Solutions Manual for

CONVECTION HEAT TRANSFER

Third Edition

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Contents

This solutions manual was organized and typed by Linda Hayes.

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Chapter 1

FUNDAMENTAL PRINCIPLES

Problem 1.1 We are told that the scales of the two major terms in the two groups of terms in eq. (1.5) or eq. (1.6) are measured experimentally:

$$\underbrace{\frac{D\rho}{Dt}}_{\left(\sim u\frac{\partial \rho}{\partial x}\right)} + \underbrace{\rho \nabla\cdot\mathbf{v}}_{\left(\sim \rho\frac{\partial u}{\partial x}\right)} = 0 \qquad \leftarrow \text{scales}$$

Therefore, if eq. (1.8) is to apply, then the first scale must be negligible,

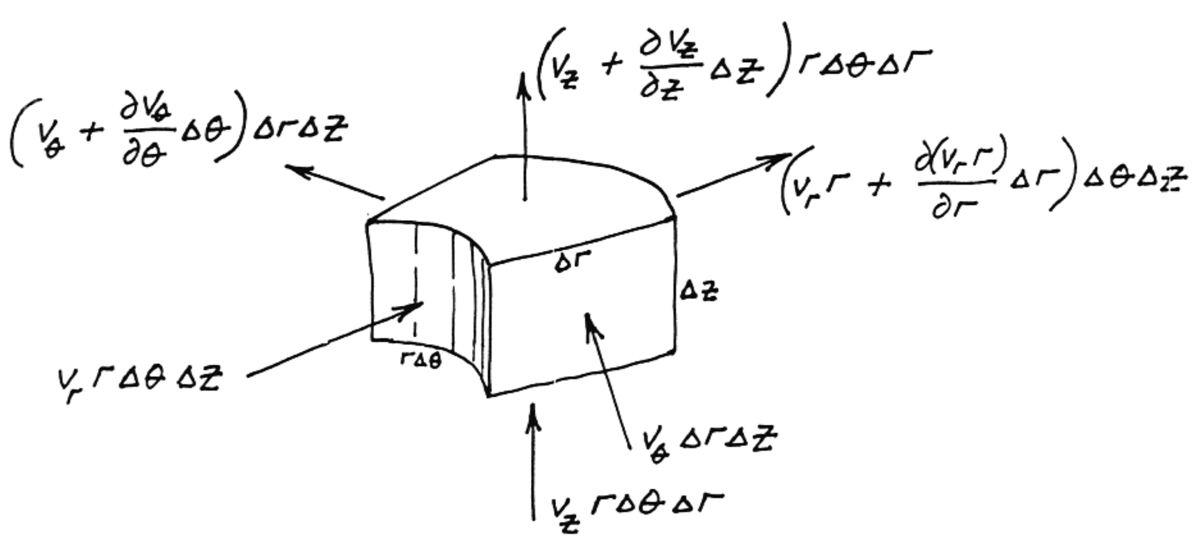
$$u\frac{\partial \rho}{\partial x} < \rho\frac{\partial u}{\partial x}$$

in other words, the relationship between $\partial\rho/\partial x$ and $\partial u/\partial x$ must be

$$\frac{\partial\rho/\partial x}{\partial u/\partial x} < \frac{\rho}{u}$$

Note that "<" means "less than, in an order-of-magnitude sense", or "negligible with respect to". The scale analysis literature often uses "<<" to say the same thing; in the present treatment I use "<", because one sign is enough when we compare orders of magnitude (the use of multiple signs such as "<<" leads to the temptation to read too much in the length of the sign, for example, by using something like "<<<" to stress the word "negligible").

Problem 1.2. Consider the control volume $(\Delta r)(r\Delta\theta)(\Delta z)$ drawn around the point (r,θ,z) in Fig. 1.1. Around this control volume we write graphically eq. (1.1):



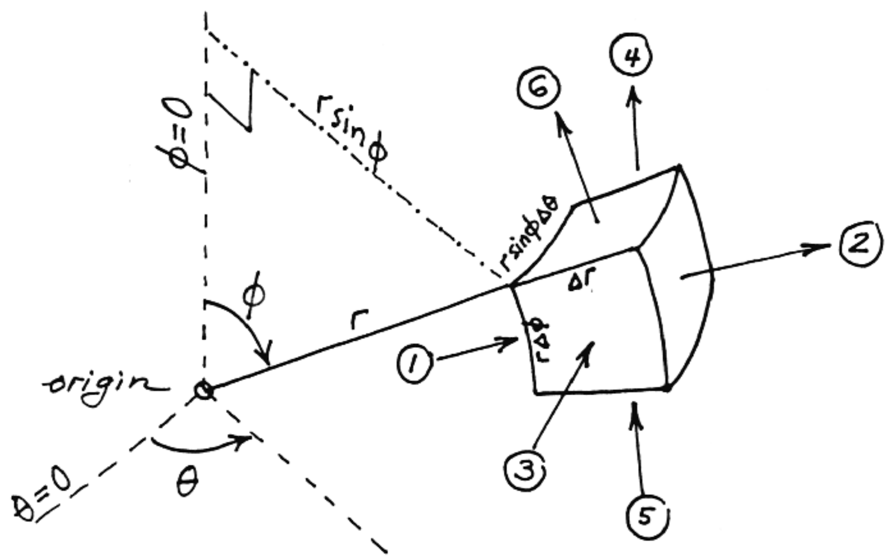
The term $\partial M_{cv}/\partial t$ is zero because ρ is constant. Note also that the "in" arrows cancel, respectively, the leading terms of the "out" arrows. Dividing the three surviving terms by the control volume $r\Delta\theta\Delta r\Delta z$, we are left with

$$\frac{1}{r}\frac{\partial}{\partial r}(v_r r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0,$$

which is the same as eq. (1.9),

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

Problem 1.3. Consider the control volume described by the point (r,θ,ϕ) in Fig. 1.1, as r, θ and ϕ change by Δr, $\Delta\theta$ and $\Delta\phi$, respectively



mass flowrates "in":

(1) $v_r\, r\Delta\phi\; r \sin\phi\Delta\theta$

(3) $v_\theta\, r\Delta\phi\, \Delta r$

(5) $v_\phi\, \Delta r\, r \sin\phi\, \Delta\theta$

mass flowrates "out":

(2) $\left(v_r r^2 + \frac{\partial}{\partial r}(v_r r^2)\Delta r\right) \sin\phi\, \Delta\phi\, \Delta\theta$

(4) $\left(v_\theta + \frac{\partial}{\partial \theta}(v_\theta)\Delta\theta\right) r\Delta\phi\, \Delta r$

(6) $\left(v_\phi \sin\phi + \frac{\partial}{\partial \phi}(v_\phi \sin\phi)\Delta\phi\right) r\, \Delta r\, \Delta\theta$

Since $\frac{\partial M_{cv}}{\partial t} = 0$, the six flowrates add up to

$$\frac{1}{r}\frac{\partial}{\partial r}(v_r r^2) + \frac{1}{\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}(v_\phi \sin\phi) = 0,$$

which is the same as eq. (1.10).

Problem 1.4. The mass conservation equation for constant-density flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \text{or} \qquad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

With this property in mind, the x-momentum equation (1.17) can be simplified:

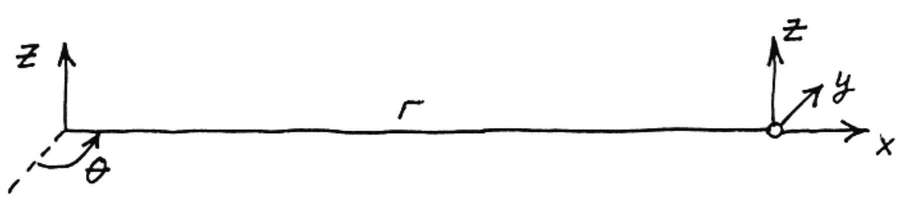
$$\rho \frac{Du}{dt} = -\frac{\partial P}{\partial x} + \underbrace{\frac{\partial}{\partial x}\Big[2\mu\frac{\partial u}{\partial x}}_{\mu\frac{\partial^2 u}{\partial x^2} - \underbrace{\mu\frac{\partial^2 v}{\partial x\,\partial y}}} - \frac{2\mu}{3}\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{0}\Big] + \underbrace{\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]}_{\mu\frac{\partial^2 u}{\partial y^2}+\underbrace{\mu\frac{\partial^2 v}{\partial y\,\partial x}}} + X$$

$$\longrightarrow 0 \longleftarrow$$

In conclusion, we obtain

$$\rho \frac{Du}{Dt} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + X \tag{1.18}$$

Problem 1.5. Graphically, the limit $r \to \infty$ and the transformation $\Delta r \to \Delta x$, $r\Delta\theta \to \Delta y$, $\Delta z \to \Delta z$ can be sketched as follows:



In eq. (1.9) we have

$$\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} + \underbrace{\frac{v_r}{r}}_{\infty} + \underbrace{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}}_{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

or, since $v_r \to u$, $v_\theta \to v$ and $v_z \to w$,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.8}$$

The momentum equations (1.21) have the same property; for example, the r equation (1.21a) can be written as

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} + \underbrace{\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}}_{\partial y} - \underbrace{\frac{v_\theta^2}{r}}_{\infty} + v_z \frac{\partial v_r}{\partial z}\right) =$$

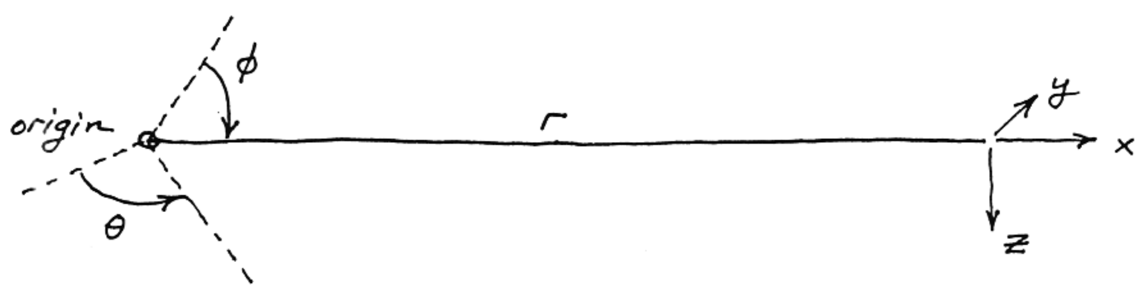
$$= -\underbrace{\frac{\partial P}{\partial r}}_{\partial x} + \mu\left(\underbrace{\frac{\partial^2 v_r}{\partial r^2}}_{\partial x^2} + \underbrace{\frac{1}{r}}_{\infty}\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} - \underbrace{\frac{v_r}{r^2}}_{\infty^2} + \underbrace{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}}_{\partial y^2} - \underbrace{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}_{\infty\, \partial y} + \frac{\partial^2 v_r}{\partial z^2}\right) + F_r,$$

in other words,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + F_r \qquad (1.19a)$$

The message of this exercise is that, through a simple transformation, the validity of equations in cylindrical coordinates may be tested based on the considerably more familiar Cartesian forms.

Problem 1.6. In the $r \to \infty$ limit, the spherical coordinates sketched in Fig. 1.1 become



in other words, $\Delta r \to \Delta x$, $r \sin\phi\, \Delta\theta \to \Delta y$ and $r\Delta\phi \to \Delta z$. The mass continuity equation (1.10) can be expanded as:

$$\underbrace{\frac{\partial v_r}{\partial r}}_{\partial x} + \underbrace{\frac{2\, v_r}{r}}_{\infty} + \underbrace{\frac{1}{r}\frac{\partial v_\phi}{\partial \phi}}_{\partial z} + \underbrace{\frac{\text{cotan}\, \phi}{r}}_{\infty} + \underbrace{\frac{1}{r \sin\phi}\frac{\partial v_\theta}{\partial \theta}}_{\partial y} = 0$$

Noting that $v_r \to u$, $v_\theta \to v$ and $v_\phi \to w$, the above equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} + \frac{\partial v}{\partial y} = 0 \qquad (1.8)$$

Following the same procedure, the momentum equation (1.22a) reduces to eq. (1.19a).

Problem 1.7. Taking the specific kinetic energy into account, e is replaced by $e + V^2/2$ in eq. (1.25). This substitution generates two more terms:

$$\{\ \}_1^* = \Delta x\, \Delta y \frac{\partial}{\partial t}\left(\rho \frac{V^2}{2}\right)$$

$$\{\ \}_2^* = -\,\Delta x\, \Delta y\left[\frac{\partial}{\partial x}\left(\rho u \frac{V^2}{2}\right) + \frac{\partial}{\partial y}\left(\rho v \frac{V^2}{2}\right)\right]$$

Together, these additional terms combine into one new term on the left-hand-side ("LHS") of eq. (1.25),

$$LHS_* = \Delta x\ \Delta y\ \rho \frac{D\left(V^2/2\right)}{Dt}$$

where we have assumed a constant-ρ flow. This contribution, i.e. LHS_*, is cancelled by the last four terms in the work transfer group $\{\ \}_5$: to see this, we combine the constitutive relations (1.15) and (1.16) with the last four work terms, and get

$$\begin{aligned}\frac{1}{\Delta x \Delta y}\left\{\begin{matrix}\text{the last}\\ \text{four terms}\end{matrix}\right\}_5 &= u\left[\frac{\partial P}{\partial x} - 2\mu \frac{\partial^2 u}{\partial x^2} + \frac{2}{3}\mu \frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{0,\ \text{cf. eq. (1.8)}}\right] \\ &\quad - \mu u \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ &\quad + v\left[\frac{\partial P}{\partial y} - 2\mu \frac{\partial^2 v}{\partial y^2} + \frac{2}{3}\mu \frac{\partial}{\partial y}\overbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}^{0}\right] \\ &\quad - \mu v \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\end{aligned}$$

In the above expression we eliminate the pressure gradients using the momentum equations (1.19a,b)

$$\frac{\partial P}{\partial x} = -\rho \frac{Du}{Dt} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial P}{\partial y} = -\rho \frac{Dv}{Dt} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

After some algebra, the last four work terms reduce to

$$\frac{1}{\Delta x \Delta y}\left\{\begin{matrix}\text{the last}\\ \text{four terms}\end{matrix}\right\}_5 = -\rho \frac{D(u^2/2)}{Dt} - \rho \frac{D(v^2/2)}{Dt} = -\frac{D(V^2/2)}{Dt}$$

which amounts to a new term on the right-hand-side of eq. (25)

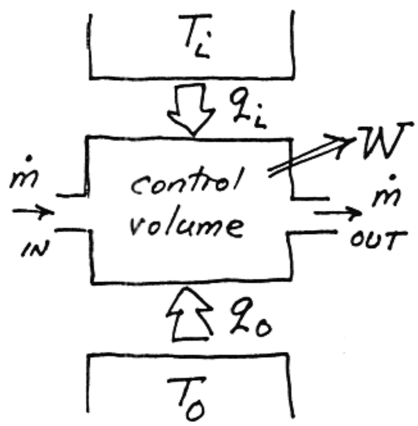
$$\text{RHS}_* = \Delta x \Delta y\, \rho\, \frac{D(V^2/2)}{Dt}$$

This additional term cancels LHS*. In conclusion, the energy equation (1.39) is valid even when the specific kinetic energy is accounted for in the derivation.

Problem 1.8. Consider the general flow system shown in the sketch. The first and second laws of thermodynamics state, respectively,

$$W = \sum q_i + q_o + \sum_{in} \dot{m}h - \sum_{out} \dot{m}h - \frac{\partial E_{cv}}{\partial t}$$

$$S_{gen} = \frac{\partial S_{cv}}{\partial t} - \sum \frac{q_i}{T_i} - \frac{q_o}{T_o} - \sum_{in} \dot{m}s + \sum_{out} \dot{m}s \geq 0$$



The reason for distinguishing q_o from the remaining heat transfer interactions q_i is that, if W is to vary with the degree of irreversibility (S_{gen}), then, according to the first law, at least one additional energy interaction must vary. Let q_o be the energy interaction that varies when W varies. In the extreme case of reversible operation, $S_{gen} = 0$, the second law requires

$$q_{o,rev} = T_o\left(-\frac{\partial S_{cv}}{\partial t} - \sum \frac{q_i}{T_i} - \sum_{in} \dot{m}s + \sum_{out} \dot{m}s\right)$$

In the same limit, the first law reads

$$W_{rev} = \sum q_i + q_{o,rev} + \sum_{in} \dot{m}h - \sum_{out} \dot{m}h - \frac{\partial E_{cv}}{\partial t}$$

The difference between the work output in the reversible limit, W_{rev}, and the actual work W is simply

$$W_{rev} - W = q_{o,rev} - q_o$$

However, from the second law for the general case ($S_{gen} > 0$), we recall that

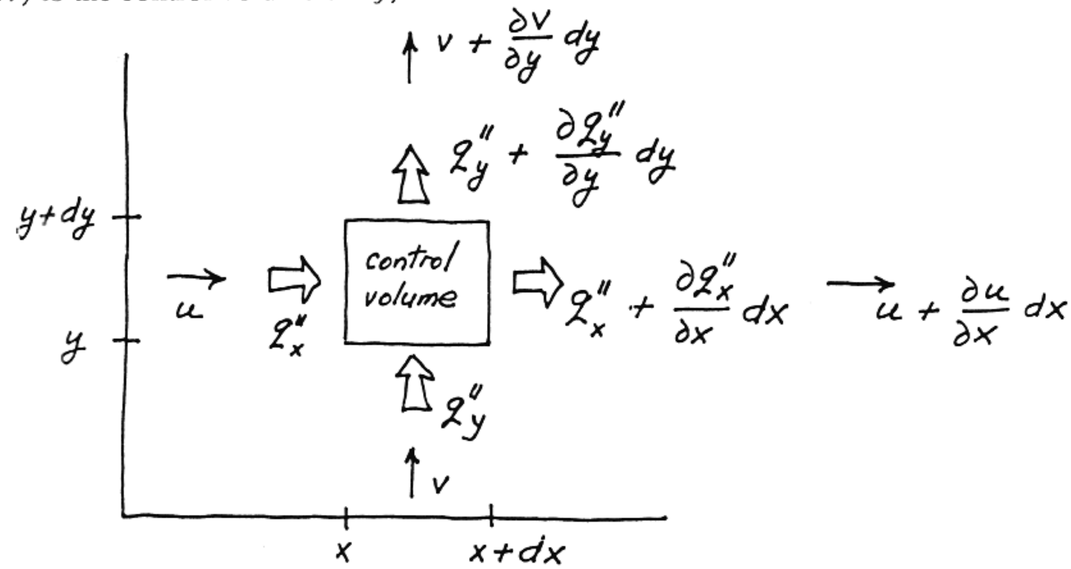
$$q_o = T_o\left(-S_{gen} + \frac{\partial S_{cv}}{\partial t} - \sum \frac{q_i}{T_i} - \sum_{in} \dot{m}s + \sum_{out} \dot{m}s\right)$$

Therefore, $q_o - q_{o,rev} = - T_o S_{gen}$, in other words

$$\underbrace{W_{rev} - W}_{W_{lost}} = T_o S_{gen} > 0$$

Note that T_o is the absolute temperature of the heat reservoir whose heat transfer rate q_o varies (floats) as W and the degree of irreversibility (S_{gen}) vary.

Problem 1.9. Consider the two-dimensional control volume of size dxdy shown below. The entropy generation rate in this system is S'''_{gen} dxdy: this quantity can be calculated by applying eq. (1.47) to the control volume dxdy,



$$S'''_{gen}dxdy = \frac{q''_x + \frac{\partial q''_x}{\partial x} dx}{T + \frac{\partial T}{\partial x} dx} dy + \frac{q''_y + \frac{\partial q''_y}{\partial y} dy}{T + \frac{\partial T}{\partial y} dy} dx - \frac{q''_x}{T} dy - \frac{q''_y}{T} dx$$

$$+ \left(s + \frac{\partial s}{\partial x} dx\right)\left(u + \frac{\partial u}{\partial x} dx\right)\left(\rho + \frac{\partial \rho}{\partial x} dx\right) dy$$

$$+ \left(s + \frac{\partial s}{\partial y} dy\right)\left(v + \frac{\partial v}{\partial y} dy\right)\left(\rho + \frac{\partial \rho}{\partial y} dy\right) dx$$

$$- su\rho\, dy - sv\rho\, dx + \frac{\partial(\rho s)}{\partial t} dxdy$$

Neglecting the terms smaller than (dxdy), we obtain

$$S'''_{gen} = \frac{1}{T}\left(\frac{\partial q''_x}{\partial x} + \frac{\partial q''_y}{\partial y}\right) - \frac{1}{T^2}\left(q''_x \frac{\partial T}{\partial x} + q''_y \frac{\partial T}{\partial y}\right) +$$

$$+ \rho\left(\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y}\right)$$

$$+ s\left[\underbrace{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{\text{zero, cf. eq. (1.5)}}\right]$$

In conclusion,

$$S'''_{gen} = \frac{1}{T}\,\nabla\cdot\mathbf{q}'' - \frac{1}{T^2}\,\mathbf{q}''\cdot\nabla T + \rho\,\frac{Ds}{Dt}$$

From the canonical relation $du = Tds - P\,d\left(\frac{1}{\rho}\right)$ we deduce that

$$\rho\,\frac{Ds}{Dt} = \frac{\rho}{T}\,\frac{Du}{Dt} - \frac{P}{\rho T}\,\frac{D\rho}{Dt}$$

and from the first law of thermodynamics, eq. (1.26),

$$\rho\,\frac{Du}{Dt} = -\,\nabla\cdot\mathbf{q}'' - P\,\nabla\cdot\mathbf{v} + \mu\Phi$$

Combining the last three equations to eliminate Ds/Dt and Du/Dt, we find

$$S'''_{gen} = -\frac{1}{T^2}\,\mathbf{q}''\cdot\nabla T + \frac{\mu}{T}\,\Phi$$

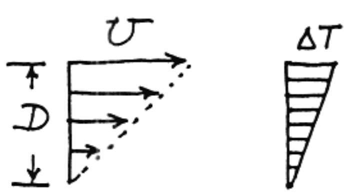
or, using the Fourier law of heat conduction, $\mathbf{q}'' = -\,k\,\nabla\,T$, in which k is constant,

$$S'''_{gen} = \frac{k}{T^2}\,(\nabla T)^2 + \frac{\mu}{T}\,\Phi \qquad (1.49)$$

Problem 1.10. In plane Couette flow we have

$$\frac{\partial u}{\partial y} = \frac{U}{D}, \quad \frac{\partial T}{\partial y} = \frac{\Delta T}{D}$$

so that eq. (1.49) becomes



$$S'''_{gen} = \underbrace{\frac{k}{T^2}\left(\frac{\Delta T}{D}\right)^2}_{\text{heat transfer irreversibility}} + \underbrace{\frac{\mu}{T}\left(\frac{U}{D}\right)^2}_{\text{fluid friction irreversibility}}$$

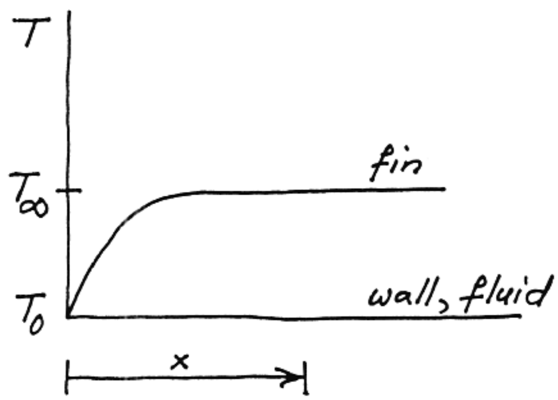
Thus, S'''_{gen} is dominated by fluid friction when

$$\frac{k\,(\Delta T)^2}{T^2\,D^2} < \frac{\mu\,U^2}{TD^2},$$

or when

$$\frac{U}{\Delta T}\left(\frac{\mu T}{k}\right)^{1/2} > 1$$

Problem 1.11. In general, the energy equation represents the competition between three effects, longitudinal conduction, lateral convection and internal heat generation,



$$\underbrace{kA\frac{d^2T}{dx^2}}_{\text{conduction}} - \underbrace{hP\,(T-T_0)}_{\substack{\text{convection}\\ \text{(plays the role of}\\ \text{"heat sink" term)}}} + \underbrace{q'''A = 0}_{\text{generation}}$$

a) In the system of length x the three competing scales are $kA\frac{\Delta T}{x^2}$, $hP\,\Delta T$ and $q'''A$, where $\Delta T = T_\infty - T_0$. In the limit $x \to \infty$, the conduction scale becomes negligible relative to convection and generation.

b) Sufficiently far from the wall the balance is $hP\,\Delta T \sim q'''A$, and this means that

$$T_\infty - T_0 \sim \frac{q'''A}{hP}$$

c) Sufficiently close to the wall we have $kA\frac{\Delta T}{\delta^2} \sim q'''A$, therefore we conclude that

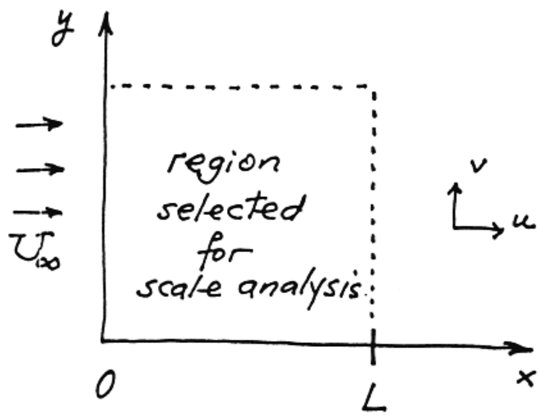
$$\delta \sim \left(\frac{k\,\Delta T}{q'''}\right)^{1/2}$$

d) The heat transfer rate into the wall is

$$q_B \sim kA\frac{\Delta T}{\delta} \sim$$
$$\sim A\,(q'''k\,\Delta T)^{1/2}.$$

Problem 1.12. The momentum equation (2.26) is

$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{inertia}} = \underbrace{-\frac{1}{\rho}\frac{dP_\infty}{dx}}_{\text{pressure}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\text{friction}}$$



In the space selected for scale analysis, we have

$$x \sim L, \quad y \sim L, \quad u \sim U_\infty$$

in other words,

$$\underset{\text{inertia}}{\frac{U_\infty^2}{L}}, \qquad \underset{\text{pressure}}{\frac{1}{\rho}\frac{d P_\infty}{dx}}, \qquad \underset{\text{friction}}{\frac{\nu U_\infty}{L^2}}$$

The ratio inertia/friction is of the order of

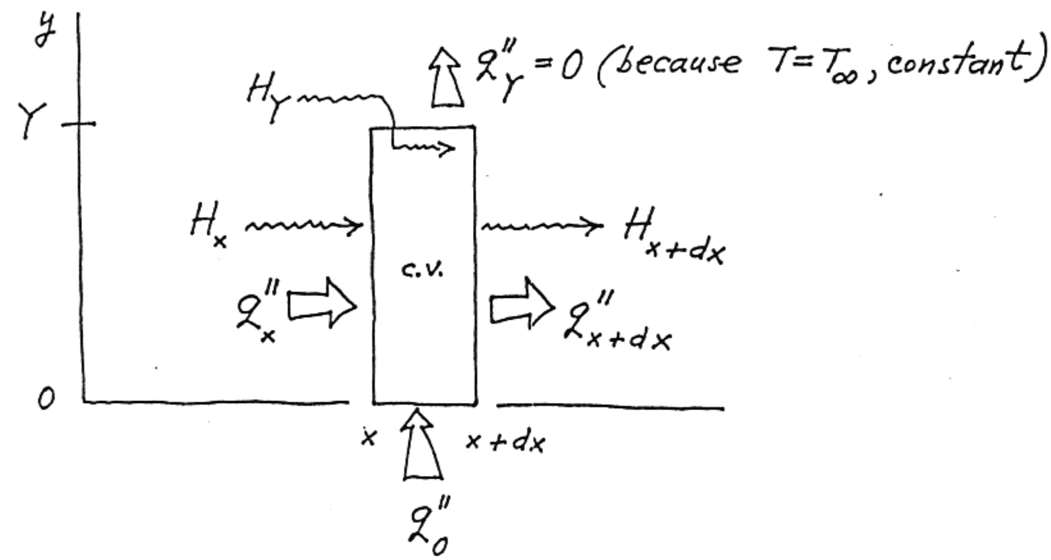
$$\frac{\text{inertia}}{\text{friction}} \sim \frac{U_\infty^2}{L}\,\frac{L^2}{\nu U_\infty} = \frac{U_\infty L}{\nu} = Re_L$$

If $Re_L = 10^3$, then inertia > friction, meaning that the friction effect can be neglected in the $L \times L$ region. In that region, the flow is ruled by the balance between the remaining effects, namely, inertia ~ pressure.

Observations: – the $L \times L$ region is not the Boundary Layer region;
– the "inertia/friction ~ Re_L" interpretation of the Reynolds number *does not apply* in the boundary layer region discussed in chapter 2.

LAMINAR BOUNDARY LAYER FLOW

Problem 2.1. In order to derive eq. (2.52), construct the energy-equivalent of the bottom portion of Fig. 2.3:



In the steady-state, the first law of thermodynamics for the control volume (c.v.) is

$$H_x + \int_0^Y q''_x \, dy - H_{x+dx} - \int_0^Y q''_{x+dx} \, dy + q''_0 \, dx + H_Y = 0$$

where

$$H_Y = c_P T_\infty d\dot{m}; \qquad \dot{m} = \int_0^Y \rho u \, dy; \qquad H_x = \int_0^Y \rho c_P u T \, dy$$

We assume T_∞ = constant. Dividing the first law by dx, we obtain

$$-\frac{d}{dx}\int_0^Y \rho c_P u T \, dy + c_P T_\infty \frac{d}{dx}\int_0^Y \rho u \, dy - \frac{d}{dx}\int_0^Y q''_x \, dy + q''_0 = 0$$

The scales of the last two terms in the last equation are, respectively,

$$\frac{1}{x} k \frac{\Delta T}{x} Y, \qquad k \frac{\Delta T}{Y}$$

Clearly, if x > Y [i.e. if the (x) × (Y) region is slender] the last scale dominates, and the third term in the energy equation can be neglected. We are left with

$$\frac{d}{dx}\int_0^Y \rho\, c_P\, u\,(T_\infty - T)\, dy = k\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

or, assuming $\rho\, c_P$ = constant,

$$\frac{d}{dx}\int_0^Y u\,(T_\infty - T)\, dy = \alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{2.52}$$

Problem 2.2. The Blasius profile problem reduces to solving

$$2f''' + ff'' = 0 \tag{2.82}$$

subject to

$$f(0) = f'(0) = 0 \tag{2.83}$$

$$f'(\infty) = 1 \tag{2.84}$$

Assuming the small-η expansion

$$f(\eta) = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + \ldots..$$

$$f'(\eta) = a_1 + 2a_2\eta + 3a_3\eta^2 + 4a_4\eta^3 + \ldots..$$

from the $\eta = 0$ conditions (2.83) we learn that

$$a_0 = a_1 = 0$$

The Blasius equation (2.82) becomes

$$2(3 \times 2 \times 1\, a_3 + 4 \times 3 \times 2\, a_4\, \eta + 5 \times 4 \times 3\, a_5\, \eta^2 + 6 \times 5 \times 4\, a_6\, \eta^3 + \ldots..)$$

$$+ (a_2\, \eta^2 + a_3\, \eta^3 + \ldots.)\,(2 \times 1\, a_2 + 3 \times 2\, a_3\, \eta + 4 \times 3\, a_4\, \eta^2 + \ldots.) = 0$$

or, in the form of a table to collect the same powers of η,

$$\left.\begin{array}{ccccccccc}
12\, a_3 & + & 48\, a_4\, \eta & + & 120\, a_5\, \eta^2 & + & 240\, a_6\, \eta^3 & + & \ldots\ldots.. \\
 & & & & 2\, a_2^2\, \eta^2 & + & 2\, a_3\, a_2\, \eta^3 & + & \ldots\ldots.. \\
 & & & & & & 6\, a_3\, a_2\, \eta^3 & + & \ldots\ldots.. \\
 & & & & & & & & \ldots\ldots..
\end{array}\right\} = 0$$

$$\begin{array}{cccc}
\downarrow & \downarrow & \downarrow & \\
a_3 = 0 & a_4 = 0 & 120\, a_5 + 2a_2^2 = 0 & \downarrow \\
 & & & 240\, a_6 + 8\, a_2 a_3 = 0 \\
 & & \downarrow & \\
 & & a_5 = -\dfrac{a_2^2}{60} & \downarrow \\
 & & & a_6 = 0
\end{array}$$

The coefficients of higher order, a_7, a_8,..., are determined in the same manner. The resulting series can be written as

$$f = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n \frac{\alpha^{n+1} C_n}{(3n+2)!} \eta^{3n+2}$$

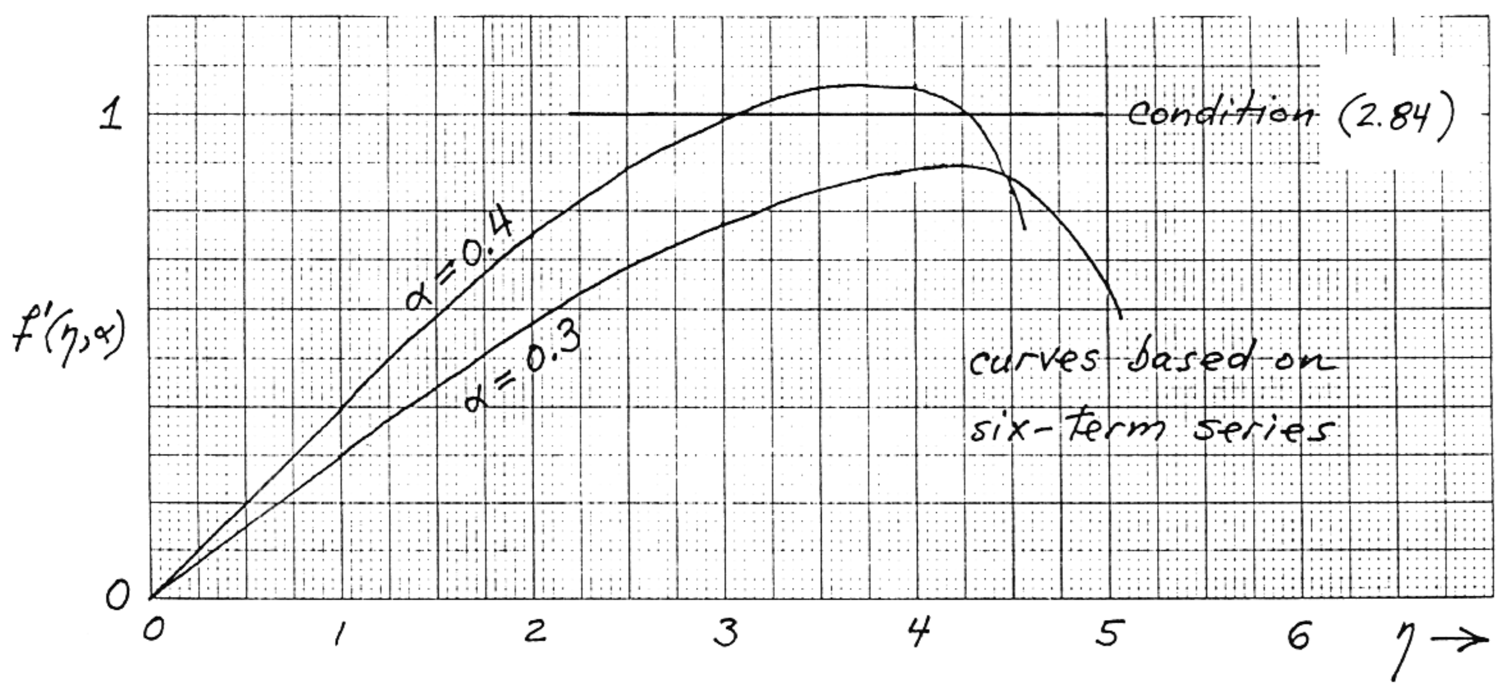
where $\alpha = f''(0)$ is the unknown curvature at the wall, and [13]

$$C_0 = 1 \qquad C_3 = 375$$

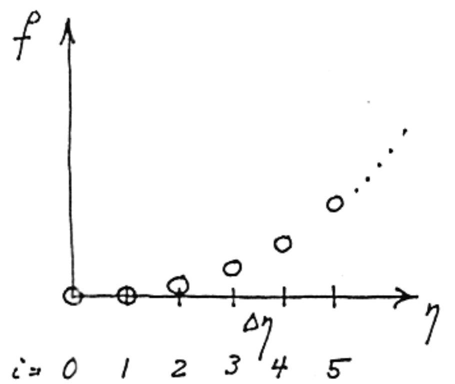
$$C_1 = 1 \qquad C_4 = 27{,}897$$

$$C_2 = 11 \qquad C_5 = 3{,}817{,}137$$

The unknown α follows from the remaining condition, eq. (2.84). Unfortunately, the six-term Blasius series listed above is acceptable only when η is small, say, $\eta < 2$ as illustrated in the sketch below. Nevertheless, the sketch of $f'(\alpha,\eta)$ shows that [$f' = 1$ at large η] is achieved when α is a number between 0.3 and 0.4. The correct value ($\alpha = 0.332$) is found by matching the small-η series with another expansion valid in the domain $\eta > O(1)$: this procedure is outlined in Ref. [13].



Problem 2.3. We divide the η space into slices of thickness $\Delta\eta$. At any node i, the Blasius equation (2.82) requires



$$2f_i''' + f_i f_i'' = 0$$

where $$f_i'' = \frac{f_{i+1} + f_{i-1} - 2f_i}{(\Delta\eta)^2}$$

$$f_i''' = \frac{f_{i+2} - 3f_{i+1} + 3f_i - f_{i-1}}{(\Delta\eta)^3}$$

Thus, the Blasius equation allows us to calculate f_{i+2} based on the values of f at the preceding three nodes,

$$f_{i+2} = 3f_{i+1} - 3f_i + f_{i-1} - \frac{\Delta\eta}{2} f_i (f_{i+1} + f_{i-1} - 2f_i)$$

The values at the first three nodes are

$f_0 = 0$, cf. condition (2.83)

$f_1 = 0$, so that $f_0' = \dfrac{f_1 - f_0}{\Delta\eta} = 0$; condition (2.83)

$f_2 = (\Delta\eta)^2\alpha$, where $\alpha = f_0''$ is the unknown curvature at the wall, which must be determined from condition (2.84)

$$f'(\infty) = 1$$

As indicated in the second part of the problem statement, it is sufficient to shoot once. Taking $\alpha = 1$, we find that the slope f' at large η's ($\eta = 10$) approaches the value

$$f'(10) = a \cong 2.088,$$

Therefore, the correct guess for the initial curvature is

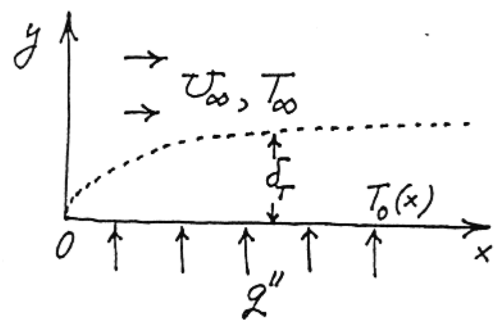
$$\alpha = 2.088^{-3/2} = 0.3314$$

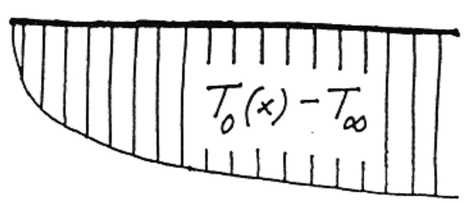
when the integration step is $\Delta\eta = 0.0125$. This value agrees very well with Blasius' well known number

$$f''(0) = 0.332$$

step size $\Delta\eta$	$f'(10)$
0.2	2.12113563
0.1	2.1019154
0.05	2.09333778
0.025	2.08946109
0.0125	2.08886862

Problem 2.4. The special feature of the heat transfer configuration in this problem is that the wall temperature T_0 must increase in the x-direction, in order for the constant flux q" to overcome the growing thermal boundary layer thickness, $\delta_T(x)$.





We are interested in the local Nusselt number

$$\mathrm{Nu} = \frac{q''}{T_0(x) - T_\infty} \frac{x}{k},$$

hence, in $T_0 - T_\infty$ or, from the graph, in $\delta_T(x)$. The energy integral for T_∞ = constant is

$$\frac{d}{dx} \int_0^Y u(T_\infty - T)\, dy = \alpha \left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (2.53)$$

subject to

$$q'' = -k \left(\frac{\partial T}{\partial y}\right)_{y=0} = \text{constant}$$

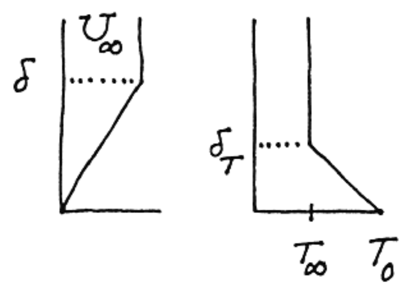
Assuming the simplest velocity and temperature profiles,

$$u = U_\infty \frac{y}{\delta} \qquad \text{and} \qquad \frac{T_0 - T}{T_0 - T_\infty} = \frac{y}{\delta_T},$$

we distinguish the following two limits:

$\delta_T < \delta$ (or $Pr > 1$). The energy equation becomes

$$\frac{d}{dx}\int_0^{\delta_T} U_\infty \frac{y}{\delta}\left(T_\infty - T_0\right)\left(1 - \frac{y}{\delta_T}\right) dy = \alpha \frac{T_\infty - T_0}{\delta_T}$$



where $$q'' = -k \frac{T_\infty - T_0}{\delta_T}, \text{ const.}$$

and, from Table 2.1,

$$\delta = 3.46\, x\, Re_x^{-1/2}$$

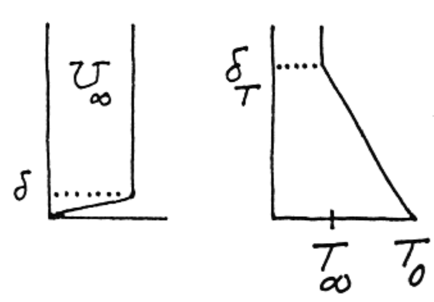
Integrating the energy equation, and noting that its right-hand-side is a constant, we obtain

$$\frac{\delta_T}{x} = 2.75\, Pr^{-1/3}\, Re_x^{-1/2}$$

In conclusion,

$$Nu = \frac{q''}{T_0 - T_\infty}\frac{x}{k} = \frac{x}{\delta_T} = 0.364\, Pr^{1/3}\, Re_x^{1/2} \qquad \text{(see Table 2.1)}$$

$\delta_T > \delta$ (or $Pr < 1$). In the extreme we have $\delta/\delta_T \to 0$, so that u may be taken as equal to U_∞ in the layer of thickness δ_T. The energy equation becomes



$$\frac{d}{dx}\int_0^{\delta_T} U_\infty (T_\infty - T_0)\left(1 - \frac{y}{\delta_T}\right) dy = \alpha \frac{T_\infty - T_0}{\delta_T}$$

where the right-hand-side is again constant. The solution is

$$\delta_T = \left(\frac{2\alpha x}{U_\infty}\right)^{1/2},$$

In other words,

$$Nu = \frac{x}{\delta_T} = 0.707\ Pr^{1/2}\ Re_x^{1/2}$$

Note that results for Pr << 1 are not listed in Table 2.1.

Problem 2.5. Since Pr > 1, the local Nusselt number given by Table 2.1 is

$$Nu = 0.332\ Pr^{1/3}\ Re_x^{1/2}$$

The Reynolds number follows from $C_{f,x} = 0.0066$ and Table 2.1,

$$C_{f,x} = 0.664\ Re_x^{-1/2} = 0.0066,$$

namely,

$Re_x = 1.01 \times 10^4$ (note that the boundary layer is entirely laminar, marginally; see Table 6.1).

In conclusion,

$$Nu = (0.332)\ (7^{1/3})\ (1.01 \times 10^4)^{1/2} = 63.9$$

Problem 2.6. The analysis that follows is a generalization of the analysis presented in the text, eqs. (2.93)-(2.107). The difference is that the boundary condition (2.95) is replaced by

$$\theta = J \frac{\partial \theta}{\partial \eta} \quad \text{at } \eta = 0 \tag{1}$$

where

$$J = \frac{k}{k_w}\left(\frac{U_\infty t^2}{\nu x}\right)^{1/2} \tag{2}$$

The J parameter is a constant if t(x) varies as

$$t = C\, x^{1/2} \tag{3}$$

The C constant is proportional to J,

$$C = J \frac{k_w}{k} \left(\frac{\nu}{U_\infty}\right)^{1/2} \tag{4}$$

The analysis up to eq. (2.97) is still valid. The new boundary condition (1) replaces eq. (2.98) with

$$\theta(\eta) - \theta(0) = \theta'(0) \int_0^\eta \exp[\,....\,]\, d\gamma \tag{5}$$

in which, cf. eq. (1),

$$\theta(0) = J\, \theta'(0) \tag{6}$$

If we set $\eta \to \infty$ and $\theta(\infty) = 1$ in eq. (5) we obtain

$$1 = \theta'(0) \left\{ J + \int_0^\infty \exp[\,....\,]\, d\eta \right\} \tag{7}$$

The integral that appears above has the value, cf. eqs. (2.99) and (2.102),

$$\int_0^\infty \exp[\,....\,]\, d\eta = \frac{1}{0.332\, Pr^{1/3}}, \qquad (Pr > 0.5) \tag{8}$$

The overall heat transfer rate through the wall of length L is, cf. eq. (2.101)

$$q' = q''_{0-L}\, L = k\,(T_\infty - T_0)\, 2\theta'(0)\, Re_L^{1/2} \tag{9}$$

The $J = 0$ limit of this result is obtained by integrating the local heat flux calculated with eq. (2.103),

$$q'\,(J = 0) = k\,(T_\infty - T_0)\, 0.664\, Pr^{1/3}\, Re_L^{1/2} \tag{10}$$

To see the effect of the solid coating (J), we estimate the ratio

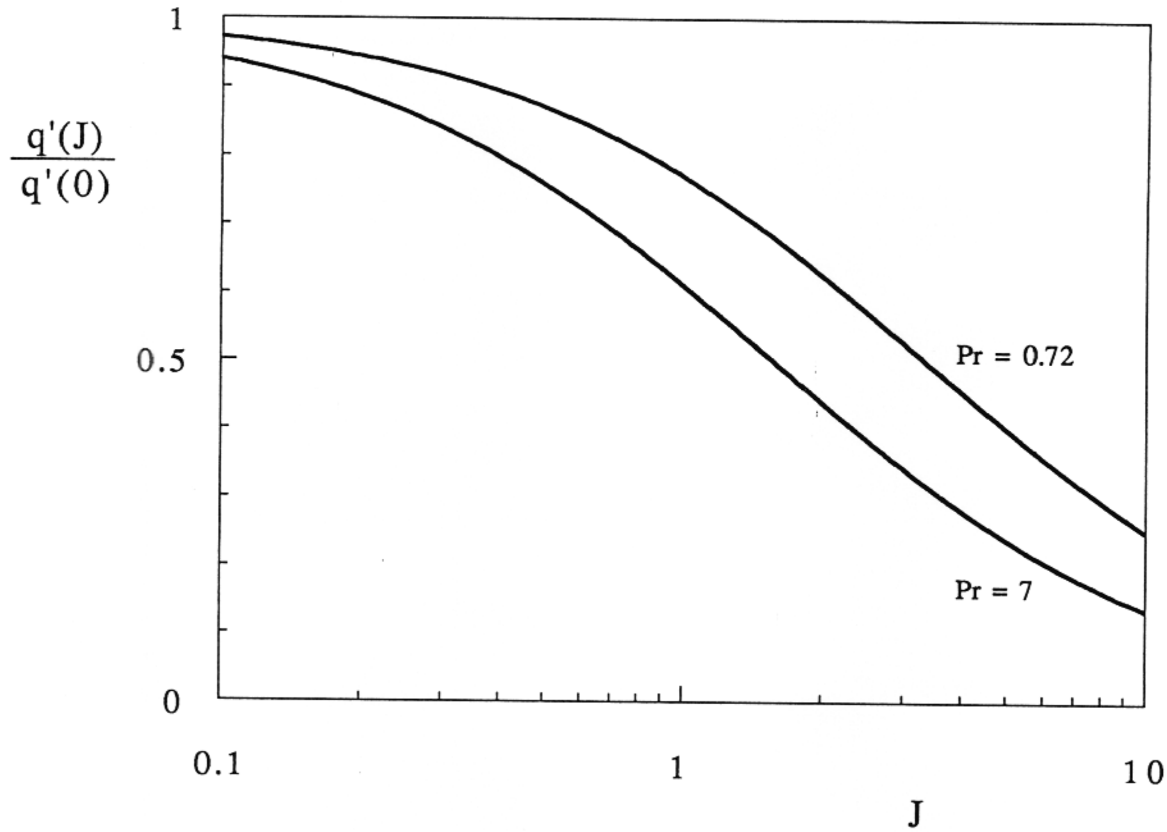
$$\frac{q'(J)}{q'(J=0)} = \frac{\theta'(0)}{0.332\, Pr^{1/3}} \tag{11}$$

$$= \frac{1}{0.332\, Pr^{1/3} \left\{ J + \int_0^\infty \exp[\,....\,]\, d\eta \right\}} \tag{12}$$

If we accept the high-Pr limit (8) this ratio becomes

$$\frac{q'(J)}{q'(J=0)} = \frac{1}{0.332\, Pr^{1/3}\, J + 1} \tag{13}$$

which is shown in the attached graph. The coating reduces the heat transfer rate when J exceeds the order of 1.



Problem 2.7. The problem statement for the thermal boundary layer over a constant-q" flat plate is

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \qquad (1)$$

$$k \frac{\partial T}{\partial y} = - q'' \quad \text{at} \quad y = 0 \qquad (2)$$

$$T \rightarrow T_\infty \quad \text{as} \quad y \rightarrow \infty \qquad (3)$$

The flow solution is known:

$$u = U_\infty f', \qquad (4)$$

$$v = \frac{1}{2} \left(\frac{\alpha U_\infty}{x} \right)^{1/2} (\eta f' - f) \qquad (5)$$

$$\eta = y \left(\frac{U_\infty}{\nu x} \right)^{1/2} \qquad (6)$$

To construct a similarity solution to problem (1)-(3), we must choose the dimensionless temperature θ in such a way that the transformed eqs. (1)-(3) depend solely on θ, η, $f(\eta)$ and Pr. The beginning of writing θ is easy,

$$\theta(\eta, Pr) = \frac{T - T_\infty}{\Delta T_{scale}} \tag{7}$$

The temperature difference scale ΔT_{scale} must be chosen such that the wall boundary condition (2) reduces to a statement involving only the similarity variables. We rewrite eq. (2) in an order of magnitude sense,

$$k\,\frac{\partial T}{\partial y} \sim k\,\frac{\Delta T_{scale}}{(\nu x/U_\infty)^{1/2}} \sim q'' \tag{8}$$

and draw the key conclusion that

$$\Delta T_{scale} = \frac{q''}{k}\left(\frac{\nu x}{U_\infty}\right)^{1/2} \tag{9}$$

Combining eqs. (7) and (9), we find the needed transformation from T to θ:

$$T = T_\infty + \theta \cdot \left(\frac{\nu x}{U_\infty}\right)^{1/2} \cdot \frac{q''}{k} \tag{10}$$

Beyond this point, the transformation of eqs. (1)-(3) is routine. The partial derivatives of eq. (1) are

$$\frac{\partial T}{\partial y} = \frac{q''}{k}\,\theta' \tag{11}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{q''}{k}\,\theta''\left(\frac{U_\infty}{\nu x}\right)^{1/2} \tag{12}$$

$$\frac{\partial T}{\partial x} = \frac{q''}{2k}\left(\frac{\nu}{U_\infty x}\right)^{1/2}(\theta - \eta\,\theta') \tag{13}$$

Equations (1)-(3) become, in order,

$$\theta'' + \frac{Pr}{2}\,(f\,\theta' - f'\,\theta) = 0 \tag{14}$$

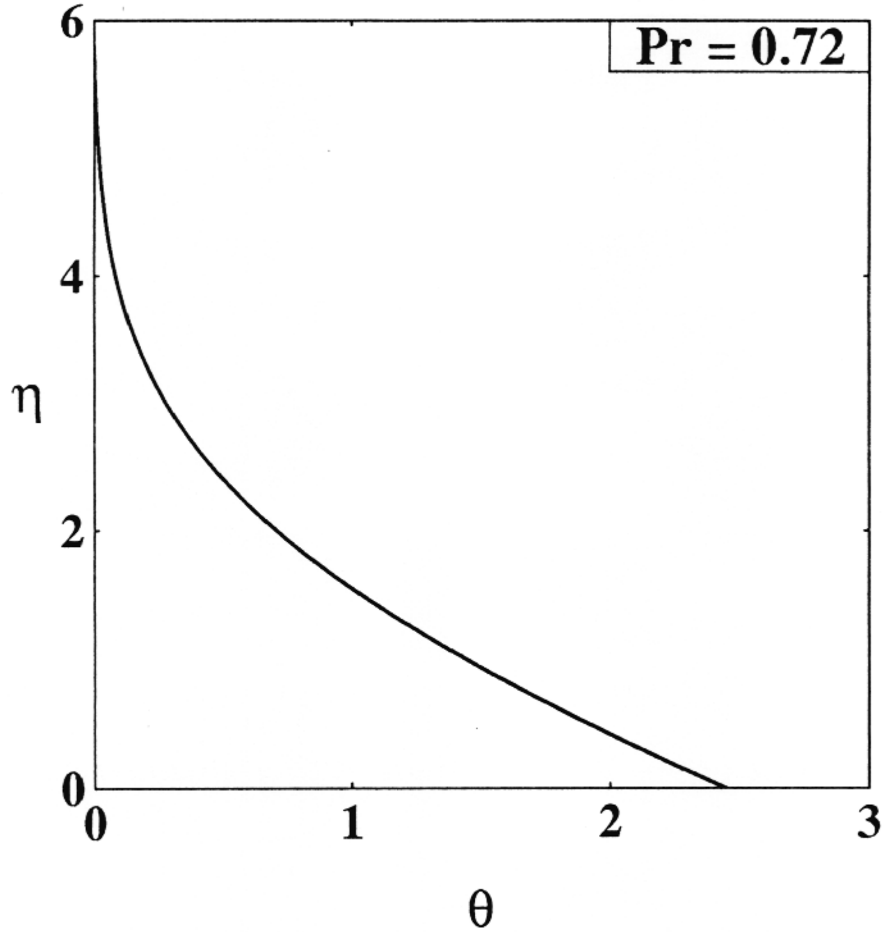
$$\theta''\,(0) = -1 \tag{15}$$

$$\theta\,(\infty) = 0 \tag{16}$$

The problem represented by eqs. (14)-(16) was solved numerically for $\theta(\eta, Pr)$. The similarity temperature profile for $Pr = 0.72$ is shown in the attached figure.

The key heat transfer result is the local Nusselt number

$$Nu = \frac{q''}{T_0(x) - T_\infty} \cdot \frac{x}{k} \tag{17}$$



which is proportional to the inverse of the local temperature difference, cf. eq. (10),

$$\mathrm{Nu} = \frac{1}{\theta(0)}\,\mathrm{Re}_x^{1/2} \tag{18}$$

For Pr > 0.5 we expect a scaling of type $\mathrm{Nu} \sim \mathrm{Pr}^{1/3}\,\mathrm{Re}_x{}^{1/2}$, which would mean that $\theta(0)$ must vary as $\mathrm{Pr}^{-1/3}$. Therefore we solve eqs. (14)-(16) for several large Pr values, and calculate the product $\theta(0)\,\mathrm{Pr}^{1/3}$, to find the constant C in the proportionality

$$\theta(0) = C\,\mathrm{Pr}^{-1/3} \tag{19}$$

The result is C = 1/0.463 if Pr > 10 [36].

For liquid metals, Pr << 0.5, we expect $\mathrm{Nu} \sim \mathrm{Pr}^{1/2}\,\mathrm{Re}_x{}^{1/2}$, which means

$$\theta(0) = K\,\mathrm{Pr}^{-1/2} \tag{20}$$

We solve eqs. (14)-(16) for two small Prandtl numbers, and obtain K → 1/0.886 as Pr → 0 [38].

In conclusion, the local Nusselt number asymptotes for laminar boundary layer flow over a constant-q" flat plate are

$$Nu = 0.463\, Pr^{1/3}\, Re_x^{1/2} \qquad (Pr > 0.5)$$

$$Nu = 0.886\, Pr^{1/2}\, Re_x^{1/2} \qquad (Pr << 0.5)$$

Problem 2.8. a) We recognize, in order, the L-averaged shear stress, the total tangential force experienced by the wall, and the mechanical power spent on dragging the wall through the fluid:

$$\bar{\tau} = 0.664\, \rho\, U^2 \left(\frac{UL}{\nu}\right)^{-1/2}$$

$$F' = \bar{\tau}\, L$$

$$\mathbf{P} = F'U = 0.664\, \rho\, U^3 L \left(\frac{\nu}{UL}\right)^{1/2}$$

$$= 0.664\, \rho\, \nu^{1/2}\, U^{5/2}\, L^{1/2}$$

If $(\ )_c$ and $(\ )_h$ represent the "cold" and "hot" flow conditions, the dissipated power changes according to the ratio:

$$\frac{\mathbf{P}_h}{\mathbf{P}_c} = \frac{\rho_h}{\rho_c}\left(\frac{\nu_h}{\nu_c}\right)^{1/2} = \left(\frac{\rho_h\, \mu_h}{\rho_c\, \mu_c}\right)^{1/2}$$

In the cold case (no heating), $T_c = 10°C$ and

$$\rho_c \cong 1\, \frac{g}{cm^3} \qquad \mu_c = 0.013\, \frac{g}{cm \cdot s}$$

In the hot case, the heat transfer from the 90°C wall to the 10°C water takes place across a boundary layer with the film temperature $T_h = 50°C$. The properties corresponding to T_h are

$$\rho_h \cong 1\, \frac{g}{cm^3} \qquad \mu_h = 0.00548\, \frac{g}{cm \cdot s}$$

The power ratio

$$\frac{\mathbf{P}_h}{\mathbf{P}_c} = \left(\frac{1}{1} \cdot \frac{0.00548}{0.013}\right)^{1/2} = 0.65$$

shows that the heating of the boundary layer decreases the dissipated power (as well as F' and $\bar{\tau}$) by 35 percent.

b) By retaining the observation that the water density is practically constant, we find that the power that has been saved by heating the boundary layer is

$$\mathbf{P}_c - \mathbf{P}_h = 0.664\ \rho\ U^{5/2}\, L^{1/2}\, \nu_c \left[1 - \left(\frac{\nu_h}{\nu_c}\right)^{1/2}\right] \tag{1}$$

The electric power spent on heating the wall is given sequentially by

$$\begin{aligned} \frac{\bar{h}L}{k} &= \frac{\overline{q''}}{\Delta T}\,\frac{L}{k} = 0.664\ \mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2} \\ \overline{q''}L &= 0.664\ k_h\,\Delta T\ \mathrm{Pr}_h^{1/3}\left(\frac{UL}{\nu_h}\right)^{1/2} \end{aligned} \tag{2}$$

in which the properties (k, Pr, ν) are evaluated at the film temperature $T_h = 50°C$,

$$k_h = 0.64\ \frac{W}{m\cdot K}, \qquad \mathrm{Pr}_h = 3.57$$

By dividing equations (1) and (2), we obtain (after some manipulation) a dimensionless measure of how effectively the heating of the wall has been converted into saved mechanical power:

$$\frac{\mathbf{P}_c - \mathbf{P}_h}{\overline{q''}L} = \frac{U^2}{c_h\,\Delta T}\,\mathrm{Pr}_h^{2/3}\left(\frac{\nu_c}{\nu_h}\right)^{1/2}\left[1 - \left(\frac{\nu_h}{\nu_c}\right)^{1/2}\right]$$

In this expression we substitute $c_h = 4.18$ kJ/kg·K, $\Delta T = 90°C - 10°C = 80°C$ and $(\nu_h/\nu_c)^{1/2} = 0.65$, and obtain

$$\frac{\mathbf{P}_c - \mathbf{P}_h}{\overline{q''}L} = \left(\frac{U}{516\ \mathrm{m/s}}\right)^2$$

This ratio is appealing (greater than 1) only when U > 516 m/s. The length L, however, must be sufficiently small if the boundary layer is to remain laminar

$$\frac{UL}{\nu_h} < 5 \times 10^5$$

$$L < 5 \times 10^5\ \frac{\nu_h}{U} < 5 \times 10^5\ \frac{\nu_h}{516\ \mathrm{m/s}}$$

$$< 5 \times 10^5\ \ \frac{0.00554\ \mathrm{cm^2/s}}{51600\ \mathrm{cm/s}}$$

$$< 0.54\ \mathrm{mm}$$

In conclusion, when L > 0.54 mm and the flow is laminar the electric power used to heat the wall is greater than the fluid-drag power saved.

Problem 2.9. The film temperature in this configuration is (40°C + 20°C) = 30°C. The calculation of the total force F proceeds in this order:

$$Re_x = \frac{U_\infty x}{\nu} 0.5 \frac{m}{s} 10m \frac{s}{0.16 \, cm^2} = 3.1 \times 10^5$$

(laminar, just barely)

$$\bar{C}_{f,x} = 1.328 \, Re_x^{-1/2} = 0.00238$$

$$A = 10m \; 20m = 200 \; m^2 \qquad \text{(roof area)}$$

$$F = A \, \bar{\tau}_{w,x} = A \, \bar{C}_{f,x} \frac{1}{2} \rho \, U_\infty^2$$

$$= 200 \, m^2 \, 0.00238 \, \frac{1}{2} \, 1.165 \frac{kg}{m^3} \left(0.5 \frac{m}{s}\right)^2$$

$$= 0.069 \, N \cong 0.016 \, lbf$$

In order to calculate the total heat transfer rate q, we must first evaluate the average heat flux:

$$\overline{Nu}_x = 0.664 \, Pr^{1/3} \, Re_x^{1/2}$$

$$= 0.664 \, (0.72)^{1/3} \left(3.1 \times 10^5\right)^{1/2} = 331.4$$

$$\bar{h}_x = \overline{Nu}_x \frac{k}{x} = 331.4 \times 0.026 \frac{W}{m \cdot K} \frac{1}{10m}$$

$$= 0.862 \frac{W}{m^2 \, K}$$

$$\bar{q}_x'' = \bar{h}_x \left(T_w - T_\infty\right) = 0.862 \frac{W}{m^2 \, K} (40 - 20) \, K$$

$$= 17.2 \frac{W}{m^2}$$

$$q = \bar{q}_x'' A = 17.2 \frac{W}{m^2} 200 \, m^2 = 3.4 \, kW$$

Problem 2.10. a) The local heat flux varies as

$$q_{w,x}'' = C \, x^n$$

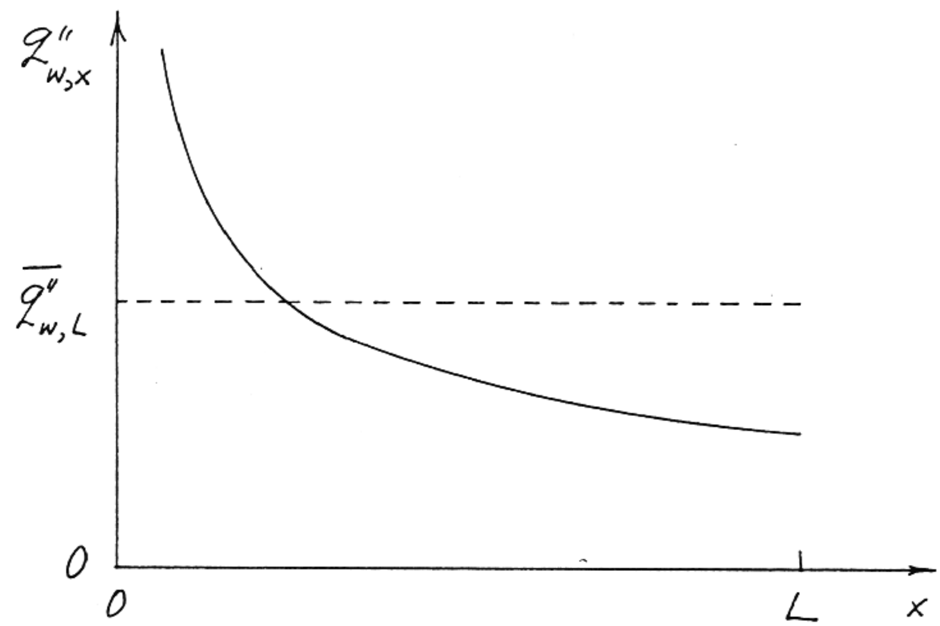
in which n = – 1/2, and C is a constant. The L-averaged heat flux is therefore

$$\overline{q''_{w,L}} = \frac{1}{1+n} q''_{w,L} = 2q''_{w,L} = 2\ CL^{-1/2}$$

The position x where the local flux matches the L-averaged value is obtained by writing

$$Cx^{-1/2} = 2\ CL^{-1/2}$$

and this yields x = L/4. The attached figure confirms this finding, because it is drawn to scale.



b) The relationship between the local heat flux in the middle and the L-averaged flux is obtained by first writing

$$q''_{w,L/2} = C\left(\frac{L}{2}\right)^{-1/2}$$
$$\overline{q}''_{w,L} = 2\ CL^{-1/2}$$

Dividing side by side, we conclude that the mid-point heat flux is about 30 percent smaller than the average flux:

$$\frac{q''_{w,L/2}}{\overline{q}''_{w,L}} = \frac{C\ 2^{1/2}\ L^{-1/2}}{2\ CL^{-1/2}} = \frac{1}{2^{1/2}} = 0.707$$

Problem 2.11. When the boundary layer is much thinner than the pipe diameter D, the wall friction force F and the total heat transfer rate q can be estimated based on the formulas for the flat wall:

$$
\begin{aligned}
F &= \pi\, DL\, \tau_{w,L} = \pi\, DL\, \bar{C}_{f,L}\, \frac{1}{2}\, \rho\, U_\infty^2 \\
&= \pi\, DL\, 1.328\, Re_L^{-1/2}\, \frac{1}{2}\, \rho\, U_\infty^2 \qquad (1) \\
&= 2.086\, \rho\, U_\infty^2\, DL\, Re_L^{-1/2}
\end{aligned}
$$

$$
\begin{aligned}
q &= \pi\, DL\, \bar{q}''_{w,L} = \pi\, DL\, \Delta T\, \bar{h}_L \\
&= \pi\, DL\, \Delta T\, \frac{k}{L}\, 0.664\, Pr^{1/3}\, Re_L^{1/2} \qquad (2) \\
&= 2.086\, k\, \Delta T\, D\, Pr^{1/3}\, Re_L^{1/2}
\end{aligned}
$$

Dividing (1) and (2) side by side we obtain

$$
\frac{q}{F} = Pr^{-2/3}\, \frac{c_P\, \Delta T}{U_\infty}
$$

Problem 2.12. We evaluate the properties of water at the film temperature $(20°C + 50°C)/2 = 35°C$, and calculate in order:

$$
Re_x = \frac{U_\infty x}{\nu} = 5\, \frac{cm}{s}\, 100\, cm\, \frac{s}{0.00725\, cm^2} = 6.9 \times 10^4 \quad \text{(laminar)}
$$

$$
\begin{aligned}
Nu_x &= 0.332\, Pr^{1/3}\, Re_x^{1/2} \\
&= 0.332\, (4.87)^{1/3} \left(6.9 \times 10^4\right)^{1/2} = 148
\end{aligned}
$$

$$
Nu_x = \frac{h_x\, x}{k}
$$

$$
h_x = 148\, \frac{k}{x} = 148 \times 0.62\, \frac{W}{m \cdot K}\, \frac{1}{1m} = 91.6\, \frac{W}{m^2K}
$$

$$
\bar{h}_x = 2\, h_x = 183.3\, \frac{W}{m^2\, K}
$$

$$
\bar{q}''_{w,x} = \bar{h}_x \Delta T = 183.3\, \frac{W}{m^2K}\, 30\, K = 5500\, \frac{W}{m^2}
$$

Since the total area of the wall of length $x = 1m$ is

$$
A = 4 \times 20\, cm \times 1m = 0.8\, m^2
$$

the total heat transfer rate is

$$
q = \bar{q}''_{w,x}\, A = 4400\, W
$$

The velocity boundary layer thickness at the same location (x)

$$\delta = 4.92 \times \mathrm{Re}_x^{-1/2} =$$
$$= 4.92 \times 1\mathrm{m}\left(6.9 \times 10^4\right)^{-1/2} = 1.9\ \mathrm{cm}$$

is much smaller than the 20 cm-side of the duct cross-section. In conclusion, the use of the flat wall boundary layer results is justified.

Problem 2.13. The narrow-strip configuration is a case of wall with non-uniform heat flux, in which

$$q''_w(\xi) = \begin{cases} 0, & 0 < \xi < x_1 \\ q'_w/\Delta x, & x_1 < \xi < x_1 + \Delta x \\ 0, & \xi > x_1 + \Delta x \end{cases}$$

Substituting this expression in eq. (2.122) we obtain, in order:

$$T_w(x) - T_\infty = \frac{0.623}{k\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}\left(\int_0^{x_1} + \int_{x_1}^{x_1+\Delta x} + \int_{x_1+\Delta x}^{x}\right)$$
$$= \frac{0.623}{k\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}\int_{x_1}^{x_1+\Delta x}\left[1 - \left(\frac{\xi}{x}\right)^{3/4}\right]^{-2/3}\frac{q'_w}{\Delta x}\,d\xi$$
$$= \frac{0.623}{k\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}\left[1 - \left(\frac{x_1}{x}\right)^{3/4}\right]^{-2/3} q'_w$$

Problem 2.14. The flux q" covers only the front half of the plate length L. We use eq. (2.122)

$$T_w(L) - T_\infty = \frac{0.623}{k\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}}\int_{\xi=0}^{L/2}\left[1 - \left(\frac{\xi}{L}\right)^{3/4}\right]^{-2/3} q''\,d\xi$$
$$= \frac{q''L}{k\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_L^{1/2}}\,\underbrace{0.623\int_0^{1/2}\left(1 - m^{3/4}\right)^{-2/3} dm}_{C = 0.682} \qquad (1)$$

The alternative calculation is based on eq. (2.121), in which we assume that the uniform flux q"/2 is spread over the entire length L:

$$\left[T_w(L) - T_\infty\right]_{\text{approx.}} = \frac{\frac{q''}{2}L}{0.453\,k\,\mathrm{Pr}^{1/3}\,\mathrm{Re}_x^{1/2}} \qquad (2)$$

The relative goodness of this approximation can be seen by dividing eq. (2) by eq. (1):

$$\frac{[T_w(L) - T_\infty]_{approx.}}{T_w\,(L) - T_\infty} = \frac{\dfrac{1}{2 \times 0.453}}{0.623 \times 0.682} = 2.60$$

In conclusion, the approximation is not very good.

Problem 2.15. The relevant properties of water at 20°C and atmospheric pressure are

$$\rho = 0.997 \, \frac{g}{cm^3} \qquad Pr = 7.07$$

$$\nu = 0.01 \, \frac{cm^2}{s} \qquad k = 0.59 \, \frac{W}{mK}$$

The L-averaged shear stress $\tau_{w,L}$ can be calculated in the following order,

$$\begin{aligned} \tau_{w,L} &= \left(\frac{1}{2}\,\rho\, U_\infty^{\;2}\right) \bar{C}_{f,L} \\ &= \left(\frac{1}{2}\,\rho\, U_\infty^{\;2}\right) 1.328 \, Re_L^{-1/2} \end{aligned} \tag{a}$$

where the Reynolds number has a value that falls in the laminar range:

$$Re_L = \frac{U_\infty L}{\nu} = \frac{0.5\ m}{s}\ \frac{1\ cm}{0.01\ cm^2/s} = 5000 \tag{b}$$

Combining (a) and (b) we obtain

$$\begin{aligned} \tau_{w,L} &= \frac{1}{2}\, 0.997 \, \frac{g}{cm^3}\, (0.5)^2 \, \frac{m^2}{s^2}\, 1.328 \, (5000)^{-1/2} \\ &= 0.00234 \, \frac{g \cdot m^2}{cm^3\, s^2} = 2340 \, \frac{N}{m^2} \end{aligned} \tag{c}$$

The L-averaged heat flux can be calculated based on a similar sequence:

$$\bar{q}''_{w,L} = \frac{k\,\Delta T}{L}\,\overline{Nu}_L$$

$$= \frac{k\,\Delta T}{L}\, 0.664 \, Pr^{1/3}\, Re_L^{1/2}$$

$$= 0.59 \, \frac{W}{m\,K}\, \frac{1\ K}{1\ cm}\, 0.664 \, (7.07)^{1/3}\, (5000)^{1/2}$$

$$= 53.17 \, \frac{W}{m \cdot cm} = 5317 \, \frac{W}{m^2} \tag{d}$$

Problem 2.16. The Prandtl number of air at 20°C is 0.72, therefore we can use eq. (2.121) as an approximate estimate of the local Nusselt number along the strip:

$$\frac{q''_w}{T_w(L) - T_\infty} \frac{L}{k} = 0.453 \, Pr^{1/3} \, Re_L^{1/2} \tag{a}$$

In this equation q''_w is the heat flux to one side of the strip, $q''_w = (300/2) W/m^2 = 150 \, W/m^2$, and $x = L$ is the position of the trailing edge where the temperature sensor is located, $T_w(L) = 30°C$. Noting further that $k = 2.5 \times 10^{-4} W/cm \, K$, we use eq. (a) to calculate $Re_L^{1/2}$:

$$Re_L^{1/2} = \frac{q''_w}{T_w(L) - T_\infty} \frac{L}{k} \frac{1}{0.453 \, Pr^{1/3}}$$

$$= \frac{150 \, W/m^2}{(30-20)°C} \frac{2 \, cm}{2.5 \times 10^{-4} \, W/cm \, K} \frac{1}{0.453 \, (0.72)^{1/3}}$$

$$= 29.56$$

From this we deduce that $Re_L = 873.5 = U_\infty L/\nu$, in which $\nu = 0.15 \, cm^2/s$. The U_∞ velocity of the free stream is therefore

$$U_\infty = 873.5 \, \frac{0.15 \, cm^2/s}{2 \, cm} = 0.66 \, \frac{m}{s}$$

Note that $Re_L = 873.5$ means also that the flow at $x = L$ is laminar, i.e. that the use of eq. (a) is justified.

Problem 2.17. a) Assume that T_0 (unknown) is the uniform temperature of the interface between the two bodies in rolling contact. Consider first the temperature distribution in body no. 1. It is a "thermal" boundary layer through which body no. 1 "flows" with the uniform speed U:

$$U \frac{\partial T}{\partial x} = \alpha_1 \frac{\partial^2 T}{\partial y^2} \tag{1}$$

$$T = T_0 \quad \text{at} \quad y = 0 \tag{2}$$

$$T \to T_1 \quad \text{as} \quad y \to \infty \tag{3}$$

The problem (1)-(3) is identical to that of the temperature distribution in a thermal boundary layer in the limit $Pr \to 0$. When Pr is very small the velocity thickness δ is much smaller than δ_T. This means that in the $Pr \to 0$ limit the fluid flows with a uniform velocity through the thermal boundary layer region. The uniform longitudinal velocity (U) is a noteworthy feature of eq. (1), therefore, for the local heat flux we can simply use eq. (2.107):

$$\frac{q''_x}{T_1 - T_0} \frac{x}{k_1} = 0.564 \left(\frac{Ux}{\alpha_1}\right)^{1/2} \tag{4}$$

A similar analysis of the thermal boundary layer that develops inside body no. 2 leads to an alternative expression for the same local heat flux q_x'':

$$\frac{q_x''}{T_0 - T_2}\frac{x}{k_2} = 0.564\left(\frac{Ux}{\alpha_2}\right)^{1/2} \tag{5}$$

b) By dividing eqs. (4) and (5) we obtain an expression for the interface temperature

$$T_0 = \frac{r}{1+r}T_1 + \frac{1}{1+r}T_2 \tag{6}$$

in which r is the dimensionless group

$$r = \frac{(\rho c k)_1^{1/2}}{(\rho c k)_2^{1/2}} \tag{7}$$

c) The local heat flux can now be calculated by eliminating T_0 between eqs. (4) and (6), or between eqs. (5) and (6):

$$q_x'' = \frac{0.564}{1+r}k_1(T_1 - T_2)\left(\frac{U}{\alpha_1}\right)^{1/2}x^{-1/2} \tag{8}$$

The L-averaged heat flux that corresponds to this result is

$$\begin{aligned}\bar{q}'' &= \frac{1}{L}\int_0^L q_x''\,dx \\ &= \frac{1.128}{1+r}k_1(T_1 - T_2)\left(\frac{U}{\alpha_1 L}\right)^{1/2}\end{aligned} \tag{9}$$

d) The preceding results apply as long as the two boundary layer regions are sufficiently slender. According to eq. (2.37), in the small-Pr limit the slenderness ratio (δ_T/L) is of order $Pe_L^{-1/2}$. There are two slenderness criteria to meet, because there are two boundary layers:

$$\left(\frac{UL}{\alpha_1}\right)^{-1/2} << 1 \tag{10}$$

$$\left(\frac{UL}{\alpha_2}\right)^{-1/2} << 1 \tag{11}$$

These conditions can be rewritten as

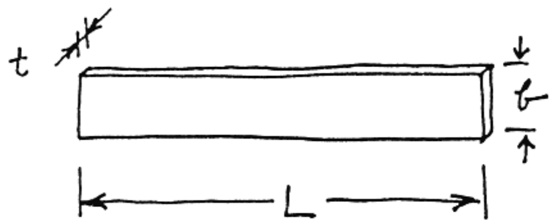
$$U >> \frac{\alpha_1}{L} \tag{12}$$

$$U >> \frac{\alpha_2}{L} \tag{13}$$

therefore, the peripheral velocity must be higher than α_m/L, where α_m is the greater of the two thermal diffusivities (α_1, α_2).

Problem 2.18. The objective of this problem is to show that if your job is to maximize a complicated function (q_B), then a good idea is to perform the analysis in *dimensionless* form. You have to maximize

$$q_B = (T_B - T_\infty)\,(hp\,kA)^{1/2} \tanh\left[L\left(\frac{hp}{kA}\right)^{1/2}\right]$$



subject to the following conditions

$$V = b\,L\,t, \text{ constant} \quad ; \quad p = 2b$$

$$t = \text{constant} \quad ; \quad A = bt$$

$$h = \frac{k_f}{b}\,0.664\,Pr^{1/3}\left(\frac{U_\infty b}{\nu}\right)^{1/2}$$

There is only one degree of freedom in this design problem, b or L; choosing b as the variable, the objective function q_B can then be arranged in dimensionless form as

$$\frac{q_B}{kt\,(T_B - T_\infty)} = \Bigg[\underbrace{1.328\,Pr^{1/3}\,\frac{k_f}{k}\,Re_t^{1/2}\left(\frac{b}{t}\right)^{3/2}}_{C}\Bigg]^{1/2} \tanh\left\{\frac{V}{t^3}\left[C\left(\frac{b}{t}\right)^{-5/4}\right]^{1/2}\right\}$$

where $Re_t = U_\infty t/\nu$ is a known constant. Setting

$$m^{5/4} = \frac{V}{t^3}\,C^{1/2}, \text{ constant,}$$

we rewrite the objective function as

$$\frac{q_B}{kt\,(T_B - T_\infty)} = \underbrace{C^{1/2}\,m^{3/4}}_{\text{constant}}\left(\frac{b}{mt}\right)^{3/4} \tanh\left[\left(\frac{b}{mt}\right)^{-5/4}\right]$$

Thus, b influences q_B in the same way that ξ influences $\xi^{3/4}\tanh(\xi^{-5/4})$, where $\xi = \frac{b}{mt}$. As shown in the graph, this function has a maximum at

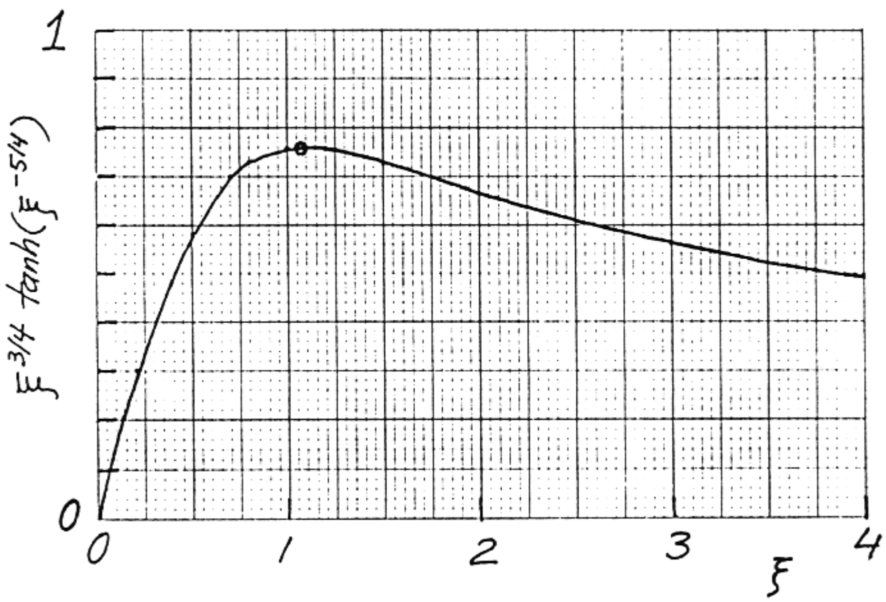
$$\xi_{opt} = 1.071,$$

which means

$$\frac{b_{opt}}{t} = 1.071\ m$$

or, recalling the definitions of m and C,

$$\frac{b_{opt}}{t} = (1.071)\left(\frac{V}{t^3}\right)^{4/5}\left(1.328\ Pr^{1/3}\,\frac{k_f}{k}\,Re_t^{1/2}\right)^{2/5}$$



Problem 2.19. Since in the limit $x \to \infty$, $\tanh x \to 1$, the q_B expression of Problem 2.18 reduces to

$$\frac{q_B}{kt\,(T_B - T_\infty)}\ \left[1.328\ Pr^{1/3}\,\frac{k_f}{k}\left(\frac{U_\infty t}{\nu}\right)^{1/2}\left(\frac{b}{t}\right)^{3/2}\right]^{1/2}$$

a) q_B is proportional to $b^{3/4}$, therefore if b increases by a factor of 2, q_B increases by a factor of $2^{3/4} = 1.68$

b) Based on the above formula for q_B, we conclude that

$$\frac{(q_B)_{water}}{(q_B)_{air}} = \left(\frac{Pr_w}{Pr_a}\right)^{1/6}\left(\frac{k_w}{k_a}\right)^{1/2}\left(\frac{\nu_a}{\nu_w}\right)^{1/4}$$

$$= \left(\frac{7}{0.72}\right)^{1/6}(23)^{1/2}\left(\frac{1}{0.07}\right)^{1/4} = 13.62$$

Problem 2.20. In the case of non-zero longitudinal pressure gradient, the momentum equation reads

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{m}{x}U_\infty^2 + \nu\frac{\partial^2 u}{\partial y^2}$$

or, introducing the streamfunction $\psi_y = u$, $\psi_x = -v$,

$$\underbrace{\psi_y\,\psi_{yx} - \psi_x\,\psi_{yy}}_{\text{LHS}} = \underbrace{\frac{m}{x}U_\infty^2 + \nu\,\psi_{yyy}}_{\text{RHS}}$$

Recalling that $U_\infty = C\,x^m$, the similarity transformation consists of

$$\psi = (U_\infty \nu x)^{1/2} f(\eta) = C^{1/2}\nu^{1/2}x^{\frac{m+1}{2}} f(\eta)$$

$$\eta = \frac{y}{x}\left(\frac{U_\infty}{\nu}\right)^{1/2} = y\,C^{1/2}\nu^{-1/2}x^{\frac{m-1}{2}}$$

One by one, we evaluate the ψ derivatives,

$$\psi_y = C\,x^m f'$$

$$\psi_{yy} = C^{3/2}\nu^{-1/2}x^{\frac{3m-1}{2}} f''$$

$$\psi_{yyy} = C^2\nu^{-1}x^{2m-1} f'''$$

$$\psi_x = \frac{m+1}{2}C^{1/2}\nu^{1/2}x^{\frac{m-1}{2}} f + \frac{m-1}{2}C^{1/2}\nu^{1/2}x^{m-1}\,y\left(\frac{C}{\nu}\right)^{1/2} f'$$

$$\psi_{xy} = Cm\,x^{m-1} f' + \frac{m-1}{2}C\,x^{\frac{3m-3}{2}}\,y\left(\frac{C}{\nu}\right)^{1/2} f''$$

Substituting these into the momentum equation, the terms containing $y\,(C/\nu)^{1/2}$ drop out, and we are left with

$$\underbrace{m(f')^2 - \frac{m+1}{2}f f''}_{\text{LHS}} = \underbrace{m + f'''}_{\text{RHS}}$$

which can be rewritten as

$$2f''' + (m+1)\,ff'' + 2m\,(1 - f'^2) = 0$$

We process the energy equation in the same way:

$$u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial y^2}$$

$$\overbrace{\psi_y\,\theta_x - \psi_\theta\,\theta_y}^{\text{LHS}} \;=\; \overbrace{\alpha\,\theta_{yy}}^{\text{RHS}}$$

$$\theta(\eta) = \frac{T - T_0}{T_\infty - T_0}$$

The required θ derivatives are

$$\theta_y = \theta'\,C^{1/2}\,\nu^{-1/2}\,x^{\frac{m-1}{2}}$$

$$\theta_{yy} = \theta''\,C\,\nu^{-1}\,x^{m-1}$$

$$\theta_x = \frac{m-1}{2}\,\theta'\,x^{\frac{m-3}{2}}\,y\left(\frac{C}{\nu}\right)^{1/2}$$

Substituting the θ and ψ derivatives into the energy equation, the terms containing $y(C/\nu)^{1/2}$ drop out from the left-hand-side of the equation,

$$-\frac{m+1}{2}\,f\,\theta' = \frac{\alpha}{\nu}\,\theta''$$

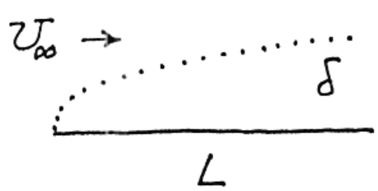
This result can be rewritten as

$$2\,\theta'' + \Pr\,(m+1)\,f\,\theta' = 0$$

Problem 2.21. The energy equation for a two-dimensional boundary layer with viscous dissipation is

$$\rho\,c_P\left(u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\Phi$$

$\uparrow$ negligible, because the boundary layer region is slender, $\delta << L$



The $\mu\Phi$ term is simplified based on the following scale analysis:

$$\mu\Phi = 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] \quad + \quad \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

$$\underbrace{\frac{U_\infty^2}{L^2} \qquad \frac{v^2}{\delta^2}}_{\substack{\text{these two terms are} \\ \text{of the same size, } U_\infty^2/L^2, \\ \text{cf. mass conservation} \\ (U_\infty/L \sim v/\delta)}} \qquad\qquad \frac{U_\infty}{\delta}, \ \frac{v}{L} \quad \text{or} \quad \frac{U_\infty}{\delta}, \frac{\delta U_\infty}{L^2}$$

$$\frac{U_\infty}{\delta} \gg \frac{\delta U_\infty}{L^2} \qquad \text{i.e. neglect } \frac{\partial v}{\partial x}$$

In conclusion, the $\mu\Phi$ term has two scales, the first of which is much smaller than the second:

$$\mu\Phi \sim \mu\frac{U_\infty^2}{L^2}, \quad \mu\frac{U_\infty^2}{\delta^2}$$

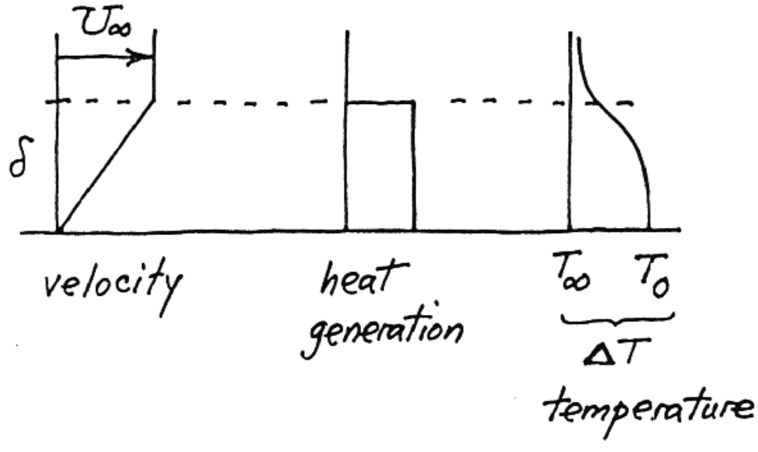
$$\mu\frac{U_\infty^2}{L^2} \ll \mu\frac{U_\infty^2}{\delta^2}$$

We conclude that in the boundary layer $\delta \times L$,

$$\mu\Phi = \mu\left(\frac{\partial u}{\partial y}\right)^2$$

in other words the energy equation reduces to

$$\underbrace{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}}_{\text{convection}} = \underbrace{\alpha\frac{\partial^2 T}{\partial y^2}}_{\text{conduction}} + \underbrace{\frac{\mu}{\rho c_P}\left(\frac{\partial u}{\partial y}\right)^2}_{\text{heat generation}}$$



Continuing with the scale analysis of the δ-thick region, we seek to determine the wall temperature rise ΔT as a function of U_∞ and fluid properties. The energy equation involves three scales:

$$U_\infty \frac{\Delta T}{L}, \qquad \alpha \frac{\Delta T}{\delta^2}, \qquad \frac{\mu}{\rho\, c_P} \frac{U_\infty^2}{\delta^2}$$

$$\text{conv.} \qquad \text{cond.} \qquad \text{heat gen.}$$

Among these, the heat generation scale can never be neglected, because without the $\mu\Phi$ term we do not have any heat transfer problem left. Two cases exist:

i) convection ~ heat generation

Writing

$$U_\infty \frac{\Delta T}{L} \sim \frac{\mu}{\rho\, c_P} \frac{U_\infty^2}{\delta^2}$$

and recalling from Blasius and Prandtl that $\delta \sim L\, Re_L^{-1/2}$, yields

$$\Delta T \sim \frac{U_\infty^2}{c_P} \quad \text{or} \quad \theta_r = O(1)$$

For what Pr values these results are valid, is shown by the conduction scale (neglected in this case):

$$1 \qquad Pr^{-1} \qquad 1$$

$$\text{conv.} \qquad \text{cond.} \qquad \text{heat gen.}$$

In conclusion, case (i) corresponds to Pr > 1 fluids.

ii) conduction ~ heat generation

Claiming now a balance between the conduction and heat generation scales,

$$\alpha \frac{\Delta T}{\delta^2} \sim \frac{\mu}{\rho\, c_P} \frac{U_\infty^2}{\delta^2}$$

we find

$$\Delta T \sim \frac{\mu\, U_\infty^2}{k} \quad \text{or} \quad \theta_r = O(Pr)$$

The three scales in the energy equation line up as follows:

$$\begin{array}{ccc} Pr & 1 & 1 \\ \text{conv.} & \text{cond.} & \text{heat gen.} \end{array}$$

Since in case (ii) the convection effect was neglected, the above results are valid for Pr < 1 fluids. The remainder of this problem is routine. The energy equation must be transformed according to the definitions

$$\theta_r(\eta) = \frac{T - T_\infty}{U_\infty^2/2\,c_P} \quad , \quad \eta = \frac{y}{\sqrt{x}} \sqrt{\frac{U_\infty}{\nu}}$$

$$u = U_\infty f'(\eta) \quad , \quad v = \frac{1}{2} \sqrt{\frac{\nu U_\infty}{x}} (\eta f' - f)$$

where $f'(\eta)$ is the Blasius profile (see Fig. 2.6). We have

$$\frac{\partial T}{\partial x} = \frac{U_\infty^2}{2\,c_P} \theta_r' \frac{-y}{2x\sqrt{x}} \left(\frac{U_\infty}{\nu}\right)^{1/2}$$

$$\frac{\partial T}{\partial y} = \frac{U_\infty^2}{2\,c_P} \theta_r' \left(\frac{U_\infty}{x\nu}\right)^{1/2}$$

and the left-hand-side of the energy equation yields

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{U_\infty^3}{2\,c_P\,x} \left(-\frac{1}{2} f \theta_r'\right)$$

On the right-hand-side we obtain

$$\alpha \frac{\partial^2 T}{\partial y^2} + \frac{\mu}{\rho\,c_P} \left(\frac{\partial u}{\partial y}\right)^2 = \alpha \frac{U_\infty^2}{2\,c_P} \theta_r'' \frac{U_\infty}{x\nu} + \frac{\mu}{\rho\,c_P} U_\infty^2 (f'')^2 \frac{U_\infty}{x\nu}$$

Equating the two sides we obtain

$$\begin{array}{ccccc} -\frac{1}{2} f \theta_r' & = & \frac{1}{Pr} \theta_r'' & + & 2f'' \\ \text{conv.} & & \text{cond.} & & \text{heat gen.} \end{array}$$

or

$$\theta_r'' + \frac{Pr}{2} f \theta_r' + 2\,Pr\,f''^2 = 0$$

This equation is linear of the first order in $\theta_r'(\eta)$: its solution is

$$\theta_r' = e^{-\int \frac{1}{2} Pr\,f\,d\eta} \left(C - \int 2\,Pr\,f''^2\, e^{\int \frac{1}{2} Pr\,f\,d\eta}\, d\eta \right)$$

If all the integrals start from $\eta = 0$, then the constant C must vanish so that the adiabatic wall condition $\theta_r' = 0$ holds at $\eta = 0$,

$$\theta_r'(\eta) = -e^{-\int_0^{\eta} \frac{1}{2} \Pr f(m)\, dm} \int_0^{\eta} 2 \Pr f''^2 (\beta)\, e^{\int_0^{\beta} \frac{1}{2} \Pr f(\gamma)\, d\gamma}\, d\beta$$

Integrating this expression from $\eta = \eta$ to $\eta = \infty$, and using the condition $\theta_r(\infty) = 0$ yields

$$\theta_r(\eta) = 2 \Pr \int_{\eta}^{\infty} \left\{ \int_0^{p} [f''(\beta)]^2\, e^{\frac{\Pr}{2} \int_0^{\beta} f(\gamma)\, d\gamma}\, d\beta \right\} e^{-\frac{\Pr}{2} \int_0^{p} f(m)\, dm}\, dp$$

The dimensionless wall temperature rise $\theta_r(0)$ can be evaluated numerically for various values of Pr. These results should be compared with the scale analysis results for θ_r obtained earlier in this solution.

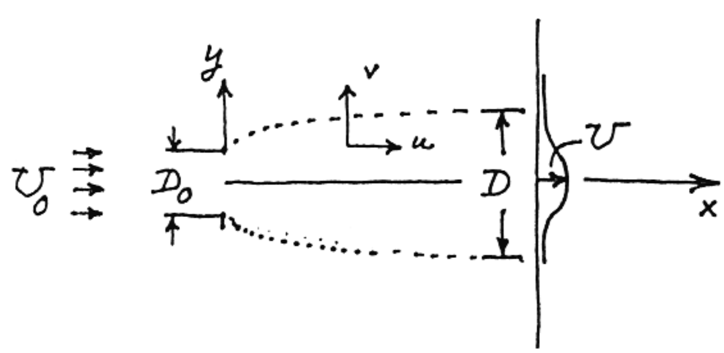
Problem 2.22. The mass conservation and longitudinal momentum equations for a two-dimensional laminar jet, as a slender flow region, are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \underbrace{\frac{dP}{dx}}_{\downarrow} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{dP}{dx} = \frac{dP_\infty}{dx} = 0$$

because $P_\infty = \text{const.}$



In terms of scales, the two equations dictate:

$$\frac{U}{x} \sim \frac{v}{D} \quad \text{and} \quad \underbrace{U\frac{U}{x}, v\frac{U}{D}}_{\text{same size}} \sim \nu \frac{U}{D^2}$$

$$\frac{U^2}{x} \sim \nu \frac{U}{D^2}$$

In conclusion, we have one relationship between the unknown scales U and D,

$$UD^2 \sim \nu x \tag{1}$$

The second relationship necessary for determining U and D uniquely follows from the HINT given in the problem statement: the x-momentum equation can be written as

$$\frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) = \nu \frac{\partial^2 u}{\partial y^2}$$

and integrated (at constant x) from y = – ∞ to y = + ∞,

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2\, dy + \underbrace{u_\infty}_{} v_\infty - \underbrace{u_{-\infty}}_{} v_{-\infty} = \nu \underbrace{\left(\frac{\partial u}{\partial y}\right)_{y=\infty}}_{} - \nu \underbrace{\left(\frac{\partial u}{\partial y}\right)_{y=-\infty}}_{}$$

zeros, because in the far-field reservoir fluid u = 0, uniform (note: $v_{\pm\infty}$ may not be zero.)

We conclude that

$$\frac{d}{dx}\int_{-\infty}^{\infty} u^2\, dy = 0,$$

or

$$\int_{-\infty}^{\infty} u^2\, dy = U_0^2 D_0, \text{ constant.}$$

The scaling law implied by this result is

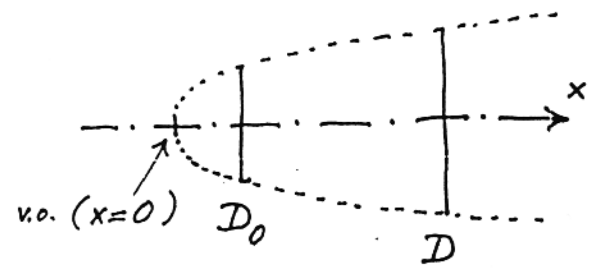
$$U^2 D \sim U_0^2 D_0 \tag{2}$$

Combining eqs. (1) and (2) we find

$$D \sim \nu^{2/3} x^{2/3} U_0^{-2/3} D_0^{-1/3}$$

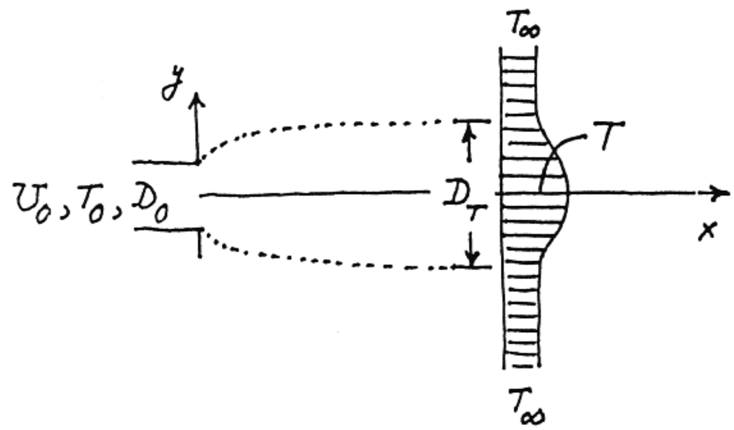
$$U \sim \nu^{-1/3} x^{-1/3} U_0^{4/3} D_0^{2/3}$$

Note: D increases as $x^{2/3}$ from a "virtual origin" (v.o.) located upstream of the nozzle.



Problem 2.23. This problem is a continuation of the preceding problem. The energy equation for the same slender flow is

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$



In the D_T-thick region, the scaling law required by the energy equation is ($\Delta T = T - T_\infty$)

$$\underbrace{u \frac{\Delta T}{x}, \; v \frac{\Delta T}{D_T}}_{\substack{\text{same size, from} \\ \text{mass conservation} \\ \text{in the x-}D_T\text{ region}}} \sim \alpha \frac{\Delta T}{D_T^2}$$

$$\underbrace{\frac{u}{x} \sim \frac{v}{D_T}}$$

$$U \frac{\Delta T}{x} \sim \alpha \frac{\Delta T}{D_T^2}$$

In conclusion,

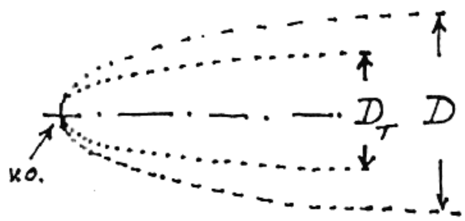
$$D_T \sim \left(\frac{\alpha x}{U}\right)^{1/2}$$

where, according to the preceding problem, $U \sim \nu^{-1/3}, x^{-1/3} U_0^{4/3} D_0^{2/3}$. Note that D_T varies as $x^{2/3}$, just like D. The ratio D_T/D is

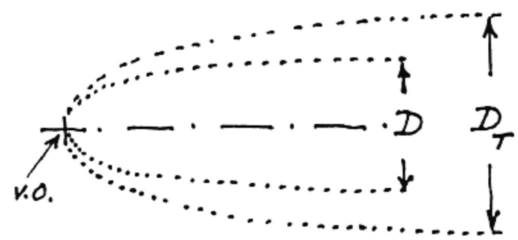
$$\frac{D_T}{D} \sim \frac{\alpha^{1/2} x^{1/2}}{\nu^{-1/6} x^{-1/6} U_0^{2/3} D_0^{1/3}} \cdot \frac{1}{\nu^{2/3} x^{2/3} U_0^{-2/3} D_0^{-1/3}} = \left(\frac{\alpha}{\nu}\right)^{1/2}$$

in other words,

$$D_T \sim Pr^{-1/2} D.$$



Pr > 1 fluids



Pr < 1 fluids
(v.o. = virtual origin)

To determine the x-variation of centerline temperature, we exploit the HINT and integrate the energy equation

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha \frac{\partial^2 T}{\partial y^2}$$

over the entire x = const. cut across the flow,

$$\frac{d}{dx}\int_{-\infty}^{\infty} u\,T\,dy + \underbrace{v_\infty T_\infty - v_{-\infty} T_\infty}_{T_\infty \underbrace{(v_\infty - v_{-\infty})}_{\substack{\text{unknown to be}\\ \text{determined from}\\ \text{the mass conservation}\\ \text{equation}}}} = \underbrace{\alpha\left(\frac{\partial T}{\partial y}\right)_{\infty} - \alpha\left(\frac{\partial T}{\partial y}\right)_{-\infty}}_{\substack{\text{zeros, because T is}\\ \text{uniform as } y \to \pm\infty}}$$

The mass conservation statement

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

means also that

$$\frac{d}{dx}\int_{-\infty}^{\infty} u\,dy + \underbrace{v_\infty - v_{-\infty}}_{\text{wanted}} = 0$$

In conclusion, the energy integral reduces to

$$\frac{d}{dx}\int_{-\infty}^{\infty} u\,T\,dy - T_\infty \frac{d}{dx}\int_{-\infty}^{\infty} u\,dy = 0,$$

which means that

$$\frac{d}{dx}\int_{-\infty}^{\infty} u\,(T - T_\infty)\,dy = 0,$$

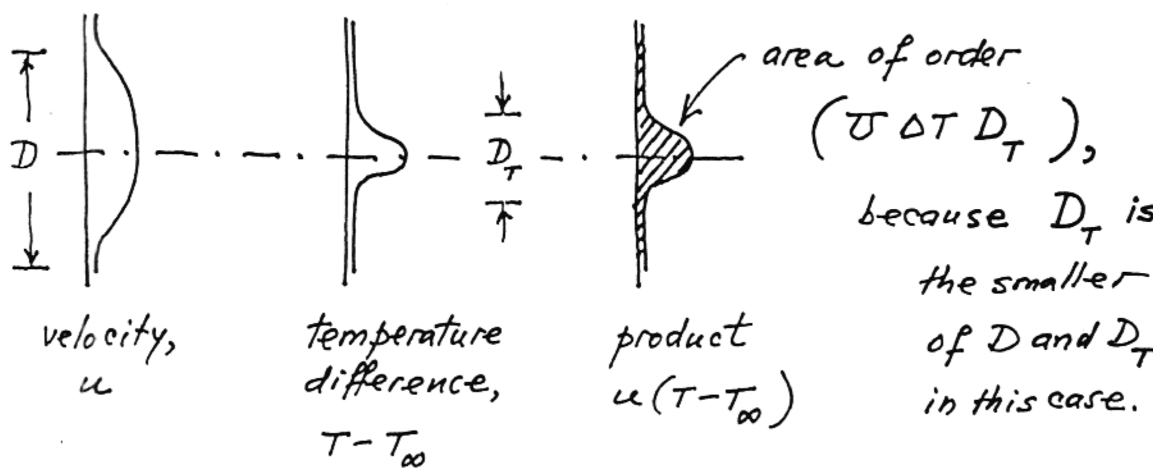
or

$$\int_{-\infty}^{\infty} u\,(T - T_\infty)\,dy = U_0\,(T_0 - T_\infty)\,D_0,\ \text{constant.}$$

In terms of scales, we have

$$(U)\,(\Delta T)\,\big(\text{the smaller of } D \text{ or } D_T\big) \sim U_0\,\Delta T_0\,D_0 \qquad (*)$$

To see why "the smaller of D or D_T" must be used in eq. (*), consider multiplying the u and ΔT profiles when D and D_T differ appreciably, say, when Pr > 1.



Therefore, in order to determine the ΔT scale we consider two separate cases:

<u>Pr > 1</u>, or $D_T < D$, so that eq. (*) becomes

$$U\,\Delta T\,D_T \sim U_0\,\Delta T_0\,D_0,$$

and since U and D_T are known already,

$$\frac{T - T_\infty}{T_0 - T_\infty} \sim \left(\frac{D_0}{x}\right)^{1/3}\left(\frac{U_0\,D_0}{\nu}\right)^{1/3} Pr^{1/2}$$

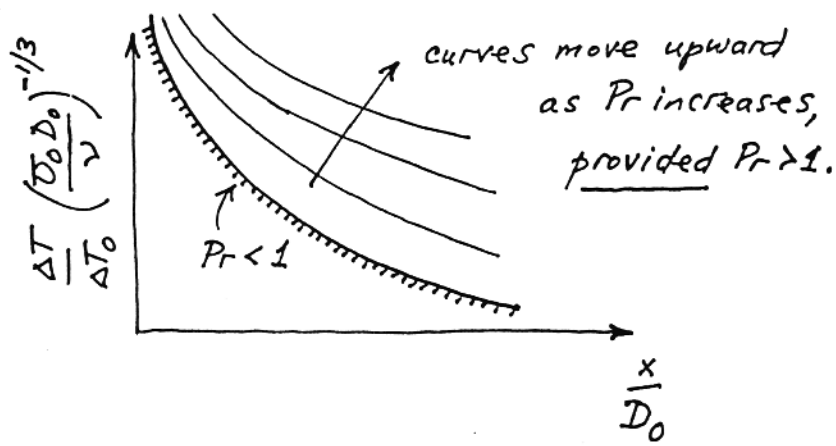
<u>Pr < 1</u>, or $D < D_T$, and eq. (*) becomes

$$U\,\Delta T\,D \sim U_0\,\Delta T_0\,D_0,$$

which yields in the end

$$\frac{T - T_\infty}{T_0 - T_\infty} \sim \left(\frac{D_0}{x}\right)^{1/3} \left(\frac{U_0 D_0}{\nu}\right)^{1/3}$$

Graphically, we see that $(T - T_\infty)$ decreases as $x^{-1/3}$ in the downstream direction.



Problem 2.24. The x-momentum equation for the liquid film, as a slender flow region is*

$$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{inertia}} = -\underbrace{\frac{1}{\rho}\frac{dP}{dx}}_{\downarrow\ \text{zero}} + \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\text{friction}} + \underbrace{g\sin\alpha}_{\text{body force}}$$

$$\left(\frac{dP}{dx} = \frac{dP_{air}}{dx} = 0\right)$$

After the terminal velocity is reached, the momentum equation reduces to

$$0 = \nu\frac{\partial^2 u}{\partial y^2} + g\sin\alpha,$$

hence

$$u = -\frac{g\sin\alpha}{\nu}\frac{y^2}{2} + c_1 y + c_2$$

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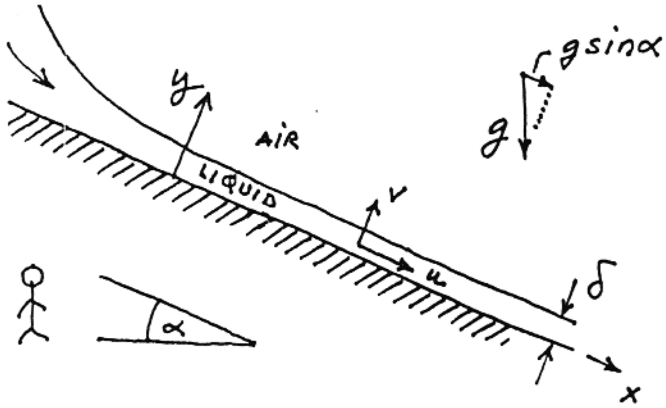
* assuming that $\rho_{liquid} >> \rho_{air}$ (for brevity $\rho_{liquid} = \rho$).



As boundary conditions, we invoke

$$u = 0 \quad \text{at} \quad y = 0 \quad \text{(no slip at the wall)}$$

$$\frac{\partial u}{\partial y} = 0 \quad \text{at} \quad y = \delta \quad \text{(zero shear free surface)}$$

The terminal velocity distribution is therefore

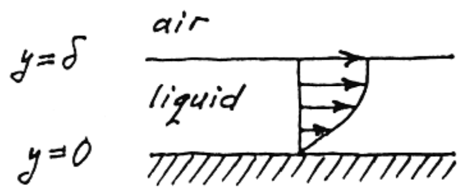
$$u(y) = \frac{g \sin \alpha}{\nu}\left(\delta y - \frac{y^2}{2}\right),$$

or, in dimensionless form,

$$\frac{u}{U} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2$$

where U is the free-surface velocity

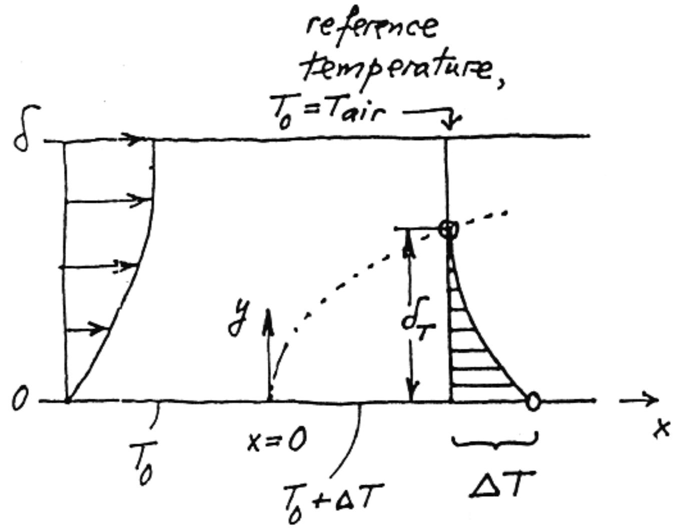
$$U = \frac{g \sin \alpha}{\nu} \cdot \frac{\delta^2}{2}$$



Moving now to the heat transfer part of the problem, the energy equation for the film is

$$u\frac{\partial T}{\partial x} + \underbrace{v}_{\downarrow}\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

zero, if terminal velocity is reached (for proof, consult the mass conservation equation)



Very close to $x = 0$, i.e. when δ_T is much smaller than δ, the velocity scale in the δ_T-thick layer is $\left(\frac{\delta_T}{\delta} U\right)$. In the δ_T-thick layer, the energy equation represents the following equivalence of scales

$$\left(\frac{\delta_T}{\delta} U\right) \frac{\Delta T}{x} \sim \alpha \frac{\Delta T}{\delta_T^2}$$

which means that

$$\delta_T \sim \left(\frac{x \alpha \delta}{U}\right)^{1/3}$$

The wall-film heat flux near $x = 0$ scales as

$$q'' \sim k \frac{\Delta T}{\delta_T} \sim k\, \Delta T \left(\frac{U}{x \alpha \delta}\right)^{1/3}$$

A more accurate result is produced by integral analysis. First we write the energy equation as

$$\frac{\partial}{\partial x}(uT) + \underbrace{\frac{\partial}{\partial y}(vT)}_{\substack{\text{zero,} \\ \text{because } v = 0}} = \alpha \frac{\partial^2 T}{\partial y^2}$$

and integrate it from $y = 0$ to $y = \delta_T$

$$\frac{d}{dx} \int_0^{\delta_T} uT\, dy = \underbrace{\alpha \left(\frac{\partial T}{\partial y}\right)_{y=\delta_T}}_{\substack{\text{zero, because the fluid above } y = \delta_T \\ \text{is isothermal } (T = T_0).}} - \alpha \left(\frac{\partial T}{\partial y}\right)_{y=0}$$

Assuming the temperature profile sketched earlier,

$$T - T_0 = \Delta T\left[1 - 2\frac{y}{\delta_T} + \left(\frac{y}{\delta_T}\right)^2\right]$$

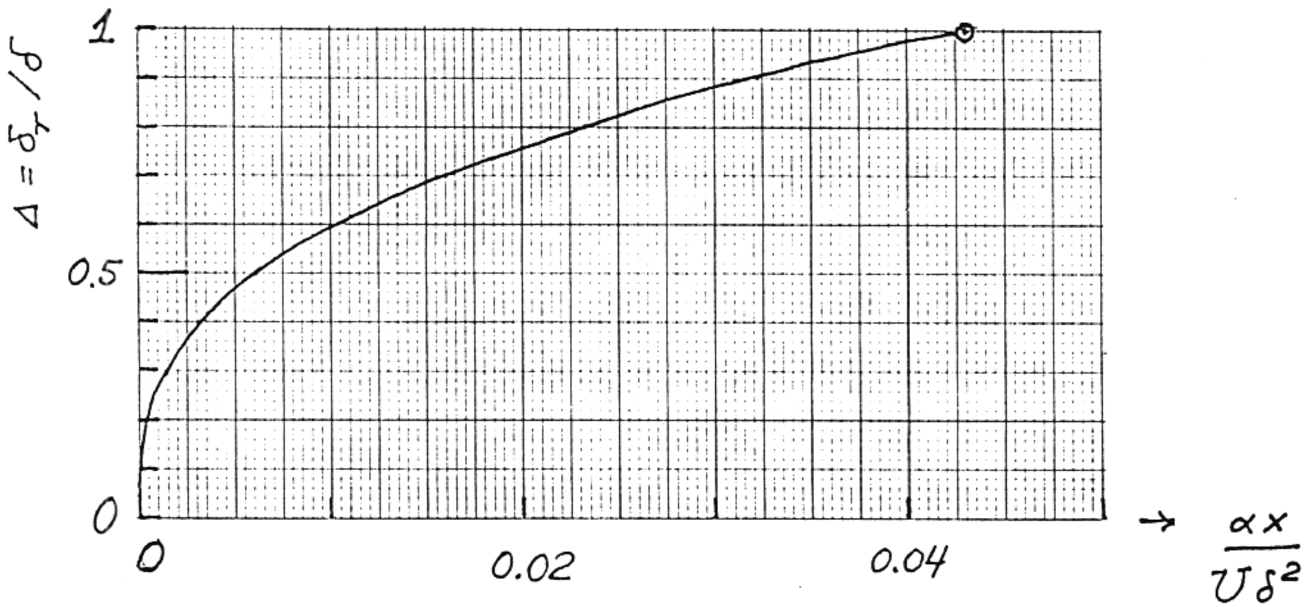
and using also the terminal velocity profile derived earlier (and skipping the algebra), yields an equation for $\delta_T(x)$:

$$\frac{d}{dx}\left(\frac{\Delta^2}{6} - \frac{\Delta^3}{30}\right) = \frac{2}{\Delta}\frac{\alpha}{U\,\delta^2}, \text{ where } \Delta = \frac{\delta_T}{\delta}$$

Integrating from x = 0, where $\Delta = 0$, we obtain

$$\frac{\Delta^3}{9} - \frac{\Delta^4}{40} = 2\frac{\alpha x}{U\,\delta^2},$$

which is plotted below.

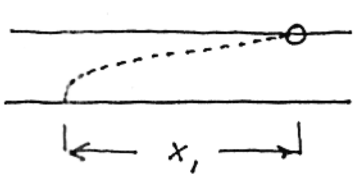


The solution expires when Δ reaches 1, i.e. when the free surface first feels the heating effect. Let $x = x_1$ where $\delta_T = \delta$:

$$\frac{1}{9} - \frac{1}{40} = 2\frac{\alpha\, x_1}{U\,\delta^2},$$

which means that

$$x_1 = \frac{31}{720}\frac{U\,\delta^2}{\alpha}$$

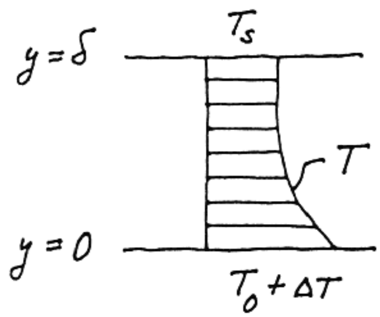


Downstream from $x = x_1$ the integral solution must be modified to allow for the temperature variation along the free surface. As new temperature profile we can use:

$$\frac{T - T_s}{T_0 + \Delta T - T_s} = 1 - 2\frac{y}{\delta} + \left(\frac{y}{\delta}\right)^2, \qquad 0 \le y \le \delta$$

where $T_s(x)$ is the free-surface temperature (T_s is the unknown in this integral analysis). Otherwise, the integral analysis is the same as for the $0 < x < x_1$ region. Note the starting condition on T_s:

$$T_s = T_0 \quad \text{at} \quad x = x_1.$$

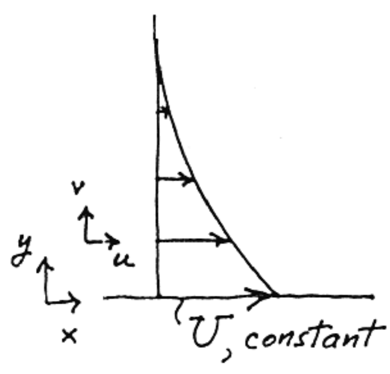


Problem 2.25. We start with the momentum equation for a transient boundary layer region

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\underbrace{\frac{dP_\infty}{dx}}_{\text{zero}} + \nu\frac{\partial^2 u}{\partial y^2}$$

$u\frac{\partial u}{\partial x} \downarrow \frac{U^2}{x} \downarrow$ zero, because $x \to \infty$

$v\frac{\partial u}{\partial y} \downarrow$ zero, because $v = 0$ when $\frac{\partial u}{\partial x} = 0$ (see the mass conservation eq.)



We are left with

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}, \quad \text{and} \quad \frac{\partial u}{\partial x} = 0$$

The scales in the layer of thickness δ are

$$\frac{U}{t} \sim \nu \frac{U}{\delta^2}, \quad \text{hence} \quad \delta \sim (\nu t)^{1/2}$$

The scaling law $\delta \sim (\nu t)^{1/2}$ suggests the similarity variable

$$\eta = \frac{1}{2} \frac{y}{(\nu t)^{1/2}}$$

and the similarity profile

$$\frac{u\,(y,t)}{U} = f(\eta)$$

By using η and f, we can write the momentum equation as

$$f'' + 2\eta f' = 0$$

Integrating once,

$$f' = c_1\, e^{-\eta^2}$$

and one more time, we obtain

$$f(\eta) - \underbrace{f(\infty)}_{\substack{\text{zero,} \\ \text{because } u \to 0 \text{ as } y \to \infty.}} = c_1 \int_{\infty}^{\eta} e^{-\eta^2}\, d\eta$$

The wall condition $f(0) = 1$ pinpoints c_1,

$$c_1 = \left(\int_{\infty}^{0} e^{-\eta^2}\, d\eta \right)^{-1} = -\frac{2}{\sqrt{\pi}},$$

therefore

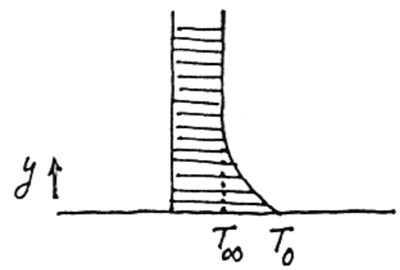
$$f = \frac{2}{\sqrt{\pi}} \int_{\eta}^{\infty} e^{-\eta^2}\, d\eta = 1 - \text{erf}\,(\eta),$$

and the near-wall velocity field is

$$\frac{u}{U} = 1 - \text{erf}\left(\frac{y}{2\sqrt{\nu t}} \right)$$

At the time $t = t_1$, the wall temperature is changed to T_0. The energy equation for the time-dependent boundary layer is

$$\frac{\partial T}{\partial t} + \underbrace{u \frac{\partial T}{\partial x}}_{\substack{\text{zero} \\ (x \to \infty)}} + \underbrace{v \frac{\partial T}{\partial y}}_{\substack{\text{zero} \\ (v=0)}} = \alpha \frac{\partial^2 T}{\partial y^2}$$



The energy equation is analytically the same as the momentum equation solved earlier. A similarity solution can be derived exactly in the same way, by selecting

$$\eta_T = \frac{1}{2} \frac{y}{[\alpha (t - t_1)]^{1/2}} \quad \text{and} \quad \frac{T - T_\infty}{T_0 - T_\infty} = \theta(\eta_T)$$

The solution is

$$\frac{T - T_\infty}{T_0 - T_\infty} = 1 - \text{erf}\left(\frac{y}{2\,[\alpha (t - t_1)]^{1/2}}\right)$$

The velocity and temperature fields derived above are valid in "boundary layer" regions, because these regions are "slender" in the extreme (the wall length is infinite), and because the flow region slenderness was invoked early in the analysis, in order to simplify the momentum and energy equations. The temperature field is independent of the velocity field, unlike in Pohlhausen's problem.

Problem 2.26. We write T_w for the uniform temperature of the substrate, and $T_0(x)$ for the nonuniform temperature of the wetted surface of the coating of thickness $t(x)$. The uniform heat flux q" can be related to the temperature difference across the coating,

$$q'' = k_w \frac{T_w - T_0(x)}{t(x)} \tag{1}$$

From this we deduce that

$$T_w - T_\infty = T_0(x) - T_\infty + \frac{q''}{k_w}\, t(x) \tag{2}$$

for which $[T_0(x) - T_\infty]$ is furnished by eq. (2.121). We find that

$$T_w - T_\infty = \frac{q''L}{k\,Pr^{1/3}\,Re_L^{1/2}} \underbrace{\left[\frac{(x/L)^2}{0.453} + \frac{k}{k_w}\,\frac{t(x)}{L}\,Pr^{1/3}\,Re_L^{1/2}\right]}_{C} \tag{3}$$

The condition that C must be constant leads to the needed t(x) relation

$$\frac{t(x)}{L} = \frac{C - \dfrac{(x/L)^{1/2}}{0.453}}{\dfrac{k}{k_w}\,Pr^{1/3}\,Re_L^{1/2}} \tag{4}$$

The L-averaged thickness of this coating is

$$\frac{\bar{t}}{L} = \frac{C - \dfrac{2/3}{0.453}}{\dfrac{k}{k_w}\,Pr^{1/3}\,Re_L^{1/2}} \tag{5}$$

Finally, by eliminating C in favor of $\bar{t}$ using eq. (5), we obtain the coating thickness variation

$$\frac{t(x)}{L} = \frac{\bar{t}}{L} + \frac{\dfrac{2}{3} - \left(\dfrac{x}{L}\right)^{1/2}}{0.453\,\dfrac{k}{k_w}\,Pr^{1/3}\,Re_L^{1/2}} \tag{4'}$$

and the temperature of the isothermal substrate,

$$T_w - T_\infty = \frac{q''\bar{t}}{k_w} + \frac{1.472\,q''L}{k\,Pr^{1/3}\,Re_L^{1/2}} \tag{3'}$$

There are two terms on the right side of eq. (3') because the transfer of q"L from T_w to T_∞ is impeded by two obstacles, the coating (the first term) and the laminar boundary layer (the second term).

Problem 2.27. When Pr = 0, δ/δ_T is zero, and this means that $u = U_\infty$ in the δ_T-thick region. In this limit the Pohlhausen problem statement is replaced by [33]

$$T(x,y) = T_\infty + \frac{q''}{k}\left(\frac{\alpha x}{U_\infty}\right)^{1/2} \tau(\zeta) \tag{1}$$

$$\zeta = y\left(\frac{U_\infty}{\alpha x}\right)^{1/2} \tag{2}$$

$$\tau'' + \frac{1}{2}(\zeta\,\tau' - \tau) = 0 \tag{3}$$

$$\tau'(0) = -1, \qquad \tau(\infty) = 0 \tag{4}$$

Separation of variables is achieved by differentiating eq. (3) once,

$$\frac{\tau'''}{\tau''} = -\frac{1}{2}\zeta \tag{5}$$

This equation is then integrated sequentially three times, and, after using eqs. (3) and (4), the analytical expression for the similarity temperature profile becomes

$$\tau(\zeta) = \frac{2}{\pi^{1/2}} \exp\left(-\frac{\zeta^2}{4}\right) - \zeta \operatorname{erfc}\left(\frac{\zeta}{2}\right) \tag{6}$$

This shows that the value at the wall is $\tau(0) = 2/\pi^{1/2} = 1/0.886$, which means that the local Nusselt number is

$$\mathrm{Nu} = \frac{q'}{T_0(x) - T_\infty}\,\frac{x}{k} = \tau(0)\left(\frac{U_\infty x}{\alpha}\right)^{1/2} = 0.886\,\mathrm{Pe}_x^{1/2}$$

Problem 2.28 From eq. (2.53) we know the integral form of the energy equation,

$$\frac{d}{dx}\int_0^{\delta_T} u\left(T_\infty - T\right)dy = \alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{1}$$

Substituting the u and T profiles

$$u = U_\infty \quad \text{for} \quad \Pr << 1 \tag{2}$$

$$\frac{T_0 - T(x)}{T_0 - T_\infty} = \frac{y}{\delta_T} \tag{3}$$

into eq. (1) we obtain

$$\delta_T \frac{d\delta_T}{dx} = 2\frac{\alpha}{U_\infty} \tag{4}$$

Integrating from $\delta_T = 0$ at $x = 0$ we find

$$\delta_T = 2\left(\frac{\alpha x}{U_\infty}\right)^{1/2} \tag{5}$$

The local heat flux is $q'' = k(T_0 - T_\infty)/\delta_T$, hence $h = k/\delta_T$ and the local Nusselt number

$$\mathrm{Nu} = \frac{hx}{k} = \frac{x}{\delta_T} = \frac{1}{2}\,\mathrm{Pr}_x^{1/2}$$

where $Pe_x = U_\infty x/\alpha$.

Problem 2.29 The L-averaged skin friction coefficient for laminar boundary layer flow is

$$\frac{\bar{\tau}}{\frac{1}{2}\rho U_\infty^2} = 1.128\left(\frac{\nu}{U_\infty L}\right)^{1/2} \tag{1}$$

From this follows the Reynolds number,

$$\frac{U_\infty L}{\nu} = 0.664^{-2/3}\left(\frac{\bar{\tau} L^2}{\mu\nu}\right)^{2/3} \tag{2}$$

which increases monotonically with the shear stress $\bar{\tau}$. Equation (2) also shows a new way of nondimensionalizing $\bar{\tau}$, namely the group $\bar{\tau} L^2/(\mu\nu)$. This group is similar to the Be number of eq. (3.133'), where the pressure difference is replaced by $\bar{\tau}$,

$$\frac{\bar{\tau} L^2}{\mu\nu} = \mathrm{Be}_\tau\ \mathrm{Pr}^{-1} \tag{3}$$

and

$$\mathrm{Be}_\tau = \frac{\bar{\tau} L^2}{\mu\nu} \tag{4}$$

The overall Nusselt number for the L-long boundary layer with Pr > 1 flow is

$$\mathrm{Nu}_{0-L} = 0.664\ \mathrm{Pr}^{1/3}\left(\frac{U_\infty L}{\nu}\right)^{1/2} \tag{5}$$

or, in view of eqs. (2) and (4),

$$\mathrm{Nu}_{0-L} = 0.664^{2/3}\,\mathrm{Be}_\tau^{1/3} \tag{6}$$

In scaling terms, the Nusselt number is the same as the Be_τ number raised to the power 1/3.

Chapter 3

LAMINAR DUCT FLOW

Problem 3.1. From eqs. (3.5) and (3.10) we have

$$\frac{x/D}{Re_D} = \frac{3}{40}\left(9m - 2 - \frac{7}{m} - 16 \ln m\right)$$

$$C_{f,x}\, Re_D = \frac{8}{3}\,\frac{m^2}{m-1}$$

where $m = U_c/U$. The above system establishes a parametric relationship between ($C_{f,x}\, Re_D$) and ($x/D\, Re_D$), via parameter m. This relationship can be plotted (as in Fig. 3.2) by first assuming values of m, and then calculating both ($C_{f,x}\, Re_D$) and ($x/D\, Re_D$). A few points along this curve are calculated below.

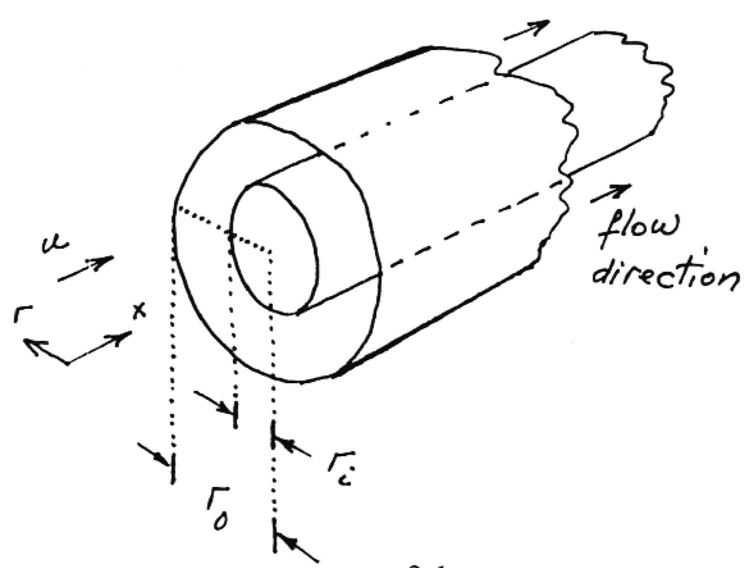
m	$\frac{x}{D\, Re_D}$	$C_{f,x}$	
1	0	∞	duct entrance
1.1	8.56×10^{-4}	32.27	
1.3	8.8×10^{-3}	15.02	
1.5	2.6×10^{-2}	12.00	beginning of fully-developed region

Problem 3.2. The fluid occupies the annular space $r_i < r < r_o$; the appropriate momentum equation for fully-developed flow is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{\mu}\frac{dP}{dx},$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\mu}\frac{dP}{dx}$$



The general solution and the two boundary conditions are

$$u(r) = \frac{r^2}{4\mu}\frac{dP}{dx} + c_1 \ln r + c_2$$

$$u = 0, \quad \text{at} \quad r = r_i$$
$$u = 0, \quad \text{at} \quad r = r_o$$

hence

$$c_1 = -\frac{1}{4\mu}\frac{dP}{dx}\frac{r_o^2 - r_i^2}{\ln\frac{r_o}{r_i}}, \qquad \text{and} \qquad c_2 = -\frac{r_o^2}{4\mu}\frac{dP}{dx} - c_1 \ln r_o$$

Omitting the algebra, the average velocity U follows from the definition

$$U\,\pi\left(r_o^2 - r_i^2\right) = \int_0^{2\pi}\int_{r_i}^{r_o} u(r)\, r\, dr\, d\theta,$$

and the result is

$$U = -\frac{r_o^2}{8\mu}\frac{dP}{dx}\left[1 + m^2 + \frac{1-m^2}{\ln(m)}\right], \quad \text{with} \quad m = r_i/r_o$$

The perimeter-averaged wall shear stress τ_{avg} is defined as

$$\tau_{avg}\underbrace{2\pi\,(r_o + r_i)}_{\substack{\text{total}\\ \text{perimeter}}} = 2\pi\, r_o\, \tau_o + 2\pi\, r_i\, \tau_i$$

where

$$\tau_i = \mu\left(\frac{\partial u}{\partial r}\right)_{r=r_i} = \frac{r_i}{2}\frac{dP}{dx} + \frac{\mu\, c_1}{r_i}$$

$$\tau_o = -\mu\left(\frac{\partial u}{\partial r}\right)_{r=r_o} = -\frac{r_o}{2}\frac{dP}{dx} - \frac{\mu\, c_1}{r_o}$$

In the end, we obtain

$$\tau_{avg} = -\frac{1}{2}\frac{dP}{dx}(r_o - r_i)$$

The above results can be condensed in the friction factor formula

$$f = \frac{\tau_{avg}}{\frac{1}{2}\rho\, U^2} = \frac{2}{\rho U}\cdot\frac{\tau_{avg}}{U}$$

namely

$$f = \frac{16}{Re_{D_h}} \frac{(1-m)^2}{1 + m^2 + \dfrac{1-m^2}{\ln(m)}},$$

where

$$Re_{D_h} = \frac{U\,D_h}{\nu}, \qquad \text{and} \qquad D_h = \frac{4\pi\left(r_o^2 - r_i^2\right)}{2\pi\left(r_o + r_i\right)} = 2r_o\,(1-m)$$

The student will find that the algebra required by the [u(r), U, τ_{avg}, f] results is lengthy and, possibly, a source of errors. It is always a good idea to check the validity of "difficult" results by reducing them to well known limits. In the case of Hagen-Poiseuille flow through an annulus, two classical limits come to mind:

i) round tube, $r_i = 0$ or $m = 0$; in this limit the friction factor reduces to

$$\lim_{m \to o} f = \frac{16}{Re_{D_h}},$$

which is the correct formula for a round tube.

ii) parallel plates, $r_i \to r_o$ or $m \to 1$, or $\varepsilon \to 0$ where $m = 1 - \varepsilon$.

$$\lim_{\varepsilon \to o} f = \frac{16}{Re_{D_h}} \cdot \frac{(1-1+\varepsilon)^2}{1 + (1-\varepsilon)^2 + \dfrac{1-(1-\varepsilon)^2}{-\varepsilon - \dfrac{\varepsilon^2}{2} - \dfrac{\varepsilon^3}{3} - \dots}}$$

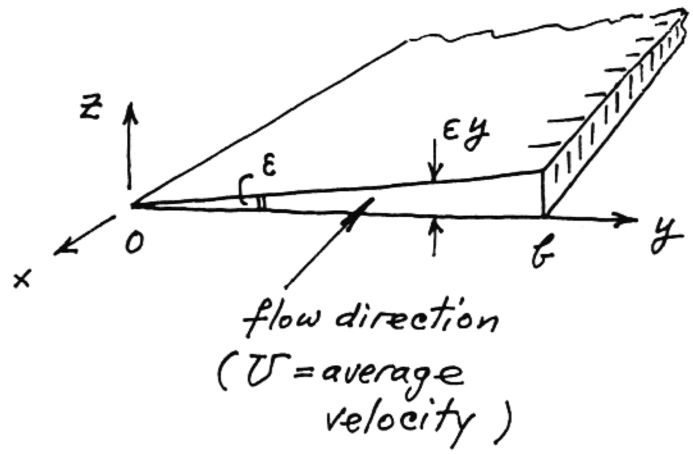
$$= \frac{16}{Re_{D_h}} \cdot \frac{3}{2} = \frac{24}{Re_{D_h}}$$

This is the correct formula for parallel plates.

Problem 3.3. The scale analysis of eq. (3.30) indicates that the $\partial^2 u/\partial y^2$ term may be neglected relative to the $\partial^2/\partial z^2$ (note that this simplification does not mean that u is not a function of y)

$$\frac{1}{\mu}\frac{dP}{dx} = \underbrace{\underset{\displaystyle \frac{U}{b^2},}{\underset{\downarrow}{\frac{\partial^2 u}{\partial y^2}}}}_{} + \underbrace{\underset{\displaystyle \frac{U}{(\varepsilon b)^2}}{\underset{\downarrow}{\frac{\partial^2 u}{\partial z^2}}}}_{}$$

$$\frac{U}{b^2} \ll \frac{U}{(\varepsilon b)^2}$$



Integrating

$$\frac{1}{\mu}\frac{dP}{dx} = \frac{\partial^2 u}{\partial z^2},$$

and claiming that

$$u = 0, \quad \text{at} \quad z = 0$$
$$u = 0, \quad \text{at} \quad z = \varepsilon y$$

yields the flow distribution

$$u(y,z) = \frac{1}{2\mu}\frac{dP}{dx}\left(z^2 - \varepsilon y z\right)$$

The cross-section-averaged velocity U can now be calculated:

$$U \underbrace{\frac{\varepsilon b^2}{2}}_{\substack{\text{cross–}\\ \text{sectional}\\ \text{area}}} = \int_0^b \left[\int_0^{\varepsilon y} u(y,z)\, dz\right] dy = -\frac{1}{48\mu}\frac{dP}{dx}\varepsilon^3 b^4$$

The perimeter-averaged wall shear stress τ_{avg} is defined by

$$\tau_{avg} \underbrace{2b}_{\substack{\text{total}\\ \text{perimeter}}} = \int_0^b \tau_{\text{bottom wall}}\, dy + \int_0^b \tau_{\text{top wall}}\, dy$$

where

$$\tau_{\text{bottom wall}} = \mu \left(\frac{\partial u}{\partial z}\right)_{z=0} = -\frac{1}{2}\frac{dP}{dx}\varepsilon y$$

$$\tau_{\text{top wall}} = \mu \left(-\frac{\partial u}{\partial z}\right)_{z=\varepsilon y} = -\frac{1}{2}\frac{dP}{dx}\varepsilon y$$

The result is

$$\tau_{avg} = -\frac{1}{4}\frac{dP}{dx}\varepsilon b$$

Finally, the friction factor turns out to be

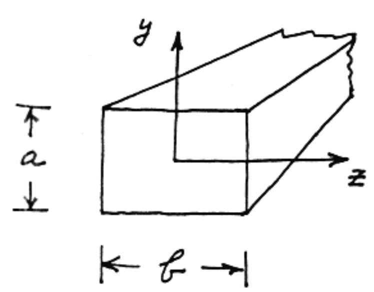
$$f = \frac{\tau_{avg}}{\frac{1}{2}\rho U^2} = \frac{2}{\rho U}\left(-\frac{1}{4}\frac{dP}{dx}\varepsilon b\right)\frac{1}{-\frac{1}{24\mu}\frac{dP}{dx}\varepsilon^2 b^2} = 12\frac{\nu}{U\varepsilon b}$$

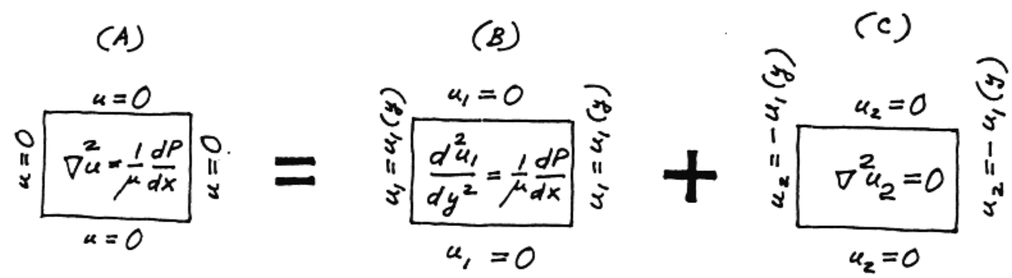
Noting that

$$D_h = \frac{4\,\frac{\varepsilon b^2}{2}}{2b} = \varepsilon b,$$

we find that the friction factor obeys the relation $f\,Re_{D_h} = 12$.

Problem 3.4. The decomposition of problem (A) into (B) + (C) can be summarized graphically as shown below [note that the differential equations and boundary conditions of (B) and (C) add up to those of (A)]:





$u(y,z) \quad = \quad u_1(y) \quad + \quad u_2(y,z)$

The (B) problem is the same as the fully-developed flow between parallel plates; the end result is

$$u_1(y) = \frac{1}{2\mu}\frac{dP}{dx}\left[y^2 - \left(\frac{a}{2}\right)^2\right]$$

The (C) problem accepts the general solution

$$u_2 = K\,(\cos\alpha y + \underbrace{\ell}_{\substack{\text{zero} \\ \left(\frac{\partial u_2}{\partial y} = 0,\ \text{at } y = 0\right)}} \sin\alpha y)\,(\cosh\alpha z + \underbrace{m}_{\substack{\text{zero} \\ \left(\frac{\partial u_2}{\partial z} = 0,\ \text{at } z = 0\right)}} \sinh\alpha z)$$

hence

$$u_2 = K\cos\alpha y\cosh\alpha z$$

From $u_2 = 0$ at $y = \pm\, a/2$, we obtain

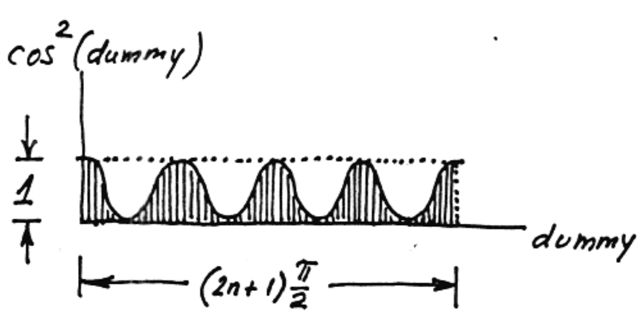
$$\alpha_n = \frac{(2n+1)\,\pi}{a},$$

in other words

$$u_2 = \sum_{n=0}^{\infty} K_n \cos\alpha_n\, y \cosh\alpha_n\, z$$

To determine K_n we multiply both sides of the above equation by $\cos\alpha_m y$, and integrate them* from $y = 0$ to $y = a/2$. The right-hand-side yields a finite term only when $m = n$:

$$\int_0^{a/2} -u_1(y)\cos\alpha_n\, y\, dy = K_n \cosh\frac{(2n+1)\,\pi b}{2a}\underbrace{\int_0^{a/2}\cos^2\alpha_n\, y\, dy}$$



$$\underbrace{\frac{1}{\alpha_n}\int_0^{(2n+1)\pi/2}\cos^2(\text{dummy})\, d\,(\text{dummy})}$$

$$\frac{1}{2}\,(1)\left[(2n+1)\,\frac{\pi}{2}\right]$$

(see dimensionless dark area in the sketch)

* while holding z constant; $z = \frac{b}{2}$ or $z = -\frac{b}{2}$.

The left-hand-side of the last equation is integrated using the earlier result for $u_1(y)$ and the formula

$$\int x^2 \cos x \, dx = 2x \cos x + \left(x^2 - 2\right) \sin x$$

The K_n result is therefore

$$K_n = \frac{(-1)^n \, 4a^2 \, \frac{dP}{dx}}{(2n+1)^3 \, \pi^3 \, \mu \cosh\left[\frac{(2n+1)\,\pi b}{2a}\right]}$$

Problem 3.5. After setting

$$u = u_0 \cos \frac{\pi y}{a} \cos \frac{\pi z}{b},$$

eqs. (3.33), (3.35) and (3.36) become:

$$ab \frac{dP}{dx} = -4\mu \, u_0 \left(\frac{b}{a} + \frac{a}{b}\right)$$

$$u_0 = \frac{\pi^2}{4} U$$

$$f = \frac{2\pi^2}{Re_{D_h}} \frac{a^2 + b^2}{(a+b)^2}$$

The friction factor has the limiting values:

$$f \, Re_{D_h} = 19.74, \text{ for flat cross-sections}$$

$$f \, Re_{D_h} = 9.87, \text{ for square cross-sections}$$

Figure 3.6 indicates that the assumed cosine form of $u(y,z)$ is not as "reasonable" as the parabolic form treated in the text.

Problem 3.6. From tables of mathematical formulas we learn that the area of a regular hexagon is

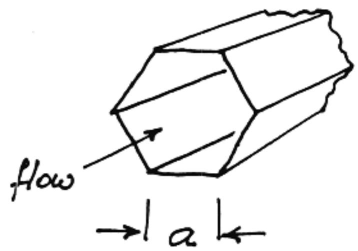
$$A_{duct} = 2.598 \, a^2,$$

hence

$$D_h = \frac{(4)\left(2.598 \, a^2\right)}{6a} = 1.732a$$

and

$$\frac{\pi D_h^2/4}{A_{duct}} = 0.907$$



The table is completed as follows

$f\, Re_{D_h} = (16)\,(0.907) = 14.51$	;	3.7% relative error
$Nu_{T_0=\text{const.}} = (3.66)\,(0.907) = 3.319$	;	1.0%
$Nu_{q''=\text{const.}} = (4.36)\,(0.907) = 3.954$	;	1.7%

The usefulness of the new group $\pi D_h^2/4/A_{duct}$ is illustrated further by Fig. 3.7, where $\pi D_h^2/4/A_{duct}$ varies from 0.5 to 1.67.

Problem 3.7. We seek the proper value of Nu so that

$$Nu = -2\,(\phi')_{r_*=1} \tag{3.73}$$

where $\phi(r_*)$ is the solution to

$$\frac{1}{r_*}\,(r_*\,\phi')' = -2\,Nu\left(1 - r_*^2\right)\phi \tag{3.71}$$

subject to conditions

$$\begin{aligned} \phi' &= 0, \quad \text{at } r_* = 0 \\ \phi &= 0, \quad \text{at } r_* = 1 \end{aligned} \tag{3.72}$$

Making the initial guess $\phi_0(r_*) = 1$, eqs. (3.71) and (3.72) yield the first corrected guess $\phi_1(r_*)$:

$$\frac{1}{r_*}\left(r_*\,\phi_1'\right)' = -2\,Nu\left(1 - r_*^2\right)(1)$$

$$\vdots$$

$$\downarrow$$

$$\phi_1 = -2\,Nu\left(\frac{1}{4}\,r_*^2 - \frac{1}{16}\,r_*^4 - \frac{3}{16}\right)$$

The second guess $\phi_2(r_*)$ is obtained by repeating the above analysis

$$\frac{1}{r_*}\left(r_* \phi_2'\right)' = -2\,\mathrm{Nu}\left(1 - r_*^2\right)\phi_1\left(r_*\right)$$

and the result is

$$\phi_2 = 4\,\mathrm{Nu}^2\left(-\frac{3}{64}\,r_*^2 + \frac{7}{256}\,r_*^4 - \frac{5}{576}\,r_*^6 + \frac{1}{1024}\,r_*^8 + 0.02724\right)$$

In order to use eq. (3.73), we calculate

$$\left(\phi_2'\right)_{r_*=1} = -\,0.1146\,\mathrm{Nu}^2$$

hence,

$$\mathrm{Nu} = -\,2\,(-\,0.1146\,\mathrm{Nu}^2)$$

or

$$\mathrm{Nu}_1 = 4.364$$

This estimate is 19% higher than the correct value 3.66.

An improved estimate can be obtained by repeating this type of analysis: the equation

$$\frac{1}{r_*}\left(r_* \phi_3'\right)' = -2\,\mathrm{Nu}\left(1 - r_*^2\right)\phi_2\left(r_*\right)$$

yields ϕ_3, which is substituted in

$$\mathrm{Nu} = -\,2\left(\phi_3'\right)_{r_*=1}$$

to calculate Nu_2, and so on.

Problem 3.8. The analysis follows the steps outlined in eqs. (3.56)-(3.65) in the text. For fully developed flow and heat transfer in the parallel plate channel of Fig. 3.1, we must solve the energy equation

$$\frac{u\,(y)}{\alpha}\,\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} \qquad (1)$$

in which

$$u = \frac{3}{2}\,U\left(1 - \eta^2\right) \qquad (2)$$

$$\eta = \frac{y}{D/2} \qquad (3)$$

$$\frac{\partial T}{\partial x} = \frac{dT_m}{dx} = \frac{2q''}{\rho\, c_P\, DU} \tag{4}$$

$$\frac{T_0(x) - T(x,y)}{T_0(x) - T_m(x)} = \phi\,(\eta) \tag{5}$$

$$T_0(x) - T_m(x) = \Delta T,\ \text{constant} \tag{6}$$

Substituting eqs. (2)-(6) into eq. (1) we obtain

$$\frac{3}{8}\,\frac{q''}{\Delta T}\,\frac{2D}{k}\left(1 - \eta^2\right) = -\,\frac{d^2\phi}{d\eta^2} \tag{7}$$

where $2D = D_h$ and $q''D_h/k\ \Delta T = Nu$. We integrate once,

$$\frac{3}{8}\,Nu\left(\eta - \frac{1}{3}\,\eta^3 + C_1\right) = -\,\frac{d\phi}{d\eta} \tag{8}$$

and invoke the symmetry condition $d\phi/d\eta = 0$ at $\eta = 0$, to find that $C_1 = 0$. We integrate eq. (8) once more,

$$\phi = \frac{3}{16}\,Nu\left(C_2 - \eta^2 + \frac{1}{6}\,\eta^4\right) \tag{9}$$

and invoke the wall condition $T = T_0$ at $y = \pm\, D/2$, which means $\phi = 0$ at $\eta = \pm\, 1$. This condition yields $C_2 = 5/6$.

The final step is to determine Nu such that the definition of the mean temperature difference holds:

$$T_0 - T_m = \frac{1}{UD}\int_{-D/2}^{D/2} u\,(T_0 - T)\,dy \tag{10}$$

Combining this with eqs. (2), (5) and (9), and dividing by ΔT we obtain

$$1 = \frac{9}{32}\,Nu \int_0^1 \left(1 - \eta^2\right)\left(\frac{5}{6} - \eta^2 + \frac{1}{6}\,\eta^4\right) d\eta \tag{11}$$

which yields

$$Nu = \frac{140}{17} = 8.235 \tag{12}$$

Problem 3.9. The energy equation for fully developed flow between two parallel plates reduces to

$$\frac{U}{\alpha}\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial y^2} \tag{1}$$

where U is the uniform longitudinal velocity (slug flow), and y is measured transversally from $y = -D/2$ to $y = D/2$. The similarity temperature profile is defined as

$$\phi(y) = \frac{T_0 - T(x,y)}{T_0 - T_m(x)} \tag{2}$$

where T_0 = constant is the wall temperature. To repeat the analysis that led to eq. (3.70) in the text, we write the first law

$$\frac{d\,T_m}{dx} = \frac{2q''(x)}{\rho\, c_P\, DU} \tag{3}$$

where $q'' = h[T_0 - T_m(x)]$, and obtain

$$\frac{T_0 - T_m(x)}{T_0 - T_m(0)} = \exp\left(-\,Nu\,\frac{x/D}{Pe_D}\right) \tag{4}$$

with

$$Nu = \frac{h \cdot 2D}{k}, \qquad Pe_D = \frac{UD}{\alpha} \tag{5}$$

The problem statement for the ϕ profile is obtained by using eqs. (2) and (4) in eq. (1),

$$-\frac{Nu}{4}\,\phi = \phi'' \tag{6}$$

$$\phi'(0) = 0, \qquad \phi(\pm 1) = 0 \tag{7}$$

where $\phi = \phi(\tilde{y})$, $(\ )' = d(\ )/d\tilde{y}$, and

$$\tilde{y} = \frac{y}{D/2} \tag{8}$$

The solution to eqs. (6)-(7) is

$$\phi = C \cdot \cos\left(\frac{\pi}{2}\,\tilde{y}\right), \quad \text{and} \quad Nu = \pi^2 \tag{9}$$

The amplitude C is determined by writing the heat flux released into the fluid by the $y = -D/2$ plate,

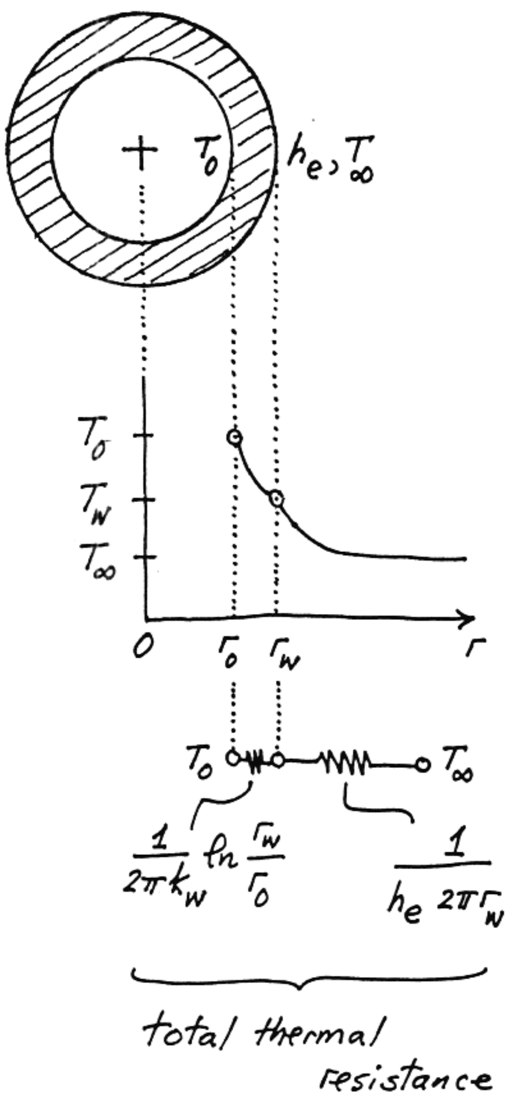
$$q'' = -k\left(\frac{\partial T}{\partial y}\right)_{y=-D/2} \tag{10}$$

which can be rearranged as

$$\frac{Nu}{4} = \phi'(-1) \tag{11}$$

Combined, eqs. (9) and (11) yield $C = \pi/2$.

Problem 3.10. With reference to the sketch shown below, the total heat transfer rate per unit pipe length q'[W/m] can be written as



$$q' = \frac{T_0 - T_\infty}{\begin{pmatrix}\text{total thermal resistance}\\ \text{per unit pipe length}\end{pmatrix}}$$

Or, keeping in mind that q" is the heat flux based on r_0,

$$2\pi\, r_0 q'' = \frac{T_0 - T_\infty}{\dfrac{\ln(r_w/r_0)}{2\pi\, k_w} + \dfrac{1}{h_e\, 2\pi\, r_w}},$$

we obtain

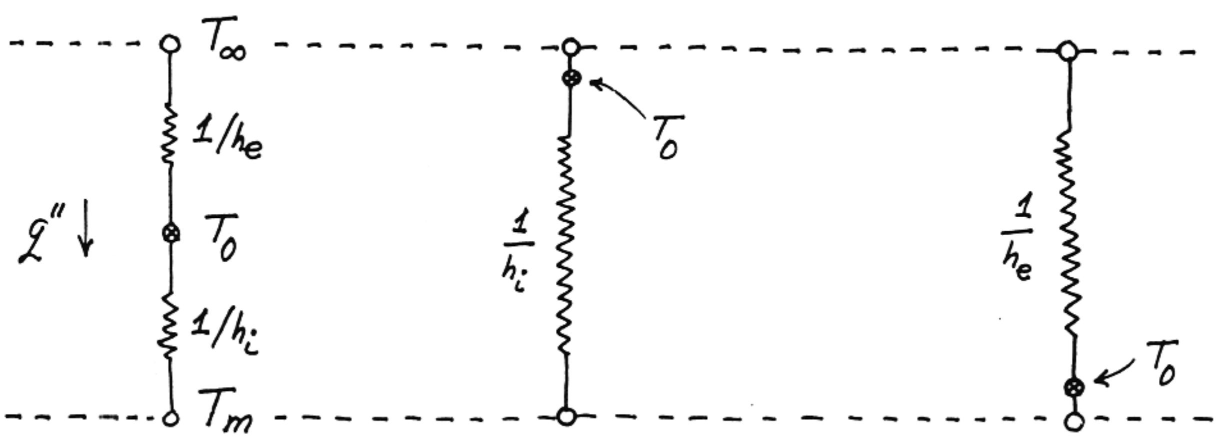
$$\frac{q''}{T_0 - T_\infty} = \frac{1}{\dfrac{r_0}{k_w}\ln\dfrac{r_w}{r_0} + \dfrac{r_0/r_w}{h_e}}$$

$$= h_{eff} \text{ of eq. (3.77)}$$

Problem 3.11. We look at Fig. 3.11 and see that at each x along the pipe the heat flux q" is impeded by two resistances in series, $1/h_e$ and $1/h_i$, where

$$\frac{1}{h_i} \sim \frac{r_0}{k}$$

because $Nu = h_i r_0/k$ and $Nu \sim 1$ (eq. 3.49). Graphically, we distinguish two limiting cases:



general case

$$\underbrace{\frac{1}{h_i} > \frac{1}{h_e}}$$

$$\frac{h_e}{k/r_0} > 1$$

and $T_0 \cong T_\infty$

i.e. T_0 = constant

$$\underbrace{\frac{1}{h_e} > \frac{1}{h_i}}$$

$$\frac{h_e}{k/r_0} < 1$$

and $q'' = \underbrace{h_e (T_\infty - T_0)}_{\text{constant}}$

Consider the last limit, where h_e is negligible compared with the fixed quantity (k/r_0). The continuity of q" through the pipe wall requires

$$q'' = h_e (T_\infty - T_0) = h_i (T_0 - T_m)$$

in which $h_i \sim k/r_0$. Therefore, if q" is finite, then $(T_\infty - T_0)$ is infinitely larger than $(T_0 - T_m)$,

$$T_\infty - T_0 > T_0 - T_m$$

The temperature variation ΔT_0 measured along a fixed length of pipe L scales as

$$\frac{\Delta T_0}{L} \sim \frac{k}{\dot{m}\, c_P} (T_0 - T_m)$$

Returning now to $q'' = h_e (T_\infty - T_0)$, we note that over the length L the temperature difference $T_\infty - T_0$ changes by ΔT_0. However, in the limit $Bi \to 0$ considered here, ΔT_0 is negligible compared with $T_\infty - T_0$ because

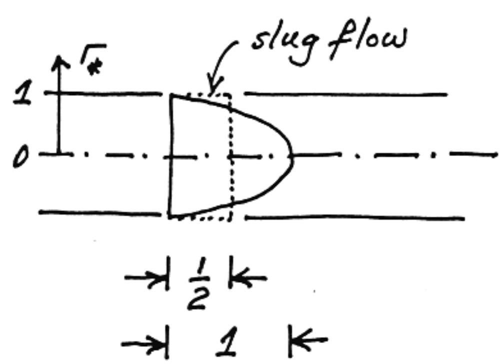
$$\frac{\Delta T_0}{T_\infty - T_0} \sim \underbrace{\frac{kL}{\dot{m}\, c_P}}_{\text{fixed}} \; \underbrace{\frac{T_0 - T_m}{T_\infty - T_0}}_{\to 0,\ \text{as Bi} \to 0}$$

In conclusion, along any finite length L the temperature difference $T_\infty - T_0$ is practically constant in the small-Bi limit, which means that

$$q'' = \text{constant}$$

Problem 3.12. In the case of slug flow, the energy eq. (3.96) assumes a simpler form

$$\frac{1}{2}\underbrace{\left(1 - r_*^2\right)}_{\substack{\text{Hagen-Poiseuille} \\ \text{profile, must be} \\ \text{replaced by 1/2}}} \frac{\partial \theta_*}{\partial x_*} = \frac{\partial^2 \theta_*}{\partial r_*^2} + \frac{1}{r_*}\frac{\partial \theta_*}{\partial r_*}$$



The complete problem statement is

(E) $$\frac{1}{4}\frac{\partial \theta_*}{\partial x_*} = \frac{\partial^2 \theta_*}{\partial r_*^2} + \frac{1}{r_*}\frac{\partial \theta_*}{\partial r_*}$$

(BC 1) $\theta_* = 0, \ \text{at} \ r_* = 1$

(BC 2) $\frac{\partial \theta_*}{\partial r_*} = 0, \ \text{at} \ r_* = 0$

(IC) $\theta_* = 1, \ \text{at} \ r_* = 0$

where

$$\theta_* = \frac{T - T_0}{T_{in} - T_0}$$

$$r_* = r/r_0$$

$$x_* = \frac{x/D}{Re_D\, Pr}$$

Separation of variables is achieved by setting

$$\theta_*\left(r_*, x_*\right) = R\left(r_*\right) \cdot \Xi\left(x_*\right)$$

and the general solution of the energy eq. (E) is

$$\theta_* = [A\, J_0(\alpha\, r_*) + \underbrace{B}_{\substack{\downarrow \\ \text{zero, because of (BC 2)}}} Y_0(\alpha\, r_*)]\, e^{-4\,\alpha^2 x_*}$$

From the first boundary condition, eq. (BC 1), we have

$$J_0(\alpha \cdot 1) = 0, \quad \text{i.e.} \quad \alpha_n = \underset{n=1}{2.4048},\ \underset{n=2}{5.5201},\ \underset{n=3}{8.6537},\ \ldots. \quad \text{etc.}$$

In conclusion,

$$\theta_*(r_*, x_*) = \sum_{n=1}^{\infty} A_n\, J_0(\alpha_n\, r_*)\, e^{-4\,\alpha_n^2 x_*}$$

To determine the A_n coefficients, we use the initial condition, eq. (IC):

$$1 = \sum_{n=1}^{\infty} A_n\, J_0(\alpha_n\, r_*)$$

Multiplying both sides by $r_*\, J_0\,(\alpha_m\, r_*)$ and integrating from $r_* = 0$ to $r_* = 1$ yields

$$\int_0^1 r_*\, J_0(\alpha_m\, r_*)\, dr_* = \sum_{n=1}^{\infty} A_n \underbrace{\int_0^1 J_0(\alpha_n\, r_*)\, J_0(\alpha_m\, r_*) \cdot r_*\, dr_*}_{\text{zero, unless } m = n}$$

or

$$\frac{1}{\alpha_n} J_1(\alpha_n) = A_n \frac{1}{2}\left[J_1^2(\alpha_n) + \underbrace{J_0(\alpha_n)}_{\text{zero}}\right]$$

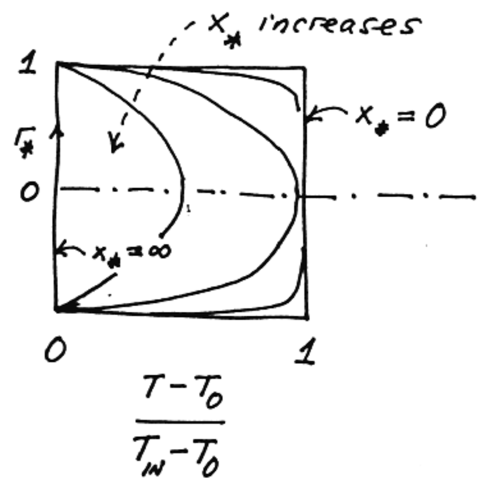
hence

$$A_n = \frac{2}{\alpha_n}\,[J_1(\alpha_n)]^{-1}$$

Putting the solution together, we can write

$$\frac{T - T_0}{T_{IN} - T_0} = 2 \sum_{n=1}^{\infty} \frac{J_0(\alpha_n\, r_*)}{\alpha_n\, J_1(\alpha_n)}\, e^{-4\,\alpha_n^2 x_*},$$

which graphically looks as follows:



Note that this problem is analytically the same as that of transient conduction in a cylindrical rod with uniform initial temperature T_{IN} and constant wall temperature T_0; the energy equation for the conduction problem is

$$\frac{1}{\alpha_{th}}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r},$$

which is the same as (E) (α_{th} is in this case the thermal diffusivity).

Problem 3.13. Let $x_* = 1$; the Nu_x and Nu_{0-x} values depend on two series:

$$S_1 = \sum_{n=0}^{\infty} G_n\, e^{-2\lambda_n^2 x_*} = (3.339)\,10^{-7} + \underbrace{(9.739)\,10^{-40} + \ldots}_{\text{negligible, especially if } x_* > 1}$$

$$S_2 = \sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2}\, e^{-2\lambda_n^2 x_*} = (4.566)\,10^{-8} + \overbrace{(2.183)\,10^{-41} + \ldots}$$

The numbers on the right-hand-side of S_1 and S_2 are from Table 3.4. Next, we evaluate one by one the quantities defined by eqs. (3.102)-(3.104):

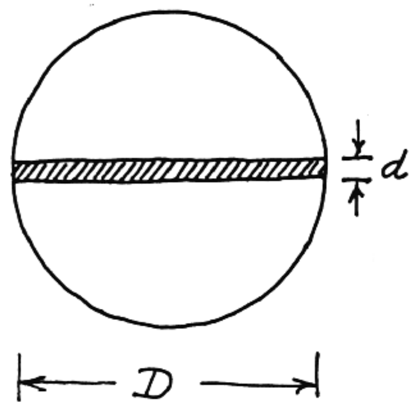
$$\theta_{*_m} \cong (8)\,(4.566)\,10^{-8} = (3.653)\,10^{-7}$$

$$Nu_x \cong \frac{(3.339)\,10^{-7}}{(2)(4.566)\,10^{-8}} = 3.656$$

$$Nu_{0-x} = \frac{1}{4}\ln\frac{1}{(3.653)\,10^{-7}} = 3.706$$

The last two values are very close to 3.66, which is the fully-developed flow Nusselt number.

Problem 3.14. Assuming that in any cross-section through the tube the twisted tape appears as a rectangle of thickness d much smaller than D, we have:



$$D_h = \frac{4\,(\text{flow cross–section})}{\text{wetted perimeter}} = \frac{4\left(\frac{\pi D^2}{4} - Dd\right)}{\pi D + 2D - 2d}$$

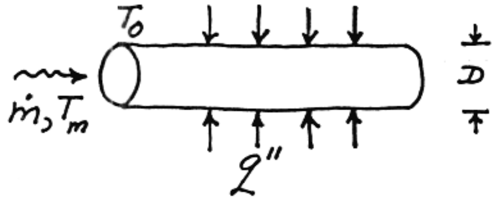
$$= \frac{4\,\frac{\pi D^2}{4}\left(1 - \frac{4}{\pi}\frac{d}{D}\right)}{(\pi + 2)\,D\left(1 - \frac{2}{\pi + 2}\frac{d}{D}\right)}$$

$$= \frac{\pi}{\pi + 2}\,D\left(1 - \frac{4}{\pi}\frac{d}{D}\right)\left(1 + \frac{2}{\pi + 2}\frac{d}{D} - \ldots\right)$$

$$= 0.611\,D\left(1 - 0.884\,\frac{d}{D} + \text{terms } O\left(\frac{d^2}{D^2}\right) \text{and higher}\right)$$

$$\cong 0.611\,D\left(1 - 0.884\,\frac{d}{D}\right)$$

Problem 3.15. We know the following quantities:



$\dot{m} = 10$ g/s

$q'' = 0.1$ W/cm^2

$r_0 = D/2 = 1$ cm

$\mu = 0.01$ g/(cm s)

$k = 0.006$ W/(cm K)

a) The Reynolds number is

$$Re = \frac{\rho U D}{\mu}$$

The product ρU follows from $\rho U \frac{\pi D^2}{4} = \dot{m}$, namely $\rho U = 3.18 \frac{g}{cm^2 s}$. We conclude that

$$Re = \frac{3.18g}{cm^2\ s} \cdot 2\ cm \cdot \frac{cm\ s}{0.01g} = 637$$

The flow through the pipe is laminar (see Table 6.1).

b) The heat transfer coefficient follows from the definition of Nusselt number

$$Nu = \frac{hD}{k} = 4.364 \text{ (Table 3.2)},$$

therefore,

$$h = (4.364)(0.006) \frac{W}{cm\ K} \frac{1}{2\ cm} = 0.0131 \frac{W}{cm^2 K}$$

c) From the definition of h, namely q" = h ($T_0 - T_m$), we deduce

$$T_0 - T_m = 0.1 \frac{W}{cm^2} \frac{cm^2\ K}{0.0131\ W} = 7.64\ K$$

Problem 3.16. a) The effectiveness is defined as

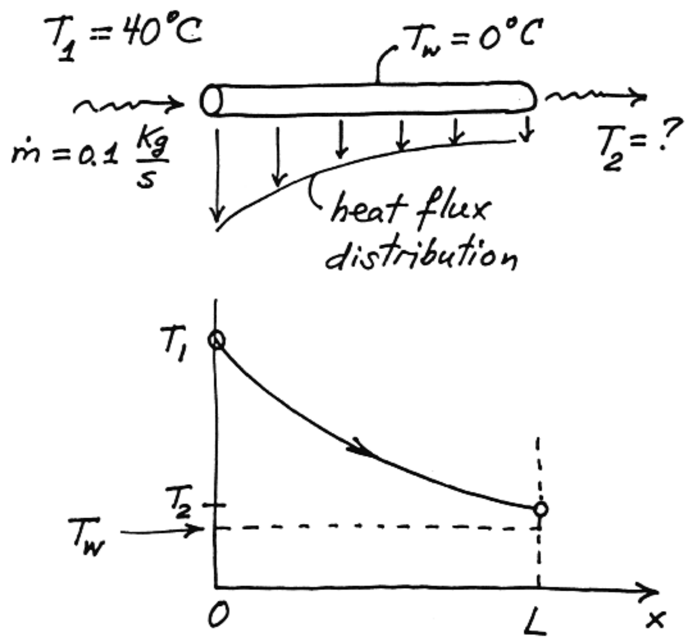
$$\varepsilon = \frac{T_1 - T_2}{T_1 - T_w}$$

hence

$$T_2 = T_1 - 0.85\ (T_1 - T_w) = 6^\circ C$$

b) Treating the tube as a control volume, we find that the total heat transfer through the tube surface is

$$Q = \int_0^L q''(x)\, \pi D\, dx = \dot{m} c\, (T_1 - T_2)$$



Numerically, $Q = 100 \frac{g}{s} 4.182 \frac{J}{gK} (40-6)K = 14.22$ kW; this calculation is based on neglecting the enthalpy change associated with pressure drop through the tube (see Table 1.1 for incompressible fluids).

c) In order to find L_1/L when $\varepsilon_1 = \varepsilon$, we must determine the relation between ε and geometry (D, L). The heat transfer per unit tube length is

$$dQ = h\,(\pi D\,dx)\,(T - T_w) = -\dot m c\,dT,$$

which integrated from $T(x = 0) = T_1$ yields

$$T(x) - T_w = (T_1 - T_w)\,e^{-\frac{h\pi Dx}{\dot m c}}$$

and, in particular,

$$T_2 = T_w + (T_1 - T_w)\,e^{-\frac{h\pi DL}{\dot m c}}$$

Based on the above, it is easy to show that

$$e = 1 - e^{-\frac{h\pi DL}{\dot{m}c}}$$

Next, we reason that if $\varepsilon = \varepsilon_1$, then

$$hDL = h_1 D_1 L_1$$

in which $hD = h_1 D_1$ because

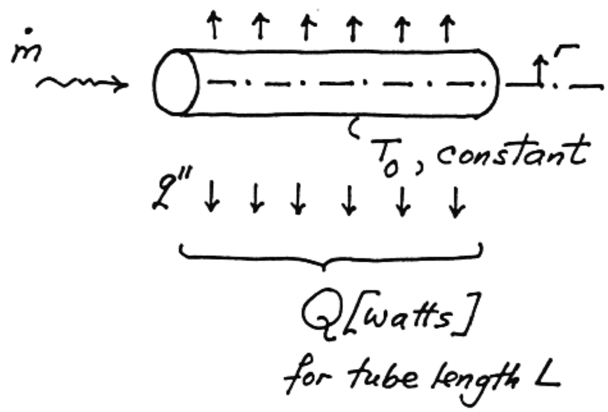
$$\frac{hD}{k} = 3.66 = \frac{h_1 D_1}{k}$$

In conclusion, the new length must be the same as the old one,

$$L = L_1$$

Problem 3.17. As the fluid is extruded through the tube it tends to heat up because of friction. The cooling effect provided along the tube maintains the tube wall temperature at T_0. To find the fluid temperature distribution T(r) and Q, we must solve the energy equation

$$\frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\left(\frac{du}{dr}\right)^2 = 0$$



subject to the boundary conditions

$$\frac{dT}{dr} = 0 \text{ at } r = 0, \qquad \text{and} \qquad T = T_0 \text{ at } r = r_0,$$

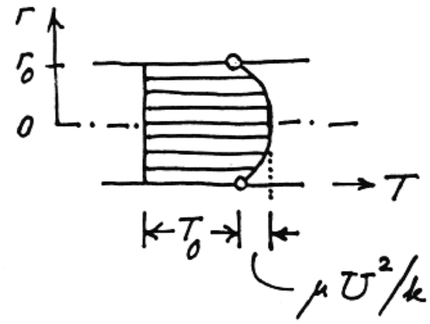
while keeping in mind that, from eqs. (3.22),

$$u = 2U\left[1 - \left(\frac{r}{r_0}\right)^2\right] \qquad \text{and} \qquad U = \frac{r_0^2}{8\mu}\left(-\frac{dP}{dx}\right)$$

Since the algebra associated with solving for T(r) is straightforward, I show only the final result:

$$T - T_0 = \frac{\mu U^2}{k}\left[1 - \left(\frac{r}{r_0}\right)^4\right]$$

$$q'' = k\left(-\frac{dT}{dr}\right)_{r=r_0} = 4\,\frac{\mu U^2}{r_0}, \text{ constant}$$



The total cooling rate for a pipe of length L is

$$Q = 2\pi\, r_0\, L\, q'' = 8\pi L\, \mu\, U^2,$$

which can be rewritten as

$$Q = 8\pi L \mu U \times U \qquad \text{note: } U = \frac{r_0^2}{8\mu}\left(-\frac{dP}{dx}\right)$$

$$= \underbrace{\pi r_0^2 U}_{\dot{m}/\rho} \quad \underbrace{L\left(-\frac{dP}{dx}\right)}_{\Delta P \text{ (pressure drop across length L)}}$$

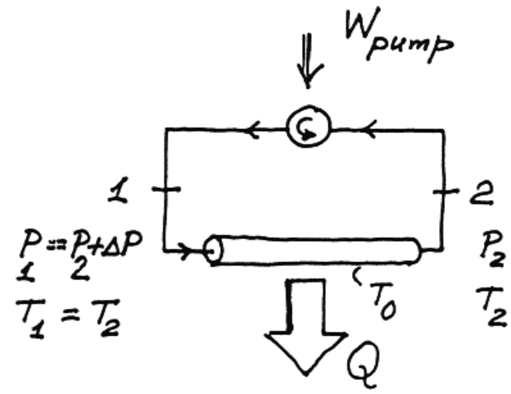
In conclusion, we arrive at

$$Q = \frac{\dot{m}\,\Delta P}{\rho} \qquad (*)$$

To see the thermodynamic significance of this result, let us assume that the fluid is incompressible (see Table 1.1), and that the pipe of length L is part of the flow circuit (cycle) sketched below. The cycle consists of two processes:

$(1 \rightarrow 2)$ = constant temperature cooling of the stream $\dot{m}$

$(2 \rightarrow 1)$ = reversible adiabatic (i.e. isentropic) compression through a pump

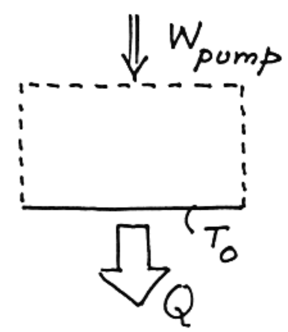


(T_1 and T_2 are bulk stream temperatures)

The first and second laws of thermodynamics for the control volume that contains the entire cycle state that:

$$Q = W_{pump} \tag{1}$$

$$S_{gen} = \frac{Q}{T_0} > 0 \tag{2}$$



It is not difficult to apply the first law to the pump alone, to show that $W_{pump} = \dot{m}\ \Delta P/\rho$; thus, eq. (1) becomes a faster way of arriving at the conclusion labeled (*) on the preceding page.

The thermodynamic significance of $Q = \dot{m}\ \Delta P/\rho$ is that it represents the pump work (per unit time) used to extrude the fluid through the pipe. This work is irreversibly lost, hence

$$Q = W_{lost},$$

and from eq. (2) above,

$$W_{lost} = T_0\, S_{gen}.$$

Problem 3.18. The relevant properties of water at 50°C are

$$k = 0.64 \frac{\text{W}}{\text{m K}} \qquad \rho = 0.988 \frac{\text{g}}{\text{cm}^3}$$

$$\nu = 0.00554 \frac{\text{cm}^2}{\text{s}} \qquad c_p = 4.18 \frac{\text{kJ}}{\text{kg K}}$$

We begin with the calculation of the Reynolds number, to be sure that the flow is laminar:

$$D_h = 2D = 2 \text{ cm}$$

$$Re_{D_h} = \frac{U D_h}{\nu} = 3.2 \frac{\text{cm}}{\text{s}} \, 2 \text{ cm} \frac{\text{s}}{0.00554 \text{ cm}^2}$$

$$= 1155 \qquad \text{(laminar)}$$

$$Nu = \frac{h D_h}{k} = 4.364 \qquad \text{(fully developed)}$$

$$h = Nu \frac{k}{D_h} = 4.364 \frac{0.64 \frac{\text{W}}{\text{m K}}}{0.02\text{m}} \cong 140 \frac{\text{W}}{\text{m}^2\text{K}}$$

$$h = \frac{q_w''}{T_w - T_m}$$

$$q_w'' = \frac{1}{2} q_{\text{one blade}}'' = 800 \frac{\text{W}}{\text{m}^2}$$

$$T_w - T_m = \frac{q_w''}{h} = 800 \frac{\text{W}}{\text{m}^2} \frac{\text{m}^2\text{K}}{140 \text{ W}} = 5.73°\text{C}$$

$$\frac{d T_m}{dx} = \frac{p}{A} \frac{q_w''}{\rho c_p U} = \frac{2W}{WD} \frac{q_w''}{\rho c_p U}$$

$$= \frac{2}{0.01\text{m}} \frac{800 \frac{\text{W}}{\text{m}^2}}{0.988 \frac{\text{g}}{\text{cm}^3} 4.18 \frac{\text{J}}{\text{g K}} 3.2 \frac{\text{cm}}{\text{s}}}$$

$$= 1.21 \frac{°\text{C}}{\text{m}}$$

For the thermal entrance length, we use eq. (3.90):

$$X_T \cong 0.1\, Re_{D_h} Pr\, D_h$$
$$\cong 0.1 \times 1155 \times 3.57 \times 2 \text{ cm} = 8.2\text{m}$$

In conclusion, the length of the parallel-plate channel must be considerably larger than 8m if the above calculations are to be valid.

Problem 3.19. a) The hydrostatic pressure distributions $P_c(y)$ and $P_h(y)$ must cross at y = H/2 so that the height-averaged pressure is the same on both sides of the door,

$$P_h\left(\frac{H}{2}\right) = P_c\left(\frac{H}{2}\right)$$

The pressure difference that drives the air leak through the bottom gap is

$$\Delta P = P_c(0) - P_h(0) = \rho_c g \frac{H}{2} - \rho_h g \frac{H}{2}$$
$$= \Delta\rho\, g\, \frac{H}{2}, \quad \text{where } \Delta\rho = \rho_c - \rho_h \tag{1}$$

The gap is a parallel-plate channel (D thin, W wide, L long), therefore

$$U = \frac{D^2}{12\mu} \frac{\Delta P}{L}$$

or

$$\Delta P = U \frac{12\, \mu L}{D^2} \frac{\rho\, DW}{\rho\, DW} = \dot{m} \frac{12\, \nu L}{D^3 W} \tag{2}$$

Combining eqs. (1) and (2) we find that

$$\dot{m} = \frac{\Delta\rho\, g\, D^3\, WH}{24\, \nu L} \tag{3}$$

The warm chamber (T_h) loses energy by convection, because of the $\dot{m}$ counterflow, warm over the top of the door, and cold under the bottom:

$$q = \dot{m}\, i_{top} - \dot{m}\, i_{bottom} \qquad (i = \text{specific enthalpy})$$
$$= \dot{m}\, c_P (T_h - T_c) \tag{4}$$

b) For the numerical part of the problem we have

$$T_c = 10°C \qquad T_h = 30°C$$

$$\rho_c = 1.247\ \frac{kg}{m^3} \qquad \rho_h = 1.165\ \frac{kg}{m^3}$$

$$H = 2.2m, \quad D = 0.5mm, \quad W = 1.5m, \quad L = 5\ cm$$

We evaluate the other air flow properties (ν, c_P, ρ) at the representative temperature of (10˚C + 30˚C)/2 = 20˚C, so that we perform the flow calculations only once (instead of doing them for each air gap separately). We obtain in order

$$\Delta\rho = \rho_c - \rho_h = 0.082\ \frac{kg}{m^3}$$

$$\Delta P = \Delta\rho\ g\ \frac{H}{2} = 0.082\ \frac{kg}{m^3}\ 9.81\ \frac{m}{s^2}\ \frac{2.2m}{2}$$

$$= 0.885\ \frac{kg}{s^2 m} = 0.885\ \frac{N}{m^2}$$

$$\dot{m} = \Delta P\ \frac{D^3\ W}{12\ \nu L}$$

$$= 0.885\ \frac{kg}{s^2 m}\ \frac{(0.0005m)^3\ 1.5m}{12 \times 0.15\ \frac{(0.01m)^2}{s}\ 0.05m}$$

$$= 1.843 \times 10^{-5}\ \frac{kg}{s}$$

$$q = \dot{m}\ c_P\ (T_h - T_c) = 1.843 \times 10^{-5}\ \frac{kg}{s}\ 1.006\ \frac{kJ}{kg\ K}\ 20°C$$

$$= 0.37W$$

It can be verified that the air flow in the gap is laminar ($Re_{D_h} \cong 1.4$), and that it is also fully developed over most of the gap length L. The important conclusion made visible by eqs. (3) and (4) is that the air leak $\dot{m}$ and the heat leak q are proportional to the gap spacing cubed (D^3). For example, if D is 2mm instead of 0.5mm, the heat leak q jumps to 24 watts.

Problem 3.20. If on either side of the hot blade the group $hA_w/\dot{m}\, c_P$ is much greater than 1 (say, greater than 3), then the outlet temperature is practically the same as T_w. For the two sides of the blade we write

$$q_1 = \dot{m}_1\, c_P\, (T_w - T_0)$$

$$q_2 = \dot{m}_2\, c_P\, (T_w - T_0)$$

In the case of fully developed laminar flow, we have

$$\dot{m}_1 = \frac{\rho\, W}{12\mu} \frac{\Delta P}{L} D_1^3$$

$$\dot{m}_2 = \frac{\rho\, W}{12\mu} \frac{\Delta P}{L} D_2^3$$

in which W is the width perpendicular to the plane $D \times L$. We write next

$$D_1 = yD \qquad \text{and} \qquad D_2 = (1 - y)D$$

and calculate the total heat transfer rate

$$q = q_1 + q_2$$

The result is

$$\frac{q}{T_w - T_0} \frac{12\mu\, L}{\rho\, c_P\, W\, D^3\, \Delta P} = y^3 + (1 - y)^3$$

The following table shows that the thermal conductance $q/(T_w - T_0)$ is the smallest (i.e. the worst) when the hot blade is positioned in the middle of the channel. The thermal conductance is four times greater than this minimum value when the blade is attached to one of the walls of the channel:

y	$\frac{q}{T_w - T_0} \frac{12\mu\, L}{\rho\, c_P\, W\, D^3\, \Delta P}$
0	1
0.25	0.44
0.5	0.25
0.75	0.44
1	1

Problem 3.21. The relation between the total heat transfer rate (q) and the largest temperature difference ($T_w - T_0$) is provided by a combination of eqs. (6.103')-(6.105),

$$q = \dot{m} c_P (T_w - T_0)\left[1 - \exp\left(-\frac{h A_w}{\dot{m} c_P}\right)\right] \tag{1}$$

where

$$\dot{m} = \frac{\rho W D^3}{12\mu} \frac{\Delta P}{L} \tag{2}$$

$$W = \text{width perpendicular to the plane } D \times L \tag{3}$$

$$\frac{h D_h}{k} = Nu = 4.86, \qquad \text{(Table 3.2)} \quad \text{(note: } D_h = 2D) \tag{4}$$

$$h = \frac{Nu}{2} \frac{k}{D}, \qquad A_w = L \times W \tag{5}$$

$$\frac{h A_w}{\dot{m} c_P} = \ldots = 6\, Nu \frac{\mu\alpha}{\Delta P L^2}\left(\frac{L}{D}\right)^4 \tag{6}$$

Next, we write the symbol Be_L for the dimensionless pressure drop

$$Be_L = \frac{\Delta P L^2}{\mu\alpha} \tag{7}$$

and substitute eqs. (2)-(7) in eq. (1):

$$q = \frac{W}{12} k (T_w - T_0) Be_L^{1/4} \delta^3 [1 - \exp(-29.16\,\delta^{-4})] \tag{8}$$

where δ is the dimensionless spacing,

$$\delta = \frac{D}{L} Be_L^{1/4} \tag{9}$$

The maximization of q with respect to δ, namely $\partial q/\partial\delta = 0$, yields the equation

$$\exp(a) = 1 + \frac{4}{3} a, \qquad \text{with} \qquad a = 29.16\,\delta^{-4} \tag{10}$$

The solution of this equation is $a_{opt} = 0.5502$, which means that $\delta_{opt} = 2.70$, and

$$\frac{D_{opt}}{L} = 2.70 \ Be_L^{-1/4} \tag{11}$$

By substituting eq. (11) in Eq. (8), we obtain the corresponding maximum thermal conductance

$$\left(\frac{q}{T_w - T_0}\right)_{max} = 0.693 \, Wk \, Be_L^{1/4} \tag{12}$$

Or, if we write $\overline{q}'' = q/WL$ for the average heat flux, we obtain finally

$$\left(\frac{\overline{q}''}{T_w - T_0}\right)_{max} \frac{L}{k} = 0.693 \ Be_L^{1/4} \tag{13}$$

In conclusion, when the pressure difference ΔP is specified, the optimal spacing has a certain value that varies as $\Delta P^{-1/4}$, and the maximum conductance increases as $\Delta P^{1/4}$.

Problem 3.22. (a) The highest temperature of the constant-q" board occurs at the trailing edge,

$$T_h = T_w \, (x = L)$$

where the relationship between the wall temperature and the fluid outlet temperature is

$$\frac{q''}{T_h - T_{out}} \frac{D_h}{k} = Nu = 5.385 \quad \text{(Table 3.2)}$$

We note that $D_h = 2D$, and conclude that

$$T_h - T_{out} = \frac{q'' 2D}{k \, Nu} \tag{1}$$

The outlet temperature can be calculated based on eq. (6.39),

$$\frac{d \, T_m}{dx} = \frac{p}{A} \frac{q''}{\rho \, c_P \, U} \tag{2}$$

where p = W, A = DW, W = width perpendicular to the plane D × L, and

$$U = \frac{\Delta P \, D^2}{12 \, \mu L} \tag{3}$$

By integrating eq. (2) from $T = T_0$ (at x = 0) to $T = T_{out}$ (at x = L) we obtain

$$T_{out} - T_0 = \frac{q'' L}{\rho \, c_P \, UD} \tag{4}$$

Next we eliminate T_{out} between eqs. (1) and (4), and then use eq. (3). The result can be arranged as follows

$$\frac{T_h - T_0}{q'' L / k} = \frac{12}{Be_L} \left(\frac{L}{D} \right)^3 + \frac{2}{Nu} \frac{D}{L} \tag{5}$$

where

$$Be_L = \frac{\Delta P \, L^2}{\mu \alpha} \tag{6}$$

The excess temperature [eq. (5)] can be minimized with respect to the spacing D, but since L is fixed, it is easier to perform the minimization with respect to the dimensionless parameter δ = D/L. The optimal value of this parameter is

$$\frac{D_{opt}}{L} = \left(18 \; Nu \; \frac{\mu \alpha}{\Delta P \, L^2} \right)^{1/4} = 3.14 \; Be_L^{-1/4} \tag{7}$$

and, after substituting it in eq. (5), the corresponding minimum T_h is given by

$$\left(\frac{T_h - T_0}{q''} \right)_{min} \frac{k}{L} = 1.554 \; Be_L^{-1/4} \tag{8}$$

Turned upside down, this result spells out the maximum heat transfer rate that can be removed by the stream when the trailing temperature of the circuit board must not exceed the (specified) ceiling temperature T_h:

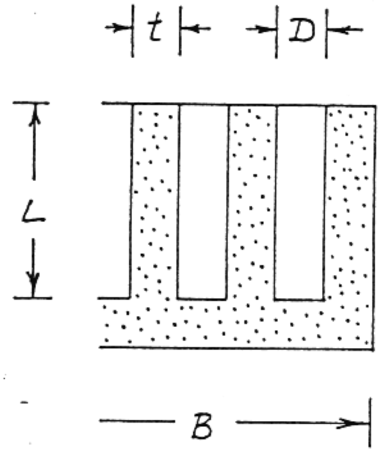
$$\left(\frac{q''}{T_h - T_0} \right)_{max} \frac{L}{k} = 0.664 \, Be_L^{1/4} \tag{9}$$

(b) The corresponding result for the board with uniform temperature T_w is (see the preceding problem)

$$\left(\frac{\overline{q}''}{T_w - T_0} \right)_{min} \frac{L}{k} = 0.693 \; Be_L^{1/4} \tag{10}$$

When the board ceiling temperature is fixed in both designs, $T_w = T_h$, the maximum heat transfer of the isothermal board exceeds by 8 percent the maximum heat transfer made possible by the board with uniform heat flux. The reason for the 8 percent difference is the leading section of the isothermal board, which is considerably warmer (i.e. higher above T_0) than the leading section of the constant-flux board.

Problem 3.23. Examine the cross-section through the flow passages and fins, which is shown in the attached figure. Assume that the heat transfer from the wall (T_w) to the fluid flow (bulk temperature T_f) is due mainly to the fins. In other words, assume that the heat transfer through the unfinned wall patches of width D is negligible.



For each fin, the heat transfer rate (per unit length normal to the figure) can be calculated with the formula

$$q'_{fin} = h \times 2 \times L\,(T_w - T_f)\,\eta \tag{1}$$

where the factor 2 means "two sides", i.e. the wetted perimeter per unit length normal to the figure. The fin efficiency is

$$\eta = \frac{\tanh\,(mL)}{mL} \tag{2}$$

where

$$mL = \left(\frac{hp}{k_w A_c}\right)^{1/2} L \tag{3}$$

In eq. (3) we substitute $p = 2W$ and $A_c = tW$, where W is the dimension normal to the figure. We also assume that the flow is laminar and fully developed, and that L >> D. According to Table 3.2, the Nusselt number

$$Nu_{D_h} = \frac{h\,D_h}{k_f} \tag{4}$$

varies between 7.54 and 8.235 depending on how we model (approximate) the thermal boundary condition over the lateral surfaces of each fin. The key here is that Nu_{D_h} is a constant (called Nu for short), whose order of magnitude is known. Remembering that $D_h = 2D$, we conclude that

$$h = Nu \frac{k_f}{2D} \tag{5}$$

and

$$mL = \left(Nu \frac{k_f}{k_w}\right)^{1/2} \frac{L}{(tD)^{1/2}} \tag{6}$$

The total number of fins spread over the wall of breadth B is B/(t + D). Therefore the total heat transfer rate released from the wall to all the channel streams is

$$\begin{aligned} q' &= \frac{B}{t+D} q'_{fin} \\ &= \frac{B}{t+D} Nu\, k_f \frac{L}{D} (T_w - T_f)\, \eta \end{aligned} \tag{7}$$

It is most convenient if we nondimensionalize the total heat transfer rate by dividing eq. (7) by $k_w(T_w - T_f)B/L$, and call it Q,

$$\frac{q'}{k_w (T_w - T_f) B/L} = Q \tag{8}$$

Equation (7) can be written sequentially as

$$\begin{aligned} Q &= Nu \frac{k_f}{k_w} \frac{(L/D)^2}{1 + \frac{t}{D}} \eta \\ &= \frac{L}{D}\left(Nu \frac{k_f}{k_w}\right)^{1/2} \frac{(t/D)^{1/2}}{1 + \frac{t}{D}} \tanh\left[\frac{L}{D}\left(Nu \frac{k_f}{k_w}\right)^{1/2} \left(\frac{D}{t}\right)^{1/2}\right] \end{aligned} \tag{9}$$

or, more succinctly,

$$Q = b \frac{x^{1/2}}{1 + x} \tanh\left(\frac{b}{x^{1/2}}\right) \tag{10}$$

where b is a number in the 0.1-10 range (because $L/D > 1$ and $k_f/k_w < 1$)

$$b = \frac{L}{D}\left(Nu \frac{k_f}{k_w}\right)^{1/2} \tag{11}$$

and x is the dimensionless fin thickness,

$$x = \frac{t}{D} \tag{12}$$

The fin thickness t (or x) is the dimension that must be chosen optimally, while holding the other parameters fixed. That Q has an optimum with respect to x can be seen by considering the following extremes. When $x \ll 1$, Q increases as $x^{1/2}$, and when $x \gg 1$, Q decreases $1/x$. The function Q of eq. (10) can be maximized numerically for a given b, and the results are

b	x_{opt}	Q_{max}	η
0.1	0.057	0.0089	0.945
0.2	0.113	0.0322	0.896
0.5	0.270	0.152	0.775
1	0.498	0.419	0.627
2	0.809	0.971	0.439
4	0.989	1.999	0.248
10	0.999	5	0.1
20	0.999	10	0.05

In conclusion, when $b \gtrsim 2$ the optimal fin thickness is approximately the same as the channel spacing D. When b is smaller, say $b \lesssim 1$, the optimal fin thickness is approximately proportional to b,

$$\frac{x_{opt}}{b} \cong 0.057 \tag{13}$$

which means that the optimal fin thickness is proportional to the L,

$$\frac{t_{opt}}{L} \cong 0.057 \left(Nu \frac{k_f}{k_w} \right)^{1/2} \tag{14}$$

Said another way, when $b \lesssim 1$ the optimal fin thickness is such that the slenderness of the fin profile equals the quantity listed on the right side of eq. (14).

Problem 3.24. When the board is positioned off center, say, $D_1 > D_2$, the $\dot{m}_1$ stream will be larger (and a better coolant) than the $\dot{m}_2$ stream. The board surface cooled by the $\dot{m}_1$ stream (surface no. 1) will have a temperature $T_1(x)$ that is lower than the corresponding (aligned) temperature of the surface cooled by the $\dot{m}_2$ stream, $T_2(x)$. The local temperature difference $T_2 - T_1$ drives a conduction heat flux through the board, from surface no. 2 to surface no. 1,

$$q_c'' = \frac{k_w}{t} (T_2 - T_1) \tag{1}$$

The heat flux generated by the electronics mounted on each surface is q" = constant. The heat flux removed by the $\dot{m}_1$ stream is larger than this, because of the q_c'' contribution,

$$q_1'' = q'' + q_c'' \tag{2}$$

For the same reason, only a portion of the electrically generated q" is removed from surface no. 2 by the $\dot{m}_2$ stream,

$$q_2'' = q'' - q_c'' \tag{3}$$

The heat fluxes (q_1'', q_2'') and surface temperatures (T_1, T_2) are functions of longitudinal position, x. The temperatures increase with x, reaching their highest values at the trailing end (x = L),

$$T_1(L) = T_{h,1}, \qquad T_2(L) = T_{h,2} \tag{4}$$

The larger of these two values is the most critical item in the design: our objective is to minimize it.

We obtain the temperature distributions $T_1(x)$ and $T_2(x)$ by making the simplifying assumption that the temperature increase along each surface [for example, $T_1(L) - T_1(0)$] is considerably greater than the local temperature difference between the surface and the corresponding stream. This assumption becomes better as the $D \times L$ channel becomes more slender. It means that we approximate the local stream temperature as being almost equal to the temperature of the neighboring spot on the surface. Consequently, the first law of thermodynamics for a dx slice of the D_1 subchannel is written as

$$\dot{m}_1 \, c_P \, dT_1 = q_1'' \, W \, dx \tag{5}$$

in which W is the width perpendicular to the $D \times L$ plane. Note that only one surface of subchannel no. 1 transfers heat to the stream. The corresponding first-law statement for the second subchannel is

$$\dot{m}_2 \, c_P \, dT_2 = q_2'' \, W \, dx \tag{6}$$

The mass flowrates $\dot{m}_1$ and $\dot{m}_2$ are driven by the same pressure drop ΔP, which is maintained across the entire assembly. Assuming that the flow is laminar and fully developed along most of the length L in both subchannels, we write

$$\dot{m}_1 = \frac{\rho W}{12\mu} \frac{\Delta P}{L} D_1^3 \tag{7}$$

$$\dot{m}_2 = \frac{\rho W}{12\mu} \frac{\Delta P}{L} D_2^3 \tag{8}$$

At this stage, we have all the ingredients that are necessary for integrating eqs. (5) and (6) away from the entrance, where

$$T_1(0) = T_2(0) \cong T_0 \tag{9}$$

This operation is considerably simpler (and safer) if we restate the problem (1)-(8) in terms of the following dimensionless variables:

$\xi = \frac{x}{L}$ longitudinal coordinate

$$\theta_1 = \frac{T_1 - T_0}{\Delta T_{scale}}$$ temperature of surface and stream no. 1

$$\theta_2 = \frac{T_2 - T_0}{\Delta T_{scale}}$$ temperature of surface and stream no. 2

$$\Delta T_{scale} = \frac{12\mu L^2 q''}{\rho c_P \Delta P D^3}$$ scale of longitudinal temperature rise [picked like this in order to "clean up" eqs. (5,6) to appear as in eqs. (10,11)]

$$y = \frac{D_1}{D}$$ spacing of subchannel no. 1

$$1 - y = \frac{D_2}{D}$$ spacing of subchannel no. 2

The problem reduces to integrating for $\theta_1(\xi)$ and $\theta_2(\xi)$ the two equations

$$y^3 \frac{d\theta_1}{d\xi} = 1 + B(\theta_2 - \theta_1) \tag{10}$$

$$(1-y)^3 \frac{d\theta_2}{d\xi} = 1 - B(\theta_2 - \theta_1) \tag{11}$$

by starting from the inlet, where $\theta_1(0) = \theta_2(0) = 0$. The dimensionless group B accounts for the transversal conductance of the board (i.e. its substrate),

$$B = \frac{k_w \Delta T_{scale}}{t q''} = 12 \frac{k_w}{k} \frac{\mu \alpha L^2}{\Delta P D^3 t} \tag{12}$$

First, we eliminate θ_1 by adding eqs. (10) and (11),

$$\theta_1 = \frac{2}{y^3} \xi - \left(\frac{1-y}{y}\right)^3 \theta_2 \tag{13}$$

We substitute this into eq. (11) and obtain a single equation for θ_2,

$$\frac{d\theta_2}{d\xi} + B\left[\frac{1}{(1-y)^3} + \frac{1}{y^3}\right]\theta_2 - \frac{1}{(1-y)^3} - \frac{2B\xi}{(1-y)^3 y^3} = 0 \tag{14}$$

The solution to this first-order linear ordinary differential equation is (recall that $\theta_2 = 0$ at $\xi = 0$)

$$\theta_2(\xi) = \left(\frac{a}{p} - \frac{b}{p^2}\right)[1 - \exp(-p\xi)] + \frac{b}{p}\xi \tag{15}$$

with the shorthand notation

$$p = B\left[\frac{1}{(1-y)^3} + \frac{1}{y^3}\right] \tag{16}$$

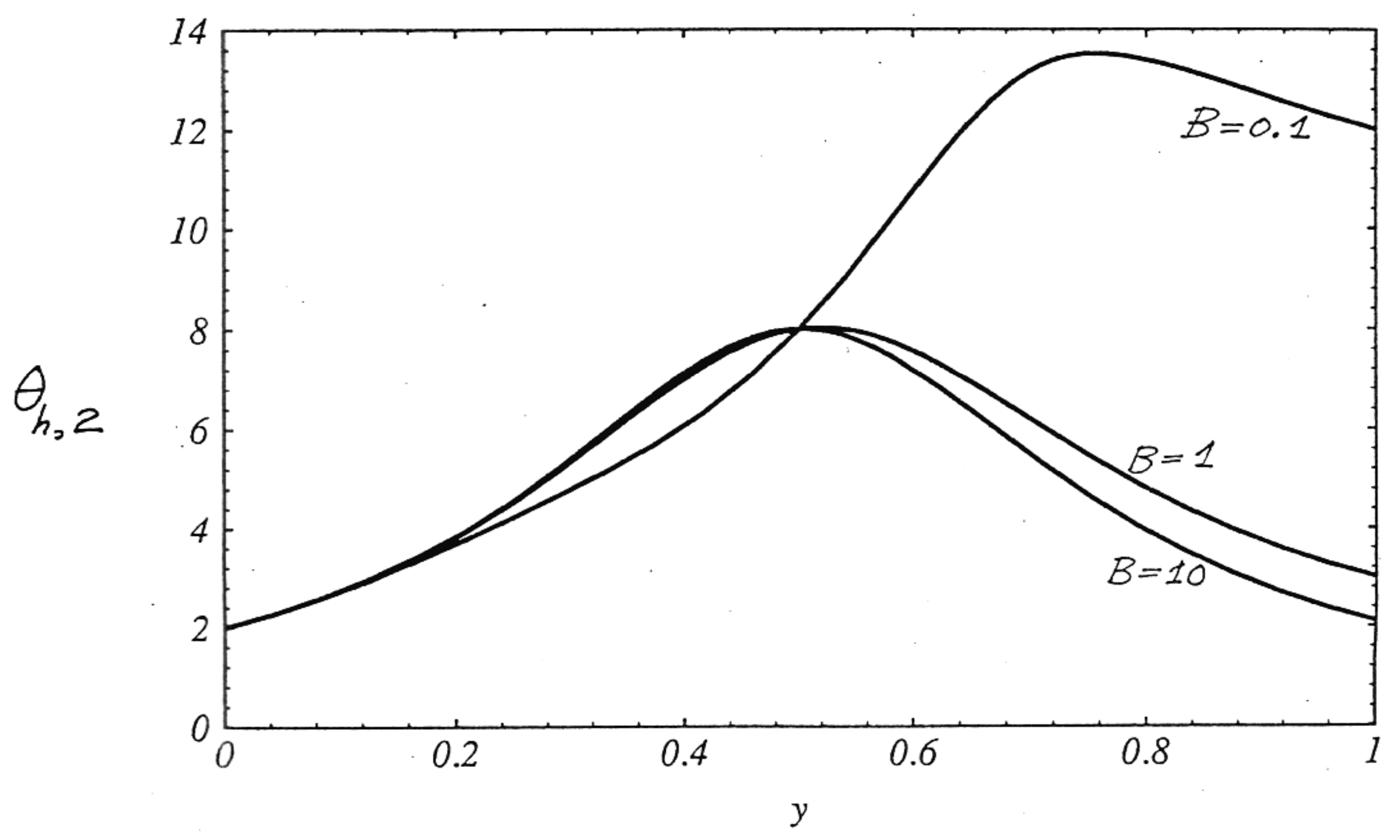
$$a = \frac{1}{(1-y)^3} \tag{17}$$

$$b = \frac{2B}{(1-y)^3 y^3} \tag{18}$$

The highest temperature of surface no. 2 is at the trailing edge, $\theta_{h,2} = \theta_2(1)$, namely

$$\theta_{h,2} = \left(\frac{a}{p} - \frac{b}{p^2}\right)[1 - \exp(-p)] + \frac{b}{p} \tag{19}$$

This temperature is a function the position of the board (y) and the board conductance number (B), as shown in the attached figure.



The highest temperature for surface no. 1, $\theta_{h,1}$, is obtained by switching y and $(1 - y)$ in the $\theta_{h,2}$ solution (19). Graphically, this is the same as superimposing on the attached figure another set of curves (for $\theta_{h,1}$) that are the mirror image of the $\theta_{h,2}$ curves (the mirror is the $y = 1/2$ vertical line). On the composite graph that results, we seek the design (board position y) that results in the lowest $\theta_{h,1}$ and $\theta_{h,2}$, when B is specified. The answer depends on whether the board substrate is a good conductor:

a) When B is of the order of 1 or larger, the $\theta_{h,1}$ and $\theta_{h,2}$ curves are bell shaped and fall on top of each other. The lowest temperatures are registered at $y = 0$ and $y = 1$, i.e. when the board is positioned close to one of the insulated walls of the channel. The worst position is in the middle of the channel, $y = 1/2$, where the highest temperature rise ($\theta_{h,1}$, or $\theta_{h,2}$) is about four times greater than when the board is mounted close to one of the insulated walls.

b) When the board is a poor thermal conductor, such that B is smaller than the order of 1, then $\theta_{h,1}$ and $\theta_{h,2}$ curves intersect forming a cusp (a V-shaped valley) at $y = 1/2$. That intersection corresponds to the lowest ($\theta_{h,1} = \theta_{h,2}$) values, indicating that the best position for the board is along the midplane of the $D \times L$ channel.

The student may wish to examine eq. (19) more closely, to determine the exact B that marks the transition from the optimal design for

a) highly conducting boards, $y_{opt} = 0,1$

to

b) poorly conducting boards, $y_{opt} = 0.5$

That "critical" B value is obtained by setting $\theta_{h,2}(1/2) = \theta_{h,2}(1)$, and solving for B.

It is absolutely fascinating that the optimal design for boards of type (b) [namely $y = 1/2$] is the same as the worst possible design for boards of type (a)! This observation stresses the importance of the dimensionless number B: this must be calculated early, to determine the problem character, (a) versus (b).

Problem 3.25. The analysis begins with the calculation of the heat transfer rate in the two asymptotic regimes noted above. Later, the optimal cylinder-to-cylinder spacing is determined by intersecting the two asymptotic estimates obtained for the tootal heat transfer between the bundle and the free stream. For the sake of concreteness we assume that the centers of adjacent cylinders form equilateral triangles, although other arrays can be treated in the same way. The approximate optimal spacing results developed here for the equilateral triangle array are applicable in and order-of-magnitude sense to any other array type.

Large spacing. Consider first the limit where the spacing S is sufficiently large that each cylinder acts as if it is alone in its own cross-flow of free-stream velocity U_∞. The total heat transfer rate experienced by the bundle is

$$q = n q_1 \tag{1}$$

where q_1 is the heat transfer associated with a single cylinder,

$$q_1 = \frac{k}{D} \mathrm{Nu} \pi D L \left(T_w - T_\infty\right) \tag{2}$$

and n is the total number of cylinders,

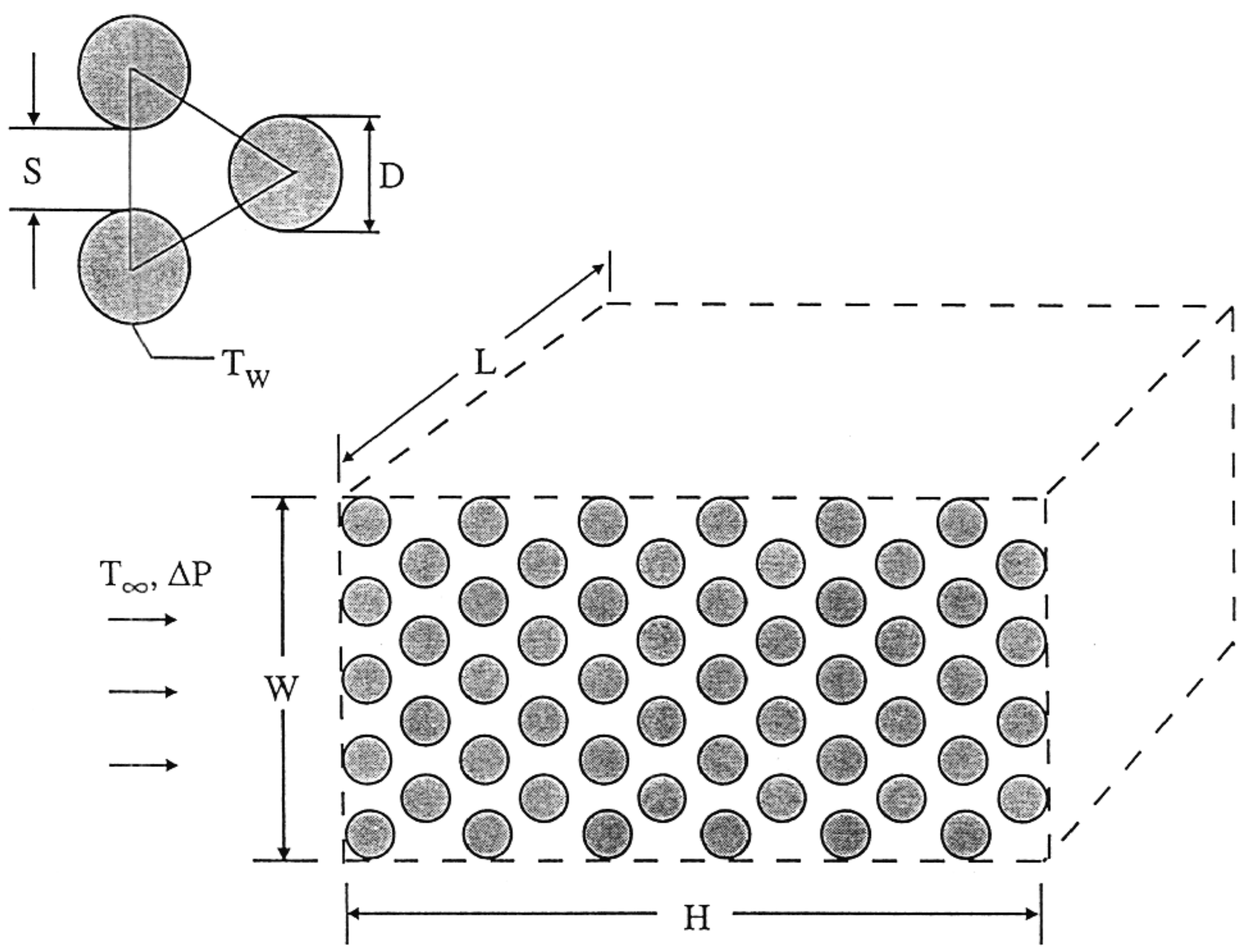
$$n = \frac{HW}{(S + D)^2 \cos 30°} \tag{3}$$

We are assuming that W is considerably greater than (S + D). In the range $0.7 < Pr < 500$ and $40 < U_\infty D/\nu < 1000$, the average Nusselt number is given by the correlation[1]

$$Nu = 0.52\, Pr^{0.37}\, Re^{1/2} \tag{4}$$

where $Re = U_\infty D/\nu$.

The free stream velocity U_∞ is not given. It is determined by the force balance on the entire bundle, $\Delta P \cdot WL = nF_1$, where F_1 is the drag force experienced by one cylinder, $F_1 = C_D DL(\rho U^2_\infty / 2)$. The drag coefficient varies from 2 to 1 in the Re range 40 - 1000, therefore in this order of magnitude analysis it is sufficient to use $C_D \sim 1.5$. The force balance yields U_∞, or Re,



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[1]Zukauskas, A., 1987, "Convective Heat Transfer in Cross Flow," Chapter 6 in Kakac, S., Shah, R.K. and Aung, W., *Handbook of Single-Phase Convective Heat Transfer*, Wiley, New York.

$$\mathrm{Re} \cong 1.1\ (\tilde{S} + 1) \left(\frac{\tilde{P}}{\tilde{H}} \right)^{1/2}, \text{ with } \tilde{S} = \frac{S}{D}, \quad \text{and} \quad \tilde{H} = \frac{H}{D} \tag{5}$$

where $\tilde{P}$ is the dimensionless pressure number

$$\tilde{P} = \frac{\Delta P \cdot D^2}{\mu \nu} = \frac{Be_D}{Pr} \tag{6}$$

where $Be_D = \Delta P \cdot D^2/(\mu\alpha)$. Combining Eqs. (1)-(6) we find that the total heat transfer rate behaves as

$$q_{largeS} \cong 2kL \frac{W}{D} (T_w - T_\infty) \frac{\tilde{H}^{3/4}\, \tilde{P}^{1/4}\, Pr^{0.37}}{(\tilde{S} + 1)^{3/2}} \tag{7}$$

In conclusion, when the spacing is sufficiently large the total heat transfer rate decreases as $S^{-3/2}$ as the spacing increases.

Small spacing. Consider now the opposite extreme when the cylinders almost touch, and the flow is almost cut off. In this limit the temperature of the coolant that exits slowly through the right end of the bundle (the plane $L \times W$) is essentially the same as the cylinder temperature T_w. The heat transfer from the bundle to the coolant is equal to the enthalpy gained by the coolant.

$$q = \dot{m}\, c_P\, (T_w - T_\infty) \tag{8}$$

where $\dot{m}$ is the mass flowrate through the $L \times W$ plane.

To obtain an order-of-magnitude estimate for the flowrate, we note that $\dot{m}$ is composed of several streams [total number $n_t \cong W/(S + D)$], each with a cross-sectional area $S \times L$ in the plane of one row of cylinder axes. The thickness of the channel traveled by each stream varies between a minimum value (S) at the row level, and a maximum value at a certain level between two rows. The volume-averaged thickness of one channel of this kind is

$$\bar{S} = S + D - 0.907 \frac{D^2}{S + D} \tag{9}$$

however, we may adjust this estimate by using 1 in place of the factor 0.907 to account for the fact that the flow must cease when the cylinders touch ($S = 0$):

$$\bar{S} = S \frac{S + 2D}{S + D} \tag{10}$$

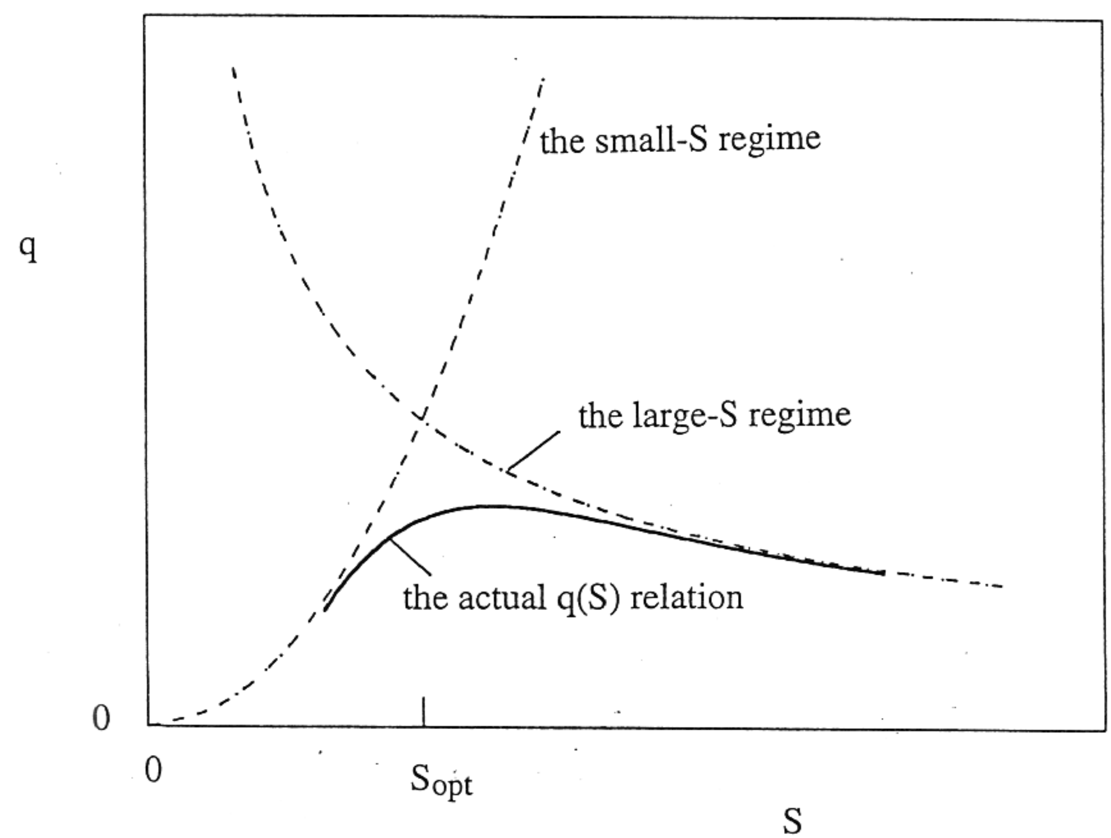
The mean velocity U through a channel of length H and cross-sectional area $\bar{S}L$ can be estimated using the solution for Hagen-Poiseuille flow through a parallel plate channel of spacing $\bar{S}$ and length H,

$$U = \frac{\bar{S}^2 \Delta P}{12 \mu H} \tag{11}$$

The mass flowrate through one channel is $\dot{m}_1 = \rho U \bar{S} L$. There are W / (S + D) channels of this kind, therefore the mass flowrate through the entire bundle is $\dot{m} = \dot{m}_1 W/(S+D)$, and Eq. (8) becomes

$$q_{smallS} \cong \frac{1}{12} k L \frac{W}{D} (T_w - T_\infty) \frac{\tilde{P}\, Pr\, \tilde{S}^3 (\tilde{S}+2)^3}{\tilde{H} (\tilde{S}+1)^4} \tag{12}$$

This estimate shows that if the spacing is small, the total heat transfer rate increases as S^2 when the spacing increases.



Intersection of the asymptotic regimes. To summarize the results determined until now, we found that q varies as $S^{-3/2}$ when S is large, and as S^2 when S is small. These asymptotic trends are sketched in the attached figure. The actual (unknown) curve q(S), which is indicated by the solid line in the figure, has a maximum where the spacing S is approximately the same as the S value obtained by intersecting the two asymptotes. The S_{opt} value obtained by eliminating q between Eqs. (7) and (12) is given implicitly by

$$\tilde{S}_{opt} \frac{\left(2 + \tilde{S}_{opt}^{-1}\right)^{6/7}}{\left(1 + \tilde{S}_{opt}^{-1}\right)^{5/7}} \cong 2.5 \frac{\tilde{H}^{1/2}}{\tilde{P}^{3/14}\, Pr^{0.18}} \tag{13}$$

This result shows that the optimal spacing increases with the length of the bundle, and decreases with the applied pressure difference and the Prandtl number. The strongest effect is due to the bundle flow length $\tilde{H}$. It is also interesting that in Eq. (13) the exponents of $\tilde{P}$ and Pr are almost the same (note that 3/14 = 0.21). This means that instead of the product $\tilde{P}^{3/4}\, Pr^{0.18}$ we may use approximately $Be_D^{3/14}$,

$$\tilde{S}_{opt} \frac{\left(2 + \tilde{S}_{opt}^{-1}\right)^{6/7}}{\left(1 + \tilde{S}_{opt}^{-1}\right)^{5/7}} \cong 2.5 \frac{\tilde{H}^{1/2}}{Be_D^{3/14}} \tag{14}$$

where the $Be_D^{3/14}$ group is defined as

$$Be_D = \frac{\Delta P \cdot D^2}{\mu\alpha} = \tilde{P}\, Pr \tag{15}$$

In this way Eq. (14) becomes similar to the dimensionless results reported for optimal plate-to-plate spacing in forced convection with $Pr \geq 1$ fluids, where the relevant dimensionless pressure group is Be_D not $\tilde{P}$.

Problem 3.26 The analysis follows the same steps as in eqs. (3.122)-(3.134) in the text. The only difference is that instead of eq. (2.110) we use the small-Pr equivalent, eq. (2.111), such that eq. (3.129) is replaced by

$$\frac{\overline{h}L}{k} = \frac{\overline{q''}L}{k(T_w - T_\infty)} = 1.128\,Pr^{1/2}\left(\frac{U_\infty L}{\nu}\right)^{1/2}$$

The rest of the analysis is replaced by

$$q_b' = 2.053\,kH(T_w - T_\infty)\frac{\Delta P^{1/3} L^{1/3}}{\alpha^{1/2}\mu^{1/6}\rho^{1/6}D^{2/3}}$$

$$\frac{D_{opt}}{L} \cong 3.33\ Be^{-1/4}\,Pr^{1/6}$$

$$q_{max}' \lesssim 0.92\ (\rho\,\Delta P)^{1/2}\,Pr^{-3/8}\,H\,c_P\,(T_w - T_\infty)$$

Problem 3.27 An analytical expression for the peak excess temperature ($T_{peak} - T_0$) can be developed when k_0 is small, and the aspect ratio H_0/L_0 is sufficiently smaller than 1 such that the conduction through the heat-generating material (k_0) is oriented perpendicularly to the fluid channel. If we also assume that $D_0 \ll H_0$, the temperature drop between the hot-spot corner (T_{peak}) and the wall spot near the channel outlet in Fig. P3.27 is $T_{peak} - T_w = q'''H_0^2/(8k_0)$, in accordance with the steady conduction analysis reported in [8]. The increase experienced by the bulk temperature of the stream from the inlet to the outlet is $T_{out} - T_0 = q'''H_0L_0/(\dot{m}_0' c_P)$. There is also a temperature difference between the bulk temperature T_{out} and the duct wall temperature (T_w) in the plane of the outlet: temperature differences of this kind are neglected in this study based on the assumption that the flow is fully developed and the channel spacing is sufficiently small.

In conclusion, the peak excess temperature is given by a two-term expression that can be nondimensionalized in the form of the overall resistance of the elemental volume,

$$\Delta\tilde{T}_0 = \frac{\tilde{H}_0}{8\tilde{L}_0} + \frac{1}{M\tilde{H}_0} \tag{2}$$

where

$$\left(\tilde{H}_0, \tilde{L}_0\right) = \frac{(H_0, L_0)}{A_0^{1/2}} \qquad \Delta\tilde{T}_0 = \frac{T_{peak} - T_0}{q'''A_0/k_0} \tag{3}$$

$$M = \dot{m}''c_P\,A_0^{1/2}/k_0,\ \text{constant} \tag{4}$$

In this notation the size constraint $H_0L_0 = A_0$ becomes $\tilde{H}_0\tilde{L}_0 = 1$. The right-hand side of eq. (2) is equal to $\tilde{H}_0^2/8 + 1/(M\tilde{H}_0)$, and shows that the overall resistance $\Delta\tilde{T}_0$ can be minimized with respect to the external shape parameter $\tilde{H}_0$. The results are

$$\tilde{H}_{0,opt} = \left(\frac{4}{M}\right)^{1/3} \qquad \tilde{L}_{0,opt} = \left(\frac{M}{4}\right)^{1/3} \tag{5}$$

$$\left(\frac{H_0}{L_0}\right)_{opt} = \left(\frac{4}{M}\right)^{2/3} \qquad \Delta\tilde{T}_{0,min} = \frac{3}{2^{5/3}M^{2/3}} \tag{6}$$

Noteworthy is the optimal external shape $(H_0/L_0)_{opt}$, which is independent of the channel size D_0. The elemental volume is more elongated when the M parameter is large, i.e., when $\dot{m}''$ and A_0 are large and k_0 is small. The starting assumption that $H_0/L_0 << 1$ means that the above solution is valid when $M >> 4$. The assumption that $T_w - T_{out}$ is negligible (relative to $T_{peak} - T_w$) means that $D_0/H_0 << Nu\, k/k_0$, where k is the thermal conductivity of the fluid and Nu is a dimensionless constant of order 1 (the Nusselt number for fully developed laminar flow in a parallel-plate channel). This last inequality comes from $T_w - T_{out} << T_{peak} - T_w$ and the definitions $Nu = 2D_0h/k$, $h = q''/(T_w - T_{out})$ and $q'' = q'''H_0/2$.

Problem 3.28 Substituting $\Delta P \sim \frac{1}{2}\rho U^2$ we find, in order,

$$Be_L = \frac{\Delta P\, L^2}{\mu\alpha} \sim \frac{1}{2}\frac{U^2 L^2}{\nu\alpha} = \frac{1}{2} Re_L^2 Pr$$

$$\frac{D_{opt}}{L} \cong 3.25\, Pr^{-1/4} Re_L^{-1/2}$$

$$q'_{max} \le 0.26\, \rho c_P H\, U\, Pr^{-1/2} (T_w - T_\infty)$$

Problem 3.29 For Pr ~ 1 fluids, the hydrodynamic and thermal entrance lengths have the same scale,

$$X \sim X_T \sim D\,Re_D \tag{1}$$

where $Re_D = UD/\nu$. In the X region the wall shear stress behaves as in laminar boundary layer flow,

$$\frac{\tau}{\rho U^2} \sim \left(\frac{\nu}{UX}\right)^{1/2} \tag{2}$$

Let ΔP be the pressure drop along the entrance region. The global force balance in the longitudinal direction requires

$$\tau X \sim D \Delta P \tag{3}$$

By eliminating U and τ between eqs. (1)-(3) we obtain

$$\frac{X}{D} \sim \left(\frac{D^2 \Delta P}{\mu \nu}\right)^{1/2} \tag{4}$$

which for Pr ~ 1 is the same as

$$\frac{X}{D} \sim Be_D^{1/2} \tag{5}$$

where according to the Be definition (3.133'),

$$Be_D = \frac{D^2\,\Delta P}{\mu \alpha} \tag{6}$$

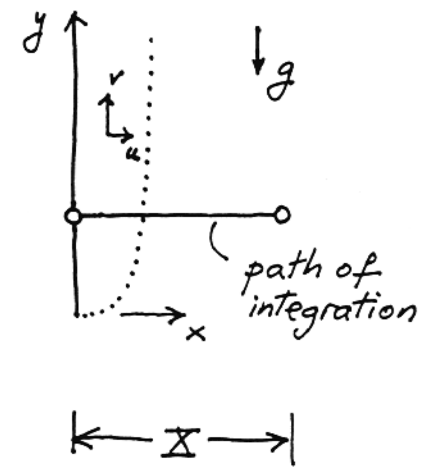
Chapter 4

EXTERNAL NATURAL CONVECTION

Problem 4.1. The momentum and energy equations for any point inside the slender boundary layer region are

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\frac{\partial^2 v}{\partial x^2} + g\beta\,(T - T_\infty)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial x^2}$$



Working on the momentum equation first, we write it as

$$\frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}\left(v^2\right) = \nu\frac{\partial^2 v}{\partial x^2} + g\beta\,(T - T_\infty)$$

and then we integrate it from x = 0 to x = X,

$$u_X \underbrace{v_X}_{0} - \underbrace{u_0}_{0}\,\underbrace{v_0}_{0} + \frac{d}{dy}\int_0^X v^2\,dx = \nu\underbrace{\left(\frac{\partial v}{\partial x}\right)_X}_{0} - \nu\left(\frac{\partial v}{\partial x}\right)_0 + g\beta\int_0^X (T - T_\infty)\,dx$$

In conclusion, we obtain

$$\frac{d}{dy}\int_0^X v^2\,dx = -\nu\left(\frac{\partial v}{\partial x}\right)_{x=0} + g\beta\int_0^X (T - T_\infty)\,dx$$

The energy equation is handled the same way,

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$u_X \underbrace{T_X}_{T_\infty} - \underbrace{u_0}_{0} T_0 + \frac{d}{dy}\int_0^X vT\,dx = \alpha \underbrace{\left(\frac{\partial T}{\partial x}\right)_X}_{0} - \alpha\left(\frac{\partial T}{\partial x}\right)_0$$

where u_X is the so-called "entrainment" velocity; we find it by integrating the mass continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

namely

$$u_X - \underbrace{u_0}_{0} + \frac{d}{dy}\int_0^X v\,dx = 0$$

Back in the energy equation, we now have

$$-T_\infty \frac{d}{dy}\int_0^X v\,dx + \frac{d}{dy}\int_0^X vT\,dx = -\alpha\left(\frac{\partial T}{\partial x}\right)_0$$

in other words,

$$\frac{d}{dy}\int_0^X v(T_\infty - T)\,dx = \alpha\left(\frac{\partial T}{\partial x}\right)_{x=0}$$

Note that immediately outside the boundary layer the reservoir fluid is not purely motionless, as the entrainment velocity is finite and negative.

Problem 4.2. The analysis consists of substituting

$$T - T_\infty = \Delta T\left(1 - \frac{x}{\delta}\right)^2 \qquad \text{and} \qquad v = V\,\frac{x}{\delta}\left(1 - \frac{x}{\delta}\right)^2$$

into the integral equations derived in Problem 4.1. Note that in the above profiles it is assumed that δ_T is the same as δ. We work first on the momentum equation:

$$\frac{d}{dy}\underbrace{\int_0^X v^2\,dx}_{V^2\delta\underbrace{\int_0^1 m^2(1-m)^4\,dm}_{\frac{1}{105}}} = -\nu\underbrace{\left(\frac{\partial v}{\partial x}\right)_0}_{\frac{V}{\delta}} + g\beta\underbrace{\int_0^X (T - T_\infty)\,dx}_{\Delta T\delta\underbrace{\int_0^1 (1-m)^2\,dm}_{\frac{1}{3}}}$$

The result is therefore

$$\frac{d}{dy}\left(\frac{V^2\delta}{105}\right) = -\nu\,\frac{V}{\delta} + \frac{1}{3}\,\Delta T\,\delta\, g\beta \qquad \text{(M)}$$

Turning now our attention to the energy equation, we have

$$\underbrace{\frac{d}{dy}\int_0^X v(T_\infty - T)\,dx}_{-V\,\Delta T\,\delta \underbrace{\int_0^1 m(1-m)^4\,dx}_{\frac{1}{30}}} = \alpha \underbrace{\left(\frac{\partial T}{\partial x}\right)_0}_{-\frac{2\,\Delta T}{\delta}}$$

in other words

$$\frac{d}{dy}(V\delta) = \frac{60}{\delta} \qquad \text{(E)}$$

At this stage we have two equations, (M) and (E), for two unknowns, V(y) and δ(y). The y-dependence of V and δ is already known from scale analysis (see Table 4.1):

$$V = C_v\, y^{1/2}, \qquad \delta = C_\delta\, y^{1/4}$$

Constants C_v and C_δ are obtained by substituting the above into (M) and (E), and the results are

$$C_v = 5.17\,\nu\left(Pr + \frac{20}{21}\right)^{-1/2}\left(\frac{g\beta\,\Delta T}{\nu^2}\right)^{1/2}$$

$$C_\delta = 3.93\,Pr^{-1/2}\left(Pr + \frac{20}{21}\right)^{1/4}\left(\frac{g\beta\,\Delta T}{\nu^2}\right)^{-1/4}$$

In particular, δ(y) can be written as

$$\frac{\delta}{y} = 3.93\left(\frac{20/21}{Pr} + 1\right)^{1/4} Ra_y^{-1/4}$$

For the local Nusselt number, we write

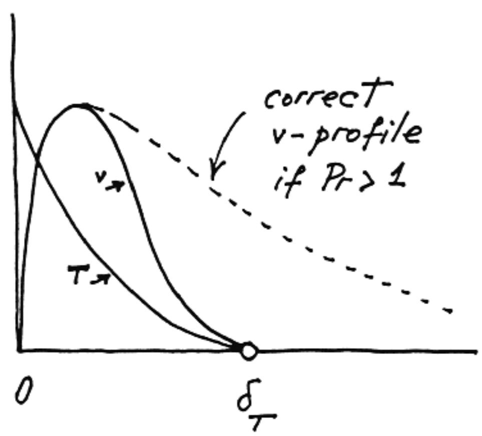
$$Nu = \frac{q''}{\Delta T}\,\frac{y}{k} = \frac{-k\,(\partial T/\partial x)_0}{\Delta T}\,\frac{y}{k} = 2\,\frac{y}{\delta}$$
$$= 0.508\left(\frac{20/21}{Pr} + 1\right)^{-1/4} Ra_y^{1/4}$$

The above is Squire's result, which is shown also in Fig. 4.4. In spite of the fact that the $\delta_T = \delta$ assumption makes Squire's analysis valid for $Pr \equiv O(1)$ only, the heat transfer results of this analysis scale correctly throughout the Pr domain (Table 4.1),

$$Nu \rightarrow 0.508\, Ra_y^{1/4}, \qquad \text{if } Pr \rightarrow \infty$$

$$Nu \rightarrow 0.514\, (Pr\, Ra_y)^{1/4}, \quad \text{if } Pr \rightarrow 0$$

This apparent contradiction is resolved if we recognize that the analysis must be judged based on its heat transfer and fluid flow results, and that it is the flow part of Squire's analysis that fails in the two Pr extremes. For example, in the case of Pr > 1 fluids the overall thickness of the velocity boundary layer should be greater than Squire's. The reason Squire's solution yields the correct Nu scales is that the temperature profile is correct (regardless of Pr), in spite of the fact that the velocity profile is not.



(The solid curves are Squire's profiles)

Problem 4.3. The similarity transformation is

$$\eta = C\, x\, y^{-1/4}, \qquad T - T_\infty = \Delta T\, \theta\,(\eta), \qquad \psi = \alpha C\, y^{3/4}\, F(\eta)$$

where

$$C = \left(\frac{g\beta\,\Delta T}{\alpha\nu}\right)^{1/4}, \text{ constant}$$

Term-by-term, the energy equation

$$\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial x^2}$$

can be written as

$$\left(\frac{3}{4}\,\alpha\, C y^{-1/4} F - \frac{1}{4}\,\alpha\, C^2 x y^{-1/2} F'\right)\left(\Delta T\,\theta'\, C y^{-1/4}\right) -$$
$$-\left(\alpha\, C^2 y^{1/2} F'\right)\left(-\frac{1}{4}\,\Delta T\,\theta' C x y^{-5/4}\right) = \alpha\,\Delta T\,\theta''\, C^2 y^{-1/2}$$

which reduces to

$$\frac{3}{4}\,\alpha\, C^2\,\Delta T\, y^{-1/2} F\,\theta' = \alpha\, C^2\,\Delta T\, y^{-1/2}\,\theta''$$

hence

$$\frac{3}{4}\, F\,\theta' = \theta''$$

Similarly, the momentum equation

$$-\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2} + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y} = -\nu\,\frac{\partial^3\psi}{\partial x^3} + g\beta\,(T - T_\infty)$$

is written term-by-term as

$$-\left(\frac{3}{4}\,\alpha\, C y^{-1/4} F - \frac{1}{4}\,\alpha\, C^2 x y^{-1/2} F'\right)\left(\alpha\, C^3 y^{1/4} F''\right) +$$
$$+\left(\alpha\, C^2 y^{1/2} F'\right)\left(\frac{1}{2}\,\alpha\, C^2 y^{-1/2} F' - \frac{1}{4}\,\alpha\, C^3 x y^{-3/4} F''\right) =$$
$$= -\nu\,\alpha\, C^4\, F''' + g\beta\,\Delta T\,\theta$$

Two terms on the left-hand-side drop out, and we are left with

$$\alpha^2 C^4\left(-\frac{3}{4}\, FF'' + \frac{1}{2}\, F'^2\right) = -\nu\,\alpha\, C^4\, F''' + \underbrace{g\beta\,\Delta T}_{\alpha\nu C^4}\,\theta$$

in other words,

$$\frac{1}{Pr}\left(-\frac{3}{4}\, FF'' + \frac{1}{2}\, F'^2\right) = -F''' + \theta \qquad \text{(M)}$$

Problem 4.4. Since air and water both have Prandtl numbers of order 1 or greater, the Nusselt number for room heat leak depends only on the Rayleigh number based on wall height (Table 4.1). So, if the air natural convection heat leak is to be simulated in a water experiment, the room and the experiment must have the same Rayleigh number. For the room, we have

$$\underset{(\text{air})}{Ra_{\text{room}}} = \frac{g\beta}{\alpha\nu}\,\Delta T\, H^3 = \frac{107}{\text{cm}^3\,\text{K}}\,(15\text{ K})(300\text{ cm})^3 = (4.3)\, 10^{10}$$

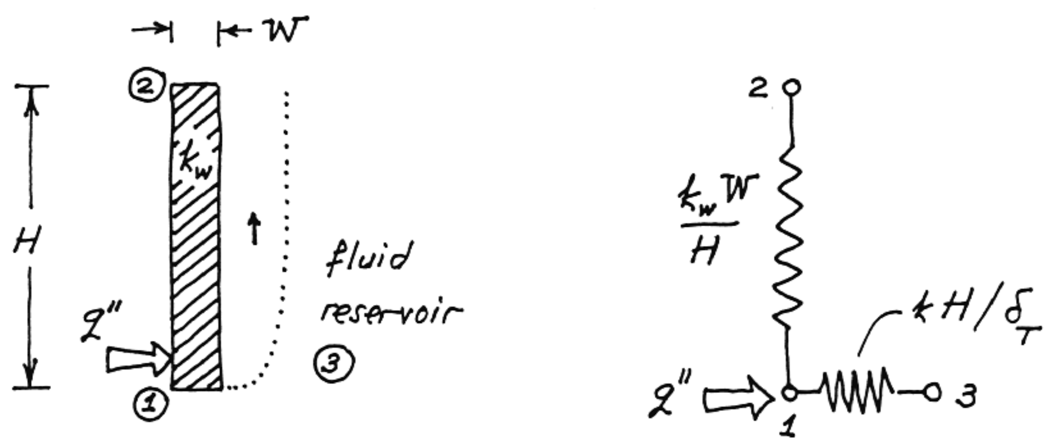
Note that this Rayleigh number is high enough to be in the transition regime, Table 6.1.

Similarly, for the water experiment we have

$$Ra_{\substack{\text{experiment}\\ \text{(water)}}} = \frac{(14.45)\,10^3}{\text{cm}^3\,\text{K}}\,(10\,\text{K})\,H^3_{water} = (4.3)\,10^{10}$$

which means that $H_{water} = 66$ cm. In conclusion, the air convection heat transfer may be simulated in a water apparatus roughly one fifth the size of the air system.

Problem 4.5. Consider the fluid pool heated from the left through a vertical wall of finite thickness, W. If the wall is to be regarded as "isothermal", then the heat flux applied to the lower end of the wall must find it easier to reach (by conduction) the upper end of the wall than the fluid reservoir. The two thermal conductances available to q" are shown below.



Therefore the wall may be modeled as isothermal when

$$\frac{k_w\,W}{H} > \frac{k\,H}{\delta_T}$$

in which $H/\delta_T \sim Ra_H^{1/4}$, therefore

$$\left(\frac{k_w}{k}\right)\left(\frac{W}{H}\right) Ra_H^{-1/4} > 1$$

The isothermal wall model deteriorates as Ra_H increases.

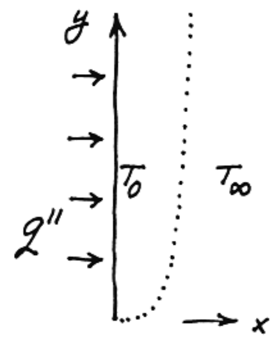
Problem 4.6. This problem is similar to Problem 4.2, the only difference being that this time the wall temperature is a function of altitude. Assuming the profiles

$$T - T_\infty = [T_0(y) - T_\infty]\left(1 - \frac{x}{\delta}\right)^2$$

$$v = V(y)\,\frac{x}{\delta}\left(1 - \frac{x}{\delta}\right)^2$$

we have the additional relation

$$q'' = \frac{2k}{\delta(y)}[T_0(y) - T_\infty] = \text{constant}$$



The integral equations are (Problem 4.2)

$$\frac{d}{dy}\left(\frac{V^2\delta}{105}\right) = -\nu\frac{V}{\delta} + \frac{g\beta\delta}{3}(T_0 - T_\infty) \qquad \text{(M)}$$

$$\frac{d}{dy}[V\delta(T_0 - T_\infty)] = \frac{60}{\delta}(T_0 - T_\infty) \qquad \text{(E)}$$

There are three unknowns (V, δ, T_0) and three equations, namely (M), (E) and the q" = constant relation. The solution form is suggested by scale analysis [see eqs. (4.69)-(4.71)],

$$\delta = C_\delta\, y^{1/5}, \qquad T_0 - T_\infty = C_T\, y^{1/5}, \qquad V = C_v\, y^{3/5}$$

and the three constants (C_δ, C_T, C_v) are determined from the three equations. In particular, the $\delta(y)$ and $T_0(y)$ results are [15]

$$\frac{\delta}{y} = 360^{1/5}\left(\frac{4/5}{Pr} + 1\right)^{1/5} Ra_{*y}^{-1/5}$$

$$T_0 - T_\infty = \frac{360^{1/5}}{2}\frac{q''y}{k}\left(\frac{4/5}{Pr} + 1\right)^{1/5} Ra_{*y}^{-1/5}$$

hence the local Nusselt number

$$Nu = 0.616\left(\frac{4/5}{Pr} + 1\right)^{-1/5} Ra_{*y}^{-1/5} \qquad (4.75)$$

In the two Pr extremes, this analysis predicts

$$Nu = 0.616\, Ra_{*y}^{-1/5}, \qquad \text{as } Pr \to \infty$$

$$Nu = 0.644\left(Pr\, Ra_{*y}\right)^{1/5}, \qquad \text{as } Pr \to 0$$

Problem 4.7. Based on the scale analysis presented in the text, for Pr > 1 fluids near a vertical wall with uniform heat flux (Fig. 4.7.b), we select the similarity variables shown below:

$$\eta = \left(\frac{g\beta q''}{\alpha \nu k}\right)^{1/5} x y^{-1/5}$$

$$T - T_\infty = \frac{q''}{k C}\, y^{1/5}\, \theta\,(\eta, \mathrm{Pr})$$

$$\psi = \alpha C y^{4/5} F\,(\eta, \mathrm{Pr}), \quad \text{where} \quad C = \left(\frac{g\beta q''}{\alpha \nu k}\right)^{1/5}$$

The energy equation

$$\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial x^2},$$

reduces to

$$\underbrace{\frac{4}{5} F\theta' - \frac{1}{5} F'\theta}_{\text{convection}} = \underbrace{\theta''}_{\text{conduction}} \qquad \text{(E)}$$

Equation (E) shows that the boundary layer is ruled by the balance between longitudinal convection and transversal thermal diffusion. Both terms in (E) are of order O(1) because the similarity transformation is based on the correct scaling laws of the flow (for more on this, see the end of this solution).

The momentum equation

$$-\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x\, \partial y} = -\nu \frac{\partial^3 \psi}{\partial x^3} + g\beta\,(T - T_\infty)$$

reduces to

$$\underbrace{\frac{1}{\mathrm{Pr}}\left[\frac{3}{5}(F')^2 - \frac{4}{5} FF''\right]}_{\text{inertia}} = \underbrace{-F'''}_{\text{friction}} + \underbrace{\theta}_{\text{buoyancy}} \qquad \text{(M)}$$

which shows how the effect of inertia decreases in importance in the momentum balance of a fluid with Pr > 1. The boundary conditions for eqs. (E) and (M) are

$$F = F' = 0 \quad \text{and} \quad \theta' = -1 \quad \text{at} \quad \eta = 0$$
$$F' \to 0 \quad \text{and} \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty$$

The scale-based similarity formulation presented above can be compared with Sparrow and Gregg's original formulation [15]:

$$\eta_{SG} = C_1\, x\, y^{-1/5}$$

$$\psi_{SG} = C_2\, y^{-4/5}\, F_0\,(\eta_{SG})$$

$$T_\infty - T_{SG} = \frac{q''}{k\, C_1}\, y^{1/5}\, \theta_0\,(\eta_{SG})$$

where

$$C_1 = \left(\frac{g\beta q''}{5k\, \nu^2}\right)^{1/5}$$

$$C_2 = \left(\frac{5^4\, g\beta q'' \nu^3}{k}\right)^{1/5}$$

The resulting equations are

$$\theta_0'' + \Pr\left(4\, \theta_0' F_0 - \theta_0 F_0'\right) = 0, \qquad (E_{SG})$$

$$F_0''' - 3\left(F_0'\right)^2 + 4\, F_0 F_0'' - \theta_0 = 0 \qquad (M_{SG})$$

Unlike eqs. (E) and (M) seen earlier, eqs. (E_{SG}) and (M_{SG}) do not reveal to the reader the actual "balances" that rule the flow.

Problem 4.8. The procedure for determining the integral momentum and energy equations is the same as in Problem 4.2 of this chapter, with the only difference that this time $T_0 - T_\infty$ of Problem 4.2 is replaced by

$$T_0 - T_\infty = \underbrace{T_0 - T_{\infty,0}}_{\Delta T,\ \text{constant}} - \gamma y$$

Dividing by ΔT, we see that $(T_0 - T_\infty)/\Delta T$ (which was equal to 1 in (M) and (E) of Problem 4.2) is this time replaced by

$$1 - \frac{\gamma y}{\Delta T} = 1 - \left(\frac{\gamma H}{\Delta T}\right)\left(\frac{y}{H}\right) = 1 - b y_*$$

The momentum and energy equations are therefore

$$\frac{d}{dy}\left(\frac{V^2 \delta}{105}\right) = -\nu \frac{V}{\delta} + \frac{1}{3}\, \Delta T \delta g \beta\, (1 - b y_*) \qquad (M)$$

$$\frac{d}{dy}\left[V\delta\,(1 - b y_*)\right] = \frac{60}{\delta}\,(1 - b y_*) \qquad (E)$$

or, introducing $\delta_* = \frac{\delta}{H} \mathrm{Ra}_{H,\Delta T}^{1/4}$ and $V_* = V \frac{H}{\alpha} \mathrm{Ra}_{H,\Delta T}^{-1/2}$,

$$\frac{1}{105\,\mathrm{Pr}} \frac{d}{dy_*}\left(V_*^2 \delta_*\right) = -\frac{V_*}{\delta_*} + \frac{\delta_*}{3}(1 - b y_*) \tag{M}$$

$$\frac{d}{dy_*}\left[V_* \delta_* (1 - b y_*)\right] = \frac{60}{\delta_*}(1 - b y_*) \tag{E}$$

In the $\mathrm{Pr} \to \infty$ limit the inertia term drops out from (M), hence, V_* can be eliminated between (M) and (E) to yield

$$\frac{d\left(\delta_*^4\right)}{dy_*} = \frac{240 + \frac{8}{3} b\, \delta_*^4}{1 - b y_*}$$

This equation can be integrated analytically from $\delta_* = 0$ at $y_* = 0$,

$$\delta_* = \left\{\frac{90}{b}\left[(1 - b y_*)^{-8/3} - 1\right]\right\}^{1/4}$$

From the formula given in the text, the overall Nusselt number can be calculated numerically

$$\frac{\mathrm{Nu}_{0-H}}{\mathrm{Ra}_H^{1/4}} = \int_0^1 \frac{2(1 - b y_*)}{\delta_*} dy_*$$

for which δ_* is provided by the preceding equation. The numerical integration is hampered somewhat by the fact that the integrand blows up at $y_* = 0$ (because $\delta_* = 0$); to get around this difficulty, we can break up the integral:

$$\int_0^1 \frac{2(1 - b y_*)}{\delta_*} dy_* = \int_0^\varepsilon \frac{2\, dy_*}{(\delta_*)_{y_* \to 0}} + \int_\varepsilon^1 \frac{2(1 - b y_*)}{\delta_*} dy_*$$

Noting that $\lim_{y_* \to 0} \delta_* = 240^{1/4} y_*^{1/4}$, we have

$$\int_0^1 \frac{2(1 - b y_*)}{\delta_*} dy_* = 0.678\, \varepsilon^{3/4} + \int_\varepsilon^1 \frac{2(1 - b y_*)}{\delta_*} dy_*$$

Evaluating the second integral by dividing the $[\varepsilon, 1]$ interval into equal steps of size $\Delta y_* = \varepsilon$, I can construct the following table for the case $b = 1$:

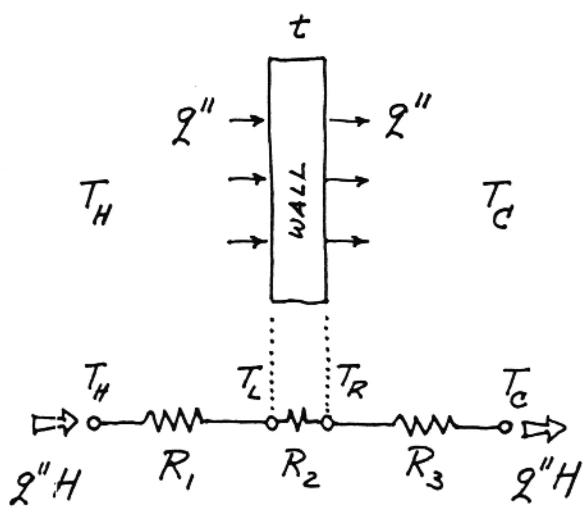
$Nu_{0-H}/Ra_H^{1/4}$	0.373	0.333	0.326	0.325
$\Delta y_* = \varepsilon$	10^{-1}	10^{-2}	10^{-3}	10^{-4}

Problem 4.9. The conjugate boundary layers and the wall may be viewed as three thermal resistances in series, hence

$$q''H = \frac{T_H - T_C}{R_1 + R_2 + R_3}$$

where

$$R_2 = \frac{t}{k_w H}$$



is the thermal resistance of the wall itself. The other resistances follow from the result for the local Nusselt number; for example, for the left face of the solid wall we can write

$$\frac{q''}{T_H - T_L(y)} \frac{y}{k} = \frac{2}{360^{1/5}} \left(1 + \frac{0.8}{Pr}\right)^{-1/5} \left(\frac{g\beta q'' y^4}{\alpha \nu k}\right)^{1/5}$$

or

$$T_H - T_L(y) = 1.623 \frac{q''}{k} \left(1 + \frac{0.8}{Pr}\right)^{1/5} \left(\frac{g\beta q''}{\alpha \nu k}\right)^{-1/5} y^{1/5}$$

Averaging this temperature difference from y = 0 to y = H we obtain

$$\overline{T_H - T_L} = 1.623\,\frac{q''}{k}\left(1+\frac{0.8}{Pr}\right)^{1/5}\left(\frac{g\beta q''}{\alpha\nu k}\right)^{-1/5}\frac{H^{6/5}}{6/5}\,\frac{1}{H}$$

$$= q''H\,\underbrace{\frac{1.352}{k}\left(1+\frac{0.8}{Pr}\right)^{1/5} Ra_{*H}^{-1/5}}_{R_1 \text{ or } R_3}$$

Putting everything together,

$$q''H = \frac{T_H - T_C}{\frac{(2)(1.352)}{k}\left(1+\frac{0.8}{Pr}\right)^{1/5} Ra_{*H}^{-1/5} + \frac{t}{k_w H}}$$

and noting the definitions

$$Nu_{0\text{-}H} = \frac{q''H}{(T_H - T_C)\,k}, \qquad \omega = \frac{t}{H}\,\frac{k}{k_w}\,Ra_H^{1/4},$$

we find that

$$Nu_{0\text{-}H} = \left[2.704\left(1+\frac{0.8}{Pr}\right)^{1/5} Ra_{*H}^{-1/5} + \omega\,Ra_H^{-1/4}\right]^{-1} \qquad (*)$$

In view of the relationship between Ra_{*H} and Ra_H,

$$Ra_{*H} = \frac{g\beta H^4 q''}{\alpha\nu k} = Ra_H\,Nu_{0\text{-}H},$$

eq. (*) provides an engineering estimate for the function $Nu_{0\text{-}H}\,(Ra_H, \omega)$. In the limit $\omega \to 0$, this function is

$$Nu_{0\text{-}H} = 0.37\left(1+\frac{0.8}{Pr}\right)^{-1/5} Ra_H^{1/5}\,Nu_{0\text{-}H}^{1/5},$$

in other words,

$$Nu_{0\text{-}H} = 0.288\left(1+\frac{0.8}{Pr}\right)^{-1/4} Ra_H^{1/4}$$

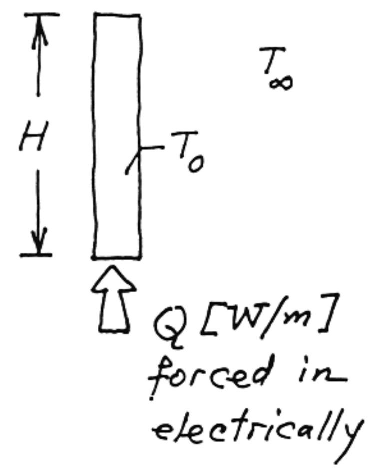
This estimate compares favorably with the alternative calculations displayed in Fig. 4.9.

Problem 4.10. a) Since $Pr_{air} = O(1)$, and assuming laminar flow (Table 6.1), from Table 4.1 we have

$$Nu \sim Ra_H^{1/4}$$

where

$$Nu = \frac{q''H}{(T_0 - T_\infty)\,k} = \frac{Q/2}{k\,\Delta T}$$



b) Rewriting the $Nu \sim Ra_H{}^{1/4}$ relation as

$$\frac{Q}{k\,\Delta T} \sim \left(\frac{g\beta H^3\,\Delta T}{\alpha\nu}\right)^{1/4}$$

we conclude that

$$\Delta T^{5/4}\,H^{3/4} = \text{constant}, \quad \text{or} \quad \Delta T = \frac{\text{const.}}{H^{3/5}}$$

Thus, if a design change calls for $H_{new}/H_{old} = 2$, then

$$\frac{\Delta T_{new}}{\Delta T_{old}} = 2^{-3/5} = 0.66$$

Problem 4.11. Let q_y'' represent the local heat flux from the natural convection boundary layer to the solid-liquid interface T_m,

$$q_y'' = \frac{k_f}{y}\,(T_\infty - T_m)\,Nu_y \qquad \text{(a)}$$

where y is measured downward, from the upper edge of the wall, and Nu_y is the local Nusselt number. The same heat flux must penetrate by conduction the solidified layer of local thickness L(y),

$$q_y'' = k_s \frac{T_m - T_w}{L} \tag{b}$$

The solid-layer thickness is therefore equal to

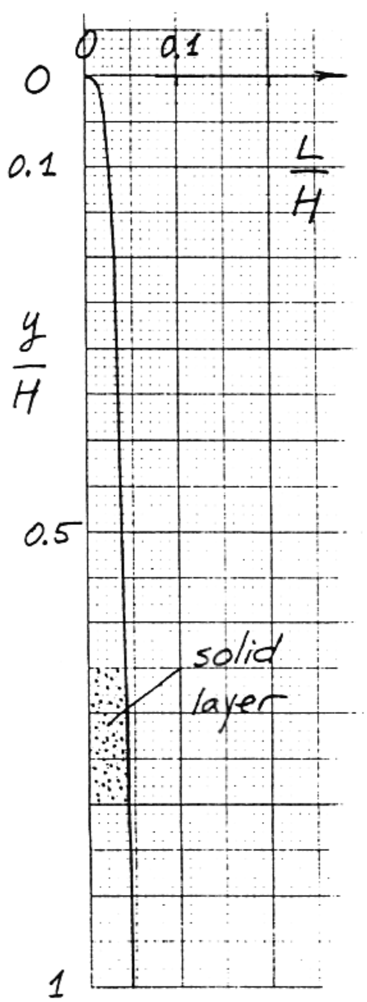
$$L = \frac{k_s}{k_f} \frac{T_m - T_w}{T_\infty - T_m} \frac{y}{Nu_y} \tag{c}$$

for which Table 4.2 recommends (recall that Pr = 55.9):

$$Nu_y = 0.487\, Ra_y^{1/4} \tag{d}$$

Equations (c) and (d) demonstrate that L is proportional to the local boundary layer thickness scale $y\, Ra_y^{-1/4}$.

Before substituting numerical values in eq. (c) it is convenient to first nondimensionalize this equation by dividing both sides by H,



$$\frac{L}{H} = \frac{k_s}{k_f} \frac{T_m - T_w}{T_\infty - T_m} \frac{1}{0.487\, Ra_H^{1/4}} \left(\frac{y}{H}\right)^{1/4} \tag{e}$$

The Rayleigh number based on the overall height is

$$Ra_H = \frac{g\beta\,(T_\infty - T_m)\,H^3}{\alpha\nu} = \frac{9.81\text{m}}{\text{s}^2} \frac{8.5 \times 10^{-4}\,\text{K}^{-1}\,(35 - 27.5)\,\text{K}\,(0.1\text{m})^3}{9 \times 10^{-4}\,\text{cm}^2/\text{s} \qquad 0.05\,\text{cm}^2/\text{s}}$$

$$= 1.39 \times 10^8, \qquad \text{(laminar flow)}$$

which means that eq. (e) becomes

$$\frac{L}{H} = \frac{0.36}{0.15} \frac{27.5 - 20}{35 - 27.5} \frac{1}{0.487\left[1.39 \times 10^8\right]^{1/4}} \left(\frac{y}{H}\right)^{1/4}$$

$$= 0.045 \left(\frac{y}{H}\right)^{1/4}$$

The shape of the L-thin solidified paraffin layer has been drawn to scale in the attached figure.

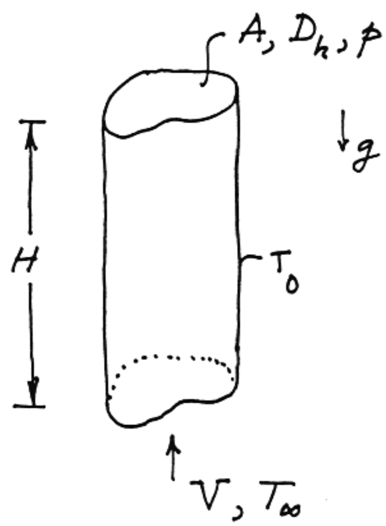
Problem 4.12. The total heat transfer rate to the stream rising through the duct is

$$Q = \rho\, c_P\, VA\, \Delta T = pH\, q''_{0-H}$$

hence

$$\frac{H\, q''_{0-H}}{k\, \Delta T} = \frac{VA}{p\alpha} = \frac{VD_h}{4\alpha}$$

The problem reduces to finding V, which is the cross-section averaged velocity through the vertical duct.



Comparing the two momentum equations for fully developed duct flows

$$\underbrace{\nabla^2 v = -\frac{g\beta\,\Delta T}{\nu}}_{\text{natural convection}} \quad \text{and} \quad \underbrace{\nabla^2 v = \frac{1}{\mu}\frac{dP}{dy}}_{\text{forced convection}}$$

we learn that the two flows are equivalent if we set

$$-\frac{g\beta\,\Delta T}{\nu} = \frac{1}{\mu}\frac{dP}{dy}$$

in other words if

$$-\frac{dP}{dy} = \rho g \beta\, \Delta T, \ \text{constant}$$

All we have to do then is exploit the solutions presented for forced flow in Chapter 3. From the definition of friction factor,

$$-\frac{dP}{dy} = f\frac{p}{A}\frac{1}{2}\rho V^2$$

we obtain

$$V^2 = \frac{g\beta\,\Delta T\,D_h}{2f} = \frac{g\beta\,\Delta T\,D_h}{2\left(f\,Re_{D_h}\right)}\cdot\frac{VD_h}{\nu}$$

or

$$V = \frac{g\beta\,\Delta T\,D_h^2}{2\left(f\,Re_{D_h}\right)\nu}$$

In conclusion, the Nusselt number formula becomes

$$\frac{H\,q''_{0\text{-}H}}{k\,\Delta T} = \frac{1}{8\left(f\,Re_{D_h}\right)}\,Ra_{D_h} = (?)\,Ra_{D_h}$$

For example, if the cross-section is circular we find that the numerical coefficient (?) is

$$(?) = \frac{1}{8(16)} = \frac{1}{128}$$

Problem 4.13. Using scale analysis only, we start with

$$(T_0 - T) < \underbrace{(T_0 - T_\infty)}_{\Delta T}$$

and with

$$q''_{0\text{-}H} \sim k\,\frac{T_0 - T}{D}$$

Combining the two equations,

$$\frac{q''_{0\text{-}H}D}{k} < \Delta T,$$

and recalling that

$$\frac{q''_{0\text{-}H}H}{k\,\Delta T} \sim Ra_D,$$

we obtain the criterion

$$Ra_D < \frac{H}{D}$$

Raising both sides of the inequality to the 1/4 power,

$$Ra_D^{1/4} < \left(\frac{H}{D}\right)^{1/4},$$

we obtain a criterion that matches within a factor of order 1 the criterion reported in eq. (4.95).

Problem 4.14. For the heat transfer to be ruled by natural convection we must have

$$(\delta_T)_{NC} < (\delta_T)_{FC},$$

where

$$(\delta_T)_{NC} \sim y\, Bo_y^{-1/4} \qquad \text{(see Table 4.1; Pr < 1)}$$
$$(\delta_T)_{FC} \sim y\, Pr^{-1/2}\, Re_y^{-1/2} \qquad \text{(see Chapter 2; Pr < 1)}$$

In conclusion, the condition is

$$\frac{Bo_y^{1/4}}{Pe_y^{1/2}} > O(1)$$

for natural convection. The reverse of this inequality characterizes the forced convection heat transfer regime.

Problem 4.15. Let us assume that the average wall temperature is sufficiently higher than $T_\infty = 25^\circ C$ so that the film temperature is 30°C. We will validate this assumption later. The relevant properties of water at 30°C are

$$Pr = 5.49 \qquad k = 0.61\ \frac{W}{m \cdot K} \qquad \frac{g\beta}{\alpha\nu} = \frac{25\ 130}{cm^3\ K}$$

The flux Rayleigh number suggests that the flow is turbulent, although we will verify the flow regime later:

$$q = 1000\ W$$
$$q_w'' = \frac{q}{A} = \frac{1000\ W}{0.5m\ 0.5m} = 4000\ W$$
$$Ra_H^* = \frac{g\beta}{\alpha\nu} H^4 \frac{q_w''}{k}$$
$$= \frac{25\ 130}{cm^3\ K} (50\ cm)^4\ 4000\ \frac{W}{m^2} \frac{m \cdot K}{0.61\ W} = 1.03 \times 10^{13}$$

For the overall Nusselt number we use eq. (4.112) in which we substitute Pr = 5.49 and neglect the 0.825 term on the right side:

$$\overline{Nu}_H = \left(... + 0.363\ Ra_H^{1/6}\right)^2 = 0.132\ Ra_H^{1/3} \qquad (1)$$

Next, we make the observation that

$$Ra_H = \frac{Ra_H^*}{\overline{Nu}_H}$$

and continue with eq. (1), which yields

$$\overline{Nu}_H^{4/3} = 0.132\ Ra_H^{*\ 1/3}$$

$$\overline{Nu}_H = 0.219\ Ra_H^{*\ 1/4} = 392.3$$

$$\overline{Nu}_H = \frac{q_w''}{\overline{\Delta T}} \frac{H}{k}$$

$$\overline{\Delta T} = \frac{q_w''}{\overline{Nu}_H} \frac{H}{k}$$

$$= \frac{1}{392.3}\, 4000 \frac{W}{m^2} \frac{0.5m}{0.61 \frac{W}{m \cdot K}} = 8.4°C$$

$$\overline{T_w} = 25°C + 8.4°C = 33.4°C$$

The correct film temperature (25°C + 33.4°C)/2 = 29.2°C is nearly the same as the value assumed at the start.

In order to verify that the boundary layer is indeed turbulent, we evaluate the $\overline{\Delta T}$-based Rayleigh number Ra_H:

$$Ra_H = \frac{Ra_H^*}{\overline{Nu}_H} = \frac{1.03 \times 10^{13}}{392.3} = 2.6 \times 10^{10}$$

and the corresponding Grashof number,

$$Gr_H = \frac{Ra_H}{Pr} = \frac{2.6 \times 10^{10}}{5.49} = 4.8 \times 10^9 \quad \text{(turbulent, barely)}$$

Problem 4.16. The film temperature is (10°C + 20°C)/2 = 15°C. The air properties that will be needed are

$$Pr = 0.72 \qquad k = 0.025 \frac{W}{m \cdot K} \qquad \frac{g\beta}{\alpha\nu} = \frac{116}{cm^3\ K}$$

The 5°C/m stratification of the room air means that the top and bottom edges of the window see the following air temperatures:

$$T_\infty\,(\text{top}) = 20°\text{C} + 0.5\text{m}\; 5\,\frac{°\text{C}}{\text{m}} = 22.5°\text{C}$$

$$T_\infty\,(\text{bottom}) = 20°\text{C} - 0.5\text{m}\; 5\,\frac{°\text{C}}{\text{m}} = 17.5°\text{C}$$

The corresponding temperature differences are

$$\Delta T\,(\text{top}) = 22.5°\text{C} - 10°\text{C} = 12.5°\text{C} = \Delta T_{max}$$

$$\Delta T\,(\text{bottom}) = 17.5°\text{C} - 10°\text{C} = 7.5°\text{C} = \Delta T_{min}$$

$$b = \frac{\Delta T_{max} - \Delta T_{min}}{\Delta T_{max}} = \frac{12.5 - 7.5}{12.5} = 0.4$$

The Pr = 0.7 curve of Fig. 4.8 shows that

$$\frac{\overline{Nu}_H}{Ra_H^{1/4}} \cong 0.44 \qquad (1)$$

in which

$$Ra_H = \frac{g\beta}{\alpha\nu} H^3\, \Delta T_{max}$$

$$= \frac{116}{\text{cm}^3\text{K}}\,(100\,\text{cm})^3\; 12.5\,\text{K} = 1.45 \times 10^9$$

The flow is transitional (barely laminar), therefore we continue to rely on eq. (1) and Fig. 4.8 while keeping in mind that the actual heat transfer rate may be slightly larger than the purely laminar estimate produced below:

$$\overline{Nu}_H = 0.44\left(1.45 \times 10^9\right)^{1/4} = 85.86$$

$$\overline{Nu}_H = \frac{\bar{q}''_w}{\Delta T_{max}}\,\frac{H}{k}$$

$$\bar{q}''_w = 85.86\,\frac{12.5°\text{C}}{1\text{m}}\; 0.025\,\frac{\text{W}}{\text{m K}} = 26.83\,\frac{\text{W}}{\text{m}^2}$$

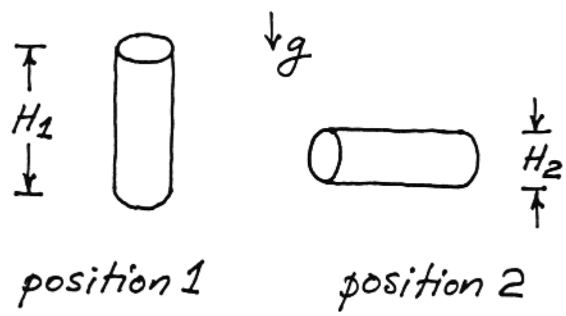
$$A = 0.8\text{m}\; 1\text{m} = 0.8\,\text{m}^2$$

$$q = \bar{q}''_w A \; = 26.83\,\frac{\text{W}}{\text{m}^2}\; 0.8\,\text{m}^2 = 21.5\,\text{W}$$

Problem 4.17. The scale of the cooldown time t is revealed by the first law of thermodynamics written for the beer bottle as a closed system,

$$\frac{dE}{dt} = - Q_{beer \rightarrow air}$$

$$(Mc)_{beer} \frac{dT_{beer}}{dt} = - hA\,(T_{beer} - T_{air}) \quad \text{(note: A = bottle area)}$$



Therefore, the cooldown time is

$$t \sim \frac{(Mc)_{beer}}{hA}$$

in which $(Mc)_{beer}/A$ is a factor independent of orientation. We conclude that

$$\frac{t_1}{t_2} = \frac{h_2}{h_1}$$

The problem becomes one of estimating the scale of the overall heat transfer coefficient h,

$$\frac{1}{h} = \frac{1}{h_{beer\,side}} + \frac{1}{h_{air\,side}}$$

where

$$h_{beer} \sim \frac{1}{H}\left(k\, Ra_H^{1/4}\right)_{beer} \quad \text{and} \quad h_{air} \sim \frac{1}{H}\left(K\, Ra_H^{1/4}\right)_{air}$$

For the same ΔT and H, the inside/ouside h-ratio is

$$\frac{h_{beer}}{h_{air}} = \frac{\left(k\,C^{1/4}\right)_{beer}}{\left(k\,C^{1/4}\right)_{air}} = \frac{0.58\,(4910)^{1/4}}{0.025\,(125)^{1/4}} = 58.1$$

where $C = g\beta/\alpha\nu$, and all numerical values are those of water (for beer) and air at 10°C. Since $h_{beer} >> h_{air}$, the overall h is

$$h \sim h_{\text{air side}} \sim \frac{1}{H}\left(k\, Ra_H^{1/4}\right)_{\text{air}} = (\text{Constant})\, H^{-1/4}$$

In conclusion,

$$\frac{t_1}{t_2} = \frac{h_2}{h_1} = \left(\frac{H_1}{H_2}\right)^{1/4} = 5^{1/4} = 1.50$$

i.e. the cooldown in the vertical position requires a time that is 50% longer than the cooldown in the horizontal position. So, if you would like to drink this beer cold and as soon as possible, choose position 2.

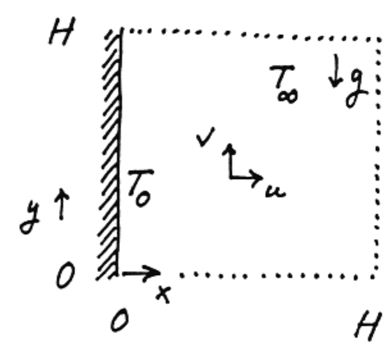
Problem 4.18. Focusing on the $H \times H$ region, and claiming the following scales

$$x \sim H, \qquad y \sim H, \qquad T - T_\infty \sim \underbrace{T_0 - T_\infty}_{\Delta T} \tag{S}$$

the momentum equation shows that

$$\underbrace{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}}_{\text{inertia}} = \underbrace{\nu\frac{\partial^2 v}{\partial x^2}}_{\text{friction}} + \underbrace{g\beta\,(T - T_\infty)}_{\text{buoyancy}}$$

$$\sim \frac{v^2}{H} \qquad \sim \frac{\nu v}{H^2} \qquad \sim g\beta\,\Delta T$$



Invoking the balance friction ~ buoyancy,

$$\frac{\nu v}{H^2} \sim g\beta\,\Delta T$$

we obtain

$$v \sim \frac{g\beta\, H^2\,\Delta T}{\nu} \tag{M}$$

and

$$\frac{\text{inertia}}{\text{friction}} \sim \frac{v^2/H}{\nu v/H^2} \sim \frac{g\beta\,\Delta T\, H^3}{\nu^2} = Gr_H$$

However, the energy equation

$$\underbrace{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}}_{\substack{\text{convection} \\ \sim \frac{v\,\Delta T}{H}}} = \underbrace{\alpha\frac{\partial^2 T}{\partial x^2}}_{\substack{\text{diffuision} \\ \sim \frac{\alpha\,\Delta T}{H^2}}}$$

suggests a different scale for v:

$$v \sim \frac{\alpha}{H} \tag{E}$$

The fact that two equations (momentum and energy) provide conflicting answers for the unknown v, should be expected: the decision to adopt scales (S) leaves only one unknown (v) for a system of two equations (momentum and energy). This case of mathematical incompatibility should tell the problem solver that one additional scale, which in (S) was assumed known, is in fact an unknown of the problem. That additional scale can only be the scale of x, i.e. the boundary layer thickness. The error in the scale analysis that culminated with (M) and (E) was the selection of the domain (x,y) in which the scale analysis was performed. (See the first rule of scale analysis, Chapter 1.)

I composed this problem in order to demonstrate the artificial origin of the Grashof number Gr_H used so often in the current literature. To claim that the Grashof number is interpreted as the parameter describing the ratio of buoyancy to viscous forces is as erroneous as claiming that the flow fills the $H \times H$ region entirely, or that the $H \times H$ region is the boundary layer.

<u>Problem 4.19</u>. In the $Pr \to \infty$ limit, eq. (4.65') yields

$$\overline{Nu}_y = 0.671\ Ra_y^{1/4} \tag{1}$$

This result holds for an isothermal wall (T_w) exposed to an isothermal reservoir (T_∞). It should <u>hold</u> approximately also in the case of the linearly stratified reservoir of Fig. 4.8, provided the $\overline{Nu}_y$ and Ra_y of eq. (1) are based on the <u>average</u> wall-reservoir temperature difference:

$$\frac{\overline{q''_{w,y}}}{\Delta T_{avg}}\frac{y}{k} \cong 0.671\left(\frac{g\beta\, y^3\, \Delta T_{avg}}{\alpha\nu}\right)^{1/4} \tag{2}$$

In order to form the groups $\overline{Nu}_H$ and Ra_H defined by eqs. (4.83) and (4.81) in the text, in the above equation we set y = H, and multiply and divide by ΔT_{max}:

$$\frac{\Delta T_{max}}{\Delta T_{avg}}\frac{\overline{q''_{w,H}}}{\Delta T_{max}}\frac{H}{k} \cong 0.671\left(\frac{g\beta\, H^3\, \Delta T_{max}}{\alpha\nu}\right)^{1/4}\left(\frac{\Delta T_{avg}}{\Delta T_{max}}\right)^{1/4} \tag{3}$$

In short,

$$\frac{\Delta T_{max}}{\Delta T_{avg}} \overline{Nu}_H \cong 0.671 \, Ra_H^{1/4} \left(\frac{\Delta T_{avg}}{\Delta T_{max}} \right)^{1/4} \tag{4}$$

which means that

$$\overline{Nu}_H \cong 0.671 \left(\frac{\Delta T_{avg}}{\Delta T_{max}} \right)^{5/4} Ra_H^{1/4} \tag{5}$$

Using the definition of b, eq. (4.81), we reach the conclusion that the ratio $\overline{Nu}_H/Ra_H^{1/4}$ must indeed decrease as the stratification parameter b increases:

$$\overline{Nu}_H \cong 0.671 \left(1 - \frac{b}{2} \right)^{5/4} Ra_H^{1/4} \tag{6}$$

The curve $\overline{Nu}_H/Ra_H^{1/4}$ recommended by eq. (6) falls below the $Pr \to \infty$ curve of Fig. 4.8, the largest discrepancy (12.8 percent) occuring at the right extremity of the graph (b = 1).

Problem 4.20. The air properties evaluated at the film temperature (20°C + 40°C)/2 = 30°C are

$$Pr = 0.72 \qquad k = 0.026 \frac{W}{m \cdot K} \qquad \frac{g\beta}{\alpha\nu} = \frac{90.7}{cm^3 \, K}$$

a) Vertical plate of height H = 4 cm:

$$\begin{aligned} Ra_H &= \frac{g\beta}{\alpha\nu} H^3 (T_w - T_\infty) \\ &= \frac{90.7}{cm^3 \, K} (4 \, cm)^3 (40 - 20) \, K = 1.16 \times 10^5 \end{aligned}$$

The Nusselt number correlation is eq. (4.107):

$$\begin{aligned} \overline{Nu}_H &= 0.68 + 0.515 \, Ra_H^{1/4} \\ &= 0.68 + 0.515 \left(1.16 \times 10^5 \right)^{1/4} = 10.19 \\ \bar{h} &= \overline{Nu}_H \frac{k}{H} \\ &= 10.19 \times 0.026 \frac{W}{m \cdot K} \frac{1}{0.04 \, m} = 6.62 \frac{W}{m^2 \, K} \end{aligned}$$

This $\bar{h}$ value applies to both sides of the plate. The total heat transfer rate is

$$\begin{aligned} q' &= \bar{h} \, (2 \text{ sides}) \, H \left(T_w - T_\infty \right) \\ &= 6.62 \frac{W}{m^2 \, K} \, 2 \times 0.04 m \, (40 - 20) \, K = 10.6 \frac{W}{m} \end{aligned}$$

b) Plate inclined at 45°C: the calculation of q' consists of repeating part (a) in which g is replaced by g cos (45°) = 0.707 g. The results are, in order,

$$Ra_H = 8.2 \times 10^4$$

$$\overline{Nu}_H = 0.68 + 0.515\, Ra_H^{1/4} = 9.4$$

$$\bar{h} = 6.1 \frac{W}{m^2\,K} \qquad \text{(this applies to both sides of the plate)}$$

$$q' = \bar{h}\,(2 \text{ sides})\, H \left(T_w - T_\infty\right) = 9.77 \frac{W}{m}$$

c) Horizontal plate: in this case we distinguish between the upper surface (hot, facing upward), and the lower surface of the plate (cold, facing downward).

For the upper surface, we begin with the characteristic length of the rectangular surface:

$$L = \frac{A}{p} = \frac{H \times \text{length}}{2 \times \text{length}} = 2 \text{ cm}$$

$$Ra_L = \frac{g\beta}{\alpha\nu} L^3 \left(T_w - T_\infty\right) = 1.45 \times 10^4$$

Equation (4.119) delivers the Nusselt number:

$$\overline{Nu}_L = 0.54\, Ra_L^{1/4} = 5.93$$

$$\bar{h}_{upper} = \overline{Nu}_L \frac{k}{L}$$

$$= 5.93 \times 0.026 \frac{W}{m \cdot K} \frac{1}{0.02 \text{ m}} = 7.7 \frac{W}{m^2\,K}$$

The lower surface has the same L and Ra_L. As Nusselt number correlation we use eq. (4.121): this choice is an <u>approximate</u> one because eq. (4.121) is more accurate when Ra_L exceeds 10^5. We obtain:

$$\overline{Nu}_L = 0.27\, Ra_L^{1/4} = 2.96$$

$$\bar{h}_{lower} = \overline{Nu}_L \frac{k}{L} = 3.85 \frac{W}{m^2\,K}$$

$$q' = q'_{upper} + q'_{lower}$$

$$= \bar{h}_{upper} H \left(T_w - T_\infty\right) + \bar{h}_{lower} H \left(T_w - T_\infty\right)$$

$$= \left(\bar{h}_{upper} + \bar{h}_{lower}\right) H \left(T_w - T_\infty\right)$$

$$= (7.7 + 3.85) \frac{W}{m^2\,K}\, 0.04 \text{ m}\, (40 - 20) \text{ K}$$

$$= 9.24 \frac{W}{m}$$

In conclusion, the best position for heat transfer is the vertical one, and the worst the horizontal. The difference between these extremes is relatively small, only 13 percent.

Problem 4.21. The properties of air at the film temperature (10°C + 30°C)/2 = 20°C are

$$\Pr = 0.72 \qquad k = 0.025\ \frac{W}{m\cdot K} \qquad \frac{g\beta}{\alpha\nu} = \frac{107}{cm^3\ K}$$

a) Sphere with the diameter D = 3m

$$\begin{aligned} Ra_D &= \frac{g\beta}{\alpha\nu} = D^3\left(T_\infty - T_w\right) \\ &= \frac{107}{cm^3\ K}\,(300\ cm)^3\,(30-10)\ K = 5.78 \times 10^{10} \end{aligned}$$

Next, we substitute this Ra_D and Pr = 0.72 in eq. (4.124):

$$\begin{aligned} \overline{Nu}_D &= 2 + 0.455\ Ra_D^{1/4} = 225.2 \\ \bar{h} &= \overline{Nu}_D \frac{k}{D} = \\ &= 225.2 \times 0.025\ \frac{W}{m\cdot K}\ \frac{1}{3m} = 1.88\ \frac{W}{m^2\ K} \\ q &= \bar{h}\ \pi D^2 \left(T_\infty - T_w\right) \\ &= 1.88\ \frac{W}{m^2\ K}\ \pi\ (3m)^2\ (30-10)\ K = 1.06\ kW \end{aligned}$$

b) Horizontal cylinder with the diameter d = 1.5m. To have the same volume as the D-sphere, the cylinder must have the length:

$$L = \frac{\frac{4}{3}\pi\left(\frac{D}{2}\right)^3}{\pi\left(\frac{d}{2}\right)^2} = \frac{2}{3}\ \frac{D^3}{d^2} = 8m$$

$$Ra_d = \frac{g\beta}{\alpha\nu}\, d^3\left(T_\infty - T_w\right) = 7.22 \times 10^9$$

The overall Nusselt number is supplied by eq. (4.122), in which Pr = 0.72:

$$\begin{aligned} \overline{Nu}_d &= \left(0.6 + 0.322\ Ra_d^{1/6}\right)^2 = 217.3 \\ \bar{h} &= \overline{Nu}_d \frac{k}{d} \\ &= 217.3 \times 0.025\ \frac{W}{m\cdot K}\ \frac{1}{1.5m} = 3.62\ \frac{W}{m^2\ K} \end{aligned}$$

The total area of the horizontal cylinder is

$$A = \pi dL + 2\pi \left(\frac{d}{2}\right)^2$$
$$= \pi d \left(L + \frac{d}{2}\right)$$
$$= \pi \, 1.5\text{m} \, (8\text{m} + 0.75\text{m}) = 41.23 \text{ m}^2$$

Since the area of the two end discs is only 8.6 percent of the total area, it is safe (i.e. a good approximation) to assume that the same $\bar{h}$ holds over the entire area:

$$q = \bar{h} \, A \left(T_\infty - T_w\right)$$
$$= 3.62 \frac{\text{W}}{\text{m·K}} \, 41.23 \text{ m}^2 \, (30 - 10) \text{ K} \cong 3 \text{ kW}$$

c) The cylinder will be heated three times faster than the sphere, therefore the preferred shape is the sphere.

Problem 4.22. The film temperature and the water properties that will be needed are

$$T_{film} = (80°\text{C} + 20°\text{C})/2 = 50°\text{C}$$

$$\text{Pr} = 3.57 \qquad k = 0.64 \frac{\text{W}}{\text{m·K}} \qquad \frac{g\beta}{\alpha\nu} = \frac{51\,410}{\text{cm}^3 \text{ K}}$$

a) Focusing on eq. (4.127), we calculate in order

$$l = 1 \text{ cm} + 2 \text{ cm} + 1 \text{ cm} = 4 \text{ cm}$$

$$Ra_l = \frac{g\beta}{\alpha\nu} l^3 \left(T_w - T_\infty\right)$$
$$= \frac{51\,410}{\text{cm}^3 \text{ K}} (4 \text{ cm})^3 \, (80 - 20) \text{ K} = 1.974 \times 10^8 \quad \text{(laminar)}$$

$$\overline{Nu}_l = 0.52 \, Ra_l^{1/4} = 61.64$$

$$\overline{Nu}_l = \frac{\bar{h} \, l}{k}$$

$$\bar{h} = 61.64 \frac{0.64 \frac{\text{W}}{\text{m·K}}}{0.04\text{m}} = 986.2 \frac{\text{W}}{\text{m}^2 \text{ K}}$$

b) If we rely on eq. (4.129) and Table 4.3, we proceed as follows:

$$L = A^{1/2} = \left[6\,(2\text{ cm})^2\right]^{1/2} = 4.9\text{ cm}$$

$$Ra_L = \frac{g\beta}{\alpha\nu} L^3 \left(T_w - T_\infty\right) = 3.63 \times 10^8 \text{ (laminar)}$$

$$\overline{Nu}_L = 3.388 + \frac{0.67 \times 0.951\, Ra_L^{1/4}}{\left[1 + \left(\frac{0.492}{3.57}\right)^{9/16}\right]^{4/9}}$$

$$= 3.388 + 0.562\, Ra_L^{1/4} = 80.95$$

$$\bar{h} = 80.95 \frac{0.64 \frac{W}{m \cdot K}}{0.049 m} = 1057.4 \frac{W}{m^2 K}$$

c) On the other hand, if we use the simpler formula (4.130), we obtain

$$\overline{Nu}_L = 3.47 + 0.51 \left(3.63 \times 10^8\right)^{1/4}$$

$$= 73.87$$

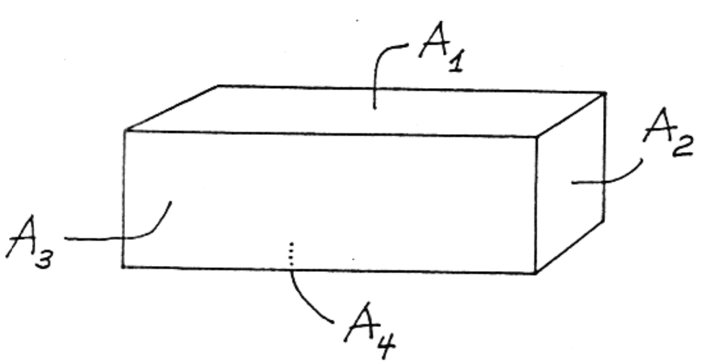
$$\bar{h} = 73.87 \frac{0.64 \frac{W}{m \cdot K}}{0.049 m} = 964.8 \frac{W}{m^2 K}$$

If we regard estimate (b) as reference, we see that estimate (a) is off (smaller) by only 6.7 percent, while estimate (c) is off by 8.8 percent. These deviations are insignificant in heat transfer calculations (think of the uncertainty in the values used for water properties), therefore all three methods are satisfactory.

Problem 4.23. The six surfaces of the parallelepiped are of four types (A_1, A_2, A_3, A_4), therefore

$$q = \overline{q_1''} A_1 + \overline{q_2''}\, 2A_2 + \overline{q_3''}\, 2A_3 + \overline{q_4''} A_4$$

$$= \left(\bar{h}_1 A_1 + 2\bar{h}_2 A_2 + 2\bar{h}_3 A_3 + \bar{h}_4 A_4\right) \Delta T$$



All the air properties are evaluated at the film temperature $(0°C + 20°C)/2 = 10°C$; namely:

$$k \cong 2.5 \times 10^{-4} \frac{W}{cm\,K}, \qquad \frac{g\beta}{\alpha\nu} \cong \frac{125}{cm^3\,K}$$

The A_1 surface is cold and faces upward. The heat transfer coefficient follows from eq. (4.121)

$$\overline{Nu}_L = 0.27\, Ra_L^{1/4}$$

in which the length scale L is

$$L = \frac{A_1}{p_1} = \frac{30\text{ cm } 1\text{m}}{2\,(1\text{m} + 30\text{ cm})} = \frac{30}{2\,(130)}\text{ m}$$

$$= 11.54\text{ cm}$$

Therefore, the Rayleigh number based on L is

$$Ra_L = \frac{g\beta L^3 \Delta T}{\alpha \nu} = \frac{125}{\text{cm}^3\text{ K}}\,(11.54)^3\text{ cm}^3\ 20\text{ K}$$

$$= 3.84 \times 10^6,$$

and, from eq. (4.121), $\overline{Nu}_L = 11.95$. The $\overline{h_1}$ coefficient is

$$\overline{h_1} = \overline{Nu}_L \frac{k}{L} = 11.95\,\frac{(2.5)\,10^{-4}\text{W}}{\text{cm K}}\,\frac{1}{11.54\text{ cm}}$$

$$= 2.59 \times 10^{-4}\,\frac{\text{W}}{\text{cm}^2\text{ K}} = 2.59\,\frac{\text{W}}{\text{m}^2\text{ K}}$$

The surfaces A_2 and A_3 are both vertical of height H = 0.3m, therefore

$$\overline{h_2} = \overline{h_3} = \overline{Nu}_H \frac{k}{H} \tag{1}$$

The average Nusselt number is given by eq. (4.106),

$$\overline{Nu}_H = 0.68 + \frac{0.67\, Ra_H^{1/4}}{\left[1 + (0.492/Pr)^{9/16}\right]^{4/9}}$$

which in the case of air (Pr = 0.72) reduces to

$$\overline{Nu}_H = 0.68 + 0.515\, Ra_H^{1/4}$$

The Rayleigh number based on H is

$$Ra_H = \frac{125}{\text{cm}^3\text{ K}}\,30^3\text{ cm}^3\ 20\text{ K} = 6.75 \times 10^7$$

therefore $\overline{Nu}_H = 47.36$, and

$$\overline{h_2} = \overline{h_3} = 47.36 \, \frac{2.5 \times 10^{-4} \, W}{cm \, K} \, \frac{1}{30 \, cm}$$

$$= 3.95 \times 10^{-4} \, \frac{W}{cm^2 \, K} = 3.95 \, \frac{W}{m^2 \, K}$$

Finally, the A_4 surface is cold and faces downward. It has the same L = 11.54 cm and $Ra_L = (3.84) \, 10^6$ as the A_1 surface, therefore eq. (4.119) yields

$$\overline{Nu}_L = 0.54 \, Ra_L^{1/4}$$
$$= 23.9$$

which translates into the average heat transfer coefficient

$$\overline{h_4} = \overline{Nu}_L \, \frac{k}{L} = 23.9 \, \frac{2.5 \times 10^{-4} \, W}{cm \, K} \, \frac{1}{11.54 \, cm}$$

$$= 5.18 \times 10^{-4} \, \frac{W}{cm^2 \, K} = 5.18 \, \frac{W}{m^2 \, K}$$

In summary, the q expression listed at the start of this solution becomes

$$q = [2.59 \times 1 \times 0.3 + 2 \times 3.95 \times (0.3)^2 +$$

$$+ 2 \times 3.95 \times 1 \times 0.3 + 5.18 \times 1 \times 0.3] \, \frac{W}{m^2 \, K} \, m^2 \, 20°C$$

$$= 108.24 \, W$$

The instantaneous melting rate is proportional to q,

$$q = \dot{m} \, h_{sf}$$

where the latent heat of melting is h_{sf} = 333.4 kJ/kg, therefore

$$\dot{m} = \frac{q}{h_{sf}} = \frac{108.24 \, W}{333.4 \, kJ/kg} = 0.325 \, \frac{W}{J/g}$$

$$= 0.325 \, \frac{g}{s}$$

Problem 4.24. In view of the elongated shape of the ice block, the length l used in Lienhard's formula

$$\overline{Nu}_l \cong 0.52\, Ra_l^{1/4} \tag{4.127}$$

can be approximated by measuring half of the perimeter of the smaller (square) cross-section:

$$l = \frac{0.3\text{m}}{2} + 0.3\text{m} + \frac{0.3\text{m}}{2} = 0.6\text{m}$$

Evaluating all the air properties at 10°C, we calculate in order

$$Ra_l = \frac{125}{\text{cm}^3\,\text{K}}\, 20°\text{C}\,(60)^3\,\text{cm}^3 = 5.4 \times 10^8$$

$$\overline{Nu}_l \cong 0.52\left[(5.4)\,10^8\right]^{1/4} = 79.27$$

$$\bar{h} = \overline{Nu}_l\,\frac{k}{l} = 79.27\,\frac{(2.5)\,10^{-4}\text{W}}{\text{cm K}}\,\frac{1}{60\text{ cm}}$$

$$= 3.3\,\frac{\text{W}}{\text{m}^2\text{K}}$$

The total area of the ice block is

$$A = 2(0.3\text{m})^2 + 4(0.3)(1)\text{m}^2 = 1.38\text{m}^2$$

and the total heat transfer rate can now be evaluated by simply writing

$$q = \bar{h}ADT = 3.3\,\frac{\text{W}}{\text{m}^2\text{K}}\,1.38\text{m}^2\,20°\text{C}$$

$$= 91.1\text{ W}$$

This quick estimate is only 16 percent lower than the more rigorous (and tedious) result of the preceding problem.

Problem 4.25. Let ΔT be the unknown of the problem, i.e. the temperature difference between wire and ambient air. The conservation of energy in each cross-section requires

$$q' = \bar{h}\,\pi D\,\Delta T = \overline{Nu}_D\,\pi k\,\Delta T$$

therefore ΔT can be calculated with the formula

$$\Delta T = \frac{q'}{\pi\, k\, \overline{Nu}_D} \tag{1}$$

The average Nusselt number is given by eq. (4.122),

$$\overline{Nu}_D = \left\{ 0.6 + \frac{0.387\ Ra_D^{1/6}}{\left[1 + (0.559/Pr)^{9/16}\right]^{8/27}} \right\}^2$$

which in the case of air (Pr = 0.72) reduces to

$$\overline{Nu}_D = \left(0.6 + 0.332\ Ra_D^{1/6}\right)^2 \qquad (2)$$

The Rayleigh number depends on the unknown ΔT,

$$Ra_D = \frac{g\beta}{\alpha\nu} D^3\ \Delta T \qquad (3)$$

If we do not have access to a programmable calculator, the temperature difference ΔT can be calculated by trial and error, executing the following steps:

i) assume a ΔT value (labeled ΔT_a),

ii) calculate sequentially Ra_D and $\overline{Nu}_D$, using eqs. (3) and (2),

iii) calculate the ΔT value (labeled ΔT_c), and compare it with the ΔT_a guess.

iv) if ΔT_c differs greatly from ΔT_a, go back and repeat the (i)-(iv) sequence.

In the first iteration of this calculation, we assume

$$\Delta T_a = 10^\circ C$$

and calculate the air properties at the reservoir temperature of 20°C,

$$k \cong 2.5 \times 10^{-4}\ \frac{W}{cm\ K}, \qquad \frac{g\beta}{\alpha\nu} \cong \frac{107}{cm^3\ K}$$

Normally, we would be evaluating these properties at the film temperature (20°C + 30°C)/2 = 25°C, however, this first iteration is "rough", and the above properties are accurate enough. From eqs. (3) and (2) we obtain:

$$Ra_D = 1.07, \qquad \text{and} \qquad \overline{Nu}_D = 0.857$$

Finally, eq. (1) yields

$$\Delta T_c = \frac{0.01\ W}{cm} \frac{cm\ K}{\pi\ 2.5 \times 10^{-4}\ W\ 0.857}$$
$$= 14.86\ K = 14.86^\circ C$$

In conclusion, $\Delta T_c > \Delta T_a$; the second iteration begins with assuming a larger ΔT_a value:

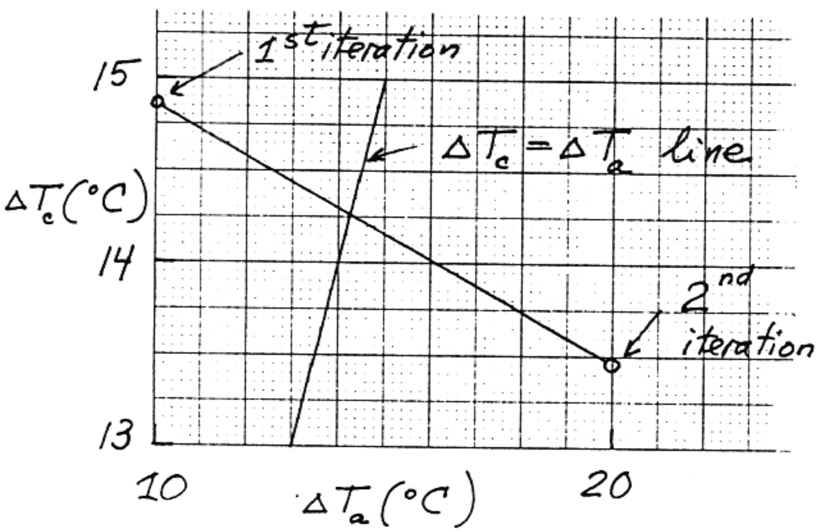
$$\Delta T_a = 20^\circ C$$

This time we evaluate the properties at the film temperature, (20° + 40°)/2 = 30°C:

$$k \cong 2.6 \times 10^{-4} \frac{W}{cm\,K}, \qquad \frac{g\beta}{\alpha\nu} \cong \frac{88}{cm^3\,K}$$

Equations (3), (2) and (1) yield in order

$$Ra_D = 1.76, \qquad \overline{Nu}_D = 0.91, \qquad \Delta T_c = 13.46°C$$



This time, the calculated ΔT is smaller than the assumed value, ΔT_a. In summary, the two iterations carried out so far allow us to draw a line in the plane (ΔT_a, ΔT_c). Intersecting this line with the theoretical line $\Delta T_a = \Delta T_c$ (see the attached sketch), we obtain the answer to the problem:

$$\Delta T \cong 14.3°C$$

If we have access to a programmable calculator, we can solve eq. (2) directly for ΔT. We still have to assume the film temperature, in order to evaluate the physical properties. To begin with, we evaluate all the properties at 20°C, and eq. (2) becomes

$$\frac{12.73°C}{\Delta T} = \left[0.6 + 0.332\left(\frac{\Delta T}{9.35°C}\right)^{1/6}\right]^2$$

with the solution $\Delta T = 13.96°C$. This answer suggests that the film temperature is higher than the assumed 20°C. Therefore, as a second try we evaluate the air properties at 30°C; eq. (2) changes slightly,

$$\frac{12.24°C}{\Delta T} = \left[0.6 + 0.332\left(\frac{\Delta T}{11.36°C}\right)^{1/6}\right]^2$$

and the new solution is $\Delta T = 13.77°C$. In summary, the table below shows that the true film temperature is approximately $20°C + \Delta T/2 \cong 26.9°C$.

T_{film}	20°C	30°C
ΔT	13.96°C	13.77°C

Interpolating between the values listed in the table, we obtain the final answer, $\Delta T = 13.83°C$, which is not far off the trial-and-error value determined in the first part of this solution, $\Delta T \cong 14.3°C$.

Problem 4.26. For the warm side of the single-pane window, Table 4.2 recommends the following formula for the height-averaged heat flux (note: $Pr = 0.72$ for air, and $\overline{Nu}_H = 4/3\ Nu_H$)

$$\overline{Nu}_H = 0.517\ Ra_H^{1/4}, \qquad (Pr = 0.72)$$

Recall that both $\overline{Nu}_H$ and Ra_H are based on the reservoir-wall temperature difference, which in this case is $T_h - T_w$:

$$\frac{\overline{q''}}{T_h - T_w}\frac{H}{k} = 0.517\left[\frac{g\beta(T_h - T_w)H^3}{\alpha\nu}\right]^{1/4}$$

Symmetry requires that the glass temperature T_w be situated half-way between T_h and T_c, therefore

$$T_h - T_w = \frac{1}{2}(T_h - T_c)$$

and the average heat flux formula based on $T_h - T_c$ becomes

$$\frac{\overline{q''}}{T_h - T_c}\frac{H}{k} = 0.217\left[\frac{g\beta(T_h - T_c)H^3}{\alpha\nu}\right]^{1/4}$$

Problem 4.27. For the warm side of the glass layer, eq. (4.75) recommends the local Nusselt number:

$$\frac{q''}{T_h - T_w(y)}\frac{y}{k} = 0.530\left(\frac{g\beta q'' y^4}{\alpha\nu k}\right)^{1/5}, \qquad (Pr = 0.72)$$

The corresponding overall Nusselt number is obtained by replacing y with H, $T_w(y)$ with $\overline{T}_w$, and the coefficient 0.53 with

$$\frac{0.53}{1 + (-1/5)} = 0.663$$

where $(-1/5)$ is the exponent n in the y-dependence of the local heat transfer coefficient,

$$\frac{q''}{T_h - T_w(y)} \sim y^n$$

In conclusion, eq. (4.75) becomes

$$\frac{q''}{T_h - \overline{T}_w}\frac{H}{k} = 0.663\left(\frac{g\beta q'' H^4}{\alpha\nu k}\right)^{1/5}$$

or, after some algebra,

$$\frac{q''}{T_h - \bar{T}_w} \frac{H}{k} = 0.598 \left[\frac{g\beta (T_h - \bar{T}_w) H^3}{\alpha \nu} \right]^{1/4}$$

The average temperature difference $T_h - \bar{T}_w$ can only be equal to half of the side-to-side temperature difference,

$$T_h - \bar{T}_w = \frac{1}{2} (T_h - T_c)$$

therefore, the wanted relationship between q" and $(T_h - T_c)$ is

$$\frac{q''}{T_h - T_c} \frac{H}{k} = 0.252 \left[\frac{g\beta (T_h - T_c) H^3}{\alpha \nu} \right]^{1/4}$$

Note that the 0.252 coefficient is only 16 percent greater than the 0.217 coefficient derived based on the constant-T_w model (see the preceding problem).

Problem 4.28. The thickness of the window glass varies linearly with altitude,

$$\delta = \bar{\delta} + b \left(\frac{1}{2} - \xi \right) \tag{1}$$

where y is measured downward,

$$\xi = \frac{y}{H} \qquad\qquad b = -\frac{d\delta}{d\xi} \tag{2}$$

The H-averaged thickness $\bar{\delta}$ is fixed, in other words, the total volume (or weight) of the window is fixed. The taper parameter b is one variable in the design of the window.

The local heat flux must overcome two resistances in series, the air boundary layer on the room side (1/h), and the glass pane itself (δ/k_w):

$$q'' = \frac{\Delta T}{\frac{1}{h} + \frac{\delta}{k_w}} \tag{3}$$

The total heat leak through the window (per unit length in the lateral direction) is

$$q' = \int_0^H \frac{\Delta T \, dy}{\frac{1}{h} + \frac{\delta}{k_w}} \tag{4}$$

Assume now that the heat transfer coefficient decreases in the downward direction, as in laminar boundary layer natural convection,

$$h = h_{min} \left(\frac{y}{H}\right)^{-n} \tag{5}$$

In this equation, $n = 1/4$, and h_{min} is the value of h at the bottom of the window, $y = H$ (i.e. the smallest value). By combining eqs. (1)-(5), we can nondimensionalize the total heat leak as

$$\tilde{q}' = \int_0^1 \frac{d\xi}{\xi^n + Bi\left[1 + S\left(\frac{1}{2} - \xi\right)\right]} \tag{6}$$

in which

$$\tilde{q}' = \frac{q'}{h_{min} H \, \Delta T} \tag{7}$$

$$Bi = \frac{h_{min}\, \bar{\delta}}{k_w} \tag{8}$$

$$S = -\frac{H}{\bar{\delta}} \frac{d\delta}{dy} \tag{9}$$

The objective is to minimize $\tilde{q}'$ with respect to the "shape" (dimensionless-taper) parameter S, while Bi is fixed by glass volume and air natural convection constraints. The integral (6) can be evaluated numerically (recall that $n = 1/4$):

S	$\tilde{q}'$		
	Bi = 0.01	Bi = 0.1	Bi = 1
0	1.31005	1.16184	0.5605
0.5	1.30860	1.15306	0.55605
0.535			***0.55603**
0.6			0.55610
1.0	1.30716	1.14540	
1.5	1.30576	1.13876	
2.0	***1.30437**	***1.13307**	

The asterisks indicate the smallest value reached by $\tilde{q}'$ as the taper parameter S changes. At low Biot numbers, $\tilde{q}'$ decreases monotonically as the taper increases, so that

$$\tilde{q}'_{min} = \tilde{q}'\,(S = 2)$$

The smallest heat leak occurs when the taper is the most accentuated (S = 2), i.e. when the glass thickness at the top of the window is $2\bar{\delta}$, and zero at the bottom.

At higher Biot numbers (e.g. Bi = 1) the $\tilde{q}'$ behavior changes. The minimum heat leak occurs at an intermediate (optimal) taper, in this case at S = 0.535.

The top line in the table ($S = 0$) shows the reference case, i.e. the heat leak through the glass pane of constant thickness. The relative merit of the tapered glass design is indicated by the ratio:

	Bi = 0.01	Bi = 0.1	Bi = 1
$\dfrac{\tilde{q}'_{min}}{\tilde{q}'_{ref}(S=0)}$	0.9957	0.9752	0.9920
% reduction	0.43	2.48	0.8

The bottom line in this table shows that the q' reduction associated with the tapering of the glass is minimal. This is especially true at small Biot numbers, which is the normal range of operation of regular-size windows. In order to see this, consider the following order of magnitude calculation:

glass pane data:

$$\bar{\delta} = 0.5 \text{ cm}$$

$$k_w = 0.81 \frac{W}{m\,K}$$

air side data:

$$Ra_H \sim 10^8$$

$$H = 1 \text{ m}$$

$$k_{air} = 0.025 \frac{W}{m\,K}$$

equation (4.65):

$$\frac{h_{min} H}{k_{air}} \sim 0.5\, Ra_H^{1/4} \sim 50$$

$$h_{min} \sim 1.25 \frac{W}{m^2\,K}$$

Biot number:

$$Bi = \frac{h_{min}\,\bar{\delta}}{k_w} \sim 0.008$$

In conclusion, if we consider the added difficulty of manufacturing tapered glass, and the unrealistic assumption that the bottom edge of the window can have a knife edge (zero thickness, or $S = 2$), the marginal heat leak reduction calculated above does not recommend the tapered-glass window as a viable energy conservation feature for building design.

Problem 4.29. We first identify the large S and small S asymptotes, and intersect the asymptotes to locate S_{opt}.

Large S. When the spacing S is sufficiently large each horizontal cylinder is coated by a distinct boundary layer. We are assuming that $(H,W) >> (D + S)$, and that $Ra_D >> 1$, where $Ra_D = g\beta D^3(T_w - T_\infty)/(\alpha\nu)$. The heat transfer from one cylinder is

$$q_1 = \frac{k}{D} Nu_D\, \pi DL\, (T_w - T_\infty) \tag{1}$$

$$Nu_D = c\, Ra_D^{1/4} \tag{2}$$

and c is a constant of order 0.5. The total number of cylinders in the bank of cross-sectional area $H \times W$ is

$$n = \frac{HW}{(S+D)^2 \cos 30^\circ} \tag{3}$$

therefore the total heat transfer from the bank is $q = nq_1$, or

$$q_{\text{large } S} = \frac{\pi c}{\cos 30^\circ} \frac{HLW}{(S+D)^2} k\, (T_w - T_\infty)\, Ra_D^{1/4} \tag{4}$$

This result shows that when the spacing is large, the thermal conductance $q/(T_w - T_\infty)$ decreases as S increases.

<u>Small S</u>. Consider now the opposite extreme when the cylinders almost touch, and the flow is almost cut off. In this limit the temperature of the coolant that exits slowly through the upper plane of the bundle ($L \times W$) is essentially the same as the cylinder temperature T_w. The heat transfer from the bundle to the coolant is equal to the enthalpy gained by the coolant, $q = \dot{m}\, c_P\, (T_w - T_\infty)$, where $\dot{m}$ is the mass flowrate through the $L \times W$ plane.

To obtain an order-of-magnitude estimate for the flowrate, we note that $\dot{m}$ is composed of several streams [total number = $W/(S + D)$], each with a cross-sectional area $S \times L$ in the plane of one horizontal row of cylinder axes. The thickness of the channel traveled upward by each stream varies between a minimum value (S) at the row level, and a maximum value at a certain level between two rows. The volume-averaged thickness of one channel is

$$\bar{S} = S + D - 0.907 \frac{D^2}{S+D} \tag{5}$$

however, we may adjust this estimate by using 1 in place of the factor 0.907 to account for the fact that the channel closes and the flow stops when the cylinders touch ($S = 0$):

$$\bar{S} = S\ \frac{S+2D}{S+D} \tag{6}$$

When $\bar{S}$ is sufficiently small, the flowrate through each channel of cross sectional area $\bar{S}L$ and flow length H is proportional to the pressure difference that drives the flow. The pressure difference is $\Delta P = \rho g H \beta (T_w - T_\infty)$, or the difference between the hydrostatic pressures under two H-tall columns of coolant, one filled with T_∞ fluid, and the other with T_w fluid. The mean velocity of the channel flow, U, can be estimated using the Hagen-Poiseuille solution for flow between two parallel plates (spacing $\bar{S}$, flow length H), namely

$$U = \frac{\bar{S}^2 \, \Delta P}{12 \mu H} \tag{7}$$

The total flowrate through the bundle, $\dot{m}(\rho U \bar{S} L) \cdot W/(S + D)$, leads to the total heat transfer rate $\dot{m}\, c_P\, (T_w - T_\infty)$, which can be summarized as

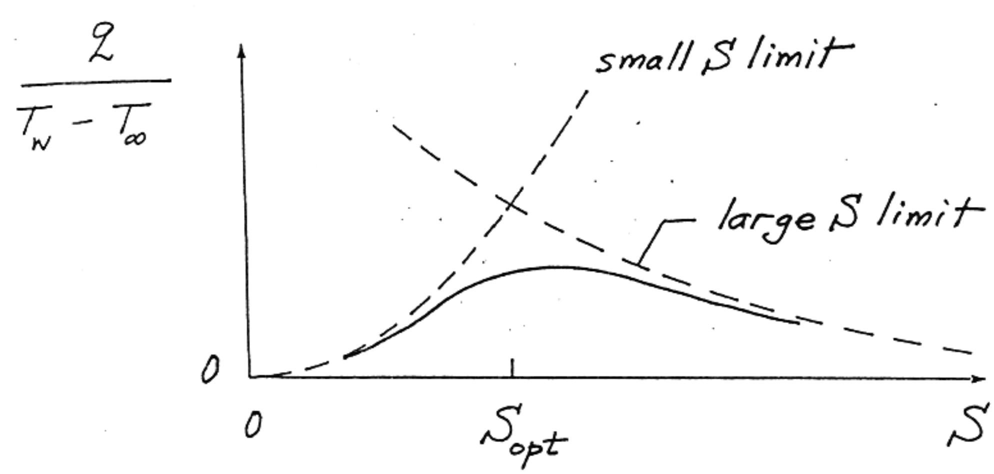
$$q_{small\,S} = \frac{\bar{S}^3 \, LW}{12\, D^3\, (S+D)} \, k\, (T_w - T_\infty)\, Ra_D \tag{8}$$

The key feature of this estimate is that when $S \to 0$ the thermal conductance $q/(T_w - T_\infty)$ decreases as S^3.

Intersection of the asymptotes. The attached figure summarizes the trends uncovered so far. The actual thermal conductance would be represented by the solid curve sketched in the figure. The peak of this curve corresponds to a spacing (S_{opt}) that can be approximated by intersecting the two asymptotes, $q_{large\,S} = q_{small\,S}$. The result of eliminating q between eqs. (4) and (8) is

$$\frac{S_{opt}}{D} \cdot \frac{2 + S_{opt}/D}{\left(1 + S_{opt}/D\right)^{2/3}} = \left(\frac{12 \pi c}{\cos 30^\circ}\right)^{1/3} \left(\frac{H}{D}\right)^{1/3} Ra_D^{-1/4} \tag{9}$$

This formula shows that the optimal spacing has a certain value (of the order of D) when the cylinder diameter and the bundle height are specified. The ratio S_{opt}/D increases almost linearly with the group $(H/D)^{1/3} Ra_D^{-1/4}$, which means that S_{opt} is approximately proportional to $H^{1/3}\, D^{-1/12}$, i.e. almost insensitive to changes in the cylinder diameter. The optimal spacing increases with the height H, and is mainly a function of H.



Problem 4.30. The first part of the problem statement has two parts. First, we are told that the stream flows isothermally:

$$T_{in} = T_{out} = T \tag{1}$$

Second, the flow system operates reversibly, i.e., with zero entropy generation

$$\dot{S}_{gen} = \dot{m}\left(s_{out} - s_{in}\right) - \frac{\dot{Q}}{T} = 0 \tag{2}$$

Next, we use what is always true: written for the same system, the first law requires that

$$\dot{Q} - \dot{W} + \dot{m}\left(h_{in} - h_{out}\right) = 0 \tag{3}$$

For ideal gases, h is a function of T only, and Eq. (1) means that

$$h_{in} = h_{out} \tag{4}$$

Consequently, the first law (3) yields

$$\dot{W} = \dot{Q} \tag{5}$$

The second law (2) yields

$$\dot{Q} = \dot{m}T\left(s_{out} - s_{in}\right) \tag{6}$$

Because the stream carries a single-phase fluid, we have

$$dh = Tds + vdP \tag{7}$$

for which

$$dh = 0 \quad \text{(ideal gas, T = constant)} \tag{8}$$

$$v = RT/P \quad \text{(ideal gas)} \tag{9}$$

Equations (7)-(9) yield

$$s_{out} - s_{in} = -R \ln \frac{P_{out}}{P_{in}} \tag{10}$$

and, in combination with Eqs. (5) and (6),

$$\dot{W} = \dot{Q} = \dot{m} RT \ln \frac{P_{in}}{P_{out}} \tag{11}$$

In the second part of the problem, the cycle executed by the atmospheric air stream consists of four processes:

1-2 isothermal heating and expansion, at T_H

2-3 isobaric cooling, at P_L

3-4 isothermal cooling and compression, at T_L

4-1 isobaric heating, at P_H

In the ideal limit, the thermal contact between the two isobaric streams is perfect,

$$T_1 = T_2 \qquad\qquad T_3 = T_4 \tag{12}$$

and the regenerative heating released by the P_L stream is absorbed completely by the P_H stream,

$$\dot{m}(h_2 - h_3) = \dot{m}(h_1 - h_4) \tag{13}$$

According to eq. (11) we can write

$$\dot{Q}_H = \dot{W}_H = \dot{m} RT_H \ln \frac{P_H}{P_L} \tag{14}$$

$$\dot{Q}_L = \dot{W}_L = \dot{m} RT_L \ln \frac{P_H}{P_L} \tag{15}$$

The net power produced by this flow system is

$$\dot{W}_{net} = \dot{W}_H - \dot{W}_L = \dot{m} R \left(T_H - T_L\right) \ln \frac{P_H}{P_L} \tag{16}$$

The heat engine efficiency is the Carnot efficiency:

$$\eta = \frac{\dot{W}_{net}}{\dot{Q}_H} = 1 - \frac{T_L}{T_H} \tag{17}$$

This is not a surprise, because the model of this flow system is based on the assumption that irreversible flows are absent. The pressure drop from 4 to 1 is zero, and so is the pressure drop from 2 to 3. Furthermore the regenerative (internal) heat transfer between the P_L stream (2-3) and P_H stream (4-1) occurs with zero temperature difference from stream to stream.

Chapter 5

INTERNAL NATURAL CONVECTION

Problem 5.1. Focusing first on the left-hand-side of eq. (5.7) we eliminate one by one the terms identified as relatively negligible:

$$
\begin{aligned}
\text{LHS} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}\right) &- \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right) \\
\frac{1}{\delta_T}\left(\frac{v}{t}, \frac{uv}{\delta_T}, \frac{v^2}{H}\right) \qquad &; \ \frac{1}{H}\left(\frac{u}{t}, \frac{u^2}{\delta_T}, \frac{vu}{H}\right)
\end{aligned}
$$

Using the mass continuity scaling $u/\delta_T \sim v/H$, the six scales shown above become

$$
\left(\underbrace{\frac{v}{t\,\delta_T}}, \underbrace{\frac{v^2}{H\,\delta_T}, \frac{v^2}{H\,\delta_T}}\right) \quad ; \quad \left(\underbrace{\frac{v\,\delta_T}{H^2 t}}, \underbrace{\frac{v^2\,\delta_T}{H^3}, \frac{v^2\,\delta_T}{H^3}}\right)
$$

very large relative to this

negligible relative to these

The surviving terms (scales) are

$$
\left(\frac{v}{t\,\delta_T}, \frac{v^2}{H\,\delta_T}, \frac{v^2}{H\,\delta_T}\right) \ ; \text{ or, multiplied by } \frac{t\,\delta_t}{v},
$$

$$
\left(1\ ,\ \underbrace{\frac{vt}{H}\ ,\ \frac{vt}{H}}_{\substack{\text{much smaller}\\ \text{than 1, as}\\ t\text{ (and }v) \to 0}} \right) ;
$$

Note that (vt) is the vertical travel along the wall, in a fluid initially at rest. In conclusion, the left-hand-side of eq. (5.7) is dominated by the first term, whose scale ia $v/(t\ \delta_T)$.

The right-hand-side of eq. (5.7) can be analyzed the same way:

$$\text{RHS} = \nu\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) - \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\right] + g\beta\frac{\partial T}{\partial x}$$

$$\nu\left[\underbrace{\frac{1}{\delta_T}\left(\frac{v}{\delta_T^2}\;,\;\frac{v}{H^2}\right)}_{\gg}\;,\;\underbrace{\frac{1}{H}\left[\left(\frac{u}{\delta_T^2}\;,\;\frac{u}{H^2}\right)\right]}_{\gg}\right] \qquad ;\; g\beta\frac{\Delta T}{\delta_T}$$

this term is always present because it "drives" the problem

$$\nu\frac{v}{\delta_T^3} \qquad\qquad \nu\frac{u}{H\,\delta_T^2}$$

very large relative to this

$$\nu\frac{v\,\delta_T/H}{H\,\delta_T^2}$$

Therefore, on the right-hand-side the surviving scales are $\nu v/\delta_T^3$ and $g\beta\Delta T/\delta_T$. Putting the pieces together, we find that the momentum equation is the competition between three scales

$$\underset{\text{inertia}}{\frac{v}{t\,\delta_T}}\;,\quad \underset{\text{friction}}{\frac{\nu v}{\delta_T^3}}\;,\quad \underset{\text{buoyancy}}{g\beta\frac{\Delta T}{\delta_T}}$$

Problem 5.2. The boundary layer is distinct in the steady state if its development as "convective" layer ends before the fluid layer is penetrated entirely by pure conduction,

$$t_f < t_{\text{conduction}}$$

$$\left(\frac{\nu H}{g\beta\,\Delta T\,\alpha}\right)^{1/2} < \frac{L^2}{\alpha}$$

Multiplying the last inequality by α/H^2 yields

$$Ra_H^{-1/2} < \left(\frac{L}{H}\right)^2,$$

which is the same as the criterion for distinct layers given in the text.

Problem 5.3. If the heat transfer rate is dominated by pure conduction, then from the energy equation we extract the following scaling law:

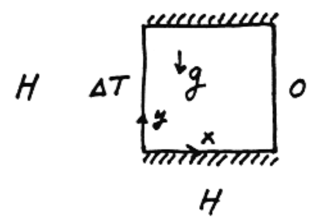
$$\text{(E)} \qquad \underbrace{u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}}_{\text{convection}} = \underbrace{\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)}_{\text{conduction}}$$

convection $\ll$ conduction

$$v\frac{\Delta T}{H} \qquad \ll \qquad \alpha\frac{\Delta T}{H^2} \tag{1}$$

Note that the above analysis refers to the entire square region (x ~ H, y ~ H) because the state of pure conduction fills the entire space between differentially heated side walls. The unknown v scale appearing in eq. (1) follows from the momentum equation, in which we invoke the balance friction ~ buoyancy [see eq. (5.8)]:

$$\nu \frac{v}{H^3} \sim \frac{g\beta\,\Delta T}{H} \tag{2}$$



Eliminating v between eqs. (1) and (2) yields

$$Ra_H << 1$$

as criterion for conduction dominated heat transfer in a square enclosure heated from the side.

Problem 5.4. Integrating the mass conservation equation across the boundary layer

$$\int_0^\infty \left(\frac{\partial u_*}{\partial x_*} + \frac{\partial v_*}{\partial y_*} = 0\right) dx_* \;\rightarrow\; u_{*\infty} = -\frac{d}{dy_*}\int_0^\infty v_*\, dx_*$$

and recalling that

$$v_* = \frac{\frac{1}{2} - T_{*\infty}}{\lambda_2^2 - \lambda_1^2}\left(-e^{-\lambda_2 x_*} + e^{-\lambda_1 x_*}\right)$$

we obtain

$$u_{*\infty} = -\frac{d}{dy_*}\left[\frac{\frac{1}{2} - T_{*\infty}}{\lambda_1 \lambda_2 (\lambda_2 + \lambda_1)}\right]$$

Using Gill's substitutions

$$\lambda_{1,2} = \frac{p}{4}(1-q)\left[1 \pm i\sqrt{1+2q}\right] \quad \text{and} \quad T_{*\infty} = \frac{q}{1+q^2}$$

where $p(y_*)$ = even function, and $q\,(y_*)$ = odd function, yields

$$u_{*\infty} = -\frac{d}{dy_*}\underbrace{\left[\frac{8}{p^3\left(1-q^4\right)}\right]}_{\text{even function}}$$

Since d/dy_* (even fcn.) is an odd function of y_*, the core velocity $u_{*\infty}$ is an odd function. This means that the horizontal core flow changes its direction as we move from positive y_*'s to negative y_*'s.

Problem 5.5. The three scales of the momentum equation are

$$\underset{\text{inertia}}{\frac{u^2}{HL}}, \quad \underset{\text{friction}}{\nu\frac{u}{H^3}}, \quad \underset{\text{buoyancy}}{g\beta\frac{\Delta T}{L}}$$

Assuming that a balance exists between inertia and buoyancy, we obtain

$$u \sim (g\beta\, H\, \Delta T)^{1/2}$$

The assumption "inertia ~ buoyancy" implies that the friction scale is negligible relative to either inertia or buoyancy; let's see whether this is true:

$$\frac{\text{friction}}{\text{buoyancy}} \sim \frac{\nu u/H^3}{g\beta\,\Delta T/L} \sim Ra_H^{-1/2}\, Pr^{1/2}\, \frac{L}{H}$$
$$\left[\text{substituting } u \sim (g\beta H\,\Delta T)^{1/2}\right]$$

We learn that in the shallow enclosure limit $H/L \to 0$ the ratio friction/buoyancy would blow up. This means that the friction scale is not negligible, hence, the assumption inertia ~ buoyancy is inadmissible.

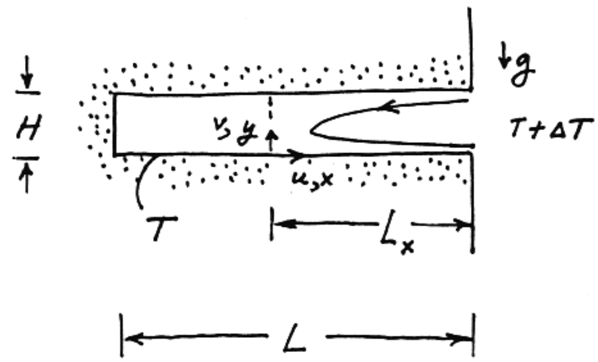
The correct assumption is the one made in the text, friction ~ buoyancy. To the mind that is still influenced by the scaling results developed in Chapter 4, the balance friction ~ buoyancy may "seem" to be valid only for $Pr > 1$. In reality, no Prandtl number restriction exists on the core flow solution sketched in Fig. 5.10; recounting the analytical steps that led to the shallow core solution, we see that it applies strictly in the $H/L \to 0$ limit, Ra_H and Pr being arbitrary but finite (fixed). Like any other flow, the core solution breaks down during the transition to turbulent flow (the stability of the S-shaped core velocity profile is discussed on page 111 of Ref. [40]). Transition criteria can be obtained quite easily by applying the scaling conclusions of Chapter 6, namely the local Reynolds number of order 10^2.

Problem 5.6. The equations for the conservation of mass, momentum and energy are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(C)}$$

$$\left.\begin{aligned} & u\frac{\partial \zeta}{\partial x} + v\frac{\partial \zeta}{\partial y} = \nu\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) + \beta g\frac{\partial T}{\partial x} \\ & \text{with } \zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}, \quad u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x} \end{aligned}\right\} \qquad \text{(M)}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{E}$$



Assuming that the flow is slender, $L_x >> H$, the balance convection ~ conduction in (E) yields

$$\frac{\psi\,\Delta T}{H\,L_x} \sim \frac{\alpha\,\Delta T}{H^2} \tag{E'}$$

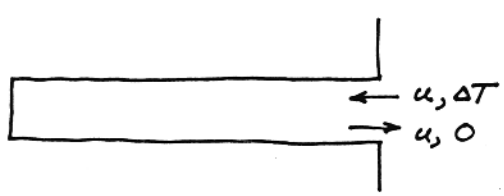
The balance friction ~ buoyancy in (M) yields

$$\frac{\nu\,\psi}{H^4} \sim \frac{\beta g\,\Delta T}{L_x} \tag{M'}$$

Eliminating ψ between the last two relations we obtain

$$\frac{L_x}{H} \sim Ra_H^{1/2}, \quad \text{where} \quad Ra_H = \frac{g\beta\,H^3\,\Delta T}{\alpha\,\nu}$$

The overall heat transfer rate between cavity and reservoir can be evaluated in two ways. One way is to focus on the counterflow through the mouth of the cavity, and to say that the heat transfer rate matches the enthalpy flowrate into the cavity,



$$\begin{aligned} Q &\sim \text{"}\dot{m}\,c_P\,\Delta T\text{"} \\ &\sim \rho\,u\,H\,c_P\,\Delta T \\ &\sim \rho\,\frac{\psi}{H}\,H\,c_P\,\Delta T \\ &\sim k\,\Delta T\,Ra_H^{1/2} \end{aligned}$$

Therefore, the Nusselt number scale is

$$Nu = \frac{Q}{k\,\Delta T} \sim Ra_H^{1/2} \tag{1}$$

Similarity solutions for this problem, as well as experiments, are presented in Ref. [60]. For example, in the limit $Pr \to \infty$ the similarity solution yields

$$\frac{L_x}{H} = 0.029 \, Ra_H^{1/2}$$

$$Nu = 0.053 \, Ra_H^{1/2}$$

These results validate the trends discovered via scale analysis.

The other way to evaluate Q is to integrate the heat flux through the horizontal wall of the cavity,

$$Q \sim k \frac{\Delta T}{H} L_x \sim k \, \Delta T \, Ra_H^{1/2}, \tag{2}$$

which is the same as eq. (1).

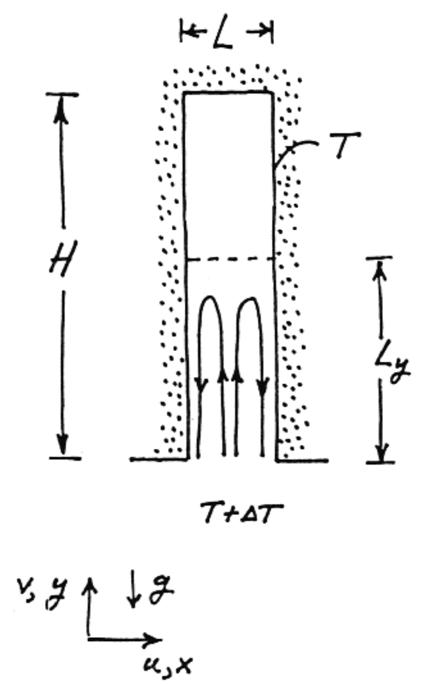
Problem 5.7. With reference to the governing equations (C), (M), (E) listed in the preceding solution, the balance friction ~ buoyancy in (M) yields

$$\frac{\nu}{L^2} \frac{\psi}{L^2} \sim g\beta \frac{\Delta T}{L}$$

in other words $\psi \sim g\beta\Delta T \, L^3/\nu$. It has been assumed that the flow region is slender, $L_y >> L$.

The balance between upward convection and lateral conduction in (E) yields

$$\frac{\psi}{L_y} \frac{\Delta T}{L} \sim \alpha \frac{\Delta T}{L^2}$$



Eliminating ψ between the last two expressions, we obtain

$$L_y \sim L\, Ra_L$$

For the overall heat transfer rate we write

$$Q \sim \underbrace{\left(k \frac{\Delta T}{L}\right) L_y}_{\substack{\text{heat flux into} \\ L_y\text{–tall wall}}} \sim k\, \Delta T\, Ra_L, \quad \text{hence} \quad Nu \sim Ra_L$$

Lighthill [61] reports an integral solution for the vertical penetration of natural convection into a vertical tube with one end open; that solution shows that $L_y \sim r_o\, Ra_{r_o}$ and $Nu \sim Ra_{r_o}$, where r_o is the tube radius.

Problem 5.8. The water properties that will be used are evaluated at the film temperature of 33.4°C

$$\alpha = 0.00148 \frac{cm^2}{s} \qquad \frac{g\beta}{\alpha\nu} = \frac{28\,982}{cm^3K}$$

Assume that in the beginning the water is motionless. The water that comes in contact with the hot surface develops a "conduction" layer the thickness of which increases as

$$\delta \sim (\alpha t)^{1/2} \tag{1}$$

This δ-tall layer is heated from below and, in accordance with a criterion similar to eq. (5.72), it becomes unstable when its Rayleigh number based on height exceeds the order of magnitude 10^3:

$$Ra_\delta \sim 10^3$$

$$\frac{g\beta}{\alpha\nu}\, \delta^3 \,(T_w - T_\infty) \sim 10^3$$

$$\delta^3 \sim 10^3 \, \frac{cm^3K}{20\,982}\, \frac{1}{(43.1 - 23.6)\,K} = 0.0018\ cm^3$$

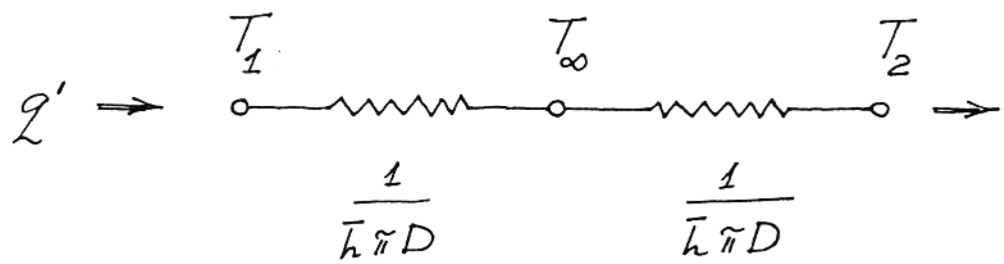
or, approximately, $\delta \sim 1$ mm. In view of eq. (1), this result means

$$\alpha t \sim 0.0146\ cm^2$$

$$t \sim \frac{0.0146\ cm^2}{0.00148\ \frac{cm^2}{s}} \cong 10\ s$$

In conclusion, thermals will rise from the same region of the heated surface at time intervals of the order of ten seconds.

Problem 5.9. Since each cylinder is surrounded by its own boundary layer, the heat transfer is impeded first by the boundary layer resistance between T_1 and the water reservoir [$T_\infty = (T_1 + T_2)/2 = 25°C$], and later by the corresponding boundary layer resistance between T_∞ and T_2. The two resistances are equal and in series. They are equal because of symmetry and the assumption that the water properties are the same (well, nearly the same) in the two boundary layers.



The heat transfer rate from the cylinder T_1 to the cylinder T_2, expressed per unit length in the direction perpendicular to the cross-section, is

$$q' = \frac{T_1 - T_2}{\dfrac{1}{\bar{h}\,\pi D} + \dfrac{1}{\bar{h}\,\pi D}} = \frac{\pi}{2}\, k\, \overline{Nu}_D\, (T_1 - T_2)$$

where $\overline{Nu}_D$ can be estimated based on eq. (4.122). The Rayleigh number in that correlation, Ra_D, is based on the temperature difference across one boundary layer, e.g. $\Delta T = T_1 - T_\infty = 5°C$, and on properties evaluated at the film temperature [$(30°C + 25°C)/2 = 27.5°C$, or $(25°C + 20°C)/2 = 22.5°C$]. We assume that the two sets of film properties are approximated well by the properties evaluated at the in-between temperature of $(27.5°C + 22.5°C)/2 = 25°C$,

$$k = 0.6\,\frac{W}{m\cdot K} \qquad Pr = 6.21 \qquad \frac{g\beta}{\alpha\nu} = \frac{19.81 \times 10^3}{cm^3\,K}$$

and calculate, in order,

$$Ra_D = \frac{g\beta}{\alpha\nu}\, D^3\, \Delta T$$
$$= \frac{19.81 \times 10^3}{cm^3\,K}\,(4\ cm)^3\, 5K = 6.34 \times 10^6$$

$$\overline{Nu}_D = \cdots = \left\{0.6 + \frac{0.387 \times 13.6}{1.07}\right\}^2 = 30.45 \qquad (4.122)$$

$$q' = \frac{\pi}{2}\, 0.6\, \frac{W}{m\cdot K}\, 30.45\,(30 - 20)\,K = 287\, \frac{W}{m}$$

Problem 5.10. An approximate way to determine the relationship between the heat flux from T_h to T_c (namely q", uniform) and the difference $T_h - T_c$ is to invoke the conclusion reached in Problem 4.26 (single-pane window, uniform q"):

$$\frac{q''}{T_h - \bar{T}_{core}} \frac{H}{k} \cong 0.252 \left[\frac{g\beta (T_h - T_{core}) H^3}{\alpha \nu}\right]^{1/4} \tag{1}$$

Here $\bar{T}_{core}$ is the average temperature of the air core contained between the two boundary layers inside the cavity. In real life, T_{core} increases from T_c at the bottom of the cavity, to T_h at the very top. Its value at midheight is clearly

$$T_{core}\left(y = \frac{H}{2}\right) = \frac{1}{2}(T_h + T_c) \tag{2}$$

The average core temperature $\bar{T}_{core}$ is also equal to $(T_h + T_c)/2$, therefore

$$T_h - \bar{T}_{core} = \frac{1}{2}(T_h - T_c) \tag{3}$$

and, in terms of $T_h - T_c$, eq. (1) becomes

$$\frac{q''}{T_h - T_c} \frac{H}{k} \cong 0.106 \left[\frac{g\beta (T_h - T_c) H^3}{\alpha \nu}\right]^{1/4} \tag{4}$$

This formula applies when the cavity is wide enough so that the innermost boundary layers are distinct. When estimating the thickness of one of the inner boundary layers, in order to see whether they are distinct,

$$\delta_T \sim H \left[\frac{g\beta (\bar{T}_w - \bar{T}_{core}) H^3}{\alpha \nu}\right]^{-1/4} \tag{5}$$

keep in mind that the glass-core temperature difference is approximately one fourth of the overall temperature difference,

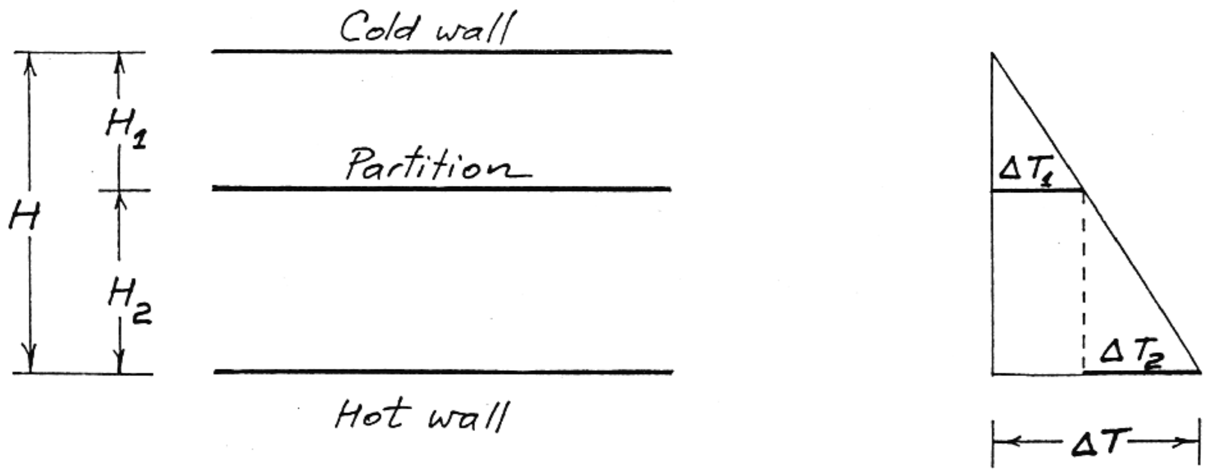
$$\bar{T}_w - \bar{T}_{core} \cong \frac{1}{4}(T_h - T_c) \tag{6}$$

Problem 5.11. The original fluid layer has the height H, and the bottom-to-top temperature difference ΔT. The horizontal partition divides this layer into two sublayers, with the following thicknesses and bottom-to-top temperature differences (see the figure)

$$H_1 = x H \qquad \Delta T_1 = x \Delta T \tag{1}$$

$$H_2 = (1 - x) H \qquad \Delta T_2 = (1 - x) \Delta T \tag{2}$$

The number x is between 0 and 1, and describes the relative position of the partition. For example, x = 1/2 means that the partition is inserted at midheight. It is important to note also that the ΔT_1 and ΔT_2 expressions written above are based on the assumption that there is no convection in either sublayer.



Our objective is to determine the optimal partition location x so that the regime of pure conduction (no convection) is extended to the largest possible Rayleigh number

$$Ra = \frac{g\,\beta}{\alpha\,\nu} H^3\,\Delta T \qquad (3)$$

This is the "external" Rayleigh number, which is based on the overall temperature difference ΔT. The objective then is to maximize the overall ΔT while preserving the state of pure conduction, i.e. least heat flux in the vertical direction. Recall that the formation of convection cells augments the heat flux relative to the heat flux present in the no-convection state.

If we assume (for simplicity) that the partition is isothermal at some intermediate temperature between the top wall and bottom wall temperatures, the top sublayer is without convection currents as long as its own Rayleigh number is smaller than the critical value,

$$\frac{g\,\beta}{\alpha\,\nu} H_1^3\,\Delta T_1 < 1708 \qquad (4)$$

The same can be said about the avoidance of convection in the lower sublayer,

$$\frac{g\,\beta}{\alpha\,\nu} H_2^3\,\Delta T_2 < 1708 \qquad (5)$$

Convection is suppressed in the entire layer (H) when conditions (4) and (5) are satisfied simultaneously. Now it is a simple matter to use the x-based formulas (1,2) and the external-Ra definition (3) to rewrite conditions (4) and (5) as

$$Ra < \frac{1708}{x^4} \qquad (6)$$

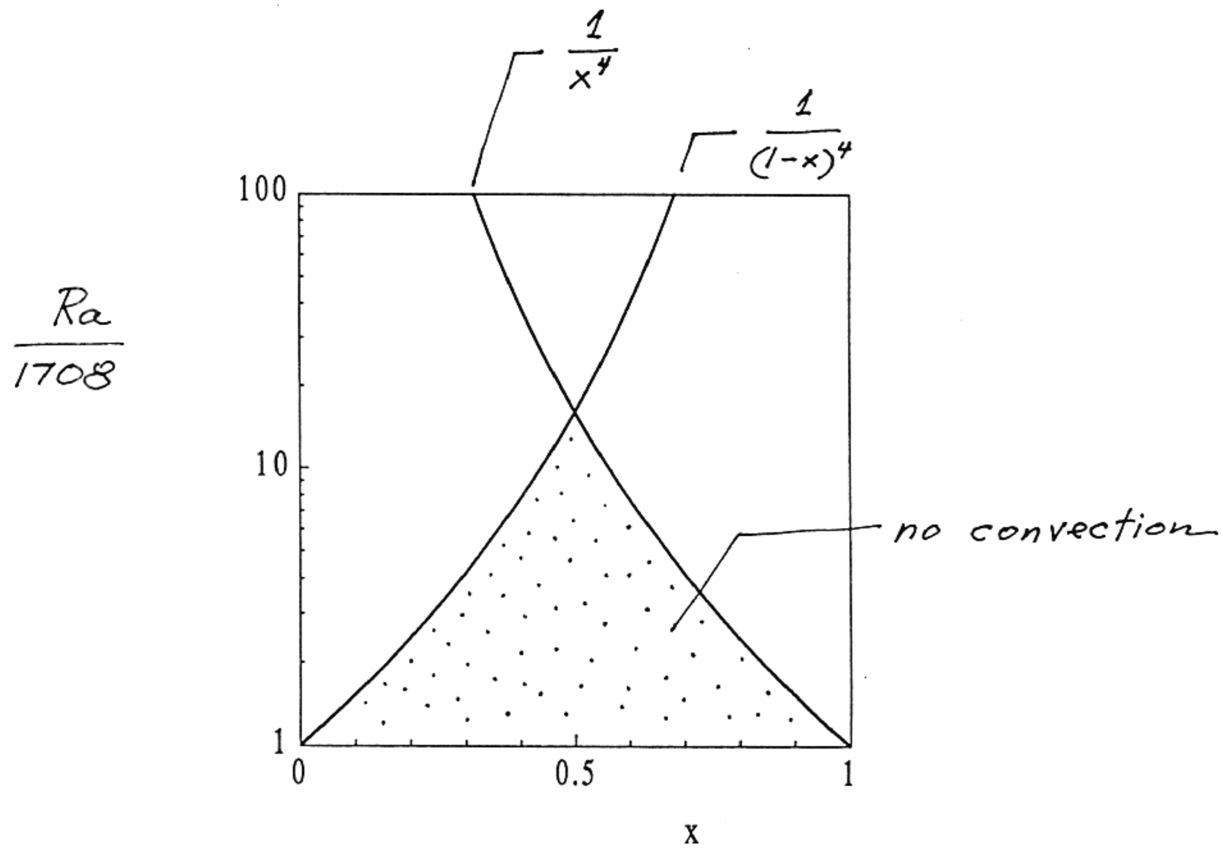
$$Ra < \frac{1708}{(1-x)^4} \qquad (7)$$

Conditions (6) and (7) are satisfied simultaneously by all the (Ra, x) points situated <u>under</u> the roof-shaped line described by $Ra = 1708/x^4$ and $Ra = 1708/(1-x)^4$ in the attached figure. It is clear that the design with the maximum external Rayleigh number for "no convection" corresponds to

$$x_{opt} = \frac{1}{2} \tag{8}$$

i.e. to a partition installed at midheight. That maximum critical external Rayleigh number is (approximately, of course)

$$Ra_{c,max} = \frac{1708}{(1/2)^4} \sim 2.7 \times 10^3 \tag{9}$$



The modelling of the thermal boundary condition on the partition does not affect the optimal location determined in eq. (8). For example, if the partition is modelled as a wall with uniform flux (instead of uniform temperature), the only item that changes in the analysis is the numerical value (of the order of 10^3) that must appear on the right-hand side of the inequalities (4) and (5). This value, however, drops out in the steps that follow en route to eq. (8).

The design can be refined by accounting for the temperature dependence of the property group $g\beta/\alpha\nu$. If this dependence is significant, the optimal location of the partition will differ from $x_{opt} = 1/2$.

Chapter 6

TRANSITION TO TURBULENCE

Problem 6.1. According to Table 6.1, the most conservative condition for the survival of laminar boundary layer flow is

$$Re_L < 2 \times 10^4, \quad \text{where} \quad Re_L = \frac{U_\infty L}{\nu}$$

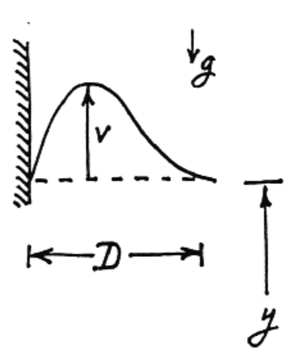
In terms of local Reynolds numbers based on local transversal length scales, the above condition becomes:

Re based on displacement thickness, δ^*	Re based on momentum thickness, θ
$\delta^* = 1.73 L\, Re_L^{-1/2}$	$\theta^* = 0.664 L\, Re_L^{-1/2}$
$Re_{\delta^*} = \frac{U_\infty \delta^*}{\nu} = Re_L \frac{\delta^*}{L}$ $= 1.73\, Re_L^{1/2}$	$Re_\theta = \frac{U_\infty \theta}{\nu} = Re_L \frac{\theta}{L}$ $= 0.664\, Re_L^{1/2}$
$Re_{\delta^*} < 1.73 \left(2 \times 10^4\right)^{1/2}$ < 245	$Re_\theta < 0.664 \left(2 \times 10^4\right)^{1/2}$ < 94

In conclusion, both results show that the boundary layer remains laminar if the local Re based on transversal length scale is less than $O(10^2)$.

Problem 6.2. From Table 4.1, the relevant scales of the wall jet of Pr > 1 fluids are

$$v \sim \frac{\alpha}{y} Ra_y^{1/2}, \qquad D \sim Pr^{1/2} y\, Ra_y^{-1/4},$$



therefore, the local Reynolds number scales as

$$\mathrm{Re} \sim \frac{vD}{\nu} \sim \mathrm{Pr}^{-1/2}\,\mathrm{Ra}_y^{1/4}$$

Table 6.1 indicates that when Pr = 6.7 the boundary layer remains laminar if

$$\mathrm{Gr}_y < 1.3 \times 10^9,$$

which means

$$\mathrm{Ra}_y < (1.3 \times 10^9)\,(6.7) = 8.71 \times 10^9,$$

in other words, when

$$\mathrm{Re} < (6.7)^{-1/2}\,(8.71 \times 10^9)^{1/4}$$

$$< 118$$

In conclusion, the transition of the buoyant wall jet from laminar to buckling (turbulent) flow occurs when the Re based on transversal length scale is of the order

$$\mathrm{Re} \sim 10^2$$

Problem 6.3. In a buoyant air plume (Pr ~ 1) the scales are

$$D \sim y\,\mathrm{Ra}_y^{-1/4}, \qquad v \sim \frac{\alpha}{y}\,\mathrm{Ra}_y^{1/2}$$

hence, the local Reynolds number is

$$\mathrm{Re} \sim \frac{vD}{\nu} \sim \mathrm{Ra}_y^{1/4} = \left(\frac{g\beta y^3\,\Delta T}{\alpha\nu}\right)^{1/4}$$

To eliminate ΔT (which depends on y), we invoke the conservation of energy,

$$q \sim \rho\, c_P\, D^2\, v\, \Delta T,$$

which yields

$$\Delta T \sim \frac{q}{ky}$$

Therefore, the local Re scales as

$$\mathrm{Re} \sim \left(\frac{g\beta\, y^2\, q}{\alpha\nu k}\right)^{1/4} = \mathrm{Ra}_q^{1/4}$$

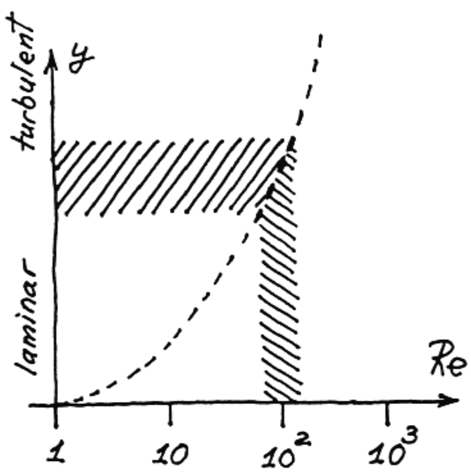
and, since Fig. 6.2 shows that the transition is marked by $Ra_q \sim 10^{10}$, the transition Re is of order

$$Re \sim (10^{10})^{1/4} = 316$$

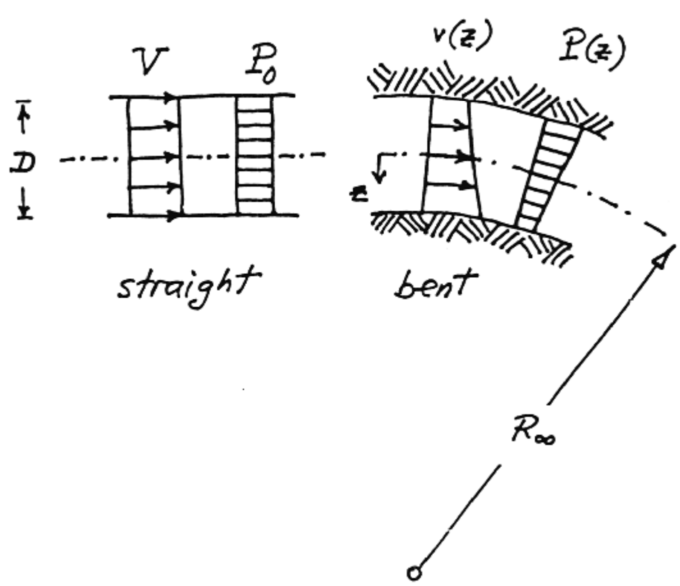
in other words,

$$Re = O(10^2).$$

An important observation concerns the dependence of the local Reynolds number Re on the longitudinal position y. Since $Re \sim Ra_q^{1/4}$, Re increases monotonically as $y^{1/2}$. This means that <u>all laminar plumes become eventually turbulent</u>, as they develop downstream. Note that the same observation applies to the laminar boundary layer of Problem 6.1, and to the laminar buoyant wall jet of Problem 6.2.



<u>Problem 6.4</u>. If the stream with constant V and P_0 is bent in a bending apparatus (duct), then the velocity distribution becomes such that the flow is the fastest near the inner wall and the pressure is the greatest near the outer wall. These features follow from the Bernoulli equation along one streamline



$$\frac{1}{2}\rho V^2 + P_0 = \frac{1}{2}\rho v^2(z) + P(z)$$

coupled with the force balance in the radial direction

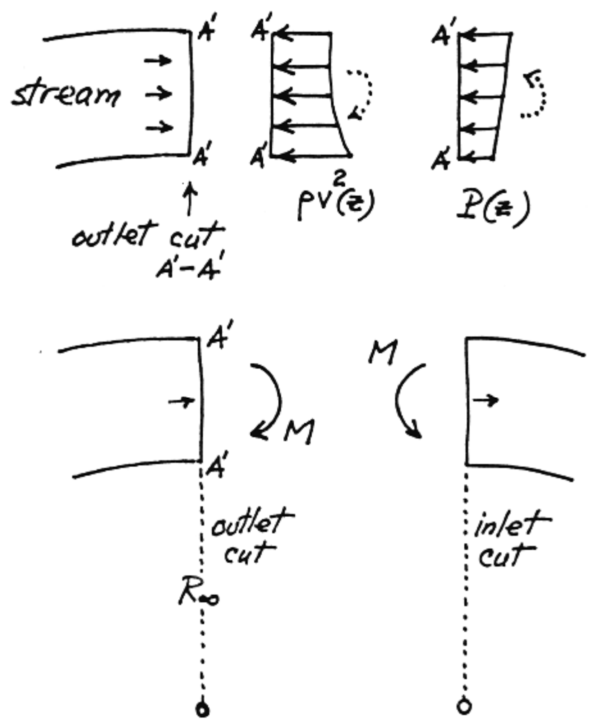
$$-\frac{\rho v^2(z)}{R_\infty} = \frac{\partial P}{\partial z},$$

where the flow has been modeled as inviscid. In the limit $z/R_\infty \to O$ (i.e. $D/R_\infty \to 0$), these two equations yield

$$v(z) = V\left(1 + \frac{z}{R_\infty} + \ldots\right)$$

$$P(z) = P_0 - \rho\frac{V^2}{R_\infty} z \ldots$$

In any cross-section, we identify <u>nonuniform</u> distributions of normal stresses due to P(z) and the reaction caused by momentum flow through the cross-section, $\rho v^2(z)$. Note that these nonuniform distributions tend to rotate the cross-section in opposing senses. The net bending moment is



$$M = \iint_A (\rho v^2 + P) z \, dA =$$

$$= \iint_A \left[\rho V^2\left(1 + \frac{2z}{R_\infty}\right) + P_0 - \frac{\rho V^2 z}{R_\infty}\right] z \, dA =$$

$$= \iint_A \frac{\rho V^2 z^2}{R_\infty} dA = \frac{\rho V^2 I}{R_\infty}, \quad \text{where} \quad I = \iint_A z^2 \, dA$$

Note that the bending moment has the same sense as in the cross-section of an elastic beam of radius of curvature R_∞ (which is why the inviscid stream possesses a buckling property analogous to that of elastic beams). The same result, $M = \rho V^2 I / R_\infty$, is obtained for an inlet cut through the stream (try it).

Problem 6.5. The solution to this problem is given as part of the solution to Problem 2.25:

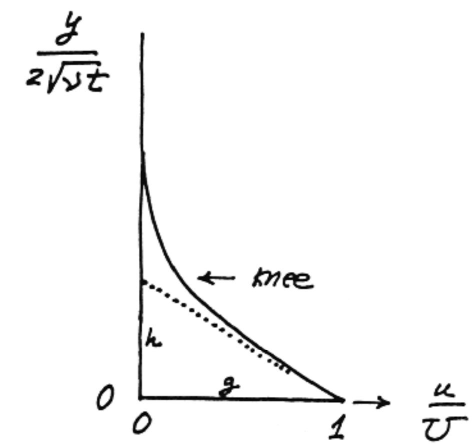
$$u = U\left[1 - \operatorname{erf}\left(\frac{y}{2(\nu t)^{1/2}}\right)\right]$$

To find the location of the knee in this velocity profile, we recall that

$$\operatorname{erf}(a) = \frac{2}{\pi^{1/2}} \int_0^a e^{-m^2} dm$$

hence

$$\frac{d}{da}[\operatorname{erf}(a)]_{a=0} = \frac{2}{\pi^{1/2}}$$



On the above figure, we have

$$\frac{g}{h} = \frac{2}{\pi^{1/2}}, \qquad \text{in other words} \qquad h = \frac{\pi^{1/2}}{2} = 0.89$$

In conclusion, the knee is located in the vicinity of

$$\frac{y}{2(\nu t)^{1/2}} = 0.89, \qquad \text{or} \qquad \frac{y}{2(\nu t)^{1/2}} = O(1).$$

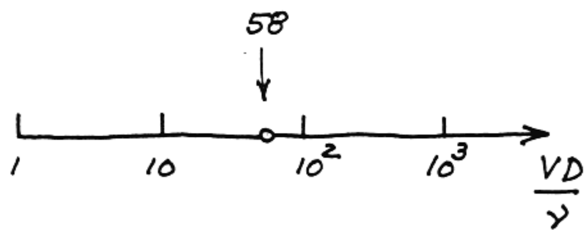
Problem 6.6. If $N_B = O(1)$, then $t_v \sim t_B$, or

$$\frac{D^2}{16\nu} \sim \frac{\lambda_B}{V/2}$$

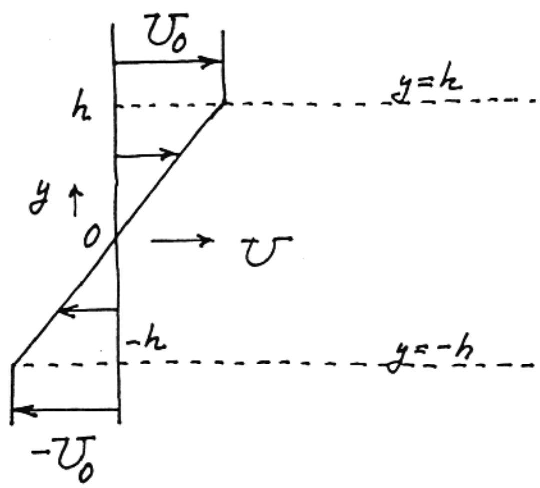
Rearranging, we obtain

$$\frac{VD}{\nu} \sim 32\,\frac{\lambda_B}{D}$$

and, if the stream is two-dimensional [$\lambda_B/D = \pi/3^{1/2}$, eq.(6.11)], $\frac{VD}{\nu} \sim 32\,\frac{\pi}{3^{1/2}} = 58$. The logarithmic scale shown below suggests that $\frac{VD}{\nu} = O\left(10^2\right)$ when $N_B = O(1)$.



Problem 6.7. **a)** The stability analysis of an inviscid broken-line shear flow may be found in Ref. [35]. The main steps are:



region (1) $\quad U = U_0$

region (2) $\quad U = U_0\,\frac{y}{h}$

region (3) $\quad U = -U_0$

The $\hat{v}$ solutions to eq. (6.22) in each of the three regions are

(1) $\quad \hat{v} = A\,e^{-ky}$

(2) $\quad \hat{v} = B\,e^{-ky} + Ce^{ky}$

(3) $\quad \hat{v} = D\,e^{ky}$

The continuity of $\hat{v}$ across the y = h and y = – h planes requires

$$A\, e^{-kh} = B\, e^{-kh} + Ce^{kh} \tag{a}$$

$$D\, e^{-kh} = B\, e^{kh} + Ce^{-kh} \tag{b}$$

Finally, the pressure continuity conditions at y = ± h, eq. (6.26), yield

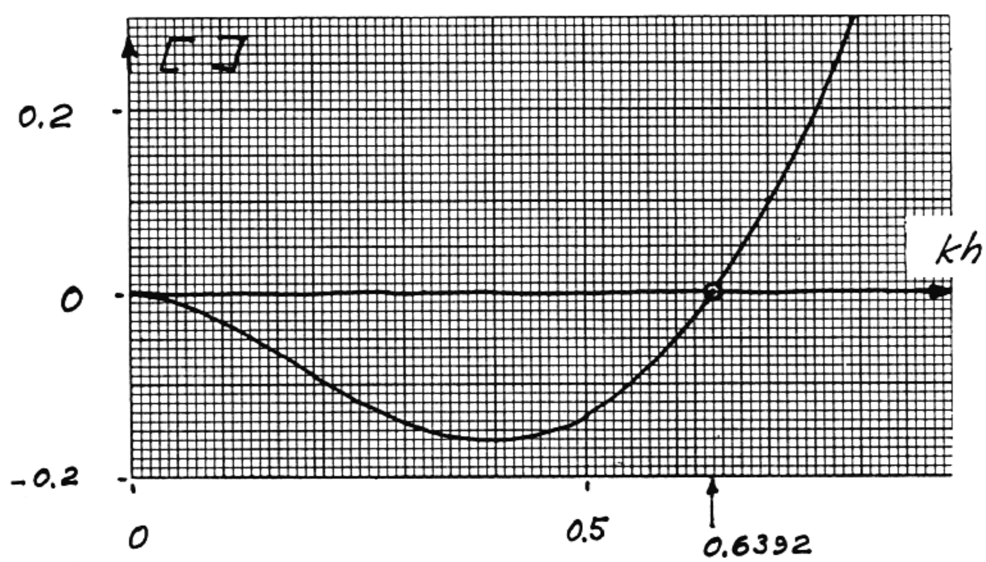
$$2\left(\sigma + k\, U_0\right) Ce^{kh} - \frac{U_0}{h}\left(B\, e^{-kh} + C\, e^{kh}\right) = 0 \tag{c}$$

$$2\left(\sigma - k\, U_0\right) Be^{kh} + \frac{U_0}{h}\left(B\, e^{kh} + C\, e^{-kh}\right) = 0 \tag{d}$$

Note that the last two equations are the result of eqs. (6.26) at y = ± h, in which A and D have been eliminated using eqs. (a) and (b). Eliminating the ratio B/C between eqs. (c) and (d) yields

$$\sigma^2 = \frac{U_0^2}{4h^2}\left[(2\, kh - 1)^2 - e^{-4kh}\right]$$

The shear flow is unstable if σ is imaginary, i.e. if the quantity in the square brackets is negative:



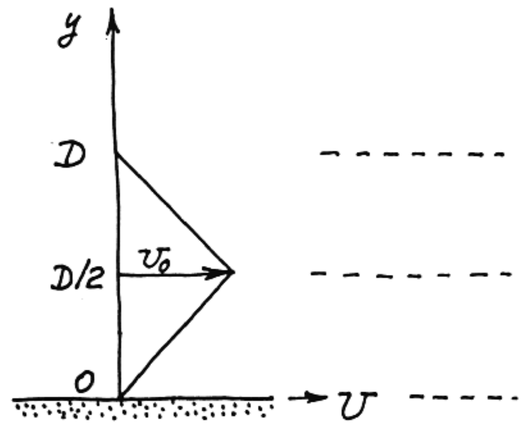
The instability condition is

$$kh < 0.6392$$

or, since $k = 2\pi/\lambda$,

$$\frac{\lambda}{(2h)} > 4.92.$$

b) The stability analysis of an inviscid broken-line wall jet is outlined in Ref. [45]. Here are the highlights:



region (1) $\quad U = 0$

region (2) $\quad U = U_0\left(2 - \frac{y}{D/2}\right)$

region (3) $\quad U = U_0 \frac{y}{D/2}$

The $\hat{v}$ solutions in each of the three regions are

(1) $\quad \hat{v} = E\, e^{-ky} + F e^{ky} \quad$, with F = 0, because $\lim_{y\to\infty} \hat{v} = \text{finite}$

(2)* $\quad \hat{v} = C\, e^{-ky} + D_0\, e^{ky}$

(3) $\quad \hat{v} = A\, e^{-ky} + B\, e^{ky}$, with A + B = 0, because $\hat{v}(0) = 0$.

The continuity of $\hat{v}$ across y = D/2 and y = D requires

$$A\, e^{-kD/2} + B\, e^{kD/2} = C\, e^{-kD/2} + D_0\, e^{kD/2} \tag{a}$$

$$C\, e^{-kD} + D_0\, e^{kD} = E\, e^{-kD} \tag{b}$$

The continuity of pressure across the same interfaces requires

$$D_0\left(\frac{\sigma D}{U_0} + 1\right) e^{kD/2} + C\, e^{-kD/2} = 0 \tag{c}$$

$$\left[1 - \frac{D}{4\,U_0}(\sigma + k\,U_0)\right] C + \left[1 + \frac{D}{4\,U_0}(\sigma + k\,U_0)\right] D_0 + \frac{D}{4\,U_0}(\sigma + k\,U_0)(A - B) = 0 \tag{d}$$

Eqs. (a)-(d) and the condition A + B = 0 are homogeneous. A non-trivial (A,B,C,D_0,E) solution is possible if the determinant of this system is zero:

$$m^2 + m\,(2\gamma^2 + kD - 3 - \gamma^4) - \gamma^4\,(1 + kD) + 2\gamma^2 = 0$$

where

$$\gamma = e^{-kD/2}$$

$$m = \frac{\sigma D}{U_0} + 1$$

Whether or not σ is imaginary depends on the character of m; since the m equation has the form $am^2 + bm + c = 0$, imaginary roots are possible if $\Delta = b^2 - 4\,ac < 0$, hence

* the coefficient D_0 should not be confused with the jet thickness D.

$$\Delta = (2\gamma^2 + kD - 3 - \gamma^4)^2 + 4\gamma^4 (1 + kD) - 8\gamma^2 < 0$$

It is found numerically that the above condition is met by

$$1.337 < kD < 3.427$$

or, since $k = 2\pi/\lambda$,

$$4.701 > \frac{\lambda}{D} > 1.833.$$

Note that the minimum unstable wavelength, 1.833D, is nearly identical to that of a free jet of triangular profile (Fig. 6.6).

Problem 6.8. From the solution to Problem 2.22 we know the following scales

$$D \sim \nu^{2/3} x^{2/3} U_0^{-2/3} D_0^{-1/3}$$

$$U \sim \nu^{-1/3} x^{-1/3} U_0^{4/3} D_0^{2/3}$$

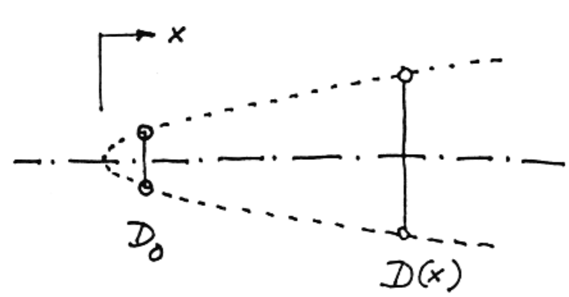
The local Reynolds number is

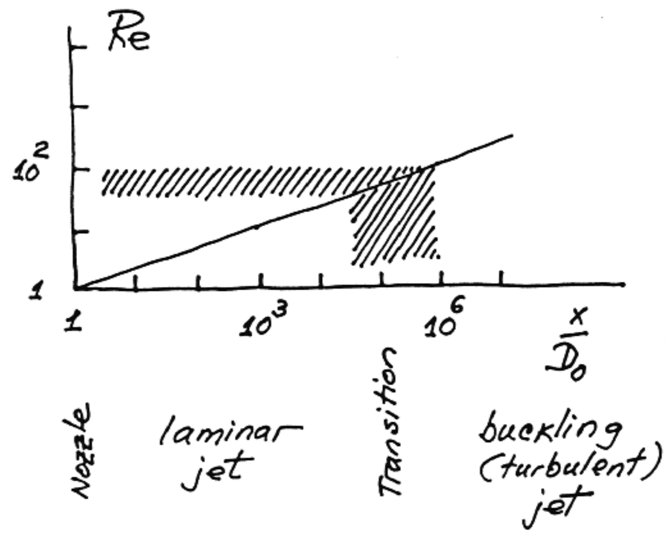
$$Re = \frac{DU}{\nu} \sim \left(\frac{x}{D_0}\right)^{1/3} \underbrace{\left(\frac{D_0 U_0}{\nu}\right)^{2/3}}_{1}$$

hence

$$Re \sim \left(\frac{x}{D_0}\right)^{1/3}$$

Since Re increases monotonically with x, the jet is always composed of a laminar section (length) followed by a turbulent section. All laminar jets buckle and become turbulent sufficiently far downstream, i.e. where $Re > 10^2$.





Problem 6.9. According to eq. (2.85), the scale of the laminar boundary layer thickness on the moving lid is

$$\delta \sim 5D\,Re_D^{-1/2} \tag{1}$$

where $Re_D = UD/\nu$. According to the local Reynolds number criterion, eq. (6.15), the flow ceases to be laminar when

$$\frac{U\delta}{\nu} \geq 10^2 \tag{2}$$

Eliminating δ between eqs. (1) and (2) we anticipate that the transition occurs when

$$Re_D \sim 400 \tag{3}$$

Chapter 7

TURBULENT BOUNDARY LAYER FLOW

Problem 7.1. The rules of time-averaging algebra are all based on the definitions

$$\bar{u} = \frac{1}{t_0}\int_0^{t_0} u\,dt$$

$$\bar{u}' = \frac{1}{t_0}\int_0^{t_0} u'\,dt = 0$$

Consequently, we can also write:

$$\overline{u+v} = \frac{1}{t_0}\int_0^{t_0} (u+v)\,dt = \frac{1}{t_0}\int_0^{t_0} u\,dt + \frac{1}{t_0}\int_0^{t_0} v\,dt = \bar{u} + \bar{v}$$

$$\overline{\bar{u}\,u'} = \frac{1}{t_0}\int_0^{t_0} \bar{u}\,u'\,dt = \bar{u}\,\frac{1}{t_0}\underbrace{\int_0^{t_0} u'\,dt}_{0} = 0$$

$$\overline{u\,v} = \frac{1}{t_0}\int_0^{t_0} uv\,dt = \frac{1}{t_0}\int_0^{t_0} \underbrace{(\bar{u}+u')(\bar{v}+v')}_{\bar{u}\bar{v}+\bar{u}\,v'+\bar{v}\,u'+u'v'}\,d\,t_0 =$$

$$= \bar{u}\bar{v}\,\underbrace{\frac{1}{t_0}\int_0^{t_0} dt}_{1} + \bar{u}\,\frac{1}{t_0}\underbrace{\int_0^{t_0} v'\,dt}_{0} + \bar{v}\,\frac{1}{t_0}\underbrace{\int_0^{t_0} u'\,dt}_{0} + \frac{1}{t_0}\int_0^{t_0} u'v'\,dt$$

$$= \bar{u}\bar{v} + \overline{u'v'};$$

When u = v, this last rule becomes

$$\overline{u^2} = (\bar{u})^2 + \overline{(u')^2}$$

Next, we have:

$$\overline{\left(\frac{\partial u}{\partial x}\right)} = \frac{1}{t_0}\int_0^{t_0} \frac{\partial u}{\partial x}\,dt = \frac{\partial}{\partial x}\left(\frac{1}{t_0}\int_0^{t_0} u\,dt\right) = \frac{\partial \bar{u}}{\partial x}$$

$\dfrac{\partial \bar{u}}{\partial t} = 0$, because $\bar{u}$ is not a function of time

$$\left(\overline{\frac{\partial u}{\partial t}}\right) = \frac{1}{t_0}\int_0^{t_0} \frac{\partial u}{\partial t}\, dt = \frac{u(t_0) - u(0)}{t_0} = 0,$$

because $t_0 \rightarrow \infty$ whereas $u(t_0)$ can only be of the same order as $u(0)$.

Problem 7.2. The time-averaging of the energy equation is analogous to that of the x-momentum equation, which is outlined in chapter 7:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

$$T\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{\partial T}{\partial t} + \frac{\partial (uT)}{\partial x} + \frac{\partial (vT)}{\partial y} + \frac{\partial (wT)}{\partial z} = \alpha\, \nabla^2 T$$

$$\frac{1}{t_0}\int_0^{t_0} [\text{---}]\, dt$$

$$0 + \frac{\partial}{\partial x}\left(\overline{uT}\right) + \frac{\partial}{\partial y}\left(\overline{vT}\right) + \frac{\partial}{\partial z}\left(\overline{wT}\right) = \alpha\, \nabla^2 \bar{T}$$

Each term of type $\partial\left(\overline{uT}\right)/\partial x$ yields three terms:

$$\frac{\partial}{\partial x}\left(\overline{uT}\right) = \frac{\partial}{\partial x}\left(\bar{u}\,\bar{T}\right) + \frac{\partial}{\partial x}\left(\overline{u'T'}\right) = \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{T}\frac{\partial \bar{u}}{\partial x} + \frac{\partial}{\partial x}\left(\overline{u'T'}\right)$$

The energy equation becomes

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z} + \bar{T}\underbrace{\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z}\right)}_{0} + \frac{\partial}{\partial x}\left(\overline{u'T'}\right) + \frac{\partial}{\partial y}\left(\overline{v'T'}\right) + \frac{\partial}{\partial z}\left(\overline{w'T'}\right) = \alpha\, \nabla^2 \bar{T}$$

In conclusion, we obtain

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{w}\frac{\partial \bar{T}}{\partial z} = \alpha\left(\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} + \frac{\partial^2 \bar{T}}{\partial z^2}\right) - \frac{\partial}{\partial x}\left(\overline{u'T'}\right) - \frac{\partial}{\partial y}\left(\overline{v'T'}\right) - \frac{\partial}{\partial z}\left(\overline{w'T'}\right)$$

Problem 7.3. The van Driest model consists of using as mixing length

$$\ell = \kappa y\left(1 - e^{-y^+/A^+}\right)$$

which yields the eddy diffusivity

$$\varepsilon_M = \kappa^2 y^2\left(1 - e^{-y^+/A^+}\right)^2\left|\frac{\partial \bar{u}}{\partial y}\right|$$

The constant-τ_{app} postulate becomes

$$\left[\nu + \kappa^2 y^2 \left(1 - e^{-y^+/A^+}\right)^2 \frac{\partial \bar{u}}{\partial y}\right] \frac{\partial \bar{u}}{\partial y} = \frac{\tau_0}{\rho}$$

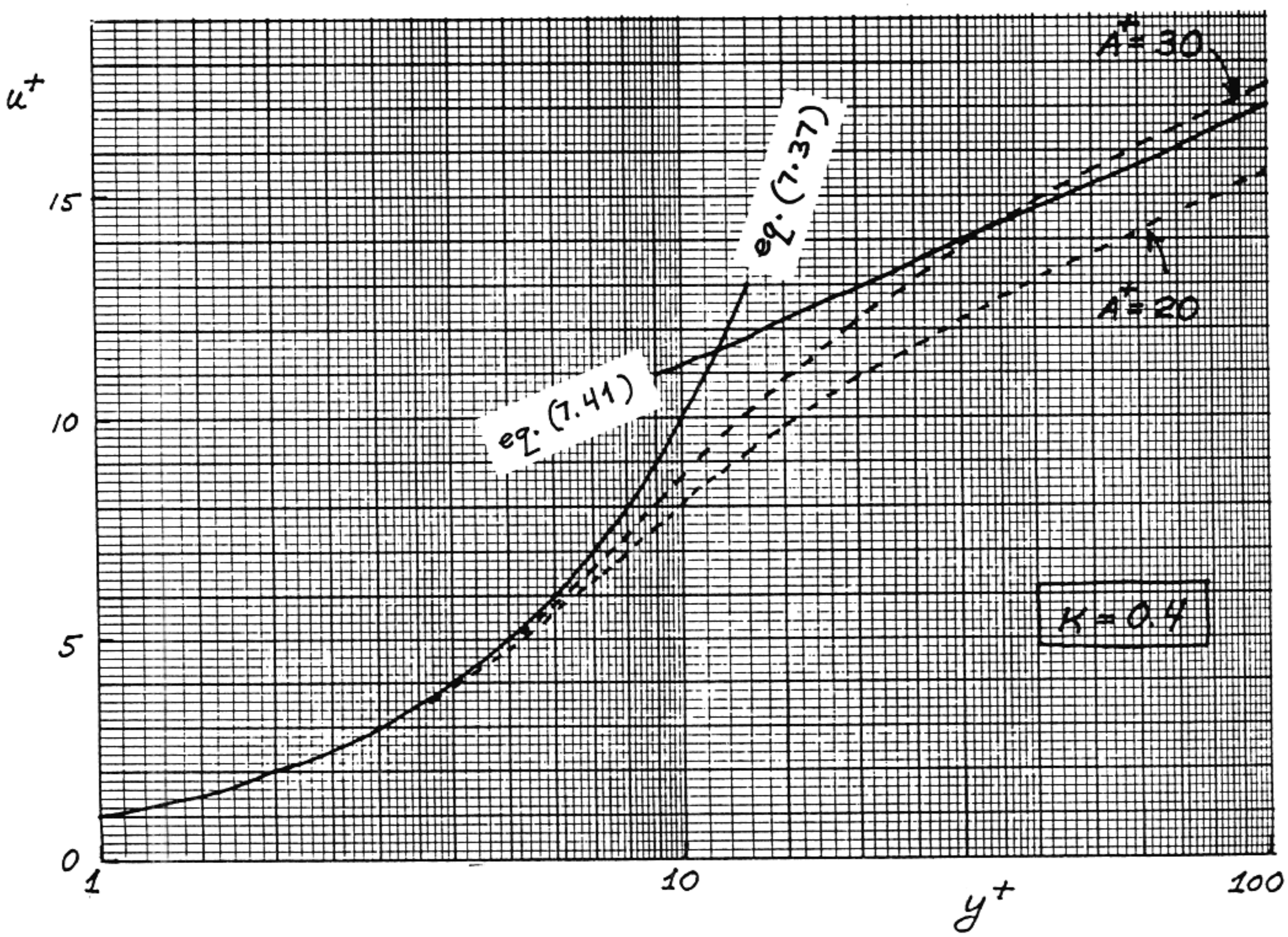
or, in wall coordinates,

$$\left[1 + \kappa^2 y^{+2} \left(1 - e^{-y^+/A^+}\right)^2 \frac{du^+}{dy^+}\right] \frac{du^+}{dy^+} = 1$$

Solving for du^+/dy^+ we obtain

$$\frac{du^+}{dy^+} = \frac{2}{1 + \left\{1 + 4\kappa^2 y^{+2} \left[1 - e^{-y^+/A^+}\right]^2\right\}^{1/2}}$$

which is also listed in Table 7.1. Integrating this numerically from $u^+(0) = 0$ yields the dash-line curves shown below. Clearly, the value of A^+ must be between 20 and 30, for the van Driest model to smooth the transition from eq. (7.37) to eq. (7.41).



Problem 7.4. If we solve for du^+/dy^+, eq. (7.47) yields

$$\frac{du^+}{dy^+} = \frac{-1 + \left(1 + 4\kappa^2 y^{+2}\right)^{1/2}}{2\kappa^2 y^{+2}}$$

Using the transformation

$$y^+ = \frac{\tan\alpha}{2\kappa}, \qquad \text{or} \qquad dy^+ = \frac{d\alpha}{2\kappa\cos^2\alpha}$$

the du^+/dy^+ expression becomes

$$\kappa du^+ = -\frac{d\alpha}{\sin^2\alpha} + \frac{d\alpha}{\cos\alpha\sin^2\alpha}$$

and

$$\left|\kappa u^+\right|_0^{u^+} = \left|\frac{\cos\alpha}{\sin\alpha} + \left(-\frac{1}{\sin\alpha} + \ln\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right)\right|_0^{\alpha}$$

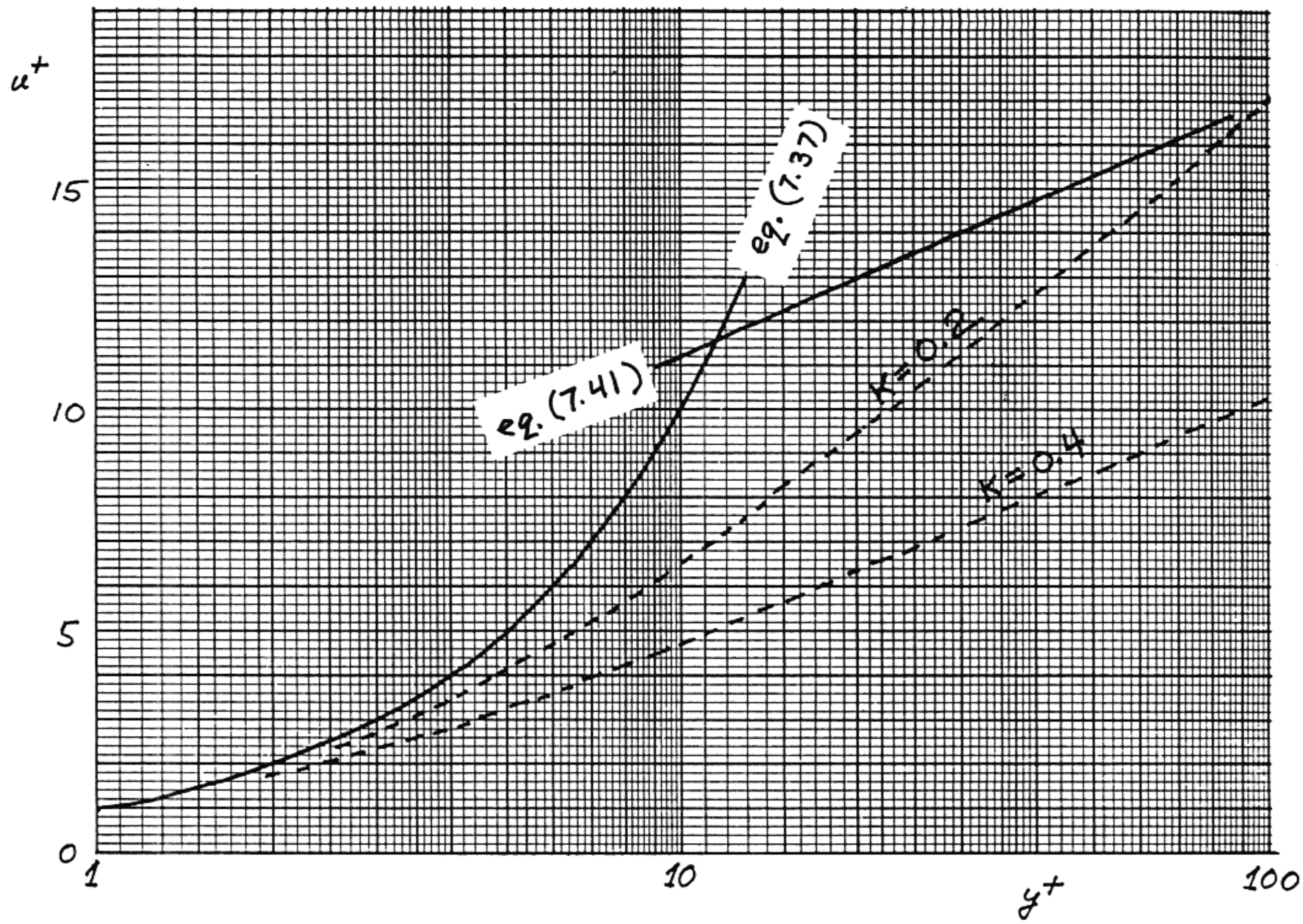
hence

$$\kappa u^+ = \frac{\cos\alpha - 1}{\sin\alpha} + \ln\left[\tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right)\right]$$

Note that

$$\lim_{\alpha\to 0}\frac{\cos\alpha - 1}{\sin\alpha} = 0$$

The above $u^+(y^+)$ formula has been plotted on the next page for $\kappa = 0.4$ and 0.2 (dash lines). It is clear that the dash lines will never mimic the profile constituted by eqs. (7.37) and (7.41), regardless of the κ value (because the dash-line curves do not have a "knee").



Problem 7.5. Using wall coordinates, we can write:

$$\int_0^{y_{VSL}} \rho\,\bar{u}\,dy = \int_0^{y_{VSL}^+} \rho\,u^+ u_*\,dy^+ \frac{\nu}{u_*} = \mu \underbrace{\int_0^{O(10)} u^+\,dy^+}_{\text{constant}}$$

Conclusion: the flowrate through the viscous sublayer is independent of the longitudinal position.

Problem 7.6. Starting out with the momentum equation (7.25),

$$\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{v}\,\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\,\frac{d\overline{P_\infty}}{dx} + \frac{1}{\rho}\,\frac{\partial \tau_{app}}{\partial y},$$

and recognizing that immediately outside the boundary layer

$$\frac{U_\infty^2}{2} + \frac{P_\infty}{\rho} = \text{constant,}$$

yields

$$\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{v}\,\frac{\partial \bar{u}}{\partial y} - U_\infty\,\frac{dU_\infty}{dx} = \frac{1}{\rho}\,\frac{\partial \tau_{app}}{\partial y} \qquad \text{(M)}$$

Invoking mass conservation, $\partial\bar{u}/\partial x + \partial\bar{v}/\partial y = 0$, the left-hand-side (LHS) can also be written as

$$\text{LHS} = \frac{\partial\left(\bar{u}^2\right)}{\partial x} + \frac{\partial\left(\bar{u}\,\bar{v}\right)}{\partial y} - U_\infty\,\frac{d\,U_\infty}{dx}$$

Performing the integral $\int_0^Y$ (LHS)dy, where Y > (boundary layer thickness), yields

$$\int_0^Y (\text{LHS})dy = \frac{d}{dx}\left[U_\infty^2 \int_0^Y \left(\frac{\bar{u}}{U_\infty}\right)^2 dy\right] + U_\infty\, v_Y - \underbrace{\bar{u}_0\,\bar{v}_0}_{\text{zero}} - U_\infty\,\frac{d\,U_\infty}{dx}\,Y$$

where, from the mass conservation integral

$$\int_0^Y \left(\frac{\partial\bar{u}}{\partial x} + \frac{\partial\bar{v}}{\partial y}\right) dy = 0$$

v_Y is given by

$$v_Y = -\int_0^Y \frac{\partial u}{\partial x}\,dy$$

In conclusion, the LHS-integral is

$$\int_0^Y (\text{LHS})\,dy = \frac{d}{dx}\left[U_\infty^2 \underbrace{\int_0^Y \left(\frac{\bar{u}}{U_\infty}\right)^2 dy}\right] - U_\infty\,\frac{d}{dx}\left[U_\infty^2 \underbrace{\int_0^Y \frac{\bar{u}}{U_\infty}\,dy}\right] - U_\infty\,\frac{dU_\infty}{dx}\,Y$$

Next, from the definitions of the displacement and momentum thicknesses,

$$\delta^* = \int_0^Y \left(1 - \frac{\bar{u}}{U_\infty}\right) dy, \qquad \theta = \int_0^Y \frac{\bar{u}}{U_\infty}\left(1 - \frac{\bar{u}}{U_\infty}\right) dy$$

we eliminate the integrals denoted by $\underbrace{\qquad}$ in the LHS-integral:

$$\int_0^Y (\text{LHS})\, dy = \frac{d}{dx}\left[U_\infty^2 \left(Y - \delta^* - \theta\right)\right] - U_\infty \frac{d}{dx}\left[U_\infty \left(Y - \delta^*\right)\right] - U_\infty \frac{d\,U_\infty}{dx} Y$$
$$= \left(-\delta^* - 2\theta\right) U_\infty \frac{d\,U_\infty}{dx} - U_\infty^2 \frac{d\theta}{dx}$$

Integrating the right-hand-side of eq. (M),

$$\int_0^Y (\text{RHS})\, dy = -\frac{\tau_0}{\rho}$$

and setting

$$\int_0^Y (\text{LHS})\, dy = \int_0^Y (\text{RHS})\, dy,$$

leads to the wanted result

$$\frac{d\theta}{dx} + \frac{\delta^* + 2\theta}{U_\infty}\frac{d\,U_\infty}{dx} = \frac{\tau_0}{\rho\, U_\infty^2}$$

This result is also valid for laminar boundary layer flow (note the identical forms of eq. (M) and the δ^*, θ definitions in laminar flow).

Problem 7.7. Taking the curve fit $f_u = 8.7(y^+)^{1/7}$, we obtain

$$u^+ = 8.7\, y^+ \text{, at any } y$$
$$U_\infty^+ = 8.7\, \delta^+, \text{ at } y = \delta$$

Dividing side by side, leads to

$$\frac{\bar{u}}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/7},$$

which is the velocity profile to be used in the momentum integral

$$\frac{d}{dx}\int_0^\delta \bar{u}\left(U_\infty - \bar{u}\right) dy = \frac{\tau_0}{\rho}$$

$$\frac{d}{dx}\left[U_\infty^2\, \delta \underbrace{\int_0^1 n^{1/7}\left(1 - n^{1/7}\right) dn}_{7/72}\right] = \frac{\tau_0}{\rho}$$

In conclusion, we obtain

$$\frac{d\delta}{dx} = \frac{72}{7} \frac{\tau_0}{\rho\, U_\infty^2}, \tag{M}$$

which is one relation between $\delta(x)$ and $\tau_0(x)$. The second relation between δ and τ_0 follows from writing $u^+ = 8.7\, y^+$ at $y = \delta$, namely

$$\frac{U_\infty}{(\tau_0/\rho)^{1/2}} = 8.7 \left(\frac{\delta (\tau_0/\rho)^{1/2}}{\nu}\right)^{1/7}$$

which yields,

$$\frac{\tau_0}{\rho} = \frac{U_\infty^{7/4}\, \nu^{1/4}}{(8.7)^{7/4}\, \delta^{1/4}} \tag{1}$$

Eliminating τ_0/ρ between eq. (1) and eq. (M) we obtain

$$\delta^{1/4} \frac{d\delta}{dx} = \left(\frac{\nu}{U_\infty}\right)^{1/4} \frac{72}{7\,(8.7)^{7/4}},$$

which, integrated from $\delta = 0$ at $x = 0$, yields

$$\frac{\delta}{x} = 0.373 \left(\frac{\nu}{U_\infty\, x}\right)^{1/5}$$

For $\tau_0/(\rho\, U_\infty{}^2)$, eq. (1) yields the following estimate

$$\frac{\tau_0}{\rho\, U_\infty^2} = (8.7)^{-7/4} \left(\frac{\nu}{U_\infty\, \delta}\right)^{1/4} = 0.0227 \left(\frac{U_\infty\, \delta}{\nu}\right)^{-1/4}$$

Finally, for δ^* and θ, we refer to their definitions

$$\delta^* = \int_0^\delta \left(1 - \frac{\bar{u}}{U_\infty}\right) dy = \delta \int_0^1 \left(1 - n^{1/7}\right) dn = \frac{\delta}{8}$$

$$\theta = \int_0^\delta \frac{\bar{u}}{U_\infty} \left(1 - \frac{\bar{u}}{U_\infty}\right) dy = \delta \int_0^1 n^{1/7} \left(1 - n^{1/7}\right) dn = \frac{7}{72}\,\delta$$

In conclusion, we obtain

$$\delta = 8\delta^* = \frac{72}{7}\,\theta$$

Problem 7.8. Taking $u^+ = C(y^+)^{1/m}$, and following exactly the same steps as in the preceding problem, we obtain

$$\frac{\bar{u}}{U_\infty} = \left(\frac{y}{\delta}\right)^{1/m}$$

$$\frac{d}{dx}\left[U_\infty^2\, \delta \int_0^1 n^{1/m}\left(1 - n^{1/m}\right) dn\right] = \frac{\tau_0}{\rho}$$

$$\frac{d\delta}{dx} = \frac{(m+1)(m+2)}{m}\, \frac{\tau_0}{\rho\, U_\infty^2} \qquad \text{(M)}$$

$$\frac{\tau_0}{\rho\, U_\infty^2} = C^{-\frac{2m}{m+1}} \left(\frac{\nu}{U_\infty\, \delta}\right)^{\frac{2}{m+1}}$$

Eliminating $\tau_0/(\rho U_\infty^2)$ between the last two equations, we have

$$\frac{\delta}{x} = \left[\frac{(m+3)(m+2)}{m}\right]^{\frac{m+1}{m+3}} C^{-\frac{2m}{m+3}} \left(\frac{U_\infty\, x}{\nu}\right)^{-\frac{2}{m+3}}$$

These results show that the numerical coefficients in the $\tau_0/(\rho U_\infty^2)$ and δ/x expressions depend on the particular curvefit, i.e. on the constants C and m.

Problem 7.9. We start with the relations

$$q''_{app} = q''_0 \qquad \text{and} \qquad \tau_{app} = \tau_0$$

and divide side by side

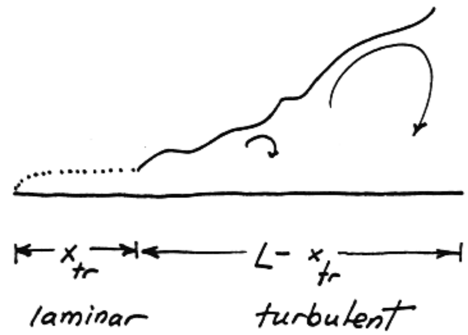
$$\frac{q''_{app}}{\tau_{app}} = \frac{q''_0}{\tau_0} = \frac{St_x\, \rho\, c_P\, U_\infty\, \Delta T}{\frac{1}{2}\, C_{f,x}\, \rho\, U_\infty^2} = \frac{c_P\, \Delta T}{U_\infty}\, Pr^{-2/3}, \text{ constant}$$

Problem 7.10. The definition of L-averaged Nusselt number tells us what we have to do:

$$Nu_{0-L} = \frac{q''_{0-L}\, L}{k\, \Delta T}$$

where

$$q''_{0-L} = \frac{1}{L}\left(\int_0^{x_{tr}} q''_{lam}\, dx + \int_0^{L-x_{tr}} q''_{turb}\, dx\right)$$



For the laminar section we write

$$\underset{Pr>1}{q''_{lam}}(x) = 0.332\,\frac{k\,\Delta T}{x}\,Pr^{1/3}\,Re_x^{1/2},$$

hence

$$\int_0^{x_{tr}} q''_{lam}\,dx = 0.664\,k\,\Delta T\,Pr^{1/3}\,Re_{x_{tr}}^{1/2}$$

We do the same for the turbulent section,

$$\underset{Pr>1}{q''_{turb}}(x) = \rho\,c_P\,U_\infty\,\Delta T\,\overbrace{St_x}^{}$$

$$\overbrace{Pr^{-2/3}\,\frac{1}{2}\,C_{f,x}}^{} \qquad \text{(Colburn)}$$

$$0.0296\,Re_x^{-1/5} \qquad \text{(Prandtl)}$$

hence

$$\int_0^{L-x_{tr}} q''_{turb}\,dx = 0.037\,\rho\,c_P\,U_\infty\,\Delta T\,Pr^{-2/3}\,(U_\infty/\nu)^{-1/5}\,(L - x_{tr})^{4/5}$$

In conclusion, we obtain

$$Nu_{0-L} = \frac{q''_{0-L}\,L}{k\,\Delta T} = Pr^{1/3}\left[0.664\,Re_{tr}^{1/2} + 0.037\,\left(Re_L - Re_{x_{tr}}\right)^{4/5}\right]$$

Problem 7.11. a) Starting with the local Stanton number definition (7.76), we write in order

$$
\begin{aligned}
St_x &= \frac{h_x}{\rho\, c_P\, U_\infty} = \frac{h_x}{\rho\, c_P\, U_\infty}\,\frac{x}{k}\,\frac{k}{x} \\
&= Nu_x\,\frac{\alpha}{U_\infty x} = Nu_x\,\frac{\nu}{U_\infty x}\,\frac{\alpha}{\nu} \qquad (1)\\
&= \frac{Nu_x}{Pe_x} = \frac{Nu_x}{Re_x\, Pr}
\end{aligned}
$$

b) According to eq. (2.92), the right side of the Colburn analogy (7.78) is

$$\frac{1}{2}\, C_{f,x} = 0.332\, Re_x^{-1/2} \qquad (2)$$

where $Re_x = U_\infty x/\nu$. The left side of the Colburn analogy can be estimated using eq. (1) and the Nu_x formula (2.103), which holds for $Pr \gtrsim 0.5$ fluids,

$$
\begin{aligned}
St_x\, Pr^{2/3} &= \frac{Nu_x}{Re_x\, Pr}\, Pr^{2/3} \\
&= \frac{0.332\, Pr^{1/3}\, Re_x^{1/2}}{Re_x\, Pr}\, Pr^{2/3} \qquad (3)\\
&= 0.332\, Re_x^{-1/2}
\end{aligned}
$$

Equations (2) and (3) show that the Colburn analogy $St_x\, Pr^{2/3} = (1/2)C_{f,x}$ applies to the laminar section of the boundary layer, provided the fluid is such that $Pr \gtrsim 0.5$.

Problem 7.12. a) The properties of water at 20°C are

$$\rho \cong 1\,\frac{g}{cm^3} \qquad c_P = 4.18\,\frac{kJ}{kg\,K} \qquad \nu = 0.01\,\frac{cm^2}{s}$$

$$k = 0.59\,\frac{W}{m\cdot K} \qquad Pr = 7.07$$

In order to calculate y,

$$y = y^+\,\nu\left(\frac{\tau_{w,x}}{\rho}\right)^{-1/2}$$

we must first evaluate $\tau_{w,x}/\rho$. For this we use eq. (7.60):

$$\frac{\tau_{w,x}}{\rho} = 0.0296\, U_\infty^2\, Re_x^{-1/5}$$

in which

$$\mathrm{Re}_x = \frac{U_\infty x}{\nu}$$
$$= 20 \frac{\text{cm}}{\text{s}} 600 \text{ cm} \frac{\text{s}}{0.01 \text{ cm}^2} = 1.2 \times 10^6 \quad \text{(turbulent)}$$

The result is

$$\frac{\tau_{w,x}}{\rho} = 0.0018 \, U_\infty^2$$
$$\left(\frac{\tau_{w,x}}{\rho}\right)^{1/2} = 0.0424 \times 20 \frac{\text{cm}}{\text{s}} = 0.849 \frac{\text{cm}}{\text{s}}$$
$$y = y^+ \nu \left(\frac{\tau_{w,x}}{\rho}\right)^{-1/2}$$
$$= 2.7 \times 0.01 \frac{\text{cm}^2}{\text{s}} \frac{\text{s}}{0.849 \text{ cm}} = 0.3 \text{ mm}$$

b) The boundary layer thickness can be evaluated based on eq. (7.58):

$$\delta = 0.37 \, x \, \mathrm{Re}_x^{-1/5}$$
$$= 0.37 \times 6\text{m} \left(1.2 \times 10^6\right)^{-1/5} = 13.5 \text{ cm}$$

In the laminar regime, the corresponding thickness would be the δ given by eq. (2.85):

$$\delta = 4.92 \, x \, \mathrm{Re}_x^{-1/2}$$
$$= 4.92 \times 6\text{m} \left(1.2 \times 10^6\right)^{-1/2} = 2.7 \text{ cm}$$

We see that the laminar boundary layer would have been much thinner (at x) than the real (turbulent) boundary layer.

c) Finally, for the x-averaged heat transfer coefficient we rely on eq. (7.78'):

$$\mathrm{Nu}_x = 0.0296 \, \mathrm{Pr}^{1/3} \, \mathrm{Re}_x^{4/5}$$
$$\overline{\mathrm{Nu}}_x = 0.037 \, \mathrm{Pr}^{1/3} \, \mathrm{Re}_x^{4/5}$$
$$= 0.037 \, (7.07)^{1/3} \left(1.2 \times 10^6\right)^{4/5}$$
$$= 5184$$
$$\bar{h} = \overline{\mathrm{Nu}}_x \frac{k}{x}$$
$$= 5184 \times 0.59 \frac{\text{W}}{\text{m} \cdot \text{K}} \frac{1}{6\text{m}} = 509 \frac{\text{W}}{\text{m}^2 \text{K}}$$

A more accurate estimate can be obtained by accounting for the laminar tip of the boundary layer, as in Problem 7.10.

Problem 7.13. a) We begin by recognizing the wall averaged shear stress, the total tangential force experienced by the wall, and the mechanical power spent on dragging the flat surface through the fluid:

$$\bar{\tau} = 0.037\ \rho\ U^2\ Re_L^{-1/5}$$

$$F' = \bar{\tau}\ L$$

$$\mathbf{P} = F'U = 0.037\ \rho\ U^3\ L\left(\frac{\nu}{UL}\right)^{1/5}$$

If $(\ )_c$ and $(\ )_h$ represent the cold-wall and hot-wall conditions, the dissipated power changes according to the ratio

$$\frac{\mathbf{P}_h}{\mathbf{P}_c} = \frac{\rho_h}{\rho_c}\left(\frac{\nu_h}{\nu_c}\right)^{1/5}$$

b) When the wall is heated, the water film temperature is (90°C + 10°C)/2 = 50°C, with the corresponding properties

$$\rho_h \cong 1\ \frac{g}{cm^3} \qquad\qquad \mu_h = 0.00548\ \frac{g}{cm\cdot s}$$

The ratio

$$\frac{\mathbf{P}_h}{\mathbf{P}_c} = \frac{1}{1}\left(\frac{0.00548}{0.013}\right)^{1/5} = 0.84$$

shows that by heating the wall to 90°C we can reduce the drag power by 16 percent.

c) The power that has been saved by heating the water boundary layer is

$$\mathbf{P}_c - \mathbf{P}_h = 0.037\ \rho\ U^{14/5}\ L^{4/5}\ \nu_c^{1/5}\left[1 - \left(\frac{\nu_h}{\nu_c}\right)^{1/5}\right] \tag{1}$$

The electric power spent on heating the wall is given sequentially by

$$\frac{\bar{h}\ L}{k_h} = 0.037\ Pr_h^{1/3}\left(\frac{UL}{\nu_h}\right)^{4/5}$$

$$\overline{q''}\ L = 0.037\ k_h\ \Delta T\ Pr_h^{1/3}\left(\frac{UL}{\nu_h}\right)^{4/5} \tag{2}$$

in which (k, Pr, ν) are also evaluated at the 50°C film temperature,

$$k_h = 0.64\ \frac{W}{m\cdot K} \qquad\qquad Pr_h = 3.57$$

By dividing eqs. (1) and (2) we obtain a dimensionless measure of how effectively the heating of the wall has been converted into **P** savings:

$$\frac{\mathbf{P_c - P_h}}{q'' L} = \frac{U^2}{c\,\Delta T}\, Pr_h^{2/3} \left(\frac{\nu_c}{\nu_h}\right)^{1/5} \left[1 - \left(\frac{\nu_h}{\nu_c}\right)^{1/5}\right]$$

In this expression we substitute c = 4.18 kJ/kg·K, $\Delta T = 90°C - 10°C = 80°C$ and $(\nu_h/\nu_c)^{1/5} = 0.84$, and obtain

$$\frac{\mathbf{P_c - P_h}}{q'' L} = \left(\frac{U}{867 \text{ m/s}}\right)^2$$

This shows that the drag power savings are much smaller than the heating power investment when the ship speed is of order 10 m/s.

Problem 7.14. The relevant properties of air at the film temperature (20°C) are

$$\rho = 1.205 \times 10^{-3} \frac{\text{g}}{\text{cm}^3}, \qquad \nu = 0.15 \frac{\text{cm}^2}{\text{s}}$$

When the iceberg drifts steadily, the sea water drag over the bottom surface matches the air drag over the top surface. Therefore, we can write in order

$$F_{D,air} = F_{D,water}$$

$$\left(\bar{\tau}_{w,L} A\right)_a = \left(\bar{\tau}_{w,L} A\right)_w$$

$$0.037\, \rho_a U_a^2 \left(\frac{\nu_a}{U_a L}\right)^{1/5} = 0.037\, \rho_w U_w^2 \left(\frac{\nu_w}{U_w L}\right)^{1/5}$$

and the last expression yields

$$\begin{aligned} \frac{U_{air}}{U_{water}} &= \left(\frac{\rho_w}{\rho_a}\right)^{5/9} \left(\frac{\nu_w}{\nu_a}\right)^{1/9} \\ &= \left[\frac{1 \text{ kg}}{(10 \text{ cm})^3} \frac{\text{cm}^3}{1.205 \times 10^{-3} \text{g}}\right]^{5/9} \left(\frac{0.015 \text{ cm}^2/\text{s}}{0.15 \text{ cm}^2/\text{s}}\right)^{1/9} \\ &= 32.4 \end{aligned}$$

and since U_{water} = 10 cm/s, we conclude that the wind velocity is

$$U_{air} = 32.4 \times 10 \frac{\text{cm}}{\text{s}} = 3.24 \frac{\text{m}}{\text{s}} = 11.7 \frac{\text{km}}{\text{h}}$$

Problem 7.15. The air properties are evaluated at the film temperature,

$$\nu = 0.15 \frac{cm^2}{s}, \qquad k = 2.5 \times 10^{-4} \frac{W}{cm\,K}, \qquad Pr = 0.72$$

The air boundary layer is turbulent,

$$Re_L = 3.24 \frac{m}{s}\, 100\,m \frac{s}{0.15\,cm^2} \cong (2.16)\, 10^7$$

therefore, the $\overline{Nu}_L$ formula is eq. (7.78"):

$$\begin{aligned}\overline{Nu}_L &\cong 0.037\, Pr^{1/3}\, Re_L^{4/5} \\ &= 0.037\,(0.72)^{1/2}\left(2.16 \times 10^7\right)^{4/5} \\ &= 2.44 \times 10^4\end{aligned}$$

The corresponding L-averaged heat flux into the top surface of the iceberg is

$$\begin{aligned}\bar{q}''_L &= \bar{h}_L\, \Delta T = \overline{Nu}_L \frac{k}{L} \Delta T \\ &= 2.44 \times 10^4\; 2.5 \times 10^{-4} \frac{W}{cm\,K} \frac{1}{100\,m}\, 40°C \\ &= 244 \frac{W}{m^2}\end{aligned}$$

The melting rate associated with this heat flux is

$$\begin{aligned}\frac{dH}{dt} &= \frac{\bar{q}''_L}{\rho\, h_{sf}} = 244 \frac{W}{m^2} \frac{(0.1\,m)^3}{1\,kg} \frac{kg}{333.4\,kJ} \\ &= 0.73 \times 10^{-6} \frac{m}{s} = 2.6 \frac{mm}{h}\end{aligned}$$

Problem 7.16. The relevant properties of air at 0°C are

$$\nu = 0.132 \frac{cm^2}{s}, \qquad \alpha = 0.184 \frac{cm^2}{s}, \qquad Pr = 0.72$$

$$\rho\, c_P = \frac{k}{\alpha} = 2.4 \times 10^{-4} \frac{W}{cm\,K} \frac{1}{0.184\,cm^2/s} = 0.0013 \frac{J}{cm^3 K}$$

In order to arrive at the heat transfer coefficient for the external surface of the window, we calculate in order:

$$Re_x = \frac{U_\infty x}{\nu} = 15\ \tfrac{m}{s}\ 60\ m\ \frac{1}{0.132\ cm^2/s}$$
$$= 6.8 \times 10^7$$

$$\frac{1}{2} C_{f,x} = 0.0296\ Re_x^{-1/5} = 8.03 \times 10^{-4}$$

$$St_x = \frac{1}{2} C_{f,x}\ Pr^{-2/3} = 8.03 \times 10^{-4}\ (0.72)^{-2/3}$$

$$\cong 0.001$$

$$h_x = St_x\ \rho\ c_p U_\infty = 0.001 \times 0.0013\ \frac{J}{cm^3 K}\ 15\ \tfrac{m}{s}$$
$$= 19.6\ \frac{W}{m^2 K}$$

This h_x value is a lower bound for the actual heat transfer coefficient because

i) equation (7.60) underpredicts the skin friction coefficient at large Reynolds numbers,

ii) The external surface of the building is not smooth, and

iii) the free stream U_∞ is not completely smooth, i.e. without eddies.

Problem 7.17. a) The properties of 10°C water are

$$\rho \cong 1\ \frac{g}{cm^3} \qquad n = 0.013\ \frac{cm^2}{s}$$

$$k = 0.58\ \frac{W}{m\ K} \qquad Pr = 9.45$$

We calculate the drag force by using Fig. 7.9:

$$Re_D = \frac{U_\infty D}{\nu} = 1\ \tfrac{m}{s}\ 0.15m\ \frac{s}{0.013 \times 10^{-4}\ m^2}$$
$$= 1.15 \times 10^5$$

$$C_D \cong 1.3 \quad \text{(Fig. 7.9)}$$

$$A = LD = 0.75m\ 0.15m$$

$$F_D = C_D A\ \frac{1}{2}\ r\ U_\infty^2$$

$$= 1.3 \times 0.75m\ 0.15m\ \frac{1}{2}\ 1\ \frac{10^{-3}\ kg}{10^{-6}\ m^3}\left(1\ \tfrac{m}{s}\right)^2 = 73\ N$$

b) How large is a force of 73 N? The weight (<u>in air</u>) of the same portion of the leg ($L \times D$, the portion that would be immersed in the river) is

$$
\begin{aligned}
W &= \rho_{meat} \frac{\pi}{4} D^2 L g \\
&\cong 1060 \frac{kg}{m^3} \frac{\pi}{4} (0.15 m)^2 \, 0.75 m \, 9.81 \frac{m}{s^2} \\
&\cong 138 \text{ N}
\end{aligned}
$$

The drag force is about half the weight of the immersed portion of the leg.

c) The heat transfer coefficient is furnished by eq. (7.100), in which we substitute $Re_D = 1.15 \times 10^5$ and $Pr = 9.45$:

$$
\begin{aligned}
\overline{Nu}_D &= 0.3 + 380 \, (1 + 0.571)^{4/5} = 545.7 \\
\bar{h} &= \overline{Nu}_D \frac{k}{D} = 545.7 \times 0.58 \frac{W}{m\,K} \frac{1}{0.15 m} \\
&= 2110 \frac{W}{m^2 K}
\end{aligned}
$$

d) $q = \bar{h} A_{lateral} \Delta T$

$$
\begin{aligned}
&= \bar{h} \, \pi D L \, \Delta T = 2110 \frac{W}{m^2 K} \pi \, 0.15 m \, 0.75 m \, 1 K \\
&= 746 \text{ W}
\end{aligned}
$$

<u>Problem 7.18</u>. a) The properties of air at 20°C are

$$
\rho = 1.205 \frac{kg}{m^3} \qquad \nu = 0.15 \frac{cm^2}{s}
$$

The drag force can be estimated based on Fig. 7.9:

$$
\begin{aligned}
Re_D &= \frac{U_\infty D}{\nu} = 35.76 \frac{m}{s} \, 0.074 m \frac{s}{0.15 \times 10^{-4} m^2} \\
&= 1.76 \times 10^5
\end{aligned}
$$

$C_D \cong 0.42$ (Fig. 7.9)

$$
\begin{aligned}
F_D &= C_D A \frac{1}{2} \rho U_\infty^2 \quad \left(\text{where } A = \frac{\pi D^2}{4}\right) \\
&= 0.42 \frac{\pi}{4} (0.074 m)^2 \frac{1}{2} \, 1.205 \frac{kg}{m^3} \left(35.76 \frac{m}{s}\right)^2 \\
&= 1.39 \text{ N}
\end{aligned}
$$

b) The weight of the baseball is comparable to the drag force:

$$mg = 0.145 \text{ kg } 9.81 \frac{\text{m}}{\text{s}^2} = 1.42 \text{ N}$$

c) Between the pitcher's mound and the catcher's mitt, the kinetic energy of the ball $[(1/2)\, mU^2]$ drops by an amount equal to the work done against the atmosphere ($F_D \cdot x$). The first law of thermodynamics for the process 1 (pitcher) $\rightarrow$ 2 (catcher) is

$$-F_D\, x = \frac{1}{2}\, m\, U_2^2 - \frac{1}{2}\, m\, U_1^2$$

$$U_1^2 - U_2^2 = \frac{2}{m}\, F_D\, x$$

$$= \frac{2}{0.145 \text{ kg}}\; 1.39 \text{ N } 18.5\text{m} = 354.7 \left(\frac{\text{m}}{\text{s}}\right)^2$$

$$U_2^2 = \left(35.76 \frac{\text{m}}{\text{s}}\right)^2 - 354.7 \left(\frac{\text{m}}{\text{s}}\right)^2 = 924.1 \left(\frac{\text{m}}{\text{s}}\right)^2$$

$$U_2 = 30.4 \frac{\text{m}}{\text{s}}$$

The final horizontal velocity is 85 percent of the initial value.

Problem 7.19. a) In eq. (7.104) we evaluate all the air properties at $T_\infty = 20^\circ\text{C}$, except $\mu_w = \mu(T_w = 30^\circ\text{C})$:

$$k_\infty = 0.025 \frac{\text{W}}{\text{m K}} \qquad \mu_\infty = 1.81 \times 10^{-5} \frac{\text{kg}}{\text{s m}}$$

$$\nu_\infty = 0.15 \frac{\text{cm}^2}{\text{s}} \qquad \mu_w = 1.86 \times 10^{-5} \frac{\text{kg}}{\text{s m}}$$

$$Pr_\infty = 0.72$$

The Reynolds number corresponds to the upper end of the Re_D range in which eq. (7.104) is valid,

$$Re_D = \frac{U_\infty D}{\nu_\infty} = 22.35 \frac{\text{m}}{\text{s}}\; 0.07\text{m}\; \frac{\text{s}}{0.15 \times 10^{-4} \text{ m}^2}$$

$$= 1.043 \times 10^5$$

therefore we obtain, in order,

$$\overline{Nu}_D = 2 + \left(0.4\, Re_D^{1/2} + 0.06\, Re_D^{2/3}\right) Pr_\infty^{0.4} \left(\frac{\mu_\infty}{\mu_w}\right)^{1/4}$$

$$= 2 + 262.1\, (0.72)^{0.4} \left(\frac{1.81}{1.86}\right)^{1/4}$$

$$= 230.3$$

$$\bar{h} = \overline{Nu}_D \frac{k_\infty}{D} = 230.3 \times 0.025 \frac{W}{m\,K} \frac{1}{0.07m}$$

$$= 82.25 \frac{W}{m^2\,K}$$

$$A = \pi D^2 = \pi\,(0.07m)^2 = 0.0154\ m^2$$

$$q = \bar{h}\,A\left(T_w - T_\infty\right) = 82.25 \frac{W}{m^2\,K}\, 0.0154\ m^2\, (30 - 20)\ K$$

$$= 12.7\ W$$

The time of travel and the total heat transfer (ball $\rightarrow$ air) are

$$t = \frac{x}{U} = \frac{18.5m}{22.35\ m/s} = 0.83s$$

$$Q = qt = 12.7\ W\ 0.83s = 10.5\ J$$

b) The depth to which the cooling effect penetrates into the leather cover is

$$\delta \sim (\alpha_{leather}\, t)^{1/2} \sim \left(0.001 \frac{cm^2}{s}\, 0.83s\right)^{1/2}$$

$$\sim 0.29\ mm$$

The temperature drop that would be experienced by the 0.29 mm skin of the ball is

$$\Delta T_w = \frac{Q}{(mc)_{skin}} = \frac{Q}{(\rho c)_{leather}\, A\delta}$$

$$\cong \frac{10.5\ J}{860 \frac{kg}{m^3}\, 1.5 \frac{kJ}{kg\,K}\, 0.0154\ m^2\, 0.00029m} \cong 1.8°C$$

This temperature drop is small compared with the ball-air temperature difference $T_w - T_\infty = 10°C$, therefore the assumption that T_w is constant between the pitcher and the catcher is reasonable.

Problem 7.20. The air properties that are required by a calculation based on eq. (7.104) are

$$\mu_w = 2 \times 10^{-5} \frac{kg}{s\,m} \qquad \nu_\infty = 0.141 \frac{cm^2}{s}$$

$$\mu_\infty = 1.76 \times 10^{-5} \frac{kg}{s\,m} \qquad k_\infty = 0.025 \frac{W}{m\,K}$$

$$Pr_\infty = 0.72$$

We begin with the Reynolds number,

$$Re_D = \frac{U_\infty D}{\nu_\infty} = 2 \frac{m}{s}\, 0.06m \frac{s}{0.141 \times 10^{-4}\, m^2}$$

$$= 8511$$

which falls in the range where eq. (7.104) is valid:

$$\overline{Nu}_D = ... = 2 + 61.91\,(0.72)^{0.4} \left(\frac{1.76}{2}\right)^{1/4} = 54.58$$

$$\bar{h} = \overline{Nu}_D \frac{k_\infty}{D} = 54.58 \times 0.025 \frac{W}{m\,K} \frac{1}{0.06m}$$

$$= 22.74 \frac{W}{m^2\,K}$$

$$A = \pi D^2 = \pi (0.06m)^2 = 0.0113\; m^2$$

$$q = \bar{h}\, A \left(T_w - T_\infty\right)$$

$$= 22.74 \frac{W}{m^2\,K}\, 0.0113\; m^2\, (60 - 10)\, K = 12.85\; W$$

Problem 7.21. When the glass bead reaches its terminal speed, the weight of the bead (mg) is balanced by the drag force (F_D). This balance provides a relationship between the drag coefficient C_D and the unknown speed U_∞,

$$F_D = C_D A \frac{1}{2} \rho_a U_\infty^2 = mg$$

$$C_D \frac{\pi}{4} D^2 \frac{1}{2} \rho_a U_\infty^2 = \rho \frac{4\pi}{3} \left(\frac{D}{2}\right)^3 g$$

$$C_D = \frac{\rho}{\rho_a} \frac{4}{3} \frac{gD}{U^2}$$

in which we substitute

$\rho = 2800\ \mathrm{kg/m^3}$ (density of glass)

$\rho_a = 1.205\ \mathrm{kg/m^3}$ (density of air at 20°C)

$g = 9.81\ \mathrm{m/s^2}$

$D = 0.0005\mathrm{m}$

The end result is

$$C_D = 15.2\left(\frac{\mathrm{m/s}}{U_\infty}\right)^2 \tag{1}$$

A second relation involving the unknown U_∞ is the definition of Re_D on the abscissa of Fig. 7.9,

$$Re_D = \frac{U_\infty D}{\nu_a}$$

in which $\nu_a = 0.15\ \mathrm{cm^2/s}$ is the kinematic viscosity of air at 20°C. Numerically, the Re_D definition reduces to

$$Re_D = 33.3\,\frac{U_\infty}{\mathrm{m/s}} \tag{2}$$

Now we must find the proper U_∞ value so that the C_D and Re_D values calculated with eqs. (1) and (2) represent a point on the "sphere" curve in Fig. 7.9. We do this by trial and error, proceeding from left to right in the following table:

pick U_∞	calculate C_D with eq. (1)	read Re_D off Fig. 7.9	calculate U_∞ with eq. (2)
3.5 m/s	1.24	85	2.44 m/s
3.8 m/s	1.05	123	3.7 m/s
3.9 m/s	1	138	4.14 m/s
4 m/s	0.95	150	4.5 m/s

The correct U_∞ value is located between the second and third guesses, more exactly at

$$U_\infty \cong 3.8\ \frac{\mathrm{m}}{\mathrm{s}}, \quad \text{for which} \quad Re_D \cong 127 \tag{2}$$

In vacuum, the bead velocity would be increasing as gt. The approximate time when the terminal speed is reached (i.e. when the air drag effect becomes important) is when gt is comparable with the just calculated U_∞,

$$gt \sim U_\infty$$

$$t \sim \frac{U_\infty}{g} = \frac{3.8 \text{ m/s}}{9.8 \text{ m/s}^2} \cong 0.4 \text{ s}$$

The bead falls for a total of at least 10m/(3.8 m/s) = 2.6s, therefore it is reasonable to assume that during most of this travel its speed has the terminal value U_∞.

We can now evaluate the average heat transfer coefficient using eq. (7.104), in which we substitute $Re_D = 127$ and

$\mu_w = 3.58 \times 10^{-5}$ kg/s·m (air viscosity at 500°C)

$\mu_\infty = 1.81 \times 10^{-5}$ kg/s·m (air viscosity at 20°C)

Pr = 0.72 (air Prandtl number at 20°C)

In the end we obtain $\overline{Nu}_D \cong 6.47$, which means that

$$h = \frac{k_a}{D} Nu = \frac{0.025 \text{ W}}{\text{m·K}} \frac{6.47}{0.0005\text{m}} = 324 \frac{\text{W}}{\text{m}^2\text{K}}$$

Treating the glass bead as a lumped capacitance of temperature T, we write the first law for the bead as a closed thermodynamic system,

$$\rho V c \frac{d(T - T_\infty)}{dt} = - hA_s (T - T_\infty)$$

and recognize that $V = (4\pi/3)(D/2)^3$, and $A_s = \pi D^2$. Integrating this equation from t = 0, where $T = T_i = 500°C$, we obtain

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp\left(- \frac{6 \, ht}{\rho c \, D}\right)$$

in which t = 2.6s and, for glass, c = 0.8 kJ/kg·K. Numerically, this yields in order

$$\frac{T - T_\infty}{T_i - T_\infty} = \exp(-4.51) = 0.011$$

$$T = 20°C + 0.011 \, (500-20)°C$$

$$= 25.3°C.$$

In conclusion, by the time it falls to the ground, the glass bead has almost the same temperature as the surrounding air.

Problem 7.22. a) The air properties are evaluated at the bulk temperature of 300°C,

$$k = 0.045 \frac{W}{m\,K} \qquad Pr = 0.68$$

$$n = 0.481 \frac{cm^2}{s} \qquad Pr_w = 0.72 \text{ (air at 30°C)}$$

Figure 7.11 shows that $C_n \cong 1$ for an array with 20 rows. For the aligned array we calculate,

$$U_{max} = U_\infty \frac{X_t}{X_t - D}$$

$$= 2 \frac{m}{s} \frac{7}{7-4} = 4.67 \frac{m}{s}$$

$$Re_D = U_{max} \frac{D}{v}$$

$$= 4.67 \times 100 \frac{cm}{s} \frac{4\ cm}{0.481\ cm^2/s} = 3881$$

$$\overline{Nu}_D = 0.27\, C_n\, Re_D^{0.63}\, Pr^{0.36} \left(\frac{Pr}{Pr_w}\right)^{1/4}$$

$$= 0.27 \times 1 \times (3881)^{0.63}\, (0.68)^{0.36} \left(\frac{0.68}{0.72}\right)^{1/4}$$

$$= 42.26$$

$$\bar{h} = \frac{k}{D} \overline{Nu}_D$$

$$= 0.045 \frac{W}{m\,K} \frac{42.26}{0.04m} = 47.54 \frac{W}{m^2K}$$

b) When the cylinders are in a staggered array, we must decide which space (flow area) is narrower, between two cylinders in the same row

$$X_t - D = 3\ cm$$

or between two cylinders aligned with the direction "a" in Fig. 7.10 (right),

$$\left[X_l^2 + \left(\frac{1}{2} X_t\right)^2\right]^{1/2} - D = 3.83\ cm$$

The space between two cylinders in the same row is the narrower; this means that U_{max} and Re_D are the same as in the first part of the problem. By using eq. (7.112) we find that the heat transfer coefficient in the staggered array is approximately 30 percent larger than in the aligned array,

$$\overline{Nu}_D = 0.35\ (3881)^{0.6}\ (0.68)^{0.36} \left(\frac{0.68}{0.72}\right)^{1/4} \left(\frac{7\ \text{cm}}{7\ \text{cm}}\right)^{0.2}$$

$$= 54.78$$

$$\bar{h} = \frac{k}{D}\ \overline{Nu}_D = 61.63\ \frac{W}{m^2\ K}$$

Problem 7.23. The number of tube rows in the longitudinal direction is

$$n_l = \frac{L}{X_l} = \frac{0.5m}{0.0203m} \cong 25$$

The tube array is characterized also by

$$X_t^* = \frac{X_t}{D} = \frac{24.8\ \text{mm}}{10.7\ \text{mm}} = 2.32$$

$$X_l^* = \frac{X_l}{D} = \frac{20.3\ \text{mm}}{10.7\ \text{mm}} = 1.90$$

$$X_t^*/X_l^* = 1.22$$

$$\chi \cong 1 \qquad \text{(Fig. 7.13 insert)}$$

The number of tubes in the transverse direction is

$$n_t = \frac{0.5m}{0.0248m} \cong 20$$

In order to calculate U_{max}, we must first determine the minimum free-flow area A_c. The latter depends on the spacings between two adjacent tubes:

transverse (vertical) spacing

$$S_t = (24.8 - 10.7)\ \text{mm} = 14.10\ \text{mm}$$

diagonal spacing

$$S_d = \left[20.3^2 + \left(\frac{1}{2}\ 24.8\right)^2\right]^{1/2} \text{mm} - 10.7\ \text{mm} = 13.09\ \text{mm}$$

The flow blade that passes through S_t must pass through $2S_d$. Since $S_t < 2\ S_d$, we conclude that the transverse spacing pinches the flow the most,

$$A_c = S_t\ (\text{width})\ n_t$$

$$= 0.0141m \times 0.5m \times 20 = 0.141\ m^2$$

$$U_{max} = \frac{\dot{m}}{\rho A_c} = \frac{1500\,kg}{3600s}\ \frac{m^3}{0.746\,kg}\ \frac{1}{0.141\,m^2}$$
$$= 3.96\,\frac{m}{s}$$

$$Re_D = \frac{U_{max} D}{\nu} = 3.96\,\frac{m}{s}\,0.0107m\,\frac{1}{0.346 \times 10^{-4}\,m^2/s}$$
$$= 1225$$

$f \cong 0.42$ (Fig. 7.13)

Now we have all we need in order to use eq. (7.113):

$$\Delta P = n_l f \chi \frac{1}{2} \rho U_{max}^2$$
$$= 25 \times 0.42 \times 1 \times \frac{1}{2} \times 0.746\,\frac{kg}{m^3}\left(3.96\,\frac{m}{s}\right)^2$$
$$= 61.4\,\frac{N}{m^2}$$

Problem 7.24. a) The properties of air at 100°C are

$$\rho = 0.946\,\frac{kg}{m^3} \qquad \nu = 0.23\,\frac{cm^2}{s}$$

The pressure drop along the square array can be calculated by using eq. (7.113) and Fig. 7.12:

$$U_{max} = U_\infty \frac{X_t}{X_t - D} = 3\,\frac{m}{s}\ \frac{9\,cm}{(9-5)\,cm} = 6.75\,\frac{m}{s}$$

$$Re_D = U_{max}\frac{D}{\nu} = 6.75\,\frac{m}{s}\ \frac{0.05m}{0.23 \times 10^{-4}\,m^2/s} = 1.47 \times 10^4$$

$$X_l^* = \frac{9\,cm}{5\,cm} = 1.8$$

$f \cong 0.25$ (Fig. 7.12)

$$\frac{X_t^* - 1}{X_l^* - 1} = 1$$

$\chi = 1$ (Fig. 7.12 insert)

$$\Delta P = n_l f \chi \frac{1}{2} \rho U_{max}^2$$

$$= 21 \times 0.25 \times 1 \times \frac{1}{2} \times 0.946 \frac{kg}{m^3} \left(6.75 \frac{m}{s}\right)^2$$

$$= 113 \frac{N}{m^2}$$

b) The staggered array with the same X_t and X_l will have the same U_{max}, Re_D and X_t^* as in part (a):

$$U_{max} = 6.75 \frac{m}{s}, \qquad Re_D = 1.47 \times 10^4, \qquad X_t^* = 1.8$$

The f and χ factors provided by Fig. 7.13 will have different values:

$$f \cong 0.33$$

$$\chi \cong 1.02, \text{ because } \frac{X_t^*}{X_l^*} = 1$$

Equation (7.113) yields

$$\Delta P = 21 \times 0.33 \times 1.02 \times \frac{1}{2} \times 0.946 \frac{kg}{m^3} \left(6.75 \frac{m}{s}\right)^2$$

$$= 152 \frac{N}{m^2}$$

In conclusion, the pressure drop along the staggered tubes is 35 percent greater than the pressure drop along the aligned tubes.

Problem 7.25. We have four unknowns (τ_0, q_0'', V_1, δ) and four equations:

$$\frac{\tau_0}{\rho} = \frac{\delta}{8} g\beta (T_0 - T_\infty) \tag{1}$$

$$0.0366 (T_0 - T_\infty) \frac{d}{dy} (V_1 \delta) = \frac{q_0''}{\rho c_P} \tag{2}$$

$$\tau_0 = 0.0225 \rho V_1^2 \left(\frac{V_1 \delta}{\nu}\right)^{-1/4} \tag{3}$$

$$\frac{q_0''}{(T_0 - T_\infty) \rho c_P V_1} Pr^{2/3} = \frac{\tau_0}{\rho V_1^2} \tag{4}$$

An expedient way to solve the problem is to set

$$V_1 = A\, y^m, \qquad \delta = B\, y^n, \qquad \tau_0 = C\, y^p, \qquad q_0'' = D\, y^q$$

and to use eqs. (1)-(4) to see what values (m, n, p, q) work. This does not take much effort, and the conclusion is

$$V_1 = A\, y^{1/2}, \qquad \delta = B\, y^{7/10}, \qquad \tau_0 = C\, y^{7/10}, \qquad q_0'' = D\, y^{1/5}$$

Using the above substitutions, the original system (1)-(4) becomes one for the new unknowns (A, B, C, D), which are all constants:

$$\begin{aligned}
&\frac{C}{B} = \frac{1}{8}\,\rho g \beta\, \Delta T &&, \text{ or } \quad \frac{C}{B} = x_1 \\
&\frac{D}{AB} = 0.0439\, \Delta T\ c_P &&, \text{ or } \quad \frac{D}{AB} = x_2 \\
&C A^{-7/4} B^{1/4} = 0.0225\, \rho\, \nu^{1/4} &&, \text{ or } \quad C A^{-7/4} B^{1/4} = x_3 \\
&\frac{DA}{C} = Pr^{-2/3}\, \Delta T\, c_P &&, \text{ or } \quad \frac{DA}{C} = x_4
\end{aligned}$$

where x_1, x_2, x_3, and x_4 are used as shorthand. The solution emerges rapidly:

$$\begin{aligned}
C &= x_1 B \\
A &= (x_1\, x_4 / x_2)^{1/2} \\
B^{5/4} &= \frac{x_3}{x_1}\left(\frac{x_1\, x_4}{x_2}\right)^{7/8} \\
D &= x_1^{2/5}\, x_2^{-1/5}\, x_3^{4/5}\, x_4^{6/5}
\end{aligned}$$

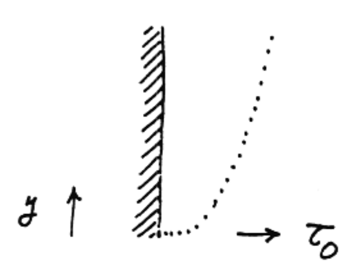
The last result is what we need to evaluate the Nusselt number:

$$\frac{q_0''}{\Delta T}\,\frac{y}{k} = \frac{y}{\Delta T\, k}\, y^{1/5} D = \ldots = 0.0391\, Pr^{-1/5}\, Ra_y^{2/5}$$

The wall shear stress τ_0 can be evaluated in the same way:

$$\tau_0 = C\, y^{7/10} = y^{7/10} \underbrace{\left(x_1^{9/10}\, x_2^{-7/10}\, x_3^{4/5}\, x_4^{7/10}\right)}_{\text{constant}}$$

Since $7/10 < 1$, the $\tau_0(y)$ distribution is as shown below.



Problem 7.26. This problem can be solved by following the steps outlined in Problem 7.25. To begin with, the left-hand-side of eq. (7.118) is

$$\frac{d}{dy}\int_0^1 V_1^2\,\delta\,(1-\beta)^8\,\beta^{2/7}\,d\beta = 0.275\,\frac{d}{dy}\left(V_1^2\,\delta\right)$$

The four equations needed for V_1, δ, τ_0 and q_0'' are

$$(2.2)\,\frac{d}{dy}\left(V_1^2\,\delta\right) = \delta\,g\beta\,\Delta T \qquad (1)$$

$$(0.0366)\,\Delta T\,\frac{d}{dy}\left(V_1\,\delta\right) = \frac{q_0''}{\rho\,c_P} \qquad (2)$$

$$\tau_0 = 0.0225\,\rho\,V_1^2\left(V_1\,\delta/\nu\right)^{-1/4} \qquad (3)$$

$$\frac{q''}{\rho\,c_P\,\Delta T} \sim 10^{-1}\,Pr^{-1/2}\left(\frac{\tau_0}{\rho}\right)^{1/2} \qquad (4)$$

Seeking power-law expressions in y, we find

$$V_1 = A\,y^{1/2}, \quad \delta = B\,y^{15/18}, \quad \tau_0 = C\,y^{2/3}, \quad q_0'' = D\,y^{1/3}$$

and, from eqs. (1)-(4),

$$\begin{aligned}
&A = 0.498\,(g\beta\,\Delta T)^{1/2} &&, \text{ or } && A = \text{known}\\
&\frac{D}{AB} = 0.0448\,\rho\,c_P\,\Delta T &&, \text{ or } && \frac{D}{AB} = x_2\\
&CA^{-7/4}\,B^{1/4} = 0.0225\,\rho\,\nu^{1/4} &&, \text{ or } && CA^{-7/4}\,B^{1/4} = x_3\\
&DC^{-1/2} \sim 0.1\,Pr^{-1/2}\,\rho^{1/2}\,c_P\,\Delta T &&, \text{ or } && DC^{-1/2} = x_4
\end{aligned}$$

Out of this system we extract D:

$$D = A^{8/9}\,x_2^{1/9}\,x_3^{4/9}\,x_4^{8/9}$$

from which we determine the Nusselt number

$$\frac{q_0''}{\Delta T}\,\frac{y}{k} = \frac{y}{\Delta T\,k}\,D\,y^{1/3} = \ldots\ldots = \left(\sim 10^{-2}\right)Pr^{1/9}\,Ra_y^{4/9}$$

which differs somewhat from Eckert & Jackson's $Pr \to 0$ limit,

$$Nu_y \to 0.0295\,Pr^{1/15}\,Ra_y^{2/5}$$

Note that both results depart from the Churchill-Chu correlation, eq. (7.116).

Problem 7.27. We must compare the empirical correlation

$$\mathrm{Nu}_{0-H} = 0.15\,\mathrm{Ra}_H^{1/3}, \ (\mathrm{Pr} > 1)$$

with the H-averaged version of eq. (7.126):

$$\mathrm{Nu}_{0-H} = \frac{5}{6}\,(0.039)\ \mathrm{Pr}^{-1/5}\,\mathrm{Ra}_H^{2/5}$$

To find the range of ± 5% agreement between these two estimates, we write in order

$$0.95 < \frac{\frac{5}{6}\,(0.039)\ \mathrm{Pr}^{-1/5}\,\mathrm{Ra}_H^{2/5}}{0.15\ \mathrm{Ra}_H^{1/3}} < 1.05$$

$$4.38 < \mathrm{Pr}^{-1/5}\,\mathrm{Ra}_H^{1/15} < 4.85$$

$$(4.19)\,10^9\,\mathrm{Pr}^3 < \mathrm{Ra}_H < (1.91)\,10^{10}\,\mathrm{Pr}^3$$

For air, Pr = 0.72, the range is

$$(1.56)\,10^9 < \mathrm{Ra}_H < (7.13)\,10^9$$

and for water, Pr = 6,

$$(9)\,10^{11} < \mathrm{Ra}_H < (4.13)\,10^{12}$$

Problem 7.28. If the horizontal round jet is driven solely by momentum, its centerline velocity decays as

$$\bar{u}_c = \frac{c_1}{x} \qquad (1)$$

where $c_1 = 7.46K^{1/2}$, and K is given by eq. (9.33).

If the round plume is vertical and driven solely by buoyancy, its centerline velocity decays as

$$\bar{v}_c = \frac{c_2}{y^{1/3}} \qquad (2)$$

where

$$c_2 = \left[\frac{25}{14\pi\hat{\alpha}^2} \quad \frac{q\,g\beta}{\rho c_P} \left(1 + \frac{b_T^2}{b^2} \right) \right]^{1/3}$$

and $\hat{\alpha} = 0.12$. The energy strength of the plume is

$$q = \rho \, \frac{\pi}{4} \, D_0^2 \, U_0 \, c_P \, (T_0 - T_\infty)$$

As an approximation of the 'bend' where the horizontal jet turns into a vertical plume, we write y(x) for the centerline trajectory in the bend region, and

$$\bar{u}_c = \frac{dx}{dt} \qquad \bar{v}_c = \frac{dy}{dt} \qquad (3, 4)$$

where t is the time of travel experienced by the fluid packets on the centerline. Combining these with eqs. (1) and (2), and integrating from $t = 0$ where $x = 0$ and $y = 0$, we obtain

$$x = (2c_1 t)^{1/2} \qquad y = \left(\frac{4}{3} \, c_2 t \right)^{3/4} \qquad (5, 6)$$

Eliminating t between eqs. (5) and (6) we find the trajectory

$$y = a\,x^{3/2} \qquad a = \left(\frac{2c_2}{3c_1} \right)^{3/4} \qquad (7, 8)$$

Let (x_1, y_1) be the location where the trajectory reaches a 45° angle,

$$\frac{dy}{dx} = 1 \qquad (9)$$

Equations (9) and (7) yield

$$x_1 = \left(\frac{2}{3a} \right)^2 \qquad y_1 = a \left(\frac{2}{3a} \right)^2 \qquad (10, 11)$$

which are related by the proportionality $y_1/x_1 = 2/3$.

TURBULENT DUCT FLOW

Problem 8.1. We begin by writing $u^+ = 8.7\, y^{+1/7}$ at $y = r_0$, namely

$$\frac{U_c}{u_*} = 8.7 \left(\frac{r_0\, u_*}{\nu}\right)^{1/7}$$

where $u_* = (\tau_0/\rho)^{1/2}$. To obtain τ_0 in terms of U (not U_c), we recall the definition of average velocity

$$\pi\, r_0^2\, U = \int_0^{2\pi} \int_0^{r_0} \bar{u}\, r\, dr\, d\theta$$

$$= 2\pi\, (8.7) \frac{u_*^{8/7}}{\nu^{1/7}}\, r_0^{15/7} \int_0^1 n^{1/7}\, (1 - n)\, dn$$

which yields

$$U_c = \frac{120}{98}\, U = 1.224\, U$$

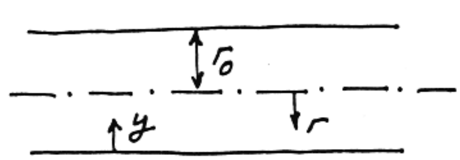
Recognizing further the f-definition,

$$f = \frac{\tau_0}{\frac{1}{2}\, \rho\, U^2},$$

the first equation on this page yields

$$f = 0.077\, Re_D^{-1/4}$$

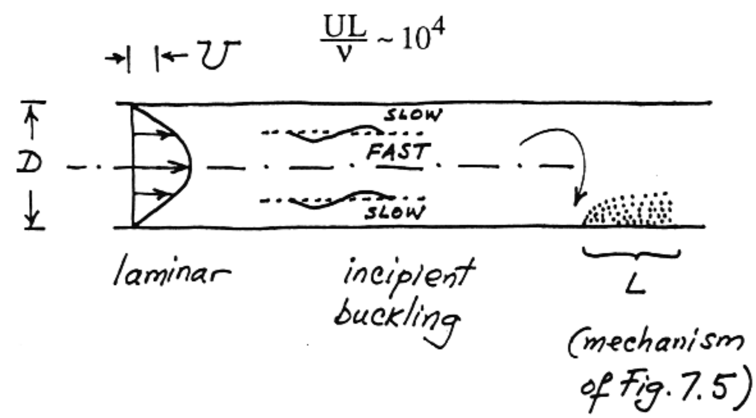
where $Re_D = UD/\nu$.



Problem 8.2. Assume a fully developed pipe flow with fixed Re_D in the range $10^3 - 10^4$. We can imagine this flow to be either laminar or coarsely turbulent (just turbulent), depending on the transition triggering mechanism. For laminar flow, the friction factor would be

$$f = \frac{16}{Re_D}$$

At transition, the flow "buckles" as a centerline jet of constant thickness, flowing through a fluid sleeve slowed down by the pipe wall. The wall friction mechanism is replaced by the intermittent wall-fluid sliding contact sketched in Fig. 7.5. Each contact spot is characterized by



and, since the Re_D range is $10^3 - 10^4$, this means that $L \sim D$. The largest eddies have diameters of order D: these eddies govern the maximum distance between direct contact spots. Therefore, during transition the density of contact spots is O(1):

$$\eta = \frac{\text{spot length } (\sim L)}{\text{spot–spot distance } (\sim D)} \sim 1$$

Note that this prediction is consistent with η values in transitional boundary layer flow (imagine $U_\infty x/\nu \sim 10^4$ to the left of Fig. 7.6). Since

$$\frac{\tau_0}{\tau_{0_{max}}} \sim \eta$$

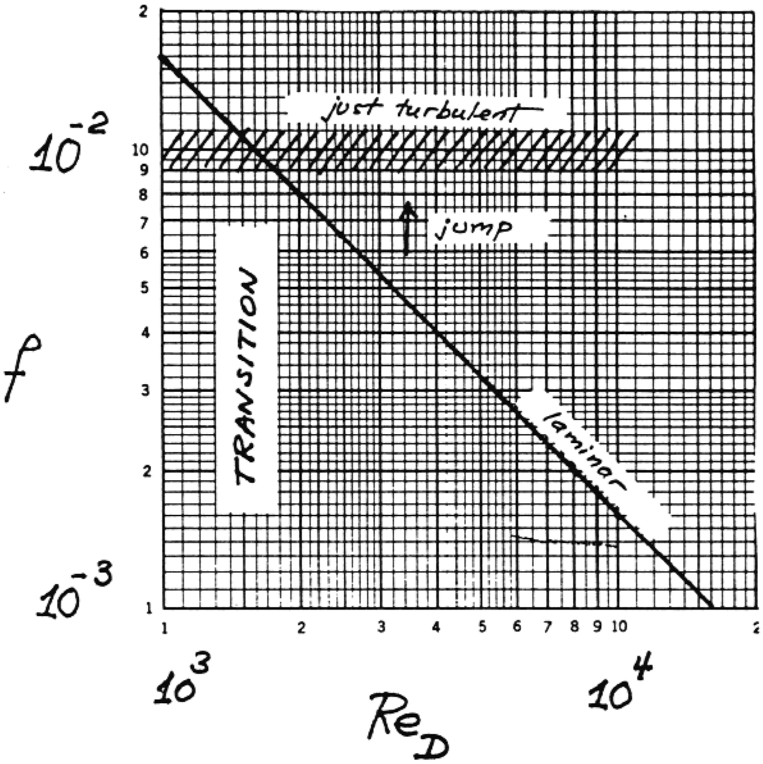
and since $\eta \sim 1$, we have

$$\tau_0 \sim \left(\frac{1}{2}\rho U^2\right) Re_L^{-1/2} \sim \left(\frac{1}{2}\rho U^2\right) 10^{-2},$$

in other words

$$f_{\text{just turbulent}} \sim 10^{-2}$$

As shown in the next figure, there will be a jump in the f value as the flow shifts from laminar to turbulent, if the laminar flow is relatively free of disturbances (i.e. if the natural transition is delayed).



Note that the above discussion is a theoretical argument for predicting the transition in laminar duct flow (Table 6.1). Note further that the predicted $f \sim 10^{-2}$ during transition, is a numerical order of magnitude supported strongly by the empirical evidence presented in the Moody chart (Fig. 8.2; note "4f" on the vertical scale).

Problem 8.3. The M function may be written as

$$M = \frac{I(r)}{I(r_0)}$$

where the integral I(r) is

$$I(r) = \frac{2}{r^2}\int_0^r \bar{u}\, r\, dr$$

In Hagen-Poiseuille flow $\bar{u}$ is replaced by $2U\left[1 - \left(\frac{r}{r_0}\right)^2\right]$,

$$I(r) = \frac{2}{r^2}\int_0^r 2U\left[1 - \left(\frac{b}{r_0}\right)^2\right] b\, db$$

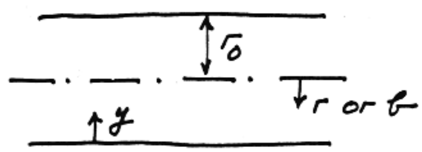
$$= 4U\left(\frac{r_0}{r}\right)^2\left[\frac{1}{2}\left(\frac{r}{r_0}\right)^2 - \frac{1}{4}\left(\frac{r}{r_0}\right)^4\right]$$

and

$$M = 2 - \left(\frac{r}{r_0}\right)^2$$

In turbulent flow with $\bar{u} \approx U_c\,(y/r_0)^{1/7}$ we have

$$I(r) = \frac{2}{r^2}\,U_c \int_0^r \left(\frac{y}{r_0}\right)^{1/7} b\,db$$



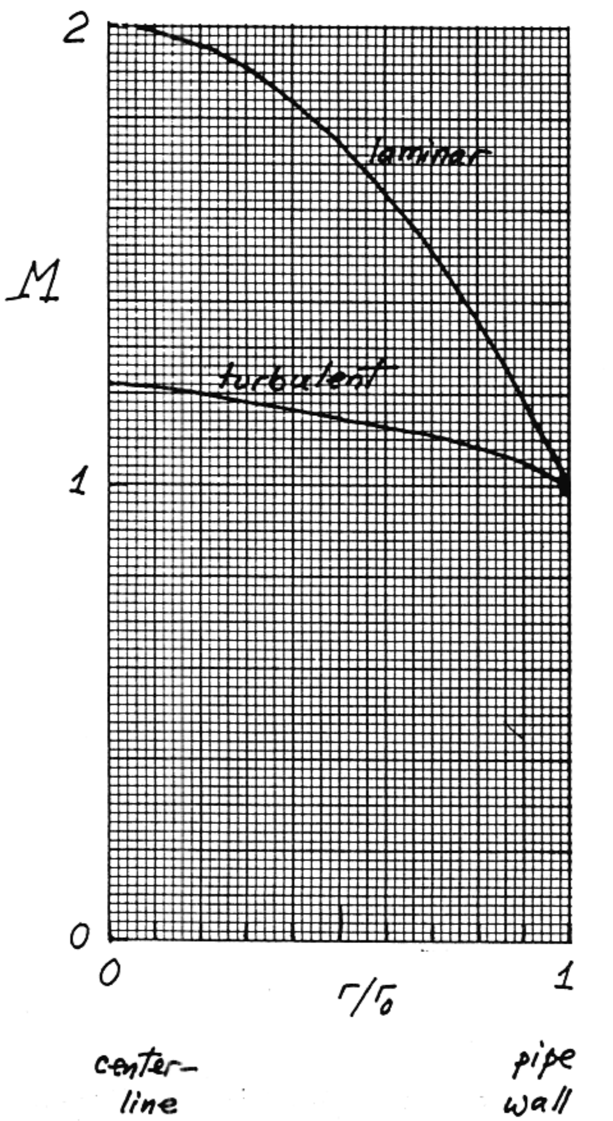
where $y = r_0 - b$. The end result for M is

$$M = \frac{1 - \frac{15}{7}\,m^{8/7} + \frac{8}{7}\,m^{15/7}}{(1-m)^2}$$

with the dimensionless notation

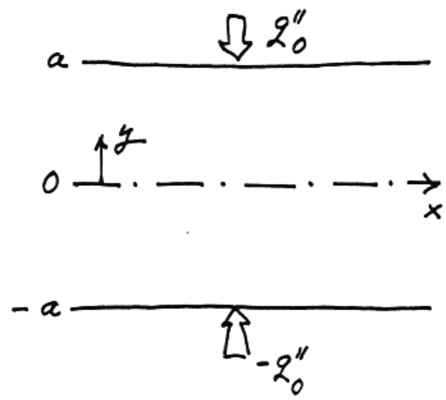
$$m = \frac{y}{r_0}$$

The two M(r) curves have been plotted below: unlike in laminar flow, the M(r) function for Prandtl's 1/7th power law velocity profile is not a strong function of r, thus justifying the $M \cong 1$ approximation used in eq. (8.24).



Problem 8.4. Consider the parallel-plate channel and the coordinate system shown below. The energy equation for fully-developed time averaged flow is

$$\rho\, c_P\, \bar{u}\, \frac{\partial \bar{T}}{\partial x} = \frac{\partial}{\partial y}\left(q''_{app}\right)$$



We integrate this equation from y = 0 to any y, and from y = 0 to y = a, while keeping in mind that symmetry requires $q''_{app} = 0$ at $y = 0$:

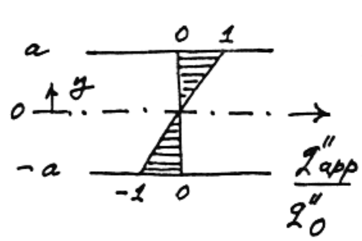
$$\int_0^y \rho\, c_P\, \bar{u}\, \frac{\partial \bar{T}}{\partial x}\, dy = q''_{app}$$

$$\int_0^a \rho\, c_P\, \bar{u}\, \frac{\partial \bar{T}}{\partial x}\, dy = q''_0$$

Dividing term by term we obtain

$$\frac{q''_{app}}{q''_0} = \frac{y}{a}\, \frac{\dfrac{1}{y}\displaystyle\int_0^y \bar{u}\, dy}{\dfrac{1}{a}\displaystyle\int_0^a \bar{u}\, dy} = \frac{y}{a}\, M$$

where, following the graphic procedure described in Problem 8.3, the new M is roughly equal to 1.



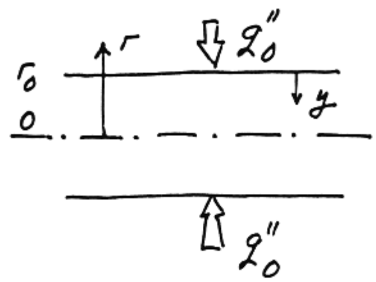
(see top sketch for q''_{app} sign convention)

Problem 8.5. If we set $\bar{u} = U$, constant, the energy equation becomes

$$\rho\, c_P\, U \frac{\partial \bar{T}}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r}\left(r\, q''_{app}\right)$$

with

$$\frac{\partial \bar{T}}{\partial x} = \frac{d\, \bar{T}_m}{dx} = \frac{2\, q''_0}{\rho\, c_P\, r_0\, U}$$



Integrating once from r = 0 to any r, and claiming that

$$q''_{app} = 0 \quad \text{at} \quad r = 0$$

yields

$$\rho\, c_P\, U \frac{d\, \bar{T}_m}{dx} \frac{r}{2} = q''_{app}$$

where

$$q''_{app} = -\rho\, c_P\, (\alpha + \varepsilon_H) \frac{\partial \bar{T}}{\partial y} \qquad (\text{note: } y = r_0 - r)$$

By writing the energy equation as

$$\frac{U}{2} \frac{d\, \bar{T}_m}{dx} \frac{r_0 - y}{\alpha + \varepsilon_H} = -\frac{\partial \bar{T}}{\partial y}$$

and integrating from y = 0 to any y, yields

$$\int_0^{y^+} \frac{1 - y/r_0}{\frac{\alpha}{\nu} + \frac{\varepsilon_H}{\nu}}\, dy^+ = T^+ \tag{1}$$

where

$$T^+ = \left(T_0 - \bar{T}\right) \frac{\rho\, c_P\, u_*}{q''_0}$$

Note also that the friction velocity is

$$u_* = \left(\frac{\tau_0}{\rho}\right)^{1/2} = U\left(\frac{f}{2}\right)^{1/2}$$

Equation (1) can be integrated as described in the problem statement,

$$T^+ = \int_0^{y^+_{CSL}} \frac{1}{Pr^{-1}}\, dy^+ + \int_{y^+_{CSL}}^{y^+} \frac{1 - y/r}{\frac{\varepsilon_H}{\varepsilon_M}\frac{\varepsilon_M}{\nu}}\, dy^+$$

with $\varepsilon_H/\varepsilon_M = Pr_t^{-1}$ and $\varepsilon_M/\nu = \kappa y^+$, and the result is

$$T^+ = Pr\, y^+_{CSL} + Pr_t\left(\frac{1}{\kappa}\ln\frac{y^+}{y^+_{CSL}} - \frac{y^+ - y^+_{CSL}}{\kappa r_0^+}\right)$$

At the tube centerline $y^+ = r_0^+$, we have

$$T_c^+ = Pr\, y^+_{CSL} + Pr_t\left(\frac{1}{\kappa}\ln\frac{r_0^+}{y^+_{CSL}} - \frac{r_0^+ - y^+_{CSL}}{\kappa r_0^+}\right), \tag{E}$$

where r_0^+ is given by

$$r_0^+ = \frac{r_0 u_*}{\nu} = \frac{DU}{\nu}\left(\frac{f}{8}\right)^{1/2} = Re_D\left(\frac{f}{8}\right)^{1/2}$$

The T_c^+ equation, labeled (E), is the source of the expression for Stanton number because

$$T_c^+ = \left(T_0 - \bar{T}_c\right)\frac{\rho c_p u_*}{q_0''} = \frac{T_0 - \bar{T}_c}{T_0 - \bar{T}_m}\cdot\frac{(f/2)^{1/2}}{St}$$

The ratio $\left(T_0 - \bar{T}_m\right)/\left(T_0 - \bar{T}_c\right)$ follows from the definition

$$\pi r_0^2\left(T_0 - \bar{T}_m\right)U = 2\pi\int_0^{r_0}\left(T_0 - \bar{T}\right)\bar{u}\, r\, dr$$

which can be combined with 1/7th power profiles for both $\bar{u}$ and $\left(T_0 - \bar{T}\right)$ to yield:

$$\frac{\bar{u}}{U_c} = \left(1 - \frac{r}{r_0}\right)^{1/7} \quad \text{and} \quad \frac{T_0 - \bar{T}}{T_0 - \bar{T}_c} = \left(1 - \frac{r}{r_0}\right)^{1/7}$$

The result is

$$\frac{T_0 - \bar{T}_m}{T_0 - \bar{T}_c} = 0.833. \qquad \left(\text{Note that Problem 8.1 yielded } \frac{U_c}{U} = \frac{120}{98}\right)$$

The Stanton number formula is therefore

$$St = \frac{(f/2)^{1/2}}{0.833\ [RHS]},$$

where [RHS] is the right-hand-side of equation (E).

Problem 8.6. The pumping power is proportional to the product $\dot{m}\,\Delta P$, namely

$$\mathbf{P} = \frac{1}{\rho}\,\dot{m}\,\Delta P \tag{1}$$

where

$$\Delta P = f\,\frac{4L}{D}\,\frac{1}{2}\,\rho\,U^2 \tag{2}$$

Since $\dot{m}$, L, D and the fluid do not change as the flow regime switches from laminar to turbulent, the Reynolds number and the mean velocity also do not change,

$$U = \frac{\dot{m}}{\rho\,\frac{\pi}{4}\,D^2} \qquad Re_D = \frac{UD}{\nu}$$

Equations (1) and (2) show that the pumping power changes in the same direction (and to the same degree) as the friction factor:

$$\frac{\mathbf{P}_{turb}}{\mathbf{P}_{lam}} = \frac{f_{turb}}{f_{lam}} \cong \frac{0.079\,Re_D^{-1/4}}{\frac{16}{Re_D}}$$

$$\cong 0.00494\,Re_D^{3/4}$$

At transition $Re_D \sim 2000$, therefore the pumping power experiences a jump of about 50 percent,

$$\frac{\mathbf{P}_{turb}}{\mathbf{P}_{lam}} \cong 1.48$$

Problem8.7. In the following chain, we see that the temperature difference $(T_w - T_m)$ varies as the inverse of the Nusselt number, because q_w'' and the fluid are fixed:

$$\frac{(T_w - T_m)_{turb}}{(T_w - T_m)_{lam}} = \frac{q_w''/h_{turb}}{q_w''/h_{lam}} = \frac{h_{lam}}{h_{turb}}$$

$$= \frac{Nu_{D,lam}}{Nu_{D,turb}} \cong \frac{4.364}{0.023\,Pr^{0.4}\,Re_D^{0.8}}$$

$$\cong 216.4\,Re_D^{-0.8} \qquad (\text{if } Pr = 0.72)$$

In the vicinity of $Re_D \sim 2500$, the temperature difference drops significantly if the laminar flow breaks down into turbulent flow:

$$\frac{(T_w - T_m)_{turb}}{(T_w - T_m)_{lam}} \cong 0.414 \qquad (\text{if } Pr = 0.72,\ Re_D = 2500)$$

Problem 8.8. The relevant properties of water at 20°C are

$$k = 0.59 \frac{W}{m\,K} \qquad \rho = 0.998 \frac{cm^2}{s}$$

$$Pr = 7.07 \qquad c_p = 4.182 \frac{kJ}{kg\,K}$$

$$\nu = 0.01004 \frac{cm^2}{s}$$

a) Pressure drop:

$$U = \frac{\dot{m}}{\rho \frac{\pi}{4} D^2} = \frac{0.5 \frac{kg}{s}}{0.998 \frac{g}{cm^3} \frac{\pi}{4} (0.02m)^2}$$

$$= 1.595 \frac{m}{s}$$

$$Re_D = \frac{UD}{\nu} = 1.595 \frac{m}{s} \frac{0.02m}{0.01004 \times 10^{-4} \frac{m^2}{s}}$$

$$= 31\,773 \qquad \text{(turbulent)}$$

$$f \cong 0.046\, Re_D^{-1/5} = 0.00579 \qquad \text{[eq. (8.14): note } Re_D \text{ range]}$$

$$\Delta P = f \frac{4L}{D} \frac{1}{2} \rho U^2$$

$$= 0.00579 \frac{4 \times 10m}{0.02m} \frac{1}{2} 0.998 \frac{g}{cm^3} \left(1.595 \frac{m}{s}\right)^2$$

$$= 14\,690 \frac{N}{m^2} = 0.145 \text{ atm}$$

b) Heat transfer coefficient based on the Colburn analogy:

$$St = \frac{\frac{1}{2} f}{Pr^{2/3}} = \frac{\frac{1}{2} 0.00579}{(7.07)^{2/3}}$$

$$= 7.86 \times 10^{-4}$$

$$St = \frac{h}{\rho\, c_p\, U}$$

$$h = St\,\rho\,c_P\,U = 7.86 \times 10^{-4}\,0.998\,\frac{g}{cm^3}\,4.182\,\frac{kJ}{kg\,K}\,1.595\,\frac{m}{s}$$

$$= 5230\,\frac{W}{m^2\,K}$$

c) Heat transfer coefficient based on the Dittus-Boelter correlation:

$$Nu_D = 0.023\,Pr^{0.4}\,Re_D^{4/5} = 0.023\,(7.07)^{0.4}\,(31\,773)^{0.8}$$

$$= 201$$

$$h = Nu_D\,\frac{k}{D} = 201\,\frac{0.59\text{ W/m K}}{0.02\text{m}}$$

$$= 5930\,\frac{W}{m^2\,K}$$

The h estimate obtained in part (b) is 12 percent lower than Dittus-Boelter prediction, which is recommended.

d) Temperature difference across the stream

$$T_w - T_m = \frac{q''_w}{h} = \frac{5 \times 10^4\text{ W/m}^2}{5930\text{ W/m}^2\text{ K}}$$

$$= 8.43\text{ K}$$

e) Temperature increase in the longitudinal direction:

$$\dot{m}\,c_P\,d\,T_m = q''_w\,\pi D\,dx$$

$$\frac{d\,T_m}{dx} = \frac{q''_w\,pD}{\dot{m}\,c_P} = \frac{5 \times 10^4\text{ W}}{m^2}\,\frac{\pi\,0.02\text{m}}{0.5\,\frac{kg}{s}}\,\frac{kg\,K}{4.182 \times 10^3\text{ J}}$$

$$= 1.5\,\frac{°C}{m}$$

$$T_{m,out} - T_{m,in} = \frac{d\,T_m}{dx}\,L = 1.5\,\frac{°C}{m}\,10\text{ m}$$

$$= 15°C$$

Problem 8.9. We begin with the properties of air, which can be evaluated at the end-to-end averaged temperature (30°C + 90°C)/2 = 60°C:

$$\nu = 0.188\ \frac{\text{cm}^2}{\text{s}} \qquad \rho = 1.06\ \frac{\text{kg}}{\text{m}^3}$$

$$Pr = 0.7 \qquad c_P = 1.008\ \frac{\text{kJ}}{\text{kg K}}$$

$$k = 0.028\ \frac{\text{W}}{\text{m K}}$$

The required length of the pipe is given by eq. (8.46), for which we first calculate the heat transfer coefficient:

$$Re_D = \frac{UD}{\nu} = 5\ \frac{\text{m}}{\text{s}}\ 0.04\text{m}\ \frac{\text{s}}{0.188 \times 10^{-4}\,\text{m}^2}$$

$$= 1.064 \times 10^4 \qquad \text{(turbulent)}$$

$$Nu_D = 0.023\ Pr^{0.4}\ Re_D^{4/5}$$

$$= 0.023\ (0.7)^{0.4}\ (1.064 \times 10^4)^{0.8} = 33.21$$

$$h = Nu_D\ \frac{k}{D}$$

$$= 33.21\ \frac{0.028\ \text{W}}{\text{m K}}\ \frac{1}{0.04\text{m}} = 23.25\ \frac{\text{W}}{\text{m}^2\,\text{K}}$$

In eq. (8.46) we substitute

$$A = \frac{\pi}{4} D^2 \qquad \text{and} \qquad \pi = \pi D$$

and obtain

$$L = \frac{\rho\, DU\, c_P}{4h} \ln \frac{T_w - T_{in}}{T_w - T_{out}} = \frac{1.06\ \frac{\text{kg}}{\text{m}^3}\ 0.04\text{m}\ 5\ \frac{\text{m}}{\text{s}}\ 1.008 \times 10^3\ \frac{\text{J}}{\text{kg K}}}{4 \times 23.25\ \frac{\text{W}}{\text{m}^2\,\text{K}}} \ln \frac{100 - 30}{100 - 90}$$

$$= 4.47\text{m}$$

Since the flow is turbulent, the entrance length

$$X \text{ (or } X_T) \sim 10D = 0.4\text{m}$$

is less than one tenth of the pipe length L. In conclusion, the assumption of full development over the entire length L is justified.

Problem 8.10. The relevant properties of water at 20°C are

$$k = 0.59 \frac{W}{m\,K} \qquad \nu = 0.01004 \frac{cm^2}{s}$$

$$Pr = 7.07 \qquad \rho = 0.998 \frac{g}{cm^3}$$

We calculate, in order, the heat transfer coefficient, the Nusselt number, the Reynolds number, and finally the mass flowrate:

$$h = \frac{q_w''}{\Delta T} = \frac{10^4\,W}{m^2}\,\frac{1}{4\,K} = 2500 \frac{W}{m^2\,K}$$

$$Nu_D = \frac{hD}{k} = 2500 \frac{W}{m^2\,K}\,0.025\,m\,\frac{m\,K}{0.59\,W}$$

$$= 105.93$$

$$Nu_D = 0.023\,Pr^{0.4}\,Re_D^{4/5}$$

$$Re_D^{4/5} = \frac{Nu_D}{0.023\,(7.07)^{0.4}} = 2106.4$$

$$Re_D = (2106.4)^{5/4} = 14\,270$$

$$U = \frac{\nu}{D} Re_D = \frac{0.01004 \frac{cm^2}{s}}{0.025m}\,14\,270$$

$$= 0.573 \frac{m}{s}$$

$$\dot{m} = \rho \frac{\pi}{4} D^2 U = 0.998 \frac{g}{cm^3} \frac{\pi}{4} (0.025m)^2\, 0.573 \frac{m}{s}$$

$$= 0.281 \frac{kg}{s}$$

Problem 8.11. The information we know amounts to the following:

$$U = 1 \text{ m/s} \qquad \nu \cong 0.01 \text{ cm}^2/\text{s (water at 20°C)}$$

$$D = 1 \text{ cm} \qquad y^+ = 11.6$$

The wall layer thickness that corresponds to $y^+ = 11.6$ is

$$y = y^+ \frac{\nu}{(\tau_w/\rho)^{1/2}}$$

or, after using eq. (8.10),

$$y = y^+ \frac{n}{U\,(f/2)^{1/2}} \qquad (1)$$

In order to determine f, we must first calculate the Reynolds number,

$$Re_D = \frac{UD}{\nu} = \frac{1\text{m}}{\text{s}}\, 1 \text{ cm}\, \frac{\text{s}}{0.01 \text{ cm}^2} = 10^4 \qquad \text{(turbulent)}$$

This Re_D value recommends the use of eq. (8.13),

$$f = 0.079\,(10^4)^{-1/4} = 0.0079$$

and, in the end, eq. (1) yields

$$y = 11.6\, \frac{0.01 \text{ cm}^2}{\text{s}}\, \frac{\text{s}}{1 \text{ m}} \left(\frac{2}{0.0079}\right)^{1/2}$$
$$= 0.18 \text{ mm}$$

Problem 8.12. a) The properties of water at 80°C and moderate (nearly atmospheric) pressures are

$$\nu = 0.00366 \frac{\text{cm}^2}{\text{s}} \qquad \rho = 0.9718 \frac{\text{g}}{\text{cm}^3}$$

The frictional pressure drop per unit length is given by

$$\frac{\Delta P}{L} = f \frac{p}{A} \frac{1}{2} \rho U^2$$
$$= f \frac{1}{D_h} 2\rho\, U^2 \qquad (1)$$

therefore, we must calculate in order

$$D_h = \frac{4A}{p} = 4\,\frac{\pi D_o^2/4 - \pi D_i^2/4}{\pi D_o + \pi D_i} = D_o - D_i =$$

$$= (22 - 16)\text{ cm} = 6\text{ cm}$$

$$U = \frac{\dot m}{\rho A} = 100\,\frac{10^3\text{kg}}{3600\text{s}}\,\frac{\text{cm}^3}{0.9718\text{g}}\,\frac{1}{(\pi/4)\left(22^2 - 16^2\right)\text{cm}^2}$$

$$= 1.596\,\frac{\text{m}}{\text{s}} = 159.6\,\frac{\text{cm}}{\text{s}}$$

$$Re_{D_h} = \frac{UD_h}{\nu} = 159.6\,\frac{\text{cm}}{\text{s}}\,6\text{ cm}\,\frac{\text{s}}{0.00366\text{ cm}^2}$$

$$= 2.62 \times 10^5, \quad \text{(turbulent)}$$

The friction factor is deduced from the Moody chart, where we know the value of Re_{D_h} on the abscissa, and the dimensionless roughness corresponding to "commercial steel",

$$\frac{k_s}{D_h} = \frac{0.05\text{ mm}}{60\text{ mm}} = 0.000833$$

therefore,

$$f \cong \frac{0.02}{4} = 0.005$$

We now have all the ingredients needed for evaluating numerically the right side of eq. (1):

$$\frac{\Delta P}{L} \cong \frac{0.005}{6\text{ cm}}\,2\,(0.9718)\,\frac{\text{g}}{\text{cm}^3}\,(159.6)^2\,\frac{\text{cm}^2}{\text{s}^2}$$

$$\cong 412.6\,\frac{\text{kg}}{\text{m}^2\text{s}^2} = 412.6\,\frac{\text{N/m}^2}{\text{m}} = 0.00407\,\frac{\text{atm}}{\text{m}}$$

If L = 200m, for example, the frictional pressure drop is 0.81 atm.

b) In the following calculations, we need two additional properties of water at 80°C and nearly atmospheric pressures:

$$c_P = 4.196\,\frac{\text{J}}{\text{gK}} \qquad \text{Pr} = 2.23$$

The mean temperature difference $\Delta T = T_w - T_m$ is given by the first law of thermodynamics, with reference to a duct element of length dx and annular flow area

$$\dot m\, c_P dT = h\,(\pi D_o\, dx)\,\Delta T$$

On the right side, πD_o is the portion of the wetted perimeter that is crossed by heat transfer (the inner wall of the annulus is insulated). In conclusion,

$$\Delta T = \left(\frac{dT}{dx}\right) \frac{\dot{m} c_P}{h \pi D_o}$$

which demands a value for the heat transfer coefficient h. Invoking the Colburn analogy,

$$St\, Pr^{2/3} = \frac{f}{2}$$

we learn first that

$$\begin{aligned} St &= Pr^{-2/3} \frac{f}{2} = (2.23)^{-2/3} \; \frac{0.005}{2} \\ &= 0.00146 \end{aligned}$$

therefore

$$\begin{aligned} h &= St\, \rho\, c_P U \\ &= 0.00146 \times \; 0.9718 \frac{g}{cm^3} \, 4.196 \frac{J}{gK} \, 159.6 \frac{cm}{s} \\ &= 0.95 \frac{W}{cm^2\, K} \end{aligned}$$

And, in view of the fact that $dT/dx = 200°C/km$, the ΔT formula yields

$$\begin{aligned} \Delta T &= 200 \frac{°C}{10^5 cm} \; 100 \frac{10^3 kg}{3600 s} \, 4.196 \frac{J}{gK} \, \frac{cm^2\, K}{0.95\, W} \, \frac{1}{\pi\, 22\, cm} \\ &= 3.55°C \end{aligned}$$

Problem 8.13. a) The pressure drop per unit length contributed by the annular space is

$$\left(\frac{\Delta P}{L}\right)_a = \frac{f_a}{D_h} 2\rho\, U_a{}^2 \tag{1}$$

where U_a is the mean velocity

$$U_a = \frac{\dot{m}}{\rho\, (\pi/4)\, (D_o^2 - D_i^2)} \tag{2}$$

and where D_h is the hydraulic diameter of the annular cross-section,

$$D_h = \frac{4A}{p} = 4 \frac{(\pi/4)\, (D_o^2 - D_i^2)}{\pi\, (D_o + D_i)} = D_o - D_i \tag{3}$$

Combined, eqs. (1-3) yield

$$\left(\frac{\Delta P}{L}\right)_a = \frac{2\rho}{D_o^5}\left(\frac{\dot{m}}{\rho}\right)^2\left(\frac{4}{\pi}\right)^2 \frac{f_a}{(1-r)^3\,(1+r)^2} \tag{4}$$

in which r is the dimensionless diameter ratio

$$r = \frac{D_i}{D_o} \tag{5}$$

Similarly, the pressure drop per unit length contributed by the adjacent flow through the inner pipe is

$$\left(\frac{\Delta P}{L}\right)_i = \frac{f_i}{D_i}\, 2\rho\, U_i^2 \tag{6}$$

with

$$U_i = \frac{\dot{m}}{r\,(\pi/4)\,D_i^2} \tag{7}$$

Equation (6) can be written as

$$\left(\frac{\Delta P}{L}\right)_i = \frac{2r}{D_o^5}\left(\frac{\dot{m}}{\rho}\right)^2\left(\frac{4}{\pi}\right)^2 \frac{f_i}{r^5} \tag{8}$$

therefore, the <u>total</u> pressure drop contributed by each unit length of coaxial heat exchanger is

$$\begin{aligned}\frac{\Delta P}{L} &= \left(\frac{\Delta P}{L}\right)_a + \left(\frac{\Delta P}{L}\right)_i \\ &= \frac{2r}{D_o^5}\left(\frac{\dot{m}}{\rho}\right)^2\left(\frac{4}{\pi}\right)^2 \Bigg[\underbrace{\frac{f_a}{(1-r)^3\,(1+r)^2} + \frac{f_i}{r^5}}_{F(r)}\Bigg]\end{aligned} \tag{9}$$

The position of the wall of diameter D_i is governed by the parameter r. It is easy to see that $F(0) = F(1) = \infty$, which means that F and $\Delta P/L$ have a minimum at a special, intermediate r value. Minimizing F(r) numerically in the case where $f_a = f_i$ = constant, we learn that the special r value is

$$r = 0.653 \tag{10}$$

b) If the flow is laminar, the friction factors are affected strongly by the respective Reynolds numbers. For the inner pipe, that relationship is

$$f_i = \frac{16}{Re_{D_i}} \tag{11}$$

The f_a relationship for the annular cross-section is considerably more complicated; when D_i is not much smaller than D_o, however, the annulus is similar to the space between two parallel plates positioned $(D_o - D_i)/2$ apart, therefore

$$f_a \cong \frac{24}{Re_{D_h}} \tag{12}$$

where $D_h = D_o - D_i$. Substituting eqs. (11,12) in the total pressure drop formula (9), we obtain

$$\frac{\Delta P}{L} = \frac{8}{\pi}\frac{\dot{m}}{r}\frac{\mu}{D_o^4}\left[\frac{24}{(1-r)^3\,(1+r)} + \frac{16}{r^4}\right] \tag{13}$$

Numerically, we find that the quantity listed inside the square brackets reaches its minimum at

$$r = 0.621 \tag{14}$$

This result is nearly the same as the one obtained in the fully-rough limit of the turbulent regime, eq. (10).

Problem 8.14. We begin with estimating the total number of tubes n, and calculate in order

$n = 11 \times 6 + 10 \times 5 = 116$ (note: the first and last rows have 6 tubes each)

$A_w = 116\,\pi\,DL$

$= 116\,\pi\,0.04\text{m}\;3\text{m} = 43.73\ \text{m}^2$

$6X_t = 42$ cm (frontal width of tube bundle)

$A = (6\,X_t)L = 42\ \text{cm}\;3\text{m} = 1.26\ \text{m}^2$ (frontal area of tube bundle)

$\dot{m} = \rho A\,U_\infty =$

$= 0.616\,\frac{\text{kg}}{\text{m}^3}\,1.26\ \text{m}^2\,2\,\frac{\text{m}}{\text{s}} = 1.555\,\frac{\text{kg}}{\text{s}}$ (air at 300°C)

$$\frac{h\,A_w}{\dot{m}\,c_p} = \frac{62\ \text{W}}{\text{m}^2\,\text{K}}\;\frac{43.73\ \text{m}^2}{1.555\ \text{kg/s}}\;\frac{\text{kg K}}{1045\ \text{J}} = 1.668$$

$$\ln\frac{\Delta T_{in}}{\Delta T_{out}} = 1.668$$

$\Delta T_{out} = 0.189\,\Delta T_{in}$

$= 0.189\,(300 - 30)°\text{C} \cong 51°\text{C}$

$T_{out} = 30°C + 51°C = 81°C$

$$q = \dot{m}\, c_P\, (T_{in} - T_{out})$$

$$= 1.555 \frac{kg}{s}\, 1045 \frac{J}{kg\, K}\, (300 - 81)\, K$$

$$\cong 356\ kW$$

Problem 8.15. At a point in the fully-turbulent sublayer of the "τ_{app} = constant" layer we have

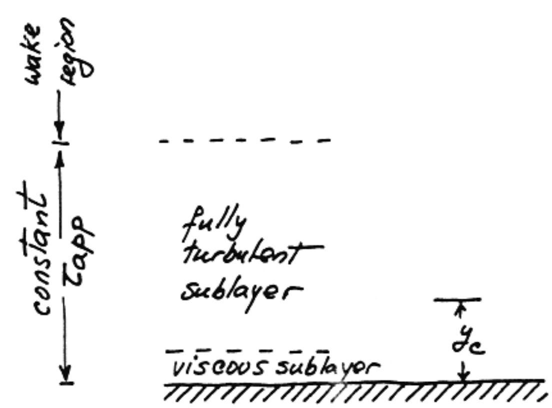
$$\frac{\bar{u}_c}{u_*} = \frac{1}{\kappa} \ln\left(\frac{u_* y_c}{\nu}\right) + B$$

von Karman's constant κ should not be confused with k (turbulence energy). From eq. (8.54), we derive in a few steps,

$$k_c = \left(\frac{\varepsilon_M}{C_\mu L}\right)^2 = \left[\frac{\nu \kappa y_c^+}{C_\mu \left(l\, C_\mu^{-3/4}\right)}\right]^2 = \ldots. = \frac{u_*^2}{C_\mu^{1/2}}$$

where $l = \kappa y_c$. Likewise, from eq. (8.60) we deduce

$$\varepsilon_c = \frac{C_\mu}{\varepsilon_M} k_c^2 = \frac{C_\mu}{\nu \kappa y_c^+} \left(\frac{u_*^2}{C_\mu^{1/2}}\right)^2 = \ldots.. = \frac{u_*^3}{\kappa y_c}$$



Problem 8.16. We will show analytically that the board-to-board spacing can be optimized, so that the overall thermal conductance of the package $q'/(T_{max} - T_\infty)$ is maximized. In the following analysis we assume that the board surfaces are isothermal at T_{max}, while the coolant supply temperature is T_∞. The pressure drop across the package, ΔP, is fixed. The board thickness t is not necessarily negligible when compared with the board-to-board spacing D. In other words, the number of boards in the stack of thickness H is $n = H/(D + t)$, where it is assumed that $n >> 1$. We follow the method shown in section 3.6.

a) When the board-to-board spacing is sufficiently small, the outlet temperature of the coolant is the same as the board temperature, and the total rate of heat transfer removed from the package is

$$q'_a = \dot{m}' c_P (T_{max} - T_\infty) \tag{1}$$

The mass flowrate is $\dot{m}' = n\rho UD$, where U is the mean velocity through each D channel with fully developed turbulent flow,

$$U = \left(\frac{D\,\Delta P}{\rho\, Lf}\right)^{1/2} \tag{2}$$

The friction factor f depends on the channel Reynolds number, as we will see in eq. (11). In conclusion, if we combine eqs. (1) and (2) with $\dot{m}' = n\rho UD$ we obtain the $D \to 0$ asymptote of the overall thermal conductance:

$$\left(\frac{q'}{T_{max} - T_\infty}\right)_{D\to 0} = \frac{c_P H}{1 + \frac{t}{D}} \left(\frac{\rho\, D\, \Delta P}{fL}\right)^{1/2} \tag{3}$$

b) In the opposite extreme each board is lined by boundary layers, while the core of the channel of spacing D is filled by coolant of temperature T_∞ and free-stream velocity U_∞. The latter is dictated by the force balance on the entire stack with fixed ΔP,

$$H\,\Delta P = 2n\,\bar{\tau}\,L \tag{4}$$

where $\bar{\tau}$ is the L-averaged shear stress on the board surface. In writing eq. (4) we have assumed that the board thickness is small enough so that the force experienced by each board is dominated by skin friction over the L-long faces. This assumption is equivalent to writing that in an order of magnitude sense,

$$\frac{1}{2}\rho\, U_\infty^2\, t << 2\,\bar{\tau}\, L \tag{5}$$

which, in view of the definition of average skin friction coefficient $C_f = \bar{\tau}/(\rho\, U_\infty^2/2)$, means that we are assuming

$$\frac{t}{L} << 2\, C_f \tag{6}$$

Combining eq. (4) with the C_f definition we obtain

$$U_\infty = \left(\frac{H\,\Delta P}{n\,\rho\, L\, C_f}\right)^{1/2} \tag{7}$$

The total heat transfer rate through one board surface (i.e. across one boundary layer) is

$$q'_1 = \overline{q''}L = St\;\, L\,\rho\, c_P\, U_\infty (T_{max} - T_\infty) \tag{8}$$

where $\bar{q}''$ is the L-averaged heat flux. The Stanton number $St = \bar{q}''/[\rho c_P U_\infty(T_{max} - T_\infty)]$ can be evaluated by invoking the Colburn analogy between momentum and heat transfer in turbulent boundary layer flow,

$$St = \frac{1}{2} C_f Pr^{-2/3} \qquad (Pr \gtrsim 0.5) \tag{9}$$

In the end, for the total heat transfer rate removed from the stack we write $q' = 2n\ q_1'$, and obtain the following asymptotic expression for the overall conductance:

$$\left(\frac{q'}{T_{max} - T_\infty}\right)_{D\to\infty} = c_P\, H\, Pr^{-2/3} \left(\frac{\rho\, L\, C_f\, \Delta P}{t + D}\right)^{1/2} \tag{10}$$

Equations (3) and (10) show that the overall conductance increases with D when D is small, and decreases when D is large. This means that $q'/(T_{max} - T_\infty)$ is maximum at an optimal intermediate spacing that is of the same order of magnitude as the D value obtained by intersecting eqs. (3) and (10). The result of this intersection is given implicitly by

$$\frac{D_{opt} / L}{\left(1 + t / D_{opt}\right)^{1/2}} = (f\, C_f)^{1/2}\, Pr^{-2/3} \qquad (Pr \gtrsim 0.5) \tag{11}$$

The corresponding maximum value of the overall thermal conductance is obtained by substituting $D = D_{opt}$ in eq. (10) or eq. (3):

$$\left[\frac{q'L}{kH\,(T_{max} - T_\infty)}\right]_{max} \lesssim \left(\frac{C_f}{f}\right)^{1/4} Pr^{1/6} \left(1 + \frac{t}{D_{opt}}\right)^{-3/4} Be_L^{1/2} \qquad (Pr \gtrsim 0.5) \tag{12}$$

where $Be_L = \Delta P \cdot L^2/(\mu\alpha)$. The inequality sign is a reminder that if q' is plotted on the ordinate and D on the abscissa, the peak of the actual q' vs. D curve is located under the intersection of the asymptotes (3) and (10). In spite of this inequality, the right side of eq. (12) represents the correct order of magnitude of the maximum overall thermal conductance.

<u>Smooth surfaces</u>. Beyond this point in the analysis we must make certain assumptions regarding the values of the friction factor and skin-friction coefficient. If all the board surfaces are smooth, we can use the standard correlations

$$f = 0.046\ Re_{D_h}^{-1/5} \qquad \left(10^4 < Re_{D_h} < 10^6\right) \tag{13}$$

$$\frac{1}{2}\, C_f = 0.037\ Re_L^{-1/5} \qquad \left(10^6 < Re_L < 10^8\right) \tag{14}$$

where $D_h = 2D$, $Re_{D_h} = 2DU/\nu$ and $Re_L = U_\infty L/\nu$. These allow us to relate U and U_∞ to ΔP, by combining eqs. (2) and (13) for U, and (7) and (14) for U_∞:

$$U = 5.98\ D^{2/3}\, \nu^{-1/9} \left(\frac{\Delta P}{\rho L}\right)^{5/9} \tag{15}$$

$$U_\infty = 4.25\ L^{-4/9}\ \nu^{-1/9}\left[\frac{\Delta P\,(D + t)}{\rho}\right]^{5/9} \tag{16}$$

Combined, eqs. (13)-(16) express f and C_f as functions of the imposed pressure drop, i.e. functions that can be substituted on the right side of eq. (11). The final expression for the optimal spacing is

$$\frac{D_{opt} / L}{(1 + t / D_{opt})^{4/11}} = 0.071\,Pr^{-5/11}\,Be_L^{-1/11} \tag{17}$$

The geometric meaning of this conclusion becomes clearer if we estimate the expected order of magnitude of the right side of eq. (17). First, note that the Re_{D_h} range of validity listed in eq. (13) can be rewritten in terms of ΔP by using eq. (15) and the assumptions that $(1 + t/D_{opt})^{4/11} \sim 1$ and $Pr = 0.72$ (air):

$$0.09 > Be_L^{-1/11} > 0.032 \tag{18}$$

Similarly, the Re_L range specified in eq. (14) can be rewritten with the help of eq. (16):

$$0.087 > Be_L^{-1/11} > 0.038 \tag{19}$$

Equations (18) and (19) show that the specified Re_{D_h} and Re_L ranges correspond to the same range of the pressure drop group Be_L. Taken together, eqs. (17)-(19) show that the slenderness ratio of each board-to-board channel (D_{opt}/L) takes values between ~ 0.003 and ~ 0.007, and is relatively insensitive to changes in the applied pressure difference.

When the surfaces are smooth [cf. eqs. (13, 14)], the maximum overall conductance expression (12) becomes

$$\left[\frac{q'L}{kH(T_{max} - T_\infty)}\right]_{max} \lesssim 0.57\ Pr^{4/99}\left(1 + \frac{t}{D_{opt}}\right)^{-67/99} Be_L^{47/99} \tag{20}$$

In the case of a fluid with Prandtl number of order 1, eq. (20) is almost the same as the more general eq. (12) with the constant 0.57 in place of $(C_f/f)^{1/4}$. In conclusion, the maximum overall conductance increases almost as $\Delta P^{1/2}$.

Problem 8.17. The pumping power is proportional to the product $\dot{m}\ \Delta P$, namely

$$\mathbf{P} = \frac{1}{\rho}\ \dot{m}\, \Delta P \tag{1}$$

where

$$\Delta P = f\ \frac{4L}{D}\ \frac{1}{2}\ \rho\, U^2 \tag{2}$$

Since $\dot{m}$, L, D and the fluid do not change as the flow regime switches from laminar to turbulent, the Reynolds number and the mean velocity also do not change,

$$U = \frac{\dot{m}}{\rho\ \frac{\pi}{4}\ D^2} \qquad\qquad Re_D = \frac{UD}{\nu}$$

Equations (1) and (2) show that the pumping power changes in the same direction (and to the same degree) as the friction factor:

$$\frac{\mathbf{P}_{turb}}{\mathbf{P}_{lam}} = \frac{f_{turb}}{f_{lam}} \cong \frac{0.079\ Re_D^{-1/4}}{\dfrac{16}{Re_D}}$$

$$\cong 0.00494\ Re_D^{3/4}$$

At transition $Re_D \sim 2000$, therefore the pumping power experiences a jump of about 50 percent,

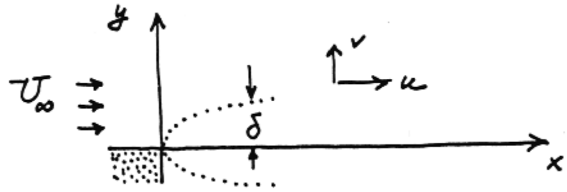
$$0.00494\ (2000)^{3/4} = 1.48$$

Chapter 9

FREE TURBULENT FLOWS

<u>Problem 9.1</u>. The laminar shear layer thickness δ follows from the momentum equation:

$$\underbrace{u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}}_{\frac{U_\infty^2}{x}} = \underbrace{\nu\frac{\partial^2 u}{\partial y^2}}_{\sim\ \nu\frac{U_\infty}{\delta^2}}$$

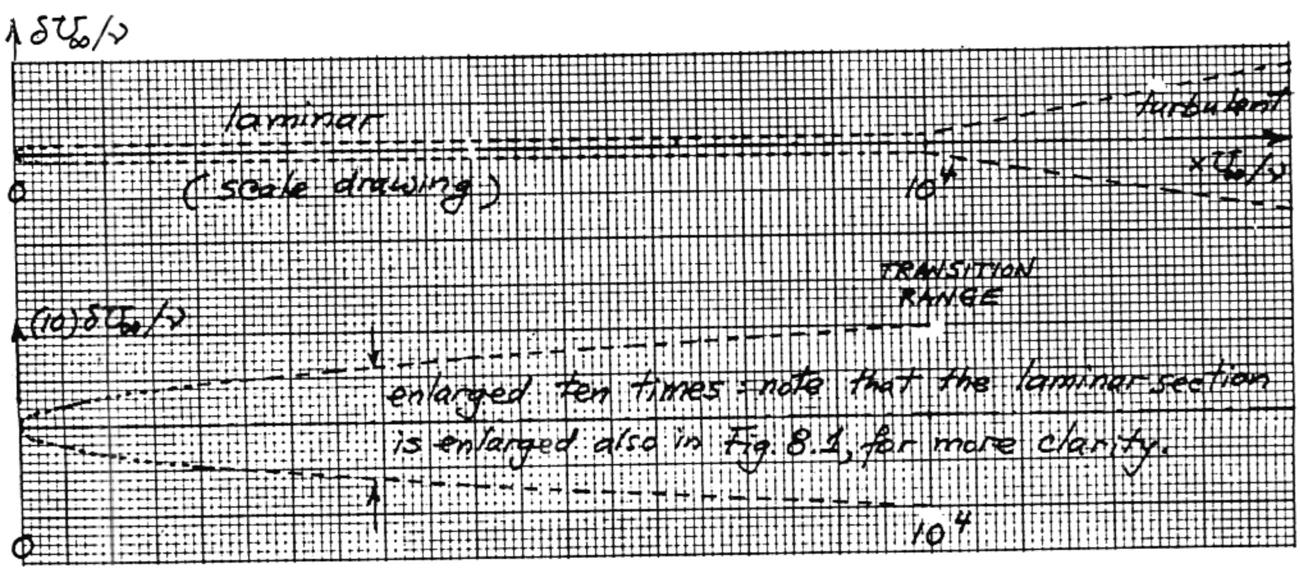


hence $\delta \sim (\nu x/U_\infty)^{1/2}$, or in dimensionless form

$$\frac{\delta\, U_\infty}{\nu} \sim \left(\frac{U_\infty\, x}{\nu}\right)^{1/2}$$

Note that $\delta U_\infty/\nu$ is the local Reynolds number identified as transition criterion in Chapter 6, therefore the laminar section prevails when

$$\frac{\delta\, U_\infty}{\nu} < 10^2 \qquad \text{or} \qquad \frac{U_\infty\, x}{\nu} < 10^4$$

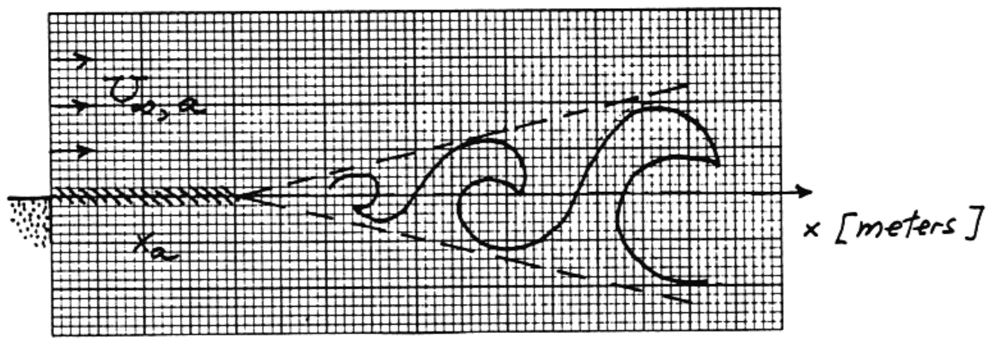


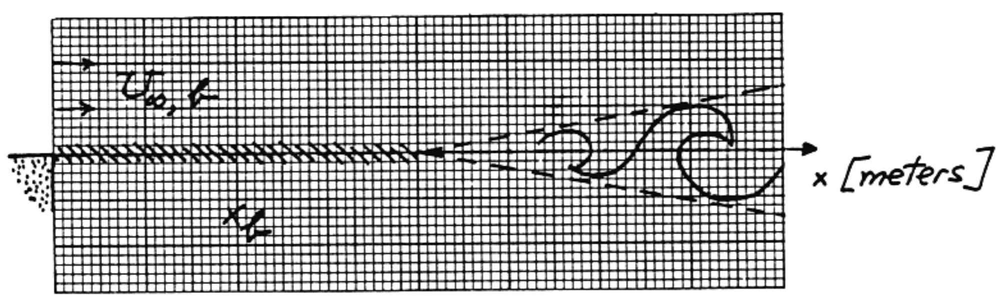
Note further that when it is drawn to scale, the virtual origin of the turbulent section falls closer to the point of transition (i.e. much closer than is shown in Fig. 9.1).

If U_∞ varies, and since $U_\infty x/\nu \sim 10^4$ (constant) at transition, the visual length of the laminar section (indicated by \\\\) must also vary,

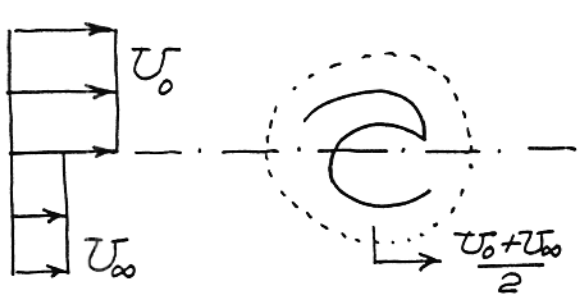
$$\left(\frac{U_\infty x}{\nu}\right)_a = \left(\frac{U_\infty x}{\nu}\right)_b,$$

hence $x_b = 2x_a$ if $U_{\infty,b} = (1/2)\, U_{\infty,a}$. This effect is illustrated below: as U_∞ decreases the laminar length increases.

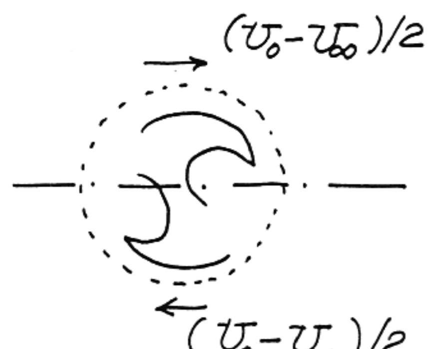




Problem 9.2. The eddies rolling up at the interface can only travel downstream at the "mixed" speed $(U_0 + U_\infty)/2$. If the observer rides on such an eddy, then he will see that the peripheral velocity of this large eddy is of order $(U_0 - U_\infty)/2$:



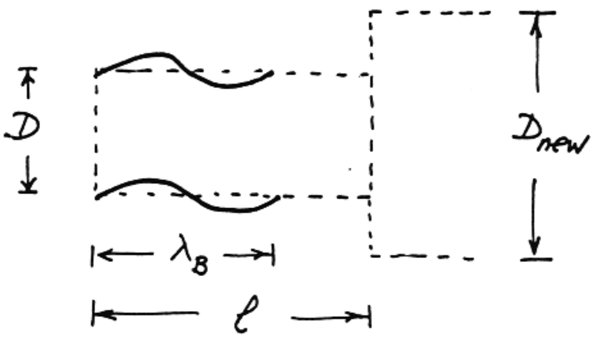
(the observer is at rest)



(the observer rides on the eddy)

If D is the thickness of the shear layer at some point in time, then the layer develops a wave of length $\lambda_B \sim 2D$. This wave will be rolled up in a time of order

$$t_B \sim \frac{\lambda_B}{(U_0 - U_\infty)/2},$$



and the resulting eddy will be born at a distance ℓ downstream

$$\ell \sim t_B \frac{U_0 + U_\infty}{2},$$

The shear layer emerges as a sequence of geometrically similar building blocks, each block having the same aspect ratio

$$\frac{\ell}{D} \sim \frac{\lambda_B}{D} \frac{U_0 + U_\infty}{U_0 - U_\infty}$$

Next, by using the scaling law

$$D_{new} \sim 2D \qquad \text{(Ref. 7, pp. 74-77)}$$

and a drawing like in Fig. 9.3, we conclude that the half-angle of visual growth must scale as

$$\frac{\alpha}{2} \sim \text{arc tan}\left(\frac{\frac{1}{2}D}{\frac{3}{2}\ell}\right) \sim \text{arctan}\left(\frac{D}{3\lambda_B} \cdot \frac{U_0 - U_\infty}{U_0 + U_\infty}\right) \quad \left(\text{note: } \frac{D}{\lambda_B} = \frac{3^{1/2}}{\pi}\right)$$

and, assuming small angles,

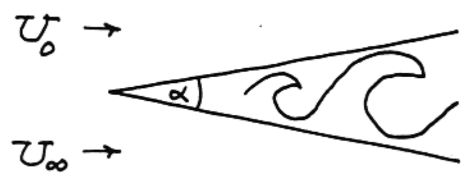
$$\alpha \sim \frac{2}{3} \frac{3^{1/3}}{\pi} \frac{U_0 - U_\infty}{U_0 + U_\infty}$$

or

$$\alpha \sim 0.37 \frac{U_0 - U_\infty}{U_0 + U_\infty} \qquad (1)$$

This result can be compared directly with Brown and Roshko's [5] observations of the visual growth rate δ_{viz}, defined as

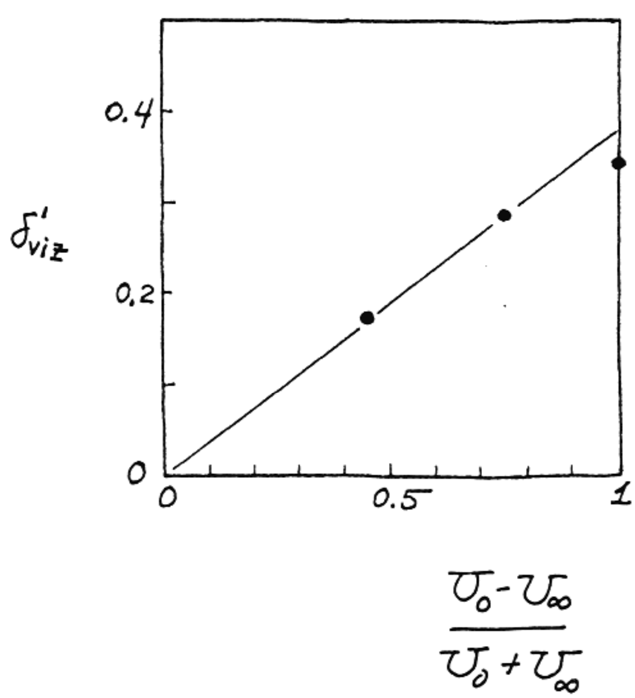
$$\delta'_{viz} = \tan(\alpha)$$



Reproduced in the attached graph is Fig. 7 of Ref. [5], which was published long before the buckling theory of inviscid streams [7]. Brown and Roshko correlated their visual growth rate measurements as

$$\delta'_{viz} \cong 0.38 \frac{U_0 - U_\infty}{U_0 + U_\infty}$$

This correlation is practically the same as the theoretical conclusion (1): The level of this agreement lends considerable support to the theory on which eq. (1) is based.

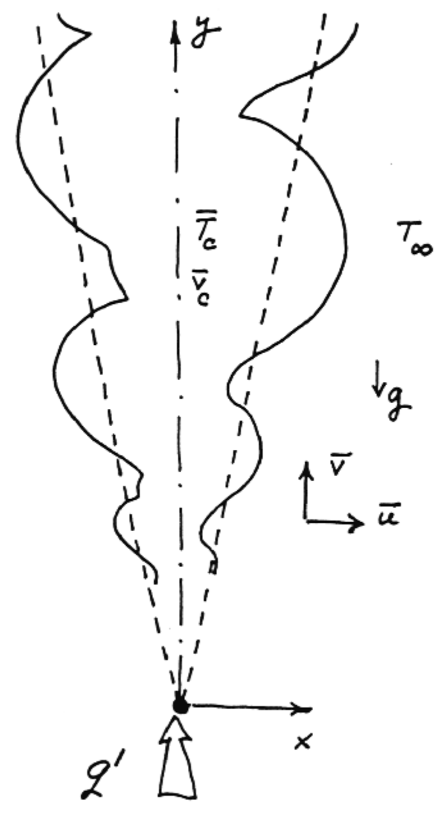


Problem 9.3. The mass continuity, momentum and energy equations are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad \text{(C)}$$

$$\frac{\partial}{\partial y}\left(\bar{v}^2\right) + \frac{\partial}{\partial x}\left(\bar{u}\,\bar{v}\right) = \frac{\partial}{\partial x}\left[\left(\nu + \varepsilon_M\right)\frac{\partial \bar{v}}{\partial x}\right] + \beta g\left(\bar{T} - T_\infty\right) \qquad \text{(M)}$$

$$\frac{\partial}{\partial y}\left(\bar{v}\,\bar{T}\right) + \frac{\partial}{\partial x}\left(\bar{u}\,\bar{T}\right) = \frac{\partial}{\partial x}\left[\left(\alpha + \varepsilon_H\right)\frac{\partial \bar{T}}{\partial x}\right] \qquad \text{(E)}$$



Integrating these equations one by one from x = 0 (centerline) to x → ∞ (outside the plume slender region) we obtain:

mass continuity:

$$\bar{u}_\infty - \underbrace{\bar{u}_0}_{\text{zero}} + \frac{d}{dy}\int_0^\infty \bar{v}\,dx = 0 \qquad (\textstyle\int\text{C})$$

momentum:

$$\frac{d}{dy}\int_0^{\infty} \bar{v}^2\, dx + \bar{u}_{\infty}\, \underbrace{\bar{v}_{\infty} - \bar{u}_0}_{\text{zeros}}\, \bar{v}_0 = \left[(\nu + \varepsilon_M)\, \underbrace{\frac{\partial \bar{v}}{\partial x}}_{\text{zero}}\right]_{\infty} - \left[(\nu + \varepsilon_M)\, \underbrace{\frac{\partial \bar{v}}{\partial x}}_{\text{zero}}\right]_0 +$$

$$+ g\beta \int_0^{\infty} (\bar{T} - T_{\infty})\, dx$$

$$\frac{d}{dy}\int_0^{\infty} \bar{v}^2\, dx = g\beta \int_0^{\infty} (\bar{T} - T_{\infty})\, dx \qquad (\int M)$$

energy:

$$\frac{d}{dy}\int_0^{\infty} \bar{v}\, \bar{T}\, dx + \bar{u}_{\infty} T_{\infty} - \underbrace{\bar{u}_0}_{\text{zero}}\, \bar{T}_0 = \left[(\alpha + \varepsilon_H)\, \underbrace{\frac{\partial \bar{T}}{\partial x}}_{\text{zero}}\right]_{\infty} - \left[(\alpha + \varepsilon_H)\, \underbrace{\frac{\partial \bar{T}}{\partial x}}_{\text{zero}}\right]_0$$

$$\frac{d}{dy}\int_0^{\infty} \bar{v}\left(\bar{T} - T_{\infty}\right) dx = 0, \quad \text{or} \quad \int_0^{\infty} \bar{v}\left(\bar{T} - T_{\infty}\right) dx = \frac{q'}{\rho\, c_P} \qquad (\int E)$$

For scale analysis, we focus on the slender region of height y and thickness D_T; from the three integral equations derived above, we deduce

mass continuity:

$$\bar{u}_{\infty} \sim \bar{v}_c \frac{D_T}{y}$$

momentum:

$$\frac{1}{y}\, \bar{v}_c^2\, D_T \sim g\beta\, (\bar{T}_c - T_{\infty})\, D_T$$

energy:

$$\bar{v}_c\, (\bar{T}_c - T_{\infty})\, D_T \sim q'/(\rho\, c_P)$$

These are 3 equations (scaling laws) for 4 unknowns [$\bar{v}_c$, $\bar{u}_{\infty}$, $(\bar{T}_c - T_{\infty})$, D_T]; the additional equation is supplied by the theoretical argument of Fig. 9.3, namely, the linear growth of all turbulent streams (Table 9.1)

$$D_T \sim y$$

or, using the mass continuity scaling ($\bar{u}_{\infty} \sim \bar{v}_c\, D_T/y$), the entrainment hypothesis

$$\bar{u}_{\infty} \sim \bar{v}_c$$

Combining the momentum and energy scaling laws yields

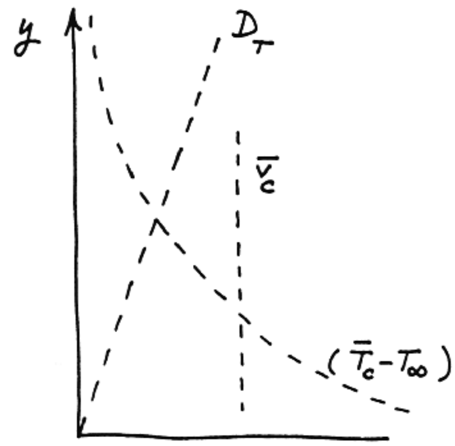
$$(\bar{T}_c - T_{\infty}) \sim \left(\frac{q'}{\rho\, c_P}\right)^{2/3} (g\beta)^{-1/3}\, y^{-1}$$

$$\bar{v}_c \sim \left(\frac{q'}{\rho\, c_p}\right)^{1/3} (g\beta)^{1/3}$$

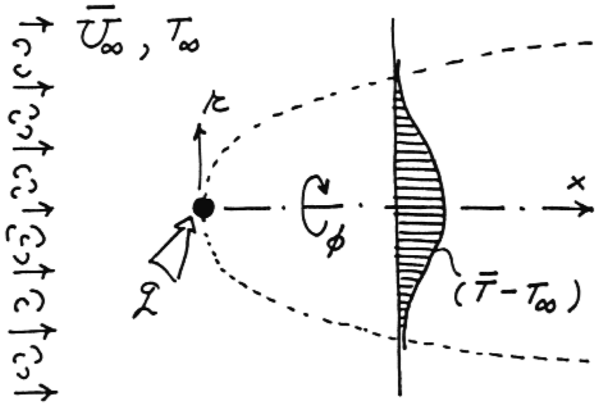
More refined integral results can be developed by combining

$$\frac{\bar{v}}{\bar{v}_c} = \exp\left[-\left(\frac{x}{my}\right)^2\right], \quad \text{and} \quad \frac{\bar{T}-T_\infty}{\bar{T}_c - T_\infty} = \exp\left[-\left(\frac{x}{my}\right)^2\right]$$

with the momentum and energy integrals, ($\int$M) and ($\int$E). (Note that "m" is a constant).



Problem 9.4. Using the cylindrical system shown in the figure, the problem statement is written



$$\bar{U}_\infty \frac{\partial \bar{T}}{\partial x} = \frac{\varepsilon_H}{r} \frac{\partial}{\partial r}\left(r \frac{\partial \bar{T}}{\partial r}\right) \qquad \text{(E)}$$

$$\bar{T} \to T_\infty \quad \text{as} \quad r \to \infty \qquad \text{(BC)}$$

$$\frac{\partial \bar{T}}{\partial r} = 0 \quad \text{at} \quad r = 0 \qquad \text{(BC)}$$

$$q = \int_0^{2\pi} \int_0^{\infty} \rho\, \bar{U}_\infty c_P (\bar{T} - T_\infty)\, r\, dr\, d\phi \qquad \text{(IC)}$$

with the notation E = energy equation, BC = boundary conditions, and IC = initial condition. The object of the analysis is to find $\overline{T}(r,x)$.

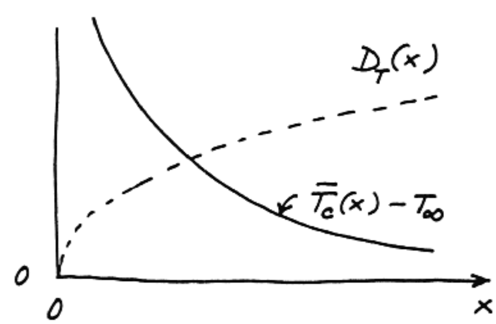
The scale analysis of the energy equation and the initial condition in the slender region $(D_T) \times (x)$ sketched above yields

$$\overline{U}_\infty \frac{\overline{T}_c - T_\infty}{x} \sim \varepsilon_H \frac{\overline{T}_c - T_\infty}{D_T^2}$$

$$q \sim \rho\, \overline{U}_\infty\, c_P \left(\overline{T}_c - T_\infty\right) D_T^2$$

where the centerline-reservoir temperature difference $\overline{T}_c - T_\infty$ is the scale of $\overline{T} - T_\infty$. Taken together, the above scaling laws dictate that

$$D_T \sim \left(\varepsilon_H x / \overline{U}_\infty\right)^{1/2} \quad \text{and} \quad \overline{T}_c - T_\infty \sim \frac{q}{\rho\, c_P\, \varepsilon_H\, x}$$



These scaling results recommend the following similarity formulation:

$$\overline{T} - T_\infty = \frac{q}{\rho\, c_P\, \varepsilon_H\, x}\, \theta(\eta)$$

$$\eta = \frac{r}{\left(\varepsilon_H\, x/\overline{U}_\infty\right)^{1/2}}$$

The problem statement becomes:

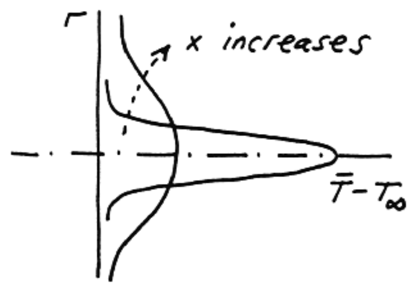
$$-\theta - \frac{\eta}{2}\theta' = \frac{\theta'}{\eta} + \theta'' \qquad \text{(E)}$$

$$\theta(\infty) = 0 \qquad \text{(BC)}$$

$$\theta'(0) = 0 \qquad \text{(BC)}$$

$$1 = 2\pi \int_0^\infty \theta\, \eta\, d\eta \qquad \text{(IC)}$$

There is more than one way to solve this; the way I prefer is to first think about the physical characteristics of $(\bar{T} - T_\infty)$ as a function of both x and r (revealed by scale analysis), and then to note that the same picture is present in the field of transient conduction in unbounded media. The attached sketch is very much like what we would be drawing for transient conduction away from a buried instantaneous line heat source, where x would be replaced by t (time). Therefore, the hint contributed by this observation is that (in transient conduction) the temperature profile <u>shape</u> is of the Gaussian type. This means that we have good reasons to try



$$\theta(\eta) = c_1\, e^{-c_2 \eta^2},$$

noting further that this form satisfies the two boundary conditions (BC). We find that this idea works, and the energy equation yields

$$c_2 = \frac{1}{4}$$

while the (IC) condition yields

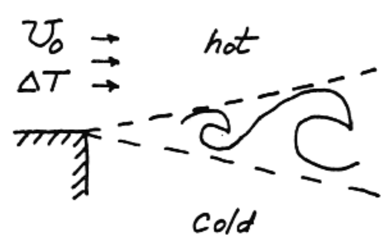
$$c_1 = \frac{1}{4\pi}$$

In conclusion, the temperature field downstream from a steady point source in a uniform stream with grid-generated eddies is

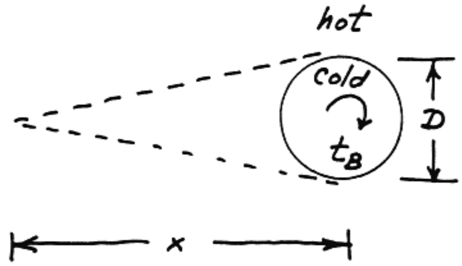
$$\bar{T}(x,r) - T_\infty = \frac{1}{4\pi} \frac{q}{\rho\, c_P\, \varepsilon_H\, x}\, e^{-\frac{1}{4}\frac{r^2 \bar{U}_\infty}{\varepsilon_H x}}$$

<u>Problem 9.5</u>. Consider the free shear layer formed between two semiinfinite fluid reservoirs at different temperatures. The velocity mixing region is as shown in Figs. 9.1 and 9.2, and predicted based on the scaling argument built around Fig. 9.3. The thickness of the velocity mixing region, D, is dictated by a large eddy of diameter D formed at a distance x from the virtual origin, such that D and x are proportional,

$$D \sim x$$

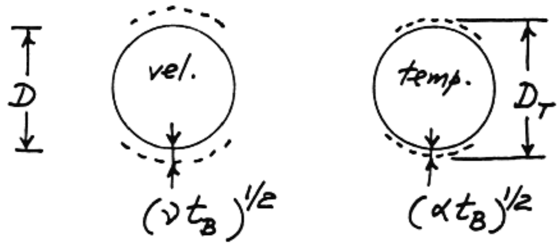


Let D_T be the thickness of the thermal mixing layer at the same x. The Ratio D_T/D depends on the Prandtl number.



Pr > 1. The rotation of the D eddy brings cold fluid in immediate contact with the hot (upper) stream. Thermal diffusion has t_B (seconds) to act, where $t_B \sim D/U_0$ is the tumbling time of the D eddy. During the t_B time the hot/cold interface between eddy fluid and reservoir fluid is smoothed over a distance (thickness) of order $(\alpha\, t_B)^{1/2}$ (note that this distance is the thermal diffusion depth associated with the time t_B). The thermal thickness of the shear layer scales as

$$D_T \sim D + (\alpha\, t_B)^{1/2}$$



Now, in Pr > 1 fluids the correction $(\alpha\, t_B)^{1/2}$ is negligible compared with D, because

$$\begin{array}{ccccc} (\alpha\, t_B)^{1/2} & \ll & (\nu\, t_B)^{1/2} & \ll & D \\ & \uparrow & & \uparrow & \\ & \text{because} & & \text{because } N_B > 1, & \\ & \text{Pr} > 1 & & \text{eq. (6.14) (note that the} & \\ & & & \text{shear layer is turbulent)} & \end{array}$$

In conclusion, in Pr > 1 fluids we must have

$$D_T \sim D$$

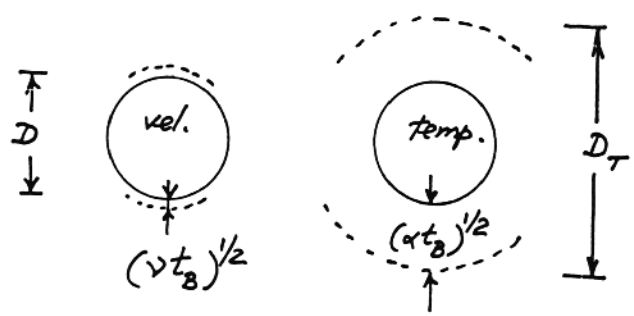
regardless of the actual value of Pr.

Pr < 1. In liquid metals the thermal penetration distance $(\alpha\, t_B)^{1/2}$ can be greater than D, however $(\nu\, t_B)^{1/2}$ is still negligible compared with D, because the flow is turbulent ($N_B > 1$). The ratio D_T/D is therefore

$$\frac{D_T}{D} \sim \frac{D + (\alpha\, t_B)^{1/2}}{D} \sim 1 + \frac{(\alpha\, D/U_0)^{1/2}}{D}$$

or recalling that D ~ x,

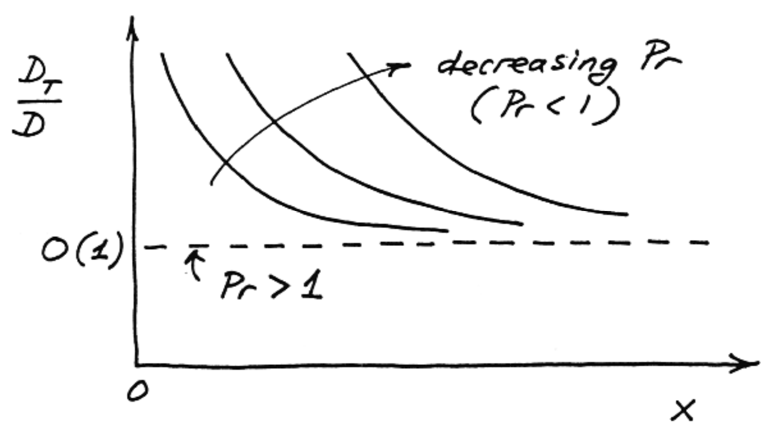
$$\frac{D_T}{D} \sim 1 + \left(\frac{U_0\, x}{\nu}\right)^{-1/2} Pr^{-1/2}$$



Keeping in mind that in scale analysis numerical coefficients of order O(1) are neglected, we conclude that D_T/D must obey a relationship of the type

$$\frac{D_T}{D} = c_1 \left[1 + c_2 \left(\frac{U_0\, x}{\nu}\right)^{-1/2} Pr^{-1/2}\right]$$

where c_1 and c_2 are of order O(1). In conclusion, in liquid metals D_T is greater than D, however, this discrepancy vanishes sufficiently far downstream from the virtual origin. The theory proposed in this problem solution remains to be tested experimentally.



Chapter 10

CONVECTION WITH CHANGE OF PHASE

Problem 10.1. With reference to the drawing that accompanies the problem statement, we write the first law for the control volume,

$$0 = h_g\,\Gamma(L) - H(L) - q' \tag{1}$$

and then use eq. (10.7) in order to evaluate the enthalpy flowrate through the bottom end of the film,

$$H(L) = \left[h_f - \frac{3}{8}\,c_{P,l}\,(T_{sat} - T_w)\right]\Gamma(L) \tag{2}$$

Combining eqs. (1) and (2) we obtain

$$\begin{aligned} q' &= \left[h_{fg} + \frac{3}{8}\,c_{P,l}\,(T_{sat} - T_w)\right]\Gamma(L) \\ &= h'_{fg}\,\Gamma(L) \end{aligned} \tag{3}$$

The actual numerical coefficient that will figure in front of $c_{P,l}(T_{sat} - T_w)$ will depend on the shapes of the u and T profiles across the bottom end of the film. Aside from this, the $q' \sim \Gamma(L)$ proportionality written as eq. (3) has general validity.

Problem 10.2. We first note that

$$q' = \bar{h}_L\,L\,(T_{sat} - T_w) \tag{10.20}$$

$$\Gamma(L) = \frac{q'}{h'_{fg}} \tag{10.21}$$

and that the elimination of q' yields

$$L = \frac{h'_{fg}\,\Gamma(L)}{\bar{h}_L\,(T_{sat} - T_w)} \tag{i}$$

Next, we evaluate $\overline{h_L}$ based on eq. (10.15), and eventually eliminate L using eq. (i). The sequence of these operations is:

$$\bar{h}_L = \frac{k_l}{L} 0.943 \left[\frac{L^3 h'_{fg}\, g \overbrace{(\rho_l - \rho_v)}^{\approx \rho_l}}{k_l \nu_l (T_{sat} - T_w)} \right]^{1/4} \tag{10.15}$$

$$\bar{h}_L^4 = k_l^3 (0.943)^4 \frac{1}{L} \frac{h'_{fg}\, g\, \rho_l}{\nu_l (T_{sat} - T_w)}$$

$$= k_l^3 (0.943)^4 \frac{\bar{h}_L (T_{sat} - T_w)}{h'_{fg}\, \Gamma} \frac{h'_{fg}\, g\, \rho_l}{\nu_l (T_{sat} - T_w)}$$

$$\bar{h}_L^3 = k_l^3 (0.943)^4 \frac{g\, \rho_l}{\Gamma\, \nu_l}$$

$$\frac{\bar{h}_L^3}{k_l^3} \frac{\nu_l^2}{g} = (0.943)^4\, 4\, \frac{\mu_l}{4\Gamma}$$

$$\frac{\bar{h}_L}{k_l} \left(\frac{\nu_l^2}{g} \right)^{1/3} = \underbrace{(0.943)^{4/3}\, 4^{1/3}}_{1.468}\ Re_L^{-1/3} \tag{10.24}$$

Problem 10.3. We begin with the laminar-film solutions

$$\frac{\bar{h}_L L}{k_l} = 0.943 \left(L^3 C \right)^{1/4} \tag{10.15}$$

$$\frac{\bar{h}_D D}{k_l} = 0.729 \left(D^3 C \right)^{1/4} \tag{10.30}$$

in which C represents the physical constant

$$C = \frac{h'_{fg}\, g\, (\rho_l - \rho_v)}{k_l \nu_l (T_{sat} - T_w)}$$

Next, we recall that q' and Γ are proportional. We equate the cooling rate of the two-sided slab with the cooling rate provided by the single horizontal cylinder:

$$q'_{slab} = q'_{cylinder}$$

$$\bar{h}_L \, 2L \, (T_{sat} - T_w) = \bar{h}_D \, \pi D \, (T_{sat} - T_w)$$

$$0.943 \, k_l \, L^{-1/4} C^{1/4} \, 2L \, (T_{sat} - T_w) = 0.729 \, k_l D^{-1/4} C^{1/4} \, \pi D \, (T_{sat} - T_w)$$

$$(0.943) \, 2 \, L^{3/4} = (0.729) \, \pi \, D^{3/4}$$

$$\frac{L}{D} = 1.296$$

In conclusion, the special diameter is $D = 0.772L$.

<u>Problem 10.4</u>. We begin with the laminar-film solutions

$$\frac{\bar{h}_L \, L}{k_l} = 0.943 \left(L^3 \, C\right)^{1/4} \tag{10.15}$$

$$\frac{\bar{h}_D \, D}{k_l} = 0.729 \left(D^3 \, C\right)^{1/4} \tag{10.30}$$

in which C represents the constant

$$C = \frac{h'_{fg} \, g \, (\rho_l - \rho_v)}{k_l \, \nu_l \, (T_{sat} - T_w)}$$

In the present case the perimeter of the flattened cross-section equals the original perimeter, therefore

$$2L = \pi D, \qquad \text{or} \qquad \frac{L}{D} = \frac{\pi}{2}$$

Next, we calculate the relative change in the total condensation rate:

$$\begin{aligned} \frac{\Gamma_{flat}}{\Gamma_{round}} = \frac{q'_{flat}}{q'_{round}} &= \frac{\bar{h}_L \, 2L \, (T_{sat} - T_w)}{\bar{h}_D \, \pi D \, (T_{sat} - T_w)} \\ &= \frac{0.943 \, k_l \, L^{-1/4} \, C^{1/4} \, 2L}{0.729 \, k_l \, D^{-1/4} \, C^{1/4} \, \pi D} \\ &= \frac{0.943}{0.729} \, \frac{2}{\pi} \left(\frac{L}{D}\right)^{3/4} = \frac{0.943}{0.729} \left(\frac{2}{\pi}\right)^{1/4} \\ &= 1.155 \end{aligned}$$

In conclusion, the flattening of the thin-walled tube promises an increase of about 15 percent in the condensation rate.

Problem 10.5. The film temperature of the condensate is (80°C + 100°C)/2 = 90°C, at which the pertinent physical properties have the following values:

$$c_{P,l} = 4.205 \frac{kJ}{kg} \quad \mu_l = 3.16 \times 10^{-4} \frac{kg}{m\,s}$$

$$k_l = 0.67 \frac{W}{m\,K} \quad n_l = 3.27 \times 10^{-7} \frac{m^2}{s}$$

$$\rho_l = 965.3 \frac{kg}{m^3} \quad Pr_l = 1.98$$

Note further that h_{fg} = 2257 kJ/kg, therefore

$$Ja = \frac{c_{P,l}\,(T_{sat} - T_w)}{h_{fg}} = 0.0373$$

$$h'_{fg} = h_{fg}\,(1 + 0.68 \times 0.0373) \cong 2314 \frac{kJ}{kg}$$

a) When the surface is tilted at 45° and looks up, we use g cos 45° = 0.707g instead of g in the B parameter of Fig. 10.6:

$$B = L\,(T_{sat} - T_w)\,\frac{4\,k_l}{\mu_l\,h'_{fg}}\left(\frac{0.707\,g}{\nu_l^2}\right)^{1/3} = 2945$$

By entering this B value and $Pr_l \cong 2$ in Fig. 10.6, we obtain

$$Re_L \cong 730$$

$$\Gamma\,(L) = 730\,\frac{\mu_l}{4} = 0.057\,\frac{kg/s}{m}$$

This condensation rate is 10 percent smaller than when the surface is vertical.

b) In the case where the L-wide surface is perfectly horizontal, we use eq. (10.33) in which (note $\rho_l >> \rho_v$),

$$\frac{L^3\,h'_{fg}\,g\,\rho_l}{k_l\,\nu_l\,(T_{sat} - T_w)} = 5 \times 10^{15}$$

therefore

$$\bar{h}_L = 1.079\,\frac{k_l}{L}\left(5 \times 10^{15}\right)^{1/5} = 997.5\,\frac{W}{m^2K}$$

$$q' = \bar{h}_L\,L\,(T_{sat} - T_w) = 19\,949\,\frac{W}{m}$$

The condensation rate collected over the width L,

$$\dot{m}' = \frac{q'}{h'_{fg}} = \frac{19\,949 \,\frac{W}{m}}{2314 \,\frac{kJ}{kg}} = 0.0086 \,\frac{kg/s}{m}$$

represents only 13.7 percent of the condensation rate produced by the same surface in the vertical position.

The film Reynolds number at the edge of the horizontal surface can be calculated by first noting the mass flowrate that spills over one edge,

$$\Gamma_{edge} = \frac{\dot{m}'}{2} = 0.0043 \,\frac{kg/s}{m}$$

therefore

$$Re = \frac{4\Gamma}{\mu_l} = \frac{4 \times 0.0043 \,\frac{kg}{s\,m}}{3.16 \times 10^{-4} \,\frac{kg}{s\,m}} \cong 54$$

This Reynolds number is of the same order as 30, therefore most of the film is in the laminar flow regime.

Problem 10.6. The properties of water at atmospheric pressure and film temperature (100°C + 60°C)/2 = 80°C are (see Appendix C):

$$c_{P,l} = 4.196 \,\frac{kJ}{kg\,K} \qquad \mu_l = 3.55 \times 10^{-4} \,\frac{kg}{m\,s}$$

$$k_l = 0.67 \,\frac{W}{m\,K} \qquad \nu_l = 3.66 \times 10^{-7} \,\frac{m^2}{s}$$

$$\rho_l = 971.8 \,\frac{kg}{m^3} \qquad Pr_l = 2.23$$

The latent heat of condensation at atmospheric pressure (or at 100°C) is h_{fg} = 2257 kJ/kg. The Jakob number is small,

$$Ja = \frac{c_{P,l}\,(T_{sat} - T_w)}{h_{fg}} = 0.074$$

therefore h'_{fg} is nearly the same as h_{fg},

$$h'_{fg} = h_{fg}\,(1 + 0.68 \times 0.074) = 2371 \,\frac{kJ}{kg}$$

The dimensionless group that appears on the right side of eq. (10.30) is (note $\rho_l >> \rho_v$),

$$\frac{D^3 h'_{fg}\, g\, \rho_l}{k_l\, \nu_l\, (T_{sat} - T_w)} = \frac{(0.02)^3\, m^3\; 2371\, \frac{10^3 J}{kg}\; 9.81\, \frac{m}{s^2}\; 971.8\, \frac{kg}{m^3}}{0.67\, \frac{W}{m\,K}\; (3.66)\, 10^{-7}\, \frac{m^2}{s}\; 40\,K}$$

$$= 1.844 \times 10^{10}$$

therefore eq. (10.30) yields

$$\overline{Nu}_D = 0.729\, (1.844 \times 10^{10})^{1/4} = 268.6$$

and

$$\bar{h}_D = \frac{k_l}{D}\, \overline{Nu}_D = 8999\, \frac{W}{m^2K}$$

a) In the first design, we see vertical columns of 3, 4 and 5 tubes. The heat transfer coefficients averaged over each type of column are, eq. (10.32),

$$\bar{h}_{D,3} = \frac{\bar{h}_D}{3^{1/4}} = 6837.6\, \frac{W}{m^2K}$$

$$\bar{h}_{D,4} = \frac{\bar{h}_D}{4^{1/4}} = 6363.1\, \frac{W}{m^2K}$$

$$\bar{h}_{D,5} = \frac{\bar{h}_D}{5^{1/4}} = 6017.9\, \frac{W}{m^2K}$$

There are two 3-tube columns, two 4-tube columns, and one 5-tube column. This means that the heat transfer coefficient averaged over all 19 tubes is

$$\bar{h}_a = \frac{2 \times 3}{19}\, \bar{h}_{D,3} + \frac{2 \times 4}{19}\, \bar{h}_{D,4} + \frac{1 \times 5}{19}\, \bar{h}_{D,5}$$

$$= 6422\, \frac{W}{m^2K}$$

and this yields the following heat transfer and condensation rates:

$$q'_a = n\,\pi D\,\bar{h}_a\,(T_{sat} - T_w)$$

$$= 19\,\pi\,0.02\text{ m}\;6422\,\frac{\text{W}}{\text{m}^2\text{K}}\;40\text{ K}$$

$$= 3.07 \times 10^5\,\frac{\text{W}}{\text{m}}$$

$$\dot{m}'_a = \frac{q'_a}{h'_{fg}} = \frac{3.07 \times 10^5\text{ W/m}}{2371 \times 10^3\text{ J/kg}}$$

$$= 0.129\,\frac{\text{kg}}{\text{s m}}$$

b) In the second design, there are vertical columns of 1, 2 and 3 tubes, therefore

$$\bar{h}_{D,1} = \frac{\bar{h}_D}{1^{1/4}} = 8999\,\frac{\text{W}}{\text{m}^2\text{K}}$$

$$\bar{h}_{D,2} = \frac{\bar{h}_D}{2^{1/4}} = 7567.1\,\frac{\text{W}}{\text{m}^2\text{K}}$$

$$\bar{h}_{D,3} = \frac{\bar{h}_D}{3^{1/4}} = 6837.6\,\frac{\text{W}}{\text{m}^2\text{K}}$$

$$\bar{h}_b = \frac{2 \times 1}{19}\,\bar{h}_{D,1} + \frac{4 \times 2}{19}\,\bar{h}_{D,2} + \frac{3 \times 3}{19}\,\bar{h}_{D,3}$$

$$= 7372\,\frac{\text{W}}{\text{m}^2\text{K}}$$

$$q'_b = n\,\pi D\,\bar{h}_b\,(T_{sat} - T_w) = 3.52 \times 10^5\,\frac{\text{W}}{\text{m}}$$

$$\dot{m}'_b = \frac{q'_b}{h'_{fg}} = 0.148\,\frac{\text{kg}}{\text{s m}}$$

Comparing $\dot{m}'_b$ with $\dot{m}'_a$, we conclude that the 90-degree rotation of the tube bank induces a 15-percent increase in the condensation rate.

Problem 10.7. The characteristic length of the horizontal strip of width L and length Z is

$$L_c = \frac{A}{p} = \frac{LZ}{2Z} = \frac{L}{2} \tag{1}$$

Using this and eq. (10.33), the average heat transfer coefficient can be rewritten as

$$\begin{aligned} \bar{h}_L &= 1.079\,L^{-2/5}\,C \\ &= 0.818\,L_c^{-2/5}\,C \end{aligned} \tag{2}$$

where C is the property group

$$C = k_l \left[\frac{h'_{fg}\, g\, (\rho_l - \rho_v)}{k_l\, \nu_l\, (T_{sat} - T_w)} \right]^{1/5} \tag{3}$$

The overall Nusselt number based on L_c is therefore

$$\overline{Nu}_{L_c} = \frac{\bar{h}_L L_c}{k_l} \cong 0.82 \left[\frac{L_c^3\, h'_{fg}\, g\, (\rho_l - \rho_v)}{k_l\, \nu_l\, (T_{sat} - T_w)} \right]^{1/5} \tag{4}$$

The same operations can be repeated for the horizontal disc facing upward:

$$\begin{aligned} L_c &= \frac{A}{p} = \frac{\pi D^2/4}{D} = \frac{D}{4} \\ \bar{h}_D &= 1.368\, D^{-2/5}\, C \\ &= 0.786\, L_c^{-2/5} C \end{aligned} \tag{10.34}$$

$$\overline{Nu}_{L_c} = \frac{\bar{h}_D L_c}{k_l} \cong 0.79 \left[\frac{L_c^3\, h'_{fg}\, g\, (\rho_l - \rho_v)}{k_l\, \nu_l\, (T_{sat} - T_w)} \right]^{1/5} \tag{5}$$

Comparing eqs. (4) and (5), we see that the use of the characteristic length L_c leads to a $\overline{Nu}_{L_c}$ formula that is almost independent of the shape of the surface. In conclusion, for an upward facing surface whose shape is somewhere between "very long" (eq. 4) and "round" (eq. 5), we can estimate the average heat transfer coefficient with the approximate formula

$$\overline{Nu}_{L_c} = \frac{\bar{h} L_c}{k_l} \cong 0.8 \left[\frac{L_c^3\, h'_{fg}\, g\, (\rho_l - \rho_v)}{k_l\, \nu_l\, (T_{sat} - T_w)} \right]^{1/5} \tag{6}$$

Problem 10.8. a) The balance of vertical forces on the hemispherical control volume requires

$$\pi r^2 P_v = \pi r^2 P_l + 2 \pi r \sigma$$

which yields

$$r = \frac{2 \sigma}{P_v - P_l} \tag{1}$$

b) Next, by assuming that the vapor is saturated, the pressure difference $(P_v - P_l)$ can be related to the corresponding temperature difference $(T_v - T_l)$ through the Clausius-Clapeyron relation

$$\frac{dP}{dT} = \frac{h_{fg}}{Tv_{fg}} \qquad (2)$$

in which $T = T_l = T_{sat}$, and

$$v_{fg} = v_g - v_f \cong v_g = \frac{1}{\rho_v}$$

$$\frac{dP}{dT} = \frac{P_v - P_l}{T_v - T_l}, \quad \text{where } T_l = T_{sat}$$

therefore eq. (2) becomes

$$\frac{P_v - P_l}{T_v - T_{sat}} = \frac{h_{fg}\,\rho_v}{T_{sat}}$$

Eliminating $(P_v - P_l)$ between eqs. (1) and (3) leads to

$$r = \frac{2\,\sigma\,T_{sat}}{h_{fg}\,\rho_v\,(T_v - T_{sat})}$$

in which T_{sat} is an absolute temperature (i.e. it is expressed in degrees Kelvin).

c) For $T_{sat} = 100°C$, $T_v - T_{sat} = 2°C$ and the properties listed for water at 100°C in Table 10.2, the bubble radius is

$$r = \frac{2 \times 0.059\,\frac{N}{m}\,373\text{ K}}{2257\,\frac{kJ}{kg}\,0.6\,\frac{kg}{m^3}\,2\text{ K}} = 16.3 \times 10^{-6}\text{m} = 0.016\text{ mm}$$

Problem 10.9. The properties of saturated helium at atmospheric pressure ($T_{sat} = 4.2$ K) are:

$$\rho_l = 125\,\frac{kg}{m^3} \qquad \sigma = 10^{-4}\,\frac{N}{m}$$

$$\rho_v = 16.9\,\frac{kg}{m^3} \qquad h_{fg} = 20.42\,\frac{kJ}{kg}$$

$$c_{P,l} = 4.98\,\frac{kJ}{kg\,K} \qquad Pr_l = 0.8$$

$$\mu_l = 3.17 \times 10^{-6}\,\frac{kg}{s\,m}$$

The excess temperature at $q''_w = 10^3\text{ W/m}^2$ can be calculated using eq. (10.43),

$$T_w - T_{sat} = \frac{h_{fg}}{c_{P,l}}\,Pr_l^s\,C_{sf}\left[\frac{q''_w}{\mu_l\,h_{fg}}\left(\frac{\sigma}{g\,(\rho_l - \rho_v)}\right)^{1/2}\right]^{1/3}$$

in which

$$\frac{q_w''}{\mu_l h_{fg}}\left(\frac{\sigma}{g(\rho_l-\rho_v)}\right)^{1/2} = \frac{10^3 \frac{W}{m^2}}{3.17\times10^{-6}\frac{kg}{s\,m}\,20.42\frac{kJ}{kg}}\left(\frac{10^{-4}\frac{N}{m}}{9.81\frac{m}{s^2}(125-16.9)\frac{kg}{m^3}}\right)^{1/2}$$

$$= 4.75$$

Assuming $C_{sf} = 0.02$ and $s = 1.7$ in eq. (10.43), we obtain finally

$$T_w - T_{sat} = \frac{20.42\frac{kJ}{kg}}{4.98\frac{kJ}{kg\,K}}\,0.8^{1.7}\,0.02\,(4.75)^{1/3}$$

$$\cong 0.1\text{ K}$$

The peak heat flux can be calculated based on eq. (10.45),

$$q_{max}'' = 0.149\, h_{fg}\, \rho_v^{1/2}[\sigma\, g\,(\rho_l-\rho_v)]^{1/4}$$

$$= 0.149\;\; 20.42\frac{kJ}{kg}(16.9)^{1/2}\frac{kg^{1/2}}{m^{3/2}}\left[10^{-4}\frac{N}{m}\,9.81\frac{m}{s^2}\,(125\text{–}16.9)\frac{kg}{m^3}\right]^{1/4}$$

$$= 7137\frac{W}{m^2}$$

to conclude that the heat leak q_w'' is roughly 14 percent of the peak nucleate boiling heat flux.

Problem 10.10. The temperature of saturated water at 4.76×10^5 N/m^2 is

$$T_{sat} = 150°C$$

The corresponding properties of water at this temperature are

$$\rho_l = 917\frac{kg}{m^3} \qquad \sigma = 0.048\frac{N}{m}$$

$$\rho_v = 2.55\frac{kg}{m^3} \qquad h_{fg} = 2114\frac{kJ}{kg}$$

$$c_{P,l} = 4.27\frac{kJ}{kg\,K} \qquad Pr_l = 1.17$$

$$\mu_l = 1.85\times10^{-4}\frac{kg}{s\,m}$$

For nucleate boiling of water on a polished copper surface, Table 10.1 suggests $C_{sf} = 0.013$ and $s = 1$, and eq. (10.44) yields

$$\frac{g\,(\rho_l - \rho_v)}{\sigma} = \frac{9.81\,\frac{m}{s^2}\,(917 - 2.55)\,\frac{kg}{m^3}}{0.048\,\frac{N}{m}}$$

$$= 1.869 \times 10^5\,\frac{1}{m^2}$$

$$\frac{c_{P,l}\,(T_w - T_{sat})}{Pr_l^s\,C_{sf}\,h_{fg}} = \frac{4.27\,\frac{kJ}{kg\,K}\,(160 - 150)\,K}{1.17 \times 0.013 \times 2114\,\frac{kJ}{kg}}$$

$$= 1.328$$

$$q_w'' = 1.85 \times 10^{-4}\,\frac{kg}{s\,m}\,2114\,\frac{kJ}{kg}\left(1.869 \times 10^5\,\frac{1}{m^2}\right)^{1/2}(1.328)^3$$

$$= 3.96 \times 10^5\,\frac{W}{m^2}$$

This heat flux level is only a fraction (17.3 percent) of the peak heat flux, which can be calculated based on eq. (10.45):

$$q_{max}'' = 0.149\,h_{fg}\,\rho_v^{1/2}\,(\sigma\,g\,\rho_l)^{1/4}$$

$$= 0.149 \times 2114\,\frac{kJ}{kg}\,(2.55)^{1/2}\,\frac{kg^{1/2}}{m^{3/2}}\left(0.048\,\frac{N}{m}\,9.81\,\frac{m}{s^2}\,917\,\frac{kg}{m^3}\right)^{1/4}$$

$$\cong 2.3 \times 10^6\,\frac{W}{m^2}$$

This means that the nucleate-boiling assumption is correct. Finally, we evaluate the total heat transfer rate

$$q_w = \frac{\pi}{4}\,D^2\,q_w'' = \frac{\pi}{4}\,(0.2)^2\,m^2\,3.96 \times 10^5\,\frac{W}{m^2}$$

$$= 1.244 \times 10^4\ W.$$

Problem 10.11. a) Writing approximately $q_w = \dot{m}\, h_{fg}$, we calculate in order

$$\dot{m} = \frac{q_w}{h_{fg}} = \frac{1.244 \times 10^4 \text{ W}}{2114 \frac{\text{kJ}}{\text{kg}}} = 0.00588 \frac{\text{kg}}{\text{s}}$$

$$m = \rho_l \frac{\pi}{4} D^2 H = 917 \frac{\text{kg}}{\text{m}^3} \frac{\pi}{4} (0.2)^2 \text{ m}^2\, 0.05 \text{ m}$$

$$= 1.44 \text{ kg}$$

$$t = \frac{m}{\dot{m}} = \frac{1.44 \text{ kg}}{0.00588 \text{ kg/s}} = 245 \text{ s} = 4.1 \text{ minutes}$$

in which m is the original liquid inventory, and t the time needed to evaporate all the liquid.

b) The vessel defines a control volume whose contents represent a thermodynamic system that operates in unsteady (time dependent) fashion. With reference to the figure attached to the problem statement, we write the first law

$$\frac{dU}{dt} = q_w - \dot{m}\, h_g \tag{1}$$

in which U is the instantaneous internal-energy inventory

$$U = m_f u_f + m_g u_g \tag{2}$$

and (m_f, m_g) are the (liquid, vapor) mass inventories. Combined, eqs. (1) and (2) yield

$$\dot{m}_f u_f + \dot{m}_g u_g = q_w - \dot{m}\, h_g \tag{3}$$

The conservation of mass requires

$$\frac{d}{dt}\left(m_f + m_g\right) = -\,\dot{m} \tag{4}$$

in other words

$$\dot{m}_f + \dot{m}_g = -\,\dot{m} \tag{5}$$

where $\dot{m}$ is the flowrate of the escaping steam. Finally, we note that the volume of the vessel remains constant,

$$V = m_f v_f + m_g v_g \tag{6}$$

or, after taking the time derivative,

$$0 = \dot{m}_f v_f + \dot{m}_g v_g$$

Equations (5) and (7) can be used to express $\dot{m}_f$ and $\dot{m}_g$ in terms of $\dot{m}$ and v_f and v_g:

$$\dot{m}_f = - \dot{m}\frac{1}{1 - v_f/v_g} \tag{8}$$

$$\dot{m}_g = \dot{m}\frac{v_f/v_g}{1 - v_f/v_g} \tag{9}$$

Substituting eqs. (8,9) into the first law (3), we obtain the final result

$$\frac{q_w}{\dot{m}} = h_g - \frac{u_f v_g - u_g v_f}{v_g - v_f} \tag{10}$$

The right hand side can be evaluated numerically using the properties of saturated water. The following results show that this quantity deviates from the h_{fg} value (assumed in part a), as the liquid-vapor mixture approaches the critical point:

T	the right side of eq. (10)	h_{fg}
150°C	2120.2 $\frac{kJ}{kg}$	2114.3 $\frac{kJ}{kg}$
360°C	990.4 $\frac{kJ}{kg}$	720.5 $\frac{kJ}{kg}$

Problem 10.12. Assuming that the dominant mode of heat transfer across the film is radiation,

$$\bar{h} \cong \bar{h}_{rad}$$

we write

$$q''_w = \bar{h}_{rad}(T_w - T_{sat})$$

$$= \frac{\sigma \varepsilon_w (T_w^4 - T_{sat}^4)}{T_w - T_{sat}} (T_w - T_{sat})$$

$$\cong \sigma T_w^4$$

because $T_w^4 << T_{sat}^4$, and $\varepsilon_w = 1$. Numerically, this means

$$T_w = \left(\frac{q''_w}{\sigma}\right)^{1/4} = \left(\frac{10^6 \ \frac{W}{m^2}}{5.669 \times 10^{-8} \ \frac{W}{m^2 K^4}}\right)^{1/4}$$

$$= 2049 \text{ K} \ = \ 1776°C$$

or an excess temperature of 1776°C – 100°C = 1676°C. This excess temperature is compatible with the values suggested in Fig. 10.15.

The actual excess temperature will be somewhat lower than the value calculated above, because the convection effect will contribute to an $\bar{h}$ value that is a bit higher than $\bar{h}_{rad}$.

Problem 10.13. a) The steam properties are evaluated at the average temperature of the vapor film, (354°C + 100°C)/2 = 227°C:

$$\rho_v = 0.435 \ \frac{kg}{m^3} \qquad c_{P,v} = 1.983 \ \frac{kJ}{kg\ K}$$

$$\nu_v = 4 \times 10^{-5} \ \frac{m^2}{s} \qquad k_v = 0.0358 \ \frac{W}{m\ K}$$

The relevant properties of saturated 100°C water are (Table 10.2),

$$\rho_l = 958 \ \frac{kg}{m^3} \qquad \sigma = 0.059 \ \frac{N}{m}$$

$$h_{fg} = 2257 \ \frac{kJ}{kg}$$

and, using eq. (10.49),

$$h'_{fg} = h_{fg} + 0.4 \ c_{P,v} \ (T_w - T_{sat})$$

$$= 2257 \ \frac{kJ}{kg} + 0.4 \times 1.983 \ \frac{kJ}{kg\ K} \ (354 - 100)\ K$$

$$= 2458.5 \ \frac{kJ}{kg}$$

The convection heat transfer coefficient for film boiling on the sphere is

$$\frac{\bar{h}_D D}{k_v} = 0.67 \left[\frac{D^3 \ h'_{fg} \ g \ (\rho_l - \rho_v)}{k_v \ \nu_v \ (T_w - T_{sat})}\right]^{1/4} \qquad (10.48)$$

$$= 0.67 \left(\frac{0.02^3 \, m^3 \, 2458.5 \, \frac{kJ}{kg} \, 9.81 \, \frac{m}{s^2} \, (958 - 0.435) \, \frac{kg}{m^3}}{0.0358 \, \frac{W}{m \cdot K} \, 4 \times 10^{-5} \, \frac{m^2}{s} \, (354 - 100) \, K} \right)$$

$$= 0.67 \left(5.08 \times 10^8\right)^{1/4} = 100.6$$

$$\bar{h}_D = 100.6 \, \frac{k_v}{D} = 100.6 \, \frac{0.0358 \, W}{m \, K} \, \frac{1}{0.02 \, m}$$

$$= 180 \, \frac{W}{m^2 \, K}$$

This value is augmented somewhat by a contribution due to radiation ($\varepsilon_w = 0.05$),

$$\bar{h}_{rad} = \frac{\sigma \, \varepsilon_w \left(T_w^4 - T_{sat}^4\right)}{T_w - T_{sat}}$$

$$= \frac{5.669 \times 10^{-8} \, W}{m^2 \, K^4} \, 0.05 \, \frac{\left(627^4 - 373^4\right) K^4}{(627 - 373) \, K}$$

$$= 1.51 \, \frac{W}{m^2 \, K}$$

$$\bar{h} = \bar{h}_D + \frac{3}{4} \, \bar{h}_{rad} = 181.1 \, \frac{W}{m^2 \, K}$$

leading to the following heat transfer rate for the entire sphere:

$$q_w = 4\pi \left(\frac{D}{2}\right)^2 q_w'' = \pi D^2 \, \bar{h} \, (T_w - T_{sat})$$

$$= \pi \, 0.02^2 \, m^2 \, 181.1 \, \frac{W}{m^2 \, K} \, (354 - 100) \, K$$

$$= 57.8 \, W$$

b) The first law for the copper ball as a lumped capacitance is

$$- \rho \, c_p V \, \frac{dT_w}{dt} = q_w$$

for which the approximate property values are:

$$\rho = 8954 \, \frac{kg}{m^3} \qquad\qquad k = 381 \, \frac{W}{m \, K}$$

$$c_p = 0.384 \, \frac{kJ}{kg \, K}$$

Calculating first the volume of the ball,

$$V = \frac{4}{3}\,\pi \left(\frac{D}{2}\right)^3 = \frac{\pi}{6}\,D^3 = \frac{\pi}{6}\,0.02^3\,\mathrm{m}^3 = 4.19 \times 10^{-6}\,\mathrm{m}^3$$

we obtain the instantaneous rate of cooling

$$-\frac{dT}{dt} = \frac{57.8\,\frac{J}{s}}{8954\,\frac{kg}{m^3}\;0.384\,\frac{10^3\,J}{kg\,K}\;4.19 \times 10^{-6}\,m^3}$$

$$= 4.01\,\frac{K}{s}$$

The time interval $\Delta t = 10$ s is therefore long enough. The temperature change after 10 seconds is

$$\Delta T_w = \frac{dT_w}{dt}\,\Delta t = -\,4.01\,\frac{K}{s}\,10\,s$$

$$\cong -\,40\,K$$

which corresponds to a new ball temperature of

$$T_w = (354 - 40)°C = 314°C$$

Problem 10.14. Relative to the coordinate system shown in the figure, the problem statement is

$$u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \tag{1}$$

$$q'' = -\,k\,\frac{\partial T}{\partial y} \quad \text{at} \quad y = 0 \tag{2}$$

$$T \rightarrow T_\infty \quad \text{as} \quad y \rightarrow \infty \tag{3}$$

The velocity components are $u = 0$ and $v = -\,V$, and, since $T(x, y) \cong T(y)$, we have $\partial^2 T/\partial x^2 = 0$. Equation (1) reduces to

$$\alpha\,\frac{d^2 T}{dy^2} + V\,\frac{dT}{dy} = 0 \tag{4}$$

We seek a solution of the type $T = c\,e^{ry}$, and find that r satisfies the equation

$$\alpha r^2 + Vr = 0 \tag{5}$$

The roots of this equation are $r_1 = 0$ and $r_2 = -V/\alpha$, such that the T solution is

$$\begin{aligned} T &= c_1 e^{r_1 y} + c_2 e^{r_2 y} \\ &= c_1 + c_2 e^{-Vy/\alpha} \end{aligned} \tag{6}$$

Combining this with the boundary condition (3) we obtain

$$T_\infty = c_1 \tag{7}$$

The boundary condition (2) requires

$$q'' = -k \left[c_2 \left(-\frac{V}{\alpha} \right) \right] \tag{8}$$

which means that

$$c_2 = \frac{q''\alpha}{kV} \tag{9}$$

In conclusion, the temperature distribution is

$$T(y) = T_\infty + \frac{q''\alpha}{kV} e^{-Vy/\alpha} \tag{10}$$

and the surface temperature is

$$T(0) = T_\infty + \frac{q''\alpha}{kV} \tag{11}$$

Let δ be the thickness of the layer of heated salt. The scale-analysis version of eq. (4) is

$$\alpha \frac{T(0) - T_\infty}{\delta^2} \sim V \frac{T(0) - T_\infty}{\delta} \tag{12}$$

which yields

$$\delta \sim \frac{\alpha}{V} \tag{13}$$

Chapter 11

MASS TRANSFER

Problem 11.1. From the definitions presented in the early part of ch. 11 we have

$$C_i = \frac{m_i}{V}, \qquad \Phi_i = \frac{m_i}{m}, \qquad x_i = \frac{n_i}{n}$$

We can rewrite Φ_i as

$$\Phi_i = \frac{m_i}{m}\frac{V}{V} = \frac{1}{\rho} C_i$$

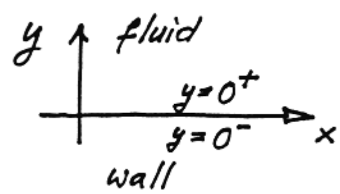
Likewise, we have

$$x_i = \frac{m_i/M_i}{m/M} = \frac{M}{M_i}\Phi_i = \frac{MC_i}{M_i\rho}$$

Putting everything together, we obtain eqs. (11.8):

$$C_i = \rho\,\Phi_i = \rho\frac{M_i}{M} x_i$$

Problem 11.2. The object of this numerical exercise is to show that by C_0 we mean the value calculated on the fluid side of the wall surface. On the wall side of the interface we have



$$(C_0)_{y=0^-} = \frac{m_{H_2O}}{V} = \frac{m_{H_2O}}{V_{H_2O}}\frac{V_{H_2O}}{V}$$

$$= 1\frac{g}{cm^3}\,0.25 = 0.25 \text{ g/cm}^3$$

For the fluid side of the interface we can write

$$(C_0)_{y=0^+} = \rho_{air}\frac{M_{H_2O}}{M_{air}} x_{H_2O}$$

From eq. (11.14), the mole fraction is P_{H_2O}/P_{atm}, where P_{H_2O} is the vapor pressure of water at 1 atm and 310 K:

$$x_{H_2O} \cong \frac{6.2 \times 10^3 \ N/m^2}{1.013 \times 10^5 \ N/m^2} \cong 0.0612$$

In conclusion, we obtain

$$(C_0)_{y=0^+} = 1.16 \times 10^{-3} \frac{g}{cm^3} \frac{18.016}{29.06} 0.0612 = 4.4 \times 10^{-5} \frac{g}{cm^3}$$

Note the difference between $(C_0)_{y=0^+}$ and $(C_0)_{y=0^-}$.

Problem 11.3. The relevant properties of atmospheric air at 25°C are

$$\rho_a = 1.185 \frac{kg}{m^3} \qquad \nu_a = 1.55 \times 10^{-5} \frac{m^2}{s}$$

The Reynolds number based on pool length shows that the boundary layer flow is turbulent,

$$Re_L = \frac{U_\infty L}{\nu_a} = 30 \frac{10^3 m}{3600s} \frac{3m}{1.55 \times 10^{-5} \ m^2/s}$$
$$= 1.61 \times 10^6 \qquad \text{(turbulent)}$$

Tables 11.1 and 11.4 show that the properties of water vapor in air are

$$D = 2.88 \times 10^{-5} \frac{m^2}{s} \qquad Sc = 0.6$$

therefore, we calculate

$$\overline{Sh} = 0.037 \ Sc^{1/3} \ Re_L^{4/5}$$
$$= 0.037 \ (0.6)^{1/3} \left(1.61 \times 10^6\right)^{4/5}$$
$$= 2882$$

$$\bar{h}_m = \overline{Sh} \frac{D}{L} = 2882 \frac{2.88 \times 10^{-5} \ m^2/s}{3m}$$
$$= 0.0277 \frac{m}{s}$$

The instantaneous water flowrate removed by the wind is

$$\dot{m} = \bar{h}_m A \left(\rho_w - \rho_\infty\right)$$

in which $A = (3m)^2 = 9m^2$. For calculating the water density at the pool surface (ρ_w) we note that the saturation pressure of water vapor at 25°C is

$$P_{sat}\,(25°C) = 3169\,\frac{N}{m^2}$$

and that the pressure of the air-water vapor mixture is 1 atm = 1.0133×10^5 N/m^2. The mole fraction of water vapor at the surface is

$$x_w = \frac{P_{sat}\,(25°C)}{1\text{ atm}} = \frac{3169}{1.0133 \times 10^5}$$
$$= 0.0313$$

The water vapor density (ρ_w) that corresponds to this mole fraction can be calculated by using eq. (11.24) and, later, eq. (11.23):

$$\rho_w = M_{H_2O}\,\frac{\rho_a}{M_a}\,x_w = 18.02\,\frac{\rho_a}{28.97}\,0.0313$$
$$= 0.0195\,\rho_a$$

Turning our attention to the calculation of ρ_∞, we recall that the relative humidity is defined as the ratio

$$\phi = \frac{P_v}{P_{sat}(T)}$$

In our case, $\phi_\infty = 0.3$ and $P_{sat}(25°C) = 3169$ N/m^2, therefore the partial pressure of water vapor outside the concentration boundary layer is

$$P_{v,\infty} = 0.3 \times 3169\,\frac{N}{m^2} = 951\,\frac{N}{m^2}$$

Beyond this point, the calculation of ρ_∞ follows the steps used earlier in the calculation of ρ_w:

$$x_\infty = \frac{951}{1.0133 \times 10^5} = 0.0094$$
$$\rho_\infty = \frac{M_{H_2O}}{M_a}\,x_\infty\,\rho_a = \frac{18.02}{28.97}\,0.0094\,\rho_a$$
$$= 0.0058\,\rho_a$$

Now we have all the necessary information for calculating the water mass transfer rate:

$$\dot{m} = \bar{h}_m A (\rho_w - \rho_\infty)$$

$$= 0.0277 \frac{m}{s} 9m^2 (0.0195 - 0.0058) 1.185 \frac{kg}{m^3}$$

$$= 0.004 \frac{kg}{s}$$

Let Δt be the time interval in which the pool water level drops by $\Delta z = 2mm$,

$$\dot{m}\, \Delta t = \rho_{water} A\, \Delta z$$

The density of water at 25°C is 997 kg/m^3, therefore

$$\Delta t = 997 \frac{kg}{m^3} \frac{9m^2\, 0.002\ m}{0.004\ kg/s} = 4487\ s = 1.24\ h$$

Problem 11.4. In the following analysis ρ (or C in the text) represents the density of water vapor (kg/m^3) in the "humid air" mixture. The flowrate of water vapor through the duct inlet is $UA_c\rho_{in}$, while the corresponding flowrate through the outlet is $UA_c\rho_{out}$. The rate at which the air stream removes water from the duct surface is therefore

$$\begin{aligned} \dot{m} &= UA_c\rho_{out} - UA_c\rho_{in} \\ &= UA_c (\rho_{out} - \rho_{in}) \end{aligned} \tag{1}$$

This result shows that we must first determine ρ_{out} if we are to calculate the water removal rate $\dot{m}$. We account first for the conservation of species (water vapor) in a duct element of length dx:

$$j_w\, p\, dx = UA_c\, d\rho \tag{2}$$

The wall mass flux j_w is proportional to the local difference $(\rho_w - \rho)$, where ρ is the local bulk density of water vapor in the stream,

$$j_w = h_m (\rho_w - \rho) \tag{3}$$

By eliminating j_w between eqs. (2) and (3) we obtain

$$\frac{dr}{\rho_w - \rho} = \frac{h_m p}{UA_c} dx \tag{4}$$

Integrating once

$$-\ln(\rho_w - \rho) = \frac{h_m p}{UA_c} x + C \tag{5}$$

and invoking the inlet condition ($\rho = \rho_{in}$ at $x = 0$),

$$-\ln\left(\rho_w - \rho_{in}\right) = 0 + C \tag{6}$$

leads to the distribution of water vapor along the stream, $\rho(x)$:

$$\frac{\rho_w - \rho}{\rho_w - \rho_{in}} = \exp\left(-\frac{h_m\, px}{UA_c}\right) \tag{7}$$

In particular, $\rho = \rho_{out}$ at $x = L$, and eq. (7) becomes

$$\frac{\rho_w - \rho_{out}}{\rho_w - \rho_{in}} = \exp\left(-\frac{h_m}{U}\,\frac{A}{A_c}\right) \tag{8}$$

where $A = pL$ is the total mass transfer area. In conclusion, eq. (8) contains the ρ_{out} answer that is needed for continuing with eq. (1):

$$\dot{m} = UA_c\left(\rho_w - \rho_{in}\right)\left[1 - \exp\left(-\frac{h_m}{U}\,\frac{A}{A_c}\right)\right] \tag{9}$$

<u>Problem 11.5.</u> a) First, we determine the characteristics of the channel flow:

$$D_h = 2d = 2 \text{ cm} \qquad \text{(hydraulic diameter)}$$

$$Re_{D_h} = \frac{UD_h}{\nu_a} = 0.1\ \frac{\text{m}}{\text{s}}\ \frac{0.02\text{ m}}{1.55 \times 10^{-5}\text{ m}^2/\text{s}}$$

$$= 129 \qquad \text{(laminar flow)}$$

$$X \cong 0.05\ D_h\ Re_{D_h} = 0.05 \times 0.02\text{m} \times 129$$

$$\cong 0.13\text{m} \qquad \text{(flow entrance length)}$$

The entrance length is much shorter than the length of the channel ($X << L$), therefore over most of the length L the flow is fully developed and laminar.

b) The heat-transfer analog of the present problem is the parallel-plate channel with isothermal walls (the penultimate entry in Table 3.2):

$$Nu_{D_h} = 7.54 = Sh_{D_h} = \frac{h_m\, D_h}{D}$$

Noting that for water vapor in atmospheric air at 25°C the mass diffusivity is $D = 2.88 \times 10^{-5}$ m^2/s, we can finally calculate the mass transfer coefficient:

$$h_m = \frac{D}{D_h} Sh_{D_h} = \frac{2.88 \times 10^{-5}\ m^2/s}{0.02m} 7.54$$
$$= 0.0109\ \frac{m}{s}$$

c) For the total rate of water removal from the channel walls we turn to the formula derived in the preceding problem,

$$\dot{m} = UA_c(\rho_w - \rho_{in})\left[1 - \exp\left(-\frac{h_m}{U}\frac{A}{A_c}\right)\right]$$

where

$$\rho_{in} = 0\ \frac{kg}{m^3} \qquad \text{(dry air at inlet)}$$
$$A = 2\ LW \qquad \text{(total mass transfer area)}$$
$$W = \text{wall width, perpendicular to the plane of the figure}$$
$$A_c = Wd \qquad \text{(flow cross-section)}$$

The water vapor density in the humid air mixture right at the wall (ρ_w) can be calculated in the following steps:

$$P_{sat}(25°C) = 3169\ \frac{N}{m^2} \qquad \text{(pressure of saturated water vapor at 25°C)}$$
$$x_w = \frac{P_{sat}(25°C)}{1\ atm} = \frac{3169\ N/m^2}{1.0133 \times 10^5\ N/m^2}$$
$$= 0.0313 \qquad \text{(mole fraction of water vapor at the wall)}$$

$$\rho_w = M_{H_2O}\frac{\rho_a}{M_a} x_w = 18.02\ \frac{\rho_a}{28.97}\ 0.0313$$
$$= 0.0195\ \rho_a = 0.0195 \times 1.185\ \frac{kg}{m^3}$$
$$= 0.0231\ \frac{kg}{m^3}$$

The $\dot{m}$ expression becomes

$$\dot{m} = 0.1\ \frac{m}{s}\ W\ 0.01m\ 0.0231\ \frac{kg}{m^3}\left[1 - \exp\left(-\ \frac{0.0109\ m/s}{0.1\ m/s}\ \frac{2W\ 1.5m}{W\ 0.01m}\right)\right]$$
$$= W\ 2.31 \times 10^{-5}\ \frac{kg/s}{m}$$

d) The total water that is initially present on the two walls is

$$m = 2WL\,\delta\,\rho_{\text{liquid water at }25^\circ C}$$

$$= 2W\ 1.5\text{m}\ 10^{-5}\text{m}\ 997\ \frac{\text{kg}}{\text{m}^3}$$

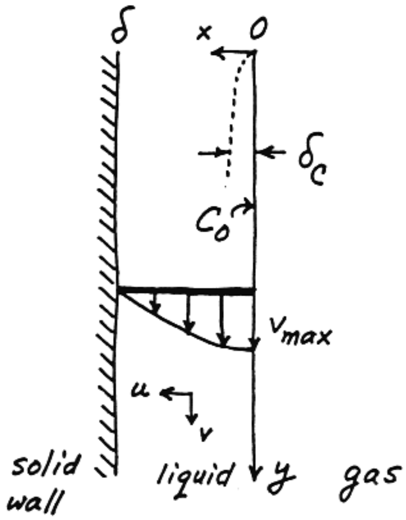
$$= W\ 0.03\ \frac{\text{kg}}{\text{m}}$$

The time needed for removing this quantity of water is

$$t = \frac{m}{\dot{m}} = \frac{W\ 0.03\ \text{kg/m}}{W\ 2.31\times 10^{-5}\ (\text{kg/m})/\text{s}}$$

$$= 1300\ \text{s} \cong 22\ \text{minutes}$$

Problem 11.6. As shown in Problem 2.24, at terminal velocity the laminar film flow attains a parabolic profile



$$\frac{v}{v_{max}} = 1 - \left(\frac{x}{\delta}\right)^2$$

where both v_{max} and δ are constant. The constituent conservation equation may be simplified:

$$\underbrace{u}_{\text{zero}}\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D\left(\frac{\partial^2 C}{\partial x^2} + \underbrace{\frac{\partial^2 C}{\partial y^2}}_{\text{negligible, if the } \delta_c \text{ layer is slender}}\right)$$

In conclusion, the complete problem statement is

$$v\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial x^2},$$

subject to

$$C = 0 \quad \text{at} \quad y \le 0$$

$$C = C_0 \quad \text{at} \quad x = 0$$

$$\frac{\partial C}{\partial x} = 0 \quad \text{at} \quad x = \delta$$

In the limit $\delta_c/\delta << 1$, we can write $v \cong v_{max}$ (constant), hence

$$v_{max}\frac{\partial C}{\partial y} = D\frac{\partial^2 C}{\partial x^2}$$

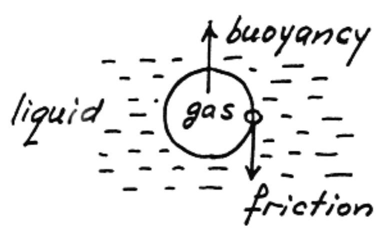
After this last simplification, the similarity solution becomes straightforward. The same analysis appears in Problem 2.25. The results are, in the order of their derivation,

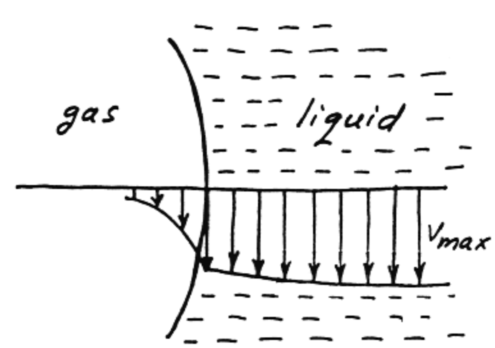
$$\frac{C}{C_0} = f(\eta), \qquad \eta = \frac{1}{2}\frac{x}{(yD/v_{max})^{1/2}}$$

$$f = 1 - \text{erf}\,(\eta)$$

$$Sh = \frac{j_0}{C_0}\frac{y}{D} = \pi^{-1/2}(v_{max}\,y/D)^{1/2}$$

Problem 11.7. When the bubble rises with terminal velocity, the buoyancy force on the bubble as a control volume is balanced by the downward viscous drag acting at the gas-liquid interface.





Relative to the observer who rides on the gas-liquid interface, the liquid pool moves downward with an approximately constant velocity v_{max} (this statement is based on the assumption that $\mu_{gas} << \mu_{liquid}$, which means that if τ is to be conserved across the interface, then the velocity gradient on the liquid side must be negligible compared with the gradient on the gas side. As shown on the right side of the sketch, the interface velocity is nearly equal to the terminal velocity v_{max}.

In conclusion, the gas-liquid interface in this problem has a liquid-side velocity distribution that approaches that of the vertical film considered in Problem 11.6. The gas transfer rate from the bubble to the liquid is therefore given by the same formula as the one that concludes Problem 11.6, with the new notation

$$y \sim D_b.$$

For the overall Sherwood number scale we can write

$$Sh = \frac{j_{0,avg}\, D_b}{C_0\, D} \sim \left(\frac{v_{max}\, D_b}{D}\right)^{1/2}$$

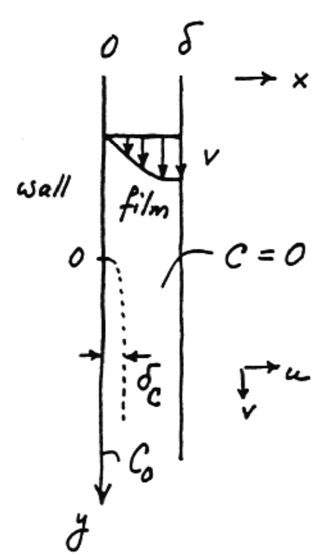
To justify this result, one needs the additional assumption that the concentration boundary layer formed on the liquid side of the interface is slender, which means that it has a thickness much smaller than D_b.

Problem 11.8. As shown in the solution to Problem 2.24, the terminal velocity distribution is

$$v = \frac{g}{\nu}\left(\delta x - \frac{x^2}{2}\right)$$

in other words,

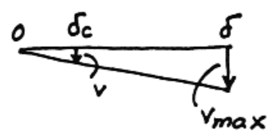
$$v_{max} \sim g\delta^2/\nu$$



The constituent conservation equation in the film region is

$$\underset{\substack{\uparrow \\ \text{zero}}}{u \frac{\partial C}{\partial x}} + v \frac{\partial C}{\partial y} = D \left(\underset{\substack{\uparrow \\ \text{negligible}}}{\frac{\partial^2 C}{\partial x^2}} + \frac{\partial^2 C}{\partial y^2} \right)$$

In the δ_c-thin layer, the v scale is



$$v \sim \frac{\delta_c}{\delta} v_{max},$$

hence, the scaling law demanded by the concentration equation

$$\left(\frac{\delta_c}{\delta} v_{max} \right) \frac{\Delta C}{y} \sim D \frac{\Delta C}{\delta_c^2}$$

The results are

$$\delta_c \sim \left(y \frac{D\nu}{g\delta} \right)^{1/3}, \qquad \text{or} \qquad j_0 \sim D \Delta C \left(y \frac{D\nu}{g\delta} \right)^{-1/3}$$

showing that the paint is eroded unevenly (j_0 varies as $y^{-1/3}$).

Problem 11.9. In order to calculate the mass-transfer Rayleigh number

$$Ra_{m,H} = \frac{g H^3}{\nu D} \beta_c (\rho_w - \rho_\infty)$$

we evaluate in order

$D = 2.6 \times 10^{-5} \frac{m^2}{s}$	(Table 11.1)
$\rho\beta_c = 0.61$	(Table 11.4)
$Sc = 0.6$	(Table 11.4)
$\nu = 0.155 \frac{cm^2}{s}$	(Dry air, Appendix D)
$\rho_w = \left(43360 \frac{cm^3}{h} \right)^{-1} = 0.0231 \frac{kg}{m^3}$	(Saturated steam tables, 25 C)

The calculation of the mass concentration in the atmosphere (40 percent relative humidity) involves these steps:

$$\phi = \frac{P_{v,\infty}}{P_{sat}\,(25°C)} = 0.4$$

$$P_{v,\infty} = 0.4 \times 3169 \,\frac{N}{m^2} = 1268 \,\frac{N}{m^2}$$

$$\rho_\infty = \frac{P_{v,\infty}}{P_{v,w}}\,\rho_w = \frac{1268\ N/m^2}{3169\ N/m^2}\,0.0231\,\frac{kg}{m^3}$$

$$= 0.00924\,\frac{kg}{m^3}$$

The density of humid air increases by roughly one percent as the relative humidity drops from 100 all the way to zero. Therefore it is reasonable to approximate the mixture density ρ (which appears in $\rho\beta_c$) in terms of the density of dry air at 25°C and 1 atm:

$$\rho \cong \rho_a = 1.185\,\frac{kg}{m^3}$$

The dimensionless group $\beta_c(\rho_w - \rho_\infty)$ that appears in the $Ra_{m,H}$ definition (in the text, C_i was used instead of ρ_i) has the value

$$\beta_c\left(\rho_w - \rho_\infty\right) = \rho\beta_c\,\frac{\rho_w - \rho_\infty}{r}$$

$$= 0.61\,\frac{0.0231 - 0.00924}{1.185} = 0.00713$$

The mass transfer Rayleigh number becomes

$$Ra_{m,H} = \frac{9.81\,\frac{m}{s^2}\,(1m)^3}{2.6\times 10^{-5}\,\frac{m^2}{s}\,0.155\times 10^{-4}\,\frac{m^2}{s}}\,0.00713 = 2.43\times 10^{10}$$

and the correlation patterned after eq. (4.105) yields

$$\overline{Sh}_H = \left(0.825 + 0.32\,Ra_{m,H}^{1/6}\right)^2 = 325.9$$

$$\bar{h}_m = \overline{Sh}_H\,\frac{D}{H}$$

$$= 325.9\,\frac{2.6\times 10^{-5}\,m^2/s}{1m} = 0.00847\,\frac{m}{s}$$

The evaporation rate produced by the $1m^2$ vertical area is

$$\dot{m} = \bar{h}_m A \left(\rho_w - \rho_\infty\right)$$

$$= 0.00847 \frac{m}{s} \, 1m^2 \, (0.0231 - 0.00924) \frac{kg}{m^3}$$

$$= 1.17 \times 10^{-4} \frac{kg}{s} = 0.42 \text{ kg/h}$$

The flow of humid air is upward, because the density of more humid air is smaller than the density of less humid air.

Problem 11.10. In laminar boundary layer flow driven by heat transfer along a vertical wall we have the following horizontal (entrainment) velocity scales:

$$u \sim \frac{\alpha}{H} Ra_H^{1/4}, \quad (Pr \gtrsim 1)$$

$$u \sim \frac{\alpha}{H} (Ra_H \, Pr)^{1/4}, \quad (Pr \lesssim 1)$$

The analogous scales for a laminar vertical boundary layer driven by mass transfer are

$$u \sim \frac{D}{H} Ra_{m,H}^{1/4}, \quad (Sc \gtrsim 1)$$

$$u \sim \frac{D}{H} \left(Ra_{m,H} \, Sc\right)^{1/4}, \quad (Sc \lesssim 1)$$

in other words, by

$$u \sim \frac{D}{H} Ra_{m,H}^{1/4} \, Sc^n$$

where

$$n = \begin{cases} 0 & \text{for} \quad Sc \gtrsim 1 \\ \frac{1}{4} & \text{for} \quad Sc \lesssim 1 \end{cases}$$

The horizontal mass flux of mixture fluid associated with this entrainment is

$$\rho u \sim \rho \frac{D}{H} Ra_{m,H}^{1/4} \, Sc^n \tag{1}$$

The mass flux through the wall can be estimated in the same way, by changing the notation in

$$\overline{Nu}_H \sim Ra_H^{1/4}, \quad (Pr \gtrsim 1)$$

$$\overline{Nu}_H \sim (Ra_H \, Pr)^{1/4}, \quad (Pr \lesssim 1)$$

We obtain

$$\overline{Sh}_H \sim Ra_{m,H}^{1/4}\ Sc^n$$

in which

$$\overline{Sh}_H = \frac{\bar{h}_m H}{D} = \frac{j_w H}{\left(\rho_{i,w} - \rho_{i,\infty}\right) D}$$

The mass flux through the wall is therefore

$$j_w \sim \frac{D}{H}\left(\rho_{i,w} - \rho_{i,\infty}\right) Ra_{m,y}^{1/4}\ Sc^n \tag{2}$$

The vertical surface may be modelled as impermeable (zero through-flow) when

$$|j_w| << \rho u$$

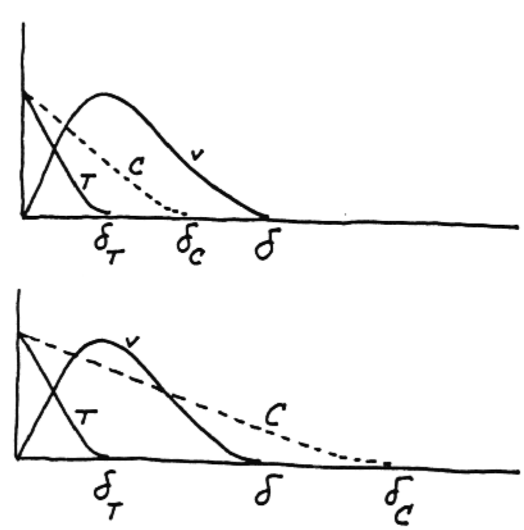
or, after using eqs. (1) and (2),

$$\frac{D}{H}\left|\rho_{i,w} - \rho_{i,\infty}\right| Ra_{m,y}^{1/4}\ Sc^n << \rho\ \frac{D}{H}\ Ra_{m,H}^{1/4}\ Sc^n$$

$$\left|\rho_{i,w} - \rho_{i,\infty}\right| << \rho$$

In conclusion, the impermeable-surface assumption is valid when the species of interest is present in small quantities (traces) in the mixture.

Problem 11.11. In the second case of Fig. 11.8, Pr > 1 and the concentration layer is thicker than the thermal layer. However, as shown in the sketch, we must consider separately the two subcases



and

$$\delta_T < \delta_c < \delta$$

$$\delta < \delta_c,$$

where $\delta \sim H\,Ra_H^{-1/4}\,Pr^{1/2}$ is the outer thickness of the velocity boundary layer (see Chapter 4). Due to the existence of these two subcases, the job of figuring out the δ_c scale based on the constituent conservation equation (11.83) must be approached with care. One way to avoid mistakes is to integrate eq. (11.83) from the wall into the reservoir

$$\frac{d}{dy}\int_0^{\infty} v\,(C - C_\infty)\,dx = -\,D\left(\frac{\partial C}{\partial x}\right)_{x=0},$$

and to recognize that this integral condition means the balance between two scales:

$$\underbrace{\frac{v\,\Delta C}{H}\,\min(\delta_c, \delta)}_{\text{vertical upflow of constituent}} \quad \sim \quad \underbrace{D\,\frac{\Delta C}{\delta_c}}_{\text{diffusive mass transfer from the side}}$$

Note that $\min(\delta_c, \delta)$ represents the thickness of the vertical constituent stream, because this stream must occupy a region that has upward flow, $0 < x < \delta$, and a high concentration of constituent, $0 < x < \delta_c$; that region can only be $0 < x < \min(\delta_c, \delta)$. Note further that the vertical velocity scale of this stream is the same as that of the velocity boundary layer itself,

$$v \sim \frac{\alpha}{H}\,Ra_H^{1/2}.$$

Therefore, if $\delta_T < \delta_c < \delta$, we can write

$$\frac{v\,\Delta C}{H}\,\delta_c \sim D\,\frac{\Delta C}{\delta_c}$$

to obtain:

$$\delta_c \sim H\,Le^{-1/2}\,Ra_H^{-1/4}$$

$$\overline{Sh} \sim Le^{1/2}\,Ra_H^{1/4}$$

The above scales are valid if $\delta_T < \delta_c < \delta$, which corresponds to

$$Le < 1 < Sc.$$

If, on the other hand, $\delta < \delta_c$, the constituent conservation scaling law becomes

$$\frac{v\,\Delta C}{H}\,\delta \sim D\,\frac{\Delta C}{\delta_c}$$

and the results (also listed in Table 11.5) are:

$$\delta_c \sim H\, Le^{-1}\, Ra_H^{-1/4}\, Pr^{-1/2}$$

$$\overline{Sh} \sim Le\, Ra_H^{1/4}\, Pr^{1/2}$$

These scales are valid if $\delta \sim \delta_c$, which means

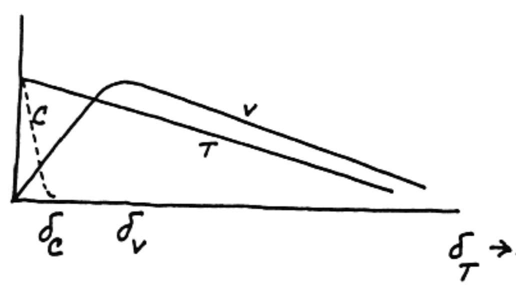
$$Sc < 1$$

Note that in this case "Pr > 1 and Sc < 1" means that the Lewis number is very small,

$$Le < 1.$$

In the third case of Fig. 11.8, δ_c is smaller than δ_v, hence the velocity scale in the δ_c-thin layer is only a fraction of the velocity scale in the velocity boundary layer,

$$v \sim \left(\frac{\delta_c}{\delta_v}\right)\left(\frac{\alpha}{H}\, Ra_H^{1/2}\, Pr^{1/2}\right)$$



From Table 4.1, we recall that the viscous shear layer thickness is $\delta_v \sim H\, Ra_H^{-1/4}\, Pr^{1/4}$. The scaling law demanded by eq. (11.83) in the concentration boundary layer is

$$v\, \frac{\Delta C}{H} \sim D\, \frac{\Delta C}{\delta_c^2}$$

which, using the v scale identified earlier, yields

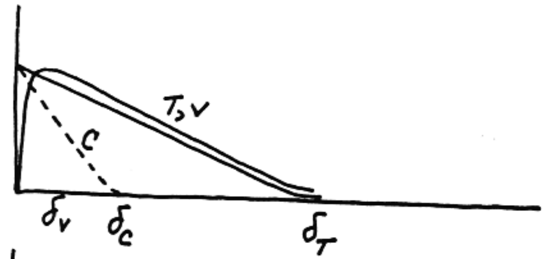
$$\delta_c \sim H\, Le^{-1/3}\, Ra_H^{-1/4}\, Pr^{-1/2}$$

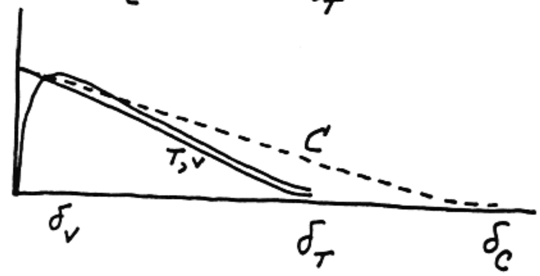
$$\overline{Sh} \sim Le^{1/3}\, Ra_H^{1/4}\, Pr^{1/2}$$

These scales are valid if $\delta_c < \delta_v$, which means

$$Sc > 1.$$

The fourth case of Fig. 11.9 can be divided into two subcases, depending on how δ_c compares with the outer thickness of Pr < 1 layers, $\delta_T \sim H\, Ra_H^{-1/4}\, Pr^{-1/4}$ (Table 4.1).





The constituent conservation integral requires

$$\frac{v\,\Delta C}{H}\min(\delta_c, \delta_T) \sim D\,\frac{\Delta C}{\delta_c}.$$

Thus, if $\delta_c < \delta_T$ we obtain:

$$\delta_c \sim H\,Le^{-1/2}\,Ra_H^{-1/4}\,Pr^{-1/4}$$

$$\overline{Sh} \sim Le^{1/2}\,Ra_H^{1/4}\,Pr^{1/4}$$

The validity domain of these scales is

$$\delta_v < \delta_c < \delta_T$$

in other words,

$$Sc < 1 < Le$$

Finally, if $\delta_c > \delta_T$, the corresponding results are

$$\delta_c \sim H\,Le^{-1}\,Ra_H^{-1/4}\,Pr^{-1/4}$$

$$\overline{Sh} \sim Le\,Ra_H^{1/4}\,Pr^{1/4},$$

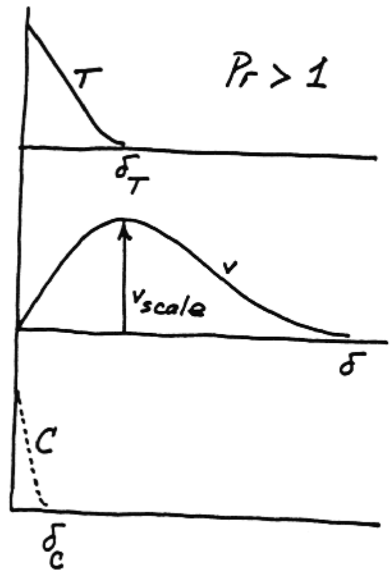
while the validity condition $\delta_c > \delta_T$ translates into

$$Le < 1.$$

Note that in this last case "$Pr < 1$ and $Le < 1$" means that the Schmidt number is small,

$$Sc < 1.$$

Problem 11.12. The scales of heat-transfer-driven natural convection near a vertical wall with uniform heat flux are presented in section 4.6 of the textbook:



$$\delta_T \sim H\, Ra_{*H}^{-1/5}$$

$$v \sim \frac{\alpha}{H} Ra_H^{1/2} \sim \ldots \sim \frac{\alpha}{H} Ra_{*H}^{2/5}$$

$$\delta \sim H\, Ra_{*H}^{-1/5}\, Pr^{1/2}$$

Following the steps outlined in Problem 11.11, we consider the following three cases of Pr > 1 fluids:

$\underline{\delta_c < \delta_T}$. In the δ_c-thin layer, the velocity scale is

$$v \sim \left(\frac{\delta_c}{\delta_T}\right)\left(\frac{\alpha}{H} Ra_{*H}^{2/5}\right),$$

hence, eq. (11.87) yields

$$\delta_c \sim H\, Le^{-1/3}\, Ra_{*H}^{-1/5}$$

$$\overline{Sh} \sim Le^{1/3}\, Ra_{*H}^{1/5}$$

From $\delta_c < \delta_T$ we conclude that the above scales are valid if

$$Le > 1$$

$\underline{\delta_T < \delta_c < \delta}$. Instead of eq. (11.87) we use

$$v \frac{\Delta C}{H} \min(\delta_c, \delta) \sim D \frac{\Delta C}{\delta_c}$$

where $v \sim (\alpha/H)\, Ra_{*H}^{2/5}$. The results are

$$\delta_c \sim H\, Le^{-1/2}\, Ra_{*H}^{-1/5}$$
$$\overline{Sh} \sim Le^{1/2}\, Ra_{*H}^{1/5}$$

and they are valid if

$$Le < 1 < Sc.$$

$\underline{\delta < \delta_c}$. Finally, the scaling law of species conservation

$$v \frac{\Delta C}{H} \delta \sim D \frac{\Delta C}{\delta_c}$$

yields the following results,

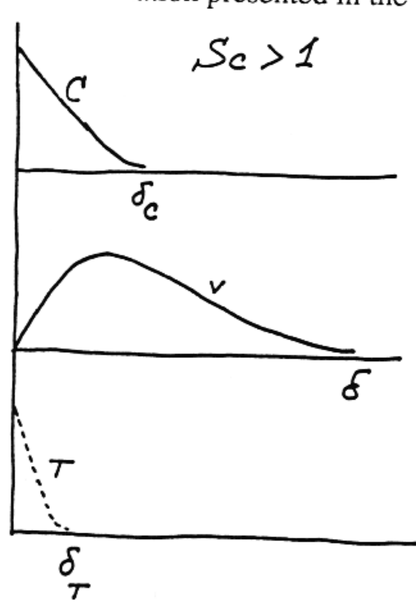
$$\delta_c \sim H\, Le^{-1}\, Ra_{*H}^{-1/5}\, Pr^{-1/2}$$
$$\overline{Sh} \sim Le\, Ra_{*H}^{1/5}\, Pr^{1/2}$$

which are valid if

$$Sc < 1.$$

The mass transfer scales of $Pr < 1$ boundary layers can be determined in the same way, this time starting with the scales listed in eq. (4.74).

Problem 11.13. The scales of mass-transfer-driven natural convection can be deduced from Table 4.1 by applying the notation transformation presented in the textbook:



$$\delta_c \sim y\, Ra_{m,y}^{-1/4}$$

$$\delta \sim y\, Ra_{m,y}^{-1/4}\, Sc^{1/2}$$

$$v \sim \frac{D}{y}\, Ra_{m,y}^{1/2}$$

To determine δ_T we focus on the energy equation

$$u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial x^2}$$

or on the scaling law

$$v\,\frac{\Delta T}{y} \sim \alpha\,\frac{\Delta T}{\delta_T^2}. \qquad \text{(E)}$$

$\underline{\delta_T < \delta_c}$. In this case the v scale inside the δ_T layer is $(\delta_T/\delta_c)\,(D/y)\,Ra_{m,y}^{1/2}$, and (E) yields

$$\delta_T \sim y\, Ra_{m,y}^{-1/4}\, Le^{1/3}$$

$$\overline{Nu} \sim Ra_{m,y}^{1/4}\, Le^{-1/3}$$

which are valid if Le < 1. The remaining cases are handled the same way, and yield:

$\underline{\delta_c < \delta_T < \delta}$. $\qquad \overline{Nu} \sim Ra_{m,y}^{1/4}\, Le^{-1/2}$, if Le > 1 and Pr > 1

$\underline{\delta < \delta_T}$. $\qquad \overline{Nu} \sim Ra_{m,y}^{1/4}\, Le^{-1}\, Sc^{1/2}$, if Le > 1 and Pr < 1

The three cases considered so far correspond to fluids with $S_c > 1$. The heat transfer scales in Sc < 1 fluids can be written immediately by noticing the symmetry between the above $\overline{Nu}$ scales and the $\overline{Sh}$ scales listed in Table 11.5 for Pr > 1. Thus for Sc < 1 fluids we have

$$\overline{Nu} \sim Le^{-1/3}\, Ra_{m,y}^{1/4}\, Sc^{1/12}, \quad \text{if } Pr > 1 \text{ and } Le < 1$$

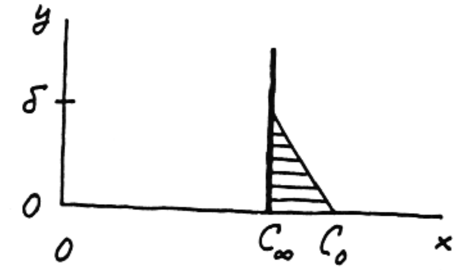
$$\overline{Nu} \sim Le^{-1/2}\, Ra_{m,y}^{1/4}\, Sc^{1/4}, \quad \text{if } Pr < 1 \text{ and } Le < 1$$

$$\overline{Nu} \sim Le^{-1}\, Ra_{m,y}^{1/4}\, Sc^{1/4}, \quad \text{if } Pr < 1 \text{ and } Le > 1$$

Problem 11.14. From eq. (11.137), the integral condition that accounts for constituent conservation is

$$\frac{d}{dx}\int_0^\infty u\,(C - C_\infty)\,dy = -D\left(\frac{\partial C}{\partial y}\right)_0 - k_n'''\int_0^\infty C^n\,dy$$

where $u = U_\infty$, $C_\infty = 0$ and, we can assume,



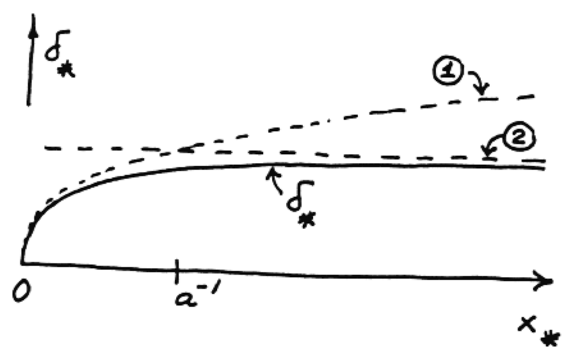
$$C = C_0 - \frac{C_0}{\delta} y, \qquad \text{if} \qquad 0 < y < \delta.$$

Following the rules of integral analysis (Chapter 2), the conservation equation can be integrated to yield

$$\delta_* = 2\left(\frac{1 - e^{-a x_*}}{a}\right)^{1/2},$$

where parameters δ_*, x_* and a are all dimensionless:

$$\delta_* = \frac{\delta U_\infty}{D}, \qquad x_* = \frac{x U_\infty}{D}, \qquad a = \frac{4D}{U_\infty^2} \frac{k_n''' C_0^{n-1}}{n+1}$$



(a) $\lim_{x_* \to 0} \delta_* = 2x_*^{1/2}$ (b) $\lim_{x_* \to \infty} \delta_* = 2a^{-1/2}$

This result shows that the effect of chemical reaction is felt sufficiently far downstream, where δ becomes a constant controlled by the reaction. "Sufficiently far" is defined as the intersection of the two limits shown above, $2x_*^{1/2} \sim 2a^{-1/2}$, which means $a x_* \sim 1$. The local Sherwood number is

$$Sh = \frac{j_0}{C_0} \frac{x}{D} = \frac{x}{\delta} = \frac{1}{2} \frac{x U_\infty}{D} \left(\frac{a}{1 - e^{-a x_*}}\right)^{1/2}$$

Limits (1) and (2) suggest that (Sh with chemical reaction) is greater than (Sh without chemical reaction) if a is positive, i.e. if the reaction consumes the species.

Problem 11.15. The concentration equation is

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)\pm k_1''' C$$

$$\underset{U_\infty}{\uparrow}\qquad \underset{\text{zero}}{\uparrow}\qquad\qquad \underset{\text{negligible}}{\uparrow}$$

We discover that the simplified equation is the same as the one solved via integral analysis in Problem 11.14, therefore the local Sherwood number is

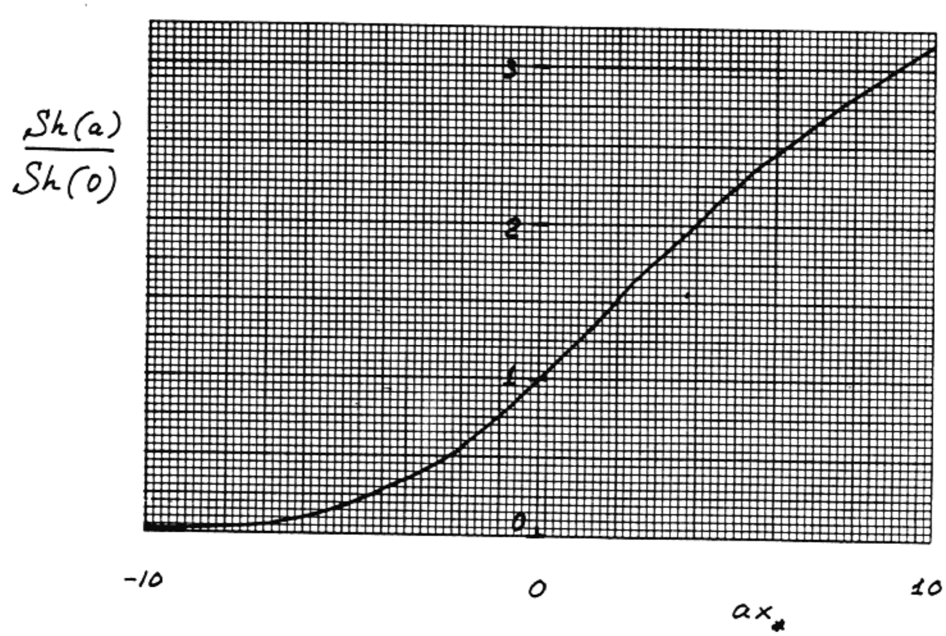
$$Sh=\frac{1}{2}\frac{x\,U_\infty}{D}\left(\frac{a}{1-e^{-ax_*}}\right)^{1/2}$$

$$x_* = x\,U_\infty/D \qquad \text{and} \qquad a=\frac{2D\,k_1'''}{U_\infty}$$

where it is assumed that the reaction is depleting the species (the sign will be "–" in the concentration equation). The integral analysis is valid regardless of the sign of $k_1''' C$, therefore, if the sign is "+" a will be negative. The Sh formula remains unchanged. To illustrate the effect of chemical reaction we calculate

$$\frac{Sh\ (a=\text{finite})}{Sh\ (a=0)}=\left(\frac{ax_*}{1-e^{-ax_*}}\right)^{1/2}$$

where ax_* can be positive or negative. The ratio Sh (with reaction)/Sh (without reaction) behaves as:



the reaction produces the species $\delta_c(a) > \delta_c(0)$ ← | → the reaction consumes the species $\delta_c(a) < \delta(0)$

Chapter 12

CONVECTION IN POROUS MEDIA

Problem 12.1. Starting with eq. (1.1),

$$\frac{\partial M_{cv}}{\partial t} = \sum_{in} \dot{m} - \sum_{out} \dot{m}$$

and considering as system the control volume shown in the lower half of Fig. 12.2, we obtain

$$\frac{\partial}{\partial t}(\rho\, \phi\, \Delta x\, \Delta y) = \rho u\, \Delta y + \rho v\, \Delta x - \left(\rho u + \frac{\partial(\rho u)}{\partial x}\Delta x\right)\Delta y -$$
$$- \left(\rho v + \frac{\partial(\rho v)}{\partial y}\Delta y\right)\Delta x$$

Dividing by $\Delta x \Delta y$, and invoking the limit $\Delta x \Delta y \to 0$, we obtain eq. (12.7).

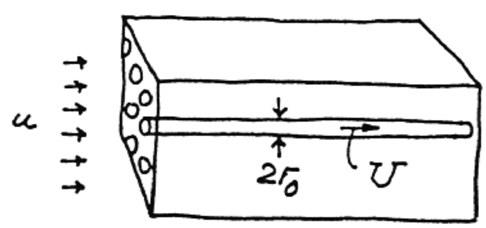
Problem 12.2. The Darcy flow law is

$$u = \frac{K}{\mu}\left(-\frac{dP}{dx}\right)$$

where u is the volume-averaged velocity

$$u = \frac{N(\pi r_o^2)}{A} U$$

and where U is the average velocity in each capillary tube.



From Chapter 3, we know that

$$U = \frac{r_o^2}{8\mu}\left(-\frac{dP}{dx}\right)$$

Putting everything together,

$$\frac{N \pi r_o^2}{A} \frac{r_o^2}{8\mu}\left(-\frac{dP}{dx}\right) = \frac{K}{\mu}\left(-\frac{dP}{dx}\right)$$

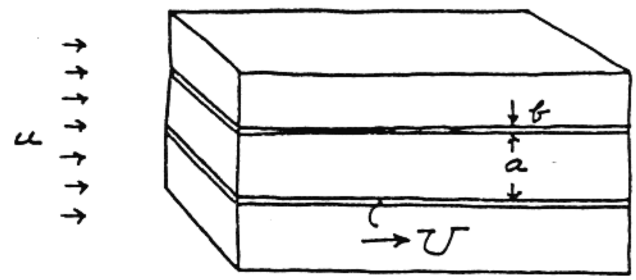
we obtain

$$K = \frac{\pi r_o^4}{8} \frac{N}{A} \qquad \text{or} \qquad K = \frac{r_o^2}{8} \phi$$

where $\phi = N\pi\ r_o^2 / A$ is the porosity.

Problem 12.3. Following the same reasoning as in Problem 12.2, we have

$$u = \frac{Ub}{a + b}$$



$$U = \frac{b^2}{12\mu} \left(- \frac{dP}{dx} \right)$$

and, from the Darcy law

$$u = \frac{K}{\mu} \left(- \frac{dP}{dx} \right) = \frac{b}{a + b} \frac{b^2}{12\mu} \left(- \frac{dP}{dx} \right)$$

In conclusion, we obtain

$$K = \frac{b^3}{(12)(a + b)} = \frac{b^2}{12} \phi$$

where ϕ is the porosity, $\phi = b/(a + b)$.

Problem 12.4. According to the Darcy law, the permeability is

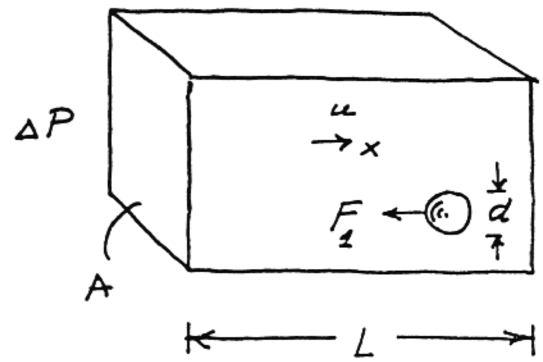
$$K = \frac{\mu u}{- dP / dx}$$

The problem consists of determining u and $-$ dP/dx as functions of the geometry shown in the figure. The average velocity is

$$u = u_p \frac{\text{void volume}}{\text{total volume}} = u_p \left(1 - N_{vol} \frac{\pi d^3}{6} \right) \tag{1}$$

The overall force balance on the bed of spheres,

$$F_1 \cdot \underbrace{(\text{total number of particles})}_{N_{vol}AL} = A\,\Delta P$$



yields

$$\frac{\Delta P}{L} = -\frac{dP}{dx} = F_1\,N_{vol} \tag{2}$$

where

$$F_1 = \frac{24\,\nu}{u_p\,d}\,\frac{1}{2}\,\rho\,u_p^2\,\frac{\pi d^2}{4} \tag{3}$$

Substituting the u and – dP/dx expressions into the K definition yields

$$K = \frac{d^2\,\phi}{18\,(1-\phi)}$$

where

$$\phi = 1 - \frac{\pi}{6}\left(\frac{d}{L}\right)^3$$

Problem 12.5. Combining eqs. (12.12) and (12.15) we obtain

$$f = \frac{K^{1/2}}{\rho u^2}\left(\frac{\mu u}{K} + b\rho u^2\right) = \underbrace{\frac{\nu}{u\,K^{1/2}}}_{Re^{-1}} + \underbrace{b\,K^{1/2}}_{\text{constant}}$$

or

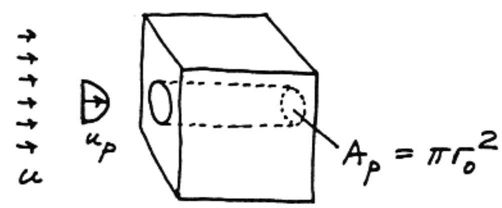
$$f = \frac{1}{Re} + C$$

Problem 12.6. Assuming that the pore is a capillary tube, we can write

$$u_p = 2U\left(1 - \frac{r^2}{r_o^2}\right)$$

$$uA = U\,\pi\, r_o^2$$

$$\Phi = \left(\frac{\partial u_p}{\partial r}\right)^2 = \frac{16\,U^2\,r^2}{r_o^4}$$



Equation (12.27) can be integrated:

$$\Delta x\,\mu \int_0^{2\pi} d\theta \int_0^{r_o} \frac{16\,U^2\,r^2}{r_o^4}\, r\,dr = \Delta x\,\mu\, 2\pi\, \frac{16\,U^2}{r_o^4}\,\frac{r_o^4}{4}$$

Next, by splitting

$$U^2 = UU = U\,\frac{r_o^2}{8\mu}\quad \underbrace{\left(-\frac{dP}{dx} + \rho_f\, g_x\right)}_{\text{pressure gradient in excess of hydrostatic}}$$

on the right-hand side of eq. (12.27), we obtain

$$\Delta x\; U\,\pi\, r_o^2\left(-\frac{dP}{dx} + \rho_f\, g_x\right) = \Delta x\; Au\left(-\frac{dP}{dx} + \rho_f\, g_x\right)$$

Problem 12.7. With reference to Fig. 12.4, the total entropy generation rate in the control volume of small size A Δx is

$$S_{gen} = \Delta x \iint_{A-A_p} S'''_{gen_{solid}}\, d\, \underbrace{\text{area}}_{a} + \Delta x \iint_{A_p} S'''_{gen_{pore}}\, d\,(\text{area})$$

$$= \Delta x \iint_{A-A_p} \frac{k}{T^2}(\nabla T)^2\, da + \Delta x \iint_{A_p} \left[\frac{k}{T^2}(\nabla T)^2 + \frac{\mu}{T}\,\Phi\right] da$$

$$= \Delta x \iint_A \frac{k}{T^2} (\nabla T)^2 \, da + \underbrace{\Delta x \iint_{A_p} \frac{\mu}{T} \Phi \, da}_{\Delta x \frac{Au}{T} \underbrace{\left(-\frac{\partial P}{\partial x} + \rho_f g_x \right)}_{\frac{\mu}{K} u}} \qquad \text{cf. eq. (12.27)} \quad \text{see eq. (12.16)}$$

$$= \Delta x \iint_A \frac{k}{T^2} (\nabla T)^2 \, da + \Delta x \, A \, \frac{\mu u^2}{TK}$$

In the limit $A \to 0$, and if the flow is three-dimensional we can write

$$S_{gen} = A \, \Delta x \, \frac{k}{T^2} (\nabla T)^2 + A \, \Delta x \, \frac{\mu}{KT} \left(\underline{\underline{v}} \right)^2,$$

which leads to eq. (12.39):

$$\frac{S_{gen}}{A \, \Delta x} = S'''_{gen} = \frac{k}{T^2} (\nabla T)^2 + \frac{\mu}{KT} (\underline{\underline{v}})^2$$

Problem 12.8. The scale analysis of eq. (12.39) indicates that

$$\frac{S'''_{gen,\,friction}}{S'''_{gen,\,heat\,transfer}} \sim \frac{\frac{\mu}{KT} U^2}{\frac{k}{T^2} \frac{(\Delta T)^2}{L^2}} \sim \underbrace{\frac{\frac{\mu U^2}{K}}{\frac{k \Delta T}{L^2}}}_{\substack{\text{less than } O(1), \\ \text{because } \frac{\mu}{k} (\underline{\underline{v}})^2 \\ \text{is negligible in the energy} \\ \text{equation (12.33)}}} \times \underbrace{\frac{T}{\Delta T}}_{\substack{\text{usually} \\ \text{much greater} \\ \text{than } O(1)}} \gtrless O(1)$$

Conclusion: the above ratio can be either large or small, depending on the additional *number* $\Delta T/T$ called *dimensionless temperature difference* τ (see A. Bejan, Entropy Generation through Heat and Fluid Flow, Wiley, New York, 1982, p. 103).

Problem 12.9. The similarity transformation is

$$\eta = \left(\frac{U_\infty}{\alpha}\right)^{1/2} \frac{y}{x^{1/2}}, \qquad \theta(\eta) = \frac{T - T_0}{\underbrace{T_\infty - T_0}_{\Delta T}}$$

hence, the energy equation becomes:

$$\underbrace{u}_{\substack{\uparrow \\ U_\infty}} \frac{\partial T}{\partial x} + \underbrace{v}_{\substack{\uparrow \\ \text{zero}}} \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}$$

$$U_\infty \, \Delta T \, \theta' \left(\frac{U_\infty}{\alpha}\right)^{1/2} y \left(-\frac{1}{2}\right) x^{-3/2} = \alpha \, \Delta T \, \theta'' \frac{U_\infty}{\alpha \, x}$$

$$-\frac{\eta}{2} \theta' = \theta''$$

Integrating once, we obtain

$$\theta' = \theta'(0) \, e^{-\eta^2/4}$$

Integrating for the second time,

$$\theta - 0 = \theta'(0) \, 2 \underbrace{\int_0^\eta e^{-\eta^2/4} \, d\left(\frac{\eta}{2}\right)}_{\frac{\pi^{1/2}}{2} \operatorname{erf}\left(\frac{\eta}{2}\right)}$$

and claiming $\theta(\infty) = 1$, yields

$$1 = \pi^{1/2} \, \theta'(0)$$

in other words, $\theta'(0) = \pi^{-1/2} = Nu_x \, Pe_x^{-1/2}$.

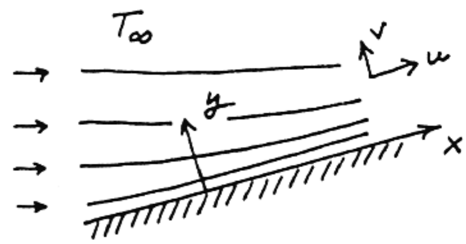
Problem 12.10. This problem can be proposed to the student as a term project involving the use of the computer. The problem statement is descriptive enough and one set of results (my own) is included. The student should construct his or her own program. While reporting the results, it is very important to document the effect of personal numerical technique (step size, shooting distance η_{max}) on the final numerical answer to the problem, $Nu_x \, Pe_x^{-1/2}$. This last requirement is not always fulfilled by the reports of numerical heat transfer results in the literature.

Problem 12.11. If near the wall $u = C x^n$, then mass conservation requires

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

hence

$$v = - Cn\, x^{n-1} y$$



Assuming the similarity profile

$$\theta(\eta) = \frac{T - T_\infty}{T_0 - T_\infty} = \frac{T - T_\infty}{A_x \lambda}, \qquad \text{where} \qquad \eta = By\, x^m,$$

the energy equation (12.42) becomes

$$C\lambda\theta - \frac{\eta}{2}(n+1)\, C\, \theta' = \alpha\, B^2\, \theta''$$

or, choosing $B = (C/\alpha)^{1/2}$,

$$\lambda\theta - \frac{\eta}{2}(n+1)\,\theta' = \theta''$$

If $\lambda = 0$ and $n = 0$ we recover the similarity energy equation solved in Problem 12.9 for an isothermal wall ($\lambda = 0$) in contact with a constant-velocity stream ($n = 0$).

If a body force acts parallel to the wall, i.e. if unlike in the momentum equations (12.41) the effect of buoyancy is present, then the condition that the similarity-transformed momentum equations must be x-independent leads to the additional restriction $n = \lambda$. For this part of the similarity analysis, the student should study Ref. [13], which deals with the combined forced and natural convection phenomenon near a wedge-shaped wall in a porous medium.

Problem 12.12. The boundary layer momentum equation is

$$\frac{\partial v}{\partial x} = \frac{Kg\beta}{\nu} \frac{\partial T}{\partial x}$$

Integrating it from any x to ∞ yields

$$v = \frac{Kg\beta}{\nu} (T - T_\infty)$$

where $v = v_0 \exp(-x/\delta_T)$. Invoking the boundary condition $T = T_0$ at $x = 0$, we find the value of v_0:

$$v_0 = \frac{Kg\beta (T_0 - T_\infty)}{\nu}$$

The integral of the energy equation is

$$\frac{d}{dy} \int_0^\infty v (T - T_\infty)\, dx = -\alpha \left.\frac{\partial T}{\partial x}\right|_{x=0}$$

Substituting the exponential profiles known (assumed) for v and $(T - T_\infty)$, and integrating the result from $\delta_T = 0$ at $y = 0$, we obtain

$$\frac{\delta_T}{y} = 2\, Ra_y^{-1/2}$$

This result confirms the scale analysis presented in the textbook (Section 12.6.2). Finally, the local Nusselt number can be derived in the usual manner

$$Nu_y = \frac{q''}{\Delta T} \frac{y}{k} = \frac{-k\,(\partial T/\partial x)_0}{\Delta T} \frac{y}{k} = \frac{y}{\delta_T} = 0.5\, Ra_y^{1/2}$$

In conclusion, the Nusselt number produced by this integral analysis comes within 13% of the similarity solution, eq. (12.95).

Problem 12.13. To complete the scale analysis (12.97)-(12.100), we find that the streamfunction must scale as [eq. (12.87)]

$$\psi \sim \alpha \left(\frac{Kg\beta y \Delta T}{\alpha \nu}\right)^{1/2} \sim \ldots . \sim \alpha\, Ra_{*y}^{1/3}$$

where

$$Ra_{*y} = \frac{Kg\beta q'' y^2}{\alpha \nu k}$$

The similarity variable is constructed in the usual way,

$$\eta = \frac{x}{\sim \delta_T} = \frac{x}{y}\, Ra_{*y}^{1/3} = x\, y^{-1/3}\, C$$

where

$$C = \left(\frac{Kg\,\beta\, q''}{\alpha \nu k}\right)^{1/3}$$

This is followed by the ψ and $T - T_\infty$ profiles:

$$f(\eta) = \frac{\psi}{\alpha\, Ra_{*y}^{1/3}} = \frac{\psi}{\alpha\, C\, y^{2/3}}$$

$$\theta(\eta) = \frac{T - T_\infty}{\frac{q''}{k}\, y\, Ra_{*y}^{-1/3}} = \frac{T - T_\infty}{\frac{q''}{k}\, y^{1/3}\, C^{-1}}$$

Making these substitutions into eqs. (12.82) and (12.83) yields, after some algebra,

$$f'' = -\,\theta' \qquad \text{(momentum)}$$

$$\frac{2}{3} f\,\theta' - \frac{1}{3} f'\theta = \theta'' \qquad \text{(energy)}$$

The appropriate boundary conditions are

$f = 0$	at	$\eta = 0$	(impermeable wall)
$\theta' = -\,1$	at	$\eta = 0$	(q'', fixed, is from the wall into the porous medium)
$f' = 0$	as	$\eta \to \infty$	(zero vertical velocity)
$\theta = 0$	as	$\eta \to \infty$	($T = T_\infty$ far enough from the wall)

Problem 12.14. From Ref. [19] we have

$$\frac{q''}{T_0(y) - T_\infty}\,\frac{y}{k} = 0.6788\left(\frac{Kg\,\beta\, y\,\overbrace{\Delta T}^{T_0(y) - T_\infty}}{\alpha\,\nu}\right)^{1/2}$$

The objective of the following analysis is to express the right-hand side in terms of Ra_{*y}, as in eqs. (12.100) and (12.101):

$$\left(\frac{q'' y}{\Delta T\, k}\right)^2 = (0.6788)^2 \frac{K g \beta y}{\alpha \nu} \Delta T$$

$$\left(\frac{q'' y}{\Delta T\, k}\right)^3 = (0.6788)^2 \frac{K g \beta y^2 q''}{\alpha \nu k}$$

$$Nu_y = \underbrace{(0.6788)^{2/3}}_{0.772} Ra_{*y}^{1/3}$$

Problem 12.15. If the partition is modeled as isothermal, the temperature difference between each porous medium and the partition is $\Delta T/2$, where $\Delta T = T_{\infty,H} - T_{\infty,L}$ is the overall temperature difference. The overall heat transfer rate $\overline{q''} H$ can be calculated based on eq. (12.96):

$$\frac{\overline{q''} H}{(\Delta T/2)\, k} = 0.888 \left(\frac{K g \beta H \Delta T/2}{\alpha \nu}\right)^{1/2}$$

$$\frac{\overline{q''} H}{\Delta T\, k} = (0.888)\, 2^{-3/2} \left(\frac{K g \beta H \Delta T}{\alpha \nu}\right)^{1/2}$$

$$\overline{Nu} = 0.314\, Ra_H^{1/2} \cdot \qquad \text{(eq. 12.106, T)}$$

If the wall is modeled as one with uniform heat flux, we can use eq. (12.102):

$$\frac{\overline{q''} H}{(\Delta T/2)\, k} = 1.03 \left(\frac{K g \beta H^2 q''}{\alpha \nu k}\right)^{1/3}$$

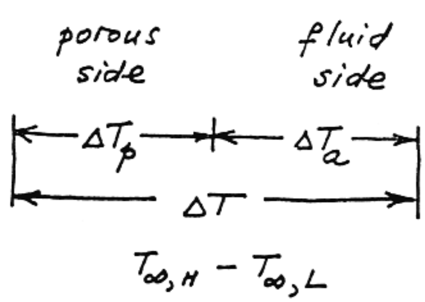
$$\frac{\overline{q''} H}{\Delta T\, k} = \left(\frac{1.03}{2}\right) \left(\frac{K g \beta H \Delta T}{\alpha \nu}\right)^{1/2}$$

$$\overline{Nu} = 0.370\, Ra_H^{1/2} \cdot \qquad \text{(eq. 12.106, } q''\text{)}$$

Comparing the two alternatives T and q" with eq. (12.106), we conclude that the uniform heat flux is a better way to model the partition.

Problem 12.16. Modeling the interface as isothermal, we have

$$\Delta T = \Delta T_p + \Delta T_a$$



The overall heat transfer rate $Q = \overline{q''} H$ can be estimated either from eq. (12.96) or from eq. (4.65').

Equation (12.96) yields

$$\frac{Q}{k\,\Delta T_p} = 0.888\left(\frac{Kg\,\beta\,H\,\Delta T_p}{\alpha\,\nu}\right)^{1/2}$$

or

$$\left(\frac{Q}{0.888\,k}\right)^{2/3}\left(\frac{\alpha\,\nu}{Kg\,\beta\,H}\right)^{1/3} = \Delta T_p \qquad (1)$$

Equation (4.65') can be rewritten the same way:

$$\frac{Q}{k_a\,\Delta T_a} = 0.671\left(\frac{g\,\beta_a\,H^3\,\Delta T_a}{\alpha_a\,\nu_a}\right)^{1/4}$$

$$\left(\frac{Q}{0.671\,k_a}\right)^{4/5}\left(\frac{\alpha_a\,\nu_a}{g\,\beta_a\,H^3}\right)^{1/5} = \Delta T_f \qquad (2)$$

By adding eqs. (1) and (2), we obtain ΔT on the right-hand side. The resulting equation can be rearranged to read

$$1.082\,B^{-2/3}\left(\frac{Nu}{Ra_{H,a}^{1/4}}\right)^{2/3} + 1.376\left(\frac{Nu}{Ra_{H,a}^{1/4}}\right)^{4/5} = 1,$$

where

$$B = \frac{k}{k_a}\,\frac{Ra_H^{1/2}}{Ra_{H,a}^{1/4}}$$

Its two asymptotes are

$$\frac{Nu}{Ra_{H,a}^{1/4}} = 0.888B, \quad \text{as} \quad B \rightarrow 0$$

$$\frac{Nu}{Ra_{H,a}^{1/4}} = 0.671, \quad \text{as} \quad B \rightarrow \infty$$

An alternative relationship between Nu $Ra_{H,a}^{1/4}$ and B can be obtained by modeling the wall as one with uniform heat flux, and using eqs. (12.102) and the wall-averaged version of eq. (4.76).

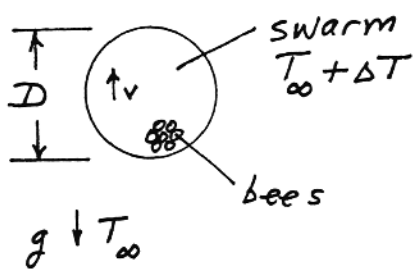
Problem 12.17. This problem is recommended as a term project. The analytical course is outlined in Section 12.6.6, and the plotting of a curve similar to the one shown in Fig. 12.10 will require numerical help.

Problem 12.18. We model the swarm as a ball of diameter D ~ 10 cm and temperature $T_\infty + \Delta T$, where T_∞ is the ambient temperature. The swarm loses heat to the ambient in two ways, internally, to the air that permeates (rises) through the bees, and, externally, to the surrounding air.

The internal heat transfer rate can be estimated by noting that the pressure difference that drives air upward (v) through the swarm is the difference

$$\Delta P \sim \rho_\infty gD - \rho\, g\, D$$

$$\sim \Delta\rho \cdot gD \sim \rho\, gD\, \beta\, \Delta T$$



This is the same as the difference between the hydrostatic pressures at the bottom of two D-tall columns, one filled with T_∞ air, and the other with $T_\infty + \Delta T$ air. The internal vertical air velocity scale is

$$v \sim \frac{K}{\mu}\frac{\Delta P}{D} \sim \frac{Kg\,\beta\,\Delta T}{\nu}$$

The mass flowrate going up is $\dot{m} \sim \rho\, v\, D^2$; the enthalpy removed by this stream from the swarm is

$$q_{internal} \sim \dot{m}\, c_P\, \Delta T$$

$$\sim k\, \Delta T\, D^2\, K\, \frac{g\,\beta\,\Delta T}{\alpha\,\nu} \qquad (1)$$

The external heat transfer rate can be estimated by noting the Nusselt number scaling for the external boundary layer,

$$\overline{Nu} = \frac{hD}{k} \sim \left(\frac{g\,\beta\,\Delta T\, D^3}{\alpha\,\nu}\right)^{1/4}$$

This leads to the external heat transfer rate

$$q_{external} \sim h\,D^2\,\Delta T$$

$$\sim k\,\Delta T\,D\left(\frac{g\,\beta\,\Delta T\,D^3}{\alpha\,\nu}\right)^{1/4} \qquad (2)$$

Finally, we set $Ra = g\beta\,\Delta T\,D^3\,/\,(\alpha\nu)$ and divide eqs. (1) and (2) to obtain a relative measure of the internal porous-medium convection effect,

$$\frac{q_{internal}}{q_{external}} \sim \frac{K}{D^2}\,Ra^{3/4} \sim \frac{1}{D^2}\,\frac{d^2\,\varphi^3}{180\,(1-\varphi)^2}\,Ra^{3/4}$$

Here we used the Carman-Kozeny model for the permeability of a bed of spheres (the bees) of diameter d.

By substituting $D \sim 10$ cm, $d \sim 1$ cm, and $\varphi \sim 0.35$, we find that

$$\frac{q_{internal}}{q_{external}} \sim 0.47$$

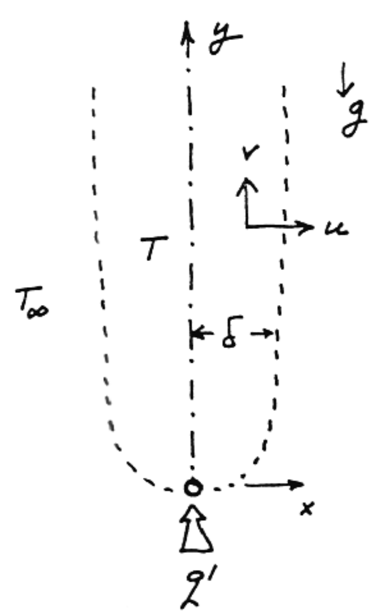
This means that the heat loss from the swarm should be calculated by accounting for boundary layer convection on the outside of the swarm, and natural convection through a porous medium (bees + air) on the inside.

Problem 12.19. We begin with the relevant boundary layer-type equations,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (C)$$

$$\frac{\partial v}{\partial x} = \frac{Kg\,\beta}{\nu}\,\frac{\partial T}{\partial x} \qquad (M)$$

$$u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial x^2} \qquad (E)$$



$$\Delta T \sim T_{x=0} - T_{x=\delta}$$

which, in the plume region of thickness δ, imply the following balances of scales:

$$\frac{u}{\delta} \sim \frac{v}{y} \tag{C}$$

$$\frac{v}{\delta} \sim \frac{Kg\,\beta}{\nu}\,\frac{\Delta T}{\delta} \tag{M}$$

$$v\,\frac{\Delta T}{y} \sim \alpha\,\frac{\Delta T}{\delta^2} \tag{E}$$

So far, we have three scaling laws for four unknown scales (u, v, δ, ΔT); the fourth scaling law is the statement that the vertical flow of enthalpy is y-independent and equal to q',

$$q' \sim \rho\, c_P\, v\, \delta\, \Delta T \tag{q'}$$

Combining the laws (C), (M), (E) and (q') we obtain the following scales

$$u \sim \frac{\alpha}{y}\,Ra^{1/3}$$

$$v \sim \frac{\alpha}{y}\,Ra^{2/3}$$

$$\delta \sim y\,Ra^{-1/3}$$

$$\Delta T \sim \frac{q'}{k}\,Ra^{-1/3}$$

where the Rayleigh number is based on q' and y,

$$Ra = \frac{Kg\,\beta\,y\,q'}{\alpha\,\nu\,k}$$

The dimensionless similarity solution is easy to construct once the scales of the flow are known. By introducing the streamfunction,

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x},$$

which means that

$$\psi \sim v\,\delta \sim \alpha\,Ra^{1/3},$$

we find that the similarity streamfunction profile must be

$$f(\eta) = \frac{\psi}{\alpha\,Ra^{1/3}}$$

where

$$\eta = \frac{x}{y}\,Ra^{1/3} \quad \text{(note that Ra depends on y)}$$

The similarity temperature profile is

$$\theta(\eta) = \frac{T - T_\infty}{(q'/k)\, Ra^{-1/3}}$$

Written in terms of f, θ and η, the streamfunction formulation of eqs. (C), (M) and (E) is:

$$f'' = -\theta' \tag{1}$$

$$\frac{1}{3}(f\theta') = \theta'' \tag{2}$$

The corresponding form of $\int_{-\infty}^{\infty} \rho\, c_P v (T - T_\infty)\, dx = q'$ is

$$\int_{-\infty}^{\infty} f'\theta = -1 \tag{3}$$

Integrating (1) subject to $f'(\infty) = 0 = \theta(\infty)$ we obtain

$$f' = -\theta \tag{4}$$

Integrating (2) subject to $\theta(\infty) = 0 = \theta'(\infty)$ we obtain further

$$\frac{1}{3} f\, \theta = \theta' \tag{5}$$

Equations (4,5) yield

$$f = -C \tanh\left(\frac{C}{6}\eta\right) \tag{6}$$

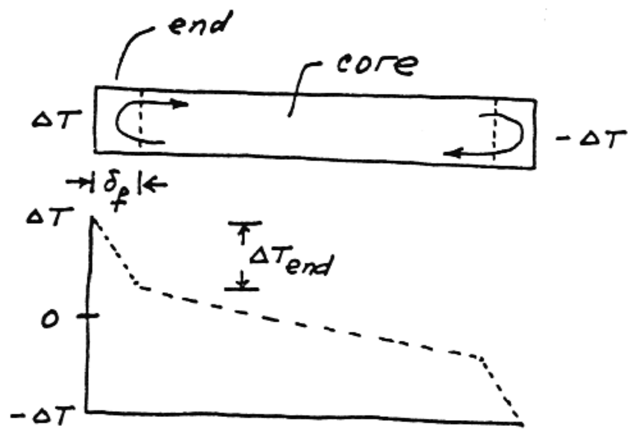
$$\theta = \frac{C^2/6}{\cosh^2\left(\frac{C}{6}\eta\right)}, \tag{7}$$

where C is the constant inherited from having integrated eq. (5). This constant follows from the q' constraint, eq. (3):

$$6C^{-3} = \int_{-\infty}^{\infty} \frac{dz}{\cosh^4 z} = \frac{4}{3}$$

Problem 12.20. Looking at the left end of the shallow layer, we have

$$Q \sim q''H \sim k\,H\,\frac{\Delta T_{end}}{\delta_f}$$



However, $\Delta T_{end} \lesssim \Delta T$, because of the presence of the long core (in the limit $H/L \to 0$, the core ΔT actually dominates ΔT_{end}). Therefore

$$Q \lesssim kH\,\frac{\Delta T}{\delta_f}$$

Problem 12.21. In the high Ra_H limit, $\delta_e << 1$, $K_1 << 1$, and eqs. (12.167)-(12.168) become

$$Nu = \frac{K_1^3}{120}\left(Ra_H\,\frac{H}{L}\right)^2 \qquad (1)$$

$$\frac{1}{120}\,\delta_e\,Ra_H^2\,K_1^3\left(\frac{H}{L}\right)^3 = 1 \qquad (2)$$

$$\frac{1}{2}\,K_1\,\frac{H}{L}\,\frac{1}{\delta_e} = 1 \qquad (3)$$

Combining eqs. (2) and (3) yields

$$K_1 = 240^{1/4}\,\frac{L}{H}\,Ra_H^{-1/2}$$

hence, eq. (1) becomes

$$Nu = \underbrace{\frac{240^{3/4}}{120}}_{0.508}\,\frac{L}{H}\,Ra_H^{1/2}$$

i.e. only 12 percent below Weber's formula, $Nu = 0.577\,(L/H)\,Ra_H^{1/2}$.

Problem 12.22 Consider first a parallel-plates channel of length L and spacing D. If the flow is in the Hagen-Poiseuille regime, the mean velocity of the flow is

$$U = \frac{D^2}{12\mu} \frac{\Delta P}{L} \tag{1}$$

The flow rate through the channel is

$$\dot{m}' = \rho D U \tag{2}$$

where the units of $\dot{m}'$ are kg s^{-1} m^{-1}. Equations (1) and (2) show that the required pressure difference is

$$\Delta P = 12\nu\, \dot{m}' \frac{L}{D^3} \tag{3}$$

Next, consider a Y-shaped assembly of three channels. One channel of size (L_1, D_1, $\dot{m}_1'$, ΔP_1) splits into two identical channels of size (L_2, D_2, $\dot{m}_2'$, ΔP_2). The overall mass flow rate is conserved

$$\dot{m}_1' = 2\,\dot{m}_2' \tag{4}$$

The overall pressure difference is

$$\Delta P = \Delta P_1 + \Delta P_2 \tag{5}$$

which according to Eqs. (3) and (4) can be written as the overall flow resistance

$$R = \frac{\Delta P}{12\,\nu\,\dot{m}_1'} = \frac{L_1}{D_1^3} + \frac{L_2}{2\,D_2^3} \tag{6}$$

The total volume occupied by the three channels is fixed. This means that the projected area of the Y-shaped construct is fixed,

$$A = L_1 D_1 + 2 L_2 D_2 \tag{7}$$

In this configuration the variables are D_1 and D_2, while L_1, L_2 and A are specified. The minimization of R with respect to D_1 and D_2 subject to constraint (7) is equivalent to minimizing the aggregate function $\Phi = R + \lambda A$, namely

$$\Phi = \frac{L_1}{D_1^3} + \frac{L_2}{2\,D_2^3} + \lambda \left(L_1 D_1 + 2\,L_2 D_2\right) \tag{8}$$

where λ is a Lagrange multiplier. Solving the system

$$\frac{\partial \Phi}{\partial D_1} = 0 \tag{9}$$

$$\frac{\partial \Phi}{\partial D_2} = 0 \tag{10}$$

we obtain the optimal ratio of spacings,

$$\frac{D_1}{D_2} = 2^{1/2} \tag{11}$$

Noteworthy is the robustness of this architectural feature. The optimal ratio of spacings is independent of L_1, L_2, A and the layout of the Y-shaped construct.

Problem 12.23 The following solution is based on elements of scale analysis, which are indicated by "~" as shorthand for "is of the same order of magnitude as". We begin with the Darcy flow part of the analysis. The mean velocity of the gas that enters one channel is

$$U \sim \frac{D_f^2}{\mu} \frac{\Delta P}{L} \tag{1}$$

The total gas mass flow rate used by the HLW volume is

$$\dot{m} = n \rho U D_f W \tag{2}$$

where n is the number of $(D_f + D_s)$ pairs present in the stack,

$$n = \frac{H}{D_f + D_s} \tag{3}$$

Combining eqs. (1)-(3), we find the relationship between mass flow rate density and geometry,

$$\dot{m}''' \frac{\mu}{\Delta P} \sim \frac{D_f^3}{L^2 (D_f + D_s)} \tag{4}$$

For the heat transfer part of the problem, we note that the heat current along one D_s blade is

$$q_1 \sim k_s D_s W \frac{\Delta T}{L} \tag{5}$$

The total heat generation rate is therefore

$$q = n q_1 \sim k_s W \Delta T \frac{H D_s}{L (D_f + D_s)} \tag{6}$$

Because of the proportionality between gas flow rate and heat generation rate,

$$q = C \dot{m} \tag{7}$$

eq. (6) is a second way of expressing the gas flow rate

$$\dot{m}''' \frac{C}{k_s \Delta T} \sim \frac{D_s}{L^2 (D_f + D_s)} \tag{8}$$

By eliminating $\dot{m}'''$ between eqs. (4) and (8), we find how the geometric parameters are related when ΔP and ΔT are specified,

$$\frac{D_f^3}{D_s} \sim N \tag{9}$$

where

$$N = \frac{k_s \Delta T}{C} \frac{\mu}{\Delta P} \tag{10}$$

The design consists of maximizing $\dot{m}'''$ of eq. (4) or eq. (8), subject to the constraint (9). One degree of freedom is represented by the ratio

$$x = \frac{D_s}{D_f} \tag{11}$$

or the porosity

$$\phi = \frac{D_f}{D_f + D_s} = \frac{1}{1 + x} \tag{12}$$

Equations (4) and (9) become

$$\dot{m}''' \frac{\mu}{\Delta P} \sim \frac{D_f^2}{L^2 (1 + x)} \tag{13}$$

$$N \sim \frac{D_f^2}{x} \tag{14}$$

They indicate that in addition to x (or D_f), another geometric degree of freedom is L. For example, if we eliminate D_f^2 between eqs. (13) and (14) we find

$$\dot{m}''' \frac{\mu}{\Delta P} \sim \frac{N}{L^2} \frac{x}{1 + x} \tag{15}$$

The effect of x on $\dot{m}'''$ is such that $\dot{m}'''$ reaches its highest values (a plateau) when $x > 1$, or when

$$D_s > D_f \tag{16}$$

This is an important and useful result: because graphite is expensive, it means that the $\dot{m}'''$ performance reached with large graphite content ($D_s > D_f$) is comparable with the performance reached with 'just enough' graphite,

$$D_s \gtrsim D_f \tag{17}$$

This means that the porosity should be

$$\phi \lesssim 0.5 \tag{18}$$

and that the 'plateau' performance of eq. (15) is given by

$$\left(\dot{m}''' \frac{\mu}{\Delta P} \right)_{max} \sim \frac{N}{L^2} \tag{19}$$

Further improvements in performance are achieved when L is made as small as possible.

The effect of the constraint is also clear: a large N is beneficial. This means that the reaction density increases in the direction of higher allowable ΔT's, because eq. (19) is the same as

$$\dot{m}'''_{max} \sim \frac{k_s \Delta T}{C L^2} \tag{20}$$
