7
Control of Vibrations

7.1 INTRODUCTION

This chapter formalizes the aspects of control theory introduced in previous chapters and applies the theory to vibration suppression of structures. This topic is usually called *structural control* and has become increasingly important, as the design of mechanisms and structures has become more precise and less tolerant of transient vibrations. Many structures, such as tall buildings, robotic manipulator arms, and flexible spacecraft, have been designed using active vibration suppression as part of the total design. Active control provides an important tool for the vibration engineer.

Control technology of linear systems is a mature discipline with many excellent texts and journals devoted to the topic. Control methods can be split into three categories: single-input, single-output frequency domain methods (classical control), state-space methods which allow multiple-input, multiple-output (MIMO) control (focused on time domain control), and modern control theory, which looks at MIMO control in the frequency domain. Like design methods, most classical control depends on being able to use low-order models of the structure (also called the plant). On the other hand, state-space control theory uses matrix theory that is compatible with the vector differential equation commonly used to describe the vibrations of structures. Hence, in this chapter, more emphasis is placed on time domain methods relying on matrix techniques. The concepts of frequency response function and other frequency domain topics common to classical control are, however, very useful in vibrations. In particular, Chapter 8, on modal testing, uses many frequency domain ideas to aid in measuring vibration properties of structures.

The words *structure* and *plant* are used interchangeably to describe the vibrating mechanical part or system of interest. The phrase *control system* refers to an actuator (or group of actuators), which is a force-generating device used to apply control forces to the structure, the sensors used to measure the response of the structure (also called the output), and the rule or algorithm that determines how the force is applied. The structure is often called the *open-loop system*, and the structure along with the control system is called the *closed-loop system*. The topic of control has been briefly introduced in Sections 1.8, 2.3, 4.10, and 6.6. The notation of these sections is summarized here as an introduction to the philosophy of active control.

Feedback control of a vibrating structure or machine requires measurements of the response (by using sensing transducers) and the application of a force to the system (by using force

Vibration with Control D. J. Inman
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transducers) on the basis of these measurements. The mathematical representation of the method of computing how the force is applied on the basis of the measurements is called the *control law*. As is often the case in modeling physical systems, there are a variety of mathematical representations of feedback control systems. Equation (2.17) represents a method of modeling the use of control forces proportional to position (denoted by $K_p\mathbf{q}$) and velocity (denoted by $K_v\dot{\mathbf{q}}$) used to shape the response of the structure. In the notation of Section 2.3, the closed-loop system is modeled by

$$M\ddot{\mathbf{q}} + (D+G)\dot{\mathbf{q}} + (K+H)\mathbf{q} = -K_p\mathbf{q} - K_v\dot{\mathbf{q}} + \mathbf{f}$$

as given in Equation (2.17) and represents *state variable feedback* (or position and velocity feedback).

Another form of feedback, called *output feedback*, is discussed in Section 4.10 and results if Equation (4.24) is substituted into Equation (4.23) to yield the closed-loop system (with $\mathbf{f} = \mathbf{0}$)

$$M\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + K\mathbf{q} = B_f\mathbf{u}$$

In this case, the control vector $\mathbf{u}$ is a function of the response coordinates of interest, denoted by the vector $\mathbf{y}$, i.e., $\mathbf{u}(t) = -G_f\mathbf{y}$. This form of control is called output feedback. The vector $\mathbf{y}$ can be any combination of state variables (i.e., position and velocities), as denoted by the output equation [Equation (4.25)], which is

$$\mathbf{y} = C_p\mathbf{q} + C_v\dot{\mathbf{q}}$$

The matrices C_p and C_v denote the locations of and the electronic gains associated with the transducers used to measure the various state variables.

Each of these two mathematical formulations can also be expressed in state-space form, as indicated in Equation (2.20) and repeated here. The difference between output feedback and state variable feedback is discussed in Section 7.2. The important point is that each of these various mathematical models is used to determine how to design a system with improved vibration performance using active feedback control, which provides an alternative to passive design methods.

Active control is most often formulated in the state space by

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

as given by Equation (2.18), with the output equation

$$\mathbf{y} = C\mathbf{x}$$

as used in the following section. The relationships between the physical coordinates (M, A_2, K, and B_f) and the state-space representation (A, B, and C) are given in Equation (2.20). Most control results are described in the state-space coordinate system. The symbols $\mathbf{y}$ and $\mathbf{u}$ in both the physical coordinate system and the state-space coordinate system are the same because they represent different mathematical models of the same physical control devices. The various relationships between the measurement or output $\mathbf{y}$ and the control input $\mathbf{u}$ determine the various types of control law, some of which are discussed in the following sections.

7.2 CONTROLLABILITY AND OBSERVABILITY

As pointed out in Section 6.6 using pole placement, it is not always possible to find a control law of a given form that causes the eigenvalues of the closed-loop system to have desired values. This inability to find a suitable control law raises the concept of *controllability*. A closed-loop system, meaning a structure and the applied control system, is said to be completely *controllable*, or state controllable, if every state variable (i.e., all positions and velocities) can be affected in such a way as to cause it to reach a particular value within a finite amount of time by some unconstrained (unbounded) control, $\mathbf{u}(t)$. If one state variable cannot be affected in this way, the system is said to be *uncontrollable*. Figures 7.1(a) and (b) illustrate two mechanical oscillators subject to the same control force $u(t)$ acting on the mass m_2. System (a) of the figure is uncontrollable because m_1 remains unaffected for any choice of $u(t)$. On the other hand, system (b) is controllable, since any nonzero choice of $u(t)$ affects both masses. Note that, if a second control force is applied to m_1 in Figure 7.1(a), then that system becomes controllable too. Hence, controllability is a function of both the system dynamics and of where and how many control actuators are applied. For instance, the system in Figure 7.1(b) is controllable with a single actuator, while that in Figure 7.1(a) requires two actuators to be controllable.

The formal definition of controllability for linear time-invariant systems is given in state space rather than in physical coordinates. In particular, consider the first-order system defined as before by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{7.1}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) \tag{7.2}$$

Recall that $\mathbf{x}(t)$ is the $2n \times 1$ state vector, $\mathbf{u}(t)$ is an $r \times 1$ input vector, $\mathbf{y}(t)$ is a $p \times 1$ output vector, A is the $2n \times 2n$ state matrix, B is a $2n \times r$ input coefficient matrix, and C is a $p \times 2n$ output coefficient matrix. The control influence matrix B is determined by the position of control devices (actuators) on the structure. The number of outputs is p, which is the same as $2s$, where s is defined in Equation (4.25) in physical coordinates as the number of sensors. In state space the state vector includes velocity and position coordinates and hence has twice the size ($p = 2s$) since the velocities can only be measured at the same locations as the position coordinates. The state, $\mathbf{x}(t)$, is said to be controllable at $t = t_0$ if there exists a piecewise continuous bounded input $\mathbf{u}(t)$ that causes the state vector to move

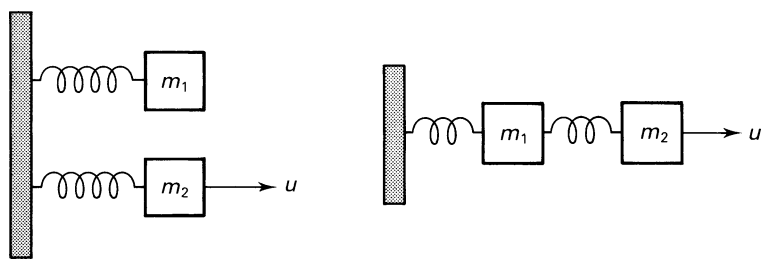

Figure 7.1 Example of (a) an uncontrollable mechanical system and (b) a controllable mechanical system.

to any final value $\mathbf{x}(t_f)$ in a finite time $t_f > t_0$. If each state $\mathbf{x}(t_0)$ is controllable, the system is said to be *completely state controllable*, which is generally what is implied when a system is said to be controllable (see, for instance, Kuo and Gholoaraghi, 2003, or Kailath, 1980).

The standard check for the controllability of a system is a rank test of a certain matrix, similar to the stability conditions used earlier for the asymptotic stability of systems with semidefinite damping. That is, the system of Equation (7.1) is completely state controllable if and only if the $2n \times 2nr$ matrix R, defined by

$$R = [B \; AB \; A^2B \cdots A^{2n-1}B] \tag{7.3}$$

has rank $2n$. In this case the pair of matrices $[A, B]$ is said to be controllable. The matrix R is called the *controllability matrix* for the matrix pair $[A, B]$.

Example 7.2.1

It is easily seen that a damped single-degree-of-freedom system is controllable. In this case, $n = 1, r = 1$. Then

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ f/m \end{bmatrix}$$

so that the controllability matrix becomes

$$R = \begin{bmatrix} 0 & f/m \\ f/m & -cf/m^2 \end{bmatrix}$$

which has rank $2 = 2n$. Thus, this system is controllable (even without damping, i.e., even if $c = 0$).

Example 7.2.2

As a second simple example, consider the state matrix of example 7.2.1 with $k/m = 1, c/m = 2$, and a control force applied in such a way as to cause $B = [1 \quad -1]^T$. Then, the controllability matrix R becomes

$$R = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

which has rank $1 \neq 2n$ and the system is not controllable. Fortunately, this choice of B is not an obvious physical choice for a control law for this system. In fact, this choice of B causes the applied control force to cancel.

A similar concept to controllability is the idea that every state variable in the system has some effect on the output of the system (response) and is called *observability*. A system is observable if examination of the response (system output) determines information about *each* of the state variables. The linear time-invariant system of Equations (7.1) and (7.2) is completely observable if, for each initial state $\mathbf{x}(t_0)$, there exists a finite time $t_f > t_0$ such that knowledge of $\mathbf{u}(t), A, B, C$, and the output $\mathbf{y}(t)$ is sufficient to determine $\mathbf{x}(t_0)$ for any

(unbounded) input $\mathbf{u}(t)$. The test for observability is very similar to that for controllability. The system described by Equations (7.1) and (7.2) is completely observable if and only if the $2np \times 2n$ matrix O defined by

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{2n-1} \end{bmatrix} \tag{7.4}$$

has rank $2n$. The matrix O is called the *observability matrix*, and the pair of matrices $[A, C]$ are said to be observable if the rank of the matrix O is $2n$. The concept of observability is also important to vibration measurement and is discussed in Chapter 8.

Example 7.2.3

Consider again the single-degree-of-freedom system of example 7.2.1. If just the position is measured, the matrix C reduces to the row vector $[1\ 0]$ and $\mathbf{y}(t)$ becomes a scalar. The observability matrix becomes

$$O = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which has rank $2(=2n)$, and this system is observable. This condition indicates that measurement of the displacement $x(t)$ (the output in this case) allows determination of both the displacement and the velocity of the system, i.e., both states are observable.

The amount of effort required to check controllability and observability can be substantially reduced by taking advantage of the physical configuration rather than using the state-space formulation of Equations (7.1) and (7.2). For example, Hughes and Skelton (1980) have examined the controllability and observability of conservative systems

$$M\ddot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{f}_f(t) \tag{7.5}$$

with observations defined by

$$\mathbf{y} = C_p\mathbf{q} + C_v\dot{\mathbf{q}} \tag{7.6}$$

Here, $\mathbf{f}_f(t) = B_f\mathbf{u}(t)$ and $\mathbf{u}(t) = -G_f\mathbf{y}(t)$ defines the input as specified in Section 4.10. In this case it is convenient to assign $G_f = I$. Then, the system of Equation (7.5) is controllable if and only if the $n \times 2n$ matrix R_n defined by

$$R_n = [\tilde{B}_f \quad \Lambda_K\tilde{B}_f \cdots \Lambda_K^{n-1}\tilde{B}_f] \tag{7.7}$$

has rank n, where $\tilde{B}_f = S_m^T B_f$ and $\Lambda_K = S_m^T K S_m$. Here, S_m is the modal matrix of Equation (7.5), and Λ_K is the diagonal matrix of eigenvalues of Equation (7.5). Thus, controllability for a conservative system can be reduced to checking the rank of a smaller-order matrix than the $2n \times 2nr$ matrix R of Equation (7.3).

This condition for controllability can be further reduced to the simple statement that system (7.5) is controllable if and only if the rank of each matrix B_q is n_q, where B_q are the partitions of the matrix $\tilde{B}_f$ according to the multiplicities of the eigenvalues of K. Here, $d < n$ denotes the number of distinct eigenvalues of the stiffness matrix, and $q = 1, 2, \ldots, d$. The integer n_q refers to the order of a given multiple eigenvalue. The matrix $\tilde{B}_f$ is partitioned into n_q rows. For example, if the first eigenvalue is repeated ($\lambda_1 = \lambda_2$), then $n_1 = 2$ and B_1 consists of the first two rows of the matrix $\tilde{B}_f$. If the stiffness matrix has distinct eigenvalues, the partitions B_q are just the rows of $\tilde{B}_f$. Thus, in particular, if the eigenvalues of K are distinct, then the system is controllable if and only if each row of $\tilde{B}_f$ has at least one nonzero entry.

For systems with repeated roots, this last result can be used to determine the minimum number of actuators required to control the response. Let d denote the number of distinct eigenvalues of K, and let n_q denote the multiplicities of the repeated roots so that $n_1 + n_2 + \cdots + n_q = n$, the number of degrees of freedom of the system, which corresponds to the partitions $\tilde{B}_q$ of $\tilde{B}_f$. Then the minimum number of actuators for the system to be controllable must be greater than or equal to the maximum of the set $\{n_1, n_2, \ldots, n_d\}$. Note that, in the case of distinct roots, this test indicates that the system could be controllable with one actuator. Similar results for general asymmetric systems can also be stated. These are discussed by Ahmadian (1985).

As in the rank conditions for stability, if the controllability or observability matrix is square, then the rank check consists of determining if the determinant is nonzero. The usual numerical question then arises concerning how to interpret the determinant having a very small value of, say, 10^{-6}. This situation raises the concept of 'degree of controllability' and 'degree of observability.' One approach to measuring the degree of controllability is to define a *controllability norm*, denoted by C_q, of

$$C_q = [\det(B_q B_q^T)]^{1/(2n_q)} \tag{7.8}$$

where q again denotes the partitioning of the control matrix $\tilde{B}_f$ according to the repeated eigenvalues of K. According to Equation (7.8), the system is controllable if and only if $C_q > 0$ for all q. In particular, the larger the value of C_q, the more controllable are the modes associated with the qth natural frequency. Unfortunately, the definition in Equation (7.8) is dependent on the choice of coordinate systems. Another more reliable measure of controllability is given later in Section 7.7, and a modal approach is given in Section 7.9.

Example 7.2.4

Consider the two-degree-of-freedom system of Figure 7.2. If k_1 is chosen to be unity and $k_2 = 0.01$, two orders of magnitude smaller, then the mass m_2 is weakly coupled to the mass m_1. Because of the increased number of forces acting on the system, a control system that acts on both m_1 and m_2 should be much more controllable than a control system acting just on the mass m_1. The following calculation, based on the controllability norm of Equation (7.8), verifies this notion. For simplicity, the masses are set at unity, i.e., $m_1 = m_2 = 1$. The equation of motion for the system of Figure 7.2 becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{\mathbf{x}} + \begin{bmatrix} 1 & -0.01 \\ -0.01 & 1.01 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

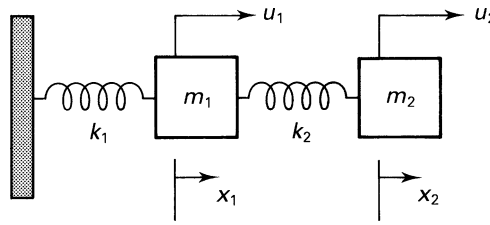

Figure 7.2 Two-degree-of-freedom structure with two control forces acting on it.

so that $\mathbf{u} = [u_1 \ u_2]^T$. In this case, $B_f = \mathbf{I}$, so that $\tilde{B}_f = S_m^T$, the transpose of the normalized modal matrix of the stiffness matrix K. Matrix K has distinct eigenvalues, so that Equation (7.8) yields $C_1 = C_2 = 1$, and both modes are controllable in agreement with the physical notion that $\mathbf{u}$ affects both x_1 and x_2. Next, consider a second control configuration with a single actuator acting on mass m_1 only. In this case, $u_2 = 0$, and B_f becomes the vector $[1 \ 0]^T$, since the vector $\mathbf{u}$ collapses to the scalar $u = u_1$. Alternatively, u could still be considered to be a vector, i.e., $\mathbf{u} = [u_1 \ 0]^T$, and B_f could then be the matrix

$$B_f = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Using either model, calculation of $\tilde{B}_f$ yields $C_1 = 0.8507$ and $C_2 = 0.5207$. Both of these numbers are smaller than 1, so the controllability measure has decreased from the two-actuator case. In addition, the second mode measure is smaller than the first mode measure ($C_2 < C_1$), so that the second mode is not as controllable as the first with the actuator placed at m_1. In addition, neither mode is as controllable as the two-actuator case. This numerical measure provides quantification of the controllability notion that, for the weakly coupled system of this example, it would be more difficult to control the response of m_2 (x_2) by applying a control force at $m_1 (C_2 = 0.5207)$. The system is still controllable, but not as easily so. Again, this is in agreement with the physical notion that pushing on m_1 will affect x_2, but not as easily as pushing on m_2 directly.

Complete controllability results if (but not only if) complete state feedback is used. Complete state feedback results if each of the state variables is used in the feedback law. In the physical coordinates of Equation (2.17), this use of *full state feedback* amounts to nonzero choices of C_p and C_v. In state space, state feedback is obtained by controls of the form

$$\mathbf{u} = -K_f \mathbf{x} \tag{7.9}$$

where K_f is a feedback gain matrix of appropriate dimension. If the control $\mathbf{u}$ is a scalar, then K_f is a row vector given by

$$\mathbf{k}_f = \left[g_1 \ g_2 \cdots g_{2n} \right] \tag{7.10}$$

and u is a scalar with the form

$$u = g_1 x_1 + g_2 x_2 + \cdots + g_{2n} \dot{x}_{2n} \tag{7.11}$$

The state equation [Equation (7.1)] becomes

$$\dot{\mathbf{x}} = (A - \mathbf{b}\mathbf{k}_f)\mathbf{x} \tag{7.12}$$

where B is reduced to a column vector $\mathbf{b}$. The product $\mathbf{b}\mathbf{k}_f$ is then a matrix (a column vector times a row vector). For complete state feedback, each gain g_i is nonzero, and the gains can be chosen such that the closed-loop system defined by the matrix $(A + \mathbf{b}\mathbf{k}_f)$ has the desired behavior (see problem 7.6).

By contrast, *output feedback* is defined for the system of Equation (7.1) by

$$\mathbf{u}(t) = -G_f\mathbf{y} \tag{7.13}$$

where G_f is the output feedback gain matrix of dimensions $r \times p$. Note that, since $\mathbf{y} = C\mathbf{x}$, where C indicates which states are measured, Equation (7.13) becomes

$$\mathbf{u}(t) = -G_f C\mathbf{x} \tag{7.14}$$

This expression appears to be similar to state feedback. The difference between state feedback and output feedback is that, unless $G_f C = K_f$, of full rank, output feedback does not use information about each state directly. On the other hand, use of the complete state variable feedback implies that a measurement of each state variable is available and is used in designing the control law. In output feedback, the output $\mathbf{y}$ is used to determine the control law, whereas in state feedback the state vector $\mathbf{x}$ is used. In general, the vector $\mathbf{y}$ is of lower order than $\mathbf{x}$. Thus, in state feedback there are more 'gains' that can be manipulated to produce the desired effect than there are in output feedback (recall example 6.6.1). Any control performance achievable by output feedback can also be achieved by complete state feedback, but the converse is not necessarily true.

In the next section, pole placement by output feedback is considered. Problem 7.6 illustrates that this task is easier with complete state feedback. Obviously, complete state feedback is the more versatile approach. However, complete state feedback requires knowledge (or measurement) of each state, which is not always possible. In addition, the hardware required to perform full state feedback is much more extensive than that required for output feedback. Section 7.5 discusses how state observers can be used to mimic state feedback when output feedback is not satisfactory for a given application, or when hardware issues do not allow for measurement of all of the states.

7.3 EIGENSTRUCTURE ASSIGNMENT

Section 6.6 points out a simple method for designing a feedback control system that causes the resulting closed-loop system to have eigenvalues (poles) specified by the designer. In this section, the concept of placing eigenvalues is improved and extended to placing mode shapes as well as natural frequencies. From an examination of the modal expansion for the forced response, it is seen that the mode shapes as well as the eigenvalues have a substantial impact on the form of the response. Hence, by placing both the eigenvalues and eigenvectors, the response of a vibrating system may be more precisely shaped.

First, the procedure of Section 6.6 is formalized into matrix form. To this end, let M_0 and K_0 define a desired mass and stiffness matrix resulting from a redesign process. Therefore, the desired system

$$\ddot{\mathbf{q}} + M_0^{-1}K_0\mathbf{q} = \mathbf{0} \tag{7.15}$$

has the *eigenstructure*, i.e., eigenvalues and eigenvectors, that are desired by design. Next, consider the closed-loop system with the existing structure (M and K) as the plant and only position feedback (so that $C_v = 0$). The system is

$$M\ddot{\mathbf{q}} + K\mathbf{q} = B_f\mathbf{u} \tag{7.16}$$

$$\mathbf{y} = C_p\mathbf{q} \tag{7.17}$$

where the various vectors and matrices have the dimensions and definitions stated for Equations (4.24) and (4.25). Recall that the constant matrix C_p represents the placement and instrument gains associated with measurement of the positions, and B_f denotes the position of the force actuators.

The class of control problems considered here uses output feedback. Output feedback uses only the output vector $\mathbf{y}$ rather than the state vector $\mathbf{x}$ in computing the gain and is defined as calculating the matrix G_f such that the control law

$$\mathbf{u}(t) = -G_f\mathbf{y}(t) \tag{7.18}$$

yields the desired response. In this case, it is desired to calculate the gain matrix G_f such that the closed-loop system has the form of Equation (7.15), which has the required eigenvalues and eigenvectors. This procedure is a form of mechanical design, as discussed in Chapter 6.

Proceeding, the closed-loop system (7.16) under output feedback becomes

$$\ddot{\mathbf{q}} + M^{-1}K\mathbf{q} = -M^{-1}B_fG_fC_p\mathbf{q}$$

or

$$\ddot{\mathbf{q}} + M^{-1}(K + B_fG_fC_p)\mathbf{q} = \mathbf{0} \tag{7.19}$$

which has the same form as Equation (4.27) without damping or velocity feedback. If Equation (7.19) is to have the same eigenstructure as the design choice given by the matrices M_0 and K_0, then comparison of Equations (7.19) and (7.15) indicates that G_f must satisfy

$$M^{-1}K + M^{-1}B_fG_fC_p = M_0^{-1}K_0 \tag{7.20}$$

Solving this expression for G_f following the rules of matrix algebra yields

$$B_fG_fC_p = M(M_0^{-1}K_0 - M^{-1}K) \tag{7.21}$$

In general, the matrices B_f and C_p are not square matrices (unless each mass has an actuator and sensor attached), so the inverses of these matrices, required to solve for G_f, do not exist. However, a *left generalized inverse*, defined by

$$B_f^l = (B_f^TB_f)^{-1}B_f^T \tag{7.22}$$

and a *right generalized inverse*, defined by

$$C_p^I = C_p^T (C_p C_p^T)^{-1} \tag{7.23}$$

can be used to 'solve' Equation (7.21). The matrices C_p^I and B_f^I are called generalized inverses in the sense that

$$B_f^I B_f = (B_f^T B_f)^{-1} B_f^T B_f = I_{r\times r} \tag{7.24}$$

and

$$C_p C_p^I = C_p C_p^T (C_p C_p^T)^{-1} = I_{s\times s} \tag{7.25}$$

where the subscripts on the identity matrices indicate their size. Note that the calculation of the generalized inverses given by Equations (7.22) and (7.23) requires that the matrices $B_f^T B_f$ and $C_p C_p^T$ both be nonsingular. Other solutions can still be found using a variety of methods (see, for instance, Golub and Van Loan, 1996). Generalized inverses are briefly discussed in Appendix B.

Premultiplying Equation (7.21) by Equation (7.22) and postmultiplying Equation (7.21) by Equation (7.23) yields a solution for the $m \times s$ gain matrix G_f to be

$$G_f = (B_f^T B_f)^{-1} B_f^T M (M_0^{-1} K_0 - M^{-1} K) C_p^T (C_p C_p^T)^{-1} \tag{7.26}$$

If this value of the gain matrix G_f is implemented, the resulting closed-loop system will have the eigenstructure and response *approximately* equal to that dictated by the design set M_0 and K_0. Note, if the system is very close to the desired system, the matrix difference $(M_0^{-1} K_0 - M^{-1} K)$ will be small and G_f will be small. However, the matrix G_f depends on where the measurements are made because it is a function of C_p and also depends on the position of the actuators because it is a function of B_f.

Example 7.3.1

Consider the two-degree-of-freedom system with original design defined by $M = I$ and

$$K = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Suppose it is desired to build a control for this system so that the resulting closed-loop system has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$ and eigenvectors given by $\mathbf{v}_1 = [1 \quad -1]^T/\sqrt{2}$ and $\mathbf{v}_2 = [1 \quad 1]^T/\sqrt{2}$, which are normalized. A system with such eigenvalues and eigenvectors can be calculated from Equation (3.21) or

$$M_0^{-1/2} K_0 M_0^{-1/2} = S \operatorname{diag}\begin{bmatrix} 2 & 4 \end{bmatrix} S^T$$

This expression can be further simplified if M_0 is taken to be the identity matrix I. Then

$$K_0 = S \operatorname{diag}\begin{bmatrix} 2 & 4 \end{bmatrix} S^T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The pair (I, K_0) then represents the desired system.

Next, based on knowledge of controllability and observability, the matrices C_p and B_f are each chosen to be identity matrices, i.e., each position is measured and each mass has a force applied to it (i.e., an actuator attached to it). This condition ensures that the system is completely controllable and observable and that the controllability and observability norms are large enough. Equation (7.26) for the gain matrix G_f becomes

$$G_f = \left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

which causes the original system with closed-loop control [i.e., Equation (7.16)] to have the desired eigenstructure. Note that the eigenvalues of K are 0.382 and 2.618 and those of $G_f + K$ are 2 and 4, as desired. In addition, the eigenvectors of $G_f + K$ are computed to be as desired:

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Although not obvious from the introductory material just presented, Wonham (1967) has shown that all the eigenvalues can be placed if and only if the system is controllable. In case more than one actuator is used, i.e., in the multi-input case, the calculated feedback gain matrix G_f is not uniquely determined by assigning just the eigenvalues (see Moore, 1976). Hence, the remaining freedom in the choice of G_f can also be used to place the mode shapes, as was the case in the preceding example. However, only mode shapes that satisfy certain criteria can be placed. These issues are discussed in detail by Andry, Shapiro, and Chung (1983), who also extended the process to damped and asymmetric systems.

7.4 OPTIMAL CONTROL

One of the most commonly used methods of modern control theory is called optimal control. Like optimal design methods, optimal control involves choosing a cost function or performance index to minimize. Although this method again raises the issue of how to choose the cost function, optimal control remains a powerful method of obtaining a desirable vibration response. Optimal control formulations also allow a more natural consideration of constraints on the state variables as well as consideration for reducing the amount of time, or final time, required for the control to bring the response to a desired level.

Consider again the control system and structural model given by Equations (4.23) and (4.24) in Section 4.10 and its state-space representation given in Equations (7.1) and (7.2). The *optimal control problem* is to calculate the control $\mathbf{u}(t)$ that minimizes some specified performance index, denoted by $J = J(\mathbf{q}, \dot{\mathbf{q}}, t, \mathbf{u})$, subject to the constraint that

$$M\ddot{\mathbf{q}} + A_2\dot{\mathbf{q}} + K\mathbf{q} = B_f\mathbf{u}$$

is satisfied, and subject to the given initial conditions $\mathbf{q}(t_0)$ and $\dot{\mathbf{q}}(t_0)$. This last expression is called a differential constraint and is usually written in state-space form. The cost function is usually stated in terms of an integral. The design process in optimal control consists of

the judicious choice of the function J. The function J must be stated in such a way as to reflect a desired performance. The best, or optimal, $\mathbf{u}$, denoted by $\mathbf{u}^*$, has the property that

$$J(\mathbf{u}^*) < J(\mathbf{u}) \tag{7.27}$$

for any other choice of $\mathbf{u}$. Solving optimal control problems extends the concepts of maximum and minimum from calculus to functionals J. Designing a vibration control system using optimal control involves deciding on the performance index J. Once J is chosen, the procedure is systematic.

Before proceeding with the details of calculating an optimal control, $\mathbf{u}^*$, several examples of common choices of the cost function J, corresponding to various design goals, will be given. The *minimum time problem* consists of defining the cost function by

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt$$

which indicates that the state equations take the system from the initial state at time t_0 [i.e., $\mathbf{x}(t_0)$] to some final state $\mathbf{x}(t_f)$ at time t_f, in a minimum amount of time.

Another common optimal control problem is called the *linear regulator problem*. This problem has specific application in vibration suppression. In particular, the design objective is to return the response (actually, all the states) from the initial state value $\mathbf{x}(t_0)$ to the system equilibrium position (which is usually $\mathbf{x}_e = \mathbf{0}$ in the case of structural vibrations). The performance index for the linear regulator problem is defined as

$$J = \frac{1}{2}\int_{t_0}^{t_f} (\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u})\, dt \tag{7.28}$$

where Q and R are symmetric positive definite *weighting matrices*. The larger the matrix Q, the more emphasis is placed by optimal control on returning the system to zero, since the value of $\mathbf{x}$ corresponding to the minimum of the quadratic form $\mathbf{x}^T Q\mathbf{x}$ is $\mathbf{x} = \mathbf{0}$. On the other hand, increasing R has the effect of reducing the amount, or magnitude, of the control effort allowed. Note that positive quadratic forms are chosen so that the functional being minimized has a clear minimum. Using both nonzero Q and R represents a compromise in the sense that, based on a physical argument, making $\mathbf{x}(t_f)$ zero requires $\mathbf{u}(t)$ to be large. The linear regulator problem is an appropriate cost function for control systems that seek to eliminate, or minimize, transient vibrations in a structure. The need to weight the control effort (R) results from the fact that no solution exists to the variational problem when constraining the control effort. That is, the problem of minimizing J with the inequality constraint $\|\mathbf{u}(t)\| < c$, where c is a constant, is not solved. Instead, R is adjusted in the cost function until the control is limited enough to satisfy $\|\mathbf{u}(t)\| < c$.

On the other hand, if the goal of the vibration design is to achieve a certain value of the state response, denoted by the state vector $\mathbf{x}_d(t)$, then an appropriate cost function would be

$$J = \int_0^{t_f} (\mathbf{x} - \mathbf{x}_d)^T Q(\mathbf{x} - \mathbf{x}_d)\, dt \tag{7.29}$$

where Q is again symmetric and positive definite. This problem is referred to as the *tracking problem*, since it forces the state vector to follow, or track, the vector $\mathbf{x}_d(t)$.

In general, the optimal control problem is difficult to solve and lends itself very few closed-form solutions. With the availability of high-speed computing, the resulting numerical solutions do not present much of a drawback. The following illustrates the problem by analyzing the linear regulator problem.

Consider the linear regulator problem for the state-space description of a structure given by Equations (7.1) and (7.2). That is, consider calculating $\mathbf{u}$ such that $J(\mathbf{u})$ given by Equation (7.28) is a minimum subject to the constraint that Equation (7.1) is satisfied. A rigorous derivation of the solution is available in most optimal control texts (see, for instance, Kirk, 1970). Proceeding less formally, assume that the form of the desired optimal control law will be

$$\mathbf{u}^*(t) = -R^{-1}B^T S(t)\mathbf{x}^*(t) \tag{7.30}$$

where $\mathbf{x}^*(t)$ is the solution of the state equation with optimal control $\mathbf{u}^*$ as input, and $S(t)$ is a symmetric time-varying $2n \times 2n$ matrix to be determined (not to be confused with the orthogonal matrix of eigenvectors S). Equation (7.30) can be viewed as a statement that the desired optimal control be in the form of state feedback. With some manipulation (see, for instance, Kirk, 1970), $S(t)$ can be shown to satisfy what is called the *matrix Riccati equation* given by

$$Q - S(t)BR^{-1}B^T S(t) + A^T S(t) + S(t)A + \frac{dS(t)}{dt} = 0 \tag{7.31}$$

subject to the final condition $S(t_f) = 0$. This calculation is a backward-in-time matrix differential equation for the unknown time-varying matrix $S(t)$. The solution for $S(t)$ in turn gives the optimal linear regulator control law (7.28), causing $J(\mathbf{u})$ to be a minimum. Unfortunately, this calculation requires the solution of $2n(2n+1)/2$ nonlinear ordinary differential equations simultaneously, backward in time (which forms a difficult numerical problem).

In most practical problems – indeed, even for very simple examples – the Riccati equation must be solved numerically for $S(t)$, which then yields the optimal control law via Equation (7.30).

The Riccati equation, and hence the optimal control problem, becomes simplified if one is interested only in controlling the steady state vibrational response and controlling the structure over a long time interval. In this case, the final time in the cost function $J(\mathbf{u})$ is set to infinity and the Riccati matrix $S(t)$ is constant for completely controllable, time-invariant systems (see, for instance, Kirk, 1970). Then, $dS(t)/dt$ is zero and the Riccati equation simplifies to

$$Q - SBR^{-1}B^T S + A^T S + SA = 0 \tag{7.32}$$

which is now a nonlinear *algebraic* equation in the constant matrix S. The effect of this method on the vibration response of a simple system is illustrated in example 7.4.1.

Example 7.4.1

This example calculates the optimal control for the infinite time linear quadratic regulator problem for a single-degree-of-freedom oscillator of the form

$$\ddot{x}(t) = 4x(t) = f(t)$$

In this case, the cost function is of the form

$$J = \int_0^\infty (\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T R\mathbf{u})\, dt$$

which is Equation (7.28) with $t_f = \infty$. A control (state feedback) is sought of the form

$$\mathbf{u} = -R^{-1}B^T S\mathbf{x}(t) = -K_f\mathbf{x}(t)$$

The state equations are

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{u}$$

Two cases are considered to illustrate the effect of the arbitrary (but positive definite) weighting matrices Q and R. The system is subject to the initial condition $\mathbf{x}(0) = [1\ 1]^T$. In the first case, let $Q = R = I$, and the optimal control is calculated from Equation (7.32) using MATLAB (Moler *et al.*, 1985) to be

$$K_f = \begin{bmatrix} 0 & 0 \\ -0.1231 & -1.1163 \end{bmatrix}$$

The resulting response and control effort are plotted in Figures 7.3 through 7.5. Figure 7.5 is the control law, $\mathbf{u}^*$, calculated by using Equation (7.30); Figures 7.3 and 7.4 illustrate the resulting position and velocity response to initial conditions under the action of the control system.

In case 2, the same problem is solved again, with the control weighting matrix set at $R = (10)I$. The result is that the new control law is given by

$$G_f = \begin{bmatrix} 0 & 0 \\ -0.0125 & -0.3535 \end{bmatrix}$$

which is 'smaller' than the first case. The resulting position response, velocity response, and control effort are plotted in Figures 7.6 through 7.8 respectively.

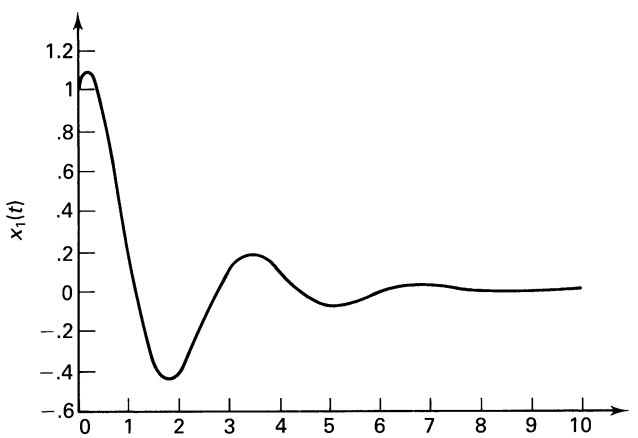

Figure 7.3 Position, $x_1(t)$, versus time for the case $Q = R = I$.

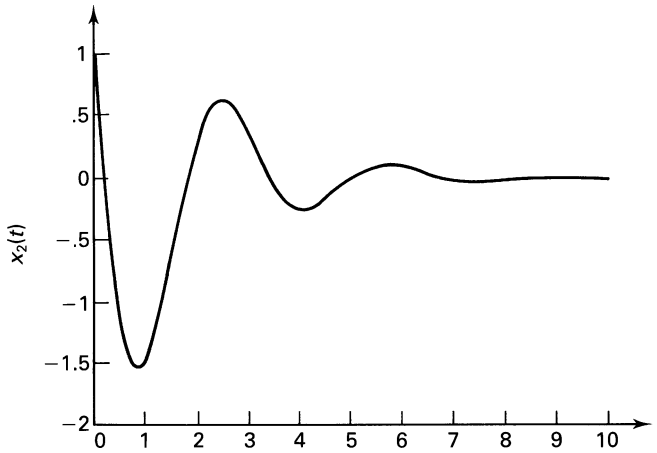

Figure 7.4 Velocity, $x_2(t)$, versus time for the case $Q = R = I$.

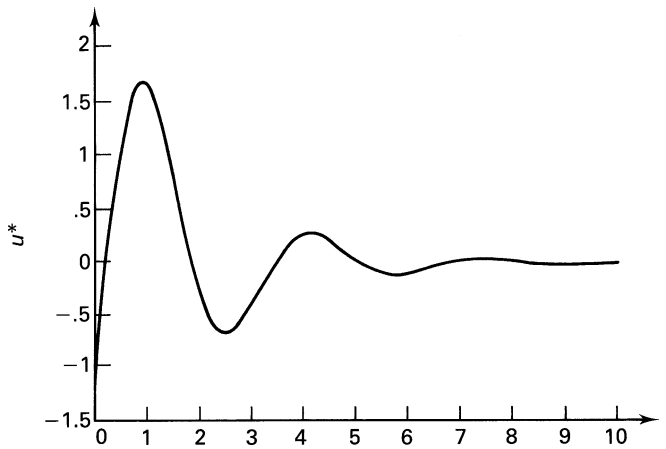

Figure 7.5 Control, $u^*(t)$, versus time for the case $Q = R = I$.

In the second case, with larger values for R, the control effort is initially much more limited, i.e., a maximum value of 0.7 units as opposed to 1.75 units for the first case. In addition, the resulting response is brought to zero faster in the case with more control effort (i.e., Figures 7.3, 7.4, and 7.6). These examples illustrate the effect that the weighting matrices have on the response. Note, a designer may have to restrict the control magnitude (hence, use large relative values of R) because the amount of control energy available for a given application is usually limited even though a better response (shorter time to settle) is obtained with larger values of control effort.

Optimal control has a stabilizing effect on the closed-loop system. Using a quadratic form for the cost function guarantees stability (bounded output) of the closed-loop system. This guaranty can be seen by considering the uncontrolled system subject to Equation (7.29)

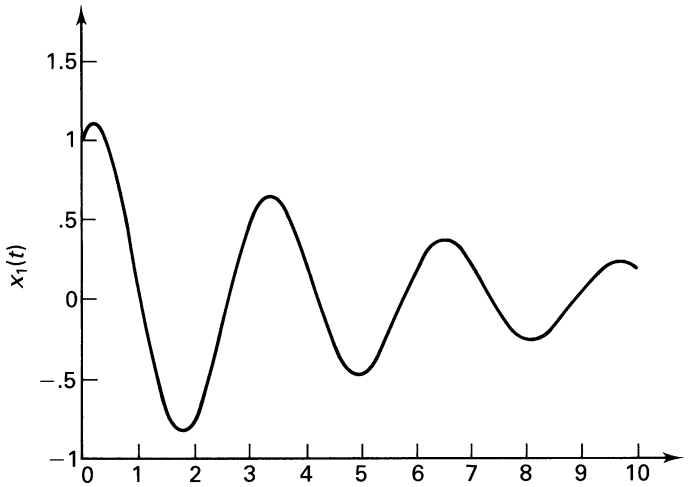

Figure 7.6 Position, $x_1(t)$, versus time for the case $Q = I, R = 10I$.

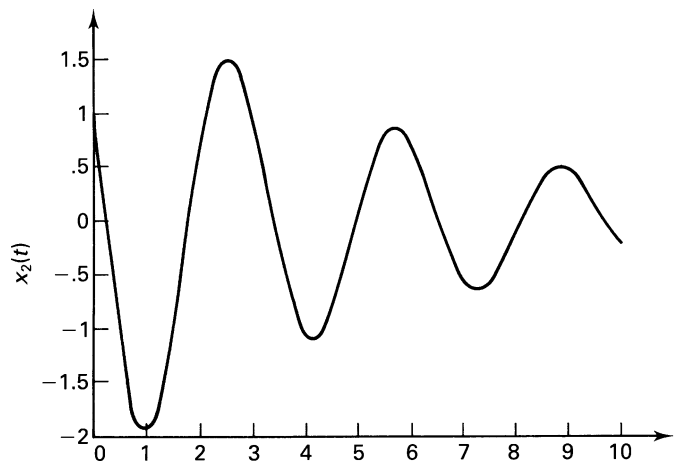

Figure 7.7 Velocity, $x_2(t)$, versus time for the case $Q = I, R = 10I$.

with $\mathbf{x}_d$ taken as the origin. Asymptotic stability requires that $\mathbf{x}$ approaches zero as time approaches infinity so that the integral

$$J = \int_0^\infty \mathbf{x}^T Q \mathbf{x}\, dt$$

converges as long as Q is positive semidefinite. Define the quadratic from $V(t) = \mathbf{x}^T P \mathbf{x}$, where P is the positive definite solution of the Lyapunov equation [Equation (4.29)]

$$PA + A^T P = -Q$$

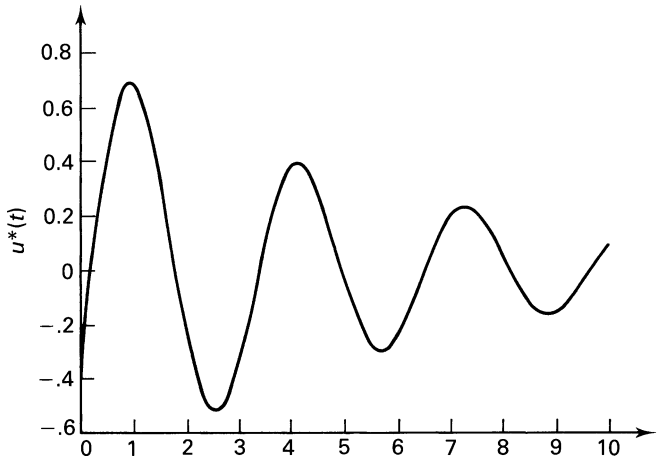

Figure 7.8 Control, $u^*(t)$, versus time for the case $Q = I, R = 10I$.

Here, A is the state matrix satisfying Equation (7.1) with $B = 0$. Computing the time derivative of $V(t)$ yields

$$\dot{V}(t) = \frac{d}{dt}(\mathbf{x}^T P\mathbf{x}) = \dot{\mathbf{x}}^T P\mathbf{x} + \mathbf{x}^T P\dot{\mathbf{x}} = \mathbf{x}^T(A^T P + PA)\mathbf{x} = -\mathbf{x}^T Q\mathbf{x} < 0$$

Hence, $V(t)$ is positive definite with a negative definite time derivative and is thus a Lyapunov function of the system of Equation (7.1). As a result, the homogeneous system is asymptotically stable (recall Section 4.10). Since the homogeneous system is asymptotically stable, the closed-loop system (forced response) will be bounded-input, bounded-output stable, as discussed in Section 5.5. Also, note that, if $\mathbf{x}_0$ is the initial state, integrating the cost function yields

$$J = \int_0^\infty \mathbf{x}^T Q\mathbf{x}\,\mathrm{d}t = \int_0^\infty \frac{d}{dt}(-\mathbf{x}^T P\mathbf{x})\,dt = \mathbf{x}_0^T P\mathbf{x}_0$$

since asymptotic stability requires that $\mathbf{x}$ approach zero at the upper limit. This calculation indicates that the value of the cost function depends on the initial conditions.

This section is not intended to provide the reader with a working knowledge of optimal control methods. It is intended to illustrate the use of optimal control as an alternative vibration suppression technique and to encourage the reader to pursue the use of optimal control through one of the references.

7.5 OBSERVERS (ESTIMATORS)

In designing controllers for vibration suppression, often not all of the velocities and displacements can be conveniently measured. However, if the structure is known and is observable, i.e., if the state matrix A is known and if the measurement matrix C is such that O has full rank, one can design a subsystem, called an *observer* (or *estimator*, in the stochastic

case), from measurements of the input and of the response of the system. The observer then provides an approximation of the missing measurements.

Consider the state-space system given by Equations (7.1) and (7.2) with output feedback as defined by Equation (7.13). To simplify this discussion, consider the case where the control $u(t)$ and the output $y(t)$ are both scalars. This is the *single-input, single-output* (SISO) case. In this situation the matrix B becomes a column vector, denoted by $\mathbf{b}$, and the matrix C becomes a row vector, denoted by $\mathbf{c}^T$.

The output $y(t)$ is proportional to a state variable or linear combination of state variables. Sometimes recovering the state vector from the measurement of $y(t)$ is trivial. For instance, if each state is measured (multiple output), C is square and nonsingular, so that $\mathbf{x} = C^{-1}\mathbf{y}$. In this section it is assumed that the state vector is not directly measured and is not easily determined from the scalar measurement $y(t)$. However, since the quantities $A, \mathbf{b}$, and $\mathbf{c}^T$, as well as measurements of the input $u(t)$ and the output $y(t)$, are known, the desired state vector can be *estimated*. Constructing this estimated state vector, denoted by $\mathbf{x}_e$, is the topic of this section. State observers can be constructed if the system is completely observable (see, for instance, Chen, 1998).

The simplest observer to implement is the *open-loop estimator*. An open-loop estimator is simply the solution of the state equations with the same initial conditions as the system under consideration. Let $\mathbf{x}_e$ denote the estimated state vector. The integration of

$$\dot{\mathbf{x}}_e = A\mathbf{x}_e + \mathbf{b}u \tag{7.33}$$

yields the desired estimated state vector. The estimated state vector can then be used to perform state feedback or output feedback. Note that integration of Equation (7.33) requires knowledge of the initial condition, $\mathbf{x}_e(0)$, which is not always available.

Unfortunately, the open-loop estimator does not work well if the original system is unstable (or almost unstable) or if the initial conditions of the unmeasured states are not known precisely. In most situations, the initial state is not known. A better observer can be obtained by taking advantage of the output of the system, $y(t)$, as well as the input. For instance, consider using the difference between the output $y(t)$ of the actual system and the output $y_e(t)$ of the estimator as a correction term in the observer of Equation (7.33). The observer then becomes

$$\dot{\mathbf{x}}_e = A\mathbf{x}_e + \mathbf{r}_e(y - y_e) + \mathbf{b}u \tag{7.34}$$

where $\mathbf{r}_e$ is the gain vector of the observer and is yet to be determined. Equation (7.34) is called an *asymptotic state estimator*, which is designed by choosing the gain vector $\mathbf{r}_e$. The error between the actual state vector $\mathbf{x}$ and the estimated state vector $\mathbf{x}_e$ must satisfy the difference between Equation (7.34) and Equation (7.1). This difference yields

$$(\dot{\mathbf{x}} - \dot{\mathbf{x}}_e) = A(\mathbf{x} - \mathbf{x}_e) + \mathbf{r}_e(y_e - y) \tag{7.35}$$

Since $y_e - y = \mathbf{c}^T(\mathbf{x}_e - \mathbf{x})$, this last expression becomes

$$\dot{\boldsymbol{e}} = (A - \mathbf{r}_e\mathbf{c}^T)\mathbf{e} \tag{7.36}$$

where the error vector $\mathbf{e}$ is defined to be the difference vector $\mathbf{e} = \mathbf{x} - \mathbf{x}_e$. This expression is the dynamic equation that determines the error between the actual state vector and the

estimated state vector. Equation (7.36) describes how the difference between the actual initial condition $\mathbf{x}(0)$ and the assumed condition $\mathbf{x}_e(0)$ evolves in time.

The idea here is to design the observer, i.e., to choose $\mathbf{r}_e$, so that the solution of Equation (7.36) remains as close to zero as possible. For instance, if the eigenvalues of $(A - \mathbf{r}_e\mathbf{c}^T)$ are chosen to have negative real parts that are large enough in absolute value, the vector $\mathbf{e}$ goes to zero quickly, and $\mathbf{x}_e$ approaches the real state $\mathbf{x}$. Thus, any difference between the actual initial conditions and the assumed initial conditions for the observer dies out with time instead of increasing with time, as could be the case with the open-loop observer.

Obviously, there is some difference between using the actual state vector $\mathbf{x}$ and the estimated state vector $\mathbf{x}_e$ in calculating a control law. This difference usually shows up as increased control effort; that is, a feedback control based on $\mathbf{x}_e$ has to exert more energy than one based on the actual state variables $\mathbf{x}$. However, it can be shown that, as far as placing eigenvalues are concerned, there is no difference in state feedback between using the actual state and the estimated state (Chen, 1970). Furthermore, the design of the observer and the control can be shown to be equivalent to performing the separate design of a control, assuming that the exact states are available, and the subsequent design of the observer (called the *separation theorem*).

To solve the control problem with state estimation, the state equation with the estimated state vector used as feedback ($u = gy = g\mathbf{c}^T\mathbf{x}_e$, recalling that $\mathbf{c}^T$ is a row vector) must be solved simultaneously with the state estimation equation [Equation (7.36)]. This solution can be achieved by rewriting the state equation as

$$\dot{\mathbf{x}} = A\mathbf{x} + g\mathbf{b}\mathbf{c}^T\mathbf{x}_e \tag{7.37}$$

and substituting the value for $\mathbf{x}_e$. Then

$$\dot{\mathbf{x}} = A\mathbf{x} + g\mathbf{b}\mathbf{c}^T(\mathbf{x} - \mathbf{e}) \tag{7.38}$$

or upon rearranging

$$\dot{\mathbf{x}} = (A + g\mathbf{b}\mathbf{c}^T)\mathbf{x} - g\mathbf{b}\mathbf{c}^T\mathbf{e} \tag{7.39}$$

Combining Equations (7.36) and (7.39) yields

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} A + g\mathbf{b}\mathbf{c}^T & -g\mathbf{b}\mathbf{c}^T \\ 0 & A - \mathbf{r}_e\mathbf{c}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{7.40}$$

Here, the zero in the state matrix is a $2n \times 2n$ matrix of zeros. Expression (7.40) is subject to the actual initial conditions of the original state equation augmented by the assumed initial conditions of the estimator. These estimator initial conditions are usually set at zero, so that $[\mathbf{x}^T(0) \quad \mathbf{e}^T(0)]^T = [\mathbf{x}^T(0) \quad \mathbf{0}]^T$. Solution of Equation (7.40) yields the solution to the state feedback problem with the states estimated, rather than directly measured.

The following example illustrates the computation of a state observer as well as the difference between using a state observer and using the actual state in a feedback control problem.

Example 7.5.1

Consider a single-degree-of-freedom oscillator ($\omega_n^2 = 4$, $\zeta = 0.25$) with displacement as the measured output. The state-space formulation for a single-input, single-output system is

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \mathbf{x} + \mathbf{b}u, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{7.41}$$

$$y = [1 \quad 0]\,\mathbf{x}$$

If output feedback is used, then $u = gy = g\mathbf{c}^T\mathbf{x}$, and the system becomes

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \mathbf{x} + g \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mathbf{x} \tag{7.42}$$

Combining yields

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ g-4 & -1 \end{bmatrix} \mathbf{x} \tag{7.43}$$

The asymptotic estimator in this case is given by

$$\dot{\mathbf{x}}_e = A\mathbf{x}_e + \mathbf{r}_e(y - \mathbf{c}^T\mathbf{x}_e) + \mathbf{b}u \tag{7.44}$$

where $\mathbf{r}_e$ is chosen to cause the eigenvalues of the matrix $(A - \mathbf{r}_e\mathbf{c}^T)$ to have negative real parts that are large in absolute value. As mentioned previously, in this case $\mathbf{c}^T = [1\ 0]$ is a row vector, so that the product $\mathbf{r}_e\mathbf{c}^T$ is a matrix. Here, the eigenvalues of $(A - \mathbf{r}_e\mathbf{c}^T)$ are chosen to be -6 and -5 (chosen only because they cause the solution of Equation (7.36) to die out quickly; other values could be chosen). These equations are equivalent to requiring the characteristic equation of $(A - \mathbf{r}_e\mathbf{c}^T)$ to be $(\lambda + 6)(\lambda + 5)$, i.e.,

$$\begin{aligned} \det(\lambda I - A + \mathbf{r}_e\mathbf{c}^T) &= \lambda^2 + (r_1 + 1)\lambda + (r_1 + r_2 + 4) \\ &= \lambda^2 + 11\lambda + 30 \end{aligned} \tag{7.45}$$

Equating coefficients of λ in this expression yields the desired values for $\mathbf{r}_e = [r_1\ r_2]^T$ to be $r_1 = 10$ and $r_2 = 16$. Thus, the estimated state is taken as the solution of

$$\dot{\mathbf{x}}_e = \begin{bmatrix} 0 & 1 \\ -4 & 1 \end{bmatrix} \mathbf{x}_e + \begin{bmatrix} 10 \\ 16 \end{bmatrix} (y - \mathbf{c}^T\mathbf{x}_e) + \mathbf{b}u \tag{7.46}$$

The solution $\mathbf{x}_e$ can now be used as feedback in the original control problem coupled with the state estimation equation. This solution yields [from Equation (7.40) with the control taken as $g = -1$]

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -1 & 1 & 0 \\ 0 & 0 & -10 & 1 \\ 0 & 0 & -20 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{7.47}$$

The plots in Figure 7.9 show a comparison between using the estimated state and the same control with the actual state used. In the figure, the control is fixed to be $g = -1$ and the actual initial conditions are taken to be $\mathbf{x}(0) = [1\ 1]^T$. The response for various different initial conditions for the observer $\mathbf{x}_e(0)$ are also plotted.

Note that in both cases the error in the initial conditions of the estimator disappears by about 1.2 time units, lasting just a little longer in Figure 7.10 than in Figure 7.9. This results because the assumed initial conditions for the observer are farther away from the actual initial condition in Figure 7.10 than in Figure 7.9. Comparison of the actual response in Figure 7.9 and Figure 7.10 with that of Figure 7.11 shows that the control law calculated using estimated state feedback is only slightly worse (takes slightly longer to decay) than those calculated using actual state feedback for this example.

In the preceding presentation of state observers, the scenario is that the state vector $\mathbf{x}$ is not available for use in output feedback control. Thus, estimated output feedback control is used instead, i.e., $u(t) = g\mathbf{c}^T\mathbf{x}_e$. A practical alternative use of an observer is to use the estimated state vector $\mathbf{x}_e$ to change a problem that is output feedback control ($g\mathbf{c}^T\mathbf{x}$) because

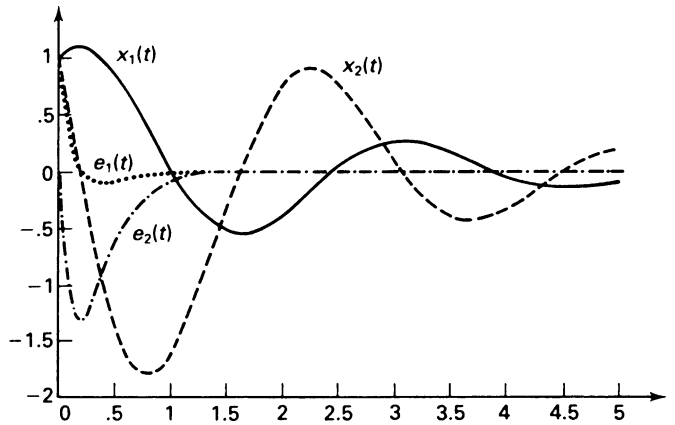

Figure 7.9 Comparison of the error vector $\mathbf{e}(t)$ versus time and the components of the state vector $\mathbf{x}(t)$ versus time for the initial condition $\mathbf{e}(0) = [1 \quad 0]^T$.

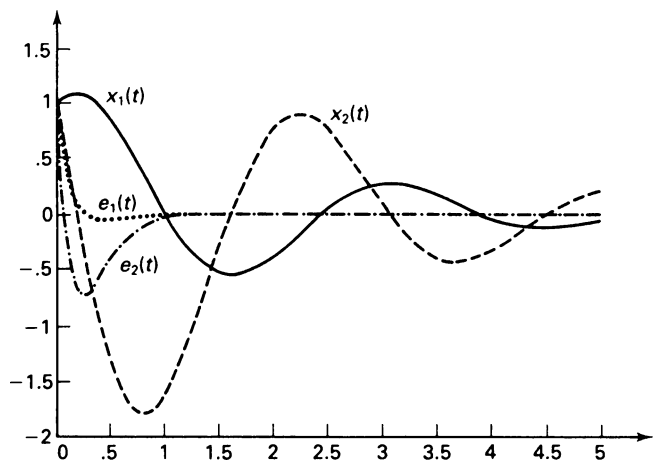

Figure 7.10 Components of the error vector $\mathbf{e}(t)$ versus time and the components of the state vector $\mathbf{x}(t)$ versus time for the initial condition $\mathbf{e}(0) = [1 \quad 1]^T$.

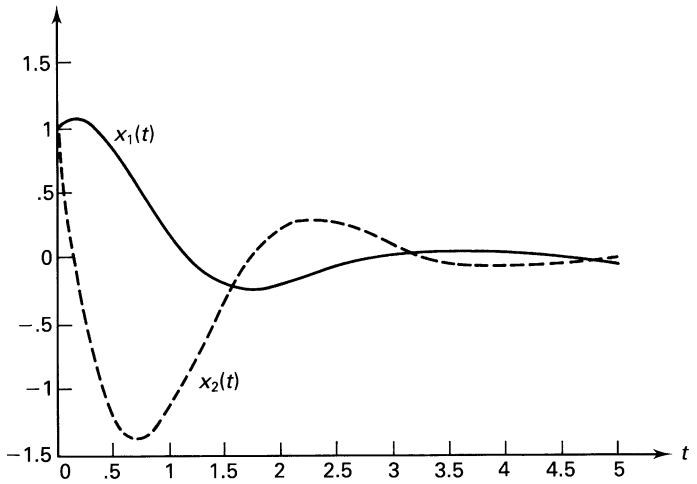

Figure 7.11 Components of the state vector $\mathbf{x}(t)$ versus time for the case of complete state feedback.

of hardware limitations to one that is augmented by the observer to use complete state variable feedback, i.e., $u(t) = gc^T\mathbf{x} + \mathbf{k}^T\mathbf{x}_e$. Here, the vector $\mathbf{k}$ makes use of each state variable, as opposed to $\boldsymbol{c}$, which uses only some state variables.

If multiple inputs and outputs are used, this analysis can be extended. The resulting observer is usually called a *Luenberger observer*. If, in addition, a noise signal is added as input to Equation (7.1), the estimation equation can still be developed. In this case they are referred to as *Kalman filters* (see, for instance, Anderson and Moore, 1979).

7.6 REALIZATION

In the preceding section, the state matrix A (and hence the coefficient matrices M, D, and K) and also the input matrix B and the output matrix C are all assumed to be known. In this section, however, the problem is to determine A, B, and C from the transfer function of the system. This problem was first introduced in Section 1.6, called plant identification, where the scalar coefficients m, c, and k were determined from Bode plots. Determining the matrices A, B, and C from the transfer function of a system is called *system realization*.

Consider again the SISO version of Equations (7.1) and (7.2). Assuming that the initial conditions are all zero, taking the Laplace transform of these two expressions yields

$$s\mathbf{X}(s) = A\mathbf{X}(s) + \mathbf{b}U(s) \tag{7.48}$$

$$y(s) = \mathbf{c}^T\mathbf{X}(s) \tag{7.49}$$

Solving Equation (7.48) for $\mathbf{X}(s)$ and substituting the result into Equation (7.49) yields

$$Y(s) = \mathbf{c}^T(sI - A)^{-1}\mathbf{b}U(s) \tag{7.50}$$

Since y and u are scalars here, the transfer function of the system is just

$$\frac{Y(s)}{U(s)} = G(s) = \mathbf{c}^T(sI - A)^{-1}\mathbf{b} \tag{7.51}$$

Here and in the following, $G(s)$ is assumed to be a rational function of s. If, in addition, $G(s)$ is such that

$$\lim_{s\to\infty} G(s) = 0 \tag{7.52}$$

then $G(s)$ is called a *proper rational function*. These two conditions are always satisfied for the physical models presented in Chapter 2. The triple A, $\mathbf{b}$, and $\mathbf{c}$ is called a *realization* of $G(s)$ if Equation (7.51) holds. The function $G(s)$ can be shown to have a realization if and only if $G(s)$ is a proper rational function. The triple $(A, \mathbf{b}, \mathbf{c})$ of minimum order that satisfies Equation (7.51) is called an *irreducible realization*, or a *minimal realization*. The triple $(A, \mathbf{b}, \mathbf{c})$ can be shown to be an irreducible realization if and only if it is both controllable and observable (see, for instance, Chen, 1998).

A transfer function $G(s)$ is said to be irreducible if and only if the numerator and denominator of G have no common factor. This statement is true if and only if the denominator of $G(s)$ is equal to the characteristic polynomial of the matrix A, and if and only if the degree of the denominator of $G(s)$ is equal to $2n$, the order of the system. While straightforward conditions are available for ensuring the existence of an irreducible realization, the realization is not unique. In fact, if $(A, \mathbf{b}, \mathbf{c})$ is an irreducible realization of the transfer function $G(s)$, then $(A', \mathbf{b}', \mathbf{c}')$ is an irreducible realization if and only if A is similar to A', i.e., there exits a nonsingular matrix P such that $A = PA'P^{-1}$, $\mathbf{b} = P\mathbf{b}'$, and $\mathbf{c} = \mathbf{c}'P^{-1}$. Hence, a given transfer function has an infinite number of realizations. This result is very important to remember when studying modal testing in Chapter 8.

There are several ways to calculate a realization of a given transfer function. The easiest method is to recall from differential equations the method of writing a $2n$-order differential equation as $2n$ first-order equations. Then, if the transfer function is given as

$$\frac{y(s)}{u(s)} = G(s) = \frac{\beta}{s^{2n} + \alpha_1 s^{2n-1} + \alpha_2 s^{2n-2} + \cdots + \alpha_{2n-1}s + \alpha_{2n}} \tag{7.53}$$

the time domain equivalent is simply obtained by multiplying this out to yield

$$s^{2n}y(s) + \alpha_1 s^{2n-1}y(s) + \cdots = \beta u(s) \tag{7.54}$$

and taking the inverse Laplace transform of Equation (7.54) to obtain

$$y^{2n} + \alpha_1 y^{(2n-1)} + \cdots + \alpha_{2n}y(t) = \beta u(t) \tag{7.55}$$

Here, $y^{(2n)}$ denotes the $2n$th time derivative of $y(t)$. Next, define the state variables by the scheme

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y^{(1)}(t) \\ &\vdots \\ x_{2n}(t) &= y^{(2n-1)}(t) \end{aligned}$$

The state equations for Equation (7.55) then become

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_{2n} & -\alpha_{2n-1} & -\alpha_{2n-2} & \cdots & -\alpha_1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \beta \end{bmatrix} u(t) \tag{7.56}$$

and

$$y(t) = [1\ 0\ 0\ \cdots]\mathbf{x}(t) \tag{7.57}$$

The triple $(A, \mathbf{b}, \mathbf{c})$ defined by Equations (7.56) and (7.57) constitutes an irreducible realization of the transfer function given by Equation (7.53).

Realization procedures are also available for *multiple-input, multiple-output* (MIMO) systems (see, for instance, Ho and Kalman, 1965). In Chapter 8, realization methods are used to determine the natural frequencies, damping ratios, and mode shapes of a vibrating structure by measuring the transfer function and using it to construct a realization of the test structure.

Example 7.6.1

Consider the transfer function of a simple oscillator, i.e.,

$$G(s) = \frac{1}{s^2 + 2\zeta\omega s + \omega^2}$$

Following Equation (7.56), a state-space realization of this transfer function is given by

$$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\zeta\omega \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

7.7 REDUCED-ORDER MODELING

Most control methods work best for structures with a small number of degrees of freedom. Many modeling techniques produce structural models of a large order. Hence, it is often necessary to reduce the order of a model before performing a control analysis and designing a control law. This topic was first introduced in Section 6.8. In that section the method of model reduction was based on knowledge that certain coordinates, such as those corresponding to a bending mode in a longitudinally excited beam, will not contribute to the response. These coordinates are then removed, producing a model of lower order. In this section, the coordinates to be removed are calculated in a more formal way as part of the reduction process, rather than specified through experience as in the Guyan reduction of Section 6.8.

The approach taken in this section is to reduce the order of a given model on the basis of deleting those coordinates, or modes, that are the least controllable and observable. The idea here is that controllability and observability of a state (coordinate) are indications of the contribution of that state (coordinate) to the response of the structure, as well as the ability of that coordinate to be excited by an external disturbance.

To implement this idea, a measure of the degree of controllability and observability is needed. One such measure of controllability is given by the controllability norm of Equation (7.8). However, an alternative, more useful measure is provided for asymptotically stable systems of the form given by Equations (7.1) and (7.2) by defining the *controllability grammian*, denoted by W_C, as

$$W_C^2 = \int_0^\infty e^{At} B B^T e^{A^T t}\, dt \tag{7.58}$$

and the *observability grammian*, denoted by W_O, as

$$W_O^2 = \int_0^\infty e^{A^T t} C^T C\, e^{At}\, dt \tag{7.59}$$

Here, the matrices A, B, and C are defined as in Equations (7.1) and (7.2). The properties of these two matrices provide useful information about the controllability and observability of the closed-loop system. If the system is controllable (or observable), the matrix W_C (or W_O) is nonsingular. These grammians characterize the degree of controllability and observability by quantifying just how far away from being singular the matrices W_C and W_O are. This is equivalent to quantifying rank deficiency. The most reliable way to quantify the rank of a matrix is to examine the singular values of the matrix, which is discussed next.

For any real $m \times n$ matrix A there exist orthogonal matrices $U_{m\times m}$ and $V_{n\times n}$ such that

$$U^T A V = \operatorname{diag}\left[\sigma_1\ \sigma_2 \cdots \sigma_p\right], \qquad p = \min(m, n) \tag{7.60}$$

where the σ_i are real and ordered via

$$\sigma_1 > \sigma_2 > \cdots > \sigma_p > 0 \tag{7.61}$$

(see, for instance, Golub and Van Loan, 1996). The numbers σ_i are called the *singular values* of matrix A. The singular values of matrix A are the nonnegative square roots of the eigenvalues of the symmetric positive definite matrix $A^T A$. The vectors $\mathbf{u}_i$ consisting of the columns of matrix U are called the *left singular vectors* of A. Likewise, the columns of V, denoted by $\mathbf{v}_i$, are called the *right singular vectors* of A. The process of calculating U, V,

and diag$[\sigma_1 \cdots \sigma_p]$ is called the *singular value decomposition* (denoted by SVD) of matrix A. Note that the singular values and vectors satisfy

$$A\mathbf{v}_i = \sigma_i \mathbf{u}_i, \qquad i = 1, \ldots, p \tag{7.62}$$

$$A^T \mathbf{u}_i = \sigma_i \mathbf{v}_i, \qquad i = 1, \ldots, p \tag{7.63}$$

Note that if A is a square symmetric positive definite matrix, then $U = V$, and the $\mathbf{u}_i$ are the eigenvectors of matrix A and the singular values of A are identical to the eigenvalues of A^2.

The real square symmetric semipositive definite matrix A is, of course, singular, or rank deficient, if and only if it has a zero eigenvalue (or singular value). This statement leads naturally to the idea that the size of the singular values of a matrix quantify how close the matrix is to being singular. If the smallest singular value, σ_n, is well away from zero, then the matrix is of full rank and 'far away' from being singular (note: unlike frequencies, we order singular values from largest to smallest).

Applying the idea of singular values as a measure of rank deficiency to the controllability and observability grammians yields a systematic model reduction method. The matrices W_{O} and W_{C} are symmetric and hence are similar to a diagonal matrix. Moore (1981) showed that there always exists an equivalent system for which these two grammians are both equal and diagonal. Such a system is then called *balanced.*

In addition, Moore showed that W_{C} and W_{O} must satisfy the two Liapunov-type equations

$$\begin{aligned} AW_{\mathrm{C}}^2 + W_{\mathrm{C}}^2 A^T &= -BB^T \\ A^T W_{\mathrm{O}}^2 + W_{\mathrm{O}}^2 A &= -C^T C \end{aligned} \tag{7.64}$$

for asymptotically stable systems.

Let the matrix P denote a linear similarity transformation, which when applied to Equations (7.1) and (7.2) yields the equivalent system

$$\begin{aligned} \dot{\mathbf{x}}' &= A'\mathbf{x}' + B'\mathbf{u} \\ \mathbf{y} &= C'\mathbf{x}' \end{aligned} \tag{7.65}$$

These two equivalent systems are related by

$$\mathbf{x} = P\mathbf{x}' \tag{7.66}$$

$$A' = P^{-1}AP \tag{7.67}$$

$$B' = P^{-1}B \tag{7.68}$$

$$C' = CP \tag{7.69}$$

Here, matrix P can be chosen such that the new grammians defined by

$$W'_{\mathrm{C}} = P^{-1} W_{\mathrm{C}} P \tag{7.70}$$

and

$$W'_{\mathrm{O}} = P^{-1} W_{\mathrm{O}} P \tag{7.71}$$

are equal and diagonal. That is,

$$W'_{\mathrm{C}} = W'_{\mathrm{O}} = \Lambda_{\mathbf{W}} = \mathrm{diag}[\sigma_1 \ \sigma_2 \cdots \sigma_{2n}] \tag{7.72}$$

where the numbers σ_i are the singular values of the grammians and are ordered such that

$$\sigma_i > \sigma_{i+1}, \qquad i = 1, 2, \ldots, 2n-1 \tag{7.73}$$

Under these circumstances, i.e., when Equations (7.72) and (7.73) hold, the system given by Equations (7.65) is said to be *internally balanced*.

Next, let the state variables in the balanced system be partitioned into the form

$$\begin{bmatrix} \dot{\mathbf{x}}'_1 \\ \dot{\mathbf{x}}'_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x}'_1 \\ \mathbf{x}'_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathbf{u} \tag{7.74}$$

$$\mathbf{y} = [C_1 \quad C_2] \begin{bmatrix} \mathbf{x}'_2 \\ \mathbf{x}'_1 \end{bmatrix} \tag{7.75}$$

where A_{11} is a $k \times k$ matrix and $\mathbf{x}'_2$ is the vector containing those states corresponding to the $(2n-k)$ smallest singular values of W_{C}. It can be shown (Moore, 1981) that the $\mathbf{x}'_2$ part of the state vector for Equations (7.74) and (7.75) affects the output much less than $\mathbf{x}'_1$ does. Thus, if σ_k is much greater than σ_{k+1}, i.e., $\sigma_k >> \sigma_{k+1}$, the $\mathbf{x}'_2$ part of the state vector does not affect the input–output behavior of the system as much as $\mathbf{x}'_1$ does.

The preceding comments suggest that a suitable low-order model of the system of Equations (7.1) and (7.2) is the subsystem given by

$$\dot{\mathbf{x}}'_1 = A_{11}\mathbf{x}'_1 + B_1\mathbf{u} \tag{7.76}$$

$$\mathbf{y} = C_1\mathbf{x}'_1 \tag{7.77}$$

This subsystem is referred to as a *reduced-order model* (often referred to as ROM). Note that, as pointed out by Moore (1981), a realization of Equations (7.1) and (7.2) should yield the reduced model of equations (7.76) and (7.77).

The reduced-order model can be calculated by first calculating an intermediate transformation matrix P_1 based on the controllability grammian. Solving the eigenvalue problem for W_{C} yields

$$W_{\mathrm{C}} = V_{\mathrm{C}} \Sigma_{\mathrm{C}}^2 V_{\mathrm{C}}^T \tag{7.78}$$

where V_C is the matrix of normalized eigenvectors of W_C and $\mathbf{\Sigma}_C^2$ is the diagonal matrix of eigenvalues of W_C. The square on $\mathbf{\Sigma}_C$ is a reminder that W_C is positive definite. Based on this decomposition, a nonsingular transformation P_1 is defined by

$$P_1 = V_C \mathbf{\Sigma}_C \tag{7.79}$$

Application of the transformation P_1 to the state equations yields the intermediate state equations defined by

$$A'' = P_1^{-1} A P_1 \tag{7.80}$$

$$B'' = P_1^{-1} B \tag{7.81}$$

$$C'' = C P_1 \tag{7.82}$$

To complete the balancing algorithm, these intermediate equations are balanced with respect to W_O''. That is, the eigenvalue problem for W_O'' yields the matrices V_O'' and $\mathbf{\Sigma}_O''^{\,2}$ such that

$$W_O'' = V_O'' \mathbf{\Sigma}_O''^{\,2} V_O''^{\,T} \tag{7.83}$$

These two matrices are used to define the second part of the balancing transformation, i.e.,

$$P_2 = V_O'' \mathbf{\Sigma}_O''^{\,-1/2} \tag{7.84}$$

The balanced version of the original state equations is then given by the product transformation $P = P_1 P_2$ in Equation (7.65). They are

$$A' = P_2^{-1} P_1^{-1} A P_1 P_2 \tag{7.85}$$

$$B' = P_2^{-1} P_1^{-1} B \tag{7.86}$$

$$C' = C P_1 P_2 \tag{7.87}$$

The balanced system is then used to define the reduced-order model of Equations (7.76) and (7.77) by determining the value of k such that $\sigma_k >> \sigma_{k+1}$. The following example illustrates an internally balanced reduced-order model.

Example 7.7.1

Consider the two-degree-of-freedom system of Figure 2.4 with $m_1 = m_2 = 1$, $c_1 = 0.2$, $c_2 = 0.1$, and $k_1 = k_2 = 1$. Let an impulse force be applied to m_2 and assume a position measurement of m_2 is available. The state matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -0.3 & 0.1 \\ 1 & -1 & 0.1 & -0.1 \end{bmatrix}$$

In addition, B becomes the vector $\mathbf{b} = [0\ 0\ 0\ 1]^T$, and the output matrix C becomes the vector $\mathbf{c}^T = [0\ 1\ 0\ 0]$.

The controllability and observability grammians can be calculated (for $t \to \infty$) from Equation (7.64) to be

$$W_C = \begin{bmatrix} 4.0569 & 6.3523 & 0.0000 & -0.3114 \\ 6.3523 & 10.5338 & 0.3114 & 0.0000 \\ 0.0000 & 0.3114 & 1.7927 & 2.2642 \\ -0.3114 & 0.0000 & 2.2642 & 4.1504 \end{bmatrix}$$

$$W_O = \begin{bmatrix} 1.8198 & 2.2290 & 0.5819 & 1.1637 \\ 2.2290 & 4.2315 & -0.8919 & 0.4181 \\ 0.5819 & -0.0819 & 4.0569 & 6.3523 \\ 1.1637 & 0.4181 & 6.3523 & 10.5338 \end{bmatrix}$$

Calculation of the matrix P that diagonalizes the two grammians yields

$$P = \begin{bmatrix} 0.4451 & -0.4975 & 0.4962 & -0.4437 \\ 0.7821 & -0.7510 & -0.2369 & 0.3223 \\ 0.2895 & 0.2895 & 0.6827 & 0.8753 \\ 0.4419 & 0.5112 & -0.4632 & -0.4985 \end{bmatrix}$$

The singular values of W_O and W_C are then $\sigma_1 = 9.3836$, $\sigma_2 = 8.4310$, $\sigma_3 = 0.2724$, and $\sigma_4 = 0.2250$. From examination of these singular values, it appears that the coordinates x_1 and x_2 associated with A' of Equation (7.85) are likely candidates for a reduced-order model (i.e., $k = 1$ in this case). Using Equations (7.85) through (7.87) yields the balanced system given by

$$A' = \begin{bmatrix} -0.0326 & 0.6166 & 0.0192 & -0.0275 \\ -0.6166 & -0.0334 & -0.0218 & 0.0280 \\ 0.0192 & 0.0218 & -0.1030 & 1.6102 \\ 0.0275 & 0.0280 & 1.6102 & -0.2309 \end{bmatrix}$$

$$B'^{\mathrm{T}} = \begin{bmatrix} 0.7821 & 0.7510 & -0.2369 & -0.03223 \end{bmatrix}^T$$

$$C' = \begin{bmatrix} 0.7821 & -0.7510 & -0.2369 & 0.3223 \end{bmatrix}$$

Given that $k = 2$ (from examination of the singular values), the coefficients in Equations (7.76) and (7.77) for the reduced-order model become

$$A_{11} = \begin{bmatrix} -0.0326 & 0.6166 \\ -0.6166 & -0.0334 \end{bmatrix}$$

$$B_1 = [0.7821 \quad 0.7510]^T$$

$$C_1 = [0.7821 \quad -0.7510]$$

Plots of the response of the full-order model and the balanced model are given in Figures 7.12 and 7.13 respectively. Note in Figure 7.13 that the two coordinates (x_3' and x_4') neglected in the reduced-order model do not contribute as much to the response and, in fact, die out after about 15 time units. However, all the coordinates in Figure 7.12 and x_1' and x_2' are still vibrating after 15 time units.

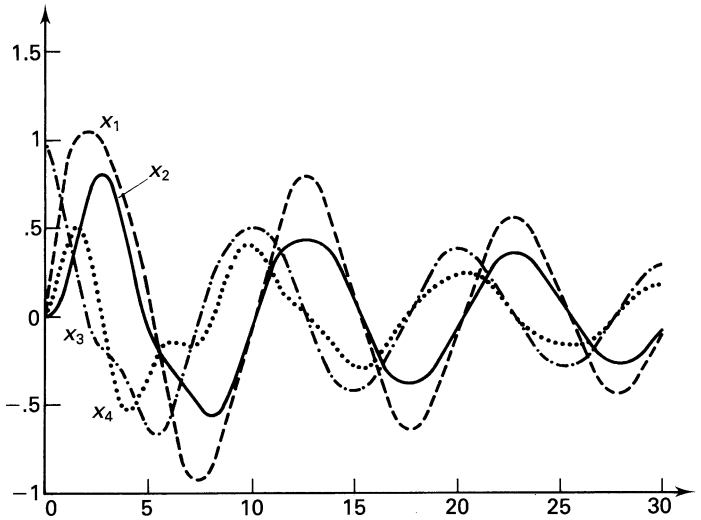

Figure 7.12 Response of the original state variables.

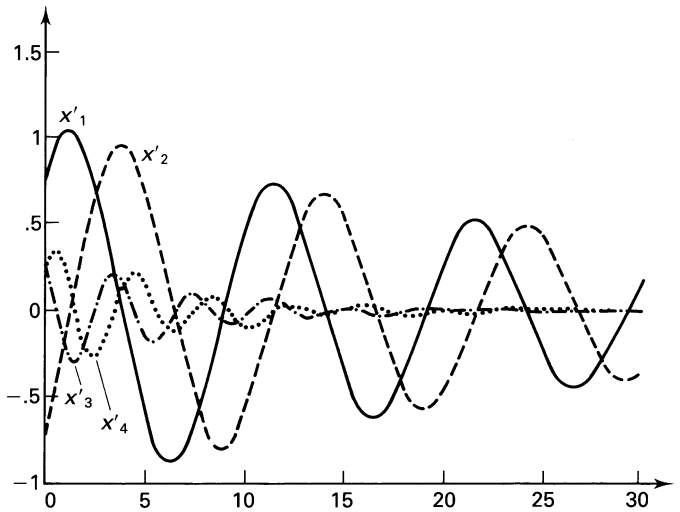

Figure 7.13 Response of the balanced state variables.

Note that the balanced reduced-order model will change if different inputs and outputs are considered, as the B and C matrices would change, altering the reduction scheme.

7.8 MODAL CONTROL IN STATE SPACE

In general, modal control refers to the procedure of decomposing the dynamic equations of a structure into modal coordinates, such as Equations (5.29) and (5.37), and designing

the control system in this modal coordinate system. In broad terms, any control design that employs a modal description of the structure is called a modal control method. Modal control works well for systems in which certain (just a few) modes of the structure dominate the response. That is, modal control works well for systems in which only a few of the modal participation factors (see Section 3.3) are large and the rest are relatively small. Modal control can be examined in either state space, via Equations (7.1) and (7.2), or in physical space, via Equations (4.23) through (4.25). This section examines modal control in the state-space coordinate system, and the following section examines modal control in the physical coordinate system.

Consider the state-space description of Equations (7.1) and (7.2). If matrix A has a diagonal Jordan form, then – following Section 5.3 – there exists a nonsingular matrix U such that

$$U^{-1}AU = \Lambda$$

where Λ is a diagonal matrix of the eigenvalues of the state matrix A. Note that the diagonal elements of Λ will be complex if the system is underdamped. Substituting $\mathbf{x} = U\mathbf{z}$ into Equation (7.1) and premultiplying by the inverse of the nonsingular matrix U yields the diagonal system

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + U^{-1}B\mathbf{u} \tag{7.88}$$

Here, the vector $\mathbf{z}$ is referred to as the *modal coordinate system*. In this form, the controllability problem and the pole placement problem become more obvious.

Consider first the controllability question for the case of a single input (i.e., $\mathbf{u}$ becomes a scalar). Then B is a vector $\mathbf{b}$ and $U^{-1}\mathbf{b}$ is a vector consisting of $2n$ elements denoted by b_i. Clearly, this system is controllable if and only if each $b_i \neq 0$, $i = 1, 2, \ldots, 2n$. If, on the other hand, b_i should happen to be zero for some index i, then the system is not controllable. With $b_i = 0$ the ith mode is not controllable, as no feedback law could possibly affect the ith mode. Thus, the vector $U^{-1}\mathbf{b}$ indicates the controllability of the system by inspection.

The form of Equation (7.88) can also be used to perform a quick model reduction (see problem 7.28). Suppose it is desired to control just the fastest modes or modes in a certain frequency range. Then these modes of interest can be taken as the reduced-order model and the others can be neglected.

Next, consider the pole placement problem. In modal form, it seems to be a trivial matter to choose the individual modal control gains b_i to place the eigenvalues of the system. For instance, suppose output feedback is used. Then $u = \mathbf{c}^T U\mathbf{z}$, and Equation (7.88) becomes

$$\dot{\mathbf{z}} = \Lambda\mathbf{z} + U^{-1}\mathbf{b}\mathbf{c}^T U\mathbf{z} = (\Lambda + U^{-1}\mathbf{b}\mathbf{c}^T U)\mathbf{z} \tag{7.89}$$

where $\mathbf{c}^T$ is a row vector. The closed-loop system then has poles determined by the matrix $(\Lambda + U^{-1}\mathbf{b}\mathbf{c}^T U)$. Suppose that the matrix $U^{-1}\mathbf{b}\mathbf{c}^T U$ is also diagonal. In this case the controls are also decoupled, and Equation (7.89) becomes the $2n$ decoupled equations

$$\dot{z}_i = (\lambda_i + u_i)z_i, \qquad i = 1, 2, \ldots, 2n \tag{7.90}$$

Here, u_i denotes the diagonal elements of the diagonal matrix $U^{-1}\mathbf{b}\mathbf{c}^T U$. Note that the vector $U^{-1}\mathbf{b}\mathbf{c}^T U\mathbf{z}$ in Equation (7.89) is identical to the vector $U^{-1}\mathbf{f}$ in Equation (5.20). The

difference between the two equations is simply that in this section the forcing function is the result of a control force manipulated by a designer to achieve a desired response. In the development of Section 5.3, the forcing term represents some external disturbance.

The matrix $U^{-1}\mathbf{b}\mathbf{c}^T U$ in Equation (7.89) may not be diagonal. In this case, Equation (7.89) is not decoupled. The controls in this situation reintroduce coupling into the system. As a result, if it is desired to change a particular modal coordinate, the control force chosen to change this mode will also change the eigenvalues of some of the other modes. Following the arguments of Section 3.5, a necessary and sufficient condition for $U^{-1}\mathbf{b}\mathbf{c}^T U$ to be diagonal is for the matrix $\mathbf{b}\mathbf{c}^T$ to commute with the state matrix A. The remainder of this section is devoted to discussing the coupling introduced by control laws and measurement points in the case where it is desired to control independently a small number of modes of a given structure.

Suppose, then, that it is desired to control independently a small number of modes. For example, it may be desired to place a small number of troublesome poles while leaving the remaining poles unaffected. Let k denote the number of modes that are to be controlled independently of the remaining $2n-k$ modes. Furthermore, assume that it is the first k modes that are of interest, i.e., the desired modes are the lowest k. Then, partition the modal equations with state feedback into the k modal coordinates that are to be controlled and the $2n-k$ modal coordinates that are to be left undisturbed. This yields

$$\begin{bmatrix} \dot{\mathbf{z}}_k \\ \dot{\mathbf{z}}_{2n-k} \end{bmatrix} = \left(\begin{bmatrix} \Lambda_k & 0 \\ 0 & \Lambda_{2n-k} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_k \mathbf{c}_k^T & \mathbf{b}_k \mathbf{c}_{2n-k}^T \\ \mathbf{b}_{2n-k} \mathbf{c}_k^T & \mathbf{b}_{2n-k} \mathbf{c}_{2n-k}^T \end{bmatrix} \right) \begin{bmatrix} \mathbf{z}_k \\ \mathbf{z}_{2n-k} \end{bmatrix} \tag{7.91}$$

Here, the matrix $U^{-1}\mathbf{b}\mathbf{c}^T U$ is partitioned into blocks defined by

$$U^{-1}\mathbf{b} = \begin{bmatrix} \mathbf{b}_k \\ \mathbf{b}_{2n-k} \end{bmatrix}$$

$$\mathbf{c}^T U = [\mathbf{c}_k^T \quad \mathbf{c}_{2n-k}^T]$$

where $\mathbf{b}_k$ denotes the first k elements of the vector $\mathbf{b}$, $\mathbf{b}_{2n-k}$ denotes the last $2n-k$ elements, and so on. Likewise, the matrices Λ_k and Λ_{2n-k} denote diagonal matrices of the first k eigenvalues and the last $2n-k$ eigenvalues respectively. Let b_i denote the elements of the vector $U^{-1}\mathbf{b}$ and c_i denote the elements of the vector $\mathbf{c}^T U$. Then, $\mathbf{b}_k \mathbf{c}_k^T$ is the $k \times k$ matrix

$$\mathbf{b}_k \mathbf{c}_k^T = \begin{bmatrix} b_1 c_1 & b_1 c_2 & \cdots & b_1 c_k \\ b_2 c_1 & b_2 c_2 & \cdots & b_2 c_k \\ \vdots & \vdots & & \vdots \\ b_k c_1 & b_k c_2 & \cdots & b_k c_k \end{bmatrix} \tag{7.92}$$

Examination of Equation (7.91) illustrates that the first k modes of the system can be controlled independently of the last $(2n-k)$ modes if and only if the two vectors $\mathbf{b}_{2n-k}$ and $\mathbf{c}_{2n-k}$ are both zero. Furthermore, the first k modes can be controlled independently of each other only if Equation (7.92) is diagonal. This, of course, cannot happen, as clearly indicated by setting the off-diagonal terms of Equation (7.92) to zero. Takahashi *et al.* (1968) discussed decoupled or ideal control in detail. In general, it is difficult to control modes independently, unless a large number of actuators and sensors are used. The vector $\mathbf{c}_{2n-k}^T$ indicates the coupling introduced into the system by measurement, and the vector $\mathbf{b}_{2n-k}$

indicates coupling due to control action. This phenomenon is known as *observation spillover* and *control spillover* (Balas, 1978).

Example 7.8.1

Consider an overdamped two-degree-of-freedom structure with the state matrix given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & -9 & 4 \\ 1 & -1 & 4 & -4 \end{bmatrix}$$

Solving the eigenvalue problem for A yields the matrices Λ, U, and U^{-1} given by

$$\Lambda = \begin{bmatrix} -10.9074 & 0 & 0 & 0 \\ 0 & -1.2941 & 0 & 0 \\ 0 & 0 & -0.5323 & 0 \\ 0 & 0 & 0 & -0.2662 \end{bmatrix}$$

$$U = \begin{bmatrix} -0.0917 & -0.4629 & 0.7491 & -0.0957 \\ 0.0512 & -0.7727 & 1.0000 & 1.0000 \\ 1.0000 & 0.5990 & -0.3988 & 0.0255 \\ -0.5584 & 1.0000 & 0.5323 & -0.2662 \end{bmatrix}$$

$$U^{-1} = \begin{bmatrix} 0.2560 & -0.1121 & 0.7846 & -0.4381 \\ 0.7382 & 0.3741 & 0.7180 & 1.1986 \\ 1.6791 & 0.3378 & 0.5368 & 0.7167 \\ -1.1218 & 0.9549 & -0.0222 & 0.2320 \end{bmatrix}$$

which constitutes a modal decomposition of the state matrix (note that $A = U\Lambda U^{-1}$). Suppose next that $\mathbf{b} = [0\ 0\ 1\ 0]^T$ and $\mathbf{c}^T = [0\ 0\ 1\ 1]$ are the control and measurement vectors respectively. In the decoupled coordinate system, the control and measurement vectors become

$$U^{-1}\mathbf{b} = \begin{bmatrix} 0.7846 & 0.7180 & 0.5368 & -0.0222 \end{bmatrix}^T$$

and

$$\mathbf{c}^T U = \begin{bmatrix} 0.44161 & 0.5990 & -0.9311 & -0.2407 \end{bmatrix}$$

Notice that these vectors are fully populated with nonzero elements. This leads to the fully coupled closed-loop system given by Equation (7.89), since the term $U^{-1}\mathbf{b}\mathbf{c}^T U$ becomes

$$U^{-1}\mathbf{b}\mathbf{c}^T U = \begin{bmatrix} 0.3465 & 1.2545 & -0.7305 & -0.1888 \\ 0.3171 & 1.1481 & -0.6685 & -0.1728 \\ 0.2371 & 0.8584 & -0.4999 & -0.1292 \\ 0.0098 & -0.0355 & 0.0207 & 0.0053 \end{bmatrix}$$

This last expression illustrates the recoupling effect caused by the control and measurement locations. Note that the matrix $\Lambda + U^{-1}\mathbf{b}\mathbf{c}^T U$ of Equation (7.89) is not diagonal but fully populated, recoupling the dynamics.

7.9 MODAL CONTROL IN PHYSICAL SPACE

As indicated in Section 3.4, the left-hand side of Equation (4.23) can be decoupled into modal coordinates if and only if $DM^{-1}K = KM^{-1}D$. This was used in Section 5.4 to decouple Equation (5.35) into modal coordinates to solve for the forced response. A similar approach is taken here, except that the input force $S_m^T\mathbf{f}$ of Equation (5.37) becomes the control force $S_m^T B_f \mathbf{u}$. As in the state space, the difficulty lies in the fact that the transformation, S_m, of the equations of motion into modal coordinates, $\mathbf{z}$, does not necessarily decouple the control input $S_m^T B_f \mathbf{u}$. (Note that in this section $\mathbf{z}$ is an $n \times 1$ vector of modal positions, whereas in the previous section $\mathbf{z}$ is a $2n \times 1$ vector of modal state variables.) The control problem in second-order physical coordinates transformed into modal coordinates is, in the notation of Sections 4.10 and 5.4,

$$I\ddot{\mathbf{z}} + \Lambda_D \dot{\mathbf{z}} + \Lambda_K \mathbf{z} = S_m^T B_f \mathbf{u} \tag{7.93}$$

$$\mathbf{y} = C_p S_m \mathbf{z} + C_v S_m \dot{\mathbf{z}} \tag{7.94}$$

Here, the dimensions of the matrices C_p and C_v are as indicated in Equation (4.25). Note that in Equation (7.93) the relative magnitudes of the elements $(S_m^T B_f \mathbf{u})_i$ are an indication of the degree of controllability for the ith mode. If $(S_m^T B_f \mathbf{u})_i$ is very small, the ith mode is hard to control. If it happens to be zero, then the ith mode is not controllable. If, on the other hand, $(S_m^T B_f \mathbf{u})_i$ is relatively large, the ith mode is very controllable.

If output feedback of the form suggested in Equation (4.26) is used, then $\mathbf{u} = -G_f \mathbf{y}$. Combining Equation (7.93) with Equation (7.94) and the control law, the system equation becomes

$$I\ddot{\mathbf{z}} + \Lambda_D \dot{\mathbf{z}} + \Lambda_K \mathbf{z} = -S_m^T B_f K_p S_m \mathbf{z} - S_m^T B_f K_v S_m \dot{\mathbf{z}} \tag{7.95}$$

where $K_p = G_f C_p$ and $K_v = G_f C_v$ represent measurement positions and B_f is taken to represent the control actuator locations. Note that, by this choice of $\mathbf{u}$, the discussion is now restricted to state variable feedback, i.e., position and velocity feedback. Equation (7.95) can be rewritten as

$$I\ddot{\mathbf{z}} + (\Lambda_D + S_m^T B_f K_v S_m)\dot{\mathbf{z}} + (\Lambda_K + S_m^T B_f K_p S_m)\mathbf{z} = \mathbf{0} \tag{7.96}$$

Now, Equation (7.96) is posed for modal pole placement control.

To cause the closed-loop system of Equation (7.96) to have the desired eigenvalues and, hence, a desired response, the control gain matrix B_f and the measurement matrices K_v and K_p must be chosen (a design process) appropriately. If, in addition, it is desired to control each mode independently, i.e., each z_i in Equation (7.96), then further restrictions must be satisfied. Namely, it can be seen from Equation (7.96) that an independent control of each mode is possible if and only if the matrices $S_m^T B_f K_v S_m$ and $S_m^T B_f K_p S_m$ are both diagonal. In the event where the matrix $B_f K_v$ and the matrix $B_f K_p$ are both symmetric, this will be true if and only if

$$B_f K_v M^{-1} D = D M^{-1} B_f K_v \tag{7.97}$$

and

$$B_f K_p M^{-1} K = K M^{-1} B_f K_p \tag{7.98}$$

Unfortunately, this puts *very* stringent requirements on the location and number of sensors and actuators. If, however, the conditions given by Equations (7.97) and (7.98) are satisfied, Equation (7.96) reduces to n single-degree-of-freedom control problems of the form

$$\ddot{z}_i + (2\zeta_i \omega_i + \alpha_i)\dot{z}_i + (\omega_i^2 + \beta_i) z_i = 0 \tag{7.99}$$

where α_i and β_i are the diagonal elements of the matrices $S_m^T B_f K_v S_m$ and $S_m^T B_f K_p S_m$ respectively. This last expression represents an independent set of equations that can be solved for α_i and β_i given desired modal response information. That is, if it is desired that the closed-loop system have a first-mode natural frequency of 10, for instance, then Equation (7.99) requires that $\omega_1^2 + \beta_1 = 10$, or $\beta_1 = 10 - \omega_1^2$. If the known open-loop system has $\omega_1^2 = 6$, then $\beta_1 = 4$ is the desired control gain.

While the modal control equations [Equations (7.99)] appear quite simple, the problem of independent control remains complicated and requires a larger number of sensor and actuator connections. This happens because the α_i and β_i of Equation (7.99), while independent, are not always capable of being independently implemented. The design problem is to choose actuator and sensor locations as well as gains such that B, K_v, and K_p satisfy Equations (7.97) and (7.98). The choice of these gains is not independent but rather coupled through the equations

$$S_m^T B_f K_v S_m = \mathrm{diag}[\alpha_i] \tag{7.100}$$

and

$$S_m^T B_f K_p S_m = \mathrm{diag}[\beta_i] \tag{7.101}$$

The modal control of a system in physical coordinates as well as the problem of performing independent modal control are illustrated in the following example.

Example 7.9.1

Consider the system of Figure 2.4 with coefficient values (dimensionless) of $m_1 = 9$, $m_2 = 1$, $c_1 = 8$, $c_2 = 1$, $k_1 = 24$, and $k_2 = 3$. This produces a system equivalent to the one of example 3.5.1, where it is shown that $DM^{-1}K = KM^{-1}D$, so that the system decouples. The matrix S_m is given by

$$S_m = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

The decoupled coordinates, listed in example 3.5.1, yield the modal frequencies and damping ratios as

$$\zeta_1 = 0.2357, \qquad \omega_1 = \sqrt{2} = 1.414 \text{ rad/s}$$
$$\zeta_2 = 0.3333, \qquad \omega_2 = 2.0 \text{ rad/s}$$

Consider the control problem of calculating a feedback law that will cause the closed-loop system (7.95) to have a response with a modal damping ratio of 0.4 in the first mode. Three cases of different sensor and actuator placements are considered.

In the first case, consider an SISO system with non-collocated control. Suppose one actuator is used to achieve the desired control and it is connected to only one mass, m_2, and one sensor is used to measure the velocity of m_1. Then

$$K_v = [g_1 \quad 0], \qquad K_p = \mathbf{0}$$
$$B_f = \begin{bmatrix} 0 \\ g_2 \end{bmatrix}$$

since u is a scalar in this case. Calculating the control and measurement quantities from Equation (7.96) yields

$$S_m^T B_f K_p S_m = 0$$
$$S_m^T B_f K_v S_m = \frac{g_1 g_2}{6} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

It should be clear from this last expression that no choice of g_1 and g_2 will allow just the first mode damping to be changed, i.e., $S_m^T B_f K_v S_n$ cannot be diagonal.

For the second case, consider using two actuators, one at each mass, and two sensors measuring the two velocities. This is a MIMO system with velocity feedback. Then

$$K_v = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}$$

and

$$B_f = \begin{bmatrix} g_3 & 0 \\ 0 & g_4 \end{bmatrix}$$

Equation (7.96) then yields

$$S_m^T B_f K_v S_m = \frac{1}{18} \begin{bmatrix} g_1 g_3 + 9 g_2 g_4 & 9 g_2 g_4 - g_1 g_3 \\ 9 g_2 g_4 - g_1 g_3 & g_1 g_3 + 9 g_2 g_4 \end{bmatrix}$$

An obvious choice for decoupled control is $9 g_2 g_4 = g_1 g_3$, as this makes the above matrix diagonal. Then

$$S_m^T B_f K_v S_m = \begin{bmatrix} g_2 g_4 & 0 \\ 0 & g_2 g_4 \end{bmatrix}$$

Comparing the desired first mode damping ratio yields

$$2\zeta_1 \omega_1 + g_2 g_4 = 2(0.4)\sqrt{2}$$

Solving this equation for the two unknown $a = g_2 g_4$ yields $g_2 g_4 = 0.164$. This choice of the gain keeps the equations of motion decoupled and assigns the first damping ratio the desired value of 0.4. Unfortunately, because of the way the gains appear in closed-loop system, the effect on the second mode velocity coefficient is

$$2\zeta_2\omega_2 + g_2 g_4 = 2\hat{\zeta}_2\omega_2$$

This results in a new damping ratio for mode 2 of $\hat{\zeta}_2 = 0.449$. Hence, although a decoupled control has been found, it is still not possible independently to change the damping ratio of mode 1 without affecting mode 2. This would require $g_1 g_3 + 9 g_2 g_4$ to be zero, which of course would also not allow ζ_1 to be controlled at all.

For the third case, then consider a MIMO controller with the velocity signal at m_1 to be fed back to mass m_2, and vice versa. This means that K_v now has the form

$$K_v = \begin{bmatrix} g_1 & g_5 \\ g_6 & g_2 \end{bmatrix}$$

In this case, calculating the velocity feedback coefficient yields

$$S_m^T B_f K_v S_m = \frac{1}{18}\begin{bmatrix} g_3 g_1 + 3 g_3 g_5 + 9 g_2 g_4 + 3 g_4 g_6 & 9 g_2 g_4 + 3 g_3 g_5 - 3 g_4 g_6 - g_3 g_1 \\ 9 g_2 g_4 - 3 g_3 g_5 + 3 g_4 g_6 - g_3 g_1 & 9 g_2 g_4 + g_3 g_1 - 3 g_4 g_6 - 3 g_3 g_5 \end{bmatrix}$$

Examining this new feedback matrix shows that *independent* modal control will be possible if and only if the off-diagonals are set to zero, decoupling the system, and the element in the (2, 2) position is zero, leaving the second mode unchanged. To change the first mode-damping ratio to 0.4, the first entry in the matrix above requires that

$$2\zeta_1\omega_1 + \frac{g_1 g_3 + 3 g_3 g_5 + 9 g_2 g_4 + 3 g_4 g_6}{18} = 2(0.4)\omega_1$$

To ensure that the controller does not recouple the equations of motion, the off-diagonal elements are set to zero, resulting in the two equations

$$9 g_2 g_4 + 3 g_3 g_5 - 3 g_4 g_6 - g_3 g_1 = 0 \qquad \text{and} \qquad 9 g_2 g_4 - 3 g_3 g_5 + 3 g_4 g_6 - g_3 g_1 = 0$$

To ensure that the second mode damping is not changed, the element in the (2, 2) position is also set to zero, resulting in the additional condition

$$9 g_2 g_4 + g_3 g_1 - 3 g_4 g_6 - 3 g_3 g_5 = 0$$

This last set of equations can be recognized as four linear equations in the four unknowns

$$a = g_1 g_3, \qquad b = g_2 g_4, \qquad c = g_3 g_5, \qquad d = g_4 g_6$$

These equations in matrix form are

$$\begin{aligned} -a + 9b + 3c + 3d &= 0 \\ -a + 9b - 3c + 3d &= 0 \\ a + 9b - 3c - 3d &= 0 \\ \frac{a}{18} + \frac{1}{2}b + \frac{1}{6}c + \frac{1}{6}d &= 2(0.4)\omega_1 - 2\zeta_1\omega_1 \end{aligned}$$

Solving this set of equations with the open-loop values of ω_1 and ζ_1 given above yields

$$a = 2.09, \qquad b = 0.232, \qquad c = 0.697, \qquad d = 0.697$$

Substitution of these values along with the free choice of $g_3 = g_4 = 1$ into the feedback matrix yields

$$S_m^T B_f K_v S_m = \begin{bmatrix} 0.464 & 0 \\ 0 & 0 \end{bmatrix}$$

The resulting closed-loop system yields a new system with the desired first mode damping ratio of 0.4 and the second mode damping ratio unchanged. Furthermore, the closed-loop system remains decoupled. Hence, independent mode control is achieved.

Note that, of the six gain values, only four can be determined. In fact, B_f could have been chosen as the identity matrix from the start with the same result. However, in practice, each sensor and actuator will have a coefficient that needs to be accounted for. In many cases each sensor and actuator will have significant dynamics, ignored here, that could change the plant by adding additional poles and zeros.

Example 7.9.1 illustrates how many control actuators and sensors must be used in order to accomplish an independent control of one mode of a simple two-degree-of-freedom system. In the case of the last example, as many sensors and actuators were required as degrees of freedom in the system. Hence, it is important in practical control design to consider the placement of actuators and sensors and not just the relationships among the various coefficient matrices for the system.

7.10 ROBUSTNESS

The concept of robust control systems, or robust systems, has been defined in many ways, not all of which are consistent. However, the basic idea behind the concept of robustness is an attempt to measure just how stable a given system is in the presence of some uncertainty or perturbation in the system. That is, if a system is stable, is it still stable after some changes have been made in the physical or control parameters of the system? This is called *stability robustness*. This same question can be asked with respect to the performance of the system. In this latter case, the question is asked in terms of a given level of acceptability of a specific performance criterion such as overshoot or settling time. For example, if it is required that a given control system have an overshoot of less than 10% and there is an uncertainty in the control gain, does the overshoot still remain less than 10% in the presence of that uncertainty? This is called *performance robustness*.

An example of performance robustness is given by Davison (1976). The steady state error of a control system is defined to be the difference between the response of the system and the desired response of the system as t becomes large in the regulator problem of Section 7.4. A logical measure of control system performance is then whether or not the steady state error is zero. A given system is then said to be *robust* if there exists a control that regulates the system with zero steady state error when subjected to perturbations in any of the matrices A, B, or C in Equations (7.1) and (7.2).

In the remainder of this section, only stability robustness is discussed. The approach presented here follows that of Patel and Toda (1980). As an example of stability robustness, consider a closed-loop system under state variable feedback of the form

$$\dot{\mathbf{x}} = (A + BK_f)\mathbf{x} = A'\mathbf{x} \tag{7.102}$$

The matrix A' contains the measurement and control matrices and is such that the closed-loop system is asymptotically stable. Let the matrix E_e denote the uncertainty in the system parameters (A) and gains (BK_f). Then, rather than having Equation (7.102), the system may be of the form

$$\dot{\mathbf{x}} = (A' + E_e)\mathbf{x} \tag{7.103}$$

In this equation the matrix E_e is not known [recall the discussion following Equation (3.93)], but rather only bounds on its elements are known. In particular, it is assumed here that each element of the uncertainty is bounded in absolute value by the same number, ε, so that

$$\left|(E_e)_{\mathrm{ij}}\right| < \varepsilon \tag{7.104}$$

for each value of the indices i and j. It was shown by Patel and Toda (1980) that the system with the uncertainty given by Equation (7.104) is asymptotically stable if

$$\varepsilon < \mu_\varepsilon$$

where

$$\mu_\varepsilon = \frac{1}{2n}\left(\frac{1}{\sigma_{\max}[F]}\right) \tag{7.105}$$

Here, F is a solution of the Lyapunov matrix equation [Equation (4.29)] in the form

$$A'^T F + FA' = -2I \tag{7.106}$$

where I is the $2n \times 2n$ identity matrix. The notation $\sigma_{\max}[F]$ refers to the largest singular value of the matrix F.

Example 7.10.1

Consider a closed-loop system defined by the augmented state matrix given by

$$A' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & -9 & 4 \\ 1 & -1 & 4 & -4 \end{bmatrix}$$

the solution of the Lyapunov equation [Equation (7.106)] yields

$$F = \begin{bmatrix} 3.5495 & -0.6593 & 0.3626 & 0.0879 \\ 0.6593 & 4.6593 & 0.6374 & 1.6374 \\ 0.3626 & 0.6374 & 0.3956 & 0.5495 \\ 0.0879 & 1.6374 & 0.5495 & 1.2088 \end{bmatrix}$$

The singular values of F are calculated to be

$$\sigma_1 = 5.5728, \qquad \sigma_2 = 3.5202, \qquad \sigma_3 = 0.6288, \qquad \sigma_4 = 0.0914$$

From Equation (7.105) the value of μ_ε becomes $(n = 2)$

$$\mu_\varepsilon = \left(\frac{1}{4}\right)\left(\frac{1}{5.5728}\right)$$

Hence, as long as the parameters are not changed more than 1.3676, i.e., as long as

$$\left|(E_e)_{\mathrm{ij}}\right| < 1.3676$$

the system defined by $(A' + E_e)$ will be asymptotically stable.

Many other formulations and indices can be used to discuss stability robustness. For instance, Rew and Junkins (1986) use the sensitivity (see Section 6.5) of the closed-loop matrix A' in Equation (7.102) to discuss robustness in terms of the condition number of A'. Kissel and Hegg (1986) have discussed stability robustness with respect to neglected dynamics in control system design. They examine how stability of a closed-loop system is affected if a reduced-order model is used in designing the feedback law. Zhou and Doyle (1997) provide a complete account of the use of robustness principles in control design for both stability and performance.

7.11 POSITIVE POSITION FEEDBACK

A popular modal control method with experimentalists and structural control engineers is called *positive position feedback* (Goh and Caughey, 1985), which adds additional dynamics to the system through the control law. A unique feature of the positive position feedback (PPF) approach is that it can be designed around an experimental transfer function of the structure and does not require an analytical model of the system or plant to be controlled. Goh and Caughey proposed using a special dynamic feedback law designed specifically for use in the second-order form, compatible with Newton's formulation of the equations of motion of a structure. These PPF control circuits are designed to roll off at higher frequency and hence are able to avoid exciting residual modes and introducing spillover as discussed in Section 7.8.

To illustrate the PPF formulation, consider the single-degree-of-freedom system (or alternatively, a single mode of the system)

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = bu \tag{7.107}$$

where ζ and ω_n are the damping ratio and natural frequency of the structure, and b is the input coefficient that determines the level of force applied to the mode of interest. The PPF control is implemented using an auxiliary dynamic system (compensator) defined by

$$\ddot{\eta} + 2\,\zeta_f\,\omega_f\,\dot{\eta} + \omega_f^2\,\eta = g\omega_f^2\,x$$
$$u = \frac{g}{b}\,\omega_f^2\,\eta \tag{7.108}$$

Here, ζ_f and ω_f are the damping ratio and natural frequency of the controller, and g is a constant. The particular form of Equation (7.108) is that of a second-order system much like a damped vibration absorber. The idea is to choose the PPF frequency and damping ratio so that the response of the structural mode has the desired damping. Combining Equations (7.107) and (7.108) gives the equations of motion in their usual second-order form, which, assuming no external force, is as follows:

$$\begin{bmatrix} \ddot{x} \\ \ddot{\eta} \end{bmatrix} + \begin{bmatrix} 2\zeta\omega_n & 0 \\ 0 & 2\zeta_f\omega_f \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} \omega_n^2 & -g\omega_f^2 \\ -g\omega_f^2 & \omega_f^2 \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.109}$$

Since the stiffness matrix couples the two coordinates, increasing the filter damping, ζ_f, will effectively add damping to the structural mode. Note also that this is a stable closed-loop system if the symmetric 'stiffness' matrix is positive definite for appropriate choices of g and ω_f, that is, if the determinant of displacement coefficient matrix is positive, which happens if

$$g^2\omega_f^2 < \omega_n^2 \tag{7.110}$$

Notice that the stability condition only depends on the natural frequency of the structure, and not on the damping or mode shapes. This is significant in practice because, when building an experiment, the frequencies of the structure are usually available with a reasonable accuracy, while mode shapes and damping ratios are much less reliable. The design of the controller then consists of choosing g and ω_f that satisfy inequality (7.110) and choosing ζ_f large enough to add significant damping to the structural mode. Note that the gains of the controller ζ_f, g, and ω_f are chosen electronically.

The stability property of PPF is also important because it can be applied to an entire structure, eliminating spillover by rolling off at higher frequencies. That is, the frequency response of the PPF controller has the characteristics of a low-pass filter. The transfer function of the controller is

$$\frac{\eta(s)}{X(s)} = \frac{g\omega_f^2}{s^2 + 2\zeta_f\omega_f s + \omega_f^2} \tag{7.111}$$

illustrating that it rolls off quickly at high frequencies. Thus, the approach is well suited to controlling a mode of a structure with frequencies that are well separated, as the controller is insensitive to the unmodeled high-frequency dynamics. If the problem is cast in the state space, the term $\mathbf{b}_{2n-k}$ in Equation (7.91) is zero, and no spillover results.

The positive position terminology in the name PPF comes from the fact that the position coordinate of the structure equation is positively fed to the filter, and the position coordinate of the compensator equation is positively fed back to the structure.

Next, suppose a multiple-degree-of-freedom analytical model of the structure is available. Following the formulation of output feedback discussed in Section 4.10 for an SISO system with no applied force yields

$$M\ddot{\mathbf{q}} + D\dot{\mathbf{q}} + K\mathbf{q} = B_f\mathbf{u} \tag{7.112}$$

Coupling this with the PPF controller in the form given in Equation (7.108) written as

$$\begin{aligned} \ddot{\eta} + 2\,\zeta_f\,\omega_f\,\dot{\eta} + \omega_f^2\,\eta &= g\omega_f^2\,B_f^T\mathbf{q} \\ u &= g\,\omega_f^2\,\eta \end{aligned} \tag{7.113}$$

yields

$$\begin{bmatrix} M & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\eta} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & 2\zeta_f\omega_f \end{bmatrix}\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\eta} \end{bmatrix} + \begin{bmatrix} K & -g\omega_f B_f \\ -g\omega_f B_f^T & \omega_f^2 \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \eta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7.114}$$

The system is SISO, so that B_f is a vector since u is a scalar. The augmented mass matrix in Equation (7.114) is symmetric and positive definite, the augmented damping matrix is symmetric and positive semidefinite, so the closed-loop stability will depend on the definiteness of the augmented stiffness matrix.

Consider then the definiteness of the augmented stiffness matrix defined by

$$\hat{K} = \begin{bmatrix} K & -g\omega_f B_f \\ -g\omega_f B_f^T & \omega_f^2 \end{bmatrix} \tag{7.115}$$

Let $\mathbf{x}$ be an arbitrary vector partitioned according to $\hat{K}$ and compute

$$\mathbf{x}^T\hat{K}\mathbf{x} = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}\begin{bmatrix} K & -g\omega_f B_f \\ -g\omega_f B_f^T & \omega_f^2 \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \mathbf{x}_1^T K\mathbf{x}_1 - g\omega_f\mathbf{x}_1^T B_f\mathbf{x}_2 - g\omega_f\mathbf{x}_2^T B_f^T\mathbf{x}_1 + \omega_f^2\mathbf{x}_2^T\mathbf{x}_2$$

Completing the square and factoring yields

$$\mathbf{x}^T\hat{K}\mathbf{x} = \mathbf{x}_1^T(K - g^2B_f^TB_f)\mathbf{x}_1 + \left(gB_f^T\mathbf{x}_1 - \omega_f\mathbf{x}_2\right)^T\left(gB_f^T\mathbf{x}_1 - \omega_f\mathbf{x}_2\right)$$

This is of the form

$$\mathbf{x}^T\hat{K}\mathbf{x} = \mathbf{x}_1^T(K - g^2B_f^TB_f)\mathbf{x}_1 + \mathbf{y}^T\mathbf{y}$$

for arbitrary $\mathbf{x}_1$ and $\mathbf{y}$. Since $\mathbf{y}^T\mathbf{y}$ is always nonnegative, $\hat{K}$ will be positive definite if the matrix $K - g^2B_f^TB_f$ is positive definite. If g is chosen such that $K - g^2B_f^TB_f$ is positive definite, then the closed-loop system will be stable (in fact asymptotically stable via the discussion in Section 4.5 since the coefficient matrices do not commute).

Example 7.11.1

Consider the two-degree-of-freedom system of example 7.9.1 and design a PPF controller to add damping to the first mode of the system without affecting the second mode.

For a single actuator at the location of the first mass, the input matrix becomes

$$B_f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The augmented mass and stiffness matrices of Equation (7.114) become

$$\hat{M} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \hat{D} = \begin{bmatrix} 9 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2\zeta_f\omega_f \end{bmatrix}, \qquad \hat{K} = \begin{bmatrix} 27 & -3 & -g\omega_f \\ -3 & 3 & 0 \\ -g\omega_f & 0 & \omega_f^2 \end{bmatrix}$$

Following the constraint given by inequality (7.110) for controlling the first mode $g^2\omega_f^2 < \omega_1^2 = 2$, one free choice is $g = \omega_f = 1$. Choosing the PPF damping of $\zeta_f = 0.5$ results in the following closed-loop damping ratios and frequencies:

$$\begin{aligned} \omega_1 &= 1.43, & \zeta_1 &= 0.237 \\ \omega_2 &= 2.00, & \zeta_2 &= 0.332 \\ \omega_f &= 0.966, & \zeta_f &= 0.531 \end{aligned}$$

as computed from the corresponding state matrix. Note that damping is added to the first mode (from 0.235 to 0.237) while mode two damping is only slightly changed and the frequency is not changed at all. All the modes change slightly because the system is coupled by the filter and, as was shown in example 7.9.1, multiple sensors and actuators are required to affect only one mode.

7.12 MATLAB COMMANDS FOR CONTROL CALCULATIONS

Most of the calculations in this chapter are easily made in MATLAB. MATLAB contains a 'toolbox' just for controls, called the Control System Toolbox (Grace *et al.*, 1992). This is a series of algorithms expressed in m-files that implements common control design, analysis, and modeling methods. Many websites are devoted to using and understanding the Control System Toolbox. Table 7.1 lists some common commands useful in implementing calculations for active control. The control commands in MATLAB assume the control problem is a linear time-invariant system of the form

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}, \qquad \mathbf{y} = C\mathbf{x} + D\mathbf{u} \tag{7.116}$$

The developments in this chapter assume that $D = 0$.

The following examples illustrate the use of some of the commands listed in Table 7.1 to perform some of the computations developed in the previous sections.

Table 7.1 MATLAB commands for control.

`ss2tf`	converts the state-space model to the system transfer function
`tf2ss`	converts a system transfer function to a state-space model
`step`	computes the step response of a system
`initial`	computes the response of a system to initial conditions
`impulse`	computes the response of a system to a unit impulse
`ctrb`	computes the controllability matrix
`obsv`	computes the observability matrix
`gram`	computes the controllability and observability grammians
`balreal`	computes the balanced realization
`place`	computes the pole placement gain matrix
`lqr`	computes the linear quadratic regulator solution
`modred`	computes the reduced-order model

Example 7.12.1

The function `initial` can be used to plot the response of the system of Equation (7.116) to a given initial condition. Consider the system in state-space form of a single-degree-of-freedom system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and plot the response using `initial` . Type the following in the command window:

```
>>clear all
>>A=[0 1;-4 -2];
>>b=[1;0];
>>c=[1 0];
>>d=[0];
>>x0=[1 0];
>>t=0:0.1:6;
>>initial(A,b,c,d,x0,t);
```

This results in the plot given in Figure 7.14.

Next, consider using MATLAB to solve the pole placement problem. MATLAB again works with the state-space model of Equation (7.116) and uses full-state feedback ($\mathbf{u} = -K\mathbf{x}$) to find the gain matrix K that causes the closed-loop system ($A - \mathbf{b}K$) to have the poles specified in the vector $\mathbf{p}$. The following example illustrates the use of the `place` command.

Example 7.12.2

Use the system of example 7.12.1 and compute the gain that causes the closed-loop system to have two real poles: -2 and -4. Using the code in Figure 7.12.1 to enter the state-space model (`A, b, c, d`), then type the following in the command window:

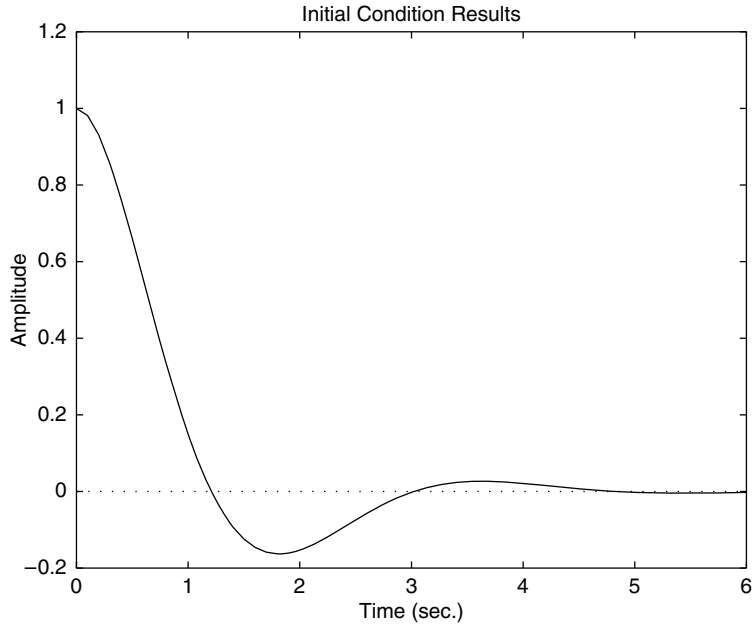

Figure 7.14 Response computed and plotted using the `initial` command.

```
>>p=[ -2 -4];
>>K=place(A,b,p)
place: ndigits= 15
K =
   4   1
```

Thus, the proper gain matrix is a vector in this case. To check that the result works, type

```
>>eig(A-b*K)
ans =
   -2
   -4
>>eig(A)
ans =
  -1.0000+ 1.7321i
  -1.0000- 1.7321i
```

This shows that the computed gain matrix causes the system to move its poles from the two complex values $-1 \pm 1.7321i$ to the two real values -2 and -4.

Next, consider solving the optimal control problem using the linear quadratic regulator problem defined by Equation (7.28). The MATLAB command `lqr` computes the solution of the

matrix Ricatta equation [Equation (7.32)], calculates the gain matrix K, and then calculates the eigenvalues of the resulting closed-loop system $(A - BK)$ for a given choice of the weighting matrices Q and R defined in Section 7.4. In this case, the output matrix B must have the same number of columns as the matrix R. The following example illustrates the procedure.

Example 7.12.3

Compute an optimal control for the system of example 7.12.1. Here, we chose **Q** to be the identity matrix and **R** to be 2**Q**. The following is typed into the command window:

```
>>A=[0 1;-4 -2];
>>B=eye(2);% the 2x2 identity matrix
>>c=[1 0];
>>d=[0];
>>Q=eye(2);R= 2*Q;% weighting matrices in the
cost function
>>[K,S,E]=lqr(A,B,Q,R)
K =
   0.5791    0.0205
   0.0205    0.1309
S =
   1.1582    0.0410
   0.0410    0.2617
E =
  -1.3550+ 1.8265i
  -1.3550- 1.8265i
```

The vector `E` contains new (closed-loop) eigenvalues of the system with the gain computed (K) from the solution of the Ricatta equation (S). Note that the closed-loop eigenvalues are more heavily damped then the open-loop eigenvalues $(-1 \pm 1.7321j)$.

Next, consider the model reduction problem of Section 7.7. The balanced realization can be obtained from the `balreal` command, and the corresponding reduced-order model can be obtained from the `modred` command. The procedure is illustrated in the next example.

Example 7.12.4

Compute the balanced realization of the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & -0.3 & 0.1 \\ 1 & -1 & 0.1 & -0.1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Use the balance realization to reduce the order of the model on the basis of its singular values. First the state-space model is typed into the command window. Then a balanced realization is formed and the singular values computed. The singular values (denoted by `g` in the code that follows) are then used with the model reduction routine to compute a reduced-order model. Type the following into the command window:

```
>>A=[0 0 1 0;0 0 0 1;-2 1 -0.3 0.1;1 -1 0.1 -0.1];
>>b=[0;0;0;1]; c=[0 1 0 0]; d=[0];
>>[ab,bb,cb,g]=balreal(A,b,c) % computes the balanced
version and singular values
ab =
   -0.0326    0.6166    0.0192   -0.0275
   -0.6166   -0.0334   -0.0218    0.0280
    0.0192    0.0218   -0.1030    1.6102
    0.0275    0.0280   -1.6102   -0.2309
bb =
    0.7821
    0.7510
   -0.2369
   -0.3223
cb =
  0.7821  -0.7510  -0.2369  0.3223
g =
   9.3836
   8.4310
   0.2724
   0.2250
>>elim=find(g<1); %eliminate states with singular values less
than 1
>>[ar,br,cr]=modred(ab,bb,cb,d,elim) % compute the reduced
order model based on eliminating the states with g<1
ar =
  -0.0319    0.6173
  -0.6173   -0.0342
br =
   0.7743
   0.7595
cr =
   0.7743  -0.7595
```

The matrix `ar` and the vectors `br` and `cr` are the reduced-order state matrix and the input and output vectors and form the reduced-order model as defined in Equations (7.76) and (7.77).

There are other commands in the Control System Toolbox that can be used to simplify the calculation of control laws and models and to perform analysis of control systems. Texts in control and the web should be consulted for additional samples and commands.

CHAPTER NOTES

This chapter attempts to introduce some control concepts useful in vibration analysis and design. Linear control engineering is, of course, a topic in its own right, and hence many texts have been written on the subject. The reader is referred to the excellent text by Kuo and Gholoaraghi (2003) for an introduction to control topics and to Kailath (1980) for a more advanced treatment of linear systems and control topics. Section 7.2 on controllability and observability stresses the use of these concepts in physical coordinate systems. However, literally hundreds of papers have been written on the topic in the state-space formulation. In addition to those references listed in the text, useful conditions for controllability and observability in second-order models are given by Laub and Arnold (1984) and Bender and Laub (1985). The early important papers in linear control and computation are presented in Patel, Laub, and Van Dooren (1994).

Section 7.3 on eigenstructure assignment extends the pole placement topics of Section 6.6 to include placing eigenvectors as well as eigenvalues. Again, hundreds of papers appear in the literature on pole placement methods in the state space. The most comprehensive treatment of eigenstructure assignment with structures in mind is that by Andry, Shapiro, and Chung (1983), which motivated the development of Section 7.3. An excellent text on eigenstructure assignment including MATLAB codes is the one by Liu and Patton (1998).

The section on optimal control, Section 7.4, should be viewed by the reader as a very brief introduction. Again, optimal control is a topic that should be studied in detail by using one of the many texts written on the subject (see, for instance, Kirk, 1970). In addition, optimal control is an intense area of current research, with several journals devoted to the topic. Optimal control is an extremely useful tool in controlling unwanted vibration. As pointed out by example 7.4.1, however, an element of art remains in optimal control by virtue of the freedom to choose a cost function. Section 7.5 on observers and estimators follows the development given by Chen (1970, 1998) and Kailath (1980). Section 7.6 on realization also closely follows the development in Chen (1998). The purpose of introducing realization is to set the stage for its use in vibration testing in Section 8.6. This illustrates a recent application of a standard control topic (realization) to a vibration problem (testing) and hence should motivate serious vibration engineers to pay more attention to the topic of linear systems and control theory as a source of alternative solutions.

Model reduction first introduced in Section 6.8 is revisited in Section 7.7 in the state space. The balancing method presented is improved numerically by Laub (1980). This is again an important area of vibration research and design that benefits from developments that have taken place in linear system theory. There are many other approaches to model reduction, notably the work of Hyland and Bernstein (1985) and Skelton, Hughes, and Hablani (1982).

Section 7.8 on modal control in the state space follows the development given by Takahashi, Rabins, and Auslander (1970). Modal control is popular and appears as a tool in a large number of papers. The section on modal control in physical coordinates is an attempt to extend the idea of modal coordinates to second-order systems. This is again a popular method in vibration control problems (see, for instance, Meirovitch and Baruh, 1982). The section on robustness is intended to provide a brief introduction to a very popular topic in control. The development follows that used by Ridgely and

Banda (1986). Note the similarity between robustness and design sensitivity discussed in Section 6.5. Much activity surrounds the topic of robust control, and space has prevented the inclusion of much material in this area. Instead the reader is referred to Zhou and Doyle (1997). The material on PPF control of Section 7.10 comes from the article by Friswell and Inman (1999), which contains additional references to PPF formulations in first-order form and connects PPF to standard output feedback and optimal control laws.

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PROBLEMS

7.1 Consider the system of example 7.3.1 and determine if the system is controllable with one actuator placed at m_1 and observable with one sensor placed at m_2.

7.2 Recalculate the gain matrix G_f for example 7.3.1 with one actuator placed at m_2 and one sensor placed at m_1 to measure $q_1(t)$ only. Does the resulting system have the desired eigenstructure?

7.3 Consider the system given by $M = I$

$$D = \text{diag}[0.1 \quad 0.2] \qquad \text{and} \qquad K = \text{diag}[1 \quad 2]$$

Is the system controllable for $B = [0 \quad 1]^T$? For $B = [1 \quad 0]^T$?

7.4 Repeat example 7.3.1 with desired eigenvectors $[1 \quad -1]$ and $[1 \quad 1]$, desired eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$, and desired mass matrix

$$M_O = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Assume the same structures as in example 7.3.1, and place the sensors and actuators as you see fit.

7.5 Calculate G_f in example 7.3.1 if the desired eigenvalues are $\lambda_1 = 10$ and $\lambda_2 = 20$ with no specification of the system eigenvectors.

7.6 Show that for $\mathbf{b} = [1 \; 0 \; 0 \; \cdots]^T$ the system of Equation (7.12) can be assigned any eigenvalues by proper choice of the gain g_i.

7.7 Show that the mode participation factors are unique for a given set of initial conditions.

7.8 Show that the control given in example 7.4.1 for the case $Q = R = I$ satisfies inequality (7.27) by calculating some other values of $\mathbf{u}$ (i.e., H_f).

7.9 Consider example 7.5.1. Let the observer poles be placed at -2 and -2 so that the speed of the observer is comparable with that of the system. Calculate $\mathbf{e}(t)$ versus time and $\mathbf{x}(t)$ versus time for $\mathbf{e}(0) = [1 \;\; 0]^T$ and plot the results. Note that in this case, the observer does not do as well.

7.10 Calculate a realization for the vibration absorber problem of Section 6.2.

7.11 Calculate W_C and W_O for the vibration absorber problem of Equation (6.14). Use $m_0 = 1$, $m = 10$, $k = 5$, and $k_0 = \omega^2$.

7.12 Calculate the singular value decomposition of the following matrices

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 6 \\ 5 & -1 \\ 1 & 0 \end{bmatrix}$$

7.13 Consider a two-degree-of-freedom system similar to that of Figure 2.4. Discuss in physical terms how such a system would have to behave if it could be reduced to a single-degree-of-freedom model very accurately.

7.14 Using available software subroutines (for things like SVD), write a program to perform model reduction using balancing. Then reproduce the reduction illustrated in example 7.7.1.

7.15 Consider example 2.4.4. Write the system of Equation (2.26) in state-space form. Let $m_1 = 1$, $m_2 = 2$, $c_1 = c_2 = 0.1$, $k_1 = 1$, and $k_2 = 4$. Use the modal control approach of Section 7.8 to design a control (i.e., choose g_1 and g_2) to raise the natural frequencies of the structure above 5 Hz. Add additional controls or measurements as necessary.

7.16 Referring to the statement of problem 7.15, calculate the minimum number of sensors and actuators required to control independently the lowest frequency; that is, for $k = 2$ in Equation (7.91), what control configuration is required for

$$\mathbf{b}_{2n-k} = \mathbf{c}_{2n-k}^T = \mathbf{0}$$

7.17 Again consider the feedback control problem for the example of Figure 2.4. Calculate B_f, K_v, and K_p for this system such that Equation (7.96) is diagonal (decoupled).

7.18 Make a physical interpretation of the calculation in problem 7.16 (i.e., where are the sensors and actuators located and does it make sense?).

7.19 Prove, or derive, that Equations (7.97) and (7.98) are in fact sufficient conditions for the closed-loop system to decouple.

7.20 Suppose that the commutivity condition assumed in Section 7.9 is not satisfied. Then the coefficient of $\dot{\mathbf{z}}$ in Equation (7.93) is *not* diagonal. Calculate conditions on the matrices B_f and K_v so that the coefficient $\dot{\mathbf{z}}$ in Equation (7.96) becomes diagonal. Such a control decouples a coupled system.

7.21 Based on your answer to problem 7.20, discuss the physical implications of the control in terms of, say, Figure 2.4.

7.22 Is the condition of problem 7.20 robust? Please discuss your answer.

7.23 Repeat example 7.10.1 for the system of problem 7.15, assuming that the coefficients c_1 and c_2 are only accurate to within 25%.

7.24 Compare the controllability norm of Equation (7.8) with the controllability grammian of Equation (7.58). Is there a mathematical relationship between them?

7.25 Show that the realization of example 7.6.1 is both controllable and observable. In general, show that Equations (7.56) and (7.57) are controllable and observable.

7.26 Using the criterion given in Equation (7.3), show that the system of Figure 7.1(a) is uncontrollable and that the system of Figure 7.1(b) is controllable.

7.27 Show that Equations (7.56) and (7.53) reduce to Equation (7.53) by using the definition of a transfer function given in Equation (7.51).

7.28 One method of performing a model reduction (Section 7.7) is to represent the open-loop system by the first few modes of the structure. Try this by using the first two (state-space) modes of example 7.7.1 as the ROM of the system. Then compare the response of this ROM to the ROM obtained in example 7.7.1 by the balancing method.

7.29 Design a PPF controller for the mode $\ddot{r}_i + 0.6\dot{r}_i + 9r_i = 0$ that increases the modal damping ratio to 0.3.

7.30 Compute the mode shapes, natural frequencies, and modal damping ratios for the closed-loop system of example 7.9.1, case 3, to see if the controller does just change the first mode damping ratio.

7.31 Consider example 7.9.1 and use pole placement to attempt to change just the first mode damping ratio (to 0.4).

7.32 Consider example 7.9.1 and use eigenstructure assignment to attempt to change just the first mode damping ratio (to 0.4).

7.33 Consider the system of example 7.9.1. Using pole placement, derive a control law that will cause the first natural frequency of the closed-loop system to be 1.6 rad/s and leave the second natural frequency fixed at 2 rad/s.

7.34 Use the eigenstructure assignment method to place the mode shapes of the system of example 7.3.1 to shift the second eigenvector to $[0.316 \quad 0.949]$.

7.35 Compute the time response of the system of example 7.9.1 using the gains from cases 1 and 3 and compare your results. Use an initial condition of zero velocity and the first coordinate displaced by 0.1 units.