Theory of
Elasticity

by

Majid Mirzaei

No originality is claimed for these notes other than selection, organization, and presentation of the material.

The major references that have been used for preparation of the lecture notes are as follows:

- Timoshenko & Goodier, "Theory of Elasticity"
- Boresi & Chong, "Elasticity in Engineering Mechanics"
- Phillip L. Gould, "Introduction to Linear Elasticity"
- Provan, J.W., "Stress Analysis Lecture Notes"

Theory of Elasticity
Lecture Notes

Majid Mirzaei, PhD
Associate Professor
Dept. of Mechanical Eng., TMU
mmirzaei@modares.ac.ir
http://www.modares.ac.ir/eng/mmirzaei

Chapter 1
Field Equations of Linear Elasticity

In many engineering applications a *pointwise* description of the field quantities, such as stress and strain, is required throughout the body. The distinctive feature of the ***theory of elasticity,*** compared to the alternative approaches like the strength of materials, is that it provides a consistent set of equations which may be solved (using the advanced techniques of applied mathematics) to obtain a unique pointwise description of the distribution of the stresses, strains, and displacements for a particular loading and geometry.

Tensors, Indicial Notation

In the theory of elasticity we usually deal with physical quantities which are independent of any particular coordinate system that may be used to describe them. At the same time, these quantities (tensors) are usually specified in a particular coordinate system by their components. Specifying the components of a *tensor* in one coordinate system determines the components in any other system. The physical laws of Elasticity are in turn expressed by tensor equations. Because tensor transformations are linear and homogeneous, such tensor equations, if they are valid in one coordinate system, are valid in any other coordinate system. This invariance of tensor equations under a coordinate transformation is one of the principal reasons for the usefulness of tensor methods in the theory of Elasticity.

In a three-dimensional Euclidean space (such as ordinary physical space) the number of components of a tensor is 3^N, where N is the order of the tensor. Accordingly a tensor of order zero is specified in any coordinate system in three-dimensional space by one component. Tensors of order zero are called *scalars*. Physical quantities having magnitude only are represented by scalars. Tensors of order one have three coordinate components in physical space and are known as *vectors*. Quantities possessing both magnitude and direction are represented by vectors. Second-order tensors are called *dyadics.* Higher order tensors such as *triadics*, or tensors of order three, and *tetradics*, or tensors of order four can also be defined.

In these notes we use the *indicial notation* for presentation of tensorial quantities and equations. Under the rules of indicial notation, a letter index may occur either once or twice in a given term. When an index occurs unrepeated in a term, that index is understood to take on the values 1,2,.. .,N where N is a specified integer that determines the range of the index. Unrepeated indices are known as *free* indices. The tensorial *rank* of a given term is equal to the number of free indices appearing in that term. Also, correctly written tensor equations have the same letters as free indices in every term. In ordinary physical space a basis is composed of three, non-coplanar vectors, and so any vector in this space is completely specified by its three components. Therefore the range on the index of a_i (which represents a vector in physical space) is 1,2,3. Accordingly the symbol a_i is understood to represent the three components a_1, a_2, a_3. For a range of three on both indices, the symbol A_{ij} represents nine components (of the second-order tensor (dyadic) **A**).

When an index appears twice in a term, that index is understood to take on all the values of its range, and the resulting terms summed. In this so-called summation convention, repeated indices are often referred to as *dummy* indices because their replacement by any other letter not appearing as a free index does not change the meaning of the term in which they occur. *In general, no index occurs more than twice in a properly written term.*

Kinematics of Deformable Solids

Fig. 1.1 shows the undeformed configuration of a material continuum at time $t = t_0$ together with the deformed configuration at a later time t = t. It is useful to refer the initial and final configurations to separate coordinate systems as shown in the figure.

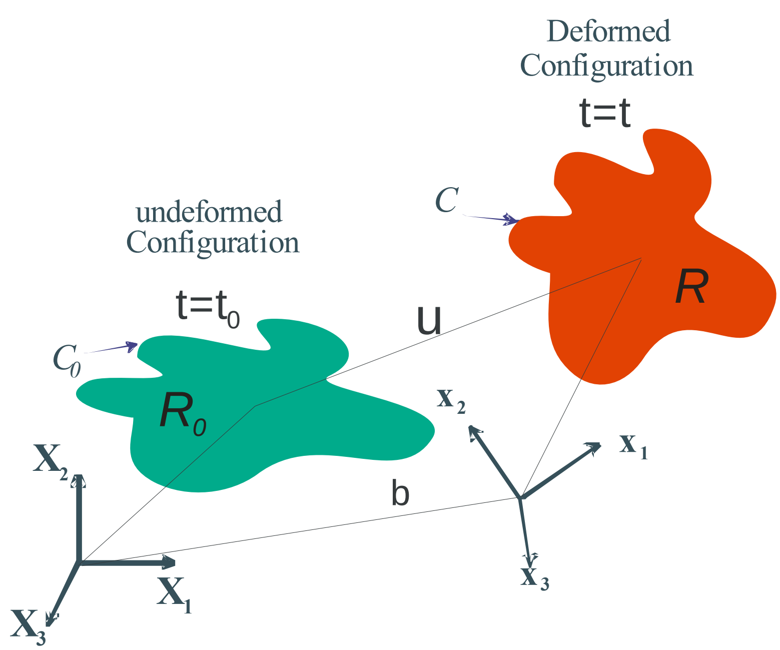

Fig. 1.1

Suppose a material point is at position **X** in the undeformed solid, and moves to a position **x** when the solid is loaded. We may describe the deformation and motion of a solid by a mapping in the following form:

$$\mathbf{x} = \chi(\mathbf{X}, t) \tag{1-1}$$

Now we consider the two coordinate systems to be coincident, as shown in Fig. 1.2. The *displacement* of a material point is:

$$\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{X} \tag{1-2}$$

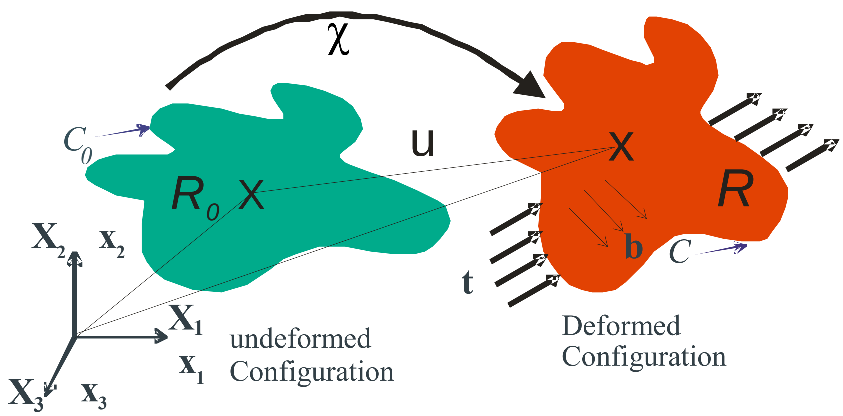

Fig. 1.2

Next, consider two straight parallel lines on the reference configuration of a solid (see Fig. 1.3). If the deformation of the solid is homogeneous, the two lines remain straight in the deformed configuration, and the lines remain parallel.

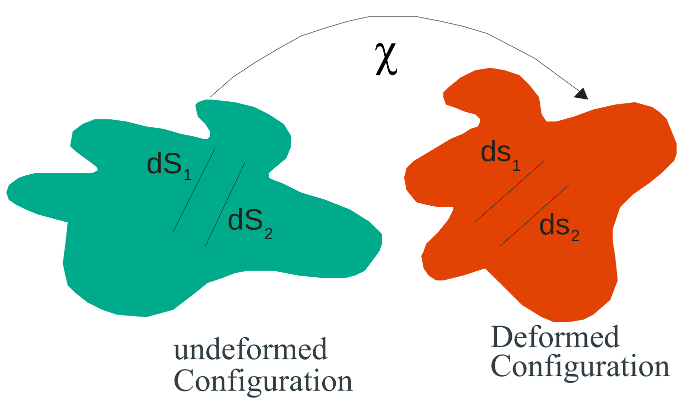

Fig. 1.3

Furthermore, the lines stretch by the same amount, i.e.:

$$\frac{ds_1}{dS_1} = \frac{ds_2}{dS_2} \tag{1-3}$$

The difference $(ds)^2 - (dS)^2$ is used to define a *measure of deformation,* which occurs in the vicinity of the particles between the initial and final configurations.

$$\begin{aligned}(ds)^2 &= d\bar{x}_1^2 + d\bar{x}_2^2 + d\bar{x}_3^2 = d\bar{x}_i d\bar{x}_i = d\mathbf{x}\cdot d\mathbf{x} \\ (dS)^2 &= d\bar{X}_1^2 + d\bar{X}_2^2 + d\bar{X}_3^2 = d\bar{X}_i d\bar{X}_i = d\mathbf{X}\cdot d\mathbf{X}\end{aligned} \tag{1-4}$$

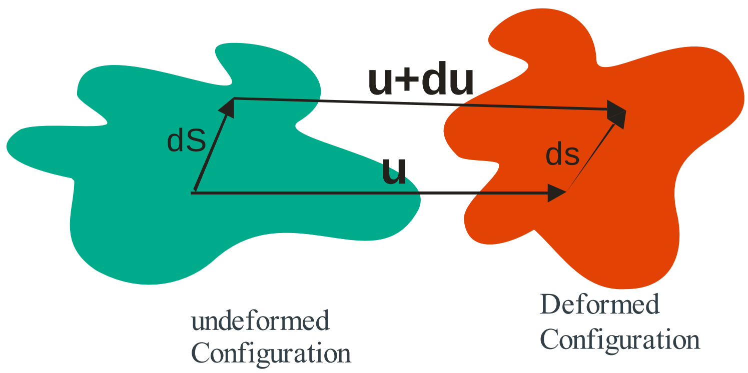

Fig. 1.4

Referring to Fig. 1.4, we may further write:

$$d\mathbf{S} + \mathbf{u} + d\mathbf{u} = \mathbf{u} + d\mathbf{s}, \quad \Rightarrow \quad d\mathbf{u} = d\mathbf{s} - d\mathbf{S} \tag{1-5}$$

We also have,

$$(ds)^2 - (dS)^2 = d\bar{x}_i d\bar{x}_i - d\bar{X}_i d\bar{X}_i \tag{1-6}$$

and,

$$\begin{aligned}\bar{x}_i &= \bar{x}_i(X_1, X_2, X_3) \\ d\bar{x}_i &= \frac{\partial \bar{x}_i}{\partial X_1} d\bar{X}_1 + \frac{\partial \bar{x}_i}{\partial X_2} d\bar{X}_2 + \frac{\partial \bar{x}_i}{\partial X_3} d\bar{X}_3 \\ d\bar{x}_i &= x_{i,j} d\bar{X}_j = \mathbf{F}\cdot d\mathbf{X}\end{aligned} \tag{1-7}$$

where the *comma* denotes differentiation with respect to a spatial coordinate. In the above $\mathbf{F}$ is known as *deformation gradient tensor*. Next, we define

$$\mathbf{u} = u_1\mathbf{e}_{x_1} + u_2\mathbf{e}_{x_2} + u_3\mathbf{e}_{x_3} = u_i\mathbf{e}_{x_i} \tag{1-8}$$

and proceed as follows:

$$\begin{aligned}(ds)^2 - (dS)^2 &= x_{i,j}d\overline{X}_j x_{i,k}d\overline{X}_k - d\overline{X}_i d\overline{X}_i = (\mathbf{F}\cdot d\mathbf{X})\cdot(\mathbf{F}\cdot d\mathbf{X}) - d\mathbf{X}\cdot d\mathbf{X} \\ &= \left(x_{i,j}x_{i,k} - \delta_{ij}\delta_{ik}\right)d\overline{X}_j d\overline{X}_k \\ &= \left[\left(X_i + u_i\right)_{,j}\left(X_i + u_i\right)_{,k} - \delta_{jk}\right]d\overline{X}_j d\overline{X}_k \\ &= \left[\left(\delta_{ij} + u_{i,j}\right)\left(\delta_{ik} + u_{i,k}\right) - \delta_{jk}\right]d\overline{X}_j d\overline{X}_k \\ &= \left[\delta_{jk} + u_{j,k} + u_{k,j} + u_{i,j}u_{i,k} - \delta_{jk}\right]d\overline{X}_j d\overline{X}_k \\ &= \left(u_{j,k} + u_{k,j} + u_{i,j}u_{i,k}\right)d\overline{X}_j d\overline{X}_k\end{aligned} \tag{1-9}$$

Rewriting the indices in a more proper form, we get

$$\begin{aligned}(ds)^2 - (dS)^2 &= \left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right)d\overline{X}_i d\overline{X}_j \\ &= 2\varepsilon_{ij}^L d\overline{X}_i d\overline{X}_j\end{aligned} \tag{1-10}$$

in which we distinguish the *Lagrangian strain tensor* ε_{ij}^L for characterization of the deformation near a point.

$$\varepsilon_{ij}^L = \frac{1}{2}\left(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right) \tag{1-11}$$

Alternatively we may write

$$\begin{aligned}&\overline{X}_i = \overline{X}_i(x_1, x_2, x_3) \\ &d\overline{X}_i = \frac{\partial\overline{X}_i}{\partial x_1}d\overline{x}_1 + \frac{\partial\overline{X}_i}{\partial x_2}d\overline{x}_2 + \frac{\partial\overline{X}_i}{\partial x_3}d\overline{x}_3 \\ &d\overline{X}_i = X_{i,j}d\overline{x}_j = \mathbf{F}\cdot d\mathbf{x}\end{aligned} \tag{1-12}$$

and proceed as follows:

$$
\begin{aligned}
(ds)^2-(dS)^2 &= d\bar{x}_i d\bar{x}_i - X_{i,j}d\bar{x}_j X_{i,k}d\bar{x}_k = d\mathbf{x}\cdot d\mathbf{x}-(\mathbf{F}\cdot d\mathbf{x})\cdot(\mathbf{F}\cdot d\mathbf{x}) \\
&= \left(\delta_{ij}\delta_{ik}-X_{i,j}X_{i,k}\right)d\bar{x}_j d\bar{x}_k \\
&= \left[\delta_{jk}-\left(x_i-u_i\right)_{,j}\left(x_i-u_i\right)_{,k}\right]d\bar{x}_j d\bar{x}_k \\
&= \left[\delta_{jk}-\left(\delta_{ij}-u_{i,j}\right)\left(\delta_{ik}-u_{i,k}\right)\right]d\bar{x}_j d\bar{x}_k \\
&= \left[\delta_{jk}-\delta_{jk}+u_{k,j}+u_{j,k}-u_{i,j}u_{i,k}\right]d\bar{x}_j d\bar{x}_k \\
&= \left(u_{j,k}+u_{k,j}-u_{i,j}u_{i,k}\right)d\bar{x}_j d\bar{x}_k
\end{aligned}
\tag{1-13}
$$

Replacing the dummy indices i and k, we have

$$
\begin{aligned}
(ds)^2-(dS)^2 &= \left(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j}\right)d\bar{x}_i d\bar{x}_j \\
&= 2\varepsilon_{ij}^{E}d\bar{x}_i d\bar{x}_j
\end{aligned}
\tag{1-14}
$$

This time, we define the *Eulerian strain tensor* ε_{ij}^{L} to characterize the deformation near a point.

$$
\varepsilon_{ij}^{E}=\frac{1}{2}\left(u_{i,j}+u_{j,i}-u_{k,i}u_{k,j}\right)
\tag{1-15}
$$

In practice, we need to make a number of assumptions to simplify the equations of linear elasticity. A major one is to assume that deformations are *infinitesimal.* In most practical circumstances it is sufficient to assume $u_{k,i}u_{k,j} \ll 1$

We use this to define a linear measure of deformation in linear elasticity, and define the infinitesimal strain tensor as:

$$
\boxed{\varepsilon_{ij}^{L}=\varepsilon_{ij}^{E}=\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right) \qquad (1-16)}
$$

In order to check the relationship between the above definition of strain and the conventional definition, we consider a one-dimensional case as follows:

$$
\begin{aligned}
(ds)^2-(dS)^2 &= 2\varepsilon_{ij}^{L}dX_i dX_j = 2\varepsilon_{22}dX_i dX_i = 2\varepsilon_{22}(dS)^2 \\
\varepsilon_{22} &= \frac{(ds)^2-(dS)^2}{2(dS)^2}=\frac{(ds+dS)(ds-dS)}{dS(ds+dS)}=\frac{ds-dS}{dS}
\end{aligned}
\tag{1-17}
$$

The linear strain tensor in Cylindrical and Spherical Coordinate Systems

Cylindrical

$$\begin{pmatrix} \varepsilon_{rr}=\dfrac{\partial u_r}{\partial r} & 2\varepsilon_{r\theta}=\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r} & 2\varepsilon_{rz}=\dfrac{\partial u_r}{\partial z}+\dfrac{\partial u_z}{\partial r} \\ & \varepsilon_{\theta\theta}=\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r} & 2\varepsilon_{\theta z}=\dfrac{\partial u_\theta}{\partial z}+\dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta} \\ & & \varepsilon_{zz}=\dfrac{\partial u_z}{\partial z} \end{pmatrix} \qquad (1\text{-}18)$$

Spherical

$$\begin{pmatrix} \varepsilon_{rr}=\dfrac{\partial u_r}{\partial r} & 2\varepsilon_{r\theta}=\dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta}+\dfrac{\partial u_\theta}{\partial r}-\dfrac{u_\theta}{r} & 2\varepsilon_{r\varphi}=\dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \varphi}+\dfrac{\partial u_\varphi}{\partial r}-\dfrac{u_\varphi}{r} \\ & \varepsilon_{\theta\theta}=\dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta}+\dfrac{u_r}{r} & 2\varepsilon_{\theta\varphi}=\dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \varphi}+\dfrac{1}{r}\dfrac{\partial u_\varphi}{\partial \theta}-\dfrac{u_\theta\cot\varphi}{r} \\ & & \varepsilon_{\varphi\varphi}=\dfrac{1}{r\sin\theta}\dfrac{\partial u_\varphi}{\partial \varphi}+\dfrac{u_r}{r}+\dfrac{u_\theta\cot\theta}{r} \end{pmatrix} \qquad (1\text{-}19)$$

Compatibility

Conditions of compatibility, imposed on the components of strain, are necessary and sufficient to insure a continuous single-valued displacement field. The procedure is to eliminate the displacements between kinematic equations to produce equations with only strain as unknowns. For instance we may write:

$$\begin{aligned} \varepsilon_{ll,mm} &= u_{l,lmm} \\ \varepsilon_{mm,ll} &= u_{m,mll} \\ \varepsilon_{lm,lm} &= \frac{1}{2}(u_{l,mlm}+u_{m,llm}) \\ &= \frac{1}{2}(u_{l,lmm}+u_{m,mll}) \\ &\Rightarrow 2\varepsilon_{lm,lm}=\varepsilon_{ll,mm}+\varepsilon_{mm,ll} \end{aligned} \qquad (1\text{-}20)$$

In general, we have

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik} \tag{1-21}$$

which represents 81 equations, only six of them are independent.

$$\begin{aligned}
&\varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12} \\
&\varepsilon_{22,33} + \varepsilon_{33,22} = 2\varepsilon_{23,23} \\
&\varepsilon_{33,11} + \varepsilon_{11,33} = 2\varepsilon_{31,31} \\
&\varepsilon_{12,13} + \varepsilon_{13,12} - \varepsilon_{23,11} = \varepsilon_{11,23} \\
&\varepsilon_{23,21} + \varepsilon_{21,23} - \varepsilon_{31,22} = \varepsilon_{22,31} \\
&\varepsilon_{31,32} + \varepsilon_{32,31} - \varepsilon_{12,33} = \varepsilon_{33,12}
\end{aligned} \tag{1-22}$$

Assignment 1:

Noting that $\nabla = (\ )_{,i}\mathbf{e}_i$ and using indicial notation, prove the identities or find the right hand side:

$$\begin{aligned}
&\nabla^2\phi = \nabla \cdot \nabla\phi = \phi_{,ii} \\
&\nabla \cdot (\phi\mathbf{v}) = \nabla\phi \cdot \mathbf{v} + \phi\nabla \cdot \mathbf{v} \\
&\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi \cdot \nabla\psi + \psi\nabla^2\phi \\
&\nabla^3(\phi\psi) = \nabla\nabla^2(\phi\psi) = ? \\
&\nabla \cdot (\phi\nabla\psi) = ?
\end{aligned}$$

Kinetics of Deformable Solids

In general, two types of forces can be applied to a solid body.
(i) External forces applied to its boundary (e.g. forces arising from contact with another solid or fluid pressure).
(ii) Externally applied forces that are distributed throughout a body (e.g. gravitational, magnetic, and inertial forces).

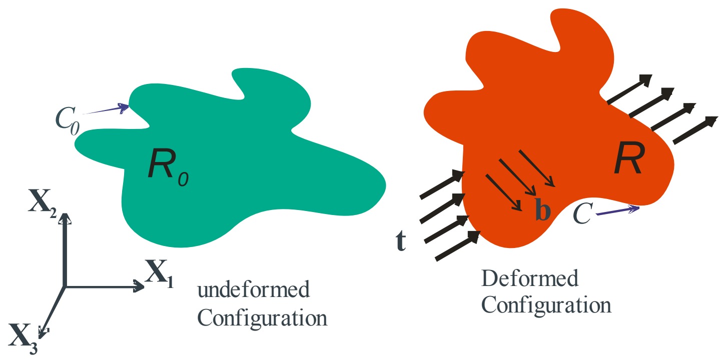

Fig. 1.5

The stress vector or *surface traction* t at a point represents the force acting on the surface per unit area and can be defined as:

$$\mathbf{t} = \lim_{dA \to 0} \frac{d\mathbf{P}}{dA} \tag{1-23}$$

in which dA is an element of area on a surface subjected to a force $d\mathbf{P}$ (see Fig. 1.6).

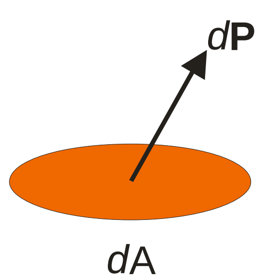

Fig. 1.6

The *body force vector* denotes the external force acting on the interior of a solid per unit volume and can be defined as:

$$\mathbf{b} = \lim_{dV \to 0} \frac{d\mathbf{P}}{dV} \tag{1-24}$$

in which dV denotes an infinitesimal volume element subjected to a force $d\mathbf{P}$ (Fig. 1.7).

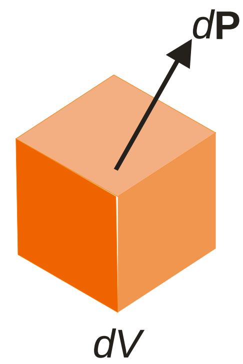

Fig. 1.7

Internal forces induced by external loading

The solid body shown in Fig. 1.8 is subjected to a balanced system of externally applied forces. As a result, *internal forces* are developed in order to keep the body together. Suppose that the body is cut into two parts. The force $\mathbf{P}_n$ represents the resultant force that acts on the two faces in order to keep the two parts in load equilibrium.

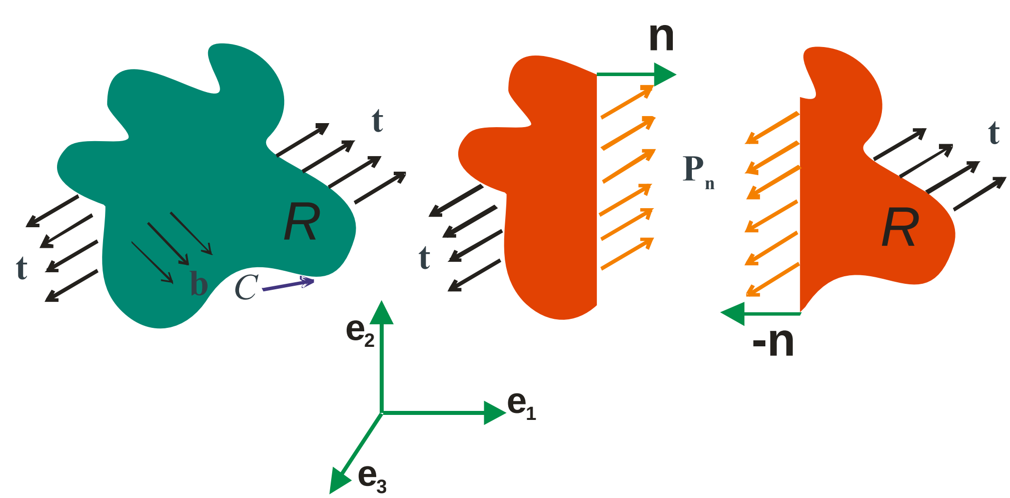

Fig. 1.8

The internal traction vector $\mathbf{T}_n$ represents the force per unit area acting on a plane with normal vector **n** inside the deformed solid an can be defined as:

$$\mathbf{T}_n = \lim_{dA \to 0} \frac{d\mathbf{P}_n}{dA} \tag{1-25}$$

in which dA is an element of area in the interior of the solid, with normal **n** (see Fig. 1.9).

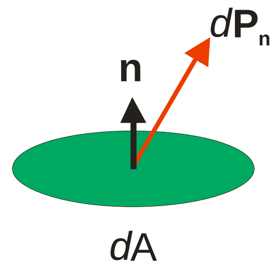

Fig. 1.9

The components of **Cauchy stress** in a given basis can be visualized as the tractions acting on planes with normals parallel to each basis vector, as depicted in Fig. 1.10.

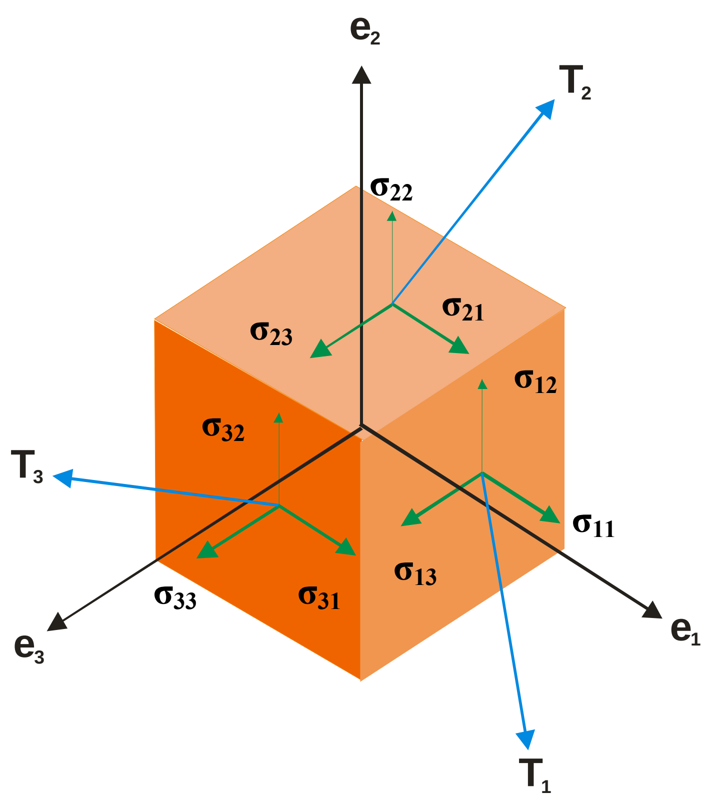

Fig. 1.10

Here, we may write:

$$\begin{aligned}\mathbf{T}_1 &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\ \mathbf{T}_2 &= \sigma_{21}\mathbf{e}_1 + \sigma_{22}\mathbf{e}_2 + \sigma_{23}\mathbf{e}_3 \qquad or \quad \mathbf{T}_i = \sigma_{ij}\mathbf{e}_j \\ \mathbf{T}_3 &= \sigma_{31}\mathbf{e}_1 + \sigma_{32}\mathbf{e}_2 + \sigma_{33}\mathbf{e}_3\end{aligned} \tag{1-26}$$

In order to find the components of the traction vector on an arbitrary plane (represented by **n)** we impose the equilibrium on a tetrahedral element, as shown in Fig. 1.11.

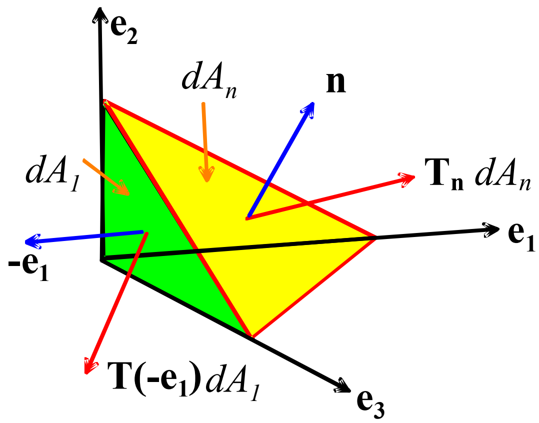

Fig. 1.11

$$\mathbf{T}_n dA_n - \mathbf{T}_i dA_i + f\left(\frac{1}{3} h dA_n\right) = 0, \qquad \not{\Sigma} n$$
$$h \to 0 \quad \Rightarrow \mathbf{T}_n dA_n - \mathbf{T}_i dA_i = 0$$
$$dA_i = dA_n \mathrm{Cos}(\mathbf{n}, \mathbf{e}_i) \tag{1-27}$$
$$\mathbf{T}_n dA_n - \mathbf{T}_i dA_n \mathbf{n} \cdot \mathbf{e_i} = 0$$
$$\mathbf{T}_n = T_i \mathbf{e}_i \;,\; \mathbf{T}_i = \sigma_{ij} \mathbf{e}_j \;,\; \mathbf{n} \cdot \mathbf{e_i} = n_i$$
$$T_i \mathbf{e}_i - \sigma_{ij} \mathbf{e}_j n_i = 0 \;,\; T_i \mathbf{e}_i = \sigma_{ij} \mathbf{e}_j n_i = \sigma_{ji} \mathbf{e}_i n_j$$

Accordingly, we obtain the following expressions which are called: *"the stress boundary equations"*

$$T_i = \sigma_{ji} n_j \tag{1-28}$$

Principle Stresses

For practical purposes, it is convenient to break down the traction vector into normal and shear components as follows:

$$\sigma_{nn} = \mathbf{T}_n \cdot \mathbf{n} = T_i e_i \cdot \mathbf{n} = T_i n_i \qquad \not{\Sigma} n \tag{1-29}$$
$$= \sigma_{ji} n_j n_i$$

and,

$$\sigma_{ns} = (T_i T_i - \sigma_{nn}{}^2)^{\frac{1}{2}} \qquad \not\Sigma n \tag{1-30}$$

In order to find the *extremum* values of the normal components of the stress tensor (the *principle stresses*) we may write:

$$\begin{aligned} \sigma_{nn} &= \sigma_{ji} n_j n_i \qquad \not\Sigma n \\ &= \sigma_{ji} n_j n_i - \lambda(n_i n_i - 1) \end{aligned} \tag{1-31}$$

where λ is the lagrangian multiplier. Next, we write:

$$\begin{aligned} \frac{\partial \sigma_{nn}}{\partial n_i} &= \sigma_{ji} n_j - \lambda n_i = 0 \qquad \not\Sigma n \\ &= \sigma_{ji} n_j - \lambda \delta_{ij} n_j = 0 \\ &= (\sigma_{ij} - \lambda \delta_{ij}) n_j = 0 \end{aligned} \tag{1-32}$$

which implies that

$$\begin{vmatrix} \sigma_{11} - \lambda & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \lambda & \sigma_{23} \\ \sigma_{31} & \sigma_{31} & \sigma_{33} - \lambda \end{vmatrix} = 0 \tag{1-33}$$

The characteristic polynomial is

$$\lambda^3 - I_I \lambda^2 + I_{II} \lambda - I_{III} = 0 \tag{1-34}$$

The roots of the above characteristic polynomial are the eigenvalues of our problem, or the principle stresses. The *invariants* of the stress tensor are defined as:

$$\begin{cases} I_I = \sigma_{ii} \\ I_{II} = \dfrac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}) \\ I_{III} = \left|\sigma_{ij}\right| \end{cases} \tag{1-35}$$

Finally, for each eigenvalue, we may find the eigenvectors which are the principle stress directions.

$$\begin{aligned}
&\lambda_1 \rightarrow (\sigma_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow n_1^1, n_2^1, n_3^1 \\
&\lambda_2 \rightarrow (\sigma_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow n_1^2, n_2^2, n_3^2 \\
&\lambda_3 \rightarrow (\sigma_{ij} - \lambda\delta_{ij})n_j = 0 \Rightarrow n_1^3, n_2^3, n_3^3
\end{aligned} \qquad (1\text{-}36)$$

Assignment 2:

For the given stress tensor, determine the maximum shear stress and show that it acts in the plane which bisects the maximum and minimum stress planes.

$$\sigma_{ij} = \begin{pmatrix} 5 & 0 & 0 \\ \cdot & -6 & -12 \\ \cdot & \cdot & 1 \end{pmatrix}$$

Conservation of Linear Momentum

In order to find the governing equations for *distribution* of the stress tensor within a body, we start by considering an arbitrary material region with volume *V* and surface *S*. Based on the general concept of conservation of linear momentum, we may write:

$$\int_S \mathbf{T}dA + \int_V \mathbf{f}dV = \int_V \rho\ddot{\mathbf{u}}dV \qquad (1\text{-}37)$$

which, using (1-28), can be written as

$$\int_S \sigma_{ji}n_j dA + \int_V f_i dV = \int_V \rho\ddot{u}_i dV \qquad (1\text{-}38)$$

Now we use the Divergence theorem of Gauss and write

$$\begin{aligned}
&\int_S \mathbf{G}\cdot\mathbf{n}dA = \int_V \nabla\cdot\mathbf{G}dV \\
&\int_V \sigma_{ji,j}dV + \int_V f_i dV \equiv \int_V \left(\sigma_{ji,j} + f_i\right)dV = \int_V \rho\ddot{u}_i dV
\end{aligned} \qquad (1\text{-}39)$$

Since we are dealing with an arbitrary volume, the three equations of motion can be derived from (1-39) as

$$\boxed{\sigma_{ji,j} + f_i = \rho\ddot{u}_i \qquad (1-40)}$$

The static form of the above equations (usually known as the *equilibrium* equations) is

$$\sigma_{ji,j} + f_i = 0 \tag{1-41}$$

The Equations of Motion in Cylindrical and Spherical Coordinate Systems can be written as:

$$
\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = \rho \ddot{u}_r \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta = \rho \ddot{u}_\theta \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = \rho \ddot{u}_z
\end{aligned} \tag{1-42}
$$

and

$$
\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{r\varphi}}{\partial \varphi} + \frac{1}{r}(2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\varphi\varphi} - \sigma_{r\theta}\cot\varphi) + f_r = \rho \ddot{u}_r \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\theta\varphi}}{\partial \varphi} + \frac{1}{r}[3\sigma_{r\theta} + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\cot\theta] + f_\theta = \rho \ddot{u}_\theta \\
&\frac{\partial \sigma_{r\varphi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\varphi}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{1}{r}(3\sigma_{r\varphi} + 2\sigma_{\varphi\theta}\cot\theta) + f_\varphi = \rho \ddot{u}_\varphi
\end{aligned} \tag{1-43}
$$

Conservation of Angular Momentum

Once again, let us consider a material region with volume V and surface S. Based on the general concept of conservation of angular momentum we may write:

$$
\begin{aligned}
&\int_S (\mathbf{X} \times \mathbf{T})dA + \int_V (\mathbf{X} \times \mathbf{f})dV = \int_V (\mathbf{X} \times \rho\ddot{\mathbf{u}})dV \\
&\int_S \varepsilon_{ijk} X_i T_j dA + \int_V \varepsilon_{ijk} X_i f_j dV = 0
\end{aligned} \tag{1-44}
$$

which, using (1-28) and the Divergence theorem of Gauss, can be written as:

$$\begin{aligned}
&\int_S \varepsilon_{ijk} X_i (\sigma_{lj} n_l) dA + \int_V \varepsilon_{ijk} X_i f_j dV = 0 \\
&\int_V \varepsilon_{ijk} (X_i \sigma_{lj})_{,l} dV + \int_V \varepsilon_{ijk} X_i f_j dV = 0 \\
&\int_V \varepsilon_{ijk} [(X_i \sigma_{lj})_{,l} + X_i f_j] dV = 0 \\
&\int_V \varepsilon_{ijk} [X_{i,l} \sigma_{lj} + X_i \sigma_{lj,l} + X_i f_j] dV = 0
\end{aligned} \tag{1-45}$$

Next, we use (1-41) to simplify the above as:

$$\begin{aligned}
&\int_V \varepsilon_{ijk} \left[X_{i,l} \sigma_{lj} + X_i \left(\overset{=0}{\sigma_{lj,l} + f_j} \right) \right] dV = 0 \\
&\int_V \varepsilon_{ijk} \delta_{il} \sigma_{lj} dV = 0
\end{aligned} \tag{1-46}$$

Since we are dealing with an arbitrary volume, we can write:

$$\begin{aligned}
&\varepsilon_{ijk} \sigma_{ij} = 0 \\
&\Rightarrow \sigma_{ij} = \sigma_{ji}
\end{aligned} \tag{1-47}$$

which shows that the stress tensor is *symmetric*.

Constitutive Equations of Linear Elasticity

For an elastic body which is gradually strained at constant temperature, the components of stress can be derived from the *strain energy density* ψ, which is a quadratic function of the strain components.

$$\sigma_{ij} = \frac{\partial \psi}{\partial e_{ij}} \tag{1-48}$$

Accordingly, we may write the most general form of the Hooke's Law as:

$$\sigma_{ij} = E_{ijkl} e_{kl} \tag{1-49}$$

in which E_{ijkl} represents 81 components. Due to the symmetry of the stress and strain tensors and also the elastic coefficient matrix, the number of independent elastic constants reduces to 21.

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1112} & 2E_{1123} & 2E_{1131} \\ & E_{2222} & E_{2233} & 2E_{2212} & 2E_{2223} & 2E_{2231} \\ & & E_{3333} & 2E_{3312} & 2E_{3323} & 2E_{3331} \\ & & & 2E_{1212} & 2E_{1223} & 2E_{1231} \\ & & & & 2E_{2323} & 2E_{2331} \\ & & & & & 2E_{3131} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{Bmatrix} \quad (1\text{-}50)$$

which can be simplified to:

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{Bmatrix} \quad (1\text{-}51)$$

The above expression represents the constitutive equations for an *anisotropic* elastic material. Most engineering materials show some degree of *isotropic* behavior as they possess properties of *symmetry* with respect to different planes or axes. We start with the plane x_2x_3 as the plane of symmetry, which implies that the x_1 axis can be reversed as shown in Fig. 1.12.

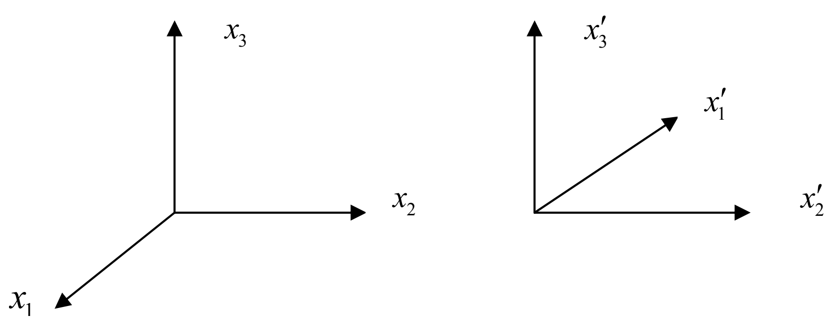

Fig. 1.12: single plane symmetry

This corresponds to a coordinate transformation with the direction cosines shown in table 1.1.

axes	x_1	x_2	x_3
x'_1	-1		
x'_2		1	
x'_3			1

Table 1.1

Using the transformation $\sigma'_{ij} = \alpha_{ik}\alpha_{jl}\sigma_{kl}$, in which α_{ik} and α_{jl} are the direction cosines, we find that:

$$\begin{aligned} &\sigma'_{11} = \sigma_{11} \quad \sigma'_{22} = \sigma_{22} \quad \sigma'_{33} = \sigma_{33} \\ &\sigma'_{12} = -\sigma_{12} \quad \sigma'_{23} = \sigma_{23} \quad \sigma'_{31} = -\sigma_{31} \end{aligned} \tag{1-52}$$

Similarly, we can show that:

$$\begin{aligned} &\varepsilon'_{11} = \varepsilon_{11} \quad \varepsilon'_{22} = \varepsilon_{22} \quad \varepsilon'_{33} = \varepsilon_{33} \\ &\varepsilon'_{12} = -\varepsilon_{12} \quad \varepsilon'_{23} = \varepsilon_{23} \quad \varepsilon'_{31} = -\varepsilon_{31} \end{aligned} \tag{1-53}$$

Hence, we may write:

$$\begin{Bmatrix} \sigma'_{11} \\ \sigma'_{22} \\ \sigma'_{33} \\ \sigma'_{12} \\ \sigma'_{23} \\ \sigma'_{31} \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ -\sigma_{12} \\ \sigma_{23} \\ -\sigma_{31} \end{Bmatrix} = \begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix} \begin{Bmatrix} \varepsilon'_{11} \\ \varepsilon'_{22} \\ \varepsilon'_{33} \\ \varepsilon'_{12} \\ \varepsilon'_{23} \\ \varepsilon'_{31} \end{Bmatrix}$$

$$= \begin{bmatrix} & & \\ & C & \\ & & \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ -\varepsilon_{12} \\ \varepsilon_{23} \\ -\varepsilon_{31} \end{Bmatrix} \tag{1-54}$$

which results in:

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{12}\\ \sigma_{23}\\ \sigma_{31}\end{Bmatrix}=\begin{bmatrix}C_{11} & C_{12} & C_{13} & -C_{14} & C_{15} & -C_{16}\\ & C_{22} & C_{23} & -C_{24} & C_{25} & -C_{26}\\ & & C_{33} & -C_{34} & C_{35} & -C_{36}\\ & & & C_{44} & -C_{45} & C_{46}\\ & & & & C_{55} & -C_{56}\\ & & & & & C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \varepsilon_{12}\\ \varepsilon_{23}\\ \varepsilon_{31}\end{Bmatrix} \quad (1\text{-}55)$$

As the constants should not change with the transformation, we must have C_{14}, C_{16}, C_{24}, C_{26}, C_{34}, C_{36}, C_{45}, and $C_{56} = 0$. Such materials are called *monoclinic* and need 13 constants to describe their elastic properties.

The double symmetry about the x_1 and x_2 axes will lead to further reduction of the constants down to 9, which represents the *orthotropic* material (Note that for 2D problems we need 4 elastic constants to describe the behavior of an orthotropic material).

axes	x_1	x_2	x_3
x_1'	-1		
x_2'		-1	
x_3'			1

We may have an additional simplification by considering the directional independence in elastic behavior by interchanging x_2 with x_3, and x_1 with x_2. Such materials are called *cubic* with three independent constants.

axes	x_1	x_2	x_3
x_1'	1	0	0
x_2'	0	0	1
x_3'	0	1	0

axes	x_1	x_2	x_3
x_1'	0	1	0
x_2'	1	0	0
x_3'	0	0	1

Finally, we may consider rotational independence in material property and define the constitutive equations for an *isotropic* elastic material with only two independent constants:

axes	x_1	x_2	x_3
x_1'	1	0	0
x_2'	0	cosθ	sinθ
x_3'	0	- sinθ	cosθ

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{bmatrix} 2\mu+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2\mu+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2\mu+\lambda & 0 & 0 & 0 \\ & & & 2\mu & 0 & 0 \\ & & & & 2\mu & 0 \\ & & & & & 2\mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{Bmatrix} \tag{1-56}$$

in which λ and μ are called the Lamé constants. The indicial form can be written as:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \tag{1-57}$$

We may invert the above equation to express the strains in terms of stresses:

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\delta_{ij}\sigma_{kk} \tag{1-58}$$

The Lamé constants are quite suitable from mathematical point of view, but they should be related to the Engineering elastic constants obtained in the laboratory (like E and υ) as well. Table 1.2 shows the relationships between different elastic constants.

	λ,μ	E,ν	μ,ν	E,μ	k,ν
λ	λ	$\frac{\nu E}{(1+\nu)(1-2\nu)}$	$\frac{2\mu\nu}{(1-2\nu)}$	$\frac{\mu(E-2\mu)}{3\mu-E}$	$\frac{3k\nu}{(1+\nu)}$
μ	μ	$\frac{E}{2(1+\nu)}$	μ	μ	$\frac{3k(1-2\nu)}{2(1+\nu)}$
k	$\lambda+\frac{2}{3}\mu$	$\frac{E}{3(1-2\nu)}$	$\frac{2\nu(1+\nu)}{3(1-2\nu)}$	$\frac{\mu E}{3(3\mu-E)}$	k
E	$\frac{\mu(3\lambda+2\mu)}{(\lambda+\mu)}$	E	$2\mu(1+\nu)$	E	$3k(1-2\nu)$
ν	$\frac{\lambda}{2(\lambda+\mu)}$	ν	ν	$\frac{E}{2\mu}-1$	ν

Table 1.2

Formulation and Solution Methods of Elasticity Problems

In order to find the fifteen unknowns (three components of displacement, six components of strain, and six components of stress) for an elasticity problem, we need fifteen equations as follows:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad \textbf{(I)}$$

$$\sigma_{ji,j} + f_i = 0 \qquad \textbf{(II)}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk} \qquad or$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\delta_{ij}\sigma_{kk} \qquad \textbf{(III)}$$

along with the compatibility and stress boundary equations:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{ik,jl} + \varepsilon_{jl,ik} \qquad \textbf{(IV)}$$

$$T_i = \sigma_{ji} n_j \qquad \textbf{(V)}$$

Classical Displacement Formulation

We start by substituting Eqs.(**I**) into the first form of Eqs.(**III**) to eliminate the strains:

$$\sigma_{ij} = \lambda\delta_{ij}u_{k,k} + \mu(u_{i,j} + u_{j,i}) \qquad (1\text{-}59)$$

Next, we substitute the above expression for the stresses in Eqs.(**II**) which resulys in:

$$\lambda\delta_{ij}u_{k,ki} + \mu(u_{i,ji} + u_{j,ii}) + f_j = 0$$

$$\lambda u_{k,kj} + \mu(u_{i,ji} + u_{j,ii}) + f_j = 0 \qquad (1\text{-}60)$$

Finally, using *i* for the dummy index in all terms we write:

$$\mu u_{j,ii} + (\lambda + \mu)u_{i,ij} + f_j = 0 \qquad (1\text{-}61)$$

or alternatively:

$$(\lambda + G)u_{j,ij} + Gu_{i,jj} + f_i = 0 \tag{1-62}$$

Equations (1-62) are three equilibrium equations in terms of displacements. They are called *Navier Equations* and constitute the *classical displacement formulation*.

Navier Displacement Equations in Cylindrical and Spherical Coordinates

Cylindrical:

$$
\begin{aligned}
&(\lambda + 2G)\frac{\partial I_e}{\partial r} - 2G\left(\frac{1}{r}\frac{\partial \omega_z}{\partial \theta} - \frac{\partial \omega_\theta}{\partial z}\right) + f_r = 0 \\
&(\lambda + 2G)\frac{\partial I_e}{r\partial \theta} - 2G\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right) + f_\theta = 0 \\
&(\lambda + 2G)\frac{\partial I_e}{\partial z} - 2G\left(\frac{\partial (r\omega_\theta)}{\partial r} - \frac{\partial \omega_r}{\partial \theta}\right) + f_z = 0 \\
&\textit{where} \\
&I_e = e_{ii} = \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial (ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \\
&2\omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \quad ; \quad 2\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \quad ; \\
&2\omega_z = \frac{1}{r}\left(\frac{\partial (ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)
\end{aligned}
\tag{1-63}
$$

Spherical:

$$(\lambda+2G)\frac{\partial I_e}{\partial r}-\frac{2G}{r\sin\theta}\left[\frac{\partial(\omega_\varphi\sin\theta)}{\partial\theta}-\frac{\partial\omega_\theta}{\partial\varphi}\right]+f_r=0$$

$$(\lambda+2G)\frac{\partial I_e}{r\partial\theta}-\frac{2G}{r}\left[\frac{1}{\sin\theta}\frac{\partial(\omega_r)}{\partial\varphi}-\frac{\partial(r\omega_\varphi)}{\partial r}\right]+f_\theta=0$$

$$\frac{(\lambda+2G)}{r\sin\theta}\frac{\partial I_e}{\partial\varphi}-\frac{2G}{r}\left[\frac{\partial(r\omega_\theta)}{\partial r}-\frac{\partial(r\omega_r)}{\partial\theta}\right]+f_\varphi=0$$

where (1-64)

$$I_e=e_{ii}=\nabla\cdot\mathbf{u}=\frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}$$

$$2\omega_r=\frac{1}{r\sin\theta}\left[\frac{\partial(u_\varphi\sin\theta)}{\partial\theta}-\frac{\partial u_\theta}{\partial\varphi}\right]\ ;\ 2\omega_\theta=\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi}-\frac{1}{r}\frac{\partial(ru_\varphi)}{\partial r}\ ;$$

$$2\omega_\varphi=\frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r}-\frac{\partial u_r}{\partial\theta}\right)$$

Classical Force Formulation

We start by substituting the RHS of the second form of Eqs. (**III**) for strains in Eqs. (**IV**) to produce 81 stress compatibility equations as follows:

$$\sigma_{ij,kl}+\sigma_{kl,ij}-\sigma_{ik,jl}-\sigma_{jl,ik}=\frac{\nu}{1+\nu}\left(\delta_{ij}\sigma_{tt,kl}-\delta_{kl}\sigma_{tt,ij}-\delta_{ik}\sigma_{tt,jl}-\delta_{jl}\sigma_{tt,ik}\right) \quad (1\text{-}65)$$

However, only six of the above equations are independent:

$$\sigma_{ij,kk}+\sigma_{kk,ij}-\sigma_{ik,jk}-\sigma_{jk,ik}=\frac{\nu}{1+\nu}\left(\delta_{ij}\sigma_{tt,kk}-\delta_{kk}\sigma_{tt,ij}-\delta_{ik}\sigma_{tt,jk}-\delta_{jk}\sigma_{tt,ik}\right) \quad (1\text{-}66)$$

In general, the above expression represents nine equations with free indices *i* and *j*. Now we substitute the equilibrium equations (**II**) in the above and simplify to have:

$$\nabla^2\sigma_{ij}+\frac{1}{1+\nu}\sigma_{mm,ij}=-\left(f_{i,j}+f_{j,i}+\frac{\nu}{1-\nu}\delta_{ij}f_{m,m}\right) \quad (1\text{-}67)$$

which are known as *Beltrami-Michell* compatibility relations and constitute the *classical force formulation*.

Assignment 3:

For a hollow sphere under internal and external pressures find the stress, strain and displacement distributions.

Beltrami-Michell Equations in Cylindrical Coordinates:

$$
\begin{aligned}
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \sigma_{rr}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\sigma_{rr}}{\partial\theta^2}+\frac{\partial^2\sigma_{rr}}{\partial z^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial r^2}=-\frac{\nu}{1-\nu}\left[\frac{1}{r}\frac{\partial(rf_r)}{\partial r}+\frac{1}{r}\frac{\partial(f_\theta)}{\partial\theta}+\frac{\partial(f_z)}{\partial z}\right]-2\frac{\partial(f_r)}{\partial r}\\
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \sigma_{\theta\theta}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\sigma_{\theta\theta}}{\partial\theta^2}+\frac{\partial^2\sigma_{\theta\theta}}{\partial z^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial \theta^2}=-\frac{\nu}{1-\nu}\left[\frac{1}{r}\frac{\partial(rf_r)}{\partial r}+\frac{1}{r}\frac{\partial(f_\theta)}{\partial\theta}+\frac{\partial(f_z)}{\partial z}\right]-2\frac{\partial(f_\theta)}{\partial \theta}\\
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \sigma_{zz}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\sigma_{zz}}{\partial\theta^2}+\frac{\partial^2\sigma_{zz}}{\partial z^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial z^2}=-\frac{\nu}{1-\nu}\left[\frac{1}{r}\frac{\partial(rf_r)}{\partial r}+\frac{1}{r}\frac{\partial(f_\theta)}{\partial\theta}+\frac{\partial(f_z)}{\partial z}\right]-2\frac{\partial(f_z)}{\partial z}\\
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \sigma_{\theta z}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\sigma_{\theta z}}{\partial\theta^2}+\frac{\partial^2\sigma_{\theta z}}{\partial z^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial \theta\partial z}=-\left[\frac{\partial(f_\theta)}{\partial z}+\frac{\partial(f_z)}{\partial\theta}\right]\\
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \sigma_{rz}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\sigma_{rz}}{\partial\theta^2}+\frac{\partial^2\sigma_{rz}}{\partial z^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial r\partial z}=-\left[\frac{\partial(f_z)}{\partial r}+\frac{\partial(f_r)}{\partial z}\right]\\
&\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \sigma_{r\theta}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\sigma_{r\theta}}{\partial\theta^2}+\frac{\partial^2\sigma_{r\theta}}{\partial z^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial r\partial \theta}=-\left[\frac{\partial(f_r)}{\partial \theta}+\frac{\partial(f_\theta)}{\partial r}\right]
\end{aligned}
\tag{1-68}
$$

$$I_t=\sigma_{rr}+\sigma_{\theta\theta}+\sigma_{zz}$$

Beltrami-Michell Equations in Spherical Coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma_{rr}}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sigma_{rr}}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\sigma_{rr}}{\partial\varphi^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial r^2}=$$
$$-\frac{\nu}{1-\nu}\left[\frac{1}{r^2}\frac{\partial\left(r^2 f_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta f_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial\left(f_\varphi\right)}{\partial\varphi}\right]-2\frac{\partial\left(f_r\right)}{\partial r}$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma_{\theta\theta}}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\sigma_{\theta\theta}}{\partial\varphi^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial\theta^2}=$$
$$-\frac{\nu}{1-\nu}\left[\frac{1}{r^2}\frac{\partial\left(r^2 f_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta f_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial\left(f_\varphi\right)}{\partial\varphi}\right]-2\frac{\partial\left(f_\theta\right)}{\partial\theta}$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma_{\varphi\varphi}}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sigma_{\varphi\varphi}}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\sigma_{\varphi\varphi}}{\partial\varphi^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial\varphi^2}=$$
$$-\frac{\nu}{1-\nu}\left[\frac{1}{r^2}\frac{\partial\left(r^2 f_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta f_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial\left(f_\varphi\right)}{\partial\varphi}\right]-2\frac{\partial\left(f_\varphi\right)}{\partial\varphi} \qquad (1\text{-}69)$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma_{\theta\varphi}}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sigma_{\theta\varphi}}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\sigma_{\theta\varphi}}{\partial\varphi^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial\theta\partial\varphi}=-\left(\frac{\partial\left(f_\theta\right)}{\partial\varphi}+\frac{\partial\left(f_\varphi\right)}{\partial\theta}\right)$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma_{r\varphi}}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sigma_{r\varphi}}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\sigma_{r\varphi}}{\partial\varphi^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial r\partial\varphi}=-\left(\frac{\partial\left(f_r\right)}{\partial\varphi}+\frac{\partial\left(f_\varphi\right)}{\partial r}\right)$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\sigma_{r\theta}}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\sigma_{r\theta}}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\sigma_{r\theta}}{\partial\varphi^2}+\frac{1}{1+\nu}\frac{\partial^2 I_t}{\partial r\partial\theta}=-\left(\frac{\partial\left(f_r\right)}{\partial\theta}+\frac{\partial\left(f_\theta\right)}{\partial r}\right)$$

$$I_t=\sigma_{rr}+\sigma_{\theta\theta}+\sigma_{\varphi\varphi}$$

Chapter 2: Some Representative Boundary Value Problems

Deformation of a Rod Standing in a Gravitational Field

As a first example in solving elasticity boundary-value problems, we consider a long rod that stands freely in a gravitational field (see Fig. 2.1).

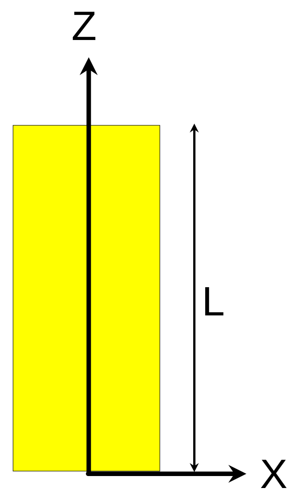

Fig. 2.1

We start with the stress boundary conditions.

$$T_i = \sigma_{ij} n_j \tag{2-1}$$

On the top surface we have:

$$\begin{aligned}
&T_i(0,0,0), \quad n_i(0,0,1) \\
&T_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \Rightarrow 0 = 0 + 0 + \sigma_{13} \Rightarrow \sigma_{13} = 0 \\
&T_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \Rightarrow 0 = 0 + 0 + \sigma_{23} \Rightarrow \sigma_{23} = 0 \\
&T_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \Rightarrow 0 = 0 + 0 + \sigma_{33} \Rightarrow \sigma_{33} = 0
\end{aligned} \tag{2-2}$$

On the lateral surfaces we have:

$$
\begin{aligned}
&T_i(0,0,0), \quad n_i(n_1,n_2,0)\\
&0=\sigma_{11}n_1+\sigma_{12}n_2+0\\
&0=\sigma_{21}n_1+\sigma_{22}n_2+0\\
&0=\sigma_{31}n_1+\sigma_{32}n_2+0
\end{aligned}
\tag{2-3}
$$

On the bottom surface we have:

$$
\begin{aligned}
&T_i(0,0,\rho gL), \quad n_i(0,0,-1)\\
&0=0+0-\sigma_{13}\\
&0=0+0-\sigma_{23}\\
&\rho gL=0+0-\sigma_{33}\Rightarrow\sigma_{33}=-\rho gL
\end{aligned}
\tag{2-4}
$$

We now turn our attention to the overall equilibrium:

$$
\begin{aligned}
&\sigma_{ij,j}+f_i=0\\
&\sigma_{11,1}+\sigma_{12,2}+\sigma_{13,3}+f_1=0 \qquad (i)\\
&\sigma_{21,1}+\sigma_{22,2}+\sigma_{23,3}+f_2=0 \qquad (ii)\\
&\sigma_{31,1}+\sigma_{32,2}+\sigma_{33,3}-\rho g=0 \qquad (iii)
\end{aligned}
\tag{2-5}
$$

Near the bottom surface we have:

$\sigma_{13}=0,\ \sigma_{23}=0,$

Thus, using (2-5 iii), we can write:

$$\sigma_{33,3}-\rho g=0\Rightarrow\sigma_{33}=\rho gz+f(1,2)$$

$$\text{B.C.: at } z=0 \text{ we have } \sigma_{33}=-\rho gL\Rightarrow f(1,2)=-\rho gL \tag{2-6}$$

$$\Rightarrow\sigma_{33}=\rho gz-\rho gL=\rho g(z-L)$$

We may therefore assume that the distribution of the stress tensor has the following form:

$$
\sigma_{ij}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&-\rho g(L-z)\end{pmatrix}
\tag{2-7}
$$

Note that this form satisfies all the equilibrium equations and the boundary conditions of the problem. Using the following constitutive equations:

$$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right] \tag{2-8}$$

we can obtain the distribution of the strain tensor throughout the rod.

$$\varepsilon_{ij} = \begin{pmatrix} \frac{\nu\rho g}{E}(L-z) & 0 & 0 \\ 0 & \frac{\nu\rho g}{E}(L-z) & 0 \\ 0 & 0 & -\frac{\rho g}{E}(L-z) \end{pmatrix} \tag{2-9}$$

Since the strain components are linear in Z, the compatibility equations are all satisfied.

Now we use the kinematic equations to obtain the displacement field:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2-10}$$

We start with the normal strain components:

$$\begin{aligned}
&\varepsilon_{11} = \frac{1}{2}\left(u_{1,1} + u_{1,1}\right) = u_{1,1} \Rightarrow u_{1,1} = \frac{\nu\rho g}{E}(L-z) \qquad \text{(i)} \\
&u_1 = \frac{\nu\rho g}{E}(L-z)x + f_1(y,z) \\
&\varepsilon_{22} = \frac{1}{2}\left(u_{2,2} + u_{2,2}\right) = u_{2,2} \Rightarrow u_{2,2} = \frac{\nu\rho g}{E}(L-z) \qquad \text{(ii)} \\
&u_2 = \frac{\nu\rho g}{E}(L-z)y + f_2(x,z) \\
&\varepsilon_{33} = \frac{1}{2}\left(u_{3,3} + u_{3,3}\right) = u_{3,3} \Rightarrow u_{3,3} = -\frac{\rho g}{E}(L-z) \qquad \text{(iii)} \\
&u_3 = -\frac{\rho g}{E}\left(Lz - \frac{1}{2}z^2\right) + f_3(x,y)
\end{aligned} \tag{2-11}$$

Now we may invoke the displacement boundary conditions:

at $x=0,\ u_1=0,\ \Rightarrow f_1=0$
at $y=0,\ u_2=0,\ \Rightarrow f_2=0$

In order to find f_3, we calculate the shear strain components and use (i, iii) of (2-11) to write:

$$\varepsilon_{13}=\frac{1}{2}\left(u_{1,3}+u_{3,1}\right)\Rightarrow u_{1,3}+u_{3,1}=0$$
$$\Rightarrow -\frac{\nu\rho g}{E}x+f_{3,1}=0\Rightarrow f_{3,1}=\frac{\nu\rho g}{E}x\Rightarrow f_3=\frac{\nu\rho g}{2E}x^2+h_1(y) \qquad (2\text{-}12)$$

Now, using the expressions (ii, iii) of Eqs. (2-11) along with (2-12), we may write:

$$\varepsilon_{23}=\frac{1}{2}\left(u_{2,3}+u_{3,2}\right)\Rightarrow u_{2,3}+u_{3,2}=0$$
$$\Rightarrow -\frac{\nu\rho g}{E}y+h_{1,2}=0\Rightarrow h_{1,2}=\frac{\nu\rho g}{E}y\Rightarrow h_1=\frac{\nu\rho g}{2E}y^2+C \qquad (2\text{-}13)$$

Hence, the final form of the displacement equations can be written as:

$$u_1=\frac{\nu\rho g}{E}(L-z)x$$
$$u_2=\frac{\nu\rho g}{E}(L-z)y$$
$$u_3=-\frac{\rho g}{E}\left(Lz-\frac{1}{2}z^2\right)+\frac{\nu\rho g}{2E}x^2+\frac{\nu\rho g}{2E}y^2+C \qquad (2\text{-}14)$$

at $x=0, y=0, z=0,$ we have $\mathbf{u}=0\Rightarrow C=0$

$$u_3=-\frac{\rho g}{E}\left(Lz-\frac{1}{2}z^2\right)+\frac{\nu\rho g}{2E}x^2+\frac{\nu\rho g}{2E}y^2$$

Rotating Discs

In this section we study the stresses induced in rotating discs. Initially, we assume that the thickness of the disc is small in comparison with its radius, so that the variation of radial and circumferential stresses over the thickness can be neglected. We also assume that $\sigma_{zz} = 0$.

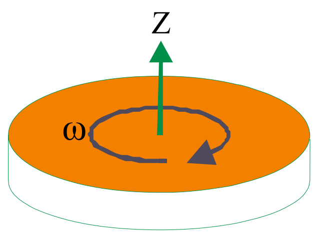

Fig. 2.2

We may start by writing the equations of motion in cylindrical coordinates:

$$\begin{aligned}
&\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + f_r = \rho \ddot{u}_r \\
&\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} + f_\theta = \rho \ddot{u}_\theta \qquad (2\text{-}15) \\
&\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = \rho \ddot{u}_z
\end{aligned}$$

Considering our *Basic* assumptions, and due to the symmetry of geometry and loading, we are left with only one equation:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} + \rho r \omega^2 = 0 \qquad (2\text{-}16)$$

which can be rearranged into the following form:

$$\frac{d(r\sigma_{rr})}{dr} - \sigma_{\theta\theta} + \rho r^2 \omega^2 = 0 \qquad (2\text{-}17)$$

Now we may set:

$$\begin{aligned}
&r\sigma_{rr} = F(r) \\
&\sigma_{\theta\theta} = \frac{dF(r)}{dr} + \rho r^2 \omega^2
\end{aligned} \qquad (2\text{-}18)$$

in which *F(r)* is a *stress function*.

The linear strain tensor in cylindrical coordinates is:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{rr} = \dfrac{\partial u_r}{\partial r} & 2\varepsilon_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} & 2\varepsilon_{rz} = \dfrac{\partial u_r}{\partial z} + \dfrac{\partial u_z}{\partial r} \\ & \varepsilon_{\theta\theta} = \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & 2\varepsilon_{\theta z} = \dfrac{\partial u_\theta}{\partial z} + \dfrac{1}{r}\dfrac{\partial u_z}{\partial \theta} \\ & & \varepsilon_{zz} = \dfrac{\partial u_z}{\partial z} \end{pmatrix} \tag{2-19}$$

which in this case reduces to:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{rr} = \dfrac{\partial u_r}{\partial r} & \varepsilon_{r\theta} = 0 & \varepsilon_{rz} = 0 \\ & \varepsilon_{\theta\theta} = \dfrac{u_r}{r} & \varepsilon_{\theta z} = 0 \\ & & \varepsilon_{zz} = 0 \end{pmatrix} \tag{2-20}$$

We may also write the constitutive equations for our problem as:

$$\begin{aligned} \varepsilon_{ij} &= \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}\right] \\ \varepsilon_{rr} &= \frac{1}{E}\left[(1+\nu)\sigma_{rr} - \nu(\sigma_{rr} + \sigma_{\theta\theta})\right] = \frac{1}{E}\left[\sigma_{rr} - \nu\sigma_{\theta\theta}\right] \\ \varepsilon_{\theta\theta} &= \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{rr}\right] \end{aligned} \tag{2-21}$$

At this point, we will eliminate the displacements between our kinematic equations to obtain a compatibility equation.

$$\frac{d\varepsilon_{\theta\theta}}{dr} = \frac{d}{dr}\left(\frac{u_r}{r}\right) = \frac{du_r}{rdr} - \frac{u_r}{r^2} = \frac{1}{r}\left[\frac{du_r}{dr} - \frac{u_r}{r}\right] = \frac{1}{r}\left[\varepsilon_{rr} - \varepsilon_{\theta\theta}\right] \tag{2-22}$$

Now we substitute the constitutive equation in the above compatibility equation:

$$\begin{aligned} \frac{d}{dr}\left[\frac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr})\right] &= \frac{1}{r}\left[\frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta} - \sigma_{\theta\theta} + \nu\sigma_{rr}\right)\right] \\ \frac{d}{dr}\sigma_{\theta\theta} - \nu\frac{d}{dr}\sigma_{rr} &= \frac{1}{r}\left[(1-\nu)\left(\sigma_{rr} - \sigma_{\theta\theta}\right)\right] \end{aligned} \tag{2-23}$$

Rewriting the above equation in terms of our stress function *F(r)*, we will have:

$$r\sigma_{rr} = F(r), \qquad \sigma_{\theta\theta} = \frac{dF(r)}{dr} + \rho r^2\omega^2, \qquad F(r) \equiv F$$

$$\frac{d^2F}{dr^2} + 2\rho\omega^2 r - \frac{\nu}{r}\frac{dF}{dr^2} + \frac{\nu F}{r^2} = \frac{(1+\nu)}{r}\left(\frac{F}{r} - \frac{dF}{dr} - \rho\omega^2 r^2\right) \tag{2-24}$$

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - F = -(3+\nu)\rho\omega^2 r^3$$

This is a nonhomogeneous, second order linear, ordinary differential equation with variable coefficients and has the following general solution:

$$F(r) = A_1 r + A_2\frac{1}{r} - \frac{(3+\nu)\rho\omega^2 r^3}{8} \tag{2-25}$$

Accordingly, the resulting stresses will be:

$$\sigma_{rr} = A_1 + \frac{A_2}{r^2} - \frac{(3+\nu)\rho\omega^2 r^2}{8}$$

$$\sigma_{\theta\theta} = A_1 - \frac{A_2}{r^2} - \frac{(1+3\nu)\rho\omega^2 r^2}{8} \tag{2-26}$$

The coefficients A_1 and A_2 can be found by imposing the appropriate boundary conditions. We consider the solution for the Solid Disc, and a Disc with stress-free circular hole.

For the solid disc we must have A_2=0 to account for the finite stresses at the center. The second B.C. is:

$$T_i = \sigma_{ij}n_j$$

$$T_i(0,0,0), \quad n_i(1,0,0) \text{ at } r = a$$

$$T_r = \sigma_{rr}n_r + \sigma_{r\theta}n_\theta + \sigma_{rz}n_z \Rightarrow 0 = \sigma_{rr}n_r + 0 + 0 \Rightarrow \sigma_{rr} = 0 \tag{2-27}$$

$$\Rightarrow A_1 = \frac{(3+\nu)\rho\omega^2 a^2}{8}$$

Accordingly, the final form of the stresses will be:

$$\sigma_{rr} = \frac{(3+\nu)\rho\omega^2}{8}\left(a^2 - r^2\right)$$
$$\sigma_{\theta\theta} = \frac{(3+\nu)\rho\omega^2}{8}a^2 - \frac{(1+3\nu)\rho\omega^2}{8}r^2 \tag{2-28}$$

The displacement field can be obtained as follows:

$$u_r = r\varepsilon_{\theta\theta} = \frac{r}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right)$$
$$= \frac{r\rho\omega^2}{8E}\left[\left(3 - 2\nu - \nu^2\right)a^2 + \left(\nu^2 - 1\right)r^2\right] \tag{2-29}$$

For a disc with stress-free circular hole, we have the following conditions at the outside radius:

$$T_i = \sigma_{ij}n_j$$
$$T_i(0,0,0), \quad n_i(1,0,0) \text{ at } r = a$$
$$T_r = \sigma_{rr}n_r + \sigma_{r\theta}n_\theta + \sigma_{rz}n_z \Rightarrow 0 = \sigma_{rr}n_r + 0 + 0 \Rightarrow \sigma_{rr} = 0 \tag{2-30}$$
$$\Rightarrow A_1 = \frac{A_2}{a^2} + \frac{\left(3+\nu\right)\rho\omega^2 a^2}{8}$$

At the inside radius we have:

$$T_i = \sigma_{ij}n_j$$
$$T_i(0,0,0), \quad n_i(-1,0,0) \text{ at } r = b$$
$$T_r = -\sigma_{rr}n_r + \sigma_{r\theta}n_\theta + \sigma_{rz}n_z \Rightarrow 0 = -\sigma_{rr}n_r + 0 + 0 \Rightarrow \sigma_{rr} = 0 \tag{2-31}$$
$$\Rightarrow A_1 = -\frac{A_2}{b^2} + \frac{\left(3+\nu\right)\rho\omega^2 b^2}{8}$$

Solving (2-30) and (2-31) for A_1 and A_2 , we will have:

$$A_1 = \frac{\left(3+\nu\right)\rho\omega^2}{8}\left(a^2 + b^2\right)$$
$$A_2 = -\frac{\left(3+\nu\right)\rho\omega^2}{8}a^2 b^2 \tag{2-32}$$

Hence the stress equations become:

$$\sigma_{rr} = \frac{(3+\nu)\rho\omega^2}{8}\left(a^2 + b^2 - \frac{a^2b^2}{r^2} - r^2\right)$$
$$\sigma_{\theta\theta} = \frac{(3+\nu)\rho\omega^2}{8}\left(a^2 + b^2 + \frac{a^2b^2}{r^2} - \frac{(1+3\nu)r^2}{(3+\nu)}\right) \tag{2-33}$$

Furthermore, we may write the maximum radial and hoop stresses as:

$$\sigma_{rr}\big|_{\max} = \frac{(3+\nu)\rho\omega^2}{8}\left(a-b\right)^2$$
$$\sigma_{\theta\theta}\big|_{\max} = \frac{(3+\nu)\rho\omega^2}{8}\left(2a^2 + \frac{2(1-\nu)b^2}{(3+\nu)}\right) \tag{2-34}$$

Also notice that for a small hole as $b \to 0$, we will have:

$$\sigma_{\theta\theta}\big|_{b\to 0}^{hole} = \frac{(3+\nu)\rho\omega^2}{8}\left(2a^2\right)$$
$$\sigma_{\theta\theta}\big|_{r\to 0}^{solid} = \frac{(3+\nu)\rho\omega^2}{8}a^2 \tag{2-35}$$
$$\Rightarrow \sigma_{\theta\theta}\big|_{b\to 0}^{hole} = 2\,\sigma_{\theta\theta}\big|_{r\to 0}^{solid}$$

The conclusion is that a *stress concentration factor* of two exists for the circular hole in our rotating disc.

Alternative solutions

There are alternative solutions to the rotating disc problem, which are in fact quite simpler than the solution presented above. For instance, we may extract our previous differential equation in terms of the stress function *F(r) directly* from the Beltrami-Michell equations in cylindrical coordinates (see Chapter 1).

We may also start with the Navier displacement equations in cylindrical coordinates:

$$(\lambda + 2G)\frac{\partial I_e}{\partial r} - 2G\left(\frac{1}{r}\frac{\partial \omega_z}{\partial \theta} - \frac{\partial \omega_\theta}{\partial z}\right) + f_r = 0$$
$$(\lambda + 2G)\frac{\partial I_e}{r\partial \theta} - 2G\left(\frac{\partial \omega_r}{\partial z} - \frac{\partial \omega_z}{\partial r}\right) + f_\theta = 0 \tag{2-36}$$
$$(\lambda + 2G)\frac{\partial I_e}{\partial z} - 2G\left(\frac{\partial (r\omega_\theta)}{\partial r} - \frac{\partial \omega_r}{\partial \theta}\right) + f_z = 0$$

In the above equations we have:

$$I_e = e_{ii} = \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$
$$2\omega_r = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} \quad ; \quad 2\omega_\theta = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} \quad ; \quad 2\omega_z = \frac{1}{r}\left(\frac{\partial(ru_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right) \tag{2-37}$$

Considering our basic assumptions and the symmetry of geometry and loading, the above equations will simplify to the following single equation:

$$r^2\frac{d^2u_r}{dr^2} + r\frac{du_r}{dr} - u = -\frac{1-\nu^2}{E}\rho\omega^2 r^3 \tag{2-38}$$

The general solution can be shown to be:

$$u_r = \frac{1}{E}\left[(1-\nu)A_1 r - (1+\nu)\frac{A_2}{r} - \frac{1-\nu^2}{8}r^2\rho\omega^2 r^3\right] \tag{2-39}$$

which upon substitution into the kinematic and constitutive equations will give the required expressions for the radial and hoop stresses.

Assignment 4:

Find the stresses in a rotating disc of variable thickness *h=f(r)*.
Hint: use the following form of the equilibrium equation:

$$\frac{d}{dr}\left(hr\sigma_{rr}\right) - h\sigma_{\theta\theta} + \rho\omega^2 hr^2 = 0$$

Saint-Venant Torsion

This section presents a general solution to the torsion of prismatic bars, known as Saint-Venant torsion problem. This solution represents a classic example of the so called *semi-inverse* approach, in which a displacement field is initially assumed and then it is shown that it satisfies the equilibrium equations and the boundary conditions.

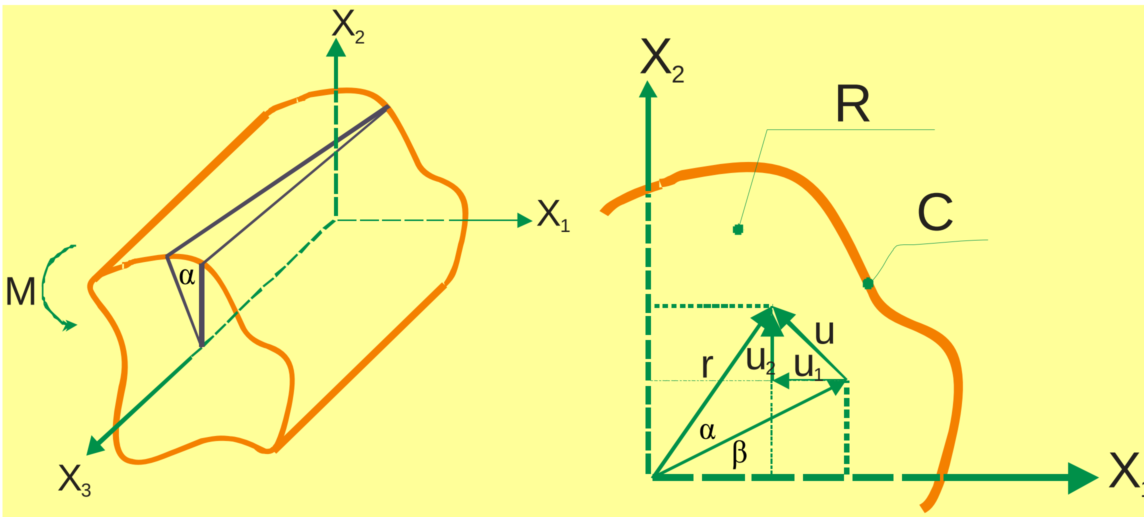

Fig. 2.3

Based on a geometric study of the schematic shown in Fig. 2.3, we may define the two in-plane components of displacement vector for an arbitrary point in the cross section of the bar by:

$$\begin{aligned} & u_1 = r\cos(\alpha+\beta) - r\cos\beta \\ & u_2 = r\sin(\alpha+\beta) - r\sin\beta \\ & \Rightarrow \begin{cases} u_1 = r\cos\alpha\cos\beta - r\sin\alpha\sin\beta - r\cos\beta \\ u_2 = r\sin\alpha\cos\beta + r\cos\alpha\sin\beta - r\sin\beta \end{cases} \end{aligned} \tag{2-40}$$

For very small values of *α*, we may assume that $\cos\alpha \approx 1$, $\sin\alpha \approx \alpha$, and rewrite the displacements as:

$$\begin{aligned} & u_1 = r\cos\beta - r\alpha\sin\beta - r\cos\beta \\ & u_2 = r\alpha\cos\beta + r\sin\beta - r\sin\beta \\ & \Rightarrow \begin{cases} u_1 = -r\alpha\sin\beta \\ u_2 = r\alpha\cos\beta \end{cases} \end{aligned} \tag{2-41}$$

which using the expressions $r\cos\beta = x_1$, $r\sin\beta = x_2$, results in,

$$\Rightarrow u_\alpha \sim (-x_2\alpha \ , \ x_1\alpha) \tag{2-42}$$

With the definition of Θ as the *torsion-angle per unit length*, we have:

$$\alpha = \Theta x_3 \tag{2-43}$$

Hence, the in-plane displacements can be written as follows:

$$u_\alpha \sim (-\Theta x_2 x_3 \ , \Theta x_1 x_3) \tag{2-44}$$

Next, we assume that the third component of displacement can be defined as:

$$u_3 = \Theta\varphi(x_1 \ , x_2) \tag{2-45}$$

where φ is called the *Warping Function*. Finally, the assumed displacement field can be summarized as:

$$u_i \sim (-\Theta x_2 x_3 \ , \Theta x_1 x_3 \ , \Theta\varphi) \tag{2-46}$$

Accordingly, the strain field becomes:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\varepsilon_{ij} \sim \begin{pmatrix} 0 & 0 & \frac{1}{2}\left(-\Theta x_2 + \Theta\varphi_{,1}\right) \\ \cdot & 0 & \frac{1}{2}\left(\Theta x_1 + \Theta\varphi_{,2}\right) \\ \cdot & \cdot & 0 \end{pmatrix} \tag{2-47}$$

and the stresses become:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{kk}$$

$$\sigma_{ij} \sim \begin{pmatrix} 0 & 0 & G\Theta\left(\varphi_{,1} - x_2\right) \\ \cdot & 0 & G\Theta\left(\varphi_{,2} + x_1\right) \\ \cdot & \cdot & 0 \end{pmatrix} \tag{2-48}$$

Now we turn our attention to the equilibrium equations:

$$\begin{aligned} &\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \\ &\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \\ &\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = G\Theta\left(\varphi_{,11} + \varphi_{,22}\right) \end{aligned} \tag{2-49}$$

which can be summarized as:

$$\nabla'^2 \varphi = 0 \quad \text{in R} \tag{2-50}$$

Now we will examine the boundary conditions.

$$\begin{aligned} &T_\alpha = \sigma_{\alpha i} n_i \\ &\qquad T_\alpha(0,0), \quad n_i(n_1, n_2, 0) \\ &\; 0 = \sigma_{11} n_1 + \sigma_{12} n_2 + 0 \\ &\; 0 = \sigma_{21} n_1 + \sigma_{22} n_2 + 0 \\ &T_3 = \sigma_{3i} n_i = 0 \\ &\; 0 = \sigma_{31} n_1 + \sigma_{32} n_2 + 0 \\ &\quad = G\Theta\left[\left(\varphi_{,1} - x_2\right) n_1 + \left(\varphi_{,2} + x_1\right) n_2\right] = 0 \\ &\;\Rightarrow \varphi_{,1} n_1 + \varphi_{,2} n_2 = x_2 n_1 - x_1 n_2 \end{aligned} \tag{2-51}$$

We may simplify the above boundary conditions based on the geometric considerations depicted in Fig. 2.4 as follows:

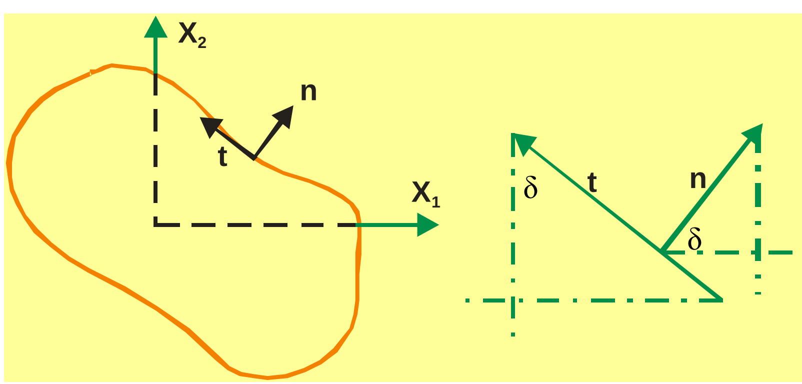

Fig. 2.4

$$n_1 = \frac{dx_1}{dn} = \cos\delta \;\;,\;\; n_2 = \frac{dx_2}{dn} = \sin\delta$$
$$n_1 = \frac{dx_2}{dt} = \cos\delta \;\;,\;\; n_2 = -\frac{dx_1}{dt} = \sin\delta \qquad \text{(2-52)}$$

Accordingly, the stress boundary condition defined in (2-51) may be rewritten as:

$$\frac{\partial\varphi}{\partial x_1}\frac{dx_1}{dn} + \frac{\partial\varphi}{\partial x_2}\frac{dx_2}{dn} = x_2\frac{dx_2}{dt} + x_1\frac{dx_1}{dt}$$
$$\Rightarrow \frac{d\varphi}{dn} = \frac{1}{2}\frac{dr^2}{dt} \quad \text{on C.} \qquad \text{(2-53)}$$

Note that the above is not yet in a suitable form because we have to deal with two variables, *n* and *t*. In order to solve this problem, we introduce the conjugate harmonic function ψ:

$$z = \varphi + i\psi \qquad \text{(2-54)}$$

Using the following Cauchy-Riemann conditions:

$$\varphi_{,1} = \psi_{,2}$$
$$\varphi_{,2} = -\psi_{,1} \qquad \text{(2-55)}$$

we will have:

$$\nabla'^2\psi = 0 \;\; \text{in R} \qquad \text{(2-56)}$$

and the boundary conditions will be:

$$\frac{\partial\psi}{\partial x_2}\frac{dx_2}{dt} + \left(-\frac{\partial\psi}{\partial x_1}\right)\left(-\frac{dx_1}{dt}\right) = \frac{1}{2}\frac{dr^2}{dt}$$
$$\Rightarrow \frac{d\psi}{dt} = \frac{1}{2}\frac{dr^2}{dt} \quad \text{on C.} \qquad \text{(2-57)}$$

Thus, for a simply connected region we have:

$$\psi = \frac{1}{2}r^2 \quad \text{on C.} \qquad \text{(2-58)}$$

and the stresses become:

$$\begin{aligned} \sigma_{13} &= G\Theta\left(\psi_{,2} - x_2\right) \\ \sigma_{23} &= G\Theta\left(-\psi_{,1} + x_1\right) \end{aligned} \tag{2-59}$$

We may further simplify our boundary value problem by another variable change, as follows:

$$\begin{aligned} &\Psi = \psi - \frac{1}{2}r^2 \\ &\Rightarrow \Psi = 0 \quad \text{on C, and} \quad \nabla'^2\Psi = -2 \quad \text{in R.} \end{aligned} \tag{2-60}$$

In fact we may solve the torsion problem for different geometries through finding the appropriate Ψ function that satisfies the conditions presented in (2-60).

Accordingly, the stresses become:

$$\sigma_{ij} \sim \begin{pmatrix} 0 & 0 & G\Theta\Psi_{,2} \\ \cdot & 0 & -G\Theta\Psi_{,1} \\ \cdot & \cdot & 0 \end{pmatrix} \tag{2-61}$$

Circular Cross Section

For a circular cross section, we choose:

$$\Psi = k(x_1^2 + x_2^2 - a^2) \tag{2-62}$$

which satisfies Eqs. (2-60). Next, we have:

$$\begin{aligned} &\Psi = 0 \quad \text{at} \ \ r = a \\ &\nabla'^2\Psi = 4k \\ &\Rightarrow k = -\frac{1}{2} \Rightarrow \Psi = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{2}a^2 \end{aligned} \tag{2-63}$$

so,

$$\begin{aligned} &\psi = \Psi + \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}a^2 = \text{constant} \\ &\Rightarrow \varphi = \text{constant} = C \end{aligned} \tag{2-64}$$

*which means that the circular cross section does not warp in torsion***.**

Accordingly, the displacements and stresses are:

$$u_i \sim (-\Theta x_2 x_3 \ , \Theta x_1 x_3 \ , \Theta C)$$
$$\sigma_{ij} \sim \begin{pmatrix} 0 & 0 & G\Theta x_2 \\ \cdot & 0 & -G\Theta x_1 \\ \cdot & \cdot & 0 \end{pmatrix} \tag{2-65}$$

Now we may calculate the total torque required to produce a certain amount of twist.

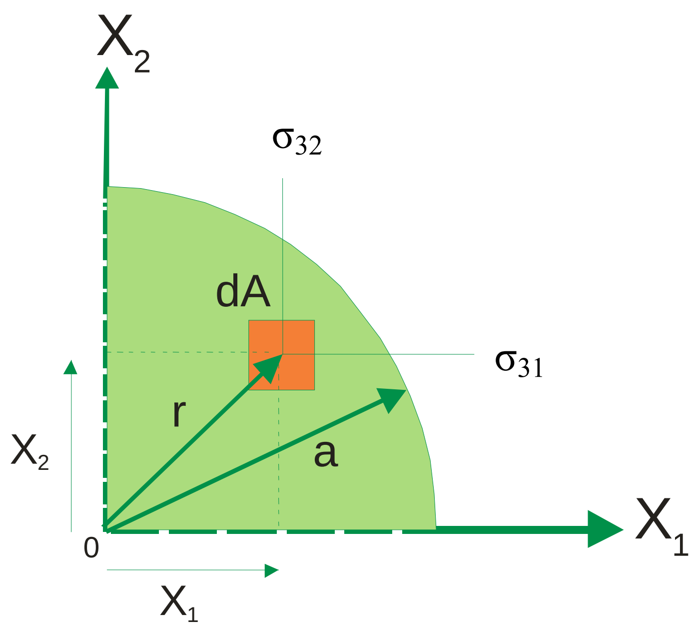

Fig. 2.5

Bases on geometric considerations depicted in Fig. 2.5, we can write,

$$\begin{aligned} M &= \int_A \left(x_1\sigma_{32} - x_2\sigma_{31}\right) dA \\ &= \int_A \left[G\Theta\left(x_1^2 + x_2^2\right)\right] dA \\ &= G\Theta \int_0^{2\pi}\int_0^a r^2 r d\theta dr \\ &= G\Theta \frac{r^4}{4}\bigg|_0^a \theta\Big|_0^{2\pi} \\ &= G\Theta \frac{a^4}{4} 2\pi \\ &= \frac{G\pi\Theta a^4}{2} \end{aligned} \tag{2-66}$$

Elliptical Cross Section

For an elliptical cross section, we choose:

$$\Psi = k\left(\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1\right) \tag{2-67}$$

which satisfies Eqs. (2-60). Next, we have,

$$\begin{aligned}
&\Psi = 0 \quad \text{on } C \\
&\nabla'^2\Psi = -2 = 2k\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \Rightarrow k = -\frac{a^2b^2}{\left(a^2+b^2\right)} \\
&\Rightarrow \Psi = \frac{a^2b^2}{\left(a^2+b^2\right)}\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right)
\end{aligned} \tag{2-68}$$

Thus the stresses become:

$$\sigma_{ij} \sim \begin{pmatrix} 0 & 0 & -\dfrac{2G\Theta a^2}{b^2+a^2}x_2 \\ \cdot & 0 & \dfrac{2G\Theta b^2}{b^2+a^2}x_1 \\ \cdot & \cdot & 0 \end{pmatrix} \tag{2-69}$$

We also have,

$$\begin{aligned}
\psi &= \Psi + \frac{1}{2}(x_1^2 + x_2^2) \\
&= \frac{a^2b^2}{\left(a^2+b^2\right)}\left(1 - \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}\right) + \frac{1}{2}(x_1^2 + x_2^2) \\
&= \frac{a^2b^2}{\left(a^2+b^2\right)} + \left[\frac{a^2-b^2}{2\left(a^2+b^2\right)}\right]x_1^2 + \left[\frac{b^2-a^2}{2\left(a^2+b^2\right)}\right]x_2^2
\end{aligned} \tag{2-70}$$

Accordingly, we may calculate the warping function using the Cauchy-Riemann conditions:

$$\begin{aligned}
&\varphi_{,1} = \psi_{,2} = \frac{b^2 - a^2}{a^2 + b^2} x_2 \\
&\Rightarrow \varphi = \frac{b^2 - a^2}{a^2 + b^2} x_1 x_2 + f(x_2) \\
&\varphi_{,2} = -\psi_{,1} \\
&\Rightarrow \frac{b^2 - a^2}{a^2 + b^2} x_1 + f_{,2} = \frac{b^2 - a^2}{a^2 + b^2} x_1 \\
&\Rightarrow f_{,2} = 0 \Rightarrow f = C \\
&\Rightarrow \varphi = \frac{b^2 - a^2}{a^2 + b^2} x_1 x_2 + C
\end{aligned} \tag{2-71}$$

Accordingly, the out-of-plane displacement is:

$$\Rightarrow u_3 = \Theta \varphi = \frac{b^2 - a^2}{a^2 + b^2} x_1 x_2 \Theta \tag{2-72}$$

Fig. 2.6 shows how an elliptical cross section *warps* due to the existence of the out-of-plane displacements.

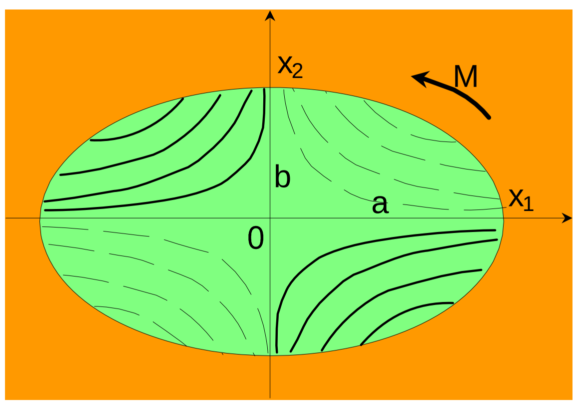

Fig. 2.6

Assignment 5:

Solve the torsion problem for a quadrilateral cross section.

Cantilever Beam under End Loading

We start this section by verification of the basic equations governing the Cantilever Beam from an elementary point of view, i.e., the Bernoulli-Euler beam theory. Later we will present the theory of elasticity point of view. The Bernoulli-Euler beam theory is built upon the following assumptions:

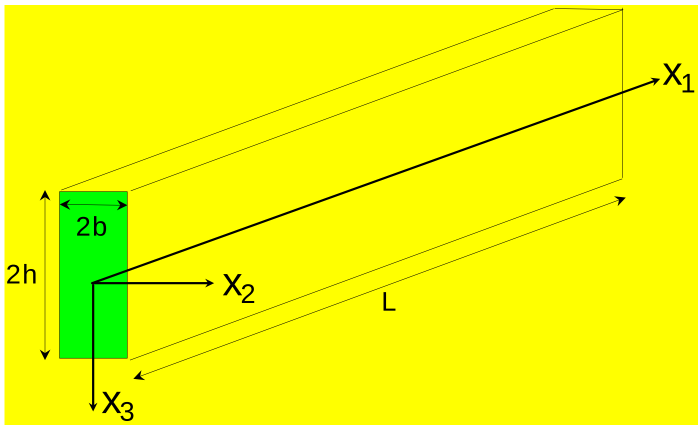

Fig. 2.7

$$
\begin{aligned}
& h << l, \quad b << l \\
& \Rightarrow \sigma_{33} << \sigma_{11} \Rightarrow \sigma_{33} \approx 0
\end{aligned}
\tag{2-73}
$$

It is further assumed that Linear Elements which are initially normal to the undeformed middle surface remain straight and normal to the deformed middle surface and suffer no extension, so we may write:

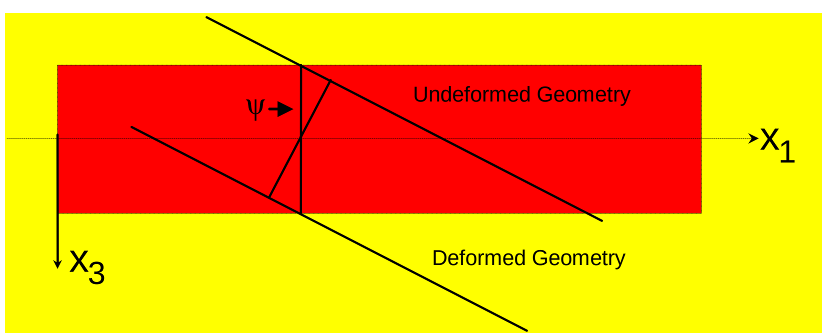

Fig. 2.8

$$\varepsilon_{ij} \sim \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \cdot & \varepsilon_{22} & \varepsilon_{23} \\ \cdot & \cdot & 0 \end{pmatrix} \qquad (2\text{-}74)$$

On the other hand, the stresses are assumed to be:

$$\sigma_{ij} \sim \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} \\ \cdot & 0 & 0 \\ \cdot & \cdot & \sigma_{33} \approx 0 \end{pmatrix} \qquad (2\text{-}75)$$

The resultant moment is:

$$M = \int_A \sigma_{11} x_3 dA \qquad (2\text{-}76)$$

and the resultant shear stress will be:

$$V = \int_A \sigma_{13} dA \qquad (2\text{-}77)$$

In order to verify the validity of the basic assumptions and the proposed stress and strain fields, we start from the assumed *Displacement* field:

$$\begin{aligned} u_1 &= -x_3 \psi(x_1) \\ u_2 &= u_2(x_i) \\ \omega = u_3 &= u_3(x_i) \end{aligned} \qquad (2\text{-}78)$$

in which ψ is the angle of rotation before and after deformation. The strains can be obtained using the kinematic equations as follows:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

$$\varepsilon_{ij} \sim \begin{pmatrix} u_{1,1} = -x_3 \psi_{,1} & \frac{1}{2} u_{2,1} & \frac{1}{2}\left(-\psi + \omega_{,1}\right) \\ \cdot & u_{2,2} & \frac{1}{2}\left(u_{2,3} + \omega_{,2}\right) \\ \cdot & \cdot & \omega_{,3} \end{pmatrix} \quad \varepsilon_{ij} \sim \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \cdot & \varepsilon_{22} & \varepsilon_{23} \\ \cdot & \cdot & 0 \end{pmatrix} \qquad (2\text{-}79)$$

which, when compared to the assumed strain field, gives the following results:

$$\varepsilon_{13}=0\Rightarrow\psi=\omega_{,1}$$
$$\varepsilon_{33}=0\Rightarrow\omega=\omega\left(x_{\alpha}\right) \tag{2-80}$$

Now we obtain the strain field from the initially assumed stresses using the following constitutive equations:

$$\varepsilon_{ij}=\frac{1}{E}\left[(1+\nu)\sigma_{ij}-\nu\delta_{ij}\sigma_{kk}\right]$$
$$\sigma_{ij}\sim\begin{pmatrix}\sigma_{11} & 0 & \sigma_{13}\\ \cdot & 0 & 0\\ \cdot & \cdot & \sigma_{33}\approx 0\end{pmatrix}\Rightarrow\varepsilon_{ij}\sim\begin{pmatrix}\dfrac{\sigma_{11}}{E} & 0 & \dfrac{(1+\nu)\sigma_{13}}{E}\\ \cdot & -\dfrac{\nu}{E}\sigma_{11} & 0\\ \cdot & \cdot & -\dfrac{\nu}{E}\sigma_{11}\end{pmatrix} \tag{2-81}$$

If we compare the resulting strains with the initially assumed strains and those obtained directly from the displacements, we will notice how the *troublesome* terms in the strain field have been *ignored* to match the stress and strain fields and obtain a workable theory.

$$\varepsilon_{ij}\sim\begin{pmatrix}\left\{\dfrac{\sigma_{11}}{E}=-x_3\psi_{,1}=-x_3\omega_{,11}\right\} & \left\{\begin{matrix}u_{2,1}=0\Rightarrow\\ u_2=u_2\left(x_2,x_3\right)\end{matrix}\right\} & \left\{\begin{matrix}\left[\dfrac{1}{2}\left(-\psi+\omega_{,1}\right)=\dfrac{(1+\nu)\sigma_{13}}{E}\right]\\ \text{is ignored}\end{matrix}\right\}\\ \cdot & \left\{-\dfrac{\nu}{E}\sigma_{11}=u_{2,2}\right\} & \left\{\begin{matrix}\dfrac{1}{2}\left(u_{2,3}+\omega_{,2}\right)=0\\ \Rightarrow u_{2,3}=-\omega_{,2}\end{matrix}\right\}\\ \cdot & \cdot & \left\{\begin{matrix}-\dfrac{\nu}{E}\sigma_{11}=0\\ (\dfrac{\nu}{E}\sigma_{11}\text{ is ignored})\end{matrix}\right\}\end{pmatrix} \tag{2-82}$$

Next, we proceed with the derivation of other equations, and write:

$$u_{2,2}=-\frac{\nu}{E}\sigma_{11},\ \sigma_{11}=-Ex_3\omega_{,11}$$
$$\Rightarrow u_2=\nu\omega_{,11}x_2x_3 \tag{2-83}$$

We also have:

$$\varepsilon_{23} = \frac{1}{2}\left(u_{2,3} + \omega_{,2}\right) = 0$$
$$\Rightarrow \omega_{,2} = -u_{2,3} = -\nu\omega_{,11}x_2 \tag{2-84}$$

and,

$$M = \int_A \sigma_{11}x_3 dA = -E\int_A \omega_{,11}x_3^2 dA = -E\omega_{,11}I$$
$$\omega_{,11} = -\frac{M}{EI} \tag{2-85}$$

Hence, we may rewrite the displacements as:

$$u_2 = -\frac{\nu M}{EI}x_2x_3 \quad ; \quad \omega_{,2} = \frac{\nu M}{EI}x_2 \tag{2-86}$$

Now we check the Equilibrium Equations:

$$\sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \quad \Rightarrow \sigma_{11,1} + \sigma_{13,3} = 0$$
$$\sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \quad \text{identically satisfied} \tag{2-87}$$
$$\sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 \quad \Rightarrow \sigma_{31,1} + \sigma_{33,3} = 0$$

Multiplying the first equilibrium equation by x_3 and integrating across the cross section, we will have:

$$\int_A x_3\sigma_{11,1}dA + \int_{-b}^{b}\left[\int_{-h}^{h} x_3\sigma_{13,3}dx_3\right]dx_2 = 0$$
$$\Rightarrow \frac{\partial}{\partial x_1}\left(\int_A x_3\sigma_{11}dA\right) + \int_{-b}^{b}\left[x_3\sigma_{13}\Big|_{-h}^{h} - \int_{-h}^{h}\sigma_{13}dx_3\right]dx_2 = 0 \tag{2-88}$$
$$\Rightarrow \frac{\partial M}{\partial x_1} + \int_{-b}^{b}\left(x_3\sigma_{13}\Big|_{-h}^{h}\right)dx_2 - V = 0$$

For problems with no surface shear force:

$$\frac{\partial M}{\partial x_1} - V = 0 \tag{2-89}$$

Integrating the third equilibrium equation across the cross section, we will have:

$$\int_A \frac{\partial \sigma_{13}}{\partial x_1} dA + \int_{-b}^{b} \left[\int_{-h}^{h} \frac{\partial \sigma_{33}}{\partial x_3} dx_3 \right] dx_2 = 0$$
$$\Rightarrow \frac{\partial}{\partial x_1}\left(\int_A \sigma_{13} dA \right) + \int_{-b}^{b} \left[\sigma_{33}\big|_h - \sigma_{33}\big|_{-h} \right] dx_2 = 0 \tag{2-90}$$

Now we define the resultant normal surface load as:

$$q = \int_{-b}^{b} \left[\sigma_{33}\big|_h - \sigma_{33}\big|_{-h} \right] dx_2 \tag{2-91}$$

and the Eq. (2-90) becomes:

$$\frac{\partial V}{\partial x_1} + q = 0 \tag{2-92}$$

We may also derive other important equations for the B.E. beam as follows:

$$\begin{aligned}
&\frac{\partial M}{\partial x_1} - V = 0 \\
&\Rightarrow \frac{\partial^2 M}{\partial x_1^2} = \frac{\partial V}{\partial x_1} = -q \quad \Rightarrow \frac{\partial^2 M}{\partial x_1^2} + q = 0 \\
&\omega_{,11} = -\frac{M}{EI} \\
&\Rightarrow \frac{\partial^4 \omega}{\partial x_1^4} = -\frac{1}{EI}\frac{\partial^2 M}{\partial x_1^2} = \frac{q}{EI} \quad \Rightarrow \frac{\partial^4 \omega}{\partial x_1^4} = \frac{q}{EI}
\end{aligned} \tag{2-93}$$

Finally, the following equations are considered to form the basis of the B.E. Beam theory.

$$\begin{aligned}
&\omega_{,11} = -\frac{M}{EI}, \\
&\frac{\partial M}{\partial x_1} - V = 0, \qquad \frac{\partial V}{\partial x_1} + q = 0, \\
&\frac{\partial^2 M}{\partial x_1^2} + q = 0, \qquad \frac{\partial^4 \omega}{\partial x_1^4} = \frac{q}{EI},
\end{aligned} \tag{2-94}$$

Solution Based on the Theory of Elasticity

We initially assume that the normal stresses are the same as those of the elementary theory and also σ_{32} is taken as zero, but no stipulation is made regarding the other shear stresses σ_{13} and σ_{12}. Accordingly, the three equilibrium equations become:

$$\begin{aligned}
&\sigma_{11,1}+\sigma_{12,2}+\sigma_{13,3}=0 \quad \Rightarrow \quad \sigma_{12,2}+\sigma_{13,3}=-\frac{Px_3}{I} \\
&\sigma_{21,1}+\sigma_{22,2}+\sigma_{23,3}=0 \quad \Rightarrow \sigma_{21,1}=0 \qquad \text{(i)} \\
&\sigma_{31,1}+\sigma_{32,2}+\sigma_{33,3}=0 \quad \Rightarrow \sigma_{31,1}=0 \qquad \text{(ii)}
\end{aligned} \tag{2-95}$$

From equations (2-95, i) and (2-95, ii), we may conclude that the shearing stresses do not depend on x_1, so they are the same for all cross sections. For this case the two relevant stress compatibility equations become:

$$\begin{aligned}
&\nabla^2\sigma_{ij}+\frac{1}{1+\nu}\sigma_{mm,ij}=-\left(f_{i,j}+f_{j,i}+\frac{\nu}{1-\nu}\delta_{ij}f_{m,m}\right) \\
&\nabla^2\sigma_{31}=-\frac{1}{1+\nu}\frac{P}{I} \qquad (i) \\
&\nabla^2\sigma_{21}=0 \qquad (ii)
\end{aligned} \tag{2-96}$$

Now we must ensure that the modified solution satisfies the boundary conditions.

$$\begin{aligned}
&\sigma_{31}(\pm h,x_2,x_1)=0 \\
&\sigma_{23}(x_3,\pm b,x_1)=0
\end{aligned} \tag{2-97}$$

At this stage we define our stress components in terms of a stress function:

$$\begin{aligned}
&\sigma_{31}=\phi_{,2}-\frac{Px_3^2}{2I}+f(x_2) \\
&\sigma_{21}=-\phi_{,3}
\end{aligned} \tag{2-98}$$

which satisfies the equilibrium equations. Then our first stress compatibility equation (Eq. (2-96, i)) becomes:

$$\begin{aligned}
&(\phi_{,233}+\phi_{,222})-\frac{P}{I}+f_{,22}=-\frac{1}{1+\nu}\frac{P}{I} \\
&\Rightarrow\left(\phi_{,33}+\phi_{,22}\right)_{,2}=\frac{\nu}{1+\nu}\frac{P}{I}-f_{,22} \\
&\Rightarrow\phi_{,33}+\phi_{,22}=\frac{\nu}{1+\nu}\frac{P}{I}x_2-f_{,2}+C
\end{aligned} \tag{2-99}$$

And the second compatibility equation (Eq. (2-96, ii)) is:

$$
\begin{aligned}
&-\phi_{,333}-\phi_{,322}=0 \\
&\Rightarrow\left(\phi_{,33}+\phi_{,22}\right)_{,3}=0
\end{aligned} \tag{2-100}
$$

Note that equation (2-99) also satisfies equation (2-100). In order to find "C", we express the rotation about the x_1 axis by:

$$
\omega_{32}=\frac{1}{2}\left(u_{3,2}-u_{2,3}\right) \tag{2-101}
$$

and its rate of change by:

$$
\begin{aligned}
\omega_{32,1}&=\frac{1}{2}\left(u_{3,2}-u_{2,3}\right)_{,1} \\
&=\frac{1}{2}\left[\left(u_{3,1}+u_{1,3}\right)_{,2}-\left(u_{2,1}+u_{1,2}\right)_{,3}\right] \\
&=\varepsilon_{31,2}-\varepsilon_{21,3}
\end{aligned} \tag{2-102}
$$

which can be written in terms of stresses using the constitutive equations.

$$
\begin{aligned}
\omega_{32,1}&=\frac{1}{2G}\left(\sigma_{31,2}-\sigma_{21,3}\right) \\
&=\frac{1}{2G}\left(\phi_{,22}+\phi_{,33}+f_{,2}\right)
\end{aligned} \tag{2-103}
$$

Also we can combine (2-103) and (2-99) to write:

$$
2G\omega_{32,1}=\frac{1}{1+\nu}\frac{P}{I}x_2+C \tag{2-104}
$$

As we are considering a rectangular cross section and the bending is symmetrical about the x_3 axis, at $x_2=0$ we should have the average rotation equal to zero, resulting in C=0.

Hence, equation (2-99) becomes:

$$
\phi_{,33}+\phi_{,22}=\frac{\nu}{1+\nu}\frac{P}{I}x_2-f_{,2} \tag{2-105}
$$

which is the *governing equation* for the stress function, provided that the resulting stresses satisfy the boundary conditions.

However, we may write the boundary conditions directly in terms of the stress function by choosing a proper form for the function $f(x_2)$, as follows:

$$f(x_2) = \frac{Ph^2}{2I} \tag{2-106}$$
$$\Rightarrow \begin{cases} \sigma_{31}(\pm h, x_2, x_1) = \phi_{,2}(\pm h, x_2, x_1) = 0 \\ \sigma_{23}(x_3, \pm b, x_1) = -\phi_{,3}(x_3, \pm b, x_1) = 0 \end{cases}$$

and the governing equation changes to:

$$\phi_{,33} + \phi_{,22} = \frac{\nu}{1+\nu}\frac{P}{I}x_2 \tag{2-107}$$

Hence, in order to solve the problem at hand, we should choose a proper form of ϕ such that it satisfies the above equation and vanishes on the boundary.

Membrane Analogy

At this stage we may draw an analogy between the above governing equation and the equation of a homogeneous membrane whose outline is the same as the cross section of our beam. First, let us start with studying the deformation of a membrane subjected to a uniform tension at the edges and a uniform lateral pressure (see Fig. 2.9).

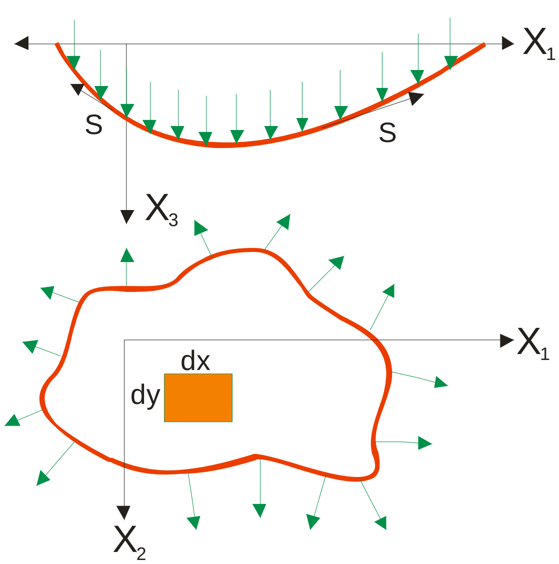

Fig. 2.9

If q is the pressure per unit area of the membrane and S is the uniform tension per unit length of its boundary, the tensile forces acting on the opposite sides (dy sides) of an infinitesimal element give a resultant in the upward direction: $-S(\partial^2 z/\partial x^2)dxdy$. In the same manner, the tensile forces acting on the other two opposite sides (dx sides) of the element give the resultant $-S(\partial^2 z/\partial y^2)dxdy$ (see Section 11.7, Kreyszig, "Advanced Engineering Mathematics") and the equation of equilibrium of the element becomes:

$$
\begin{aligned}
& qdxdy + S\frac{\partial^2 z}{\partial x^2}dxdy + S\frac{\partial^2 z}{\partial y^2}dxdy = 0 \\
& \Rightarrow \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -\frac{q}{S}
\end{aligned}
\tag{2-108}
$$

At the boundary, the deflection of the membrane is zero. Comparing the above equation with the governing equation of our beam problem, it is clear that the solution of the beam problem reduces to the determination of the deflections of the membrane which are produced by a continuous load with the intensity proportional to $\left[-\nu/(1+\nu)(Px_2/I)\right]$.

The curve *mnp* in Fig. 2.10 represents the intersection of the membrane with the x_1x_2 plane.

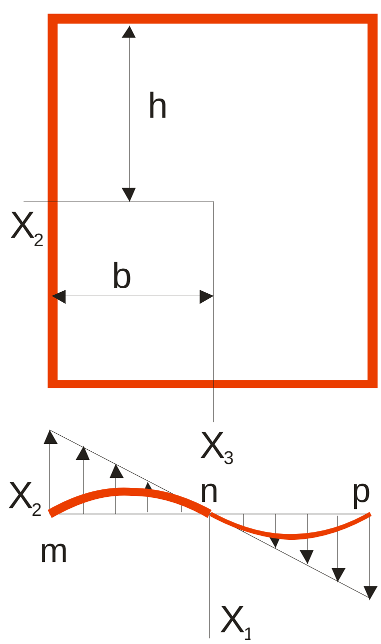

Fig. 2.10

Now we rewrite the stresses from equation (2-98) (incorporating $f(x_2) = Ph^2/2I$):

$$\begin{aligned}\sigma_{31} &= \phi_{,2} - \frac{Px_3^2}{2I} + \frac{Ph^2}{2I} \\ &= \frac{P(h^2 - x_3^2)}{2I} + \phi_{,2} \\ \sigma_{21} &= -\phi_{,3}\end{aligned} \tag{2-109}$$

From the above equations it is clear that the deviations of the shearing stresses from the *elementary theory* are proportional to the first derivatives or slope of the membrane at any point. For example, the correction to σ_{31} shows a maximum positive value at the two sides, a maximum negative value in the center, and is zero at the quarter points (See Fig. 2.10). From the condition of loading of the membrane it is clear that ϕ is an even function of x_3 and an odd function of x_2. Hence, it is appropriate to take the stress function in the form of the Fourier series:

$$\phi = \sum_{m=0}^{m=\infty} \sum_{n=1}^{n=\infty} A_{2m+1,n} \cos \frac{(2m+1)\pi x_3}{2h} \sin \frac{n\pi x_2}{b} \tag{2-110}$$

Substituting ϕ from above expression into our governing equation we may find the coefficient of the Fourier series and obtain the final form of our stress function as:

$$\phi = -\frac{\nu}{1+\nu} \frac{P}{I} \frac{8b^3}{\pi^4} \sum_{m=0}^{m=\infty} \sum_{n=1}^{n=\infty} \frac{(-1)^{m+n+1} \cos \frac{(2m+1)\pi x_3}{2h} \sin \frac{n\pi x_2}{b}}{(2m+1)n\left[(2m+1)^2 \frac{b^2}{4h^2} + n^2\right]} \tag{2-111}$$

which can be used to calculate the shearing stresses using equation (2-98).

Note that the above membrane analogy and Eq. (2-108) can also be used to obtain solutions to the TORSION problems discussed before.

Approximate Solutions

If the depth of the beam is large compared with the width *h*>>*b*, we may consider a cylindrical shape for the surface of the membrane at the points sufficiently far from the short sides at $x_3 = \pm h$ (assume $\phi = \phi(x_2)$), so we may write:

$$\begin{aligned}
&\phi = \phi(x_2) \\
&\Rightarrow \phi_{,22} = \frac{\nu}{1+\nu}\frac{Px_2}{I} \\
&\Rightarrow \phi(x_2) = \frac{\nu}{1+\nu}\frac{P}{6I}\left(x_2^3 + C_1 x_2 + C_2\right) \\
&\begin{cases} \phi(0) = 0 \Rightarrow C_2 = 0 \\ \phi(\pm b) = 0 \Rightarrow C_1 = -b^2 \end{cases} \\
&\Rightarrow \phi(x_2) = \frac{\nu}{1+\nu}\frac{P}{6I}\left(x_2^3 - b^2 x_2\right) \\
&\Rightarrow \sigma_{31}(x_3, x_2) = \frac{P}{2I}\left[\left(h^2 - x_3^2\right) + \frac{\nu}{1+\nu}\left(x_2^2 - \frac{b^2}{3}\right)\right]
\end{aligned} \tag{2-112}$$

On the other hand, if the width of the beam is large compared to the depth (*b*>>*h*), for the points far from the short sides at $x_2 = \pm b$, we may consider the deflections of the membrane as a linear function of x_2 and write:

$$\begin{aligned}
&\Rightarrow \phi_{,33} = \frac{\nu}{1+\nu}\frac{Px_2}{I} \\
&\Rightarrow \phi_{,3} = \frac{\nu}{1+\nu}\frac{Px_2 x_3}{I} + C_1 \\
&\Rightarrow \phi(x_3, x_2) = \frac{\nu}{1+\nu}\frac{Px_2}{2I}x_3^2 + C_1 x_3 + C_2 \\
&\phi(\pm h, x_2) = 0 \Rightarrow C_2 = \frac{-\nu}{1+\nu}\frac{Px_2 h^2}{2I} \quad , \quad C_1 = 0 \\
&\Rightarrow \phi(x_3, x_2) = \frac{\nu}{1+\nu}\frac{Px_2}{2I}\left(x_3^2 - h^2\right) \\
&\Rightarrow \sigma_{31} = \frac{P\left(h^2 - x_3^2\right)}{2I}\left[1 - \frac{\nu}{1+\nu}\right] = \frac{1}{1+\nu}\frac{P\left(h^2 - x_3^2\right)}{2I} \\
&\sigma_{21} = \frac{-\nu}{1+\nu}\frac{Px_3 x_2}{2I}
\end{aligned} \tag{2-113}$$

Anti-Plane Shear Problem

Consider a cylindrical solid with arbitrary cross-section, as shown in the Fig. 2.11. Also assume that the length of the cylinder greatly exceeds any cross sectional dimension. We may have states of *anti-plane shear* in a solid by an appropriate loading, as we will see in the following example.

The governing equations and boundary conditions for anti-plane shear problems are very simple and lead to solving the Laplace's equation (for which we have many powerful solution techniques).

We consider the following boundary value problem, with body force $\mathbf{b} = b_3^*(x_1, x_2)\mathbf{e_3}$

$$\begin{aligned} &T_1 = T_2 = 0 && \text{on} \quad C \\ &u_3 = u_3^*(x_1, x_2) && \text{on} \quad C_u \\ &T_3 = T_3^*(x_1, x_2) && \text{on} \quad C_T \end{aligned} \qquad (2\text{-}114)$$

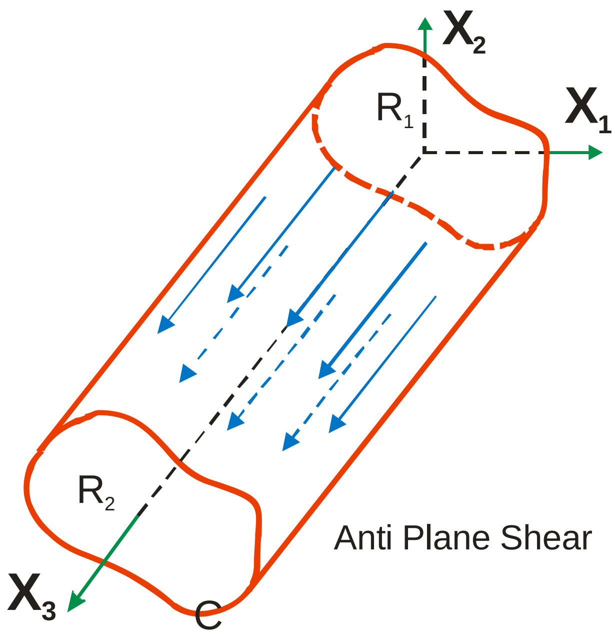

Fig. 2.11

In addition, for a traction boundary value problem we must ensure the existence of a static equilibrium solution:

$$\int_C T_3^*(x_1, x_2)ds + \int_R b_3^*(x_1, x_2)dV = 0 \qquad (2\text{-}115)$$

To accept any solution with zero resultant force and moment acting on the ends of the cylinder, we set the boundary conditions on R_α $(\alpha = 1,2)$ as:

$$\int_{R_\alpha} \mathbf{n}\cdot\boldsymbol{\sigma} dA = 0$$
$$\int_{R_\alpha} \mathbf{r}\times\mathbf{n}\cdot\boldsymbol{\sigma} dA = 0 \tag{2-116}$$

Recall the field equations:

$$\begin{aligned} \varepsilon_{ij} &= \tfrac{1}{2}(u_i,_j + u_j,_i) \\ \sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \\ \sigma_{ij},_j + b_i &= 0 \end{aligned} \tag{2-117}$$

Since all forces and boundary displacements act in the x_3 direction, it is natural to assume that the displacements are in the x_3 direction everywhere. Let's assume a solution of the form:

$$\begin{aligned} u_1 &= u_2 = 0 \\ u_3 &= \tilde{u}_3(x_1, x_2) \end{aligned} \tag{2-118}$$

The strains and stresses follow as:

$$\begin{aligned} &\varepsilon_{\alpha\beta} = 0, \quad \varepsilon_{33} = 0 \quad \varepsilon_{3\alpha} = \tfrac{1}{2}\tilde{u}_3,_\alpha \\ &\sigma_{\alpha\beta} = 0, \quad \sigma_{33} = 0, \quad \sigma_{3\alpha} = \mu\tilde{u}_3,_\alpha \end{aligned} \tag{2-119}$$

where Greek subscripts range from 1 to 2. The equilibrium equations reduce to:

$$\begin{aligned} &\sigma_{3\alpha},_\alpha + b_3 = 0 \\ &\Rightarrow \mu\tilde{u}_3,_{\alpha\alpha} + b_3^* = 0 \\ &\Rightarrow \tilde{u}_3,_{\alpha\alpha} = -\frac{b_3^*}{\mu} \end{aligned} \tag{2-120}$$

The boundary conditions may be re-written as,

$$\begin{aligned} &\tilde{u}_3 = u_3^*(x_1, x_2) \qquad \text{on} \quad C_u \\ &T_3 = \sigma_{3\alpha}n_\alpha = \mu\tilde{u}_3,_\alpha n_\alpha = T_3^*(x_1, x_2) \quad \text{on} \quad C_T \end{aligned} \tag{2-121}$$

As mentioned before, the Laplace's equation can be solved in different ways. Here we will use the complex variable method. This approach is based on the fact that both the real and imaginary parts of an analytic function satisfy the Laplace's equation. We start with the following definitions:

$$\begin{aligned} f(z) &= v(x_1, x_2) + iw(x_1, x_2) \\ z &= x_1 + ix_2 \qquad i = \sqrt{-1} \end{aligned} \tag{2-122}$$

If $f(z)$ is analytic, the derivative with respect to z is path independent, which requires:

$$\frac{\partial v}{\partial x_1} = \frac{\partial w}{\partial x_2} \qquad \frac{\partial w}{\partial x_1} = -\frac{\partial v}{\partial x_2} \tag{2-123}$$

These are known as the Cauchy-Riemann conditions. Hence, we have:

$$v,_{\alpha\alpha} = \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} = \frac{\partial}{\partial x_1}\left(\frac{\partial w}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial w}{\partial x_1}\right) = 0 \tag{2-124}$$

or, in general:

$$v,_{\alpha\alpha} = w,_{\alpha\alpha} = 0 \tag{2-125}$$

Thus, both the real and imaginary parts of an analytic function satisfy the Laplace equation.

Now, let $z = x_1 + ix_2 = re^{i\theta}$ characterize the position of a point in the plane of the solid of interest. Then let $f(z)$ denote any analytic function of position. We can set:

$$\tilde{u}_3 = \Re e(f(z)) = v(x_1, x_2) \tag{2-126}$$

Note that $\tilde{u}_3$ automatically satisfies the equilibrium equation $\tilde{u}_{3,\alpha\alpha} = 0$. We can solve our elasticity problem by finding an analytic function that satisfies appropriate boundary conditions. Of course, we may also use:

$$\tilde{u}_3 = \operatorname{Im}(f(z)) = w(x_1, x_2) \tag{2-127}$$

Now, we can determine the stresses directly from *f(z)*. Note that

$$\begin{aligned} f(z) &= v + iw \\ f'(z) &= v,_1 + iw,_1 = v,_1 - iv,_2 = w,_2 + iw,_1 \end{aligned} \tag{2-128}$$

Suppose we choose $\tilde{u}_3 = \Re e(f(z))$, then using (2-119) we have:

$$f'(z) = \tilde{u}_{3,1} - i\tilde{u}_{3,2}$$
$$\Rightarrow \mu f'(z) = \sigma_{31} - i\sigma_{32} \qquad (2\text{-}129)$$
$$\Rightarrow \mu \overline{f'(z)} = \sigma_{31} + i\sigma_{32}$$

Where $\overline{f'(z)}$ denotes the complex conjugate.

Similar expressions can be determined for $\tilde{u} = \Im m(f(z))$. In this case

$$f'(z) = \tilde{u}_{3,2} + i\tilde{u}_{3,1}$$
$$\Rightarrow \mu f'(z) = \sigma_{32} + i\sigma_{31} \qquad (2\text{-}130)$$

As an example we consider the fundamental solution for an infinite solid and find the displacement and stress fields induced by a line load $\mathbf{F} = F\mathbf{e}_3$ acting at the origin.
We start by generating the solution from the following function:

$$f(z) = C\log(z) \qquad (2\text{-}131)$$

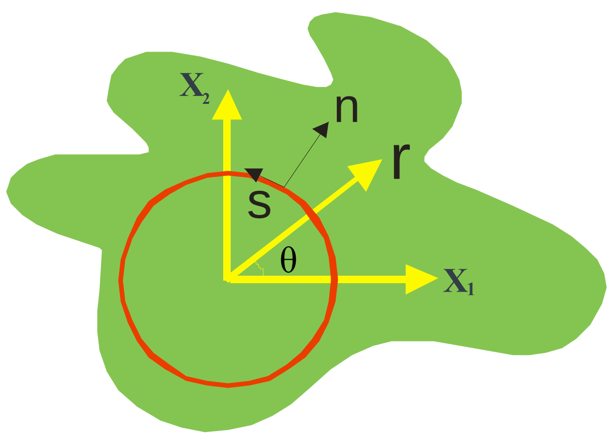

Fig. 2.12

This is analytic everywhere except the origin. Next we choose:

$$\tilde{u}_3 = \Re e(f(z)) \qquad (2\text{-}132)$$

and compute the resultant force acting on a circular arc enclosing the origin:

$$F_3 = \int_A^B (\sigma_{31} n_1 + \sigma_{32} n_2) ds = \int_0^{2\pi} \left(\sigma_{31} \cos\theta + \sigma_{23} \sin\theta \right) r d\theta \tag{2-133}$$

Here we may write:

$$\begin{aligned} \sigma_{31} \cos\theta + \sigma_{32} \sin\theta &= \Re e\left[\left(\sigma_{31} + i\sigma_{32} \right)(\cos\theta - i\sin\theta) \right] \\ &= \Re e(\mu \overline{f'(z)} e^{-i\theta}) \end{aligned} \tag{2-134}$$

which using (2-131), can be written as:

$$\begin{aligned} \sigma_{31} \cos\theta + \sigma_{32} \sin\theta &= \Re e\left[\mu C \frac{e^{-i\theta}}{\overline{z}} \right] \\ &= \Re e\left[\mu C \frac{e^{-i\theta}}{re^{-i\theta}} \right] = C \frac{\mu}{r} \end{aligned} \tag{2-135}$$

Substituting (2-135) into the right hand side of (2-133), results in:

$$\begin{aligned} &F_3 = 2\pi\mu C \\ &\Rightarrow C = \frac{F_3}{2\pi\mu} \end{aligned} \tag{2-136}$$

Therefore, the solution can be found and simplified as follows:

$$\begin{aligned} &\tilde{u}_3 = \frac{F_3}{2\pi\mu} \Re e\{\log(z)\} \\ &\tilde{u}_3 = \frac{F_3}{2\pi\mu} \Re e[\log(re^{i\theta})] = \frac{F_3}{2\pi\mu} \log(r) \\ &r = \sqrt{x_\alpha x_\alpha} \end{aligned} \tag{2-137}$$

Assignment 6:
Find the stress field for a **Screw Dislocation problem.**
Hint: **A**ssume $\tilde{u} = \dfrac{b}{2\pi} \Im m\{\log(z)\}$

The Mode III Crack Problem

The analysis of Mode III crack problem is relatively simple because we may assume that $u_1=u_2=0$ and $u_3 = u_3(x_1 , x_2)$, which clearly represent an Anti-Plane Shear problem.

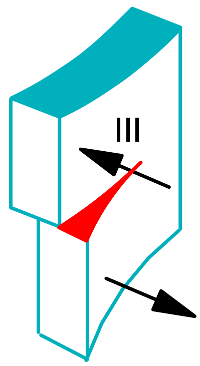

Fig. 2.13

Referring to the material covered in the above sections, and ignoring the body forces in our crack problem, we may write:

$$\tilde{u}_{3,\alpha\alpha} = \nabla^2 \tilde{u}_3 = 0 \tag{2-138}$$

Here, we consider a solution in the following form:

$$\tilde{u}_3 = \frac{1}{\mu}\left[f(z) + \overline{f(z)} \right] \tag{2-139}$$

Hence, the strains become:

$$\begin{aligned} \varepsilon_{31} &= \frac{1}{2\mu}\left[f'(z) + \overline{f'(z)} \right] \\ \varepsilon_{32} &= \frac{i}{2\mu}\left[f'(z) - \overline{f'(z)} \right] \end{aligned} \tag{2-140}$$

Using the constitutive equations we find that:

$$\sigma_{31} - i\sigma_{32} = 2 f'(z) \tag{2-141}$$

Now, let the origin of the coordinate system be located at the tip of a crack lying along the negative x_1 axis as shown in Fig. 2.14.

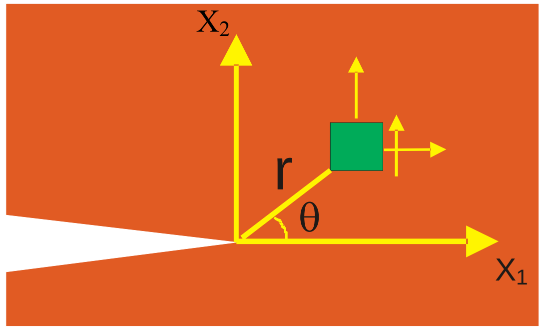

Fig. 2.14

Now, we will focus our attention upon a small region (denoted as D) containing the crack tip and consider the holomorphic function:

$$f(z) = Cz^{\lambda+1}, \quad C = A + iB \tag{2-142}$$

where *A, B,* and λ are real undetermined constants. For finite displacements at the crack tip we must have: $(|z| = r = 0), \lambda \succ -1$). Then we will have:

$$\sigma_{31} - i\sigma_{32} = 2(\lambda+1)Cz^{\lambda} = 2(\lambda+1)r^{\lambda}(A+iB)(\cos\lambda\theta + i\sin\lambda\theta) \tag{2-143}$$

where,

$$\begin{aligned} \sigma_{31} &= 2(\lambda+1)r^{\lambda}(A\cos\lambda\theta - B\sin\lambda\theta) \\ \sigma_{32} &= -2(\lambda+1)r^{\lambda}(A\sin\lambda\theta + B\cos\lambda\theta) \end{aligned} \tag{2-144}$$

The boundary condition that the crack surfaces be traction free requires that $\sigma_{32} = 0$ on $\theta = \pm\pi$. Consequently we have:

$$\begin{aligned} A\sin\lambda\pi + B\cos\lambda\pi &= 0 \\ A\sin\lambda\pi - B\cos\lambda\pi &= 0 \end{aligned} \tag{2-145}$$

To avoid the trivial solution, the determinant of the coefficients of the above equations must vanish. This leads to:

$$\sin 2\lambda\pi = 0 \tag{2-146}$$

which for λ> −1 has the following roots:

$$\lambda = -\frac{1}{2}, n/2, \quad n = 0,1,2,... \tag{2-147}$$

Of the infinite set of functions of the form of the above equation that yield traction-free crack surfaces within *D,* the function with λ = –1/2 for which *A* = 0, provides the most significant contribution to the crack-tip fields. For this case the stresses and displacements become, respectively:

$$\begin{Bmatrix} \sigma_{31} \\ \sigma_{32} \end{Bmatrix} = \frac{K_{III}}{(2\pi r)^{1/2}} \begin{Bmatrix} \sin(\theta/2) \\ \cos(\theta/2) \end{Bmatrix} \tag{2-148}$$

and,

$$\tilde{u}_3 = \frac{2K_{III}}{\mu}\left(\frac{r}{2\pi}\right)^{\frac{1}{2}} \sin(\theta/2) \tag{2-149}$$

where *B* has been chosen such that:

$$K_{III} = \lim_{r\to 0}\left\{(2\pi r)^{1/2}\sigma_{32}\big|_{\theta=0}\right\} \tag{2-150}$$

The quantity K_{III} is referred to as the Mode III stress intensity factor, **which is established by the far field boundary conditions and is a function of the applied loading and the geometry of the cracked body**. Whereas the stresses associated with the other values of λ are finite at the crack tip, the stress components associated with λ = –1/2 have an inverse square root singularity at the crack tip. It is clear that the latter components will dominate as the crack tip is approached and represent the **asymptotic** forms of the elastic stress and displacement fields.

Chapter 3
2D Static Boundary Value Problems: Plane Elasticity

Chinese Proverb *- It is better to ask a question and look like a fool for five minutes, than not to ask a question at all and be a fool for the rest of your life.*

Two Dimensional Elastostatic Problems

Plane Strain Formulation

Consider a long prismatic bar loaded with surface tractions and body forces normal to the x_3 axis as depicted in Fig 3.1.

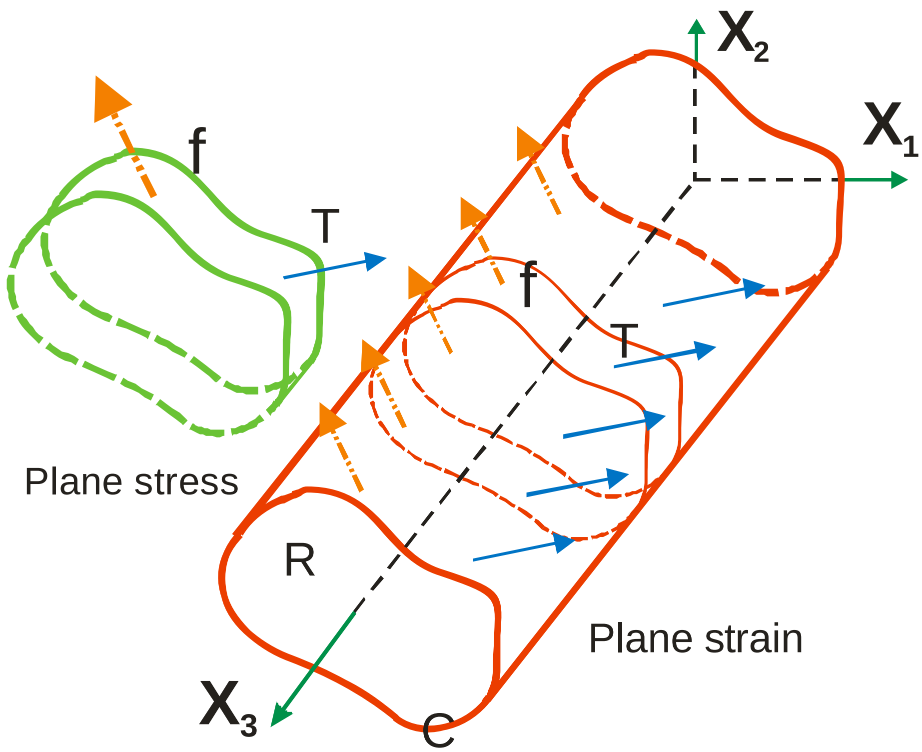

Fig. 3.1

Also assume that the magnitude of loadings does not vary along this axis. Under these conditions we may write:

$$u = u_\alpha\left(x_\alpha\right)\begin{cases} u_1 = u_1\left(x_1, x_2\right) \\ u_2 = u_2\left(x_1, x_2\right) \end{cases} \quad \alpha, \beta = 1, 2$$
$$u_3 = \text{constant}, \Rightarrow \text{ generalized plane strain}$$
$$u_3 = 0, \Rightarrow \text{ plane strain} \tag{3.1}$$

Accordingly, the strain field becomes:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \sim \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \cdot & \varepsilon_{22} & 0 \\ \cdot & \cdot & 0 \end{pmatrix} \tag{3.2}$$

In order to calculate the stress field we use the following constitutive equations:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij} \tag{3.3}$$

Partitioning the stress tensor, we have

$$\sigma_{ij} \sim \begin{pmatrix} \sigma_{\alpha\beta} & \sigma_{\alpha 3} \\ \sigma_{3\beta} & \sigma_{33} \end{pmatrix} \tag{3.4}$$

Here we have three constitutive equations:

$$\sigma_{\alpha\beta} = \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} + 2G\varepsilon_{\alpha\beta} \qquad \text{(i)}$$
$$\sigma_{3\alpha} = \sigma_{\beta 3} = 0$$
$$\sigma_{33} = \lambda\varepsilon_{\gamma\gamma} = \lambda\left(\varepsilon_{11} + \varepsilon_{22}\right) \qquad \text{(ii)} \tag{3.5}$$

Contracting on α in equation (3.5i) we have:

$$\sigma_{\alpha\alpha} = \lambda\varepsilon_{\gamma\gamma}\cdot 2 + 2G\varepsilon_{\alpha\alpha}$$
$$\Rightarrow \varepsilon_{\alpha\alpha} = \frac{1}{2\left(\lambda + G\right)}\sigma_{\alpha\alpha} \tag{3.6}$$

Using the above equation and substituting for $\varepsilon_{\gamma\gamma}$ in equation (3.5ii) we can write:

$$\sigma_{33} = \frac{\lambda}{2\left(\lambda + G\right)}\sigma_{\alpha\alpha} = \nu\sigma_{\alpha\alpha} \tag{3.7}$$

which shows that in plane strain condition the stress in the x_3 direction is a dependent quantity. Accordingly, the strains can be written as:

$$\begin{aligned}\varepsilon_{\alpha\beta} &= \frac{1}{2G}\left[\sigma_{\alpha\beta} - \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right] \\ &= \frac{1}{2G}\left[\sigma_{\alpha\beta} - \frac{\lambda}{2(\lambda+G)}\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right] \\ &= \frac{1}{2G}\left[\sigma_{\alpha\beta} - \nu\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right]\end{aligned} \tag{3.8}$$

It is important to note that the above equation cannot be obtained from the original constitutive equation by simply replacing *i,j* by *α,β*.

$$\varepsilon_{\alpha\beta} \sim \begin{pmatrix} \frac{1}{2G}\left[\sigma_{11} - \nu(\sigma_{11}+\sigma_{22})\right] & \frac{1}{2G}\sigma_{12} \\ . & \frac{1}{2G}\left[\sigma_{22} - \nu(\sigma_{11}+\sigma_{22})\right] \end{pmatrix} \tag{3.9}$$

The equilibrium equations in the plane strain case become:

$$\begin{aligned}&\sigma_{\alpha\beta,\alpha} + f_\beta = 0 \\ &\begin{cases}\sigma_{11,1} + \sigma_{12,2} + f_1 = 0 \\ \sigma_{12,1} + \sigma_{22,2} + f_2 = 0\end{cases}\end{aligned} \tag{3.10}$$

Now, we assume that the body force f_β can be derivable from a potential, *V*. Hence, we may write:

$$\begin{aligned}&f_\beta = -V_{,\beta} \\ &\Rightarrow \sigma_{\alpha\beta,\alpha} - V_{,\beta} = 0 \\ &\begin{cases}(\sigma_{11} - V)_{,1} + \sigma_{12,2} = 0 \\ (\sigma_{22} - V)_{,2} + \sigma_{12,1} = 0\end{cases}\end{aligned} \tag{3.11}$$

The above equations can be satisfied by assuming the existence of two functions $A(x_\alpha)$ and $B(x_\alpha)$, such that:

$$\begin{aligned}&\sigma_{11} - V = A_{,2} \;\;;\;\; \sigma_{12} = -A_{,1} \\ &\sigma_{22} - V = B_{,1} \;\;;\;\; \sigma_{12} = -B_{,2} \\ &\Rightarrow -B_{,2} + A_{,1} = 0\end{aligned} \tag{3.12}$$

The latter is satisfied by assuming the existence of a stress function such that:

$$B = \varphi_{,1} \;\;;\;\; A = \varphi_{,2} \tag{3.13}$$

Accordingly, the stresses can be written in terms of the *Airy stress function* φ as:

$$\begin{aligned}\sigma_{11} - V &= \varphi_{,22} \\ \sigma_{22} - V &= \varphi_{,11} \\ \sigma_{12} = \sigma_{21} &= -\varphi_{,12}\end{aligned} \tag{3.14}$$

The stress boundary conditions for the plane strain reduce to:

$$T_{\alpha} = \sigma_{\alpha\beta} n_{\beta} \tag{3.15}$$

From the six stress compatibility equations we have only one important non-trivial equation, which upon substitution of the stresses by the stress function becomes:

$$\begin{aligned}&\nabla'^2 \sigma_{33} + \frac{1}{1+\nu}\sigma_{mm,33} = \frac{\nu}{1-\nu}\nabla'^2 V \\ &\nabla'^2 = (\ )_{,\alpha\alpha} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \\ &\Rightarrow \nu\nabla'^2 \sigma_{\gamma\gamma} = \frac{\nu}{1-\nu}\nabla'^2 V \\ &\Rightarrow \nabla'^2 \left(V + \varphi_{,22} + V + \varphi_{,11}\right) = \frac{1}{1-\nu}\nabla'^2 V \\ &\Rightarrow \nabla'^2 \left(\varphi_{,22} + \varphi_{,11}\right) = \left(\frac{1}{1-\nu} - 2\right)\nabla'^2 V \\ &\Rightarrow \nabla'^2 \nabla'^2 \varphi = -\left(\frac{1-2\nu}{1-\nu}\right)\nabla'^2 V \\ &\Rightarrow \nabla'^4 \varphi = -\left(\frac{1-2\nu}{1-\nu}\right)\nabla'^2 V\end{aligned} \tag{3.16}$$

The above is the governing partial differential equation for the Airy Stress Function. If the potential function is harmonic, i.e., $\nabla'^2 V = 0$, then φ satisfies the homogeneous biharmonic equation:

$$\begin{aligned}&\nabla'^4 \varphi = 0 \\ &\nabla'^4 = \nabla'^2 \nabla'^2 = (\ )_{,\alpha\alpha\beta\beta} = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 x_2^2} + \frac{\partial^4}{\partial x_2^4}\end{aligned} \tag{3.17}$$

Plane Stress Formulation

Let us consider a short prismatic body loaded with surface tractions and body forces normal to the x_3 axis as depicted in Fig. 3.1. Under these conditions we may write for the stresses:

$$\bar{\sigma}_{ij} = \frac{1}{h}\int_{-h/2}^{h/2} \sigma_{ij} dx_3 \tag{3.18}$$

The *bar* sign over a quantity indicates that it has been averaged over the thickness. The stress tensor is:

$$\bar{\sigma}_{ij} \sim \begin{pmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{12} & 0 \\ \cdot & \bar{\sigma}_{22} & 0 \\ \cdot & \cdot & 0 \end{pmatrix} \tag{3.19}$$

Also, the equilibrium equations can be written as:

$$\begin{aligned} &\bar{f}_\beta = -\bar{V}_{,\beta} \\ &\Rightarrow \bar{\sigma}_{\alpha\beta,\alpha} - \bar{V}_{,\beta} = 0 \\ &\begin{cases} \left(\bar{\sigma}_{11} - \bar{V}\right)_{,1} + \bar{\sigma}_{12,2} = 0 \\ \left(\bar{\sigma}_{22} - \bar{V}\right)_{,2} + \bar{\sigma}_{12,1} = 0 \end{cases} \end{aligned} \tag{3.20}$$

which are exactly the same as those derived for plane strain condition, except for the mean value interpretation. Thus we may similarly write:

$$\begin{aligned} &\bar{\sigma}_{11} - \bar{V} = \bar{\varphi}_{,22} \\ &\bar{\sigma}_{22} - \bar{V} = \bar{\varphi}_{,11} \\ &\bar{\sigma}_{12} = \bar{\sigma}_{21} = -\bar{\varphi}_{,12} \end{aligned} \tag{3.21}$$

and the related constitutive equations are:

$$\begin{aligned} &\bar{\varepsilon}_{\alpha\beta} = -\frac{\nu}{E}\bar{\sigma}_{\gamma\gamma}\delta_{\alpha\beta} + \frac{1+\nu}{E}\bar{\sigma}_{\alpha\beta} \qquad \text{(i)} \\ &\bar{\varepsilon}_{3\alpha} = 0 \\ &\bar{\varepsilon}_{33} = -\frac{\nu}{E}\bar{\sigma}_{\gamma\gamma} \qquad \text{(ii)} \end{aligned} \tag{3.22}$$

From Eq. (3.22i) we have:

$$\bar{\varepsilon}_{\gamma\gamma} = -\frac{\nu}{E}\bar{\sigma}_{\gamma\gamma}\cdot 2 + \frac{1+\nu}{E}\bar{\sigma}_{\gamma\gamma} = \frac{1-\nu}{E}\bar{\sigma}_{\gamma\gamma}$$

$$\Rightarrow \bar{\sigma}_{\gamma\gamma} = \frac{E}{1-\nu}\bar{\varepsilon}_{\gamma\gamma} \tag{3.23}$$

$$(3.22\text{ii}) \Rightarrow \bar{\varepsilon}_{33} = -\frac{\nu}{1-\nu}\bar{\varepsilon}_{\gamma\gamma}$$

which shows that in plane stress condition the strain in the x_3 direction is a dependent quantity. We may also write stresses in terms of strains:

$$\bar{\sigma}_{\alpha\beta} = \frac{E}{1+\nu}\left[\bar{\varepsilon}_{\alpha\beta} + \frac{\nu}{E}\bar{\sigma}_{\gamma\gamma}\delta_{\alpha\beta}\right] \tag{3.24}$$

Substituting for $\bar{\sigma}_{\gamma\gamma}$ we will have:

$$\begin{aligned}\bar{\sigma}_{\alpha\beta} &= \frac{E}{1+\nu}\left[\bar{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta}\right] \\ &= 2G\bar{\varepsilon}_{\alpha\beta} + \frac{2G\lambda}{\lambda+2G}\bar{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta}\end{aligned} \tag{3.25}$$

If we compare the above equations with those used in plane strain, we conclude that the entire system of equations in plane stress are analogous to those for plane strain, provided that we replace the *barred* quantities with the *unbarred* and define the followings:

$$\begin{aligned}\bar{G} &= G \\ \bar{\lambda} &= \frac{2G\lambda}{\lambda+2G}\end{aligned} \tag{3.26}$$

Hence, we can derive the fundamental biharmonic boundary value problem for the plane stress conditions as:

$$\begin{aligned}\nabla'^4\bar{\varphi} &= -\left(\frac{1-2\bar{\nu}}{1-\bar{\nu}}\right)\nabla'^2\bar{V} \\ \bar{\nu} &= \frac{\nu}{1+\nu}\end{aligned} \tag{3.27}$$

Assignment 7:

Find the stress field for a narrow rectangular beam under end loading.

Airy Stress Functions in Cylindrical Coordinates

It is quite beneficial to solve the boundary value problems involving cylindrical regions using appropriate cylindrical coordinates.

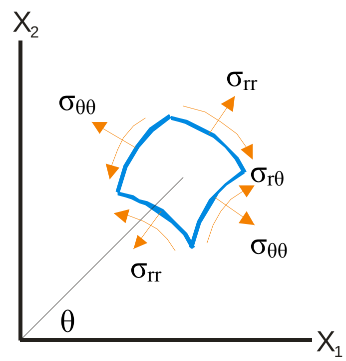

Fig. 3.2

In case of zero or constant body forces, the state of stress is related to the Airy function using the following transformations:

$$\begin{aligned}\sigma_{rr} &= \alpha_{ri}\alpha_{rj}\sigma_{ij}\\ \sigma_{\theta\theta} &= \alpha_{\theta i}\alpha_{\theta j}\sigma_{ij}\\ \sigma_{r\theta} &= \alpha_{ri}\alpha_{\theta j}\sigma_{ij}\end{aligned} \tag{3.28}$$

which, by appropriate substitutions for the stress function φ, lead to the following expressions for the stresses:

$$\begin{aligned}\sigma_{rr} &= \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = \frac{1}{r}\varphi_{,r} + \frac{1}{r^2}\varphi_{,\theta\theta}\\ \sigma_{\theta\theta} &= \frac{\partial^2\varphi}{\partial r^2} = \varphi_{,rr}\\ \sigma_{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\theta}\right) = \frac{1}{r^2}\varphi_{,\theta} - \frac{1}{r}\varphi_{,r\theta}\end{aligned} \tag{3.29}$$

The governing equations for this coordinate system will be:

$$\begin{aligned}\nabla^2\varphi &= \varphi_{,rr} + \frac{1}{r}\varphi_{,r} + \frac{1}{r^2}\varphi_{,\theta\theta}\\ \nabla^4\varphi &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\varphi_{,rr} + \frac{1}{r}\varphi_{,r} + \frac{1}{r^2}\varphi_{,\theta\theta}\right) = 0\end{aligned} \tag{3.30}$$

As an example, we will obtain the stress field due to a line load of magnitude *P* per unit out-of-plane length, acting on the surface of a homogeneous isotropic half-space.

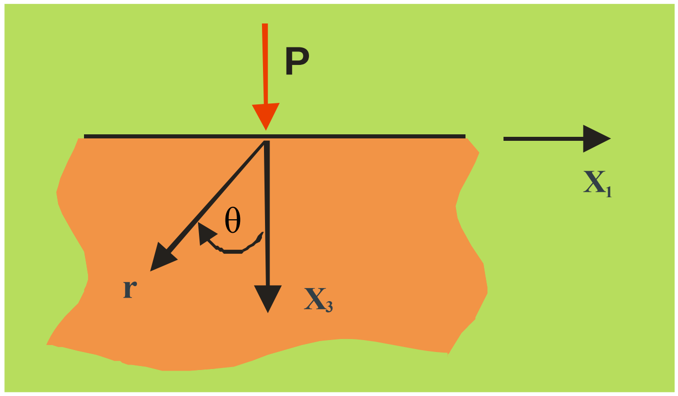

Fig. 3.3

Assume the Airy function in the form of:

$$\phi = -\frac{P}{\pi} r\theta \sin\theta \tag{3.31}$$

The resulting state of stress can be written as:

$$\sigma_{rr} = -\frac{2P}{\pi}\frac{\cos\theta}{r} \qquad \sigma_{\theta\theta} = \sigma_{r\theta} = 0 \tag{3.32}$$

In order to find the displacement field we need to determine the strains by substituting the expression for stress into the constitutive relation and then integrating the strains to calculate the displacements. For the point force solution, one we can show that:

$$\begin{aligned} u_r &= \frac{1-\nu}{\pi\mu} P\cos\theta \log r - \frac{1-2\nu}{2\pi\mu} P\theta \sin\theta \\ u_\theta &= \frac{1-\nu}{\pi\mu} P\sin\theta \log r + \frac{\nu}{2\pi\mu} P\sin\theta \end{aligned} \tag{3.33}$$

Axisymmetric Stress Distribution

When we have symmetrical stress distribution about an axis, in the absence of body forces, we may write:

$$\begin{aligned}
\sigma_{rr} &= \frac{1}{r}\frac{\partial\varphi}{\partial r} = \frac{1}{r}\varphi_{,r} \\
\sigma_{\theta\theta} &= \frac{\partial^2\varphi}{\partial r^2} = \varphi_{,rr} \\
\sigma_{r\theta} &= 0
\end{aligned} \tag{3.34}$$

and also:

$$\begin{aligned}
\nabla^2\varphi &= \varphi_{,rr} + \frac{1}{r}\varphi_{,r} \\
\nabla^4\varphi &= \frac{1}{r}\left\{r\left[\frac{1}{r}\left(r\varphi_{,r}\right)_{,r}\right]_{,r}\right\}_{,r} = 0
\end{aligned} \tag{3.35}$$

The above expression can successively be integrated with respect to r to give the general form of the stress function for axisymmetric stress distribution in cylindrical coordinates as follows:

$$\begin{aligned}
&\left\{r\left[\frac{1}{r}\left(r\varphi_{,r}\right)_{,r}\right]_{,r}\right\}_{,r} = 0 \quad \Rightarrow \left\{r\left[\frac{1}{r}\left(r\varphi_{,r}\right)_{,r}\right]_{,r}\right\} = C_1 \\
&\Rightarrow \left[\frac{1}{r}\left(r\varphi_{,r}\right)_{,r}\right]_{,r} = \frac{C_1}{r} \quad \Rightarrow \frac{1}{r}\left(r\varphi_{,r}\right)_{,r} = C_1 \ln r + C_2 \\
&\Rightarrow \left(r\varphi_{,r}\right)_{,r} = C_1 r \ln r + C_2 r \\
&\Rightarrow r\varphi_{,r} = C_1\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + C_2\frac{r^2}{2} + C_3 = C_1' r^2 \ln r + C_2' r^2 + C_3 \\
&\Rightarrow \varphi_{,r} = C_1' r \ln r + C_2' r + \frac{C_3}{r} \\
&\Rightarrow \varphi = C_1'\left(\frac{r^2}{2}\ln r - \frac{r^2}{4}\right) + C_2'\frac{r^2}{2} + C_3 \ln r + C_4 \\
&\Rightarrow \varphi = C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4
\end{aligned} \tag{3.36}$$

Accordingly the stresses become:

$$\begin{aligned}
\sigma_{rr} &= C_1\left(1 + 2\ln r\right) + 2C_2 + C_3\frac{1}{r^2} \\
\sigma_{\theta\theta} &= C_1\left(3 + 2\ln r\right) + 2C_2 - C_3\frac{1}{r^2} \\
\sigma_{r\theta} &= 0
\end{aligned} \tag{3.37}$$

Finally, the displacements are:

$$u_r = \frac{1}{E}\left\{C_1 r\left[(1-\nu)(2\ln r - 1) - 2\nu\right] + 2C_2(1-\nu)r - C_3\frac{1+\nu}{r} + C_4\sin\theta + C_5\cos\theta\right\}$$
$$u_\theta = \frac{1}{E}\left[4C_1 r\theta + C_4\cos\theta - C_5\sin\theta + C_6 r\right] \qquad (3.38)$$

Axisymmetric Stress and Displacement fields

When we also have axisymmetric displacements in the absence of body forces, the displacement values do not depend on θ, so we have $C_1=0$ and we may write:

$$\sigma_{rr} = C_2 + C_3\frac{1}{r^2}$$
$$\sigma_{\theta\theta} = C_2 - C_3\frac{1}{r^2} \qquad (3.39)$$
$$\sigma_{r\theta} = 0$$

Assignment 8:

Find the stress field for a cylindrical pressure vessel under internal and external pressures.

Curved Beam under constant bending moment

As an example for two-dimensional problems in cylindrical coordinates we consider the solution of a curved beam with rectangular cross section, under a constant bending moment M.

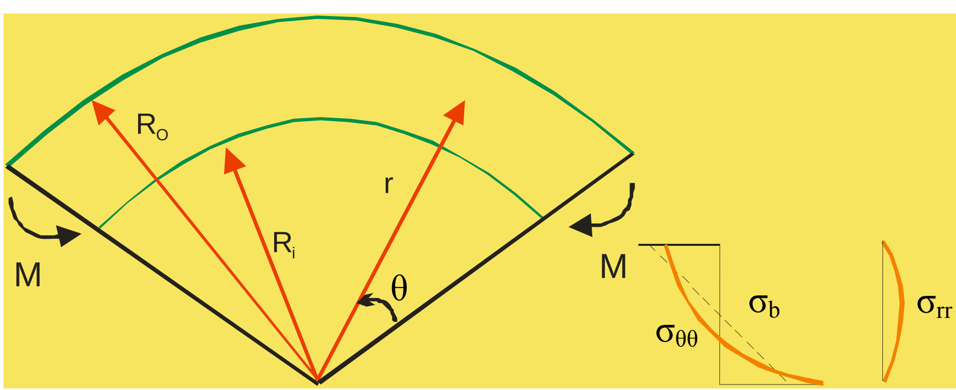

Fig. 3.4

For the upper and lower faces, we can write:

$$\begin{aligned}\sigma_{rr}(R_0)=\sigma_{rr}(R_i)=0\\ \sigma_{r\theta}(R_0)=\sigma_{r\theta}(R_i)=0\end{aligned} \tag{3.40}$$

On the other hand, the equilibrium conditions dictate that:

$$\begin{aligned}&\int_A \sigma_{\theta\theta}dA=0 \qquad \text{(i)}\\ &\int_A r\sigma_{\theta\theta}dA=M \qquad \text{(ii)}\end{aligned} \tag{3.41}$$

Notice that in this problem the stress field is axisymetric but the displacement field is not, so the stresses can be written as:

$$\begin{aligned}\sigma_{rr}&=C_1\left(1+2\ln r\right)+2C_2+C_3\frac{1}{r^2}\\ \sigma_{\theta\theta}&=C_1\left(3+2\ln r\right)+2C_2-C_3\frac{1}{r^2}\\ \sigma_{r\theta}&=0\end{aligned} \tag{3.42}$$

Next, we write the expression for σ_{rr} at the boundaries:

$$\begin{aligned}C_1\left(1+2\ln R_0\right)+2C_2+C_3\frac{1}{R_0^2}=0\\ C_1\left(1+2\ln R_i\right)+2C_2+C_3\frac{1}{R_i^2}=0\end{aligned} \tag{3.43}$$

Note that we still need a third equation to be able to determine our three constants. In order to find it, we write equation (3.41i) in terms of the stress function as follows:

$$\int_A \sigma_{\theta\theta}dA=B\int_{R_i}^{R_o}\varphi_{,rr}dr=B\left[\varphi_{,r}\right]_{R_i}^{R_o}=0 \tag{3.44}$$

Since we have $\sigma_{rr}=\frac{1}{r}\varphi_{,r}$, the above expression is consistent with our first boundary condition and indicates that $\varphi_{,r}=0$ on the boundary.

Also, the second equilibrium condition (3.41ii) can be written as:

$$
\begin{aligned}
&\int_A r\sigma_{\theta\theta} dA = B\int_{R_i}^{R_o} r\varphi_{,rr} dr = M \\
&\Rightarrow B\int_{R_i}^{R_o} r\varphi_{,rr} dr = B\, r\varphi_{,r}\Big|_{R_i}^{R_o} - B\int_{R_i}^{R_o} \varphi_{,r} dr = M \\
&\Rightarrow \varphi(R_o) - \varphi(R_i) = -\frac{M}{B}
\end{aligned}
\tag{3.45}
$$

Considering the general form of the stress function in cylindrical coordinates:

$$
\varphi = C_1 r^2 \ln r + C_2 r^2 + C_3 \ln r + C_4 \tag{3.46}
$$

we may write (3.45) as:

$$
C_1\left(R_o^{\,2} \ln R_o - R_i^2 \ln R_i\right) + C_2\left(R_o^{\,2} - R_i^{\,2}\right) + C_3 \ln\frac{R_o}{R_i} = -\frac{M}{B} \tag{3.47}
$$

which gives the required third equation in terms of the three constants of the problem. Accordingly, we will have:

$$
\begin{aligned}
&C_1 = \frac{2M}{NB}\left(R_o^{\,2} - R_i^{\,2}\right) \\
&C_2 = -\frac{M}{NB}\left[\left(R_o^{\,2} - R_i^{\,2}\right) + 2\left(R_o^{\,2} \ln R_o - R_i^2 \ln R_i\right)\right] \\
&C_3 = \frac{4M}{NB} R_o^{\,2} R_i^2 \ln\frac{R_o}{R_i}
\end{aligned}
\tag{3.48}
$$

where

$$
N = \left(R_o^{\,2} - R_i^{\,2}\right) - 4R_o^{\,2} R_i^2 \left(\ln\frac{R_o}{R_i}\right)^2
$$

Hence, the stresses can be found as:

$$
\begin{aligned}
&\sigma_{rr} = \frac{4M}{NB}\left(R_o^{\,2} R_i^2 \ln\frac{R_o}{R_i}\frac{1}{r^2} + R_o^{\,2} \ln\frac{r}{R_i} + R_i^2 \ln\frac{R_i}{r}\right) \\
&\sigma_{\theta\theta} = \frac{4M}{NB}\left(R_o^{\,2} R_i^2 \ln\frac{R_o}{R_i}\frac{1}{r^2} - R_o^{\,2} \ln\frac{r}{R_o} - R_i^2 \ln\frac{R_i}{r} - R_o^{\,2} + R_i^{\,2}\right) \\
&\sigma_{r\theta} = 0
\end{aligned}
\tag{3.49}
$$

Sheet with a Circular Hole under Uniaxial Stress

As an example for two-dimensional problems in cylindrical coordinates we consider a thin sheet, with a circular hole, under uniaxial uniform tension as follows:

$$\begin{aligned} \sigma_{11} &= T_1 \\ \sigma_{22} &= \sigma_{12} = 0 \end{aligned} \tag{3.50}$$

The above uniform stress is disturbed by the presence of the circular hole. Also note that it is more convenient to define the local stress distribution near the hole using cylindrical coordinates. Hence, we use the St.Venant's principle and argue that the stress distribution near the hole remains unchanged if we construct a hypothetical circle, with a diameter equal to the sheet width, and apply equivalent forces on the boundary of this circle, as shown in Fig. 3.5a.

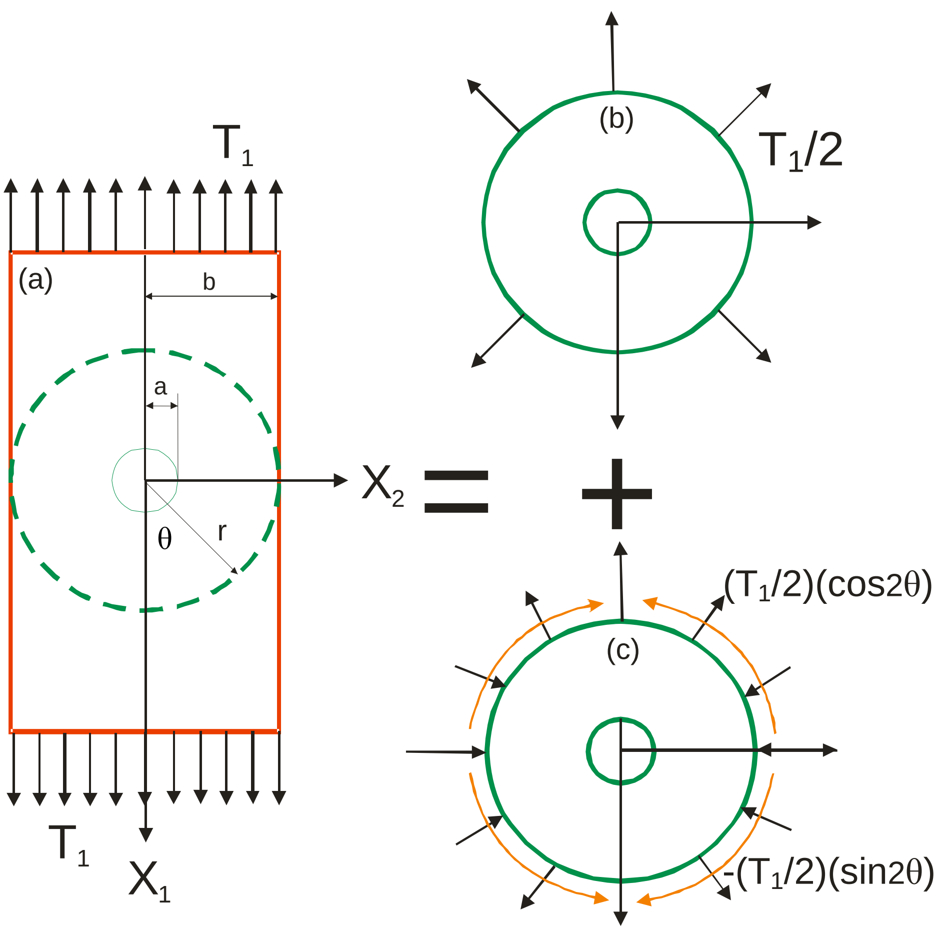

Fig. 3.5

We start by a transformation of the Cartesian components of stress into the Polar components as:

$$\begin{aligned}
\sigma_{rr} &= \alpha_{ri}\alpha_{rj}\sigma_{ij} = \sigma_{11}\cos^2\theta + 2\sigma_{12}\sin\theta\cos\theta + \sigma_{22}\sin^2\theta \\
\sigma_{\theta\theta} &= \alpha_{\theta i}\alpha_{\theta j}\sigma_{ij} = \sigma_{11}\sin^2\theta - 2\sigma_{12}\sin\theta\cos\theta + \sigma_{22}\cos^2\theta \\
\sigma_{r\theta} &= \alpha_{ri}\alpha_{\theta j}\sigma_{ij} = \left(\sigma_{22} - \sigma_{11}\right)\sin\theta\cos\theta + \sigma_{12}\left(\cos^2\theta - \sin^2\theta\right)
\end{aligned} \tag{3.51}$$

which, in combination with the stress state expressed by equation (3.50), give the stresses at the location of the hypothetical circle as follows:

$$\begin{aligned}
\sigma_{rr}\left(b,\theta\right) &= T_1\cos^2\theta = \frac{T_1}{2}\left(1+\cos 2\theta\right) \\
\sigma_{\theta\theta}\left(b,\theta\right) &= T_1\sin^2\theta = \frac{T_1}{2}\left(1-\cos 2\theta\right) \\
\sigma_{r\theta}\left(b,\theta\right) &= -T_1\sin\theta\cos\theta = -\frac{T_1}{2}\sin 2\theta
\end{aligned} \tag{3.52}$$

As depicted in Fig. 3.5, we may separate the stresses along the outer boundary into two parts. For the first part (shown in Fig. 3.5b) we have the axisymmetric stress state:

$$\begin{aligned}
\sigma_{rr}^{(1)}\left(b,\theta\right) &= \frac{T_1}{2} \\
\sigma_{r\theta}^{(1)}\left(b,\theta\right) &= 0
\end{aligned} \tag{3.53}$$

Note that $\sigma_{\theta\theta}$ does not act on the boundary. The second part is defined for the circle depicted in Fig. 3.5c in the form of a θ-dependent stress state:

$$\begin{aligned}
\sigma_{rr}^{(2)}\left(b,\theta\right) &= \frac{T_1}{2}\cos 2\theta \\
\sigma_{r\theta}^{(2)}\left(b,\theta\right) &= -\frac{T_1}{2}\sin 2\theta
\end{aligned} \tag{3.54}$$

The solution for the axisymmetric stress state (represented by Eqs. (3.53)) can be easily obtained from the solution of a circular disc under radial load (see Eq. (3.39)) as follows:

$$\begin{aligned}
\sigma_{rr} &= C_2 + C_3\frac{1}{r^2} \\
\sigma_{\theta\theta} &= C_2 - C_3\frac{1}{r^2} \\
\sigma_{r\theta} &= 0
\end{aligned} \tag{3.55}$$

We may find the constants using the following boundary conditions:

$$\begin{aligned} &\sigma_{rr}^{(1)}(b,\theta)=\frac{T_1}{2} \\ &\sigma_{r\theta}^{(1)}(b,\theta)=0 \\ &\sigma_{rr}^{(1)}(a,\theta)=0 \\ &\sigma_{r\theta}^{(1)}(a,\theta)=0 \end{aligned} \tag{3.56}$$

Accordingly, the stresses can be written as;

$$\begin{aligned} &\sigma_{rr}^{(1)}(r,\theta)=\frac{T_1}{2}\frac{b^2}{b^2-a^2}-\frac{T_1}{2}\frac{a^2b^2}{b^2-a^2}\frac{1}{r^2} \\ &\sigma_{\theta\theta}^{(1)}(r,\theta)=\frac{T_1}{2}\frac{b^2}{b^2-a^2}+\frac{T_1}{2}\frac{a^2b^2}{b^2-a^2}\frac{1}{r^2} \end{aligned} \tag{3.57}$$

In order to find the solution to the second part, we start with the general form of the biharmonic equation. Due to the periodicity of the boundary conditions we consider a proper form for the stress function and proceed as follows:

$$\begin{aligned} &\nabla^4\varphi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\varphi_{,rr}+\frac{1}{r}\varphi_{,r}+\frac{1}{r^2}\varphi_{,\theta\theta}\right)=0 \\ &\varphi=F(r)\cos 2\theta \\ &\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[\cos 2\theta\left(F_{,rr}+\frac{1}{r}F_{,r}-\frac{4}{r^2}F\right)\right]=0 \\ &\Rightarrow F_{,rrrr}+\frac{2}{r}F_{,rrr}-\frac{9}{r^2}F_{,rr}+\frac{9}{r^3}F_{,r}=0 \end{aligned} \tag{3.58}$$

The above differential equation can be solved for F and will eventually result in the following expression for our stress function:

$$\varphi=\left(C_1+C_2r^2+C_3r^4+C_4\frac{1}{r^2}\right)\cos 2\theta \tag{3.59}$$

Equation (3.59) along with equations (3.29) can be used to calculate the stresses as follows:

$$
\begin{aligned}
\sigma_{rr}^{(2)} &= -\left(4C_1\frac{1}{r^2}+2C_2+6C_4\frac{1}{r^4}\right)\cos 2\theta \\
\sigma_{\theta\theta}^{(2)} &= \left(2C_2+12C_3r^2+6C_4\frac{1}{r^4}\right)\cos 2\theta \\
\sigma_{r\theta}^{(2)} &= \left(-2C_1\frac{1}{r^2}+2C_2+6C_3r^2-6C_4\frac{1}{r^4}\right)\sin 2\theta
\end{aligned}
\tag{3.60}
$$

The boundary conditions are:

$$
\begin{aligned}
&\sigma_{rr}^{(2)}(b,\theta)=\frac{T_1}{2}\cos 2\theta \\
&\sigma_{r\theta}^{(2)}(b,\theta)=-\frac{T_1}{2}\sin 2\theta \\
&\sigma_{rr}^{(2)}(a,\theta)=0 \\
&\sigma_{r\theta}^{(2)}(a,\theta)=0
\end{aligned}
\tag{3.61}
$$

which can be rewritten as:

$$
\begin{bmatrix}
-\frac{4}{a^2} & -2 & 0 & -\frac{6}{a^4} \\
-\frac{2}{a^2} & 2 & 6a^2 & -\frac{6}{a^4} \\
-\frac{4}{b^2} & -2 & 0 & -\frac{6}{b^4} \\
-\frac{2}{b^2} & 2 & 6b^2 & -\frac{6}{b^4}
\end{bmatrix}
\begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ \frac{T_1}{2} \\ -\frac{T_1}{2} \end{Bmatrix}
\tag{3.62}
$$

The above four equations can be solved for the four unknown constants. However, we may simplify Eqs.(3.62) for the case where the hole is small compared to the width of the plate. First, we multiply the forth equation by $(a/b)^2$ and let $a/b \to 0$, which gives $C_3=0$. Next, we multiply the third equation by a^2, let $a/b \to 0$, and get $C_2=-T_1/4$. Hence, we may write the first and the second equations as:

$$
\begin{aligned}
&-\frac{4}{a^2}C_1+\frac{T_1}{2}-\frac{6}{a^4}C_4=0 \\
&-\frac{2}{a^2}C_1-\frac{T_1}{2}-\frac{6}{a^4}C_4=0
\end{aligned}
\tag{3.63}
$$

from which we find:

$$C_1 = \left(\frac{a^2}{2}\right)T_1 \text{ and } C_4 = -\left(\frac{a^4}{4}\right)T_1 \tag{3.64}$$

Finally, we obtain the stresses using Eq. (3.60):

$$\begin{aligned}
\sigma_{rr}^{(2)} &= -\left(2\frac{a^2}{r^2} - \frac{1}{2} - \frac{3}{2}\frac{a^4}{r^4}\right)T_1 \cos 2\theta \\
\sigma_{\theta\theta}^{(2)} &= -\left(\frac{1}{2} + \frac{3}{2}\frac{a^4}{r^4}\right)T_1 \cos 2\theta \\
\sigma_{r\theta}^{(2)} &= \left(-\frac{a^2}{r^2} - \frac{1}{2} + \frac{3}{2}\frac{a^4}{r^4}\right)\sin 2\theta
\end{aligned} \tag{3.65}$$

In order to obtain the total solution, we reevaluate Eq. (3.57), by dividing the numerator and denominator of each term by b^2, letting $a/b \to 0$, and then adding the results to Eq. (3.65):

$$\begin{aligned}
\sigma_{rr}(r,\theta) &= \frac{T_1}{2}\left(1 - \frac{a^2}{r^2}\right) + \frac{T_1}{2}\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta \\
\sigma_{\theta\theta}(r,\theta) &= \frac{T_1}{2}\left(1 + \frac{a^2}{r^2}\right) - \frac{T_1}{2}\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta \\
\sigma_{r\theta}(r,\theta) &= -\frac{T_1}{2}\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\sin 2\theta
\end{aligned} \tag{3.66}$$

Now we may examine the stresses at particular points that are most important. For instance, the maximum hoop stress that occurs at $r = a$ is:

$$\sigma_{\theta\theta}(a,\theta) = T_1 - 2T_1 \cos 2\theta \tag{3.67}$$

Note that its largest value occurs at $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ with the magnitude of:

$$\sigma_{\theta\theta}\left(a, \frac{\pi}{2}\right) = 3T_1 \tag{3.68}$$

which indicates that we have a *stress concentration factor* of 3 for the specified location.

Assignment 9:

Find the stress concentration factor for a small circular hole in a large sheet subjected to a remote tensile stress T_1 and a compressive stress T_2.

General Solution of Plane Problems using Complex Variable Functions

In this section we will use complex variable functions to obtain general solutions to the plane elasticity problems. Starting with the biharmonic equation we can show:

$$
\begin{aligned}
&\nabla'^4\varphi = 0 \\
&\nabla'^4 = \nabla'^2\nabla'^2 = (\ )_{,\alpha\alpha\beta\beta} = \frac{\partial^4}{\partial x_1^4} + 2\frac{\partial^4}{\partial x_1^2 x_2^2} + \frac{\partial^4}{\partial x_2^4} \\
&\Rightarrow \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2}\right) = 0 \\
&\Rightarrow \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\left(\sigma_{22} + \sigma_{11}\right) = 0 \\
&\Rightarrow \left(\frac{\partial^2 \mathrm{P}}{\partial x_1^2} + \frac{\partial^2 \mathrm{P}}{\partial x_2^2}\right) = \nabla'^2\mathrm{P} = \mathrm{P}_{,\alpha\alpha} = 0
\end{aligned}
\tag{3.69}
$$

The conclusion is that that P is harmonic. Now we assume that there is a complex variable function which has P as its real part:

$$f(z) = \mathrm{P} + iQ \tag{3.70}$$

Since $f(z)$ is analytic, its integral is also be analytic. Accordingly, we may write:

$$
\begin{aligned}
&\int f(z)dz = \int \left(\mathrm{P} + iQ\right)dz = \text{ lets say } 4\Omega(z) \\
&\Omega(z) = p + iq \\
&\Rightarrow \Omega(z) = \frac{1}{4}\int f(z)dz \\
&\Omega_{,1} = \frac{d\Omega(z)}{dz} z_{,1} = \Omega'(z) \\
&\Rightarrow \Omega'(z) = p_{,1} + iq_{,1} = \frac{1}{4} f(z) \\
&\Rightarrow p_{,1} + iq_{,1} = \frac{1}{4}\left(\mathrm{P} + iQ\right) \quad \Rightarrow \quad p_{,1} = \frac{\mathrm{P}}{4} \ , \ q_{,1} = \frac{Q}{4}
\end{aligned}
\tag{3.71}
$$

Invoking the Cauchy-Riemann conditions we will have:

$$\begin{cases} p_{,1} = q_{,2} = \dfrac{\mathrm{P}}{4} \quad \text{(i)} \\ p_{,2} = -q_{,1} = -\dfrac{Q}{4} \end{cases}$$
$$\text{(i)} \Rightarrow \nabla'^2\varphi = \mathrm{P} = 4p_{,1} = 2p_{,1} + 2q_{,2}$$
$$\Rightarrow \nabla'^2\varphi - 2p_{,1} - 2q_{,2} = 0$$
$$\Rightarrow \nabla'^2\left(\varphi - x_1 p - x_2 q\right) = 0 \tag{3.72}$$

Note that $\left(\varphi - x_1 p - x_2 q\right)$ is a solution of the Laplace's equation, so we may assume it to be the real part of a complex variable function:

$$\left(\varphi - x_1 p - x_2 q\right) = p_1 \quad \Rightarrow \quad \varphi = x_1 p + x_2 q + p_1 \tag{3.73}$$

where we have:

$$\omega(z) = p_1 + iq_1 \tag{3.74}$$

Moreover, $\varphi = x_1 p + x_2 q + p_1$ may be assumed to be the real part of the following analytic function:

$$\left(x_1 - ix_2\right)\left(p + iq\right) + p_1 + iq_1 \quad \text{or}$$
$$\varphi = x_1 p + x_2 q + p_1 = \Re e\left[\left(x_1 - ix_2\right)\left(p + iq\right) + p_1 + iq_1\right] \tag{3.75}$$

Finally, we may write the general form of the Muskhelishvili stress function in terms of two complex potentials Ω and ω as follows:

$$\varphi = \Re e\left[\bar{z}\Omega(z) + \omega(z)\right] \tag{3.76}$$

Plane Stress Formulation

Starting with the constitutive equations for plane stress we have:

$$\bar{\varepsilon}_{\alpha\beta} = -\frac{\nu}{E}\bar{\sigma}_{\gamma\gamma}\delta_{\alpha\beta} + \frac{1+\nu}{E}\bar{\sigma}_{\alpha\beta} \tag{3.77}$$

which may be rewritten as:

$$
\begin{aligned}
&Eu_{1,1} = \sigma_{11} - \nu\sigma_{22} \qquad \text{(i)}\\
&Eu_{2,2} = \sigma_{22} - \nu\sigma_{11} \qquad \text{(ii)}\\
&\mathrm{G}\left(u_{1,2} + u_{2,1}\right) = \sigma_{12} \qquad \text{(iii)}
\end{aligned}
\tag{3.78}
$$

Now, using Eqs. (3.72) we may write:

$$
\begin{aligned}
&\begin{cases} Eu_{1,1} = -(1+\nu)\varphi_{,11} + \mathrm{P} \\ Eu_{2,2} = -(1+\nu)\varphi_{,22} + \mathrm{P} \end{cases}\\
&\mathrm{P} = 4p_{,1} = 4q_{,2}\\
&\Rightarrow \begin{cases} 2Gu_{1,1} = -\varphi_{,11} + \left(\dfrac{4}{1+\nu}\right)p_{,1} \\[2ex] 2Gu_{2,2} = -\varphi_{,22} + \left(\dfrac{4}{1+\nu}\right)q_{,2} \end{cases}
\end{aligned}
\tag{3.79}
$$

The above equations can be integrated to give the displacements:

$$
\begin{cases} 2Gu_{1} = -\varphi_{,1} + \left(\dfrac{4}{1+\nu}\right)p + f(x_2) \\[2ex] 2Gu_{2} = -\varphi_{,2} + \left(\dfrac{4}{1+\nu}\right)q + f(x_1) \end{cases}
\tag{3.80}
$$

If we substitute the above equations into Eq. (3.78iii), it can be shown that $f(x_2)$ and $f(x_1)$ are constants. Now we may substitute for the φ with our general complex variable stress function (Eq. (3.76)) but first we note:

$$
\begin{aligned}
&\begin{cases} f(z) = \alpha + i\beta \\ \overline{f(z)} = \alpha - i\beta \end{cases} \Rightarrow f(z) + \overline{f(z)} = 2\alpha = 2\Re e\left(f(z)\right)\\
&\text{so from (3.76)} \Rightarrow \bar{z}\Omega(z) + \omega(z) + z\overline{\Omega(z)} + \overline{\omega(z)} = 2\varphi
\end{aligned}
\tag{3.81}
$$

If we substitute the above expression for the φ into Eq. (3.80), we will obtain the following general expression for the displacements in terms of the two complex potentials Ω and ω:

$$
2G(u_1 + iu_2) = \frac{3-\nu}{1+\nu}\Omega(z) - z\overline{\Omega'(z)} - \overline{\omega'(z)}
\tag{3.82}
$$

It should be noted that we can obtain the above equation for the plane strain condition by replacing ν with $\dfrac{\nu}{1-\nu}$.

Having found the displacements, the general expressions for the stresses can be easily obtained as follows:

$$\begin{aligned}
&\sigma_{11}+\sigma_{22}=4\Re e\left(\Omega'(z)\right) \\
&\sigma_{22}-\sigma_{11}-2i\sigma_{12}=2\left[z\overline{\Omega''(z)}+\overline{\omega''(z)}\right] \quad \text{or} \\
&\sigma_{22}-\sigma_{11}+2i\sigma_{12}=2\left[\bar{z}\Omega''(z)+\omega''(z)\right]
\end{aligned} \tag{3.83}$$

The final conclusion is that the solution to a particular two-dimensional elasticity problem can be easily obtained if we can find the appropriate complex variable potentials for that problem. However, finding the required potentials is not usually that easy!

The Mode I Crack Problem

As an example for the solution of plane problems using complex variable potentials, we consider the Mode I crack problem. Here, we may assume the displacement field as $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$, and $u_3 = 0$.

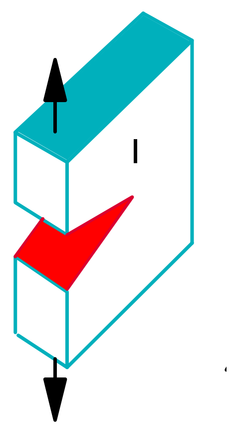

Fig. 3.6

We directly start from the general solution with the following form of Eq. (3.83):

$$\begin{aligned}
&\sigma_{11}+\sigma_{22}=4\Re e\left(\Omega'(z)\right)=2\left[\Omega'(z)+\overline{\Omega'(z)}\right] \\
&\sigma_{22}-\sigma_{11}-2i\sigma_{12}=2\left[z\overline{\Omega''(z)}+\overline{\omega''(z)}\right]
\end{aligned} \tag{3.84}$$

Due to symmetry with respect to the crack plane we choose a solution of the form:

$$\Omega = Az^{\lambda+1}, \quad \omega = Bz^{\lambda+1} \tag{3.85}$$

where *A, B,* and λ are real constants. In order to have nonsingular displacements at the crack tip we must have $\lambda > -1$. The substitution of Eq. (3.85) into Eq. (3.84) yields:

$$\begin{aligned}\sigma_{22} - i\sigma_{12} = (\lambda+1)r^{\lambda}\{A[2\cos\lambda\theta + \lambda\cos(\lambda-2)\theta] \\ +B\cos\lambda\theta - i[A\lambda\sin(\lambda-2)\theta + B\sin\lambda\theta]\}\end{aligned} \tag{3.86}$$

The boundary conditions at the crack tip dictate that the stresses in Eq. (3.86) vanish for $\theta = \pm\pi$. Consequently we have:

$$\begin{aligned}A(2+\lambda)\cos\lambda\pi + B\cos\lambda\pi = 0 \\ A\lambda\sin\lambda\pi + B\sin\lambda\pi = 0\end{aligned} \tag{3.87}$$

for which a nontrivial solution exists if:

$$\sin 2\lambda\pi = 0 \tag{3.88}$$

which for $\lambda > -1$ has the following roots:

$$\lambda = -\frac{1}{2}, n/2, \quad n = 0,1,2,... \tag{3.89}$$

The dominant contribution to the crack-tip stress and displacement fields occurs for $\lambda = -1/2$ for which $A = 2B$. An inverse square root singularity in the stress field exists at the crack tip. Substituting Eq. (3.87) with $A = 2B$ and $\lambda = -1/2$ into Eqs. (3.86) and (3.82), we can show that:

$$\begin{Bmatrix}\sigma_{11}\\ \sigma_{12}\\ \sigma_{22}\end{Bmatrix} = \frac{K_I}{(2\pi r)^{1/2}}\cos(\theta/2)\begin{Bmatrix}1-\sin(\theta/2)\sin(3\theta/2)\\ \sin(\theta/2)\cos(\theta/2)\\ 1+\sin(\theta/2)\sin(3\theta/2)\end{Bmatrix} \tag{3.90}$$

and also:

$$\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \frac{K_I}{2\mu}\left(\frac{r}{2\pi}\right)^{\frac{1}{2}}\begin{Bmatrix}\cos(\theta/2)\left[\kappa-1+2\sin^2(\theta/2)\right]\\ \sin(\theta/2)\left[\kappa+1-2\cos^2(\theta/2)\right]\end{Bmatrix} \tag{3.91}$$

The Mode I stress intensity factor *K,* which can be obtained using the global boundary conditions of the problem, is defined by:

$$K_I = \lim_{r\to 0}\left\{(2\pi r)^{1/2}\sigma_{22}\big|_{\theta=0}\right\} \tag{3.92}$$

General Solutions in Curvilinear coordinates

For many two-dimensional elasticity problems the solution procedure requires that we define our previously derived general solutions using *curvilinear coordinates*. In general, we may specify the position of a point in a plane as the intersection of two curves. This constitutes the curvilinear coordinates. The two curves may be represented by:

$$\begin{cases} F_1(x_1,x_2)=\xi \\ F_2(x_1,x_2)=\eta \end{cases} \tag{3.93}$$

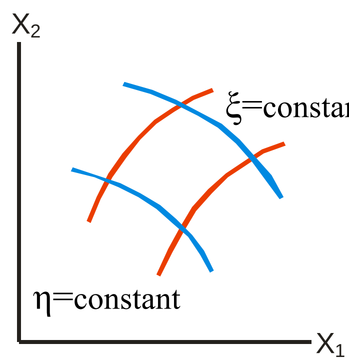

Fig. 3.7

For instance, the usual polar coordinates can be considered as a form of curvilinear coordinates which specify the position of our point as an intersection of a radial line, at the angle θ from the initial line, with a circle of radius r. Then, we have:

$$\begin{cases} F_1(x_1,x_2)=\sqrt{x_1^2+x_2^2}=r \\ F_2(x_1,x_2)=\arctan\left(\dfrac{x_2}{x_1}\right)=\theta \end{cases} \tag{3.94}$$

We may also solve Eqs. (3.93) for x_1, x_2 and obtain:

$$\begin{cases} x_1=f_1(\xi,\eta) \\ x_2=f_2(\xi,\eta) \end{cases} \tag{3.95}$$

By choosing different functions for f_1 and f_2 we can find various curvilinear coordinates. Let us assume:

$$\begin{cases} x_1=f_1(\xi,\eta)=C\cosh\xi\cos\eta \\ x_2=f_2(\xi,\eta)=C\sinh\xi\sin\eta \end{cases} \tag{3.96}$$

If we eliminate η between the above two equations, we can show that:

$$\frac{x_1^2}{C^2\cosh^2\xi}+\frac{x_2^2}{C^2\sinh^2\xi}=1 \tag{3.97}$$

which for different values of ξ gives different ellipses with the same foci, i.e., a family of confocal ellipses. Alternatively, if we eliminate ξ between the two equations, we will have:

$$\frac{x_1^2}{C^2\cos^2\eta}-\frac{x_2^2}{C^2\sin^2\eta}=1 \tag{3.98}$$

which for different values of η gives different hyprbolas with the same foci, or a family of confocal hyperbolas.

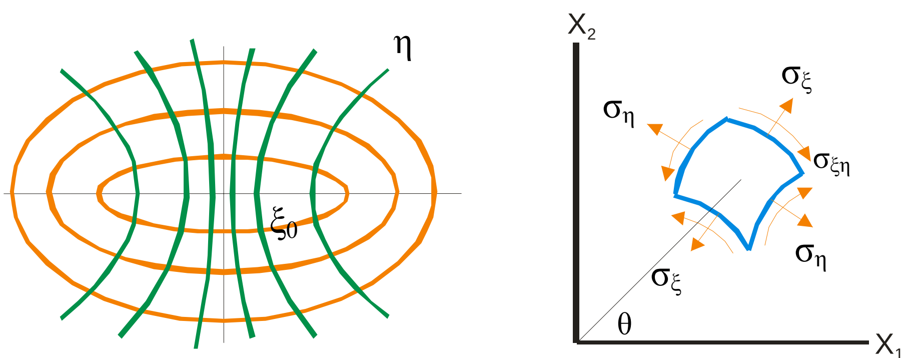

Fig. 3.8

We may also define the complex variable $\zeta=\xi+i\eta$ in terms of our curvilinear coordinates by writing the following equalities using Eq. (3.96):

$$\begin{aligned}&x_1+ix_2=C\cosh(\xi+i\eta)\\&\Rightarrow z=C\cosh\zeta\\&\Rightarrow z=f(\zeta)\end{aligned} \tag{3.99}$$

Now we express the stresses in the curvilinear coordinates in terms of the stresses in the cartesian coordinates by a simple transformation caused by a rotation of θ degrees:

$$\begin{aligned}
\sigma_{\xi} &= \frac{\sigma_{11}+\sigma_{22}}{2}+\frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta+\sigma_{12}\sin 2\theta \\
\sigma_{\eta} &= \frac{\sigma_{11}+\sigma_{22}}{2}-\frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta-\sigma_{12}\sin 2\theta \\
\sigma_{\xi\eta} &= \sigma_{12}\cos 2\theta-\frac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta
\end{aligned} \tag{3.100}$$

which leads to:

$$\begin{aligned}
&\sigma_{\xi}+\sigma_{\eta}=\sigma_{11}+\sigma_{22} \\
&\sigma_{\eta}-\sigma_{\xi}+2i\sigma_{\xi\eta}=e^{2i\theta}\left(\sigma_{22}-\sigma_{11}+2i\sigma_{12}\right)
\end{aligned} \tag{3.101}$$

Finally, we use Eqs. (3.83) and (3.82) and write the general expressions for the stresses and displacements in terms of complex potentials in the curvilinear coordinates as:

$$\begin{aligned}
&\sigma_{\xi}+\sigma_{\eta}=4\Re e\left(\Omega'(z)\right) \\
&\sigma_{\eta}-\sigma_{\xi}+2i\sigma_{\xi\eta}=2e^{2i\theta}\left[\bar{z}\Omega''(z)+\omega''(z)\right] \\
&2G(u_{\xi}-iu_{\eta})=e^{i\theta}\left[\frac{3-\nu}{1+\nu}\overline{\Omega(z)}-\bar{z}\Omega'(z)-\omega'(z)\right]
\end{aligned} \tag{3.102}$$

In order to find the factor $e^{2i\theta}$, we write:

$$\begin{aligned}
&f'(\zeta)=\frac{df(\zeta)}{d\zeta} \\
&f'(\zeta)=j\cos\theta+ij\sin\theta=je^{i\theta} \\
&\Rightarrow \overline{f'(\zeta)}=je^{-i\theta} \\
&e^{2i\theta}=\frac{f'(\zeta)}{\overline{f'(\zeta)}}
\end{aligned} \tag{3.103}$$

For example, for the elliptic coordinates we can show that:

$$e^{2i\theta}=\frac{\sinh\zeta}{\sinh\bar{\zeta}} \tag{3.104}$$

Uniformly Stressed Sheet with an Elliptic Hole

Consider a large sheet, containing an elliptic hole with semiaxes, a and b, subjected to a far-field uniform stress σ_0 as depicted in Fig. 3.9. We may define our elliptic coordinates as:

$$z = C\cosh\zeta \quad ; \quad \zeta = \xi + i\eta \quad ; \quad \frac{dz}{d\zeta} = C\sinh\zeta$$
$$\begin{cases} x_1 = C\cosh\xi\cos\eta \\ x_2 = C\sinh\xi\sin\eta \end{cases} \quad ; \quad \begin{cases} C\cosh\xi_0 = a \\ C\sinh\xi_0 = b \end{cases} \tag{3.105}$$

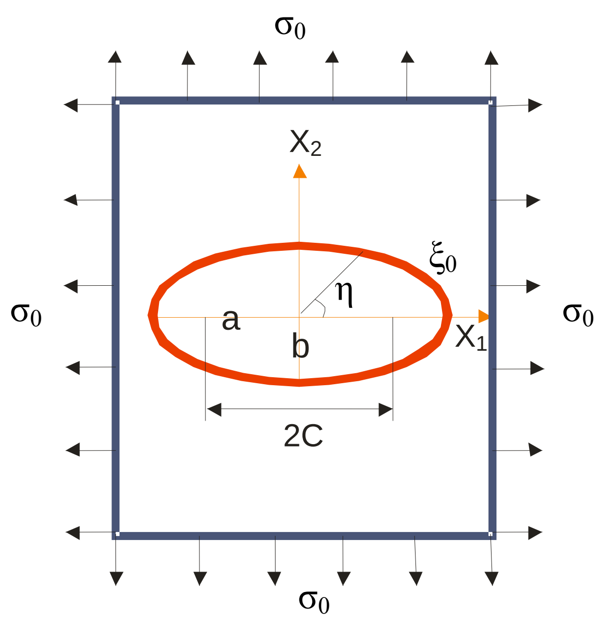

Fig. 3.9

For this case, it is appropriate to use the general expressions for the stresses and displacements in terms of complex potentials in curvilinear coordinates (Eqs. (3.102)) with the following boundary conditions:

$$x_1, x_2 \to \infty \; (\xi \to \infty) \Rightarrow \sigma_{11} = \sigma_{22} = \sigma_0 \quad \text{(i)}$$
$$\xi = \xi_0 \Rightarrow \sigma_\xi = \sigma_{\xi\eta} = 0 \quad \text{(ii)} \tag{3.106}$$

From Eqs. (3.102) we have:

$$\begin{aligned} &\sigma_\xi + \sigma_\eta = 4\Re e\left(\Omega'(z)\right) && \text{(i)} \\ &\sigma_\eta - \sigma_\xi + 2i\sigma_{\xi\eta} = 2e^{2i\theta}\left[\bar{z}\Omega''(z) + \omega''(z)\right] && \text{(ii)} \end{aligned} \tag{3.107}$$

which, in combination with our first boundary condition, indicates that we must have:

$$\begin{cases} \sigma_\xi + \sigma_\eta = 4\Re e\left(\Omega'(z)\right) = 2\sigma_0 \;\Rightarrow\; \Re e\left(\Omega'(z)\right) = \dfrac{\sigma_0}{2} \\ \bar{z}\Omega''(z) + \omega''(z) = 0 \end{cases} \quad \text{at infinity} \tag{3.108}$$

In order to find the appropriate complex potentials, we note that they should conform to our elliptic boundary and be periodic in θ. Hence, we assume the following potentials and check if our boundary conditions are satisfied:

$$\begin{aligned} &\begin{cases} \Omega(z) = AC\sinh\zeta \\ \omega(z) = BC^2\zeta \end{cases} \\ &\Omega'(z) = AC\cosh\zeta\frac{d\zeta}{dz} = A\frac{\cosh\zeta}{\sinh\zeta} = A\coth\zeta \\ &\xi \to \infty \Rightarrow \coth\zeta \to 1 \Rightarrow \Re e\Omega'(z) = A = \frac{\sigma_0}{2} \\ &\omega'(z) = \frac{BC}{\sinh\zeta} \Rightarrow \omega''(z) = -B\frac{\cosh\zeta}{\sinh^3\zeta} \\ &\Omega''(z) = -\frac{A}{C}\frac{1}{\sinh^3\zeta} \\ &\xi \to \infty \Rightarrow \zeta \to \infty \Rightarrow \bar{z}\Omega''(z) + \omega''(z) = 0 \end{aligned} \tag{3.109}$$

It is clear that with $A = \dfrac{\sigma_0}{2}$ the far field boundary conditions are satisfied. In order to find B, we proceed with the local boundary conditions. From Eq. (3.107) we have:

$$\begin{aligned} &\text{(i)} - \text{(ii)} \Rightarrow \sigma_\xi - i\sigma_{\xi\eta} = 2\Re e\left(\Omega'(z)\right) - e^{2i\theta}\left[\bar{z}\Omega''(z) + \omega''(z)\right] \\ &\Rightarrow \sigma_\xi - i\sigma_{\xi\eta} = \Omega'(z) + \bar{\Omega}'(\bar{z}) - e^{2i\theta}\left[\bar{z}\Omega''(z) + \omega''(z)\right] \end{aligned} \tag{3.110}$$

where $e^{2i\theta}$ can be found from Eq. (3.104) as:

$$e^{2i\theta} = \frac{\sinh\zeta}{\sinh\bar{\zeta}} \tag{3.111}$$

Accordingly, Eq. (3.110) becomes:

$$
\begin{aligned}
\sigma_\xi - i\sigma_{\xi\eta} &= A\left(\frac{\cosh\zeta}{\sinh\zeta} + \frac{\cosh\bar{\zeta}}{\sinh\bar{\zeta}}\right) + \frac{\sinh\zeta}{\sinh\bar{\zeta}}\left(A\frac{\cosh\bar{\zeta}}{\sinh^3\zeta} + B\frac{\cosh\zeta}{\sinh^3\zeta}\right) \\
&= \frac{1}{\sinh^2\zeta\sinh\bar{\zeta}}\left\{A\left[\sinh\zeta\sinh\left(\zeta+\bar{\zeta}\right) + \cosh\bar{\zeta}\right] + B\cosh\zeta\right\}
\end{aligned}
\tag{3.112}
$$

which may be written at the boundary of the ellipse as:

$$
\begin{gathered}
\begin{cases} \xi = \xi_0 \\ \zeta + \bar{\zeta} = 2\xi_0 \Rightarrow \bar{\zeta} = 2\xi_0 - \zeta \end{cases} \\
\sigma_\xi - i\sigma_{\xi\eta} = \frac{1}{\sinh^2\zeta\sinh\bar{\zeta}}\left(A\cosh 2\xi_0 + B\right)\cosh\zeta
\end{gathered}
\tag{3.113}
$$

Invoking the second boundary condition in Eq. (3.106ii) we have:

$$
\begin{gathered}
B = -A\cosh 2\xi_0 = -\frac{1}{2}\sigma_0\cosh 2\xi_0 \\
\Rightarrow \begin{cases} \Omega(z) = \frac{1}{2}\sigma_0 C\sinh\zeta \\ \omega(z) = \left(-\frac{1}{2}\sigma_0\cosh 2\xi_0\right)C^2\zeta \end{cases}
\end{gathered}
\tag{3.114}
$$

Although all the boundary conditions are now satisfied, we must make sure that there would be no discontinuity in the resulting displacement field. Hence, we use Eq. (3.102) to obtain the Cartesian components of displacement as:

$$
2G(u+iv) = \frac{3-\nu}{1+\nu}AC\sinh\zeta - AC\cosh\zeta\coth\bar{\zeta} - \frac{BC}{\sinh\bar{\zeta}}
\tag{3.115}
$$

which shows that displacements are periodic in η and there would be no discontinuity in displacements. The most interesting component of the stress is in fact the largest value of σ_η at the hole and can be obtained from Eq. (3.107) as follows:

$$
\begin{gathered}
\sigma_\xi + \sigma_\eta = 4\Re e\left(\Omega'(z)\right) = 2\sigma_0\Re e\coth\zeta \\
\coth\zeta = \frac{\sinh 2\xi - i\sinh 2\eta}{\cosh 2\xi - \cos 2\eta} \quad \Rightarrow \sigma_\xi + \sigma_\eta = \frac{2\sigma_0\sinh 2\xi}{\cosh 2\xi - \cos 2\eta} \\
\Rightarrow \left(\sigma_\eta\right)_{\xi=\xi_0} = \frac{2\sigma_0\sinh 2\xi_0}{\cosh 2\xi_0 - \cos 2\eta} \\
\Rightarrow \left(\sigma_\eta\right)_{\eta=0,\pi}^{\max} = \frac{2\sigma_0\sinh 2\xi_0}{\cosh 2\xi_0 - 1}
\end{gathered}
\tag{3.116}
$$

Using Eq. (3.105) we may show that:

$$C^2 = a^2 - b^2$$
$$\sinh 2\xi_0 = \frac{2ab}{C^2} \tag{3.117}$$
$$\cosh 2\xi_0 = \frac{a^2 + b^2}{C^2}$$

which, in combination with Eq. (3.116), results in:

$$\left(\sigma_\eta\right)_{\eta=0,\pi}^{\max} = 2\sigma_0 \frac{a}{b}$$
$$\left(\sigma_\eta\right)_{\eta=\frac{\pi}{2},\frac{3\pi}{2}}^{\min} = 2\sigma_0 \frac{b}{a} \tag{3.118}$$

Eqs. (3.118) show that as $b \to 0$ (as the ellipse becomes a crack) a stress singularity develops at the crack tip. For a=b, the ellipse becomes a circle with a stress concentration factor of two, which is in agreement with the results we already obtained for the *sheet with a circular hole*.

Assignment 10:

Find the maximum stresses for an elliptic hole in a large sheet subjected to remote tensile stress T_1. The major axis of the elliptic hole makes an angle β with the x_1 direction.

Chapter 4

Linear Thermoelasticity

Temperature change can often be a source of significant stresses in elastic bodies. In general, we should define a *reference* temperature at which we consider the body to be free of thermal stresses. As the temperature is raised or lowered with respect to the reference temperature the body expands or contracts and thermal stresses may arise.

It should be note that for a body which *is not constrained*, a uniform temperature change with respect to the reference temperature will not cause thermal stresses. However, this is not the case for a *constrained* body, where the thermal stresses are developed even if the temperature changes are uniform throughout the body. In either case, the existence of temperature *gradients* within an elastic body will result in the development of thermal stresses which are generally tensile at locations of lower temperatures and compressive at locations of higher temperatures. Although later we will examine some special cases where the thermal stresses can be absent also for bodies under temperature gradients.

Usually, the solution of the thermoelasticity problem begins with finding the time and position dependent temperature field. We may write the general form of the heat conduction equation as:

$$\frac{\partial \Theta}{\partial t} = a\Theta_{,ii} + \frac{S}{\rho C_v} \quad ; \quad a = \frac{k}{\rho C_v} \tag{4.1}$$

in which Θ is the temperature field , S is the strength of the heat sources in the field , ρ is the density, C_v is the specific heat, and k is the thermal conductivity. For a stationary field with no heat sources we have:

$$\Theta_{,ii} = 0 \tag{4.2}$$

Since we are dealing with linear thermoelasticity, the strain tensor is defined by:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{4.3}$$

and also the equilibrium equations are:

$$\sigma_{ij,j} + f_i = 0 \tag{4.4}$$

However, we may split the total strain tensor into the mechanical and thermal parts as:

$$\varepsilon_{ij} = \varepsilon_{ij}{}^{m} + \varepsilon_{ij}{}^{T} \tag{4.5}$$

For the mechanical stresses we have:

$$\begin{aligned} &\varepsilon_{ij}{}^{m} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \\ &G = \frac{E}{2(1+\nu)} \\ &\Rightarrow \varepsilon_{ij}{}^{m} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right] \end{aligned} \tag{4.6}$$

In order to obtain an expression for the thermal strains we may write:

$$dS_T = (1+\alpha\Theta)\, dS_0 \tag{4.7}$$

Here dS_0 is the initial length of a line element, dS_T is its length as a result of a temperature change Θ, and α is the coefficient of thermal expansion. Accordingly, the thermal strains are:

$$\varepsilon_{ij}{}^{T} = \alpha\Theta\delta_{ij} \tag{4.8}$$

We can combine Eqs. (4.6) and (4.8) to obtain the total strains, resulting in the so-called Duhamel-Neumann Constitutive equations:

$$\varepsilon_{ij} = \varepsilon_{ij}{}^{m} + \varepsilon_{ij}{}^{T} = \frac{1}{2G}\left[\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right] + \alpha\Theta\delta_{ij} \tag{4.9}$$

In order to find the stresses in terms of the strains, we first contract on *i* and *j* indices to obtain:

$$\begin{aligned} &\varepsilon_{ll} = \frac{1}{2G}\left(\frac{1-2\nu}{1+\nu}\right)\sigma_{ll} + 3\alpha\Theta \\ &\Rightarrow \sigma_{ll} = \frac{2G(1+\nu)}{1-2\nu}\left(\varepsilon_{ll} - 3\alpha\Theta\right) \end{aligned} \tag{4.10}$$

Substituting back into Eq. (4.9) we have:

$$\begin{aligned}\varepsilon_{ij} &= \frac{1}{2G}\left[\sigma_{ij} - \frac{2G\nu}{1-2\nu}\left(\varepsilon_{ll} - 3\alpha\Theta\right)\delta_{ij}\right] + \alpha\Theta\delta_{ij} \\ &= \frac{1}{2G}\sigma_{ij} - \frac{\nu}{1-2\nu}\varepsilon_{ll}\delta_{ij} + \left[\frac{\nu+1}{1-2\nu}\right]\alpha\Theta\delta_{ij}\end{aligned} \tag{4.11}$$

Hence, the stresses are:

$$\sigma_{ij} = 2G\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{ll}\delta_{ij}\right] - \frac{2G\left(1+\nu\right)}{1-2\nu}\alpha\Theta\delta_{ij} \tag{4.12}$$

Based on our new set of constitutive equations and implementing the procedure we followed before, we may obtain a *generalized* form for the Navier displacement equations as follows:

$$u_{i,jj} + \frac{1}{1-2\nu}u_{j,ij} + \frac{1}{G}f_i = \frac{2\left(1+\nu\right)}{\left(1-2\nu\right)}\alpha\Theta_{,i} \tag{4.13}$$

We may also derive a *generalized* form for the Beltrami-Michell equations:

$$\begin{aligned}\left(1+\nu\right)\sigma_{ij,kk} + \sigma_{kk,ij} = &-\left(1+\nu\right)\left(f_{i,j} + f_{j,i} + \frac{\nu}{1-\nu}f_{k,k}\delta_{ij}\right) \\ &-2G\alpha\left(1+\nu\right)\left(\Theta_{,ij} + \frac{1+\nu}{1-\nu}\Theta_{,kk}\delta_{ij}\right)\end{aligned} \tag{4.14}$$

A Thin Circular Disc Problem

As a first example in solving thermoelasticity problems, we will find the thermal stresses in a thin circular disc. We assume that the temperature field is symmetrical about the center of the disc and does not vary over the thickness. For this case the equilibrium equations simplify to:

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \tag{4.15}$$

We now repeat the procedure implemented above for determination of the thermoelastic constitutive equations for our specific problem. Starting with the strains we write:

$$\begin{aligned}\varepsilon_{rr} - \alpha\Theta &= \frac{1}{E}\left(\sigma_{rr} - \nu\sigma_{\theta\theta}\right) \\ \varepsilon_{\theta\theta} - \alpha\Theta &= \frac{1}{E}\left(\sigma_{\theta\theta} - \nu\sigma_{rr}\right)\end{aligned} \tag{4.16}$$

which can be written in terms of the stresses as:

$$\sigma_{rr} = \frac{E}{1-\nu^2}\left[\varepsilon_{rr} + \nu\varepsilon_{\theta\theta} - \left(1+\nu\right)\alpha\Theta\right]$$
$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[\varepsilon_{\theta\theta} + \nu\varepsilon_{rr} - \left(1+\nu\right)\alpha\Theta\right] \tag{4.17}$$

Substituting the above expressions into our equilibrium equation (Eq. (4.15)) we have:

$$r\frac{d}{dr}\left(\varepsilon_{rr} + \nu\varepsilon_{\theta\theta}\right) + \left(1-\nu\right)\left(\varepsilon_{rr} - \varepsilon_{\theta\theta}\right) = \left(1+\nu\right)\alpha r\frac{d\Theta}{dr} \tag{4.18}$$

which can be rewritten in terms of displacements to give:

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = \left(1+\nu\right)\alpha\frac{d\Theta}{dr}$$
$$\Rightarrow \frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right] = \left(1+\nu\right)\alpha\frac{d\Theta}{dr} \tag{4.19}$$

This is in fact nothing but the simplified form of the *generalized* Navier equations for the current problem. The above differential equation can be solved for the radial displacement to give:

$$u_r = \left(1+\nu\right)\alpha\frac{1}{r}\int_a^r \Theta r dr + C_1 r + \frac{C_2}{r} \tag{4.20}$$

Accordingly, the stresses become:

$$\sigma_{rr} = -\alpha E\frac{1}{r^2}\int_a^r \Theta r dr + \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right) - C_2\left(1-\nu\right)\frac{1}{r^2}\right]$$
$$\sigma_{\theta\theta} = \alpha E\frac{1}{r^2}\int_a^r \Theta r dr - \alpha E\Theta + \frac{E}{1-\nu^2}\left[C_1\left(1+\nu\right) - C_2\left(1-\nu\right)\frac{1}{r^2}\right] \tag{4.21}$$

The constants can be found from the boundary conditions. For a solid disc, "*a*" can be taken as zero and we also have:

$$\lim_{r\to 0}\frac{1}{r}\int_0^r \Theta r dr = 0 \tag{4.22}$$

Note that Eq. (4.20) implies that we must have $C_2 = 0$ to ensure that the displacements at the center of the disc have finite values. Moreover, at the outer edge of the disc we have $\sigma_{rr} = 0$, so we may write:

$$C_1 = (1-\nu)\frac{\alpha}{b^2}\int_0^b \Theta r dr \tag{4.23}$$

Finally, the stresses can be shown to be:

$$\begin{aligned} \sigma_{rr} &= \alpha E\left(\frac{1}{b^2}\int_0^b \Theta r dr - \frac{1}{r^2}\int_0^r \Theta r dr\right) \\ \sigma_{\theta\theta} &= \alpha E\left(-\Theta + \frac{1}{b^2}\int_0^b \Theta r dr + \frac{1}{r^2}\int_0^r \Theta r dr\right) \end{aligned} \tag{4.24}$$

Assignment 11:

Find the thermal stresses for a hollow sphere for which the inner and outer surfaces are kept at constant temperatures T_i and T_o respectively.

Instantaneous Point Source in an Infinite Body

In this example we consider a quantity of heat Q deposited at time t = 0 at $X_i = 0$ inside an infinite linear elastic body. The heat will gradually spread into the body, and every point will experience an increase and later decrease in temperature with time. Eventually, the body will return to its initial temperature after a long period of time. In order to obtain the resulting thermal stresses, we must first find the time and position dependent temperature field. The problem can be treated as spherically symmetric, so we may start with the proper form of the equation of heat conduction in spherical coordinates:

$$\frac{\partial\Theta}{\partial t} = a\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Theta}{\partial r}\right) \tag{4.25}$$

Defining the Laplace transform as:

$$\hat{\Theta}(s) = \int_0^\infty \Theta(t)e^{-st}dt \tag{4.26}$$

we have:

$$£\left(\frac{\partial\Theta}{\partial \text{t}}\right) = s\hat{\Theta} \tag{4.27}$$

The Laplace transform of Eq. (4.25) results in:

$$s\hat{\Theta} = \frac{a}{r^2}\frac{d}{dr}\left(r^2\frac{d\hat{\Theta}}{dr}\right) \tag{4.28}$$

The solution to the above differential equation can be shown to be:

$$\hat{\Theta} = \frac{A}{r}e^{-\left(\sqrt{\frac{s}{a}}r\right)} + \frac{B}{r}e^{\left(\sqrt{\frac{s}{a}}r\right)} \tag{4.29}$$

Note that we should set $B = 0$ to ensure that $\hat{\Theta}$ remains bounded as $r \to \infty$.
In order to find the constant *A,* we write:

$$Q = \int_0^\infty 4\pi r^2 \rho C_v \Theta dr \tag{4.30}$$

We also define Q using the Heaviside step function as:

$$Q = QH(t) \tag{4.31}$$

Next, we take the Laplace transform of both sides of Eq. (4.30) and proceed as follows:

$$\begin{aligned} Q\frac{1}{s} &= \int_0^\infty 4\pi r^2 \rho C_v \hat{\Theta} dr \\ &= 4\pi\rho C_v A\int_0^\infty e^{-\left(\sqrt{\frac{s}{a}}r\right)} r dr \\ &= 4\pi\rho C_v A\frac{a}{s} \end{aligned} \tag{4.32}$$

Accordingly, we find the constant *A* as:

$$A = \frac{Q}{4\pi\rho C_v a} \tag{4.33}$$

Finally, we find the time and position dependent temperature field for our problem as follows:

$$\begin{aligned} \hat{\Theta} &= \frac{Q}{4\pi\rho C_v a r}e^{-\left(\sqrt{\frac{s}{a}}r\right)} \\ \Theta &= \frac{Q}{(4\pi a t)^{3/2}\rho C_v}e^{-\left(\frac{r^2}{4at}\right)} \end{aligned} \tag{4.34}$$

The next step in solving our thermoelasticity problem is the determination of the displacement field. We start with the generalized form of the Navier equations (Eqs. (4.13)):

$$u_{i,jj} + \frac{1}{1-2\nu} u_{j,ij} + \frac{1}{G} f_i = \frac{2(1+\nu)}{(1-2\nu)} \alpha \Theta_{,i} \tag{4.35}$$

The solution to the above equation consists of the general solution u_i^h plus the particular integral u_i^p. Now, we may assume that the particular solution can be derived from the thermoelastic potential Φ as follows:

$$u_i^p = \Phi_{,i} \tag{4.36}$$

If we substitute the above expression into the Generalized Navier equations, we will have:

$$\begin{aligned} &\left[\Phi_{,jj} + \frac{1}{1-2\nu} \Phi_{,jj} - \frac{2(1+\nu)}{1-2\nu} \alpha \Theta \right]_{,i} = 0 \\ &\Rightarrow \Phi_{,jj} = \nabla^2 \Phi = \frac{(1+\nu)}{(1-\nu)} \alpha \Theta \end{aligned} \tag{4.37}$$

Assuming a non-stationary source-free temperature distribution, a solution for the above equation can be written as:

$$\Phi = \frac{(1+\nu)}{(1-\nu)} \alpha a \int_0^t \Theta dt + \Phi_0 + t\Phi_1 \tag{4.38}$$

where $\nabla^2 \Phi_1 = 0$, and Φ_0 represents the initial thermoelastic potential.

We now turn our attention to the homogeneous form of the Navier equations. For the current problem, since there are no body forces or surface tractions, we may assume:

$$u_i^h = 0 \tag{4.39}$$

Substituting the obtained temperature field into the general solution for the thermoelastic potential, we can write:

$$\Phi = \frac{(1+\nu) Q \alpha a}{(1-\nu)(4\pi a)^{3/2} \rho C_v} \int_0^t \frac{e^{-\left(\frac{r^2}{4at}\right)}}{t^{3/2}} dt + \Phi_0 + t\Phi_1 \tag{4.40}$$

In order to ease the formulations, we may further consider;

$$\begin{cases} V = \dfrac{r}{2\sqrt{at}} \quad \Rightarrow \quad V^2 = \dfrac{r^2}{4at} \\ t = \dfrac{r^2}{4aV^2} \quad \Rightarrow \quad t^{3/2} = \dfrac{r^3}{8a^{3/2}V^3} \\ \qquad dt = -\dfrac{r^2}{2aV^3}dV \end{cases} \tag{4.41}$$

which upon substitution into Eq. (4.40), results in:

$$\begin{aligned} \Phi - \Phi_0 - t\Phi_1 &= \frac{(1+\nu)Q\alpha}{(1-\nu)4\pi r\rho C_v}\left(-\frac{2}{\sqrt{\pi}}\int_{\infty}^{\frac{r}{2\sqrt{at}}} e^{-V^2}dV\right) \\ &= \frac{(1+\nu)Q\alpha}{(1-\nu)4\pi r\rho C_v}\left(\frac{2}{\sqrt{\pi}}\int_{\frac{r}{2\sqrt{at}}}^{0} e^{-V^2}dV + \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-V^2}dV\right) \\ &= \frac{(1+\nu)Q\alpha}{(1-\nu)4\pi r\rho C_v}\left(-\frac{2}{\sqrt{\pi}}\int_0^{\frac{r}{2\sqrt{at}}} e^{-V^2}dV + \frac{2}{\sqrt{\pi}}\frac{1}{2}\sqrt{\pi}\right) \\ &= \frac{(1+\nu)Q\alpha}{(1-\nu)4\pi r\rho C_v}\left[1 - erf\left(\frac{r}{2\sqrt{at}}\right)\right] \end{aligned} \tag{4.42}$$

The error function is:

$$erf\left(\frac{r}{2\sqrt{at}}\right) = \frac{2}{\sqrt{\pi}}\int_0^{\frac{r}{2\sqrt{at}}} e^{-V^2}dV \tag{4.43}$$

Now we will determine Φ_1 and Φ_0 values of the thermoelastic potential. After a long period of time the displacements and stresses return to zero, so we may write:

$$\begin{cases} \qquad \Phi_1 = 0 \\ \Phi_0 = -\Phi\big|_{t=\infty} = \dfrac{(1+\nu)Q\alpha}{(1-\nu)4\pi r\rho C_v} \end{cases} \tag{4.44}$$

$$\Rightarrow \Phi = -\frac{(1+\nu)Q\alpha}{(1-\nu)4\pi r\rho C_v} erf\left(\frac{r}{2\sqrt{at}}\right)$$

Having found the thermoelastic potential, the displacement, strain, and stress fields can be determined as follows:

$$
\begin{aligned}
& u_i = u_i^h + u_i^p = \Phi_{,i} \\
& \varepsilon_{ij} = \Phi_{,ij} \\
& \sigma_{ij} = 2G\left[\Phi_{,ij} + \frac{\nu}{1-2\nu}\Phi_{,kk}\delta_{ij} - \frac{(1+\nu)}{1-2\nu}\alpha\Theta\delta_{ij}\right]
\end{aligned} \tag{4.45}
$$

Two-Dimensional Thermoelastic Problems

Plane strain

Here we start by reviewing some of the plane strain formulations:

$$
\begin{cases}
u = u_\alpha(x_\alpha)\begin{cases} u_1 = u_1(x_1, x_2) \\ u_2 = u_2(x_1, x_2) \end{cases} \quad \alpha, \beta = 1, 2 \\
\qquad u_3 = 0, \text{constant} \\
\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \square \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \cdot & \varepsilon_{22} & 0 \\ \cdot & \cdot & 0 \end{pmatrix} \\
\qquad \sigma_{\alpha\beta,\alpha} + f_\beta = 0 \\
\qquad f_\alpha = -V_{,\alpha} \\
\qquad \sigma_{11} - V = \varphi_{,22} \\
\qquad \sigma_{22} - V = \varphi_{,11} \\
\qquad \sigma_{12} = \sigma_{21} = -\varphi_{,12}
\end{cases} \tag{4.46}
$$

The two-dimensional form of the Duhamel-Neumann constitutive relations is:

$$
\begin{aligned}
& \sigma_{\alpha\beta} = 2G\left[\left(\varepsilon_{\alpha\beta} + \frac{\nu}{1-2\nu}\varepsilon_{\gamma\gamma}\delta_{\alpha\beta}\right) - \frac{(1+\nu)}{1-2\nu}\alpha\Theta\delta_{\alpha\beta}\right] && \text{(i)} \\
& \sigma_{\alpha 3} = 2G\varepsilon_{\alpha 3} = 0 && \text{(ii)} \\
& \sigma_{33} = 2G\left[\frac{\nu}{1-2\nu}\varepsilon_{\gamma\gamma} - \frac{(1+\nu)}{1-2\nu}\alpha\Theta\right] && \text{(iii)}
\end{aligned} \tag{4.47}
$$

In order to find the appropriate expression for σ_{33} we need to find $\varepsilon_{\gamma\gamma}$. We start by contracting on α and β indices in Eq. (4.47i), as follows:

$$\sigma_{\alpha\alpha} = 2G\left[\left(\varepsilon_{\alpha\alpha} + \frac{2\nu}{1-2\nu}\varepsilon_{\gamma\gamma}\right) - \frac{2(1+\nu)}{1-2\nu}\alpha\Theta\right]$$
$$\Rightarrow \varepsilon_{\alpha\alpha} = \frac{1-2\nu}{2G}\sigma_{\alpha\alpha} + 2(1+\nu)\alpha\Theta \tag{4.48}$$

Substituting for $\varepsilon_{\gamma\gamma}$ in Eq. (4.47iii) results in:

$$\sigma_{33} = 2G\left[\frac{\nu}{2G}\sigma_{\gamma\gamma} - (1+\nu)\alpha\Theta\right]$$
$$\Rightarrow \sigma_{33} = \nu\sigma_{\gamma\gamma} - E\alpha\Theta \tag{4.49}$$

which indicates that σ_{33} may be determined from a knowledge of $\sigma_{\gamma\gamma}$ and Θ. Furthermore, we have:

$$\varepsilon_{\alpha\beta} = \frac{1}{2G}\left[\sigma_{\alpha\beta} - \nu\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right] + (1+\nu)\alpha\Theta\delta_{\alpha\beta} \tag{4.50}$$

which is the two-dimensional form of the Duhamel-Neumann relation. Now, using the above expression, and implementing the same approach we followed in Chapter 3, we can derive the biharmonic equation for linear thermoelasticity as follows:

$$\nabla'^4\varphi = -\left(\frac{1-2\nu}{1-\nu}\right)\nabla'^2 V - 2G\left(\frac{1+\nu}{1-\nu}\right)\alpha\nabla'^2\Theta \tag{4.51}$$

If body forces are negligible or constant we may write:

$$\nabla'^4\varphi = -2G\left(\frac{1+\nu}{1-\nu}\right)\alpha\nabla'^2\Theta \tag{4.52}$$

The solution to the above biharmonic equation is:

$$\varphi = \varphi^h + \varphi^p \tag{4.53}$$

where φ^h can be found from the elasticity part of the problem without considering the thermal effects. In order to find the particular integral, we use the definition of the thermoelastic potential, Eqs. (4.36), in two-dimensional form:

$$u_\alpha^p = \Phi_{,\alpha} \tag{4.54}$$

which satisfies the two-dimensional form of Eq. (4.37):

$$\Phi_{,\alpha\alpha} = \nabla'^2\Phi = \frac{(1+\nu)}{(1-\nu)}\alpha\Theta \tag{4.55}$$

If we differentiate the above equation twice we will have:

$$\nabla'^4\Phi = \frac{(1+\nu)}{(1-\nu)}\alpha\nabla'^2\Theta \tag{4.56}$$

Substituting into our biharmonic equation (Eq. (4.52)) we can write:

$$\begin{aligned} &\nabla'^4\varphi^p = -2G\nabla'^4\Phi \\ &\Rightarrow \varphi^p = -2G\Phi \end{aligned} \tag{4.57}$$

Naturally, the isothermal deformation satisfies the homogeneous form of Eq. (4.52), and as a result the complete solution for the stress function is:

$$\varphi = \varphi^h + \varphi^p = \varphi^h - 2G\Phi \tag{4.58}$$

Plane Stress Formulation

We start by reviewing some of the plane stress formulations:

$$\begin{aligned} &\bar{\sigma}_{ij} = \frac{1}{h}\int_{-h/2}^{h/2}\sigma_{ij}dx_3 \\ &\bar{\sigma}_{ij} \sim \begin{pmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{12} & 0 \\ \cdot & \bar{\sigma}_{22} & 0 \\ \cdot & \cdot & 0 \end{pmatrix} \end{aligned} \tag{4.59}$$

The *bar* sign over a quantity indicates that it has been averaged over the thickness. The equilibrium equations can be written as:

$$\begin{aligned} &\bar{f}_\alpha = -\bar{V}_{,\alpha} \\ &\Rightarrow \bar{\sigma}_{\alpha\beta,\alpha} - \bar{V}_{,\beta} = 0 \end{aligned} \tag{4.60}$$

which are exactly the same as those derived for plane strain condition, except for the mean value interpretation, so we may similarly write:

$$\begin{aligned} &\bar{\sigma}_{11} - \bar{V} = \bar{\varphi}_{,22} \\ &\bar{\sigma}_{22} - \bar{V} = \bar{\varphi}_{,11} \\ &\bar{\sigma}_{12} = \bar{\sigma}_{21} = -\bar{\varphi}_{,12} \end{aligned} \tag{4.61}$$

The constitutive equations are:

$$\begin{aligned} &\bar{\varepsilon}_{\alpha\beta} = \frac{1}{2G}\left[\bar{\sigma}_{\alpha\beta} - \frac{\nu}{1+\nu}\bar{\sigma}_{\gamma\gamma}\delta_{\alpha\beta}\right] + \alpha\bar{\Theta}\delta_{\alpha\beta} \\ &\bar{\varepsilon}_{\alpha 3} = \frac{1}{2G}\bar{\sigma}_{\alpha 3} = 0 \\ &\bar{\varepsilon}_{33} = \frac{1}{2G}\left[-\frac{\nu}{1+\nu}\bar{\sigma}_{\gamma\gamma}\right] + \alpha\bar{\Theta} \end{aligned} \tag{4.62}$$

which can be inverted in a standard manner to give:

$$\bar{\sigma}_{\alpha\beta} = \frac{2G}{1-\nu}\left[(1-\nu)\bar{\varepsilon}_{\alpha\beta} + \nu\bar{\varepsilon}_{\gamma\gamma}\delta_{\alpha\beta} - (1+\nu)\alpha\bar{\Theta}\delta_{\alpha\beta}\right] \tag{4.63}$$

Now, similar to our plane strain formulation, upon substitution for the strains into the only nonzero compatibility equation we find that:

$$\nabla'^4\bar{\varphi} = -E\alpha\nabla'^2\bar{\Theta} \tag{4.64}$$

The solution to the above biharmonic equation is:

$$\bar{\varphi} = \bar{\varphi}^h + \bar{\varphi}^p \tag{4.65}$$

where φ^h can be found from the corresponding isothermal elasticity problem. In order to find the particular integral, we use the plane stress form of the thermoelastic potential expression to write:

$$\begin{aligned} &\nabla'^2\bar{\Phi} = \bar{\Phi}_{,\alpha\alpha} = (1+\nu)\alpha\bar{\Theta} \\ &\nabla'^4\bar{\Phi} = (1+\nu)\alpha\nabla'^2\bar{\Theta} \\ &\text{using (4.49)} \Rightarrow \nabla'^4\bar{\varphi} = -\frac{E}{(1+\nu)}\nabla'^4\bar{\Phi} \\ &\Rightarrow \bar{\varphi}^p = -2G\bar{\Phi} \end{aligned} \tag{4.66}$$

Solution Strategies for Two-Dimensional Problems

As mentioned before, the solution of the thermoelasticity problems begins with finding the time and position dependent temperature field. For the two-dimensional problems, the next step is the solution of either of the following expressions for the thermoelastic potential:

$$\nabla'^2\Phi = \Phi_{,\alpha\alpha} = \frac{(1+\nu)}{(1-\nu)}\alpha\Theta \quad ; \quad \nabla'^2\bar{\Phi} = \bar{\Phi}_{,\alpha\alpha} = (1+\nu)\alpha\bar{\Theta} \tag{4.67}$$

which using $\varphi^p = -2G\Phi$ will give the particular solution.

The solution to the isothermal elasticity problem can be found using the methods introduced earlier, or it can be constructed as a finite or infinite series of properly chosen biharmonic functions. In Cartesian coordinates we may use:

$$e^{\pm\lambda x_2}\cos\lambda x_1 \quad ; \quad x_1 e^{\pm\lambda x_2}\cos\lambda x_1 \quad ; \quad x_2 e^{\pm\lambda x_2}\cos\lambda x_1 \tag{4.68}$$

where λ is an arbitrary constant. All other combinations obtained by interchanging x_1 and x_2 and/or replacing $\cos\lambda x_1$ by $\sin\lambda x_1$ can also be used.

In Polar coordinates, a complete set of biharmonic functions which are periodic in θ are:

$$\begin{gathered} r^2 \quad ; \quad \log r \quad ; \quad r^2\log r \\ r\log r\cos\theta \quad ; \quad r^{\pm n}\cos n\theta \quad ; \quad r^{2\pm n}\cos n\theta \quad ; \quad n = 0,1,2,\ldots \end{gathered} \tag{4.69}$$

and the corresponding expressions with $\cos n\theta$ replaced by $\sin n\theta$ can also be used.

Assignment 12:

Using the stress function approach, find the thermal stresses for a long circular hollow cylinder for which the inner and outer surfaces are kept at constant temperatures T_i and T_o respectively. The cylinder is also under internal pressure P_i.

Two-Dimensional Simply-Connected Regions with Steady Heat Flow

Here, we will show that there are special cases where the thermal stresses can be absent even if the body experiences temperature gradients. Let us consider a steady heat flow which is confined to the x_1x_2 plane, for example, consider a long cylinder with no variation of temperature in the axial direction. The temperature field will satisfy the 2D form of the Laplace's equation:

$$\Theta_{,\alpha\alpha} = 0 \tag{4.70}$$

and the constitutive equations are:

$$\begin{aligned} \varepsilon_{11} - (1+\nu)\alpha\Theta &= \frac{1-\nu^2}{E}\left(\sigma_{11} - \frac{\nu}{1-\nu}\sigma_{22}\right) \\ \varepsilon_{22} - (1+\nu)\alpha\Theta &= \frac{1-\nu^2}{E}\left(\sigma_{22} - \frac{\nu}{1-\nu}\sigma_{11}\right) \end{aligned} \tag{4.71}$$

Now, in order to maintain $\sigma_{11} = \sigma_{22} = 0$, we should have:

$$\begin{aligned} \varepsilon_{11} &= (1+\nu)\alpha\Theta \\ \varepsilon_{22} &= (1+\nu)\alpha\Theta \end{aligned} \tag{4.72}$$

The above strains should satisfy the compatibility equations, which in this case simplify to:

$$\varepsilon_{11,22} + \varepsilon_{22,11} = 0 \tag{4.73}$$

which is identically satisfied if we substitute Eqs. (4.72) into Eq. (4.70). Hence we may conclude that the thermal stresses can be absent for two-dimensional simply-connected regions with steady heat flow (even for bodies under temperature gradients). As we will see in the next example the situation is different for non-simply-connected regions.

Insulated Circular Hole inside an Infinite Plate subjected to Uniform Heat Flow

In this example we assume a uniform heat flow in the direction of the negative x_2 axis in an infinite plate. For this case we may write:

$$\bar{\Theta} = q x_2 \tag{4.74}$$

where q is the constant temperature gradient. As depicted in Fig. 4.1, the uniform heat flow is disturbed by the presence of an insulated circular hole.

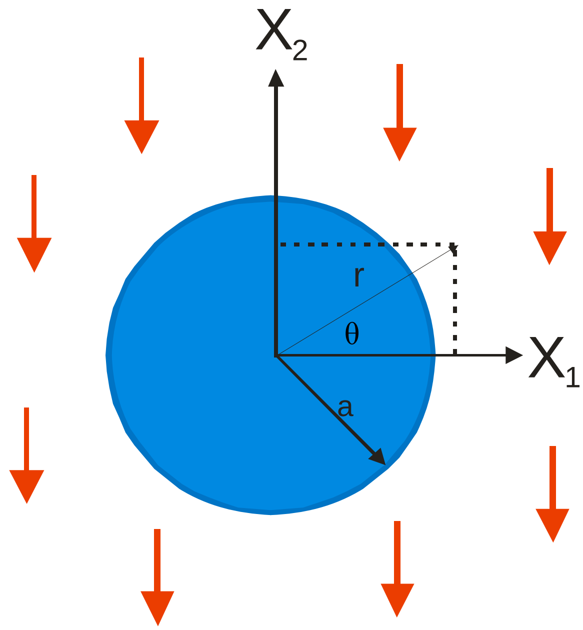

Fig. 4.1

In order to obtain the thermal stresses, we should first find the temperature field. The two-dimensional form of Eq. (4.2) for plane stress is:

$$\bar{\Theta}_{,\alpha\alpha} = 0 \tag{4.75}$$

which can be written in cylindrical coordinates as:

$$\frac{\partial^2 \bar{\Theta}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\Theta}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{\Theta}}{\partial \theta^2} = 0 \tag{4.76}$$

We may solve the above partial differential equation using the separation of variables method as follows:

$$\bar{\Theta} = R(r)X(\theta) \tag{4.77}$$

Accordingly we may write:

$$\begin{aligned} \frac{\partial \bar{\Theta}}{\partial r} &= \frac{dR}{dr}X \quad ; \quad \frac{\partial \bar{\Theta}}{\partial \theta} = R\frac{dX}{d\theta} \\ \frac{\partial^2 \bar{\Theta}}{\partial r^2} &= \frac{d^2R}{dr^2}X \quad ; \quad \frac{\partial^2 \bar{\Theta}}{\partial \theta^2} = R\frac{d^2X}{d\theta^2} \end{aligned} \tag{4.78}$$

Substituting the above expressions into Eq. (4.76), we have:

$$\frac{d^2R}{dr^2}X+\frac{1}{r}\frac{dR}{dr}X+\frac{R}{r^2}\frac{d^2X}{d\theta^2}=0$$
$$\Rightarrow \frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}+\frac{1}{X}\frac{d^2X}{d\theta^2}=0 \tag{4.79}$$

which upon splitting into two parts results in:

$$\frac{r^2}{R}\frac{d^2R}{dr^2}+\frac{r}{R}\frac{dR}{dr}=k^2 \qquad \text{(i)}$$
$$\frac{1}{X}\frac{d^2X}{d\theta^2}=-k^2 \qquad \text{(ii)} \tag{4.80}$$

Equation (4.80i) can be rearranged as the Cauchy-Euler's equation:

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-k^2R=0 \tag{4.81}$$

which can be solved using the following considerations:

$$r=e^z \quad ; \quad z=\ln r$$
$$\Rightarrow \frac{dR}{dr}=\frac{dR}{dz}\frac{dz}{dr}=\frac{1}{r}\frac{dR}{dz}$$
$$\Rightarrow \frac{d^2R}{dr^2}=-\frac{1}{r^2}\frac{dR}{dz}+\frac{1}{r^2}\frac{d^2R}{dz^2} \tag{4.82}$$

With proper substitutions into Eq. (4.81), the solution can be obtained as follows:

$$\frac{d^2R}{dz^2}=k^2R$$
$$\Rightarrow R=A_1e^{kz}+A_2e^{-kz}$$
$$\Rightarrow R=A_1r^k+A_2r^{-k} \tag{4.83}$$

Now we turn our attention to Eq. (4.80ii), which can be solved as the Euler's equation:

$$\frac{d^2X}{d\theta^2}+k^2X=0$$
$$\Rightarrow X=B_1e^{ik\theta}+B_2e^{ik\theta}$$
$$\Rightarrow X=C_1\sin k\theta+C_2\cos k\theta \tag{4.84}$$

Finally, the temperature field can be written as:

$$\bar{\Theta} = RX = \left(A_1 r^k + A_2 r^{-k}\right)\left(C_1 \sin k\theta + C_2 \cos k\theta\right) \tag{4.85}$$

for which the four constants can be determined from the boundary conditions. Also note that we have:

$$\frac{\partial \bar{\Theta}}{\partial r} = k\left(A_1 r^{k-1} - A_2 r^{-k-1}\right)\left(C_1 \sin k\theta + C_2 \cos k\theta\right) \tag{4.86}$$

The first temperature boundary condition is:

$$\begin{aligned} &\theta = 0 \quad ; \quad \bar{\Theta} = 0 \quad \forall r \\ &\Rightarrow C_2 = 0 \end{aligned} \tag{4.87}$$

Thus, Eqs. (4.85) and (4.86) can be rewritten as:

$$\begin{aligned} &\bar{\Theta} = \left(A_1 C_1 r^k + A_2 C_1 r^{-k}\right)\sin k\theta = \left(D_1 r^k + D_2 r^{-k}\right)\sin k\theta \\ &\frac{\partial \bar{\Theta}}{\partial r} = \left(D_1 r^{k-1} - D_2 r^{-k-1}\right) k \sin k\theta \end{aligned} \tag{4.88}$$

The second temperature boundary condition is:

$$\begin{aligned} &r = a \quad ; \quad \frac{\partial \bar{\Theta}}{\partial r} = 0 \quad \forall \theta \\ &\Rightarrow D_2 = D_1 a^{k-1+k+1} = D_1 a^{2k} \end{aligned} \tag{4.89}$$

This time, Eq. (4.88) can be rewritten as:

$$\begin{aligned} &\bar{\Theta} = D_1 r^k \left(1 + \frac{a^{2k}}{r^{2k}}\right)\sin k\theta \\ &\frac{\partial \bar{\Theta}}{\partial r} = D_1 k r^{k-1}\left(1 - \frac{a^{2k}}{r^{2k}}\right)\sin k\theta \end{aligned} \tag{4.90}$$

The third temperature boundary condition is:

$$\begin{aligned} &\theta = \frac{\pi}{2} \quad ; \quad \frac{\partial \bar{\Theta}}{\partial r} = \frac{\partial \bar{\Theta}}{\partial x_2} = q \\ &\Rightarrow D_1 k r^{k-1}\left(1 - \frac{a^{2k}}{r^{2k}}\right) = q \end{aligned} \tag{4.91}$$

The above expression must remain valid as $r \to \infty$ so we may conclude that:

$$\begin{cases} k-1=0 \quad \Rightarrow \quad k=1 \\ \qquad \Rightarrow D_1 = q \end{cases} \tag{4.92}$$

Finally, the temperature distribution can be found as:

$$\begin{aligned} \bar{\Theta} &= q\left(r\sin\theta + \frac{a^2}{r}\sin\theta \right) \\ &= q\left(x_2 + \frac{a^2}{r}\sin\theta \right) \end{aligned} \tag{4.93}$$

The above expression shows that the uniform temperature field qx_2 remains unchanged at the far field and the effect of the circular hole is represented by the $q\frac{a^2}{r}\sin\theta$ term. As discussed before, the uniform temperature field has no influence on the thermal stresses and may be disregarded in the process of determination of the thermoelastic potential from its governing equation as follows:

$$\begin{aligned} &\nabla'^2\Phi = (1+\nu)\alpha\bar{\Theta} \\ &\frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = (1+\nu)\alpha q\frac{a^2}{r}\sin\theta \\ &\Rightarrow \Phi = \frac{(1+\nu)\alpha q a^2}{2} r\log r\sin\theta \end{aligned} \tag{4.94}$$

Having found the thermoelastic potential, we may easily calculate $\bar{\varphi}^p$ from $\bar{\varphi}^p = -2G\bar{\Phi}$. In order to have some clue to the proper form of $\bar{\varphi}^h$, we may proceed as follows:

$$\begin{aligned} &\bar{\varphi} = \bar{\varphi}^h + \bar{\varphi}^p = \bar{\varphi}^h - 2G\bar{\Phi} \Rightarrow \\ &\bar{\sigma}_{rr} = \bar{\sigma}_{rr}^h + \bar{\sigma}_{rr}^p \\ &\quad = \bar{\sigma}_{rr}^h - 2G\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \right)\bar{\Phi} \\ &\quad = \bar{\sigma}_{rr}^h - \frac{2G(1+\nu)\alpha q a^2}{2}\left(\frac{1}{r}\sin\theta \right) \\ &\quad = \bar{\sigma}_{rr}^h - \frac{E\alpha q a^2}{2r}\sin\theta \end{aligned} \tag{4.95}$$

Now we may apply our first ***overall*** stress boundary condition:

$$\begin{aligned}&\bar{\sigma}_{rr}=0 \quad \text{at} \quad r=a\\&\Rightarrow \bar{\sigma}_{rr}^{h}\Big|_{r=a}=\frac{E\alpha qa}{2}\sin\theta\\&\Rightarrow \left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^{2}}\frac{\partial^{2}}{\partial\theta^{2}}\right)\bar{\varphi}^{h}\Big|_{r=a}=\frac{E\alpha qa}{2}\sin\theta\end{aligned} \tag{4.96}$$

which suggests the following form for $\bar{\varphi}^{h}$:

$$\bar{\varphi}^{h}=\frac{E\alpha qa}{2}f(r)\sin\theta \quad ; \quad f(a)=1 \tag{4.97}$$

Similarly, for the shear stresses we have:

$$\begin{aligned}\bar{\sigma}_{r\theta}&=\bar{\sigma}_{r\theta}^{h}+\bar{\sigma}_{r\theta}^{p}\\&=\bar{\sigma}_{r\theta}^{h}-2G\left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\right]\bar{\Phi}\\&=\bar{\sigma}_{r\theta}^{h}+\frac{E\alpha qa^{2}}{2r}\cos\theta\end{aligned} \tag{4.98}$$

Applying our second ***overall*** stress boundary condition, we will have:

$$\begin{aligned}&\bar{\sigma}_{r\theta}=0 \quad \text{at} \quad r=a\\&\Rightarrow \bar{\sigma}_{r\theta}^{h}\Big|_{r=a}=-\frac{E\alpha qa}{2}\cos\theta\\&\Rightarrow -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right)\bar{\varphi}^{h}\Big|_{r=a}=-\frac{E\alpha qa}{2}\cos\theta\end{aligned} \tag{4.99}$$

which approves the previously suggested form for $\bar{\varphi}^{h}$. However, according to our third ***overall*** boundary condition all the stresses should vanish as $r\to\infty$. Accordingly, we may modify the expression (4.97) as:

$$\bar{\varphi}^{h}=\frac{E\alpha qa}{2}f\left(\frac{1}{r}\right)\sin\theta \quad ; \quad f(a)=1 \tag{4.100}$$

Here, we consult the list of the biharmonic functions in polar coordinates and finalize our expression as:

$$\bar{\varphi}^{h}=\frac{E\alpha qa}{2}Ar^{-1}\sin\theta \tag{4.101}$$

Accordingly, we can write:

$$\left.\begin{aligned}
&\left.\bar{\sigma}_{rr}^{h}\right|_{r=a} = \frac{E\alpha q a}{2}\sin\theta && \text{(i)}\\
&\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left.\bar{\varphi}^{h}\right|_{r=a} = -\frac{AE\alpha q a}{a^3}\sin\theta && \text{(ii)}
\end{aligned}\right\}\text{(i),(ii)} \Rightarrow A = -\frac{a^3}{2}$$

$$\Rightarrow \bar{\sigma}_{rr}^{h} = \frac{E\alpha q a}{2}\left(\frac{a}{r}\right)^3 \sin\theta$$

$$\Rightarrow \bar{\sigma}_{rr} = -\frac{E\alpha q a}{2}\left(\frac{a}{r}-\frac{a^3}{r^3}\right)\sin\theta \tag{4.102}$$

The other stress components can be calculated using $\bar{\varphi}^{h}$ as follows:

$$\bar{\varphi}^{h} = -\frac{E\alpha q a^4}{4r}\sin\theta$$

$$\Rightarrow \bar{\sigma}_{r\theta} = \frac{E\alpha q a}{2}\left(\frac{a}{r}-\frac{a^3}{r^3}\right)\cos\theta \tag{4.103}$$

$$\Rightarrow \bar{\sigma}_{\theta\theta} = -\frac{E\alpha q a}{2}\left(\frac{a}{r}+\frac{a^3}{r^3}\right)\sin\theta$$

Zero Thermal Stresses in a Three-Dimensional Body

As mentioned before, there are some special cases where the thermal stresses can be absent, even if the body experiences temperature gradients. Previously, we examined the steady heat flow in two-dimensional simply-connected regions. We may also arrive at zero thermal stresses for an unrestrained body when the temperature distribution is a linear function of *rectangular space coordinates*. For the stresses all to be zero, the six stress compatibility equations result in:

$$\alpha\Theta_{,11} = \alpha\Theta_{,22} = \alpha\Theta_{,33} = \alpha\Theta_{,12} = \alpha\Theta_{,23} = \alpha\Theta_{,31} = 0 \tag{4.104}$$

Hence, the thermal strain distribution must be a linear function of the rectangular space coordinates like:

$$\alpha\Theta = a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 \tag{4.105}$$

Note that the coefficients may be time dependent. Accordingly, if the thermal properties of the material are constant, the zero stress condition dictates a linear temperature distribution.

Assignment 13:

Consider an unrestrained circular disc for which we have $\Theta = Ar$. Based on our previous discussions concerning the special cases for zero thermal stress, can we conclude that the disc is stress free? Why?

Chapter 5
Variational Principles and Applications

Leonhard Euler: *Since the fabric of the universe is most perfect, and is the work of a most wise creator, nothing whatsoever takes place in the universe in which some relation of maximum and minimum does not appear. Wherefore, there is absolutely no doubt that every effect in universe can be explained as satisfactorily from final causes, by the aid of the method of Maxima and Minima, as it can from the effective causes themselves.* Ref: Timoshenko, S., History of Strength of Materials (McGraw-Hill Book Company, Inc., New York, 1953)

Energy Principles

As clearly stated by Leonhard Euler in the 18th century, the field equations of the theory of elasticity may also be developed from the energy considerations. In particular, these methods provide powerful tools for obtaining approximate solutions and implementation of numerical solution techniques.

Principle of Virtual Work

The virtual work can be defined as "the work done on a deformable body, by all the forces acting on it, as the body is given a small hypothetical displacement which is consistent with the constraints present". The virtual displacements are represented by the symbol "δ", known as the variation operator. In general, the loadings consist of body forces and surface tractions. The later are prescribed over a part of the boundary designated by S_σ. Over the remaining boundary (designated by S_u) the displacement field $\mathbf{u}$ is prescribed. However, it must be ensured that $\delta\mathbf{u} = 0$ on S_u to avoid violating the constraints.

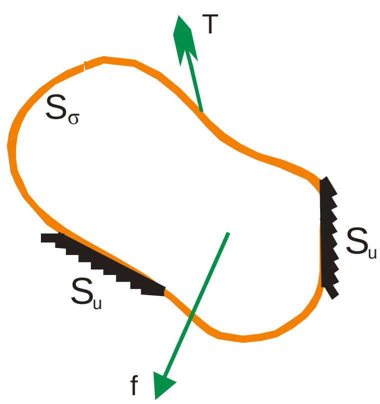

Fig. 5.1

Accordingly, the virtual work is defined by:

$$\begin{aligned}\delta W_{virt} &= \int_S \mathbf{T}.\delta\mathbf{u} + \int_V \mathbf{f}.\delta\mathbf{u} \\ &= \int_S T_i \delta u_i dS + \int_V f_i \delta u_i dV\end{aligned} \tag{5.1}$$

Now we may invoke the stress boundary relations and implement the divergence theorem to obtain:

$$\begin{aligned}\int_S T_i \delta u_i dS + \int_V f_i \delta u_i dV &= \int_S \sigma_{ij} n_j \delta u_i dS + \int_V f_i \delta u_i dV \\ &= \int_V \left[\left(\sigma_{ij} \delta u_i \right)_{,j} + f_i \delta u_i \right] dV \\ &= \int_V \left[\left(\overset{=0}{\sigma_{ij,j} + f_i} \right) \delta u_i + \sigma_{ij} \left(\delta u_{i,j} \right) \right] dV\end{aligned} \tag{5.2}$$

The first grouping of terms within the last integral can be set to zero to ensure the equilibrium. The product of the symmetric stress tensor with the skew-symmetric part of $\delta u_{i,j}$ is also zero. Since the symmetric part of $\delta u_{i,j}$ is nothing but a variation in the strain tensor, we may write the following expression, known as the principle of virtual work, PVW:

$$\int_S T_i \delta u_i dS + \int_V f_i \delta u_i dV = \int_V \sigma_{ij} \delta \varepsilon_{ij} dV \tag{5.3}$$

It is interesting to note that the above equation is independent of any constitutive law and can be applied to all continuous materials within the limitations of small deformations.

Illustrative Example I

Consider a planar truss as depicted in Fig. 5.2. It is assumed that all the members have the same cross-sectional area and Young's modulus.

If we neglect the gravitational effects, the PVW for this problem can be written as:

$$P_\alpha \delta u_\alpha = \int_V \sigma_{ij} \delta \varepsilon_{ij} dV \tag{5.4}$$

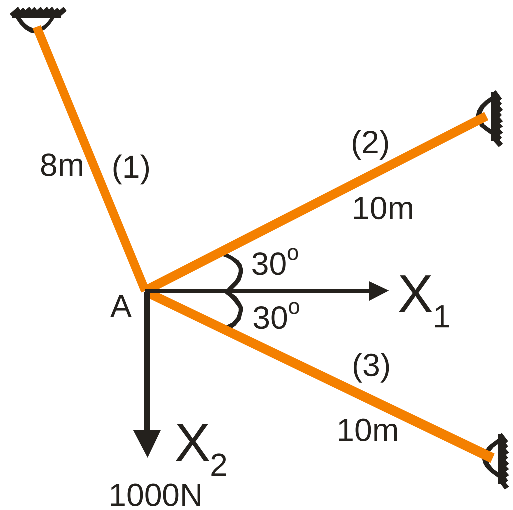

Fig. 5.2

For this problem, we may only consider virtual displacements for the point A. The changes in the length of each rod due to the imposed virtual displacements δu_α are:

$$\begin{aligned} {}^{X_1}\delta L_1 &= \delta u_1 \cos 60 \quad ; \quad {}^{X_2}\delta L_1 = \delta u_2 \sin 60 \\ {}^{X_1}\delta L_2 &= -\delta u_1 \cos 30 \quad ; \quad {}^{X_2}\delta L_2 = \delta u_2 \sin 30 \\ {}^{X_1}\delta L_3 &= -\delta u_1 \cos 30 \quad ; \quad {}^{X_2}\delta L_3 = -\delta u_2 \sin 30 \end{aligned} \tag{5.5}$$

In the above, ${}^{X_\alpha}\delta L_i$ represent the components of the virtual elongation (or contraction) of the members in the X_α directions. Note that, δu_α have been chosen positive in the positive direction of the axes. Accordingly, we can calculate the components of the virtual strains as:

$$\begin{aligned} {}^{X_1}\delta\varepsilon_1 &= \delta u_1 \frac{\cos 60}{8} \quad ; \quad {}^{X_2}\delta\varepsilon_1 = \delta u_2 \frac{\sin 60}{8} \\ {}^{X_1}\delta\varepsilon_2 &= -\delta u_1 \frac{\cos 30}{10} \quad ; \quad {}^{X_2}\delta\varepsilon_2 = \delta u_2 \frac{\sin 30}{10} \\ {}^{X_1}\delta\varepsilon_3 &= -\delta u_1 \frac{\cos 30}{10} \quad ; \quad {}^{X_2}\delta\varepsilon_3 = -\delta u_2 \frac{\sin 30}{10} \end{aligned} \tag{5.6}$$

where, ${}^{X_\alpha}\delta\varepsilon_i$ represent the components of the virtual strain of the members in the X_α directions.

Now, we will calculate the *real* stress in each rod, using the *real* axial strain caused by the *real* displacement of the point A.

$$\begin{aligned}
\bar{\varepsilon}_1 &= \bar{u}_1 \frac{\cos 60}{8} + \bar{u}_2 \frac{\sin 60}{8} &&\Rightarrow \quad \bar{\sigma}_1 = E\bar{u}_1 \frac{\cos 60}{8} + E\bar{u}_2 \frac{\sin 60}{8} \\
\bar{\varepsilon}_2 &= -\bar{u}_1 \frac{\cos 30}{10} + \bar{u}_2 \frac{\sin 30}{10} &&\Rightarrow \quad \bar{\sigma}_2 = -E\bar{u}_1 \frac{\cos 30}{10} + E\bar{u}_2 \frac{\sin 30}{10} \\
\bar{\varepsilon}_3 &= -\bar{u}_1 \frac{\cos 30}{10} - \bar{u}_2 \frac{\sin 30}{10} &&\Rightarrow \quad \bar{\sigma}_3 = -E\bar{u}_1 \frac{\cos 30}{10} - E\bar{u}_2 \frac{\sin 30}{10}
\end{aligned} \tag{5.7}$$

Accordingly, we can apply the PVW in the horizontal direction as follows:

$$\begin{aligned}
P_1\delta u_1 = {} & 8AE\left[\bar{u}_1 \frac{\cos 60}{8}\left(\delta u_1 \frac{\cos 60}{8}\right) + \bar{u}_2 \frac{\sin 60}{8}\left(\delta u_1 \frac{\cos 60}{8}\right)\right] \\
& +10AE\left[-\bar{u}_1 \frac{\cos 30}{10}\left(-\delta u_1 \frac{\cos 30}{10}\right) + \bar{u}_2 \frac{\sin 30}{10}\left(-\delta u_1 \frac{\cos 30}{10}\right)\right] \\
& +10AE\left[-\bar{u}_1 \frac{\cos 30}{10}\left(-\delta u_1 \frac{\cos 30}{10}\right) - \bar{u}_2 \frac{\sin 30}{10}\left(-\delta u_1 \frac{\cos 30}{10}\right)\right]
\end{aligned} \tag{5.8}$$

which may be simplified to give:

$$\frac{P_1}{AE} = 0.181\bar{u}_1 + 0.054\bar{u}_2 \tag{5.9}$$

and also vertically as:

$$\begin{aligned}
P_2\delta u_2 = {} & 8AE\left[\bar{u}_1 \frac{\cos 60}{8}\left(\delta u_2 \frac{\sin 60}{8}\right) + \bar{u}_2 \frac{\sin 60}{8}\left(\delta u_2 \frac{\sin 60}{8}\right)\right] \\
& +10AE\left[-\bar{u}_1 \frac{\cos 30}{10}\left(\delta u_2 \frac{\sin 30}{10}\right) + \bar{u}_2 \frac{\sin 30}{10}\left(\delta u_2 \frac{\sin 30}{10}\right)\right] \\
& +10AE\left[-\bar{u}_1 \frac{\cos 30}{10}\left(-\delta u_2 \frac{\sin 30}{10}\right) - \bar{u}_2 \frac{\sin 30}{10}\left(-\delta u_2 \frac{\sin 30}{10}\right)\right]
\end{aligned} \tag{5.10}$$

which upon simplifying gives the second required equation:

$$\frac{P_2}{AE} = 0.054\bar{u}_1 + 0.143\bar{u}_2 \tag{5.11}$$

Now, we may set: $P_\alpha = (0 \ , \ 1000)$, and solve Eqs. (5.9) and (5.11) to give:

$$\bar{u}_1 = -\frac{2342}{AE} \quad ; \quad \bar{u}_2 = \frac{7846}{AE} \tag{5.12}$$

Principle of Complementary Virtual Work

In order to establish another useful variational principle, we may vary the stress field and applied forces while keeping the displacement field fixed.

$$\begin{aligned}\delta W^*_{virt} &= \int_S \mathbf{u}.\delta\mathbf{T}dS + \int_V \mathbf{u}.\delta\mathbf{f}dV \\ &= \int_S u_i\delta T_i dS + \int_V u_i\delta f_i dV\end{aligned} \qquad (5.13)$$

Using an approach similar to what used for PVW, we may write the Principle of Complementary Virtual Work, PCVW, as:

$$\int_S u_i\delta T_i dS + \int_V u_i\delta f_i dV = \int_V \varepsilon_{ij}\delta\sigma_{ij} dV \qquad (5.14)$$

Illustrative Example II

Consider a simple pin-connected structure as depicted in Fig. 5.3. The aim is to determine the vertical movement of the point C. We assume linear elastic behavior, identical cross section, and same modulus of elasticity for all the members.

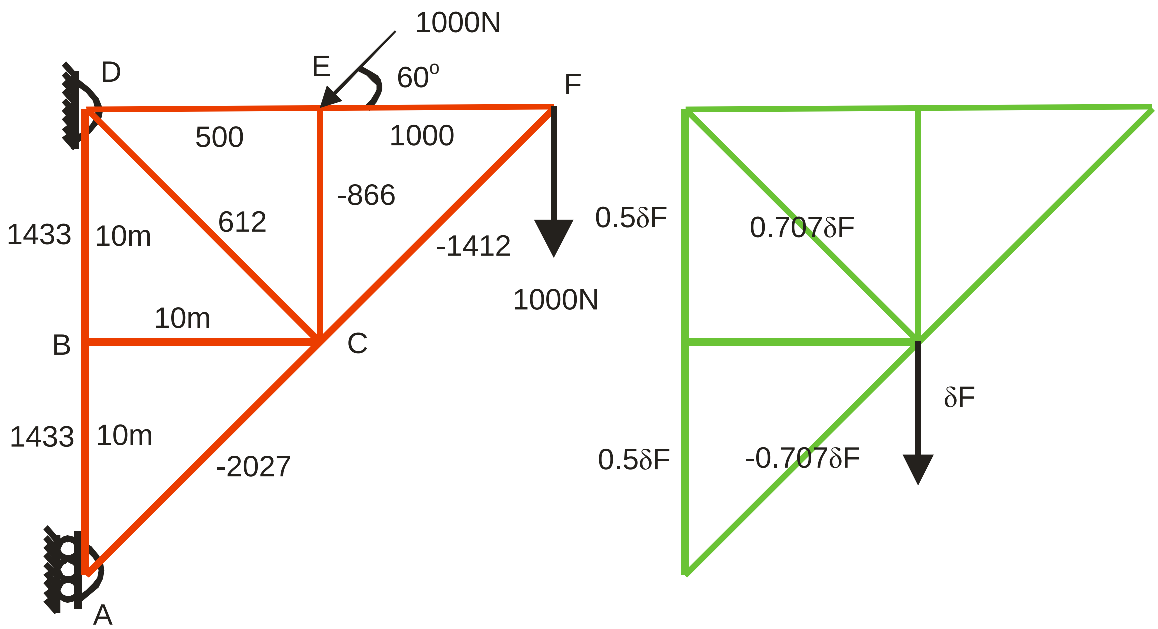

Fig. 5.3

Since we require the vertical displacement of the point C, we apply a virtual vertical load at this point. Next, we will obtain the virtual stresses due to this virtual load.

$$\begin{aligned} \delta\sigma_{CD} &= \frac{0.707\delta F}{A} \\ \delta\sigma_{AC} &= -\frac{0.707\delta F}{A} \\ \delta\sigma_{AB} = \delta\sigma_{BD} &= \frac{0.5\delta F}{A} \end{aligned} \tag{5.15}$$

Now we obtain the real strains in the relevant members. Note that we only consider those members which are affected by the virtual load.

$$\begin{aligned} \varepsilon_{CD} &= \frac{612}{AE} \\ \varepsilon_{AC} &= -\frac{2027}{AE} \\ \varepsilon_{AB} = \varepsilon_{BD} &= \frac{1433}{AE} \end{aligned} \tag{5.16}$$

Accordingly, the principle of complementary virtual work will be:

$$\begin{aligned} \delta F v = &\left(\frac{612}{AE}\right)\left(\frac{0.707\delta F}{A}\right)14.14A + \left(\frac{-2027}{AE}\right)\left(\frac{-0.707\delta F}{A}\right)14.14A \\ &+2\left(\frac{1433}{AE}\right)\left(\frac{0.5\delta F}{A}\right)10A \end{aligned} \tag{5.17}$$

which can be solved for the displacement to give:

$$v = \frac{40720}{AE} \tag{5.18}$$

Principle of Total Potential Energy

The general form of the constitutive expressions for an elastic continuum can be written as:

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial \varepsilon_{ij}} \tag{5.19}$$

in which ρ is the density and ψ is the strain energy density. If we substitute the above expression for stresses in Eq. (5.3), we will have:

$$\begin{aligned} &\int_S T_i \delta u_i dS + \int_V f_i \delta u_i dV = \int_V \rho \frac{\partial \psi}{\partial \varepsilon_{ij}} \delta \varepsilon_{ij} dV \\ &\int_S T_i \delta u_i dS + \int_V f_i \delta u_i dV = \int_V \rho \delta \psi dV = \delta \int_V \rho \psi dV = \delta U \end{aligned} \tag{5.20}$$

where U is the total strain energy stored in the body. The left side of Eq. (5.20) may be defined as a variation in the potential energy, $-V$, so we may write:

$$\begin{aligned} &-\delta V = \delta U \\ &\Rightarrow \delta(U+V) = 0 \\ &or \quad \delta \Pi = 0 \end{aligned} \tag{5.21}$$

in which Π is the total potential energy of the body.

Strain Energy

For elastic bodies the strain energy can be defined as:

$$U = \int_V \left(\int_0^{\varepsilon_{ij}} \sigma_{ij} d\varepsilon_{ij} \right) dV \tag{5.22}$$

If we substitute the constitutive equation for the linear elastic response of isotropic materials into Eq. (5.22), we have:

$$U = \int_V \left(\frac{\lambda}{2} \varepsilon_{kk} \varepsilon_{jj} + \mu \varepsilon_{ij} \varepsilon_{ij} \right) dV \tag{5.23}$$

for which the one-dimensional form is:

$$U = \int_V \int_0^e E e de dV = \frac{1}{2} \int_V E e^2 dV = \frac{1}{2} \int_V \sigma e dV \tag{5.24}$$

Assignment 14:

Solve the illustrative example I, using the principle of total potential energy.

Principle of Total Complementary Energy

The principle of total complementary energy can be derived from the principle of complementary virtual work by considering a specific complementary strain energy density as a function of stresses such that:

$$\varepsilon_{ij} = \rho \frac{\partial \psi^*}{\partial \sigma_{ij}} \tag{5.25}$$

Accordingly, we may write:

$$\begin{aligned} \int_S u_i \delta T_i dS + \int_V u_i \delta f_i dV &= \int_V \rho \frac{\partial \psi^*}{\partial \sigma_{ij}} \delta \sigma_{ij} dV \\ &= \int_V \rho \delta \psi^* dV = \delta \int_V \rho \psi^* dV = \delta U^* \end{aligned} \tag{5.26}$$

where U^* is the complementary strain energy stored in the body. The left side of Eq. (5.26) may be defined as a variation in the potential energy, $-V^*$, so we may write:

$$\begin{aligned} &-\delta V^* = \delta U^* \\ &\Rightarrow \delta(U^* + V^*) = 0 \\ &or \quad \delta \Pi^* = 0 \end{aligned} \tag{5.27}$$

in which Π^* is the total complementary energy of the body.

Complementary Strain Energy

For elastic bodies the complementary strain energy can be defined as:

$$U^* = \int_V \left(\int_0^{\sigma_{ij}} \varepsilon_{ij} d\sigma_{ij} \right) dV \tag{5.28}$$

If we substitute the constitutive equation for the linear elastic response of isotropic materials into Eq. (5.28), we will have:

$$U^* = \int_V \left(-\frac{\nu}{2E} \sigma_{kk} \sigma_{jj} + \frac{1}{G} \sigma_{ij} \sigma_{ij} \right) dV \tag{5.29}$$

for which the one-dimensional form is:

$$U^* = \int_V \int_0^\sigma \frac{\sigma}{E} d\sigma dV = \frac{1}{2}\int_V \frac{\sigma^2}{E} dv = \frac{1}{2}\int_V \sigma e dV \tag{5.30}$$

It is clear that ***only*** for linear elasticity we have $U = U^*$.

The First Castigliano Theorem

In this section we will derive the first Castigliano theorem using the principle of total potential energy. Consider an elastic body maintained in a state of static or dynamic equilibrium as depicted in Fig. 5.4. The body rests on a system of rigid supports while being under the action of externally applied loads.

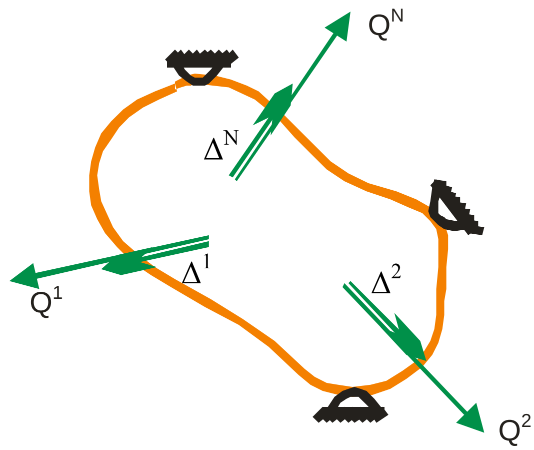

Fig. 5.4

Hence, we may write:

$$\Pi = U\left(\Delta^i\right) - \sum_{j=1}^{N} Q^j \Delta^j \quad ; \quad V = -\sum_{j=1}^{N} Q^j \Delta^j \tag{5.31}$$

Implementing the principle of total potential energy, we have:

$$\begin{aligned} &\delta\Pi = \sum_{i=1}^{N} \frac{\partial \Pi}{\partial \Delta^i} \delta\Delta^i = 0 \\ &\Rightarrow \sum_{i=1}^{N} \left(\frac{\partial U}{\partial \Delta^i} - Q^i \right) \delta\Delta^i = 0 \\ &\Rightarrow Q^i = \frac{\partial U}{\partial \Delta^i} \quad ; \quad i = 1, \ldots, N \end{aligned} \tag{5.32}$$

The above equation shows that if the strain energy can be computed as a function of displacements, the related force or torque for a particular displacement component can be determined by differentiation.

Illustrative Example III

Consider the problem shown in Fig. 5.5. We intend to determine the force required to cause the joint to descend a given distance Δ.

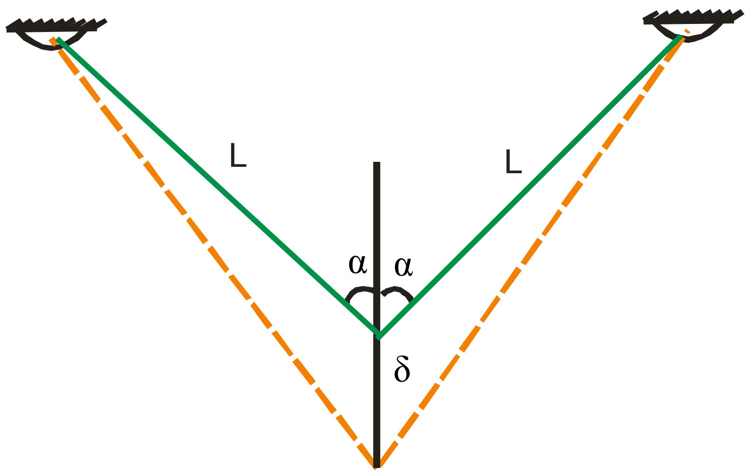

Fig. 5.5

The deformed length of each member is:

$$L + \Delta \cos\alpha \tag{5.33}$$

Hence, we may write:

$$e = \frac{\Delta \cos\alpha}{L} \Rightarrow$$
$$U = 2\int_V \int_0^e \sigma de dV = 2ALE\frac{e^2}{2} \tag{5.34}$$
$$= \frac{EA}{L}\cos^2\alpha\Delta^2$$

Now, using the first castigliano's theorem, we have:

$$P = \frac{dU}{d\Delta} = \frac{2AE\Delta}{L}\cos^2\alpha \tag{5.35}$$

The Second Castigliano Theorem

Here, we utilize the principle of total complementary energy.

$$\Pi^* = U^*\left(Q^i\right) - \sum_{j=1}^{N} Q^j \Delta^j \quad ; \quad V^* = -\sum_{j=1}^{N} Q^j \Delta^j \tag{5.36}$$

Accordingly, we may write:

$$\begin{aligned} &\delta\Pi^* = \sum_{i=1}^{N} \frac{\partial \Pi^*}{\partial Q^i} \delta Q^i = 0 \\ &\Rightarrow \sum_{i=1}^{N} \left(\frac{\partial U^*}{\partial Q^i} - \Delta^i \right) \delta Q^i = 0 \\ &\Rightarrow \Delta^i = \frac{\partial U^*}{\partial Q^i} \quad ; \quad i = 1, \ldots, N \end{aligned} \tag{5.37}$$

The above equation shows that if the complementary strain energy can be computed as a function of forces, the related displacement for a particular force component can be determined by differentiation.

An Introduction to Energy Functionals

In general, we may present a partial differential equation by:

$$Lu = f \tag{5.38}$$

where L is an operator acting on the dependent variable u, and f is a function of the independent variable and is called the driving or forcing function. We may also define the inner product of two functions, $\mathbf{g}$ and $\mathbf{h}$, over the domain of the problem, *V*, as:

$$\left(\mathbf{g}, \mathbf{h}\right) = \int_V \mathbf{g} \cdot \mathbf{h} dV \quad \text{or} \quad \left(g, h\right) = \int_V gh dV \tag{5.39}$$

If the operator L is self-adjoint or symmetric, we will have:

$$\left(Lu, v\right) = \left(u, Lv\right) \tag{5.40}$$

In our discussions we will also require that L be positive definite for all functions u satisfying homogenous boundary conditions:

$$\left(Lu, u\right) > 0 \tag{5.41}$$

The general expression for homogenous boundary condition is:

$$\alpha u_i + \beta \frac{\partial u_i}{\partial n} = 0 \tag{5.42}$$

where α and β are constants and n is the normal to the boundary. Now we may define a function $I(u)$, called a *quadratic functional*, such that:

$$I(u) = \frac{1}{2}\left(Lu, u\right) - (u, f) \tag{5.43}$$

It can be shown that if the equation $Lu = f$ has a solution, and if L is a self-adjoint positive definite operator, then the function u that minimizes $I(u)$ is the solution to $Lu = f$. The minimization procedure can also be used to find approximate solutions.

Variational Principles Applied to Beams

We start with the definition of the strain energy for a linear elastic beam:

$$\begin{aligned} U = U^* &= \int_V \int_0^{\sigma_{ij}} \varepsilon_{ij} d\sigma_{ij} dV \\ &= \frac{1}{2} b \int_0^l \int_{-h/2}^{h/2} \sigma_{11}\varepsilon_{11} dx_3 dx_1 \\ &= \frac{1}{2} bE \int_0^l \int_{-h/2}^{h/2} \varepsilon_{11}^2 dx_3 dx_1 \\ &= \frac{1}{2} bE \int_0^l \int_{-h/2}^{h/2} \left(-x_3 \omega_{,11}\right)^2 dx_3 dx_1 \\ &= \frac{EI}{2} \int_0^l \left(\omega_{,11}\right)^2 dx_1 \end{aligned} \tag{5.44}$$

The potential V is defined by:

$$V = -\int_V f_i u_i dV - \int_S T_i u_i dS = -\int_0^l q\omega dx_1 \tag{5.45}$$

Finally, the total potential energy can be written as:

$$\Pi = U + V = \int_0^l \left[\frac{EI}{2}\left(\omega_{,11}\right)^2 - q\omega \right] dx_1 \tag{5.46}$$

which has the same form as Eq. (5.43).

The Ritz Technique

Based on the above definitions it may be shown, either by the application of variational principles to the Navier displacement equations of equilibrium or by comparing Eq.(5.46) with Eq.(5.43), that for homogenous boundary conditions the quadratic functional is:

$$I(u) = \Pi \tag{5.47}$$

Hence, in order to find approximate solutions to our beam problem, we may apply the ***Ritz technique*** to the total potential energy. The procedure starts with the approximate displacement field defined as:

$$u_i^{(n)} = \sum_{j=1}^{n} a_i^{(j)} \varphi_i^{(j)} \qquad \not{\Sigma} i \tag{5.48}$$

The $a_i^{(j)}$ are called the **Ritz coefficients** and $\varphi_i^{(j)}$ are known as **coordinate functions**. The coefficients $a_i^{(j)}$ are determined so as to minimize the total potential energy $\Pi^{(n)}$, which implies that:

$$\frac{\partial \Pi^{(n)}}{\partial a_i^{(j)}} = 0 \quad , \quad j = 1, \cdots, n \tag{5.49}$$

The above differentiation results in *3n* equations for the unknown coefficients.

Illustrative Example IV

Let us find approximate expressions for the deflection of the beam shown in Fig. 5.6.

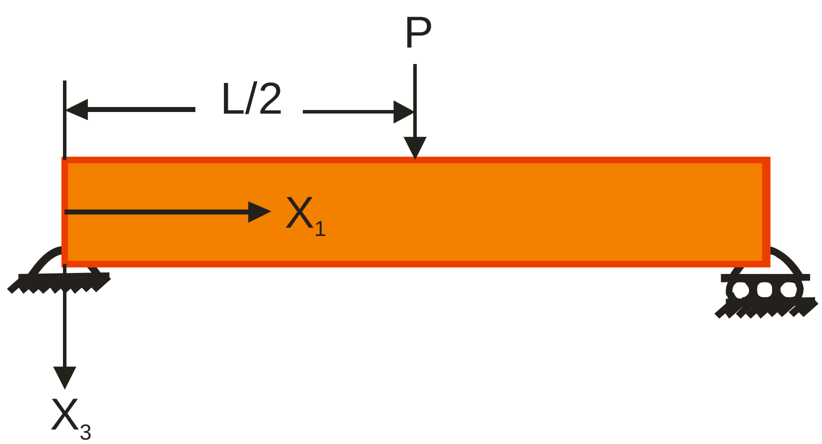

Fig. 5.6

The exact solution to this problem is:

$$\omega = \frac{PL^3}{48EI}\left[3\left(\frac{x_1}{L}\right) - 4\left(\frac{x_1}{L}\right)^3\right] \quad ; \quad x_1 < \frac{L}{2} \tag{5.50}$$

We start by finding the total potential energy as follows:

$$\Pi = U + V = \int_0^L \left[\frac{EI}{2}\left(\omega_{,11}\right)^2\right] dx_1 - P\omega\Big|_{x_1=\frac{L}{2}} \tag{5.51}$$

We may choose our coordinate function as:

$$\omega^{(1)} = a^{(1)} \sin\frac{\pi x_1}{L} \tag{5.52}$$

Accordingly the total potential energy can be written as:

$$\Pi^{(1)} = \frac{EI}{2}\left(a^{(1)}\right)^2 \left(\frac{\pi}{L}\right)^4 \frac{L}{2} - Pa^{(1)} \tag{5.53}$$

Now we may find the coefficients through the minimization of $\Pi^{(1)}$ and determine the displacement function as follows:

$$\begin{aligned} &\frac{\partial \Pi^{(1)}}{\partial a^{(1)}} = \frac{EIL}{2} a^{(1)} \left(\frac{\pi}{L}\right)^4 - P = 0 \\ &\Rightarrow a^{(1)} = \frac{2PL^3}{\pi^4 EI} \\ &\Rightarrow \omega^{(1)} = \frac{2PL^3}{\pi^4 EI} \sin\frac{\pi x_1}{L} \end{aligned} \tag{5.54}$$

Finally, we compare the obtained result with the exact solution at two different points:

	$\frac{L}{4}$	$\frac{L}{2}$
$\frac{EI\omega}{PL^3}$ (exact)	0.01432	0.02083
$\frac{EI\omega^{(1)}}{PL^3}$ (Ritz)	0.01452	0.02053

which shows a very good agreement.