-1/2}\mathbf{u}_1$ allows the interpretation that vector $\mathbf{v}_1$ is the first *mode shape* of the system, or the mode shape corresponding to the first natural frequency. This can clearly be repeated for each of the subscripts k, so that $\mathbf{v}_k$ is the kth mode shape. Hence, the transformed eigenvectors are referred to as the *modes* of vibration of the system. Since eigenvectors are arbitrary to within a multiplicative constant, so are the mode shapes. If the arbitrary constant is chosen so that $\mathbf{v}_k$ is normalized, i.e., so that $\mathbf{v}_k^T\mathbf{v}_k = 1$, and the vector $\mathbf{v}_\mathrm{k}$ is real, the $\mathbf{v}_\mathrm{k}$ is called a *normal mode* of the system. The constants c_k in Equation (3.42) are called *modal participation*

factors because their relative magnitudes indicate how much the indexed mode influences the response of the system.

The procedure just described constitutes a theoretical *modal analysis* of the system of Equation (3.22). Some researchers refer to Equations (3.39) and (3.42) as the *expansion theorem*. They depend on the completeness of the eigenvectors associated with the system, i.e., of the matrix $M^{-1/2}KM^{-1/2}$.

Example 3.3.1

It should be obvious from Equations (3.27) and (3.32) how to calculate the eigenvalues and hence the natural frequencies of the system as they are the roots of the characteristic polynomial following from $\det(\tilde{K} - \lambda I) = 0$. How to calculate the eigenvectors, however, may not be as obvious; thus, calculation is illustrated in this example. Let λ_1 be an eigenvalue of A; then λ_1 and $\mathbf{u}_1 = [x_1 \;\; x_2]^T$ satisfy the vector equation

$$\begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.48}$$

This represents two dependent equations in x_1 and x_2, the two components of the eigenvector $\mathbf{u}_1$. Hence, only their ratio can be determined. Proceeding with the first equation in system (3.48) yields

$$(a_{11} - \lambda_1)x_1 + a_{12}x_2 = 0$$

which is solved for the ratio x_1/x_2. Then, the vector $\mathbf{u}_1$ is 'normalized' so that $\mathbf{u}_1^T\mathbf{u}_1 = x_1^2 + x_2^2 = 1$. The normalization yields specific values for x_1 and x_2. As a consequence of the singularity of $(A - \lambda I)$, the second equation in system (3.48), $a_{21} + x_1 + (a_{22} - \lambda)x_2 = 0$, is dependent on the first and does not yield new information.

Example 3.3.2

This example illustrates the procedure for calculating the free vibrational response of a multiple-degree-of-freedom system by using a *modal expansion*. The procedure is illustrated by a two-degree-of-freedom system, since the procedure for a larger number of degrees of freedom is the same. The purpose of the example is to develop an understanding of the eigenvector problem, and it is not intended to imply that this is the most efficient way to calculate the time response of a system (it is not).

Consider the system described in Figure 2.4 with $c_1 = c_2 = 0$, $m_1 = 9$, $m_2 = 1$, $k_1 = 24$, and $k_2 = 3$. Then, the equation of motion becomes

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \mathbf{q} = \mathbf{0}$$

subject to the initial condition $\mathbf{q}(0) = [1 \quad 0]^T$ and $\dot{\mathbf{q}}(0) = [0 \quad 0]^T$ in some set of consistent units. The matrix $\tilde{K} = M^{-1/2}KM^{-1/2}$ becomes

$$\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

The characteristic equation [equation (3.32)] becomes

$$\lambda^2 - 6\lambda + 8 = 0$$

which has roots $\lambda_1 = 2, \lambda_2 = 4$. The corresponding normalized eigenvectors are computed to be $\mathbf{u}_1 = [1/\sqrt{2} \quad 1/\sqrt{2}]^T$ and $\mathbf{u}_2 = [-1/\sqrt{2} \quad 1/\sqrt{2}]^T$, so that the orthogonal matrix of eigenvectors is

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{3.49}$$

Also note that $S^T(M^{-1/2}KM^{-1/2})S = \text{diag}[2 \quad 4]$, as it should. The transformed initial conditions become

$$\mathbf{r}(0) = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

and of course $\dot{\mathbf{r}}(0) = [0 \quad 0]^T$. The values of the constants in Equation (3.42) are found from

$$\phi_1 = \tan^{-1}\left[\frac{\omega_1 \mathbf{u}_1^T \mathbf{r}(0)}{\mathbf{u}_1^T \dot{\boldsymbol{r}}(0)}\right] = \tan^{-1}\infty = \frac{\pi}{2}$$

$$c_1 = \frac{\mathbf{u}_1^T \mathbf{r}(0)}{\sin\phi_1} = \frac{3}{\sqrt{2}}$$

$$\phi_2 = \tan^{-1}\left[\frac{\omega_2 \mathbf{u}_2^T \mathbf{r}(0)}{\mathbf{u}_2^T \dot{\mathbf{r}}(0)}\right] = \tan^{-1}\infty = \frac{\pi}{2}$$

$$c_2 = \frac{\mathbf{u}_2^T \mathbf{r}(0)}{\sin\phi_2} = \frac{-3}{\sqrt{2}}$$

Hence, the solution $\mathbf{r}(t)$ is given by

$$\mathbf{r}(t) = 1.5\cos\sqrt{2}t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 1.5\cos 2t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

In the original coordinates this becomes $\mathbf{q}(t) = M^{-1/2}\mathbf{r}(t)$

$$\mathbf{q}(t) = 1.5\cos\sqrt{2}t \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} - 1.5\cos 2t \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

Multiplying this out yields the motion of the individual masses:

$$q_1(t) = 0.5\cos\sqrt{2}t + 0.5\cos 2t$$

$$q_2(t) = 1.5\cos\sqrt{2}t - 1.5\cos 2t$$

The two mode shapes are ($\mathbf{v}_i = M^{1/2}\mathbf{u}_i$)

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$$

If K is not symmetric, i.e., if we have a system of the form

$$M\ddot{\mathbf{q}}(t) + (K + H)\mathbf{q}(t) = \mathbf{0}$$

then proceed by solving an eigenvalue problem of the form

$$M^{-1}(K+H)\mathbf{x}=\lambda\mathbf{x}$$

or

$$A\mathbf{x}=\lambda\mathbf{x}$$

where A is not a symmetric matrix. In this case the eigenvalues λ_i and the eigenvectors $\mathbf{x}_i$ are in general complex numbers. Also, because of the asymmetry, matrix A has a *left eigenvector*, $\mathbf{y}_k$, which satisfies

$$\mathbf{y}_k^T A=\lambda_k\mathbf{y}_k^T$$

and, in general, may not equal the *right eigenvector*, $\mathbf{x}_k$. Now, let $\mathbf{y}_k$ be a left eigenvector and $\mathbf{x}_i$ be a right eigenvector. Then

$$A\mathbf{x}_i=\lambda_i\mathbf{x}_i \qquad \text{or} \qquad \mathbf{y}_k^T A\mathbf{x}_i=\lambda_i\mathbf{y}_k^T\mathbf{x}_i \tag{3.50}$$

and

$$\mathbf{y}_k^T A=\lambda_k\mathbf{y}_k^T \qquad \text{or} \qquad \mathbf{y}_k^T A\mathbf{x}_i=\lambda_k\mathbf{y}_k^T\mathbf{x}_i \tag{3.51}$$

where the ith eigenvalues of $\mathbf{x}_i$ and $\mathbf{y}_i$ are the same. Subtracting Equation (3.51) from Equation (3.50) yields $(\lambda_i-\lambda_k)\mathbf{y}_k^T\mathbf{x}_i=0$, so that, if $\lambda_i\neq\lambda_k$, then $\mathbf{y}_k^T\mathbf{x}_i=0$. This is called *biorthogonality.*

For distinct eigenvalues, the right and left eigenvectors of A each form a linearly independent set and can then be used to express any $n\times 1$ vector, i.e., an expansion theorem still exists. These relations are useful for treating gyroscopic systems, systems with constraint damping, systems with follower forces, and feedback control systems.

3.4 CANONICAL FORMS

The diagonal matrix of eigenvalues of Section 3.3 is considered a canonical, or simple, form of a symmetric matrix. This is so because of the ease of manipulation of a diagonal matrix. For instance, the square root of a diagonal matrix is just the diagonal matrix with nonzero elements equal to the square root of the diagonal elements of the original matrix.

From the point of view of vibration analysis, the diagonal form provides an immediate record of natural frequencies of vibration of systems. In addition, the similarity transformation equation [Equation (3.12)] can be used to solve the undamped vibration problem of Equation (3.33). To see this, let S be the orthogonal similarity transformation associated with the symmetric matrix $\tilde{K}=M^{-1/2}KM^{-1/2}$. Substitution of $\mathbf{r}(t)=S\mathbf{y}(t)$ into Equation (3.33) and premultiplying by S^T yields

$$\ddot{\mathbf{y}}(t)+\Lambda\mathbf{y}(t)=0 \tag{3.52}$$

where Λ is diagonal. Thus, Equation (3.52) represents n scalar equations, each of the form

$$\ddot{y}_i(t) + \omega_i^2 y_i(t) = 0, \qquad i = 1, 2, \ldots, n \tag{3.53}$$

These expressions can be integrated separately using the initial conditions $\mathbf{y}(0) = S^T\mathbf{r}(0)$ and $\dot{\mathbf{y}}(0) = S^T\dot{\mathbf{r}}(0)$ to yield a solution equivalent to Equation (3.42). This argument forms the crux of what is called *modal analysis* and is repeated many times in the following chapters.

Unfortunately, not every square matrix is similar to a diagonal matrix. However, every square matrix is similar to an upper triangular matrix. That is, let matrix A have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$; there then exists a nonsingular matrix P such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & t_{12} & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & t_{23} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & t_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{3.54}$$

The matrix $P^{-1}AP$ is said to be *upper triangular*. If the matrix is symmetric, then the t_{ij} in Equation (3.54) are all zero, and the upper triangular matrix becomes a diagonal matrix.

A classic result in the theory of matrices is known as *Jordan's theorem* and states the following. Let A be $n \times n$ with eigenvalues λ_i of multiplicities m_i, so that

$$\det(A - \lambda I) = \prod_{i=1}^{k} (\lambda_i - \lambda)^{m_i}, \qquad \text{where} \quad \sum_{i=1}^{k} m_i = n$$

Then, every matrix A is similar to a block-diagonal matrix of the form

$$J = \begin{bmatrix} \Lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \Lambda_3 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \Lambda_i \end{bmatrix} \tag{3.55}$$

where each block Λ_i is of the form

$$\Lambda_i = \begin{bmatrix} \lambda_i & a & 0 & \cdots & 0 \\ 0 & \lambda_i & a & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \lambda_i & a \\ 0 & 0 & \cdots & 0 & \lambda_i \end{bmatrix}$$

Here $a = 0$ or 1, depending on whether or not the associated eigenvectors are dependent. The value of a is determined as follows. If λ_i are distinct, then $a = 0$, always. If λ_i is repeated m_i times but has m_i linearly independent eigenvectors, then $a = 0$. If the eigenvector $\mathbf{x}_i$ is

dependent (degenerate), then $a = 1$. If the preceding matrix describes a vibration problem, the value of a determines whether or not a given system can be diagonalized. Note, then, that in general it is eigenvector 'degeneracy' that causes problems in vibration analysis – not just repeated eigenvalues.

Next, recall again that the determinant of a matrix is invariant under a similarity transformation. Expanding the determinant yields

$$\det(A - \lambda I) = (-1)^n(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

which is the characteristic polynomial and hence is equal to

$$\det(A - \lambda I) = (-1)^n(\lambda^n + c_1\lambda^{n-1} + \cdots + c_{n-1}\lambda + c_n) \tag{3.56}$$

Thus, the coefficients c_i of the characteristic polynomial must also be invariant under similarity transformations. This fact is used to some advantage.

The *trace* of a matrix A is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii} \tag{3.57}$$

That is, the trace is the sum of the diagonal entries of the matrix. Some manipulation yields

$$c_1 = -\mathrm{tr}(A) \tag{3.58}$$

and

$$\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i \tag{3.59}$$

Thus, the trace of a matrix is invariant under similarity transformations. Some additional properties of the trace are

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \tag{3.60}$$

For nonsingular matrix P

$$\mathrm{tr}(A) = \mathrm{tr}(P^{-1}AP) \tag{3.61}$$

For α and β scalars

$$\mathrm{tr}(\alpha A + \beta B) = \alpha\mathrm{tr}(A) + \beta\mathrm{tr}(B) \tag{3.62}$$

and

$$\mathrm{tr}(A) = \mathrm{tr}(A^T) \tag{3.63}$$

It is interesting to note that the $\mathrm{tr}(A)$ and $\det(A)$ can be used for a check of computational accuracy because they are invariant under similarity transformations.

3.5 LAMBDA MATRICES

Since many structures exhibit velocity-dependent forces, the ideas of Section 3.4 need to be extended to equations of the form

$$A_1\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + A_3\mathbf{q} = \mathbf{0} \tag{3.64}$$

Of course, this expression could be placed in the state-space form of Equation (2.20), and the methods of Section 3.4 can be applied. In fact, many numerical algorithms do exactly that. However, the second-order form does retain more of the physical identity of the problem and hence is worth developing.

Again, assume solutions of Equation (3.64) of the form $\mathbf{q}(t) = \mathbf{u}\,e^{\lambda t}$, where $\mathbf{u}$ is a nonzero vector of constants. Then Equation (3.64) becomes

$$(\lambda^2 A_1 + \lambda A_2 + A_3)\mathbf{u}\,e^{\lambda t} = \mathbf{0}$$

or, since $e^{\lambda t}$ is never zero,

$$(\lambda^2 A_1 + \lambda A_2 + A_3)\mathbf{u} = \mathbf{0}$$

This last expression can be written as

$$D_2(\lambda)\mathbf{u} = \mathbf{0} \tag{3.65}$$

where $D_2(\lambda)$ is referred to as a *lambda matrix* and $\mathbf{u}$ is referred to as a *latent vector*. In fact, in this case $\mathbf{u}$ is called the right latent vector (Lancaster, 1966).

Here, it is important to distinguish between the concept of eigenvalues and eigenvectors of a *matrix* [Equation (3.1)] and eigenvalues and eigenvectors of a *system* [Equation (3.65)] Lancaster (1966) has suggested referring to λ and $\mathbf{u}$ of the system as latent roots and latent vectors respectively, in order to make this distinction clear. Unfortunately, this did not catch on in the engineering literature. In order to be compatible with the literature, the distinction between eigenvectors (of a single matrix) and latent vectors (of the system) must be made from context. Equation (3.65) expresses the system eigenvectors and occasionally is referred to as a nonlinear eigenvalue problem, a matrix polynomial problem, or a lambda matrix problem.

For the existence of nonzero solutions of Equation (3.65), the matrix $D_2(\lambda)$ must be singular, so that

$$\det(D_2(\lambda)) = 0 \tag{3.66}$$

The solutions to this $2n$-degree polynomial in λ are called *latent roots*, eigenvalues, or characteristic values and contain information about the natural frequencies of the system. Note that the solution of Equation (3.66) and the solution of $\det(A - \lambda I) = 0$ are the same. Here, A is the state matrix [see Equation (2.20)] given by

$$A = \begin{bmatrix} 0 & I \\ -A_1^{-1}A_3 & -A_1^{-1}A_2 \end{bmatrix} \tag{3.67}$$

Also, the eigenvectors of A are just $[\mathbf{u}_i \quad \lambda_i \mathbf{u}_i]^T$, where $\mathbf{u}_i$ are the latent vectors of Equation (3.65) and λ_i are the solutions of Equation (3.66).

An $n \times n$ lambda matrix, $D_2(\lambda)$, is said to be *simple* if A_1^{-1} exists and if, for each eigenvalue (latent root) λ_i satisfying Equation (3.65), the rank of $D_2(\lambda_i)$ is $n - \alpha_i$, where α_i is the multiplicity of the eigenvalue λ_i. If this is not true, then $D_2(\lambda)$ is said to be *degenerate*. If each of the coefficient matrices are real and symmetric and if $D_2(\lambda)$ is simple, the solution of Equation (3.64) is given by

$$\mathbf{q}(t) = \sum_{i=1}^{2n} c_i \mathbf{u}_i \, e^{\lambda_i t} \tag{3.68}$$

Here the c_i are $2n$ constants to be determined from the initial conditions, and the $\mathbf{u}_i$ are the right eigenvectors (latent vectors) of $D_2(\lambda)$. Note that, if $A_2 = 0$, Equation (3.65) collapses to the eigenvalue problem of a matrix. The definitions of degenerate and simple still hold in this case.

Since, in general, $\mathbf{u}_i$ and λ_i are complex, the solution $\mathbf{q}(t)$ will be complex. The physical interpretation is as follows. The displacement is the real part of $\mathbf{q}(t)$, and the velocity is the real part of $\dot{\mathbf{q}}(t)$. The terms *modes* and *natural frequencies* can again be used if care is taken to interpret their meaning properly. The damped natural frequencies of the system are again related to the λ_i in the sense that, if the initial conditions $\mathbf{q}(0)$ and $\dot{\mathbf{q}}(0)$ are chosen such that $c_i = 0$ for all values of i except $i = 1$, each coordinate $q_i(t)$ will oscillate (if underdamped) at a frequency determined by λ_i. Furthermore, if the $\mathbf{u}_i$ are normalized, i.e., if $\mathbf{u}_i^* \mathbf{u}_i = 1$, then the elements of $\mathbf{u}_i$ indicate the relative displacement and phase of each mass when the system vibrates at that frequency. Here, $\mathbf{u}^*$ denotes the complex conjugate of the transpose of vector $\mathbf{u}$.

In many situations, the coefficient matrices are symmetric and the damping matrix D is chosen to be of a form that allows the solution (2.13) to be expressed as a linear combination of the normal modes, or eigenvectors, of the matrix $\tilde{K}$, which, of course, are real. In this case, the matrix of eigenvectors decouples the equations of motion. In fact the main reason for this assumption is the convenience offered by the analysis of systems that decouple. The advantage in the normal mode case is that the eigenvectors are all real valued. To this end, consider the symmetric damped system of Equation (2.13) and note the following:

1. If $D = \alpha M + \beta K$, where α and β are any real scalars, then the eigenvectors (latent vectors) of Equation (3.65) are the same as the eigenvectors of the same eigenvalue problem with $D = 0$.
2. If $D = \sum_{i=1}^{n} \beta_{i-1} K^{i-1}$, where β_i are real scalars, then the eigenvectors of Equation (2.13) are the same as the eigenvectors of the undamped system ($D = 0$).
3. The eigenvectors of Equation (2.13) are the same as those of the undamped system (with $D = 0$) if and only if $DM^{-1}K = KM^{-1}D$ (Caughey and O'Kelly, 1965).

Systems satisfying any of the above rules are said to be *proportionally damped*, to have *Rayleigh damping*, or to be *normal mode systems*. Such systems can be decoupled by the modal matrix associated with matrix $\tilde{K}$.

Of the cases just mentioned, the third is the most general and includes the other two as special cases. It is interesting to note that case 3 follows from a linear algebra theorem that

states that two symmetric matrices have the same eigenvectors if and only if they commute (Bellman, 1970), i.e., if and only if there exists a similarity transformation simultaneously diagonalizing both matrices. It is also worth noting that, in the normal mode case, the eigenvectors are real, but the reverse is not true (see the discussion of overdamping below). That is, some structures with real-valued eigenvectors are not normal mode systems because the matrix of modal vectors does not decouple the equations of motion (i.e., diagonalize the coefficient matrices). The significance of complex eigenvectors is that the elements are not in phase with each other as they are in the normal mode case. Some researchers have incorrectly stated that, if the damping is small in value, normal modes can be assumed. However, even small amounts of damping can cause condition 3 above to be violated, resulting in complex mode shapes (see, for example, Lallament and Inman, 1995).

As a generic illustration of a normal mode system, let S_m be the matrix of eigenvectors of K normalized with respect to the mass matrix M (i.e., $S_m = M^{-1/2}S$) so that

$$\begin{aligned} S_m^T M S_m &= I \\ S_m^T K S_m &= \Lambda_K = \mathrm{diag}[\omega_i^2] \end{aligned} \tag{3.69}$$

where ω_i^2 are the eigenvalues of matrix K and correspond to the square of the natural frequencies of the undamped system. If case 3 holds, then the damping is also diagonalized by the transformation S_m, so that

$$S_m^T D S_m = \mathrm{diag}[2\zeta_i\omega_i] \tag{3.70}$$

where ζ_i are called the modal damping ratios. Then, Equation (3.64) can be transformed into a diagonal system via the following. Let $\mathbf{q}(t) = S_m\mathbf{y}(t)$ in Equation (2.13) and premultiply by S_m^T to get

$$\ddot{y}_i(t) + 2\zeta_i\omega_i\dot{y}_i(t) + \omega_i^2 y_i(t) = 0, \qquad i = 1, 2, \ldots, n \tag{3.71}$$

where $y_i(t)$ denotes the ith component of vector $\mathbf{y}(t)$. Each of the n equations of system (3.71) is a scalar, which can be analyzed by the methods of Chapter 1 for single-degree-of-freedom systems. In this case the ζ_i are called *modal damping ratios* and the ω_i are the undamped *natural frequencies*, or modal frequencies.

Alternatively, the modal decoupling described in the above paragraph can be obtained by using the mass normalized stiffness matrix. To see this, substitute $\mathbf{q} = M^{-1/2}\mathbf{r}$ into Equation (2.12), multiply by $M^{-1/2}$ to form $\tilde{K} = M^{-1/2}KM^{-1/2}$, compute the normalized eigenvectors of $\tilde{K}$, and use these to form the columns of the orthogonal matrix S. Next, use the substitution $\mathbf{r} = S\mathbf{y}$ in the equation of motion, premultiply by S^T, and Equation (3.71) results. This procedure is illustrated in the following example.

Example 3.5.1

Let the coefficient matrices of Equation (2.13) have the values

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 9 & -1 \\ -1 & 1 \end{bmatrix}, \qquad K = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$$

Calculating $DM^{-1}K$ yields

$$DM^{-1}K = \begin{bmatrix} 30 & -6 \\ -6 & \dfrac{10}{3} \end{bmatrix}$$

which is symmetric and hence equal to $KM^{-1}D$, so that condition 3 is satisfied. From example 3.3.2, the eigenvectors and eigenvalues of matrix $\tilde{K}$ are as follows: $\mathbf{u}_1^T = [1 \quad 1]/\sqrt{2}$, $\lambda_1 = 2$, $\mathbf{u}_2^T = [-1 \quad 1]/\sqrt{2}$, and $\lambda_2 = 4$.

Then $S^T M^{-1/2} D M^{-1/2} S = \text{diag}[2/3 \quad 4/3]$, and $S^T M^{-1/2} K M^{-1/2} S = \text{diag}[2 \quad 4]$. Hence, Equation (2.13) with $\mathbf{f} = \boldsymbol{0}$ is equivalent to the two scalar equations given by

$$\dot{y}_1(t) + (2/3)\dot{y}_1(t) + 2y_1(t) = 0$$

and

$$\ddot{y}_2(t) + (4/3)\dot{y}_2(t) + 4y_2(t) = 0$$

each of which can easily be solved by the methods of Chapter 1. From the displacement coefficient, the frequencies are

$$\omega_1 = \sqrt{2} \text{ rad/s} \qquad \text{and} \qquad \omega_2 = \sqrt{4} = 2 \text{ rad/s}$$

and from the velocity coefficients the damping ratios are

$$\zeta_1 = \frac{2}{3}\frac{1}{2\omega_1} = \frac{1}{3\sqrt{2}} \qquad \text{and} \qquad \zeta_2 = \frac{4}{3}\frac{1}{2\omega_2} = \frac{1}{3}$$

3.6 OSCILLATION RESULTS

The definition of critical damping, overdamping, and underdamping, stated for single-degree-of-freedom systems in Chapter 1, can be extended to some of the lumped-parameter systems of this chapter. In particular, consider the symmetric positive definite system given by

$$\ddot{\mathbf{r}}(t) + \tilde{D}\dot{\mathbf{r}}(t) + \tilde{K}\mathbf{r}(t) = \mathbf{0} \tag{3.72}$$

Here, $\tilde{D} = M^{-1/2} D M^{-1/2}$, $\tilde{K} = M^{-1/2} K M^{-1/2}$, and $\mathbf{r}(t) = M^{1/2}\mathbf{q}(t)$ in Equation (2.13). In a form imitating the single-degree-of-freedom case, a critical damping matrix is defined as $\tilde{D}_{\text{cr}} = 2\tilde{K}^{1/2}$. Then, the following classifications can be derived (Inman and Andry, 1980, and Barkwell and Lancaster, 1992):

1. If $\tilde{D} = \tilde{D}_{\text{cr}}$, then Equation (3.72) is said to be a *critically damped system*, each mode of vibration is critically damped, and each eigenvalue of Equation (3.72) is a repeated negative real number. The response of such systems will not oscillate, and all the eigenvectors are real.

2. If the matrix $\tilde{D} - \tilde{D}_{\text{cr}}$ is positive definite and $\tilde{D}\tilde{K} = \tilde{K}\tilde{D}$, then Equation (3.72) is said to be an *overdamped system*, each 'mode' of the structure is overdamped, and each eigenvalue is a negative real number. The response of such systems will not oscillate, and all the eigenvectors are real.
3. If the matrix $\tilde{D}_{\text{cr}} - \tilde{D}$ is positive definite, then Equation (3.72) is said to be an *underdamped system*, each mode of vibration is underdamped, and each eigenvalue is a complex conjugate pair with a negative real part. The response of such systems oscillates with decaying amplitude and the eigenvectors are, in general, complex (unless the matrix $DM^{-1}K$ is symmetric).

A fourth possibility exists for the matrix case. That is, the matrix $\tilde{D} - \tilde{D}_{\text{cr}}$ could be indefinite. In this case, Equation (3.72) is said to exhibit *mixed damping*, and at least one mode oscillates and at least one mode does not oscillate. In addition, if A is the state matrix associated with Equation (3.72), then the system is overdamped if and only if A can be factored into the product of two positive definite Hermitian matrices (Nicholson, 1983). In order to relax the condition of normal modes in the overdamped case (case 2 above), Barkwell and Lancaster (1992) showed that Equation (3.72) has all negative real eigenvalues if $\beta_1 > 2\omega_n$, where β_1 is the smallest eigenvalue of the damping matrix $\tilde{D}$ and ω_n is the largest undamped natural frequency.

The determinant condition of Section 3.2 for the positive definiteness of a matrix can be used on the matrix $\tilde{D} - \tilde{D}_{\text{cr}}$ to provide a system of nonlinear inequalities in the physical parameters m_i, c_i, and k_i of a given structure. These inequalities can be solved for low-order systems to yield choices of m_i, c_i, and k_i that will cause the system to be overdamped or underdamped as desired. The following example illustrates the process.

Example 3.6.1

Consider the two-degree-of-freedom system of Figure 2.4, which has equations of motion given by

$$\begin{bmatrix} m_1 & 0 \\ 0 & m \end{bmatrix} \ddot{\mathbf{q}}(t) + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \dot{\mathbf{q}}(t) + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \mathbf{q}(t) = \mathbf{0} \tag{3.73}$$

where $\mathbf{q}(t) = [x_1(t) \quad x_2(t)]^T$.

To form the matrix $2\tilde{K}^{1/2}$ requires the computation of the square root of a matrix. This computational burden can be reduced by noting that Bellman (1968) has shown that, if $4\tilde{K} - \tilde{D}^2$ is positive definite, so is the matrix $2\tilde{K}^{1/2} - \tilde{D}$. Hence, it is sufficient to calculate only the square of a matrix instead of the square root of a matrix. To proceed, calculation of the square of the damping matrix in terms of the generic values of the system parameters yields the following:

$$\begin{aligned}
(4\tilde{K} - \tilde{D}^2)_{11} &= 4\frac{k_1 + k_2}{m_1} - \frac{(c_1 + c_2)^2}{m_1^2} - \frac{c_2^2}{m_1 m_2} \\
(4\tilde{K} - \tilde{D}^2)_{12} &= -\frac{4k_2}{\sqrt{m_1 m_2}} + \frac{c_1 c_2 + c_2^2}{m_1\sqrt{m_1 m_2}} + \frac{c_2^2}{m_2\sqrt{m_1 m_2}} \\
(4\tilde{K} - D^2)_{22} &= \frac{4k_2}{m_2} - \frac{c_2^2}{m_2^2} - \frac{c_2^2}{m_1 m_2}
\end{aligned} \tag{3.74}$$

Applying the determinant condition to the matrix defined by Equation (3.74) yields the inequalities

$$4\frac{k_1+k_2}{m_1} > \frac{(c_1+c_2^2)}{m_1^2} + \frac{c_2^2}{m_1 m_2} \tag{3.75}$$

$$\left[\frac{4k_2}{(m_1 m_2)^{1/2}} - \frac{c_1 c_2 + c_2^2}{(m_1^3 m_2)^{1/2}} - \frac{c_2^2}{(m_1 m_2^3)^{1/2}}\right]^2 > \frac{c_1+c_2}{m_1^2} + \frac{c_2^2}{m_1 m_2} - 4\frac{k_1+k_2}{m_1}\frac{c_2^2}{m_2^2} + \frac{c_2^2}{m_1 m_2} - 4\frac{k_2}{m_2}$$

These inequalities have many solutions. One possibility is to choose $m_1 = m_2 = 1$, $c_1 = 2$, $c_2 = 1$, $k_1 = 5$, and $k_2 = 4$. With this choice, the motion should oscillate.

To check to see that this is, in fact, the case, these values of m_i, c_i, and k_i can be substituted into Equation (3.73). The characteristic equation then becomes

$$\lambda^4 + 4\lambda^3 + 15\lambda^2 + 13\lambda + 20 = 0 \tag{3.76}$$

This has roots

$$\left.\begin{aligned}\lambda_1 &= -0.312 - 1.306j\\ \lambda_2 &= -0.312 + 1.306j\end{aligned}\right\} \Rightarrow \omega_1 = 1.343 \text{ rad/s} \quad \text{and} \quad \zeta_1 = 0.232 < 1$$

$$\left.\begin{aligned}\lambda_3 &= -1.688 - 2.870j\\ \lambda_4 &= -1.688 + 2.870j\end{aligned}\right\} \Rightarrow \omega_2 = 3.33 \text{ rad/s} \quad \text{and} \quad \zeta_2 = 0.507 < 1$$

This clearly indicates that the system oscillates as indicated by the theory. Here, the natural frequencies and damping ratios are determined from the complex eigenvalues by solving the two equations $\lambda_{1,2} = -\zeta_1\omega_1 \pm \omega_1\sqrt{1-\zeta_1^2}j$ for the two unknowns ζ_1 and ω_1. Note that the matrix elements in Equation (3.74) and the determinant in Equation (3.75) can be derived using symbolic computations in Mathcad, MATLAB, or Mathematica.

The condition of critical damping is a very special situation and is not easily obtainable. In fact, unlike single-degree-of-freedom structures, not all multiple-degree-of-freedom systems can be made critically damped by adjusting the spring, mass, and/or damping parameters. For instance, consider the example in Figure 2.4. In order for this system to be critically damped, each of the elements of matrix (3.74) must be zero. Since the matrix is symmetric, this yields the three equalities

$$\frac{(c_1+c_2)^2}{m_1^2} + \frac{c_2^2}{m_1 m_2} = 4\frac{k_1+k_2}{m_1} \tag{3.77}$$

$$\frac{c_2^2 + c_1 c_2}{m_1} + \frac{c_2^2}{m_2} = 4k_2 \tag{3.78}$$

$$\frac{c_2^2}{m_2^2} + \frac{c_2^2}{m_1 m_2} = 4\frac{k_2}{m_2} \tag{3.79}$$

Manipulation of these equations shows that all three equalities can be satisfied if and only if one of the pairs (k_1, c_1) or (k_2, c_2) is zero. This means critical damping can result only

if the system is reduced to a single degree of freedom, or perhaps by adding additional components.

If structural changes are allowed, the two-degree-of-freedom system in Figure 2.4 can be made critically damped. For example, consider adding one more dashpot, c_3, and one more spring, k_3, to the system in Figure 2.4 by attaching them from m_2 to ground. The equation of motion then becomes

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \dot{\mathbf{q}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \mathbf{q} = \mathbf{0} \tag{3.80}$$

Choosing the mass matrix to be the identity matrix, the three equalities resulting from setting $\tilde{D}^2 = 4\tilde{K}$ become

$$\begin{aligned} (c_1 + c_2)^2 + c_2^2 &= 4(k_1 + k_2) \\ c_2(c_1 + c_3 + 2c_2) &= 4k_2 \\ (c_2 + c_3)^2 + c_2^2 &= 4(k_2 + k_3) \end{aligned} \tag{3.81}$$

One solution for this system is

$$\begin{aligned} c_1 &= 4, \qquad & k_1 &= 4 \\ c_2 &= 2, \qquad & k_2 &= 6 \\ c_3 &= 4, \qquad & k_3 &= 4 \end{aligned}$$

The characteristic equation then becomes

$$\lambda^4 + 12\lambda^3 + 52\lambda^2 + 96\lambda + 64 = 0 \tag{3.82}$$

which has roots

$$\begin{aligned} \lambda_{1,2} &= -2 \Rightarrow \omega_1 = 2 \qquad \text{and} \qquad \zeta_1 = 1 \\ \lambda_{3,4} &= -4 \Rightarrow \omega_2 = 4 \qquad \text{and} \qquad \zeta_2 = 1 \end{aligned}$$

Hence, each mode is critically damped, as predicted by the theory.

The preceding methods of defining critical damping, overdamping, and underdamping are based on a 'permode' concept of critical damping. That is, a critically damped system is one in which each mode is critically damped. However, as pointed out in problem 1.5, critical damping can be viewed as the smallest value of the damping rate such that the system does not oscillate. This latter approach, taken by Beskos and Boley (1980), can be used for multiple-degree-of-freedom systems to generate critical damping surfaces in spaces defined by the damping parameters of the system. These surfaces can be calculated for two-degree-of-freedom systems of the same structure as in Figure 2.4. Such curves are computed by finding solutions for values of c_1 and c_2 that satisfy

$$\frac{d}{db}[\det(Mb^2 - Db + K)] = 0 \tag{3.83}$$

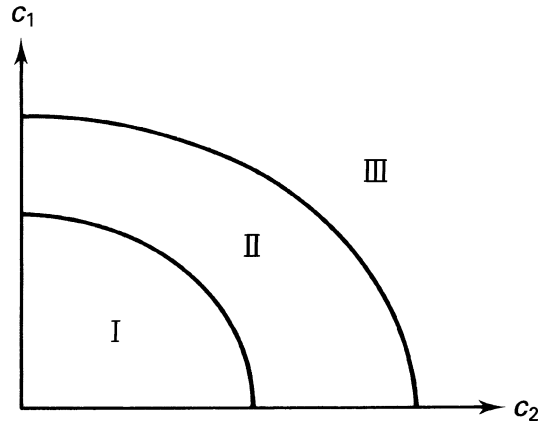

Figure 3.1 Critical damping curves for a two-degree-of-freedom system.

where b is restricted to be a positive real number and d/db indicates the derivative with respect to b. The curves are given in Figure 3.1. Systems with values of c_1 and c_2 lying in region I exhibit oscillation in both modes. In region II, one mode oscillates and one does not. In region III, neither mode oscillates. The two curves, called *critical damping curves*, are the solutions to Equation (3.83) for fixed values of m_i and k_i.

Several extensions of the preceding ideas have been developed in the literature. Papargyri-Beskou, Thessaloniki, and Beskos (2002) present the latest discussion of critical damping and examine a system with an indefinite damping matrix, followed by a comparison of the published definitions. The interest in calculating the critical damping matrix is for comparison and design, as is often the case for single-degree-of-freedom systems.

3.7 EIGENVALUE ESTIMATES

In many instances it is enough to know an approximate value, or estimate, of a particular eigenvalue or how changes in certain parameters affect the natural frequencies. Methods that require less computation than solving the characteristic equation of a given system but yield some information about the eigenvalues of the system may be useful. As an example, consider the single-degree-of-freedom spring–mass system driven by $F_0 \sin \omega t$. If, in a given design situation, one wanted to avoid resonance, it would be enough to know that the natural frequency is less than the driving frequency ω. Also, since the free response of the system is a function of the eigenvalues, estimates of eigenvalues yield some estimates of the nature of the free response of the structure and may lead to design inequalities.

One of the most basic estimates of the eigenvalues of a symmetric matrix is given by Rayleigh's principle. This principle states that, if $\lambda_{\min}$ is the smallest eigenvalue of the symmetric matrix A and $\lambda_{\max}$ is its largest, then for any nonzero vector $\mathbf{x}$

$$\lambda_{\min} < \frac{\mathbf{x}^T A \mathbf{x}}{\mathbf{x}^T \mathbf{x}} < \lambda_{\max} \tag{3.84}$$

This quotient defines what is called the *Rayleigh quotient* for matrix A; i.e., the Rayleigh quotient is defined as the scalar ratio $R(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}/\mathbf{x}^T\mathbf{x}$ (see, for instance, Huseyin (1978) for a proof).

The variational characterization of Rayleigh's quotient can also be used to characterize the other eigenvalues of A. If the minimization of the Rayleigh quotient is carried out over all vectors orthogonal to the first eigenvector, the second eigenvalue results. The ith eigenvalue is calculated by

$$\lambda_i = \min_{\substack{\mathbf{x}^T\mathbf{x}=1 \\ \mathbf{x}^T\mathbf{x}_k=0}} (\mathbf{x}^T A\mathbf{x}), \qquad k = 1, 2, \ldots, i-1 \tag{3.85}$$

which states that the ith eigenvalue is obtained by taking the minimum value of $\mathbf{x}^T A\mathbf{x}$ over all vectors $\mathbf{x}$ that satisfy $\mathbf{x}^T\mathbf{x} = 1$ and that are orthogonal to the first $(i-1)$ eigenvectors.

To apply Rayleigh's quotient to the vibration problem of a conservative system

$$M\ddot{\mathbf{q}} + K\mathbf{q} = \mathbf{0} \tag{3.86}$$

requires little manipulation. Recall that the eigenvalue problem for Equation (3.86) can be written as

$$\lambda M\mathbf{u} = K\mathbf{u}$$

or

$$R(\lambda, \mathbf{u}) = \frac{\mathbf{u}^T K\mathbf{u}}{\mathbf{u}^T M\mathbf{u}} \tag{3.87}$$

where the notation $R(\lambda, \mathbf{u})$ denotes the Rayleigh quotient. Equation (3.87) can be examined for all vectors such that $\mathbf{u}^T M\mathbf{u} = 1$. Alternatively, $R(\lambda, \mathbf{u})$ can be formed for system of equations (3.32) to yield

$$R(\lambda, \mathbf{q}) = \mathbf{q}^T M^{-1/2} K M^{-1/2}\mathbf{q} \tag{3.88}$$

which can be examined for all vectors $\mathbf{q}$ with $\|\mathbf{q}\| = \sqrt{\mathbf{q}^T\mathbf{q}} = 1$, called the *norm* of $\mathbf{q}$.

Example 3.7.1

Consider the system in Figure 2.4 with $c_1 = c_2 = 0$, $m_1 = 1$, $m_2 = 4$, $k_1 = 2$, and $k_2 = 1$. The nondimensional equation of motion is then given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{q}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{q} = 0$$

where $\mathbf{q} = [x_1 \quad x_2]^T$. Since M is diagonal,

$$M^{-1/2} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

and $R(\lambda, \mathbf{q})$ from Equation (3.88) becomes

$$R(\lambda, \mathbf{q}) = \mathbf{q}^T \begin{bmatrix} 3 & -1/2 \\ -1/2 & 1/4 \end{bmatrix} \mathbf{q}$$

If a trial vector is chosen (out of thin air and then normalized) of $\mathbf{q} = [0.243 \quad 0.970]^T$, then $R(\lambda, \mathbf{q}) = 0.176$. Since the actual value is $\lambda_1 = 0.1619$, the Rayleigh quotient appears to be a reasonable estimate.

Again, note that the Rayleigh method provides an estimate of λ_1 without having to solve for the roots of the characteristic equation. It should also be noted that the method is not as accurate as it may sometimes appear from the usual textbook examples. If the trial vector $\mathbf{q}$ is 'near' the first eigenvector, the estimate will be fairly close. If not, the estimate will not be as good. For instance, if $\mathbf{q} = [1 \quad 0]^T$ is chosen in the preceding example, then $R(\lambda, \mathbf{q}) = 3$, which is not a very good estimate of λ_1. However, while of little computational value for computing frequencies, the Rayleigh quotient is of use in analysis and design.

Several other results of interest involving eigenvalue inequalities are useful in vibration analysis. One is a method for determining the effect of truncating the degrees of freedom of a system on the eigenvalues of the system. Let the symmetric matrix A be $n \times n$ with eigenvalues, λ_i, ordered as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

and let matrix B be formed from matrix A by deleting a row and column. Hence, B is $(n-1)$ by $(n-1)$, so it will have $n-1$ eigenvalues, which are denoted by

$$\gamma_1 < \gamma_2 < \cdots < \gamma_{n-1}$$

It can be shown that these two sets of eigenvalues are *interlaced*, i.e., that

$$\lambda_1 < \gamma_1 < \lambda_2 < \gamma_2 < \lambda_3 < \cdots < \gamma_{n-1} < \lambda_n \tag{3.89}$$

This last statement shows that the natural frequencies of a system decrease as the number of degrees of freedom increase. In fact, if A_r denotes a symmetric $r \times r$ matrix and $\lambda_i(A_r)$ denotes the ith eigenvalue of matrix A_r, then

$$\lambda_i(A_{r+1}) < \lambda_i(A_r) < \lambda_{i+1}(A_{r+1}) \tag{3.90}$$

This is referred to as a *Sturmian separation theorem* (see, for instance, Bellman, 1970) and is useful in illustrating how the order of a vibration model affects the natural frequencies, such as when model reduction is used (defined in Section 6.8).

Another useful result reported by Bellman (1970) is that, if A and B are $n \times n$ symmetric matrices, then

$$\lambda_k(A+B) > \lambda_k(A), \qquad k = 1, 2, \ldots, n \tag{3.91}$$

if B is positive semidefinite, and

$$\lambda_k(A+B) > \lambda_k(A), \qquad k = 1, 2, \ldots, n \tag{3.92}$$

if B is positive definite. Here, $\lambda_k(A+B)$ refers to the kth eigenvalue of the matrix $A+B$, and so on.

The physical parameters of a system are often known only to a certain precision. For instance, mass and stiffness coefficients may be measured quite accurately for most systems, but viscous damping coefficients are very hard to measure and are not always known to a high degree of accuracy.

A symmetric matrix with error in its elements can be written as the sum

$$B = A + E_e \tag{3.93}$$

where B is a known symmetric matrix with known eigenvalues

$$\beta_1 < \beta_2 < \cdots < \beta_n$$

and A is a symmetric matrix with unknown eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

and E_e is a symmetric matrix representing the errors in matrix B. The objective is to estimate λ_i given the numbers β_i, without knowing too much about matrix E_e. It can be shown that

$$|\beta_i - \lambda_i| < \|E_e\| \tag{3.94}$$

where $\|E_e\|$ denotes the Euclidian norm of matrix E_e, defined as the square root of the sum of the squares of each element of E_e. It is easy to see that $\|E_e\| < n\varepsilon$, where n is the dimension of E_e and ε is the absolute value of the largest element in matrix E_e. Combining these two inequalities yields

$$|\beta_i - \lambda_i| < n\varepsilon \tag{3.95}$$

Inequality (3.95) can be used to measure the effects of errors in the parameters of a physical system on the eigenvalues of the system. For instance, let $\tilde{K}$ be the mass normalized stiffness matrix of the actual system associated with Equation (3.33), which is measured by some experiment. Let B denote the matrix consisting of all measured values, and let E_e be the matrix consisting of all the measured errors. Then, from expression (3.95), with $A = \tilde{K}$ and with eigenvalues ω_i^2, the inequality becomes $|\beta_i - \omega_i^2| < n\varepsilon$, or $-n\varepsilon < \omega_i^2 < \beta_i + n\varepsilon$, which in turn can be written as

$$\beta_i - n\varepsilon < \omega_i^2 < \beta_i + n\varepsilon \tag{3.96}$$

This last expression indicates how the *actual* natural frequencies, ω_i, are related to the calculated natural frequencies, $\beta_i^{1/2}$, and the measurement error, ε. Note that the assumption of symmetry will be satisfied for the matrix E_i since each element is the sum of the errors of the stiffness elements in that position so that the ijth element of E_i will contain the same measurement error as the jith element of E_i.

A fundamental theorem from linear algebra that yields simple estimates of the eigenvalues of a matrix from knowledge only of its elements is attributed to Gerschgorin (Todd, 1962). Simply stated, let a_{ij} denote the ijth element of a matrix A. Then every eigenvalue of A lies inside at least one of the circles in the complex plane centered at a_{ii} of radius

$$r_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \tag{3.97}$$

If a disc has no point in common with any other disc, it contains only one eigenvalue. The following example serves to illustrate the statement of Gerschgorin's theory for a symmetric matrix.

Example 3.7.2

Let matrix A be

$$A = \begin{bmatrix} 2.5 & -1 & 0 \\ -1 & 5 & -\sqrt{2} \\ 0 & -\sqrt{2} & 10 \end{bmatrix}$$

Then, using formula (3.97), define three circles in the plane. The first one has its center at 2.5 and a radius $r_1 = |a_{12}| + |a_{13}| = 1$, the second has its center at 5 with a radius $r_2 = |a_{21}| + |a_{23}| = (1 + \sqrt{2})$, and the third is centered at 10 with a radius of $\sqrt{2}$. The circles are illustrated in Figure 3.2. The actual eigenvalues of the system are

$$\lambda_1 = 2.1193322$$
$$\lambda_2 = 5.00$$
$$\lambda_3 = 10.380678$$

which lie inside the Gerschgorin circles, as illustrated in Figure 3.2.

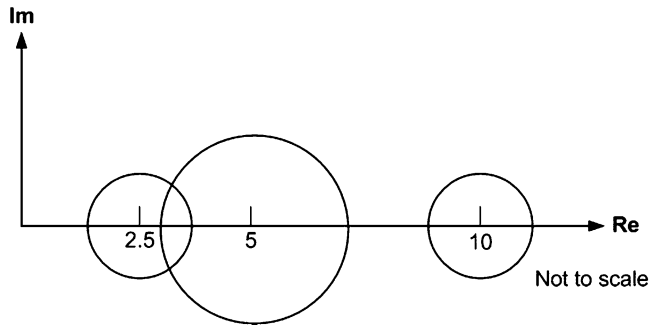

Figure 3.2 Gerschgorin circles and eigenvalues.

In the course of the development of a prototype, a system is built, analyzed, and finally tested. At that point, small adjustments are made in the design to fine-tune the system so that the prototype satisfies all the response specifications. Once these design changes are made, it may not be desirable or efficient to recalculate the eigensolution. Instead, a perturbation technique may be used to show how small changes in the elements of a matrix affect its eigensolution.

Perturbation methods are based on approximations of a function obtained by writing down a Taylor series expansion (see any introductory calculus text) for a function about some point. The equivalent statement for matrix and vector functions is more difficult to derive. However, with proper assumptions, a similar expansion can be written down for the eigenvalue problem.

In the following, let A denote an $n \times n$ symmetric matrix with distinct eigenvalues, denoted by μ_i, and refer to A as the *unperturbed matrix*. Define the matrix $A(\varepsilon)$ by $A(\varepsilon) = A + \varepsilon B$. Matrix $A(\varepsilon)$ is called the *perturbed matrix*. Note that $A(0) = A$. Furthermore, denote the eigenvalues of $A(\varepsilon)$ by $\lambda_i(\varepsilon)$ and the corresponding eigenvectors by $\mathbf{x}_i(\varepsilon)$. It is clear that, as ε approaches zero, $\lambda_i(\varepsilon)$ approaches μ_i and $\mathbf{x}_i(\varepsilon)$ approaches $\mathbf{x}_i$ for each value of index i. Here, μ_i and $\mathbf{x}_i$ are the eigenvalues and eigenvectors of A respectively (see, for instance, Lancaster, 1969). For sufficiently small ε and symmetric A and B, the expansions for $\lambda_i(\varepsilon)$ and $\mathbf{x}_i(\varepsilon)$ are

$$\lambda_i(\varepsilon) = \lambda_i + \varepsilon\lambda_i^{(1)} + \varepsilon^2\lambda_i^{(2)} + \cdots \tag{3.98}$$

and

$$\mathbf{x}_i(\varepsilon) = \mathbf{x}_i + \varepsilon\mathbf{x}_i^{(1)} + \varepsilon^2\mathbf{x}_i^{(2)} + \cdots \tag{3.99}$$

where $\mathbf{x}_i^T\mathbf{x}_i = 1$. Here, the parenthetical superscript (k) denotes the kth derivative, with respect to the parameter ε, evaluated at $\varepsilon = 0$ and multiplied by $(1/k!)$. That is,

$$\lambda_i^{(k)} = \left(\frac{1}{k!}\right)\left[\frac{d^k\lambda_i}{d\varepsilon^k}\right]_{\varepsilon=0}$$

Here, differentiation of vector $\mathbf{x}$ is defined by differentiating each element of $\mathbf{x}$.

Next, consider the ith eigenvalue problem for the perturbed matrix

$$A(\varepsilon)\mathbf{x}_i(\varepsilon) = \lambda_i(\varepsilon)\mathbf{x}_i(\varepsilon) \tag{3.100}$$

Substitution of Equations (3.98) and (3.99) into Equation (3.100) yields

$$(A + \varepsilon B)(\mathbf{x}_i + \varepsilon\mathbf{x}_i^{(1)} + \varepsilon^2\mathbf{x}_i^{(2)} + \cdots) = (\lambda_i + \varepsilon\lambda_i^{(1)} + \varepsilon^2\lambda_i^{(2)} + \cdots)(\mathbf{x}_i + \varepsilon\mathbf{x}_i^{(1)} + \cdots) \tag{3.101}$$

Multiplying out this last expression and comparing coefficients of the powers of ε yields several useful relationships. The result of comparing the coefficients of ε^0 is just the eigenvalue problem for the unperturbed system. The coefficient of ε^1, however, yields the expression

$$(\lambda_i I - A)\mathbf{x}_i^{(1)} = (B - \lambda_i^{(1)} I)\mathbf{x}_i \tag{3.102}$$

Premultiplying this by $\mathbf{x}_i^T$ (suppressing the index) results in

$$\mathbf{x}^T(B-\lambda^{(1)}I)\mathbf{x}=\mathbf{x}^T(\lambda I-A)\mathbf{x}^{(1)}=0 \tag{3.103}$$

The last term in Equation (3.103) is zero, since $\mathbf{x}^T$ is the left eigenvector of A, i.e., $\mu\mathbf{x}_i^T=\mathbf{x}_i^T A$. Hence, the first term in the perturbation of the eigenvalue (recall that $\mathbf{x}^T\mathbf{x}=1$) becomes

$$\lambda_i^{(1)}=\mathbf{x}_i^T B\mathbf{x}_i \tag{3.104}$$

Equation (3.104) indicates how the eigenvalues of a matrix, and hence the natural frequencies of an undamped system, change as the result of a small change, εB, in the matrix values. This is illustrated in example 3.7.3. The preceding formulae can be used to calculate the eigenvalues of the perturbation matrix in terms of the perturbation matrix itself and the known eigensolution of the unperturbed system defined by A. Equation (3.98) can be used to yield the eigenvalues of the 'new,' or perturbed, system by making the approximations $\lambda_i(\varepsilon)=\mu_i+\varepsilon\lambda_i^{(1)}$ and using Equation (3.104). This method is good for small values of ε.

Perturbation schemes can also be used to calculate the effect of the perturbation on the eigenvectors as well. In addition, the method can be easily used for nongyroscopic conservative systems of the forms given in Equation (3.32). It has also been used for damped systems and for systems with gyroscopic forces. Example 3.7.3 illustrates its use for systems in the form of Equation (3.33).

Example 3.7.3

This example illustrates the use of perturbation calculations to find the result of making a small perturbation to a given system [here A is perturbed to $A(\varepsilon)$]

$$M^{-1/2}KM^{-1/2}=A=\begin{bmatrix}3 & -1 & 0\\ -1 & 1 & -1\\ 0 & -1 & 5\end{bmatrix} \quad \text{and} \quad A(\varepsilon)=\begin{bmatrix}3.1 & -1.1 & 0\\ -1.1 & 1.1 & -1\\ 0 & -1 & 5\end{bmatrix}$$

Suppose the eigensolution of A is known, i.e.,

$$\begin{aligned}
\lambda_1&=0.3983\\
\lambda_2&=3.3399\\
\lambda_3&=5.2618\\
x_i&=[0.3516 \quad 0.9148 \quad 0.1988]^T\\
x_2&=[-0.9295 \quad 0.3159 \quad 0.1903]^T\\
x_3&=[0.1113 \quad -0.2517 \quad 0.9614]^T
\end{aligned}$$

Given this information, the eigensolution of the new system $A(\varepsilon)$ is desired, where

$$\varepsilon B=A(\varepsilon)-A=(0.1)\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Here, $\varepsilon = 0.1$ is small, so that the series in Equation (3.98) converges and can be truncated. Equation (3.104) yields

$$\varepsilon\lambda_1^{(1)} = \mathbf{x}_1^T \varepsilon B \mathbf{x}_1 = 0.03172$$
$$\varepsilon\lambda_2^{(1)} = \mathbf{x}_2^T \varepsilon B \mathbf{x}_2 = 0.15511$$
$$\varepsilon\lambda_3^{(1)} = \mathbf{x}_3^T \varepsilon B \mathbf{x}_3 = 0.01317$$

Then, the new (perturbed) eigenvalues are $\lambda_i(\varepsilon) = \lambda_i + \varepsilon\lambda_i^{(1)}$

$$\lambda_1(\varepsilon) = 0.43002 \quad (0.4284)$$
$$\lambda_2(\varepsilon) = 3.55410 \quad (3.4954)$$
$$\lambda_3(\varepsilon) = 5.27497 \quad (5.2762)$$

Here, the actual values are given in parentheses for comparison.

The methods presented in this section are not really needed to compute eigenvalues. Rather, the methods of the following section should be used for computing accurate eigenvalues and modal data. The eigenvalue approximations and bounds presented in this section are significant analytical tools that can be used in design and redesign to understand how changes in the system or system model affect modal data.

3.8 COMPUTATION EIGENVALUE PROBLEMS IN MATLAB

The availability of cheap, high-speed computing and the subsequent development of high-level mathematically oriented computer codes (MATLAB, Mathcad, and Mathematic in particular) almost negate the need for eigenvalue approximation methods and schemes presented in the previous section. The very nature of many computational schemes demands that the analytical formulation change. The following presents some alternative formulations to matrix-related computations based on the available codes. The details of the various algorithms used in these codes are left to the references (Meirovitch, 1980; Golub and Van Loan, 1996; Datta, 1995). Table 3.1 lists various MATLAB commands useful in computing natural frequencies, damping ratios, and mode shapes.

The best way to compute a matrix inverse is not to. Rather, Gaussian elimination can be used effectively to solve for the inverse of a matrix. The matrix inverse can be thought of as the solution to a system of n linear equations in n variables written in the matrix form $A\mathbf{x} = \mathbf{b}$. Solving this by Gaussian elimination yields the effective inverse $\mathbf{x} = A^{-1}\mathbf{b}$.

The best way to compute the eigenvalues and eigenvectors of a matrix is to use one of the many eigenvalue routines developed by the numerical linear algebra community and packaged nicely in a variety of commercial codes. These are both numerically superior to computing the roots of the polynomial derived from $\det(\lambda I - A)$ and applicable to systems of much larger order.

Table 3.1 Sample MATLAB matrix commands for solving the eigenvalue problem.

`M = [1 0; 0 4]`	creates the mass matrix of example 3.7.1
`Chol(M)`	computes the Cholesky factor of matrix M
`Sqrtm(M)`	computes the matrix square root of M
`inv(M)`	computes the inverse of matrix M
`M\I`	computes the inverse of matrix M, using Gaussian elimination
`d = eig(A)`	returns a vector $\mathbf{d}$ containing the eigenvalues of A
`[V,D] = eig(A)`	returns a matrix V of eigenvectors and a matrix D of eigenvalues
`[V,D] = eig(A,'nobalance')`	returns a matrix V of eigenvectors and a matrix D of eigenvalues without balancing
`d = eig(A,B)`	returns a vector $\mathbf{d}$ of eigenvalues, using the generalized problem $A\mathbf{x} = \lambda B\mathbf{x}$ (works for a singular B matrix)
`[V,D] = eig(A,B)`	returns a matrix D of eigenvalues and a matrix V of mode shapes, solving the generalized problem $A\mathbf{x} = \lambda B\mathbf{x}$

The matrix square root can be computed by using the function of a matrix approach, which is trivial for diagonal matrices (as is often, but not always, the case for the mass matrix). However, for nondiagonal matrices, the square root involves solving the eigenvalue problem for the matrix. This is given in Equation (3.31) and repeated here. If M is a positive definite matrix, then its eigenvalues μ_i are all positive numbers, and its eigenvectors $\mathbf{u}_i$ form an orthonormal set and can be used to form an orthogonal matrix $S = [\mathbf{u}_1 \mathbf{u}_2 \cdots \mathbf{u}_n]$ such that $S^T MS = \text{diag}(\mu_i)$. Then, any scalar function f of matrix M can be computed by

$$f(M) = S\text{diag}[f(\mu_1) \quad f(\mu_2) \ \cdots \ f(\mu_n)]S^T \tag{3.105}$$

In particular, the inverse and matrix square root of any positive definite matrix can be computed with Equation (3.105).

An alternative to the eigenvalue decomposition of Equation (3.105) is to use the Cholesky decomposition, or Cholesky factors, of a positive definite matrix. Cholesky noted that every positive definite matrix can be factored into the product of an upper triangular matrix R and its transpose: $M = R^T R$. In this case it follows that

$$(R^T)^{-1} M R^{-1} = I$$

Hence, the Cholesky factor R behaves like a square root. In fact, if M is diagonal, $R = R^T$ is the square root of M.

The most efficient way to compute the undamped eigenvalues is to use the Cholesky factors. In this case the transformations of Equations (3.33) and (3.72) become

$$\tilde{K} = (R^T)^{-1} K R^{-1} \qquad \text{and} \qquad \tilde{C} = (R^T)^{-1} C R^{-1}$$

So far, several different approaches to computing the natural frequencies and mode shapes of a conservative system have been presented. These are summarized in Table 3.2, along with a computational 'time' measured by listing the floating-point operations per second (flops) for a given example in MATLAB.

Note from Table 3.2 that using the Cholesky factor R requires the least flops to produce the eigenvalues and eigenvectors. The next 'fastest calculation' is using Gaussian elimination

Table 3.2 Comparison of the computing 'time' required to calculate eigenvalues and eigenvectors for the various methods for a conservative system.

Method	Flops
`inv (R')*K*inv (R)`	118
`M\K`	146
`inv (M)*K`	191
`inv (sqrtm(M))*K* inv (sqrtm(M))`	228
`[V,D]=eig (K,M)`	417

to compute $M^{-1}K$, but this becomes an asymmetric matrix so that the eigenvectors are not orthogonal, and hence an additional computational step is required.

The eigenvalue problem can also be placed into a number of state matrix forms, and these are now presented. The first and most common case is given by Equation (2.20). The associated eigenvalue problem for the state matrix is asymmetric and in general gives complex eigenvalues and eigenvectors. In addition, the eigenvectors of the state matrix are twice as long and related to the eigenvectors $\mathbf{u}_i$ in second-order form by

$$A\mathbf{z} = \lambda\mathbf{z}, \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \Rightarrow \mathbf{z}_i = \begin{bmatrix} \mathbf{u}_i \\ \lambda_i \mathbf{u}_i \end{bmatrix} \tag{3.106}$$

The eigenvalues, however, are exactly the same.

Other state-space approaches can be formulated by rearranging the equations of motion in state-space form. For instance, in Equation (3.64) let

$$\mathbf{y}_1 = \mathbf{q} \qquad \text{and} \qquad \mathbf{y}_2 = \dot{\mathbf{q}}$$

This then implies that

$$\dot{\mathbf{y}}_1 = \mathbf{y}_2$$

and hence

$$-K\dot{\mathbf{y}}_1 = -K\mathbf{y}_2$$

Then the equation of motion can be written as

$$M\dot{\mathbf{y}}_2 = -C\mathbf{y}_2 - K\mathbf{y}_1$$

Combining the last two expressions yields the state-space system and symmetric generalized eigenvalue problem:

$$\underbrace{\begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}}_{A} \begin{bmatrix} \dot{\mathbf{y}}_1 \\ \dot{\mathbf{y}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -K \\ -K & -C \end{bmatrix}}_{B} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \Rightarrow \lambda A\mathbf{y} = B\mathbf{y}$$

which does not require a matrix inverse.

Alternative forms of solving the eigenvalue problem can be useful for special cases, such as a nearly singular mass matrix. Such formulae can also be useful for analysis. Once the state-space eigenvalue problem is solved, the data need to be related to natural frequencies, damping ratios, and mode shapes of the physical system. This can be done in the case of an underdamped system by representing all of the eigenvalues as the complex pairs

$$\lambda_i = -\zeta_i\omega_i - \omega_i\sqrt{1-\zeta_i^2}j \qquad \text{and} \qquad \lambda_{i+1} = -\zeta_i\omega_i + \omega_i\sqrt{1-\zeta_i^2}j$$

Comparing this form with the complex form $\lambda_i = \alpha_i + \beta_i j = \mathrm{Re}(\lambda_i) + \mathrm{Im}(\lambda_i)j$ shows that the modal frequencies and damping ratios can be determined by

$$\begin{aligned} \omega_i &= \sqrt{\alpha_i^2 + \beta_i^2} = \sqrt{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2} \\ \zeta_i &= \frac{-\alpha_i}{\sqrt{\alpha_i^2 + \beta_i^2}} = \frac{-\mathrm{Re}(\lambda_i)}{\sqrt{\mathrm{Re}(\lambda_i)^2 + \mathrm{Im}(\lambda_i)^2}} \end{aligned} \tag{3.107}$$

The mode shapes are taken as the first n values of the $2n$ state vector by the relationship given in Equation (3.106). The mode shapes in this case are likely to be complex valued even if the condition for normal modes to exist is satisfied ($DM^{-1}K = KM^{-1}D$). In this case there will be a normalizing condition on $\mathbf{u}$ in Equation (3.106) that will normalize the modes to be real valued. If, however, $DM^{-1}K \neq KM^{-1}D$, then vector $\mathbf{u}$ will be complex, meaning that the masses pass through their equilibrium out of phase with each other.

3.9 NUMERICAL SIMULATION OF THE TIME RESPONSE IN MATLAB

The time response can be computed by calculating the eigenvalues and eigenvectors of the system and then forming the summation of modes as outlined in example 3.3.2. This same procedure also works for the damped case as long as the damping is proportional. However, for systems that do not have proportional damping (the nonsymmetric $KM^{-1}C$ matrix), the modal summations are overcomplex values, which can occasionally lead to confusion. In these cases, numerical simulation can be performed to compute the time response directly without computing the eigenvalues and eigenvectors. The method follows directly from the material in Section 1.10 with the state-space model of Equations (2.20) and (3.106). For any class of second-order systems, the equations of motion can be written in state-space form as given in Equation (2.20) and repeated here (for the free response case, $\mathbf{f}(t) = \mathbf{0}$):

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

where

$$\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 0 & I \\ -M^{-1}(K+H) & -M^{-1}(D+G) \end{bmatrix}$$

To solve this using numerical integration, the Runge–Kutta ode command in MATLAB is used. The ode command uses a fifth-order Runge–Kutta automated time step method for numerically integrating the equation of motion (see, for instance, Inman, 2001). The following example illustrates the procedure.

Example 3.9.1

Compute the response of the system

$$M=\begin{bmatrix}4 & 0\\ 0 & 3\end{bmatrix},\qquad D=\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix},\qquad G=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix},\qquad K=\begin{bmatrix}10 & -4\\ -4 & 4\end{bmatrix}$$

to the initial conditions

$$\mathbf{x}(0)=\begin{bmatrix}0.1\\ 0\end{bmatrix}\text{ m},\qquad \dot{\mathbf{x}}(0)=\begin{bmatrix}0\\ 0\end{bmatrix}\text{ m/s}$$

using MATLAB numerical integration.

In order numerically to integrate the equations of motion in MATLAB using Runge–Kutta, an m-file containing the system dynamics must first be created and stored (see example 1.10.2). The following file sets up the equations of motion in state-space form:

```
function v=f391(t,x)
M=[4 0; 0 3];D=[2 -1;-1 1];G=[0 1; -1 0];K=[10 -4;-4 4];
A=[zeros(2) eye(2);-inv(M)*K -inv(M)*(D+G)];
v=A*x;
```

This function must be saved under the name f391.m. Note that the command `zeros(n)` produces an $n \times n$ matrix of zeros and that the matrix `eye(n)` creates an $n \times n$ identity matrix. Once this is saved, the following is typed in the command window:

```
EDU>clear all
EDU>xo=[0.1;0;0;0];
EDU>ts=[0 40];
EDU>[t,x]=ode45('f391',ts,xo);
EDU>plot(t,x(:,1),t,x(:,2),'--')
```

This returns the plot shown in Figure 3.3. Note that the command `x(:,1)` pulls off the record for $x_1(t)$ and the command ode45 calls a fifth-order Runge–Kutta program. The command `ts=[0 40];` tells the code to integrate from 0 to 40 time units (seconds in this case).

The plot illustrated in Figure 3.3 can also be labeled and titled using additional plotting commands in MATLAB. For instance, typing `,title('displacement versus time')` after the plot command in the code in example 3.9.1 would add a title to the plot.

This numerical solution technique also still applies if the system is nonlinear. In this case the state-space formulation becomes a nonlinear vector rather than a matrix. This form was illustrated in Equations (1.66) and (1.67), and again in Section 2.7. An example of the state-space form of a nonlinear system is given in example 2.7.2.

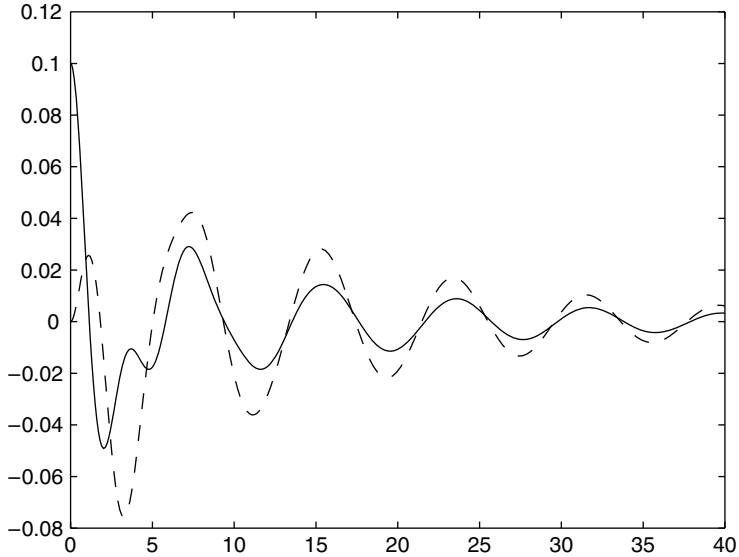

Figure 3.3 Response $q_1(t)$ versus time (solid line) and response $q_2(t)$ versus time (dashed line) as computed in MATLAB using numerical integration.

CHAPTER NOTES

The material of Section 3.2 can be found in any text concerning linear algebra or matrices, such as Lancaster (1969). An excellent quick summary of relevant matrix results is available in the first chapter of Huseyin (1978). A very good historical account and development can be found in Bellman (1960, 1970). An explanation of mode shapes and undamped natural frequencies in Section 3.3 can be found in any modern vibration text. Most linear algebra and matrix texts devote several chapters to canonical forms (Section 3.4); for instance, both Lancaster (1966) and Bellman (1970) do. The development of lambda matrices of Section 3.5 stems mostly from the book and work of Lancaster (1966), who has published extensively in that area. The idea of decoupling the equations of motion is based on the result of commuting matrices discussed in Bellman (1960) and was set straight in the engineering literature by Caughey and O'Kelly (1965). The extension of critical damping and the like to multiple-degree-of-freedom systems of Section 3.6 comes directly from Inman and Andry (1980), which contains all the references up to that date. Since then, several results have appeared that examine more efficient means of computing a critical damping matrix. Nicholson and Inman (1983) provide a review of oscillation results. Barkwell and Lancaster (1992) corrected the overdamping condition by pointing out that the result initially reported (Inman and Andry, 1980) was only a local condition. Papargyri-Beskou, Thessaloniki, and Beskos (2002) provide interesting examples and results regarding critical damping. The material of Section 3.7 follows the pattern presented in Meirovitch (1980); however, Rayleigh quotients are discussed in every vibration text and most texts on matrices – in particular, Bellman (1970) and Lancaster (1969). Bellman (1970) also treats the lacing of eigenvalues in a rigorous fashion. Gerschgorin's result is also to be found in many texts on matrices. An excellent treatment of perturbation methods can be found in Kato (1966).

The results presented in Section 3.7 on perturbation of eigenvalues are due to Lancaster (1969). Other applications of perturbation results to vibration problems are presented in Hagedorn (1983) and Meirovitch and Ryland (1979). Key papers in the development of linear systems and control using linear algebra can be found in Patel, Laub, and Van Dooren (1994). Information and sample codes for solving dynamics problems in MATLAB can be found in Soutas-Little and Inman (1999) or by simply typing 'MATLAB' into Google.

REFERENCES

Barkwell, L. and Lancaster P. (1992) Overdamped and gyroscopic vibrating systems. *Trans. ASME, Journal of Applied Mechanics*, **59**, 176–81.

Bellman, R. (1960) *Introduction to Matrix Analysis*, 1st ed, McGraw-Hill, New York.

Bellman, R. (1968) Some inequalities for the square root of a positive definite matrix. *Linear Algebra and Its Applications*, **1**, 321–4.

Bellman, R. (1970) *Introduction to Matrix Analysis*, 2nd ed, McGraw-Hill, New York.

Beskos, D.E. and Boley, B.A. (1980) Critical damping in linear discrete dynamic systems. *Trans. ASME, Journal of Applied Mechanics*, **47**, 627–30.

Caughey, T.K. and O'Kelley, M.E.J. (1965) Classical normal modes in damped linear dynamic systems. *Trans. ASME, Journal of Applied Mechanics*, **32**, 583–8.

Datta, B.N. (1995) *Numerical Linear Algebra and Applications*, Brooks/Cole, Pacific Grove, California.

Golub, G.H. and Van Loan, C.F. (1996) *Matrix Computations*, 3rd ed, Johns Hopkins University Press, Baltimore.

Hagedorn, P. (1983) The eigenvalue problem for a certain class of discrete linear systems: a perturbation approach. In Proceedings of 4th Symposium on *Dynamics and Control of Large Structures*, Blacksfurg, Virginia, 355–72.

Huseyin, K. (1978) *Vibrations and Stability of Multiple Parameter Systems*, Sijthoff Noordhoff International Publishers, Alphen aan den Rijn.

Inman, D.J. (2001) *Engineering Vibrations*, 2nd ed, Prentice-Hall, Upper Saddle River, New Jersey.

Inman, D.J. and Andry Jr, A.N. (1980) Some results on the nature of eigenvalues of discrete damped linear systems. *Trans. ASME, Journal of Applied Mechanics*, **47** (4), 927–30.

Kato, T. (1966) *Perturbation Theory for Linear Operators*, Springer-Verlag, New York.

Lallament, G. and Inman, D.J. (1995) A tutorial on complex modes. In Proceedings of 13th International Modal Analysis Conference, Society of Experimental Mechanics, Nashville, Tennessee, 490–5.

Lancaster, P. (1966) *Lambda Matrices and Vibrating Systems*, Pergamon Press, Elmsford, New York.

Lancaster, P. (1969) *Theory of Matrices*, Academic Press, New York.

Meirovitch, L. (1980) *Computational Methods in Structural Dynamics*, Sijthoff & Noordhoff International Publishers, Alphen aan den Rijn.

Meirovitch, L. and Ryland, G. (1979) Response of slightly damped gyroscopic system. *Journal of Sound and Vibration*, **46** (1), 149.

Nicholson, D.W. (1983) Overdamping of a linear mechanical systems. *Mechanics Research Communications*, **10**, 67–76.

Nicholson, D.W. and Inman, D.J. (1983) Stable response of damped linear systems. *The Shock and Vibration Digest*, **15** (II), 19–25.

Patel, R.V., Laub, A.J., and Van Dooren, P.M. (eds) (1994) *Numerical Linear Algebra Techniques for Systems and Control*, Institute of Electrical and Electronic Engineers, Inc., New York.

Papargyri-Beskou, S., Thessaloniki, S., and Beskos, D.E. (2002), On critical viscous damping determination in linear discrete dynamic systems. *Acta Mechanica*, **153**, 33–45.

Soutas-Little, R.W. and Inman, D.J. (1999) *MATLAB Supplement to Engineering Mechanics, Dynamics*, Prentice-Hall, Upper Saddle River, New Jersey.

Todd, J. (1962) *Survey of Numerical Analysis*, McGraw-Hill, New York.

PROBLEMS

3.1 Check if the four vectors given by $\mathbf{x}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$, $\mathbf{x}_2 = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T$, $\mathbf{x}_3 = \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^T$, and $\mathbf{x}_4 = \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}^T$ are independent.

3.2 Select a basis for R^3, which denotes the set of all 3×1 vectors with real elements, from the vectors $\mathbf{x}_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, $\mathbf{x}_2 = \begin{bmatrix} 2 & -1 & 1 \end{bmatrix}^T$, $\mathbf{x}_3 = \begin{bmatrix} 0 & 3 & 1 \end{bmatrix}^T$, and $\mathbf{x}_4 = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$.

3.3 Determine whether the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

is singular or not by calculating the value of its determinant.

3.4 Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & 3 & 1 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

3.5 Consider the following system:

$$\begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = 0$$

with initial conditions $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\dot{\mathbf{x}}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

(a) Calculate the eigenvalues of the system.
(b) Calculate the eigenvectors and normalize them.
(c) Use (a) and (b) to write the solution $\mathbf{x}(t)$ for the preceding initial conditions.
(d) Sketch $x_1(t)$ versus t and $x_2(t)$ versus $\mathbf{t}$.
(e) What is the solution if $\mathbf{x}(0) = [0 \quad 0]^T$ and $\dot{\mathbf{x}}(0) = [0 \quad 0]^T$?

3.6 Calculate the natural frequencies of the following system:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \mathbf{x} = 0$$

3.7 Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

and calculate its eigenvalues and eigenvectors. Are the left and right eigenvectors the same? Are they orthogonal? Are they biorthogonal?

3.8 Does the following system have normal modes (i.e., does it decouple)?

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 15 & -3 \\ -3 & 3 \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x} = 0$$

3.9 Does the system in problem 3.8 oscillate? Why or why not?

3.10 Consider the following system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \dot{\mathbf{x}} + \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \mathbf{x} = 0$$

(a) Calculate the eigenvalues of the system.
(b) Calculate the system eigenvectors and normalize them.
(c) Show that the eigenvectors can be used to diagonalize the system.
(d) Calculate the modal damping ratios and damped and undamped natural frequencies.
(e) Calculate the free response for $\mathbf{x}^T(0) = [1\ 0], \dot{\mathbf{x}}^T(0) = [0\ 0]$.
(f) Plot the responses $x_1(t)$ and $x_2(t)$ as well as $||\mathbf{x}(t)||$.

3.11 Calculate the eigenvalues for the matrix

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

what are the eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & -1 \\ -1 & 4 \end{bmatrix}?$$

Think before you calculate anything.

3.12 For the matrix in problem 3.11, calculate $\mathbf{x}^T A \mathbf{x}_1 / \mathbf{x}_1^T \mathbf{x}_1$ and $\mathbf{x}_2^T A \mathbf{x}_2 / \mathbf{x}_2^T \mathbf{x}_2$, where $\mathbf{x}_1$ and $\mathbf{x}_2$ are the eigenvectors of A. Next, choose *five* different values of vector $\mathbf{x}$ and calculate the five scalars $\mathbf{x}^T A \mathbf{x} / \mathbf{x}^T \mathbf{x}$ for your five choices. Compare all of these numbers with the values of the eigenvalues computed in problem 3.11. Can you draw any conclusions?

3.13 Consider the following model of a machine part that has equations of motion given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \mathbf{x} = 0$$

Let $k_1 = 2$ and $k_2 = 1$. The elements of M are known precisely, whereas the elements of K are known only to within 0.01 at worst. (Everything here is dimensionless.) Note that the machine will fail if it is disturbed by a driving frequency equal to one of the natural frequencies of the system. If there is a disturbance to this system of frequency $\sqrt{0.15}\, j$ ($\lambda = 0.15$) will this system fail? Why or why not? Try to work this out with a minimum of calculation.

3.14 Referring to problem 3.13, suppose that, in order to satisfy a given manufacturing change, the spring coefficient k_1 is required to change from 2 to 2.1 units. How will this affect the natural frequencies of the system? Give a quantitative answer without recalculating the eigenvalues, that is, use perturbation results.

3.15 If m_1 is neglected in problem 3.13, i.e., if the order of the model is reduced by one, by what would you expect the natural frequency of the new system to be bounded? Check your result by calculation.

3.16 Show that Gerschgorin's theory works for the matrices

$$R = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

3.17 Show that the solution of Equations (3.77) through (3.79) requires either $c_1 = k_1 = 0$ or $c_2 = k_2 = 0$.

3.18 Prove that, if A is similar to a diagonal matrix, then the eigenvectors of A form a linearly independent set.

3.19 Derive the relationship between S_m of Equation (3.20) and matrix S of Equation (3.17).

3.20 Show that the matrices $M, M^{1/2}, M^{-1}$ and M^2 all have the same eigenvectors. How are the eigenvalues related?

3.21 Prove that, if a real symmetric matrix has positive eigenvalues, then it must be positive definite.

3.22 Derive Equation (3.40). Let $n = 3$, and solve symbolically for the constants of integration.

3.23 Derive Equation (3.42) from Equation (3.39).

3.24 Let S be the matrix of eigenvectors of the symmetric matrix A. Show that $S^T AS$ is diagonal and compare it with SAS^T.

3.25 Derive the relationship between the modal matrix S of example 3.3.2 and the matrix S_m of Equation (3.21).

3.26 Use perturbation to calculate the effect on the eigenvalues of matrix A given in example 3.7.2 by making the following changes in A: change a_{11} by 0.1, a_{12} and a_{21} by 0.1, and a_{22} by 0.2.

3.27 A geometric interpretation of the eigenvector problem for a 2×2 matrix is that the eigenvectors determine the principal axis of an ellipse. Calculate matrix A for the quadratic form $2x_1^2 + 2x_1x_2 + 2x_2^2 = 3 = \mathbf{x}^T A\mathbf{x}$. Then use the eigenvector of A to determine the principal axis for the ellipse.

3.28 Show that the eigenvalues for the first-order form [Equation (2.20)] are equivalent to the latent roots of Equation (3.65) by noting that

$$\det \begin{bmatrix} A & D \\ C & B \end{bmatrix} = \det A \ \det[B - CA^{-1}D]$$

as long as A^{-1} exists, for the case where $G = H = 0$.

3.29 Show that the generic system of Equation (3.73) has normal modes if and only if

$$\frac{c_1}{c_2} = \frac{k_1}{k_2}.$$

3.30 Consider the system defined by the following coefficient matrices:

$$M = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 200 \end{bmatrix}, \qquad K = \begin{bmatrix} 2000 & -1000 & 0 \\ -1000 & 2000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix}$$

Compute the eigenvalues, eigenvectors, and natural frequencies.

3.31 Consider again the system of problem 3.30 and determine the effects of damping. Suppose a damping matrix of the form

$$C = \begin{bmatrix} 10 & -10 & 0 \\ -10 & 30 & -20 \\ 0 & -20 & 20 \end{bmatrix}$$

is added to the system of problem 3.30. Is the system overdamped, underdamped, critically damped, or does it exhibit mixed damping? Does the system have normal modes or not?

3.32 Compute the eigenvalues and eigenvectors for the system of problem 3.31. Also compute the natural frequencies and mode shapes. If you worked out problem 3.31, do your computations agree with the results obtained there?

3.33 Compute the response of the system defined in problem 3.31 to the initial displacement $\mathbf{x}(0) = \begin{bmatrix} 0.01 & 0 & 0 & -0.01 \end{bmatrix}^T$ and zero initial velocity.

3.34 Consider the system of problem 3.30 with a gyroscopic term added of the form

$$G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

Compute the eigenvalues and eigenvectors. What are the natural frequencies?

3.35 Compute the time response of the system of problem 3.34 to the initial displacement $\mathbf{x}(0) = \begin{bmatrix} 0.01 & 0 & 0 & -0.01 \end{bmatrix}^T$ and zero initial velocity.

3.36 Show that the coefficient c_i in Equation (3.42) can be written as

$$c_i = \pm \frac{1}{\omega_i} \sqrt{\mathbf{r}^T(0)\mathbf{u}_i\mathbf{u}_i^T\mathbf{r}(0) + \dot{\mathbf{r}}^T(0)\mathbf{u}_i\mathbf{u}_i^T\dot{\mathbf{r}}(0)}$$