

# INTRODUCTION TO FUNCTIONAL ANALYSIS

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ABSTRACT. This is lecture notes for several courses **on Functional Analysis** at **School of Mathematics** of **University of Leeds**. They are based on the notes of Dr. Matt Daws, **Prof. Jonathan R. Partington** and Dr. David Salinger used in the previous years. However all misprints, omissions, and errors are only my responsibility. I am very grateful to Filipa Soares de Almeida, Eric Borgnet, Pasc Gavruta for pointing out some of them. Please **let me know** if you find more.

The notes are available also for download in **PDF**.

The suggested textbooks are [1,5,7,8]. The other nice books with many interesting problems are [2,6].

Exercises with stars **are not** a part of mandatory material but are nevertheless worth to hear about. And they are not necessarily difficult, try to solve them!

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## NOTATIONS AND ASSUMPTIONS

$\mathbb{Z}_+, \mathbb{R}_+$  denotes non-negative integers and reals.

$x, y, z, \dots$  denotes vectors.

$\lambda, \mu, \nu, \dots$  denotes scalars.

$\Re z, \Im z$  stand for real and imaginary parts of a complex number  $z$ .

**Integrability conditions.** In this course, the functions we consider will be real or complex valued functions defined on the real line which are *locally Riemann integrable*. This means that they are Riemann integrable on any finite closed interval  $[a, b]$ . (A complex valued function is Riemann integrable iff its real and imaginary parts are Riemann-integrable.) In practice, we shall be dealing mainly with bounded functions that have only a finite number of points of discontinuity in any finite interval. We can relax the boundedness condition to allow improper Riemann integrals, but we then require the integral of the absolute value of the function to converge.

We mention this right at the start to get it out of the way. There are many fascinating subtleties connected with Fourier analysis, but those connected with technical aspects of integration theory are beyond the scope of the course. It turns out that one needs a "better" integral than the Riemann integral: the Lebesgue integral, and I commend the module, Linear Analysis 1, which includes an introduction to that topic which is available to MM students (or you could look it up in *Real and Complex Analysis* by Walter Rudin). Once one has the Lebesgue integral, one can start thinking about the different classes of functions to which Fourier

analysis applies: the modern theory (not available to Fourier himself) can even go beyond functions and deal with generalized functions (distributions) such as the Dirac delta function which may be familiar to some of you from quantum theory.

From now on, when we say “function”, we shall assume the conditions of the first paragraph, unless anything is stated to the contrary.

## 1. MOTIVATING EXAMPLE: FOURIER SERIES

**1.1. Fourier series: basic notions.** Before proceed with an abstract theory we consider a motivating example: Fourier series.

**1.1.1.  $2\pi$ -periodic functions.** In this part of the course we deal with functions (as above) that are periodic.

We say a function  $f : \mathbb{R} \rightarrow \mathbb{C}$  is *periodic* with *period*  $T > 0$  if  $f(x + T) = f(x)$  for all  $x \in \mathbb{R}$ . For example,  $\sin x$ ,  $\cos x$ ,  $e^{ix} (= \cos x + i \sin x)$  are periodic with period  $2\pi$ . For  $k \in \mathbb{R} \setminus \{0\}$ ,  $\sin kx$ ,  $\cos kx$ , and  $e^{ikx}$  are periodic with period  $2\pi/|k|$ . Constant functions are periodic with period  $T$ , for any  $T > 0$ . We shall specialize to periodic functions with period  $2\pi$ : we call them  $2\pi$ -periodic functions, for short. Note that  $\cos nx$ ,  $\sin nx$  and  $e^{inx}$  are  $2\pi$ -periodic for  $n \in \mathbb{Z}$ . (Of course these are also  $2\pi/|n|$ -periodic.)

Any half-open interval of length  $T$  is a *fundamental domain* of a periodic function  $f$  of period  $T$ . Once you know the values of  $f$  on the fundamental domain, you know them everywhere, because any point  $x$  in  $\mathbb{R}$  can be written uniquely as  $x = w + nT$  where  $n \in \mathbb{Z}$  and  $w$  is in the fundamental domain. Thus  $f(x) = f(w + (n - 1)T + T) = \dots = f(w + T) = f(w)$ .

For  $2\pi$ -periodic functions, we shall usually take the fundamental domain to be  $]-\pi, \pi]$ . By abuse of language, we shall sometimes refer to  $[-\pi, \pi]$  as the fundamental domain. We then have to be aware that  $f(\pi) = f(-\pi)$ .

**1.1.2. Integrating the complex exponential function.** We shall need to calculate  $\int_a^b e^{ikx} dx$ , for  $k \in \mathbb{R}$ . Note first that when  $k = 0$ , the integrand is the constant function 1, so the result is  $b - a$ . For non-zero  $k$ ,  $\int_a^b e^{ikx} dx = \int_a^b (\cos kx + i \sin kx) dx = (1/k)[\sin kx - i \cos kx]_a^b = (1/ik)[(\cos kx + i \sin kx)]_a^b = (1/ik)[e^{ikx}]_a^b = (1/ik)(e^{ikb} - e^{ika})$ . Note that this is exactly the result you would have got by treating  $i$  as a real constant and using the usual formula for integrating  $e^{ax}$ . Note also that the cases  $k = 0$  and  $k \neq 0$  have to be treated separately: this is typical.

**Definition 1.1.** Let  $f : \mathbb{R} \rightarrow \mathbb{C}$  be a  $2\pi$ -periodic function which is Riemann integrable on  $[-\pi, \pi]$ . For each  $n \in \mathbb{Z}$  we define the *Fourier coefficient*  $\hat{f}(n)$  by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

**Remark 1.2.** (i)  $\hat{f}(n)$  is a complex number whose modulus is the amplitude and whose argument is the phase (of that component of the original function).

- (ii) If  $f$  and  $g$  are Riemann integrable on an interval, then so is their product, so the integral is well-defined.
- (iii) The constant before the integral is to divide by the length of the interval.
- (iv) We could replace the range of integration by any interval of length  $2\pi$ , without altering the result, since the integrand is  $2\pi$ -periodic.
- (v) Note the minus sign in the exponent of the exponential. The reason for this will soon become clear.

- Example 1.3.** (i)  $f(x) = c$  then  $\hat{f}(0) = c$  and  $\hat{f}(n) = 0$  when  $n \neq 0$ .  
 (ii)  $f(x) = e^{ikx}$ , where  $k$  is an integer.  $\hat{f}(n) = \delta_{nk}$ .  
 (iii)  $f$  is  $2\pi$  periodic and  $f(x) = x$  on  $]-\pi, \pi]$ . (Diagram) Then  $\hat{f}(0) = 0$  and, for  $n \neq 0$ ,

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x e^{-inx} dx = \left[ \frac{-x e^{-inx}}{2\pi i n} \right]_{-\pi}^{\pi} + \frac{1}{in} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-inx} dx = \frac{(-1)^n i}{n}.$$

**Proposition 1.4 (Linearity).** *If  $f$  and  $g$  are  $2\pi$ -periodic functions and  $c$  and  $d$  are complex constants, then, for all  $n \in \mathbb{Z}$ ,*

$$(cf + dg)\hat{\phantom{f}}(n) = c\hat{f}(n) + d\hat{g}(n).$$

**Corollary 1.5.** *If  $p(x) = \sum_{-k}^k c_n e^{inx}$ , then  $\hat{p}(n) = c_n$  for  $|n| \leq k$  and  $= 0$ , for  $|n| \geq k$ .*

$$p(x) = \sum_{n \in \mathbb{Z}} \hat{p}(n) e^{inx}.$$

This follows immediately from Ex. 1.3(ii) and Prop.1.4.

- Remark 1.6.* (i) This corollary explains why the minus sign is natural in the definition of the Fourier coefficients.  
 (ii) The first part of the course will be devoted to the question of how far this result can be extended to other  $2\pi$ -periodic functions, that is, for which functions, and for which interpretations of infinite sums is it true that

$$(1.1) \quad f(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}.$$

**Definition 1.7.**  $\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}$  is called the *Fourier series* of the  $2\pi$ -periodic function  $f$ .

For real-valued functions, the introduction of complex exponentials seems artificial: indeed they can be avoided as follows. We work with (1.1) in the case of a finite sum: then we can rearrange the sum as

$$\begin{aligned} & \hat{f}(0) + \sum_{n>0} (\hat{f}(n) e^{inx} + \hat{f}(-n) e^{-inx}) \\ &= \hat{f}(0) + \sum_{n>0} [(\hat{f}(n) + \hat{f}(-n)) \cos nx + i(\hat{f}(n) - \hat{f}(-n)) \sin nx] \\ &= \frac{a_0}{2} + \sum_{n>0} (a_n \cos nx + b_n \sin nx) \end{aligned}$$

Here

$$\begin{aligned} a_n &= (\hat{f}(n) + \hat{f}(-n)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) (e^{-inx} + e^{inx}) dx \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx \end{aligned}$$

for  $n > 0$  and

$$b_n = i(\hat{f}(n) - \hat{f}(-n)) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

for  $n > 0$ .  $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$ , the constant chosen for consistency.

The  $a_n$  and  $b_n$  are also called Fourier coefficients: if it is necessary to distinguish them, we may call them *Fourier cosine* and *sine coefficients*, respectively.

We note that if  $f$  is real-valued, then the  $a_n$  and  $b_n$  are real numbers and so  $\Re \hat{f}(n) = \Re \hat{f}(-n)$ ,  $\Im \hat{f}(n) = -\Im \hat{f}(-n)$ : thus  $\hat{f}(-n)$  is the complex conjugate of  $\hat{f}(n)$ . Further, if  $f$  is an even function then all the sine coefficients are 0 and if  $f$  is an odd function, all the cosine coefficients are zero. We note further that the sine and cosine coefficients of the functions  $\cos kx$  and  $\sin kx$  themselves have a particularly simple form:  $a_k = 1$  in the first case and  $b_k = 1$  in the second. All the rest are zero.

For example, we should expect the  $2\pi$ -periodic function whose value on  $] -\pi, \pi]$  is  $x$  to have just sine coefficients: indeed this is the case:  $a_n = 0$  and  $b_n = i(\hat{f}(n) - \hat{f}(-n)) = (-1)^{n+1}2/n$  for  $n > 0$ .

The above question can then be reformulated as “to what extent is  $f(x)$  represented by the Fourier series  $a_0/2 + \sum_{n>0}(a_n \cos x + b_n \sin x)$ ?” For instance how well does  $\sum (-1)^{n+1}(2/n) \sin nx$  represent the  $2\pi$ -periodic sawtooth function  $f$  whose value on  $] -\pi, \pi]$  is given by  $f(x) = x$ . The easy points are  $x = 0$ ,  $x = \pi$ , where the terms are identically zero. This gives the ‘wrong’ value for  $x = \pi$ , but, if we look at the periodic function near  $\pi$ , we see that it jumps from  $\pi$  to  $-\pi$ , so perhaps the mean of those values isn’t a bad value for the series to converge to. We could conclude that we had defined the function incorrectly to begin with and that its value at the points  $(2n + 1)\pi$  should have been zero anyway. In fact one can show (ref. ) that the Fourier series converges at all other points to the given values of  $f$ , but I shan’t include the proof in this course. The convergence is not at all uniform (it can’t be, because the partial sums are continuous functions, but the limit is discontinuous.) In particular we get the expansion

$$\frac{\pi}{2} = 2(1 - 1/3 + 1/5 - \dots)$$

which can also be deduced from the Taylor series for  $\tan^{-1}$ .

**1.2. The vibrating string.** In this subsection we shall discuss the formal solutions of the wave equation in a special case which Fourier dealt with in his work.

We discuss the wave equation

$$(1.2) \quad \frac{\partial^2 y}{\partial x^2} = \frac{1}{K^2} \frac{\partial^2 y}{\partial t^2},$$

subject to the boundary conditions

$$(1.3) \quad y(0, t) = y(\pi, t) = 0,$$

for all  $t \geq 0$ , and the initial conditions

$$\begin{aligned} y(x, 0) &= F(x), \\ y_t(x, 0) &= 0. \end{aligned}$$

This is a mathematical model of a string on a musical instrument (guitar, harp, violin) which is of length  $\pi$  and is plucked, i.e. held in the shape  $F(x)$  and released at time  $t = 0$ . The constant  $K$  depends on the length, density and tension of the string. We shall derive the formal solution (that is, a solution which assumes existence and ignores questions of convergence or of domain of definition).

**1.2.1. Separation of variables.** We first look (as Fourier and others before him did) for solutions of the form  $y(x, t) = f(x)g(t)$ . Feeding this into the wave equation (1.2) we get

$$f''(x)g(t) = \frac{1}{K^2} f(x)g''(t)$$

and so, dividing by  $f(x)g(t)$ , we have

$$(1.4) \quad \frac{f''(x)}{f(x)} = \frac{1}{K^2} \frac{g''(t)}{g(t)}.$$

The left-hand side is an expression in  $x$  alone, the right-hand side in  $t$  alone. The conclusion must be that they are both identically equal to the same constant  $C$ , say.

We have  $f''(x) - Cf(x) = 0$  subject to the condition  $f(0) = f(\pi) = 0$ . Working through the method of solving linear second order differential equations tells you that the only solutions occur when  $C = -n^2$  for some positive integer  $n$  and the corresponding solutions, up to constant multiples, are  $f(x) = \sin nx$ .

Returning to equation (1.4) gives the equation  $g''(t) + K^2n^2g(t) = 0$  which has the general solution  $g(t) = a_n \cos Knt + b_n \sin Knt$ . Thus the solution we get through separation of variables, using the boundary conditions but ignoring the initial conditions, are

$$y_n(x, t) = \sin nx(a_n \cos Knt + b_n \sin Knt),$$

for  $n \geq 1$ .

1.2.2. *Principle of Superposition.* To get the general solution we just add together all the solutions we have got so far, thus

$$(1.5) \quad y(x, t) = \sum_{n=1}^{\infty} \sin nx(a_n \cos Knt + b_n \sin Knt)$$

ignoring questions of convergence. (We can do this for a finite sum without difficulty because we are dealing with a linear differential equation: the iffy bit is to extend to an infinite sum.)

We now apply the initial condition  $y(x, 0) = F(x)$  (note  $F$  has  $F(0) = F(\pi) = 0$ ). This gives

$$F(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

We apply the reflection trick: the right-hand side is a series of odd functions so if we extend  $F$  to a function  $G$  by reflection in the origin, giving

$$G(x) := \begin{cases} F(x) & , \text{ if } 0 \leq x \leq \pi; \\ -F(-x) & , \text{ if } -\pi < x < 0. \end{cases}$$

we have

$$G(x) = \sum_{n=1}^{\infty} a_n \sin nx,$$

for  $-\pi \leq x \leq \pi$ .

If we multiply through by  $\sin rx$  and integrate term by term, we get

$$a_r = \frac{1}{\pi} \int_{-\pi}^{\pi} G(x) \sin rx \, dx$$

so, assuming that this operation is valid, we find that the  $a_n$  are precisely the sine coefficients of  $G$ . (Those of you who took Real Analysis 2 last year may remember that a sufficient condition for integrating term-by-term is that the series which is integrated is itself uniformly convergent.)

If we now assume, further, that the right-hand side of (1.5) is differentiable (term by term) we differentiate with respect to  $t$ , and set  $t = 0$ , to get

$$(1.6) \quad 0 = y_t(x, 0) = \sum_{n=1}^{\infty} b_n Kn \sin nx.$$

This equation is solved by the choice  $b_n = 0$  for all  $n$ , so we have the following result

**Proposition 1.8** (Formal). *Assuming that the formal manipulations are valid, a solution of the differential equation (1.2) with the given boundary and initial conditions is*

$$(2.11) \quad y(x, t) = \sum_1^{\infty} a_n \sin nx \cos Knt ,$$

where the coefficients  $a_n$  are the Fourier sine coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} G(x) \sin nx \, dx$$

of the  $2\pi$  periodic function  $G$ , defined on  $]-\pi, \pi[$  by reflecting the graph of  $F$  in the origin.

*Remark 1.9.* This leaves us with the questions

- (i) For which  $F$  are the manipulations valid?
- (ii) Is this the only solution of the differential equation? (which I'm not going to try to answer.)
- (iii) Is  $b_n = 0$  all  $n$  the only solution of (1.6)? This is a special case of the **uniqueness** problem for trigonometric series.

**1.3. Historic: Joseph Fourier.** *Joseph Fourier*, Civil Servant, Egyptologist, and mathematician, was born in 1768 in Auxerre, France, son of a tailor. Debarred by birth from a career in the artillery, he was preparing to become a Benedictine monk (in order to be a teacher) when the French Revolution violently altered the course of history and Fourier's life. He became president of the local revolutionary committee, was arrested during the Terror, but released at the fall of Robespierre.

Fourier then became a pupil at the Ecole Normale (the teachers' academy) in Paris, studying under such great French mathematicians as Laplace and Lagrange. He became a teacher at the Ecole Polytechnique (the military academy).

He was ordered to serve as a scientist under Napoleon in Egypt. In 1801, Fourier returned to France to become Prefect of the Grenoble region. Among his most notable achievements in that office were the draining of some 20 thousand acres of swamps and the building of a new road across the alps.

During that time he wrote an important survey of Egyptian history ("a masterpiece and a turning point in the subject").

In 1804 Fourier started the study of the theory of heat conduction, in the course of which he systematically used the sine-and-cosine series which are named after him. At the end of 1807, he submitted a memoir on this work to the Academy of Science. The memoir proved controversial both in terms of his use of Fourier series and of his derivation of the heat equation and was not accepted at that stage. He was able to resubmit a revised version in 1811: this had several important new features, including the introduction of the Fourier transform. With this version of his memoir, he won the Academy's prize in mathematics. In 1817, Fourier was finally elected to the Academy of Sciences and in 1822 his 1811 memoir was published as "Théorie de la Chaleur".

For more details see *Fourier Analysis* by T.W. Körner, 475-480 and for even more, see the biography by J. Herivel *Joseph Fourier: the man and the physicist*.

*What is Fourier analysis.* The idea is to analyse functions (into sine and cosines or, equivalently, complex exponentials) to find the underlying frequencies, their strengths (and phases) and, where possible, to see if they can be recombined (synthesis) into the original function. The answers will depend on the original properties of the functions, which often come from physics (heat, electronic or sound



waves). This course will give basically a mathematical treatment and so will be interested in mathematical classes of functions (continuity, differentiability properties).

## 2. BASICS OF LINEAR SPACES

A person is solely the concentration of an infinite set of inter-relations with another and others, and to separate a person from these relations means to take away any real meaning of the life.

VI. Soloviev

A space around us could be described as a three dimensional Euclidean space. To single out a point of that space we need a fixed *frame of references* and three real numbers, which are *coordinates* of the point. Similarly to describe a pair of points from our space we could use six coordinates; for three points—nine, end so on. This makes it reasonable to consider Euclidean (linear) spaces of an arbitrary finite dimension, which are studied in the courses of **linear algebra**.

The basic properties of Euclidean spaces are determined by its *linear* and *metric* structures. The *linear space* (or *vector space*) structure allows to **add and subtract vectors** associated to points as well as to **multiply vectors** by real or complex numbers (scalars).

The *metric space* structure assign a *distance*—non-negative real number—to a pair of points or, equivalently, defines a *length of a vector* defined by that pair. A metric (or, more generally a topology) is essential for definition of the core analytical notions like **limit** or **continuity**. The importance of linear and metric (topological) structure in analysis sometime encoded in the formula:

$$(2.1) \quad \text{Analysis} = \text{Algebra} + \text{Geometry} .$$

On the other hand we could observe that many sets admit a sort of linear *and* metric structures which are linked each other. Just few among many other examples are:

- The set of convergent sequences;
- The set of continuous functions on  $[0, 1]$ .

It is a very *mathematical way of thinking* to declare such sets to be *spaces* and call their elements *points*.

But shall we lose all information on a particular element (e.g. a sequence  $\{1/n\}$ ) if we represent it by a shapeless and size-less “point” without any inner configuration? Surprisingly not: all properties of an element could be now retrieved not from its *inner configuration* but from interactions with other elements through linear and metric structures. Such a “sociological” approach to all kind of mathematical objects was codified in the abstract **category theory**.

Another surprise is that starting from our three dimensional Euclidean space and walking far away by a road of abstraction to infinite dimensional Hilbert spaces we are arriving just to yet another picture of the surrounding space—that time on the language of *quantum mechanics*.

The distance from Manchester to Liverpool is 35 miles—just about the mileage in the opposite direction!

*A tourist guide to England*

**2.1. Banach spaces (basic definitions only).** The following definition generalises the notion of *distance* known from the everyday life.

**Definition 2.1.** A *metric* (or *distance function*)  $d$  on a set  $M$  is a function  $d : M \times M \rightarrow \mathbb{R}_+$  from the set of pairs to non-negative real numbers such that:

- (i)  $d(x, y) \geq 0$  for all  $x, y \in M$ ,  $d(x, y) = 0$  implies  $x = y$ .

- (ii)  $d(x, y) = d(y, x)$  for all  $x$  and  $y$  in  $M$ .
- (iii)  $d(x, y) + d(y, z) \geq d(x, z)$  for all  $x, y,$  and  $z$  in  $M$  (*triangle inequality*).

**Exercise 2.2.** Let  $M$  be the set of UK's cities are the following function are metrics on  $M$ :

- (i)  $d(A, B)$  is the price of 2nd class railway ticket from  $A$  to  $B$ .
- (ii)  $d(A, B)$  is the off-peak driving time from  $A$  to  $B$ .

The following notion is a useful specialisation of metric adopted to the linear structure.

**Definition 2.3.** Let  $V$  be a (real or complex) vector space. A *norm* on  $V$  is a real-valued function, written  $\|x\|$ , such that

- (i)  $\|x\| \geq 0$  for all  $x \in V$ , and  $\|x\| = 0$  implies  $x = 0$ .
- (ii)  $\|\lambda x\| = |\lambda| \|x\|$  for all scalar  $\lambda$  and vector  $x$ .
- (iii)  $\|x + y\| \leq \|x\| + \|y\|$  (*triangle inequality*).

A vector space with a norm is called a *normed space*.

The connection between norm and metric is as follows:

**Proposition 2.4.** If  $\|\cdot\|$  is a norm on  $V$ , then it gives a metric on  $V$  by  $d(x, y) = \|x - y\|$ .

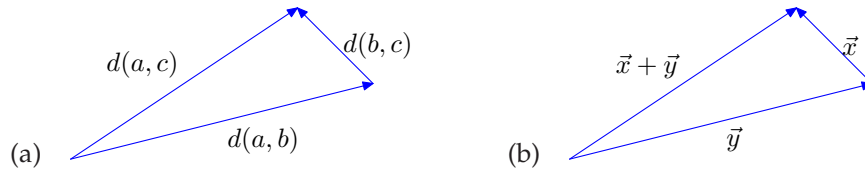


FIGURE 1. Triangle inequality in metric (a) and normed (b) spaces.

*Proof.* This is a simple exercise to derive items 2.1(i)–2.1(iii) of Definition 2.1 from corresponding items of Definition 2.3. For example, see the Figure 1 to derive the triangle inequality.  $\square$

An important notions known from real analysis are **limit and convergence**. Particularly we usually wish to have enough limiting points for all “reasonable” sequences.

**Definition 2.5.** A sequence  $\{x_k\}$  in a metric space  $(M, d)$  is a *Cauchy sequence*, if for every  $\epsilon > 0$ , there exists an integer  $n$  such that  $k, l > n$  implies that  $d(x_k, x_l) < \epsilon$ .

**Definition 2.6.**  $(M, d)$  is a *complete metric space* if every Cauchy sequence in  $M$  converges to a limit in  $M$ .

For example, the set of integers  $\mathbb{Z}$  and reals  $\mathbb{R}$  with the natural distance functions are complete spaces, but the set of rationals  $\mathbb{Q}$  is not. The complete normed spaces deserve a special name.

**Definition 2.7.** A *Banach space* is a complete normed space.

**Exercise\* 2.8.** A convenient way to define a norm in a Banach space is as follows. The *unit ball*  $U$  in a normed space  $B$  is the set of  $x$  such that  $\|x\| \leq 1$ . Prove that:

- (i)  $U$  is a *convex set*, i.e.  $x, y \in U$  and  $\lambda \in [0, 1]$  the point  $\lambda x + (1 - \lambda)y$  is also in  $U$ .
- (ii)  $\|x\| = \inf\{\lambda \in \mathbb{R}_+ \mid \lambda^{-1}x \in U\}$ .
- (iii)  $U$  is closed if and only if the space is Banach.

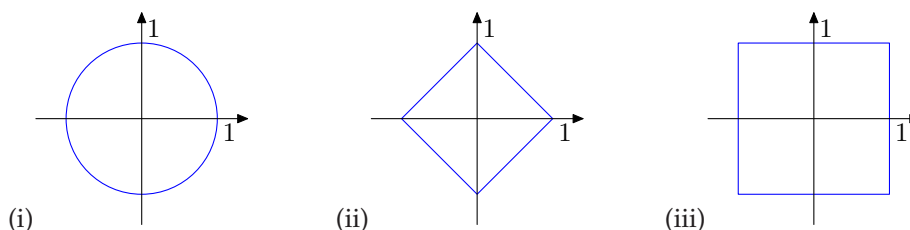


FIGURE 2. Different unit balls defining norms in  $\mathbb{R}^2$  from Example 2.9.

**Example 2.9.** Here is some examples of normed spaces.

- (i)  $\ell_2^n$  is either  $\mathbb{R}^n$  or  $\mathbb{C}^n$  with norm defined by

$$\|(x_1, \dots, x_n)\|_2 = \sqrt{|x_1|^2 + |x_2|^2 + \dots + |x_n|^2}.$$

- (ii)  $\ell_1^n$  is either  $\mathbb{R}^n$  or  $\mathbb{C}^n$  with norm defined by

$$\|(x_1, \dots, x_n)\|_1 = |x_1| + |x_2| + \dots + |x_n|.$$

- (iii)  $\ell_\infty^n$  is either  $\mathbb{R}^n$  or  $\mathbb{C}^n$  with norm defined by

$$\|(x_1, \dots, x_n)\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|).$$

- (iv) Let  $X$  be a topological space, then  $C_b(X)$  is the space of continuous bounded functions  $f : X \rightarrow \mathbb{C}$  with norm  $\|f\|_\infty = \sup_X |f(x)|$ .

- (v) Let  $X$  be any set, then  $\ell_\infty(X)$  is the space of all bounded (not necessarily continuous) functions  $f : X \rightarrow \mathbb{C}$  with norm  $\|f\|_\infty = \sup_X |f(x)|$ .

All these normed spaces are also complete and thus are Banach spaces. Some more examples of both complete and incomplete spaces shall appear later.

—We need an extra space to accommodate this product!  
*A manager to a shop assistant*

**2.2. Hilbert spaces.** Although metric and norm capture important geometric information about linear spaces they are not sensitive enough to represent such geometric characterisation as angles (particularly *orthogonality*). To this end we need a further refinements.

From courses of linear algebra known that the scalar product  $\langle x, y \rangle = x_1y_1 + \dots + x_ny_n$  is important in a space  $\mathbb{R}^n$  and defines a norm  $\|x\|^2 = \langle x, x \rangle$ . Here is a suitable generalisation:

**Definition 2.10.** A *scalar product* (or *inner product*) on a real or complex vector space  $V$  is a mapping  $V \times V \rightarrow \mathbb{C}$ , written  $\langle x, y \rangle$ , that satisfies:

- (i)  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0$  implies  $x = 0$ .
- (ii)  $\langle x, y \rangle = \overline{\langle y, x \rangle}$  in complex spaces and  $\langle x, y \rangle = \langle y, x \rangle$  in real ones for all  $x, y \in V$ .
- (iii)  $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ , for all  $x, y \in V$  and scalar  $\lambda$ . (What is  $\langle x, \lambda y \rangle$ ?)
- (iv)  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ , for all  $x, y$ , and  $z \in V$ . (What is  $\langle x, y + z \rangle$ ?)

Last two properties of the scalar product is oftenly encoded in the phrase: “it is linear in the first variable if we fix the second and anti-linear in the second if we fix the first”.

**Definition 2.11.** An *inner product space*  $V$  is a real or complex vector space with a scalar product on it.

**Example 2.12.** Here is some examples of inner product spaces which demonstrate that expression  $\|x\| = \sqrt{\langle x, x \rangle}$  defines a norm.

- (i) The inner product for  $\mathbb{R}^n$  was defined in the beginning of this section. The inner product for  $\mathbb{C}^n$  is given by  $\langle x, y \rangle = \sum_1^n x_j \bar{y}_j$ . The norm  $\|x\| = \sqrt{\sum_1^n |x_j|^2}$  makes it  $\ell_2^n$  from Example 2.9(i).
- (ii) The extension for infinite vectors: let  $\ell_2$  be

$$(2.2) \quad \ell_2 = \{ \text{sequences } \{x_j\}_1^\infty \mid \sum_1^\infty |x_j|^2 < \infty \}.$$

Let us equip this set with operations of term-wise addition and multiplication by scalars, then  $\ell_2$  is closed under them. Indeed it follows from the **triangle inequality** and properties of **absolutely convergent series**. From the standard **Cauchy–Bunyakovskii–Schwarz inequality** follows that the series  $\sum_1^\infty x_j \bar{y}_j$  **absolutely converges** and its sum defined to be  $\langle x, y \rangle$ .

- (iii) Let  $C_b[a, b]$  be a space of continuous functions on the interval  $[a, b] \in \mathbb{R}$ . As we learn from Example 2.9(iv) a normed space it is a normed space with the norm  $\|f\|_\infty = \sup_{[a, b]} |f(x)|$ . We could also define an inner product:

$$(2.3) \quad \langle f, g \rangle = \int_a^b f(x) \bar{g}(x) dx \quad \text{and} \quad \|f\|_2 = \left( \int_a^b |f(x)|^2 dx \right)^{\frac{1}{2}}.$$

Now we state, probably, the most important inequality in analysis.

**Theorem 2.13** (Cauchy–Schwarz–Bunyakovskii inequality). *For vectors  $x$  and  $y$  in an inner product space  $V$  let us define  $\|x\| = \sqrt{\langle x, x \rangle}$  and  $\|y\| = \sqrt{\langle y, y \rangle}$  then we have*

$$(2.4) \quad |\langle x, y \rangle| \leq \|x\| \|y\|,$$

with equality if and only if  $x$  and  $y$  are scalar multiple each other.

*Proof.* For any  $x, y \in V$  and any  $t \in \mathbb{R}$  we have:

$$0 < \langle x + ty, x + ty \rangle = \langle x, x \rangle + 2t\Re \langle y, x \rangle + t^2 \langle y, y \rangle,$$

Thus the discriminant of this quadratic expression in  $t$  is non-positive:  $(\Re \langle y, x \rangle)^2 - \|x\|^2 \|y\|^2 \leq 0$ , that is  $|\Re \langle x, y \rangle| \leq \|x\| \|y\|$ . Replacing  $y$  by  $e^{i\alpha} y$  for an arbitrary  $\alpha \in [-\pi, \pi]$  we get  $|\Re(e^{i\alpha} \langle x, y \rangle)| \leq \|x\| \|y\|$ , this implies the desired inequality.  $\square$

**Corollary 2.14.** *Any inner product space is a normed space with norm  $\|x\| = \sqrt{\langle x, x \rangle}$  (hence also a metric space, Prop. 2.4).*

*Proof.* Just to check items 2.3(i)–2.3(iii) from Definition 2.3.  $\square$

Again complete inner product spaces deserve a special name

**Definition 2.15.** A complete inner product space is *Hilbert space*.

The relations between spaces introduced so far are as follows:

$$\begin{array}{ccccc} \text{Hilbert spaces} & \Rightarrow & \text{Banach spaces} & \Rightarrow & \text{Complete metric spaces} \\ \downarrow & & \downarrow & & \downarrow \\ \text{inner product spaces} & \Rightarrow & \text{normed spaces} & \Rightarrow & \text{metric spaces.} \end{array}$$

How can we tell if a given norm comes from an inner product?

**Theorem 2.16** (Parallelogram identity). *In an inner product space  $H$  we have for all  $x$  and  $y \in H$  (see Figure 3):*

$$(2.5) \quad \|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

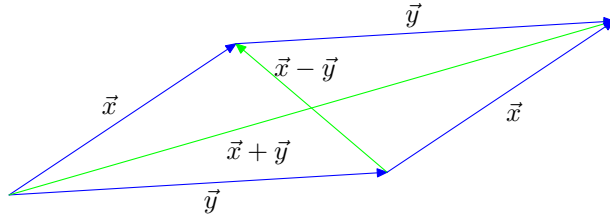


FIGURE 3. To the parallelogram identity.

*Proof.* Just by linearity of inner product:

$$\langle x + y, x + y \rangle + \langle x - y, x - y \rangle = 2 \langle x, x \rangle + 2 \langle y, y \rangle,$$

because the cross terms cancel out. □

**Exercise 2.17.** Show that (2.5) is also a sufficient condition for a norm to arise from an inner product. Namely, for a norm on a complex Banach space satisfying to (2.5) the formula

$$\begin{aligned} (2.6) \quad \langle x, y \rangle &= \frac{1}{4} \left( \|x + y\|^2 - \|x - y\|^2 + i \|x + iy\|^2 - i \|x - iy\|^2 \right) \\ &= \sum_0^3 i^k \|x + i^k y\|^2 \end{aligned}$$

defines an inner product. What is a suitable formula for a real Banach space?

Divide and rule!

*Old but still much used recipe*

**2.3. Subspaces.** To study Hilbert spaces we may use the traditional mathematical technique of *analysis* and *synthesis*: we split the initial Hilbert spaces into smaller and probably simpler subsets, investigate them separately, and then reconstruct the entire picture from these parts.

As known from the linear algebra, a *linear subspace* is a subset of a linear space is its subset, which inherits the linear structure, i.e. possibility to add vectors and multiply them by scalars. In this course we need also that subspaces inherit topological structure (coming either from a norm or an inner product) as well.

**Definition 2.18.** By a *subspace* of a normed space (or inner product space) we mean a linear subspace with the same norm (inner product respectively). We write  $X \subset Y$  or  $X \subseteq Y$ .

- Example 2.19.**
- (i)  $C_b(X) \subset \ell_\infty(X)$  where  $X$  is a metric space.
  - (ii) Any linear subspace of  $\mathbb{R}^n$  or  $\mathbb{C}^n$  with any norm given in Example 2.9(i)–2.9(iii).
  - (iii) Let  $c_{00}$  be the *space of finite sequences*, i.e. all sequences  $(x_n)$  such that exist  $N$  with  $x_n = 0$  for  $n > N$ . This is a subspace of  $\ell_2$  since  $\sum_1^\infty |x_j|^2$  is a finite sum, so finite.

We also wish that the both inhaled structures (linear and topological) should be in agreement, i.e. the subspace should be complete. Such inheritance is linked to the property be closed.

A subspace need *not* be closed—for example the sequence

$$x = (1, 1/2, 1/3, 1/4, \dots) \in \ell_2 \quad \text{because} \quad \sum 1/k^2 < \infty$$

and  $x_n = (1, 1/2, \dots, 1/n, 0, 0, \dots) \in c_{00}$  converges to  $x$  thus  $x \in \overline{c_{00}} \subset \ell_2$ .

- Proposition 2.20.** (i) Any closed subspace of a Banach/Hilbert space is complete, hence also a Banach/Hilbert space.  
(ii) Any complete subspace is closed.  
(iii) The closure of subspace is again a subspace.

*Proof.* (i) This is true in any metric space  $X$ : any Cauchy sequence from  $Y$  has a limit  $x \in X$  belonging to  $\bar{Y}$ , but if  $Y$  is closed then  $x \in Y$ .

(ii) Let  $Y$  is complete and  $x \in \bar{Y}$ , then there is sequence  $x_n \rightarrow x$  in  $Y$  and it is a Cauchy sequence. Then completeness of  $Y$  implies  $x \in Y$ .

(iii) If  $x, y \in \bar{Y}$  then there are  $x_n$  and  $y_n$  in  $Y$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . From the **triangle inequality**:

$$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \rightarrow 0,$$

so  $x_n + y_n \rightarrow x + y$  and  $x + y \in \bar{Y}$ . Similarly  $x \in \bar{Y}$  implies  $\lambda x \in \bar{Y}$  for any  $\lambda$ .

□

Hence  $c_{00}$  is an *incomplete* inner product space, with inner product  $\langle x, y \rangle = \sum_{k=1}^{\infty} x_k y_k$  (this is a finite sum!) as it is not closed in  $\ell_2$ .

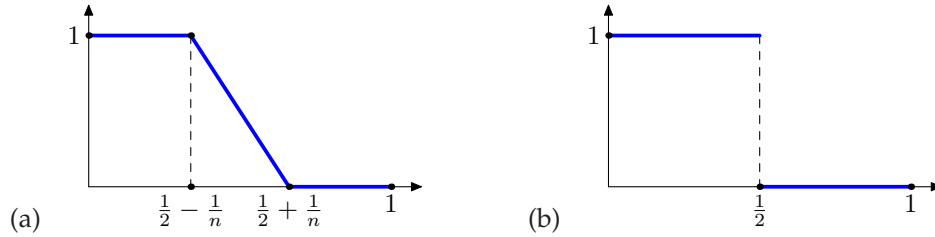


FIGURE 4. Jump function on (b) as a  $L_2$  limit of continuous functions from (a).

Similarly  $C[0, 1]$  with inner product norm  $\|f\| = \left( \int_0^1 |f(t)|^2 dt \right)^{1/2}$  is incomplete—take the large space  $X$  of functions continuous on  $[0, 1]$  except for a possible jump at  $\frac{1}{2}$  (i.e. left and right limits exists but may be unequal and  $f(\frac{1}{2}) = \lim_{t \rightarrow \frac{1}{2}^+} f(t)$ ). Then the sequence of functions defined on Figure 4(a) has the limit shown on Figure 4(b) since:

$$\|f - f_n\| = \int_{\frac{1}{2} - \frac{1}{n}}^{\frac{1}{2} + \frac{1}{n}} |f - f_n|^2 dt < \frac{2}{n} \rightarrow 0.$$

Obviously  $f \in \overline{C[0, 1]} \setminus C[0, 1]$ .

**Exercise 2.21.** Show alternatively that the sequence of function  $f_n$  from Figure 4(a) is a Cauchy sequence in  $C[0, 1]$  but has no continuous limit.

Similarly the space  $C[a, b]$  is *incomplete* for any  $a < b$  if equipped by the inner product and the corresponding norm:

$$(2.7) \quad \langle f, g \rangle = \int_a^b f(t) \bar{g}(t) dt$$

$$(2.8) \quad \|f\|_2 = \left( \int_a^b |f(t)|^2 dt \right)^{1/2}.$$

**Definition 2.22.** Define a Hilbert space  $L_2[a, b]$  to be the smallest complete inner product space containing space  $C[a, b]$  with the restriction of inner product given by (2.7).

It is practical to realise  $L_2[a, b]$  as a certain space of “functions” with the inner product defined via an integral. There are several ways to do that and we mention just two:

- (i) Elements of  $L_2[a, b]$  are equivalent classes of Cauchy sequences  $f^{(n)}$  of functions from  $C[a, b]$ .
- (ii) Let integration be extended from the **Riemann definition** to the wider *Lebesgue integration* (see Section 13). Let  $L$  be a set of square integrable in Lebesgue sense functions on  $[a, b]$  with a finite norm (2.8). Then  $L_2[a, b]$  is a quotient space of  $L$  with respect to the equivalence relation  $f \sim g \Leftrightarrow \|f - g\|_2 = 0$ .

**Example 2.23.** Let the *Cantor function* on  $[0, 1]$  be defined as follows:

$$f(t) = \begin{cases} 1, & t \in \mathbb{Q}; \\ 0, & t \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function is *not* integrable in the Riemann sense but *does* have the Lebesgue integral. The later however is equal to 0 and as an  $L_2$ -function the Cantor function equivalent to the function identically equal to 0.

- (iii) The third possibility is to map  $L_2(\mathbb{R})$  onto a space of “true” functions but with an additional structure. For example, in *quantum mechanics* it is useful to work with the *Segal–Bargmann space* of **analytic functions on  $\mathbb{C}$  with the inner product**:

$$\langle f_1, f_2 \rangle = \int_{\mathbb{C}} f_1(z) \bar{f}_2(z) e^{-|z|^2} dz.$$

**Theorem 2.24.** *The sequence space  $\ell_2$  is complete, hence a Hilbert space.*

*Proof.* Take a Cauchy sequence  $x^{(n)} \in \ell_2$ , where  $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, x_3^{(n)}, \dots)$ . Our proof will have three steps: identify the limit  $x$ ; show it is in  $\ell_2$ ; show  $x^{(n)} \rightarrow x$ .

- (i) If  $x^{(n)}$  is a Cauchy sequence in  $\ell_2$  then  $x_k^{(n)}$  is also a Cauchy sequence of numbers for any fixed  $k$ :

$$\left| x_k^{(n)} - x_k^{(m)} \right| \leq \left( \sum_{k=1}^{\infty} \left| x_k^{(n)} - x_k^{(m)} \right|^2 \right)^{1/2} = \|x^{(n)} - x^{(m)}\| \rightarrow 0.$$

Let  $x_k$  be the limit of  $x_k^{(n)}$ .

- (ii) For a given  $\epsilon > 0$  find  $n_0$  such that  $\|x^{(n)} - x^{(m)}\| < \epsilon$  for all  $n, m > n_0$ . For any  $K$  and  $m$ :

$$\sum_{k=1}^K \left| x_k^{(n)} - x_k^{(m)} \right|^2 \leq \|x^{(n)} - x^{(m)}\|^2 < \epsilon^2.$$

Let  $m \rightarrow \infty$  then  $\sum_{k=1}^K \left| x_k^{(n)} - x_k \right|^2 \leq \epsilon^2$ .

Let  $K \rightarrow \infty$  then  $\sum_{k=1}^{\infty} \left| x_k^{(n)} - x_k \right|^2 \leq \epsilon^2$ . Thus  $x^{(n)} - x \in \ell_2$  and because  $\ell_2$  is a linear space then  $x = x^{(n)} - (x^{(n)} - x)$  is also in  $\ell_2$ .

- (iii) We saw above that for any  $\epsilon > 0$  there is  $n_0$  such that  $\|x^{(n)} - x\| < \epsilon$  for all  $n > n_0$ . Thus  $x^{(n)} \rightarrow x$ .

Consequently  $\ell_2$  is complete. □

All good things are covered by a thick layer of chocolate (well, if something is not yet—it certainly will)

**2.4. Linear spans.** As was explained into introduction 2, we describe “internal” properties of a vector through its relations to other vectors. For a detailed description we need sufficiently many external reference points.

Let  $A$  be a subset (finite or infinite) of a normed space  $V$ . We may wish to upgrade it to a linear subspace in order to make it subject to our theory.

**Definition 2.25.** The *linear span* of  $A$ , write  $\text{Lin}(A)$ , is the intersection of all linear subspaces of  $V$  containing  $A$ , i.e. the smallest subspace containing  $A$ , equivalently the set of all finite linear combination of elements of  $A$ . The *closed linear span* of  $A$  write  $\text{CLin}(A)$  is the intersection of all *closed* linear subspaces of  $V$  containing  $A$ , i.e. the smallest *closed* subspace containing  $A$ .

**Exercise\* 2.26.** (i) Show that if  $A$  is a subset of finite dimension space then  $\text{Lin}(A) = \text{CLin}(A)$ .

(ii) Show that for an infinite  $A$  spaces  $\text{Lin}(A)$  and  $\text{CLin}(A)$  could be different. (*Hint: use Example 2.19(iii).*)

**Proposition 2.27.**  $\overline{\text{Lin}(A)} = \text{CLin}(A)$ .

*Proof.* Clearly  $\overline{\text{Lin}(A)}$  is a closed subspace containing  $A$  thus it should contain  $\text{CLin}(A)$ . Also  $\text{Lin}(A) \subset \text{CLin}(A)$  thus  $\overline{\text{Lin}(A)} \subset \overline{\text{CLin}(A)} = \text{CLin}(A)$ . Therefore  $\overline{\text{Lin}(A)} = \text{CLin}(A)$ .  $\square$

Consequently  $\text{CLin}(A)$  is the set of all limiting points of finite linear combination of elements of  $A$ .

**Example 2.28.** Let  $V = C[a, b]$  with the sup norm  $\|\cdot\|_\infty$ . Then:

$\text{Lin}\{1, x, x^2, \dots\} = \{\text{all polynomials}\}$

$\text{CLin}\{1, x, x^2, \dots\} = C[a, b]$  by the **Weierstrass approximation theorem** proved later.

The following simple result will be used later many times without comments.

**Lemma 2.29** (about Inner Product Limit). *Suppose  $H$  is an inner product space and sequences  $x_n$  and  $y_n$  have limits  $x$  and  $y$  correspondingly. Then  $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$  or equivalently:*

$$\lim_{n \rightarrow \infty} \langle x_n, y_n \rangle = \left\langle \lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n \right\rangle.$$

*Proof.* Obviously by the **Cauchy-Schwarz inequality**:

$$\begin{aligned} |\langle x_n, y_n \rangle - \langle x, y \rangle| &= |\langle x_n - x, y_n \rangle + \langle x, y_n - y \rangle| \\ &\leq |\langle x_n - x, y_n \rangle| + |\langle x, y_n - y \rangle| \\ &\leq \|x_n - x\| \|y_n\| + \|x\| \|y_n - y\| \rightarrow 0, \end{aligned}$$

since  $\|x_n - x\| \rightarrow 0$ ,  $\|y_n - y\| \rightarrow 0$ , and  $\|y_n\|$  is bounded.  $\square$

### 3. ORTHOGONALITY

Pythagoras is forever!

The catchphrase from TV commercial of **Hilbert Spaces** course

As was mentioned in the introduction the Hilbert spaces is an analog of our 3D Euclidean space and theory of Hilbert spaces similar to plane or space geometry. One of the primary result of Euclidean geometry which still survives in high



school curriculum despite its continuous nasty de-geometrisation is Pythagoras' theorem based on the notion of *orthogonality*<sup>1</sup>.

So far we were concerned only with distances between points. Now we would like to study angles between vectors and notably *right angles*. Pythagoras' theorem states that if the angle C in a triangle is right then  $c^2 = a^2 + b^2$ , see Figure 5.

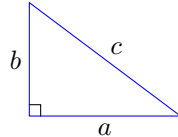


FIGURE 5. The Pythagoras' theorem  $c^2 = a^2 + b^2$

It is a very *mathematical way of thinking* to turn this *property* of right angles into their *definition*, which will work even in infinite dimensional Hilbert spaces.

Look for a triangle, or even for a right triangle

A universal advice in solving problems from elementary geometry.

**3.1. Orthogonal System in Hilbert Space.** In inner product spaces it is even more convenient to give a definition of orthogonality not from Pythagoras' theorem but from an equivalent property of inner product.

**Definition 3.1.** Two vectors  $x$  and  $y$  in an inner product space are *orthogonal* if  $\langle x, y \rangle = 0$ , written  $x \perp y$ .

An *orthogonal sequence* (or *orthogonal system*)  $e_n$  (finite or infinite) is one in which  $e_n \perp e_m$  whenever  $n \neq m$ .

An *orthonormal sequence* (or *orthonormal system*)  $e_n$  is an orthogonal sequence with  $\|e_n\| = 1$  for all  $n$ .

**Exercise 3.2.** (i) Show that if  $x \perp x$  then  $x = 0$  and consequently  $x \perp y$  for any  $y \in H$ .

(ii) Show that if all vectors of an orthogonal system are non-zero then they are linearly independent.

**Example 3.3.** These are orthonormal sequences:

- (i) Basis vectors  $(1, 0, 0), (0, 1, 0), (0, 0, 1)$  in  $\mathbb{R}^3$  or  $\mathbb{C}^3$ .
- (ii) Vectors  $e_n = (0, \dots, 0, 1, 0, \dots)$  (with the only 1 on the  $n$ th place) in  $\ell_2$ . (Could you see a similarity with the previous example?)
- (iii) Functions  $e_n(t) = 1/(\sqrt{2\pi})e^{int}$ ,  $n \in \mathbb{Z}$  in  $C[0, 2\pi]$ :

$$(3.1) \quad \langle e_n, e_m \rangle = \int_0^{2\pi} \frac{1}{2\pi} e^{int} e^{-imt} dt = \begin{cases} 1, & n = m; \\ 0, & n \neq m. \end{cases}$$

**Exercise 3.4.** Let  $A$  be a subset of an inner product space  $V$  and  $x \perp y$  for any  $y \in A$ . Prove that  $x \perp z$  for all  $z \in \text{CLin}(A)$ .

**Theorem 3.5 (Pythagoras').** If  $x \perp y$  then  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ . Also if  $e_1, \dots, e_n$  is orthonormal then

$$\left\| \sum_1^n a_k e_k \right\|^2 = \left\langle \sum_1^n a_k e_k, \sum_1^n a_k e_k \right\rangle = \sum_1^n |a_k|^2.$$

<sup>1</sup>Some more "strange" types of orthogonality can be seen in the paper *Elliptic, Parabolic and Hyperbolic Analytic Function Theory-1: Geometry of Invariants*.

*Proof.* A one-line calculation. □

The following theorem provides an important property of Hilbert spaces which will be used many times. Recall, that a subset  $K$  of a linear space  $V$  is *convex* if for all  $x, y \in K$  and  $\lambda \in [0, 1]$  the point  $\lambda x + (1 - \lambda)y$  is also in  $K$ . Particularly any subspace is convex and any unit ball as well (see Exercise 2.8(i)).

**Theorem 3.6** (about the Nearest Point). *Let  $K$  be a non-empty convex closed subset of a Hilbert space  $H$ . For any point  $x \in H$  there is the unique point  $y \in K$  nearest to  $x$ .*

*Proof.* Let  $d = \inf_{y \in K} d(x, y)$ , where  $d(x, y)$ —the distance coming from the norm  $\|x\| = \sqrt{\langle x, x \rangle}$  and let  $y_n$  a sequence points in  $K$  such that  $\lim_{n \rightarrow \infty} d(x, y_n) = d$ . Then  $y_n$  is a Cauchy sequence. Indeed from the **parallelogram identity** for the parallelogram generated by vectors  $x - y_n$  and  $x - y_m$  we have:

$$\|y_n - y_m\|^2 = 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - \|2x - y_n - y_m\|^2.$$

Note that  $\|2x - y_n - y_m\|^2 = 4\|x - \frac{y_n + y_m}{2}\|^2 \geq 4d^2$  since  $\frac{y_n + y_m}{2} \in K$  by its convexity. For sufficiently large  $m$  and  $n$  we get  $\|x - y_m\|^2 \leq d + \epsilon$  and  $\|x - y_n\|^2 \leq d + \epsilon$ , thus  $\|y_n - y_m\| \leq 4(d^2 + \epsilon) - 4d^2 = 4\epsilon$ , i.e.  $y_n$  is a Cauchy sequence.

Let  $y$  be the limit of  $y_n$ , which exists by the completeness of  $H$ , then  $y \in K$  since  $K$  is closed. Then  $d(x, y) = \lim_{n \rightarrow \infty} d(x, y_n) = d$ . This shows the existence of the nearest point. Let  $y'$  be another point in  $K$  such that  $d(x, y') = d$ , then the parallelogram identity implies:

$$\|y - y'\|^2 = 2\|x - y\|^2 + 2\|x - y'\|^2 - \|2x - y - y'\|^2 \leq 4d^2 - 4d^2 = 0.$$

This shows the uniqueness of the nearest point. □

**Exercise\* 3.7.** The essential rôle of the parallelogram identity in the above proof indicates that the theorem does not hold in a general Banach space.

- (i) Show that in  $\mathbb{R}^2$  with either norm  $\|\cdot\|_1$  or  $\|\cdot\|_\infty$  from Example 2.9 the nearest point could be non-unique;
- (ii) Could you construct an example (in Banach space) when the nearest point does not exist?

Liberte, Egalite, Fraternite!

A longstanding ideal approximated in the real life by something completely different

**3.2. Bessel's inequality.** For the case then a convex subset is a subspace we could characterise the nearest point in the term of orthogonality.

**Theorem 3.8** (on Perpendicular). *Let  $M$  be a subspace of a Hilbert space  $H$  and a point  $x \in H$  be fixed. Then  $z \in M$  is the nearest point to  $x$  if and only if  $x - z$  is orthogonal to any vector in  $M$ .*

*Proof.* Let  $z$  is the nearest point to  $x$  existing by the **previous Theorem**. We claim that  $x - z$  orthogonal to any vector in  $M$ , otherwise there exists  $y \in M$  such that  $\langle x - z, y \rangle \neq 0$ . Then

$$\begin{aligned} \|x - z - \epsilon y\|^2 &= \|x - z\|^2 - 2\epsilon \Re \langle x - z, y \rangle + \epsilon^2 \|y\|^2 \\ &< \|x - z\|^2, \end{aligned}$$

if  $\epsilon$  is chosen to be small enough and such that  $\epsilon \Re \langle x - z, y \rangle$  is positive, see Figure 6(i). Therefore we get a contradiction with the statement that  $z$  is closest point to  $x$ .

On the other hand if  $x - z$  is orthogonal to all vectors in  $H_1$  then particularly  $(x - z) \perp (z - y)$  for all  $y \in H_1$ , see Figure 6(ii). Since  $x - y = (x - z) + (z - y)$  we got by the **Pythagoras' theorem**:

$$\|x - y\|^2 = \|x - z\|^2 + \|z - y\|^2.$$

So  $\|x - y\|^2 \geq \|x - z\|^2$  and they are equal if and only if  $z = y$ . □

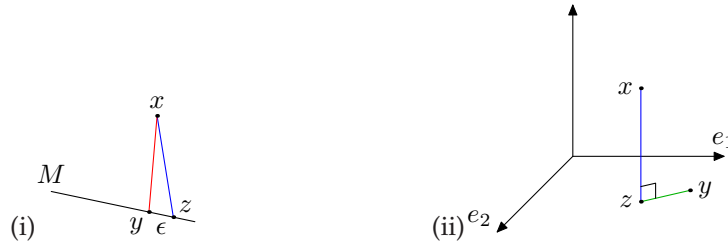


FIGURE 6. (i) A smaller distance for a non-perpendicular direction; and (ii) Best approximation from a subspace

**Exercise 3.9.** The above proof does not work if  $\langle x - z, y \rangle$  is an imaginary number, what to do in this case?

Consider now a basic case of *approximation*: let  $x \in H$  be fixed and  $e_1, \dots, e_n$  be orthonormal and denote  $H_1 = \text{Lin}\{e_1, \dots, e_n\}$ . We could try to approximate  $x$  by a vector  $y = \lambda_1 e_1 + \dots + \lambda_n e_n \in H_1$ .

**Corollary 3.10.** *The minimal value of  $\|x - y\|$  for  $y \in H_1$  is achieved when  $y = \sum_1^n \langle x, e_i \rangle e_i$ .*

*Proof.* Let  $z = \sum_1^n \langle x, e_i \rangle e_i$ , then  $\langle x - z, e_i \rangle = \langle x, e_i \rangle - \langle z, e_i \rangle = 0$ . By the **previous Theorem**  $z$  is the nearest point to  $x$ . □

**Example 3.11.** (i) In  $\mathbb{R}^3$  find the best approximation to  $(1, 0, 0)$  from the plane  $V : \{x_1 + x_2 + x_3 = 0\}$ . We take an orthonormal basis  $e_1 = (2^{-1/2}, -2^{-1/2}, 0)$ ,  $e_2 = (6^{-1/2}, 6^{-1/2}, -2 \cdot 6^{-1/2})$  of  $V$  (Check this!). Then:

$$z = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2 = \left(\frac{1}{2}, -\frac{1}{2}, 0\right) + \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right) = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right).$$

(ii) In  $C[0, 2\pi]$  what is the best approximation to  $f(t) = t$  by functions  $a + be^{it} + ce^{-it}$ ? Let

$$e_0 = \frac{1}{\sqrt{2\pi}}, \quad e_1 = \frac{1}{\sqrt{2\pi}} e^{it}, \quad e_{-1} = \frac{1}{\sqrt{2\pi}} e^{-it}.$$

We find:

$$\langle f, e_0 \rangle = \int_0^{2\pi} \frac{t}{\sqrt{2\pi}} dt = \left[ \frac{t^2}{2} \frac{1}{\sqrt{2\pi}} \right]_0^{2\pi} = \sqrt{2}\pi^{3/2};$$

$$\langle f, e_1 \rangle = \int_0^{2\pi} \frac{te^{-it}}{\sqrt{2\pi}} dt = i\sqrt{2\pi} \quad (\text{Check this!})$$

$$\langle f, e_{-1} \rangle = \int_0^{2\pi} \frac{te^{it}}{\sqrt{2\pi}} dt = -i\sqrt{2\pi} \quad (\text{Why we may not check this one?})$$

Then the best approximation is (see Figure 7):

$$\begin{aligned} f_0(t) &= \langle f, e_0 \rangle e_0 + \langle f, e_1 \rangle e_1 + \langle f, e_{-1} \rangle e_{-1} \\ &= \frac{\sqrt{2}\pi^{3/2}}{\sqrt{2\pi}} + ie^{it} - ie^{-it} = \pi - 2 \sin t. \end{aligned}$$

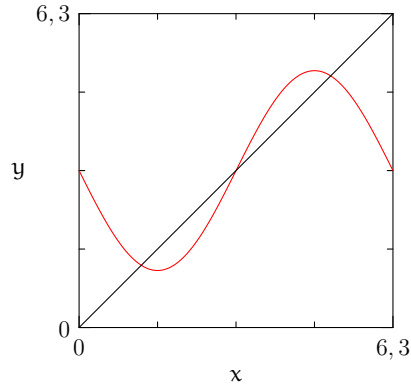


FIGURE 7. Best approximation by three trigonometric polynomials

**Corollary 3.12** (Bessel's inequality). *If  $(e_i)$  is orthonormal then*

$$\|x\|^2 \geq \sum_{i=1}^n |\langle x, e_i \rangle|^2.$$

*Proof.* Let  $z = \sum_{i=1}^n \langle x, e_i \rangle e_i$  then  $x-z \perp e_i$  for all  $i$  therefore by Exercise 3.4  $x-z \perp z$ . Hence:

$$\begin{aligned} \|x\|^2 &= \|z\|^2 + \|x-z\|^2 \\ &\geq \|z\|^2 = \sum_{i=1}^n |\langle x, e_i \rangle|^2. \end{aligned}$$

□

—Did you say “rice and fish for them”?  
A student question

**3.3. The Riesz–Fischer theorem.** When  $(e_i)$  is orthonormal we call  $\langle x, e_n \rangle$  the  $n$ th Fourier coefficient of  $x$  (with respect to  $(e_i)$ , naturally).

**Theorem 3.13** (Riesz–Fisher). *Let  $(e_n)_1^\infty$  be an orthonormal sequence in a Hilbert space  $H$ . Then  $\sum_{n=1}^\infty \lambda_n e_n$  converges in  $H$  if and only if  $\sum_{n=1}^\infty |\lambda_n|^2 < \infty$ . In this case  $\|\sum_{n=1}^\infty \lambda_n e_n\|^2 = \sum_{n=1}^\infty |\lambda_n|^2$ .*

*Proof. Necessity:* Let  $x_k = \sum_{n=1}^k \lambda_n e_n$  and  $x = \lim_{k \rightarrow \infty} x_k$ . So  $\langle x, e_n \rangle = \lim_{k \rightarrow \infty} \langle x_k, e_n \rangle = \lambda_n$  for all  $n$ . By the **Bessel's inequality** for all  $k$

$$\|x\|^2 \geq \sum_{n=1}^k |\langle x, e_n \rangle|^2 = \sum_{n=1}^k |\lambda_n|^2,$$

hence  $\sum_{n=1}^k |\lambda_n|^2$  converges and the sum is at most  $\|x\|^2$ .

*Sufficiency:* Consider  $\|x_k - x_m\| = \left\| \sum_{n=m}^k \lambda_n e_n \right\| = \left( \sum_{n=m}^k |\lambda_n|^2 \right)^{1/2}$  for  $k > m$ . Since  $\sum_{n=1}^k |\lambda_n|^2$  converges  $x_k$  is a Cauchy sequence in  $H$  and thus has a limit  $x$ . By

the **Pythagoras' theorem**  $\|x_k\|^2 = \sum_1^k |\lambda_n|^2$  thus for  $k \rightarrow \infty$   $\|x\|^2 = \sum_1^\infty |\lambda_n|^2$  by the **Lemma about inner product limit**.  $\square$

Observation: the **closed linear span** of an orthonormal sequence in any Hilbert space looks like  $\ell_2$ , i.e.  $\ell_2$  is a universal model for a Hilbert space.

By **Bessel's inequality** and the **Riesz–Fisher theorem** we know that the series  $\sum_1^\infty \langle x, e_i \rangle e_i$  converges for any  $x \in H$ . What is its limit?

Let  $y = x - \sum_1^\infty \langle x, e_i \rangle e_i$ , then

$$(3.2) \quad \langle y, e_k \rangle = \langle x, e_k \rangle - \sum_1^\infty \langle x, e_i \rangle \langle e_i, e_k \rangle = \langle x, e_k \rangle - \langle x, e_k \rangle = 0 \quad \text{for all } k.$$

**Definition 3.14.** An orthonormal sequence  $(e_i)$  in a Hilbert space  $H$  is *complete* if the identities  $\langle y, e_k \rangle = 0$  for all  $k$  imply  $y = 0$ .

A complete orthonormal sequence is also called *orthonormal basis* in  $H$ .

**Theorem 3.15** (on Orthonormal Basis). *Let  $e_i$  be an orthonormal basis in a Hilbert space  $H$ . Then for any  $x \in H$  we have*

$$x = \sum_{n=1}^\infty \langle x, e_n \rangle e_n \quad \text{and} \quad \|x\|^2 = \sum_{n=1}^\infty |\langle x, e_n \rangle|^2.$$

*Proof.* By the **Riesz–Fisher theorem**, equation (3.2) and **definition of orthonormal basis**.  $\square$

There are constructive existence theorems in mathematics.  
An example of pure existence statement

**3.4. Construction of Orthonormal Sequences.** Natural questions are: Do orthonormal sequences always exist? Could we construct them?

**Theorem 3.16** (Gram–Schmidt). *Let  $(x_i)$  be a sequence of linearly independent vectors in an inner product space  $V$ . Then there exists orthonormal sequence  $(e_i)$  such that*

$$\text{Lin}\{x_1, x_2, \dots, x_n\} = \text{Lin}\{e_1, e_2, \dots, e_n\}, \quad \text{for all } n.$$

*Proof.* We give an explicit algorithm working by induction. The *base* of induction: the first vector is  $e_1 = x_1 / \|x_1\|$ . The *step* of induction: let  $e_1, e_2, \dots, e_n$  are already constructed as required. Let  $y_{n+1} = x_{n+1} - \sum_{i=1}^n \langle x_{n+1}, e_i \rangle e_i$ . Then by (3.2)  $y_{n+1} \perp e_i$  for  $i = 1, \dots, n$ . We may put  $e_{n+1} = y_{n+1} / \|y_{n+1}\|$  because  $y_{n+1} \neq 0$  due to linear independence of  $x_k$ 's. Also

$$\begin{aligned} \text{Lin}\{e_1, e_2, \dots, e_{n+1}\} &= \text{Lin}\{e_1, e_2, \dots, y_{n+1}\} \\ &= \text{Lin}\{e_1, e_2, \dots, x_{n+1}\} \\ &= \text{Lin}\{x_1, x_2, \dots, x_{n+1}\}. \end{aligned}$$

So  $(e_i)$  are orthonormal sequence.  $\square$

**Example 3.17.** Consider  $C[0, 1]$  with the usual inner product (2.7) and apply orthogonalisation to the sequence  $1, x, x^2, \dots$ . Because  $\|1\| = 1$  then  $e_1(x) = 1$ . The continuation could be presented by the table:

$$\begin{aligned} e_1(x) &= 1 \\ y_2(x) &= x - \langle x, 1 \rangle 1 = x - \frac{1}{2}, \quad \|y_2\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \frac{1}{12}, \quad e_2(x) = \sqrt{12} \left(x - \frac{1}{2}\right) \\ y_3(x) &= x^2 - \langle x^2, 1 \rangle 1 - \left\langle x^2, x - \frac{1}{2} \right\rangle \left(x - \frac{1}{2}\right) \cdot 12, \quad \dots, \quad e_3 = \frac{y_3}{\|y_3\|} \\ &\dots\dots\dots \end{aligned}$$

**Example 3.18.** Many famous sequences of **orthogonal polynomials**, e.g. **Chebyshev**, **Legendre**, **Laguerre**, **Hermite**, can be obtained by orthogonalisation of  $1, x, x^2, \dots$  with various inner products.

(i) *Legendre polynomials* in  $C[-1, 1]$  with inner product

$$(3.3) \quad \langle f, g \rangle = \int_{-1}^1 f(t) \overline{g(t)} dt.$$

(ii) *Chebyshev polynomials* in  $C[-1, 1]$  with inner product

$$(3.4) \quad \langle f, g \rangle = \int_{-1}^1 f(t) \overline{g(t)} \frac{dt}{\sqrt{1-t^2}}$$

(iii) *Laguerre polynomials* in the space of polynomials  $P[0, \infty)$  with inner product

$$\langle f, g \rangle = \int_0^{\infty} f(t) \overline{g(t)} e^{-t} dt.$$

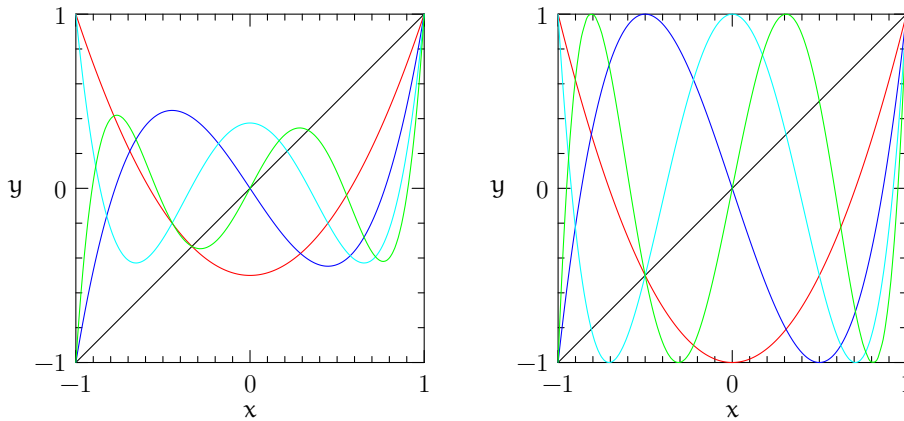


FIGURE 8. Five first Legendre  $P_i$  and Chebyshev  $T_i$  polynomials

See Figure 8 for the five first Legendre and Chebyshev polynomials. Observe the difference caused by the different inner products (3.3) and (3.4). On the other hand note the similarity in oscillating behaviour with different “frequencies”.

Another natural question is: When is an orthonormal sequence complete?

**Proposition 3.19.** *Let  $(e_n)$  be an orthonormal sequence in a Hilbert space  $H$ . The following are equivalent:*

- (i)  $(e_n)$  is an orthonormal basis.
- (ii)  $\text{CLin}((e_n)) = H$ .
- (iii)  $\|x\|^2 = \sum_1^{\infty} |\langle x, e_n \rangle|^2$  for all  $x \in H$ .

*Proof.* Clearly 3.19(i) implies 3.19(ii) because  $x = \sum_1^{\infty} \langle x, e_n \rangle e_n$  in  $\text{CLin}((e_n))$  and  $\|x\|^2 = \sum_1^{\infty} \langle x, e_n \rangle \langle x, e_n \rangle$  by Theorem 3.15.

If  $(e_n)$  is not complete then there exists  $x \in H$  such that  $x \neq 0$  and  $\langle x, e_k \rangle = 0$  for all  $k$ , so 3.19(iii) fails, consequently 3.19(ii) implies 3.19(i).

Finally if  $\langle x, e_k \rangle = 0$  for all  $k$  then  $\langle x, y \rangle = 0$  for all  $y \in \text{Lin}((e_n))$  and moreover for all  $y \in \text{CLin}((e_n))$ , by the **Lemma on continuity of the inner product**. But then  $x \notin \text{CLin}((e_n))$  and 3.19(ii) also fails because  $\langle x, x \rangle = 0$  is not possible. Thus 3.19(ii) implies 3.19(i).  $\square$

**Corollary 3.20.** A separable Hilbert space (i.e. one with a countable dense set) can be identified with either  $\ell_2^n$  or  $\ell_2$ , in other words it has an orthonormal basis  $(e_n)$  (finite or infinite) such that

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n \quad \text{and} \quad \|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2.$$

*Proof.* Take a countable dense set  $(x_k)$ , then  $H = \text{CLin}((x_k))$ , delete all vectors which are a linear combinations of preceding vectors, make orthonormalisation by Gram–Schmidt the remaining set and apply the **previous proposition**.  $\square$

Most pleasant compliments are usually orthogonal to our real qualities.

An advise based on observations

### 3.5. Orthogonal complements.

**Definition 3.21.** Let  $M$  be a subspace of an inner product space  $V$ . The *orthogonal complement*, written  $M^\perp$ , of  $M$  is

$$M^\perp = \{x \in V : \langle x, m \rangle = 0 \forall m \in M\}.$$

**Theorem 3.22.** If  $M$  is a closed subspace of a Hilbert space  $H$  then  $M^\perp$  is a closed subspace too (hence a Hilbert space too).

*Proof.* Clearly  $M^\perp$  is a subspace of  $H$  because  $x, y \in M^\perp$  implies  $ax + by \in M^\perp$ :

$$\langle ax + by, m \rangle = a \langle x, m \rangle + b \langle y, m \rangle = 0.$$

Also if all  $x_n \in M^\perp$  and  $x_n \rightarrow x$  then  $x \in M^\perp$  due to **inner product limit Lemma**.  $\square$

**Theorem 3.23.** Let  $M$  be a closed subspace of a Hilbert space  $H$ . Then for any  $x \in H$  there exists the unique decomposition  $x = m + n$  with  $m \in M$ ,  $n \in M^\perp$  and  $\|x\|^2 = \|m\|^2 + \|n\|^2$ . Thus  $H = M \oplus M^\perp$  and  $(M^\perp)^\perp = M$ .

*Proof.* For a given  $x$  there exists the unique closest point  $m$  in  $M$  by the **Theorem on nearest point** and by the **Theorem on perpendicular**  $(x - m) \perp y$  for all  $y \in M$ .

So  $x = m + (x - m) = m + n$  with  $m \in M$  and  $n \in M^\perp$ . The identity  $\|x\|^2 = \|m\|^2 + \|n\|^2$  is just Pythagoras' theorem and  $M \cap M^\perp = \{0\}$  because null vector is the only vector orthogonal to itself.

Finally  $(M^\perp)^\perp = M$ . We have  $H = M \oplus M^\perp = (M^\perp)^\perp \oplus M^\perp$ , for any  $x \in (M^\perp)^\perp$  there is a decomposition  $x = m + n$  with  $m \in M$  and  $n \in M^\perp$ , but then  $n$  is orthogonal to itself and therefore is zero.  $\square$

**Corollary 3.24** (about Orthoprojection). *There is a linear map  $P_M$  from  $H$  onto  $M$  (the orthogonal projection or orthoprojection) such that*

$$(3.5) \quad P_M^2 = P_M, \quad \ker P_M = M^\perp, \quad P_{M^\perp} = I - P_M.$$

*Proof.* Let us define  $P_M(x) = m$  where  $x = m + n$  is the decomposition from the previous theorem. The linearity of this operator follows from the fact that both  $M$  and  $M^\perp$  are linear subspaces. Also  $P_M(m) = m$  for all  $m \in M$  and the image of  $P_M$  is  $M$ . Thus  $P_M^2 = P_M$ . Also if  $P_M(x) = 0$  then  $x \perp M$ , i.e.  $\ker P_M = M^\perp$ . Similarly  $P_{M^\perp}(x) = n$  where  $x = m + n$  and  $P_M + P_{M^\perp} = I$ .  $\square$

**Example 3.25.** Let  $(e_n)$  be an orthonormal basis in a Hilbert space and let  $S \subset \mathbb{N}$  be fixed. Let  $M = \text{CLin}\{e_n : n \in S\}$  and  $M^\perp = \text{CLin}\{e_n : n \in \mathbb{N} \setminus S\}$ . Then

$$\sum_{k=1}^{\infty} a_k e_k = \sum_{k \in S} a_k e_k + \sum_{k \notin S} a_k e_k.$$

*Remark 3.26.* In fact there is a one-to-one correspondence between closed linear subspaces of a Hilbert space  $H$  and orthogonal projections defined by identities (3.5).

#### 4. FOURIER ANALYSIS

All bases are equal, but some are more equal than others.

As we saw already any separable Hilbert space possesses an orthonormal basis (infinitely many of them indeed). Are they equally good? This depends from our purposes. For solution of differential equation which arose in mathematical physics (wave, heat, Laplace equations, etc.) there is a proffered choice. The fundamental formula:  $\frac{d}{dx}e^{ax} = ae^{ax}$  reduces the derivative to a multiplication by  $a$ . We could benefit from this observation if the orthonormal basis will be constructed out of exponents. This helps to solve differential equations as was demonstrated in Subsection 1.2.

7.40pm Fourier series: Episode II  
Today's TV listing

**4.1. Fourier series.** Now we wish to address questions stated in Remark 1.9. Let us consider the space  $L_2[-\pi, \pi]$ . As we saw in Example 3.3(iii) there is an orthonormal sequence  $e_n(t) = (2\pi)^{-1/2}e^{int}$  in  $L_2[-\pi, \pi]$ . We will show that it is an orthonormal basis, i.e.

$$f(t) \in L_2[-\pi, \pi] \Leftrightarrow f(t) = \sum_{k=-\infty}^{\infty} \langle f, e_k \rangle e_k(t),$$

with convergence in  $L_2$  norm. To do this we show that  $\text{CLin}\{e_k : k \in \mathbb{Z}\} = L_2[-\pi, \pi]$ .

Let  $\text{CP}[-\pi, \pi]$  denote the continuous functions  $f$  on  $[-\pi, \pi]$  such that  $f(\pi) = f(-\pi)$ . We also define  $f$  outside of the interval  $[-\pi, \pi]$  by periodicity.

**Lemma 4.1.** *The space  $\text{CP}[-\pi, \pi]$  is dense in  $L_2[-\pi, \pi]$ .*

*Proof.* Let  $f \in L_2[-\pi, \pi]$ . Given  $\epsilon > 0$  there exists  $g \in C[-\pi, \pi]$  such that  $\|f - g\| < \epsilon/2$ . From continuity of  $g$  on a compact set follows that there is  $M$  such that  $|g(t)| < M$  for all  $t \in [-\pi, \pi]$ . We can now replace  $g$  by periodic  $\tilde{g}$ , which coincides with

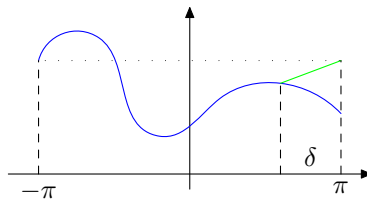


FIGURE 9. A modification of continuous function to periodic

$g$  on  $[-\pi, \pi - \delta]$  for an arbitrary  $\delta > 0$  and has the same bounds:  $|\tilde{g}(t)| < M$ , see Figure 9. Then

$$\|g - \tilde{g}\|_2^2 = \int_{\pi - \delta}^{\pi} |g(t) - \tilde{g}(t)|^2 dt \leq (2M)^2 \delta.$$

So if  $\delta < \epsilon^2/(4M)^2$  then  $\|g - \tilde{g}\| < \epsilon/2$  and  $\|f - \tilde{g}\| < \epsilon$ .  $\square$

Now if we could show that  $\text{CLin}\{e_k : k \in \mathbb{Z}\}$  includes  $\text{CP}[-\pi, \pi]$  then it also includes  $L_2[-\pi, \pi]$ .



**Notation 4.2.** Let  $f \in CP[-\pi, \pi]$ , write

$$(4.1) \quad f_n = \sum_{k=-n}^n \langle f, e_k \rangle e_k, \quad \text{for } n = 0, 1, 2, \dots$$

the *partial sum of the Fourier series* for  $f$ .

We want to show that  $\|f - f_n\|_2 \rightarrow 0$ . To this end we define  $n$ th *Fejér sum* by the formula

$$(4.2) \quad F_n = \frac{f_0 + f_1 + \dots + f_n}{n + 1},$$

and show that

$$\|F_n - f\|_\infty \rightarrow 0.$$

Then we conclude

$$\|F_n - f\|_2 = \left( \int_{-\pi}^{\pi} |F_n(t) - f|^2 \right)^{1/2} \leq (2\pi)^{1/2} \|F_n - f\|_\infty \rightarrow 0.$$

Since  $F_n \in \text{Lin}((e_n))$  then  $f \in \text{CLin}((e_n))$  and hence  $f = \sum_{-\infty}^{\infty} \langle f, e_k \rangle e_k$ .

*Remark 4.3.* It is **not** always true that  $\|f_n - f\|_\infty \rightarrow 0$  even for  $f \in CP[-\pi, \pi]$ .

**Exercise 4.4.** Find an example illustrating the above Remark.

It took 19 years of his life to prove this theorem

#### 4.2. Fejér's theorem.

**Proposition 4.5** (Fejér, age 19). *Let  $f \in CP[-\pi, \pi]$ . Then*

$$(4.3) \quad F_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(x-t) dt, \quad \text{where}$$

$$(4.4) \quad K_n(t) = \frac{1}{n+1} \sum_{k=0}^n \sum_{m=-k}^k e^{imt},$$

is the Fejér kernel.

*Proof.* From notation (4.1):

$$\begin{aligned} f_k(x) &= \sum_{m=-k}^k \langle f, e_m \rangle e_m(x) \\ &= \sum_{m=-k}^k \int_{-\pi}^{\pi} f(t) \frac{e^{-imt}}{\sqrt{2\pi}} dt \frac{e^{imx}}{\sqrt{2\pi}} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{m=-k}^k e^{im(x-t)} dt. \end{aligned}$$

Then from (4.2):

$$\begin{aligned}
 F_n(x) &= \frac{1}{n+1} \sum_{k=0}^n f_k(x) \\
 &= \frac{1}{n+1} \frac{1}{2\pi} \sum_{k=0}^n \int_{-\pi}^{\pi} f(t) \sum_{m=-k}^k e^{im(x-t)} dt \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{1}{n+1} \sum_{k=0}^n \sum_{m=-k}^k e^{im(x-t)} dt,
 \end{aligned}$$

which finishes the proof.  $\square$

**Lemma 4.6.** *The Fejér kernel is  $2\pi$ -periodic,  $K_n(0) = n+1$  and:*

$$(4.5) \quad K_n(t) = \frac{1}{n+1} \frac{\sin^2 \frac{(n+1)t}{2}}{\sin^2 \frac{t}{2}}, \quad \text{for } t \notin 2\pi\mathbb{Z}.$$

*Proof.* Let  $z = e^{it}$ , then:

$$\begin{aligned}
 K_n(t) &= \frac{1}{n+1} \sum_{k=0}^n (z^{-k} + \dots + 1 + z + \dots + z^k) \\
 &= \frac{1}{n+1} \sum_{j=-n}^n (n+1-|j|)z^j,
 \end{aligned}$$

by switch from counting in rows to counting in columns in Table 1. Let  $w = e^{it/2}$ ,

			1		
		$z^{-1}$	1	$z$	
	$z^{-2}$	$z^{-1}$	1	$z$	$z^2$
$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

TABLE 1. Counting powers in rows and columns

i.e.  $z = w^2$ , then

$$\begin{aligned}
 K_n(t) &= \frac{1}{n+1} (w^{-2n} + 2w^{-2n+2} + \dots + (n+1) + nw^2 + \dots + w^{2n}) \\
 (4.6) &= \frac{1}{n+1} (w^{-n} + w^{-n+2} + \dots + w^{n-2} + w^n)^2 \\
 &= \frac{1}{n+1} \left( \frac{w^{-n-1} - w^{n+1}}{w^{-1} - w} \right)^2 \quad \text{Could you sum a geometric progression?} \\
 &= \frac{1}{n+1} \left( \frac{2i \sin \frac{(n+1)t}{2}}{2i \sin \frac{t}{2}} \right)^2,
 \end{aligned}$$

if  $w \neq \pm 1$ . For the value of  $K_n(0)$  we substitute  $w = 1$  into (4.6).  $\square$

The first eleven Fejér kernels are shown on Figure 10, we could observe that:

**Lemma 4.7.** *Fejér's kernel has the following properties:*

- (i)  $K_n(t) \geq 0$  for all  $t \in \mathbb{R}$  and  $n \in \mathbb{N}$ .
- (ii)  $\int_{-\pi}^{\pi} K_n(t) dt = 2\pi$ .

(iii) For any  $\delta \in (0, \pi)$

$$\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi} K_n(t) dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

*Proof.* The **first property** immediately follows from the explicit formula (4.5). In contrast the **second property** is easier to deduce from expression with double sum (4.4):

$$\begin{aligned} \int_{-\pi}^{\pi} K_n(t) dt &= \int_{-\pi}^{\pi} \frac{1}{n+1} \sum_{k=0}^n \sum_{m=-k}^k e^{imt} dt \\ &= \frac{1}{n+1} \sum_{k=0}^n \sum_{m=-k}^k \int_{-\pi}^{\pi} e^{imt} dt \\ &= \frac{1}{n+1} \sum_{k=0}^n 2\pi \\ &= 2\pi, \end{aligned}$$

since the formula (3.1).

Finally if  $|t| > \delta$  then  $\sin^2(t/2) \geq \sin^2(\delta/2) > 0$  by monotonicity of sinus on  $[0, \pi/2]$ , so:

$$0 \leq K_n(t) \leq \frac{1}{(n+1) \sin^2(\delta/2)}$$

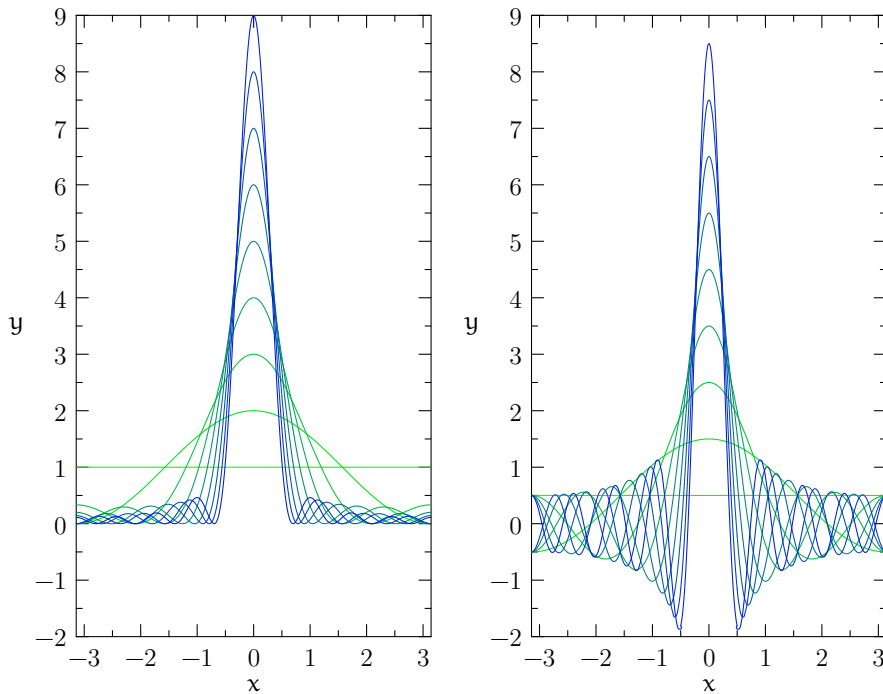


FIGURE 10. A family of Fejér kernels with the parameter  $m$  running from 0 to 9 is on the left picture. For a comparison unregularised Fourier kernels are on the right picture.

implying:

$$0 \leq \int_{\delta \leq |t| \leq \pi} K_n(t) dt \leq \frac{1(\pi - \delta)}{(n+1) \sin^2(\delta/2)} \rightarrow 0 \quad \text{as } n \rightarrow 0.$$

Therefore the **third property** follows from the **squeeze rule**.  $\square$

**Theorem 4.8** (Fejér Theorem). *Let  $f \in CP[-\pi, \pi]$ . Then its Fejér sums  $F_n$  (4.2) converges in supremum norm to  $f$  on  $[-\pi, \pi]$  and hence in  $L_2$  norm as well.*

*Proof.* Idea of the proof: if in the formula (4.3)

$$F_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) K_n(x-t) dt,$$

$t$  is long way from  $x$ ,  $K_n$  is small (see Lemma 4.7 and Figure 10), for  $t$  near  $x$ ,  $K_n$  is big with total "weight"  $2\pi$ , so the weighted average of  $f(t)$  is near  $f(x)$ .

Here are details. Using property 4.7(ii) and periodicity of  $f$  and  $K_n$  we could express trivially

$$f(x) = f(x) \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} K_n(x-t) dt = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(x) K_n(x-t) dt.$$

Similarly we rewrite (4.3) as

$$F_n(x) = \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} f(t) K_n(x-t) dt,$$

then

$$\begin{aligned} |f(x) - F_n(x)| &= \frac{1}{2\pi} \left| \int_{x-\pi}^{x+\pi} (f(x) - f(t)) K_n(x-t) dt \right| \\ &\leq \frac{1}{2\pi} \int_{x-\pi}^{x+\pi} |f(x) - f(t)| K_n(x-t) dt. \end{aligned}$$

Given  $\epsilon > 0$  split into three intervals:  $I_1 = [x - \pi, x - \delta]$ ,  $I_2 = [x - \delta, x + \delta]$ ,  $I_3 = [x + \delta, x + \pi]$ , where  $\delta$  is chosen such that  $|f(t) - f(x)| < \epsilon/2$  for  $t \in I_2$ , which is possible by continuity of  $f$ . So

$$\frac{1}{2\pi} \int_{I_2} |f(x) - f(t)| K_n(x-t) dt \leq \frac{\epsilon}{2} \frac{1}{2\pi} \int_{I_2} K_n(x-t) dt < \frac{\epsilon}{2}.$$

And

$$\begin{aligned} \frac{1}{2\pi} \int_{I_1 \cup I_3} |f(x) - f(t)| K_n(x-t) dt &\leq 2 \|f\|_{\infty} \frac{1}{2\pi} \int_{I_1 \cup I_3} K_n(x-t) dt \\ &= \frac{\|f\|_{\infty}}{\pi} \int_{\delta < |u| < \pi} K_n(u) du \\ &< \frac{\epsilon}{2}, \end{aligned}$$

if  $n$  is sufficiently large due to property 4.7(iii) of  $K_n$ . Hence  $|f(x) - F_n(x)| < \epsilon$  for a large  $n$  independent of  $x$ .  $\square$

We almost finished the demonstration that  $e_n(t) = (2\pi)^{-1/2} e^{int}$  is an orthonormal basis of  $L_2[-\pi, \pi]$ :

**Corollary 4.9** (Fourier series). *Let  $f \in L_2[-\pi, \pi]$ , with Fourier series*

$$\sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} c_n e^{int} \quad \text{where} \quad c_n = \frac{\langle f, e_n \rangle}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-int} dt.$$

*Then the series  $\sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} c_n e^{int}$  converges in  $L_2[-\pi, \pi]$  to  $f$ , i.e*

$$\lim_{k \rightarrow \infty} \left\| f - \sum_{n=-k}^k c_n e^{int} \right\|_2 = 0.$$

*Proof.* This follows from the **previous Theorem**, Lemma 4.1 about density of CP in  $L_2$ , and Theorem 3.15 on orthonormal basis.  $\square$

**4.3. Parseval's formula.** The following result first appeared in the framework of  $L_2[-\pi, \pi]$  and only later was understood to be a general property of inner product spaces.

**Theorem 4.10** (Parseval's formula). *If  $f, g \in L_2[-\pi, \pi]$  have Fourier series  $f = \sum_{n=-\infty}^{\infty} c_n e^{int}$ ,  $g = \sum_{n=-\infty}^{\infty} d_n e^{int}$  then*

$$(4.7) \quad \langle f, g \rangle = \int_{-\pi}^{\pi} f(t) \overline{g(t)} dt = 2\pi \sum_{n=-\infty}^{\infty} c_n \overline{d_n}.$$

*More generally if  $f$  and  $g$  are two vectors of a Hilbert space  $H$  with an orthonormal basis  $(e_n)_{n=-\infty}^{\infty}$  then*

$$\langle f, g \rangle = \sum_{k=-\infty}^{\infty} c_k \overline{d_k}, \quad \text{where} \quad c_n = \langle f, e_n \rangle, \quad d_n = \langle g, e_n \rangle,$$

*are the Fourier coefficients of  $f$  and  $g$ .*

*Proof.* In fact we could just prove the second, more general, statement—the first one is its particular realisation. Let  $f_n = \sum_{k=-n}^n c_k e_k$  and  $g_n = \sum_{k=-n}^n d_k e_k$  will be partial sums of the corresponding Fourier series. Then from orthonormality of  $(e_n)$  and linearity of the inner product:

$$\langle f_n, g_n \rangle = \left\langle \sum_{k=-n}^n c_k e_k, \sum_{k=-n}^n d_k e_k \right\rangle = \sum_{k=-n}^n c_k \overline{d_k}.$$

This formula together with the facts that  $f_k \rightarrow f$  and  $g_k \rightarrow g$  (following from Corollary 4.9) and **Lemma about continuity of the inner product** implies the assertion.  $\square$

**Corollary 4.11.** *A integrable function  $f$  belongs to  $L_2[-\pi, \pi]$  if and only if its Fourier series is convergent and then  $\|f\|^2 = 2\pi \sum_{-\infty}^{\infty} |c_k|^2$ .*

*Proof.* The necessity, i.e. implication  $f \in L_2 \Rightarrow \langle f, f \rangle = \|f\|^2 = 2\pi \sum |c_k|^2$ , follows from the **previous Theorem**. The sufficiency follows by **Riesz–Fisher Theorem**.  $\square$

**Remark 4.12.** The actual rôle of the Parseval's formula is shadowed by the orthonormality and is rarely recognised until we meet the *wavelets* or *coherent states*. Indeed the equality (4.7) should be read as follows:

**Theorem 4.13** (Modified Parseval). *The map  $W : H \rightarrow \ell_2$  given by the formula  $[Wf](n) = \langle f, e_n \rangle$  is an isometry for any orthonormal basis  $(e_n)$ .*

We could find many other systems of vectors  $(e_x)$ ,  $x \in X$  (very different from orthonormal bases) such that the map  $\mathcal{W} : H \rightarrow L_2(X)$  given by the simple universal formula

$$(4.8) \quad [\mathcal{W}f](x) = \langle f, e_x \rangle$$

will be an isometry of Hilbert spaces. The map (4.8) is often called *wavelet transform* and most famous is the *Cauchy integral formula* in complex analysis. The majority of wavelets transforms are linked with *group representations*, see our post-graduate course *Wavelets in Applied and Pure Maths*.

Heat and noise but not a fire?

Answer: „Application of Fourier Series“

**4.4. Some Application of Fourier Series.** We are going to provide now few examples which demonstrate the importance of the Fourier series in many questions. The first two (Example 4.14 and Theorem 4.15) belong to pure mathematics and last two are of more applicable nature.

**Example 4.14.** Let  $f(t) = t$  on  $[-\pi, \pi]$ . Then

$$\langle f, e_n \rangle = \int_{-\pi}^{\pi} t e^{-int} dt = \begin{cases} (-1)^n \frac{2\pi i}{n}, & n \neq 0 \\ 0, & n = 0 \end{cases} \quad (\text{check!}),$$

so  $f(t) \sim \sum_{-\infty}^{\infty} (-1)^n (i/n) e^{int}$ . By a direct integration:

$$\|f\|_2^2 = \int_{-\pi}^{\pi} t^2 dt = \frac{2\pi^3}{3}.$$

On the other hand by the **previous Corollary**:

$$\|f\|_2^2 = 2\pi \sum_{n \neq 0} \left| \frac{(-1)^n i}{n} \right|^2 = 4\pi \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Thus we get a beautiful formula

$$\sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Here is another important result.

**Theorem 4.15 (Weierstrass Approximation Theorem).** *For any function  $f \in C[a, b]$  and any  $\epsilon > 0$  there exists a polynomial  $p$  such that  $\|f - p\|_{\infty} < \epsilon$ .*

*Proof.* Change variable:  $t = 2\pi(x - \frac{a+b}{2})/(b-a)$  this maps  $x \in [a, b]$  onto  $t \in [-\pi, \pi]$ . Let  $P$  denote the subspace of polynomials in  $C[-\pi, \pi]$ . Then  $e^{int} \in \bar{P}$  for any  $n \in \mathbb{Z}$  since Taylor series converges uniformly in  $[-\pi, \pi]$ . Consequently  $P$  contains the closed linear span in (supremum norm) of  $e^{int}$ , any  $n \in \mathbb{Z}$ , which is  $CP[-\pi, \pi]$  by the **Fejér theorem**. Thus  $\bar{P} \supseteq CP[-\pi, \pi]$  and we extend that to non-periodic function as follows (why we could not make use of Lemma 4.1 here, by the way?).

For any  $f \in C[-\pi, \pi]$  let  $\lambda = (f(\pi) - f(-\pi))/(2\pi)$  then  $f_1(t) = f(t) - \lambda t \in CP[-\pi, \pi]$  and could be approximated by a polynomial  $p_1(t)$  from the above discussion. Then  $f(t)$  is approximated by the polynomial  $p(t) = p_1(t) + \lambda t$ .  $\square$

It is easy to see, that the rôle of exponents  $e^{int}$  in the above prove is rather modest: they can be replaced by any functions which has a Taylor expansion. The real glory of the Fourier analysis is demonstrated in the two following examples.

**Example 4.16.** The modern history of the Fourier analysis starts from the works of Fourier on the heat equation. As was mentioned in the introduction to this part, the exceptional role of Fourier coefficients for differential equations is explained by the simple formula  $\partial_x e^{inx} = ine^{inx}$ . We shortly review a solution of the *heat equation* to illustrate this.

Let us have a rod of the length  $2\pi$ . The temperature at its point  $x \in [-\pi, \pi]$  and a moment  $t \in [0, \infty)$  is described by a function  $u(t, x)$  on  $[0, \infty) \times [-\pi, \pi]$ . The mathematical equation describing a dynamics of the temperature distribution is:

$$(4.9) \quad \frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} \quad \text{or, equivalently,} \quad (\partial_t - \partial_x^2) u(t, x) = 0.$$

For any fixed moment  $t_0$  the function  $u(t_0, x)$  depends only from  $x \in [-\pi, \pi]$  and according to Corollary 4.9 could be represented by its Fourier series:

$$u(t_0, x) = \sum_{n=-\infty}^{\infty} \langle u, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} c_n(t_0) e^{inx},$$

where

$$c_n(t_0) = \frac{\langle u, e_n \rangle}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(t_0, x) e^{-inx} dx,$$

with Fourier coefficients  $c_n(t_0)$  depending from  $t_0$ . We substitute that decomposition into the heat equation (4.9) to receive:

$$(4.10) \quad \begin{aligned} (\partial_t - \partial_x^2) u(t, x) &= (\partial_t - \partial_x^2) \sum_{n=-\infty}^{\infty} c_n(t) e^{inx} \\ &= \sum_{n=-\infty}^{\infty} (\partial_t - \partial_x^2) c_n(t) e^{inx} \\ &= \sum_{n=-\infty}^{\infty} (c_n'(t) + n^2 c_n(t)) e^{inx} = 0. \end{aligned}$$

Since function  $e^{inx}$  form a basis the last equation (4.10) holds if and only if

$$(4.11) \quad c_n'(t) + n^2 c_n(t) = 0 \quad \text{for all } n \text{ and } t.$$

Equations from the system (4.11) have general solutions of the form:

$$(4.12) \quad c_n(t) = c_n(0) e^{-n^2 t} \quad \text{for all } t \in [0, \infty),$$

producing a general solution of the heat equation (4.9) in the form:

$$(4.13) \quad u(t, x) = \sum_{n=-\infty}^{\infty} c_n(0) e^{-n^2 t} e^{inx} = \sum_{n=-\infty}^{\infty} c_n(0) e^{-n^2 t + inx},$$

where constant  $c_n(0)$  could be defined from boundary condition. For example, if it is known that the initial distribution of temperature was  $u(0, x) = g(x)$  for a function  $g(x) \in L_2[-\pi, \pi]$  then  $c_n(0)$  is the  $n$ -th Fourier coefficient of  $g(x)$ .

The general solution (4.13) helps produce both the analytical study of the heat equation (4.9) and numerical simulation. For example, from (4.13) obviously follows that

- the temperature is rapidly relaxing toward the thermal equilibrium with the temperature given by  $c_0(0)$ , however never reach it within a finite time;
- the “higher frequencies” (bigger thermal gradients) have a bigger speed of relaxation; etc.

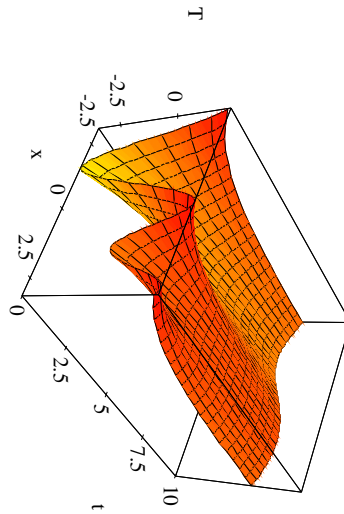


FIGURE 11. The dynamics of a heat equation:  
 $x$ —coordinate on the rod,  
 $t$ —time,  
 $T$ —temperature.

The example of numerical simulation for the initial value problem with  $g(x) = 2 \cos(2 * u) + 1.5 \sin(u)$ . It is clearly illustrate our above conclusions.

**Example 4.17.** Among the oldest periodic functions in human culture are acoustic waves of musical tones. The mathematical theory of musics (including rudiments of the Fourier analysis!) is as old as mathematics itself and was highly respected already in *Pythagoras' school* more 2500 years ago.

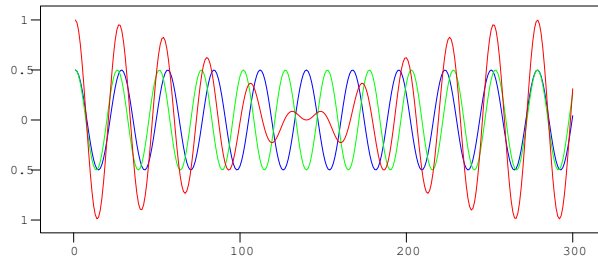


FIGURE 12. Two oscillation with unharmonious frequencies and the appearing dissonance. Click to listen the **blue** and **green** pure harmonics and **red dissonance**.

The earliest observations are that

- (i) The musical sounds are made of pure harmonics (see the blue and green graphs on the Figure 12), in our language  $\cos$  and  $\sin$  functions form a basis;
- (ii) Not every two pure harmonics are compatible, to be their frequencies should make a simple ratio. Otherwise the dissonance (red graph on Figure 12) appears.

The musical tone, say G5, performed on different instruments clearly has something in common and different, see Figure 13 for comparisons. The decomposition into the pure harmonics, i.e. finding Fourier coefficient for the signal, could provide the complete characterisation, see Figure 14.



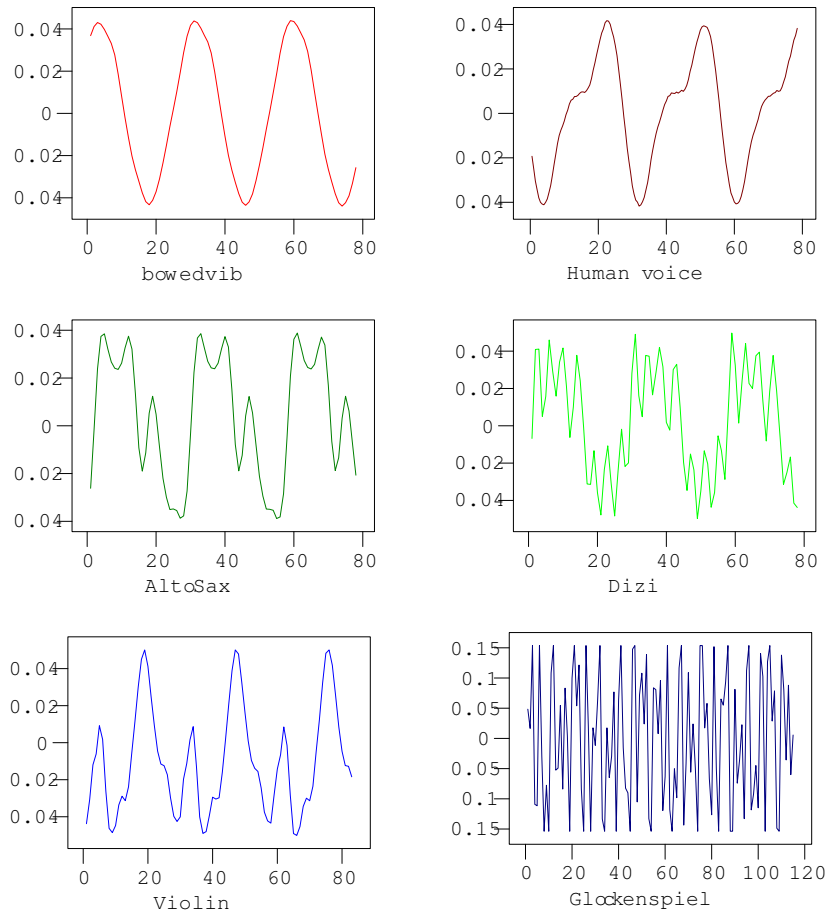


FIGURE 13. Graphics of G5 performed on different musical instruments (click on picture to hear the sound). Samples are taken from [Sound Library](#).

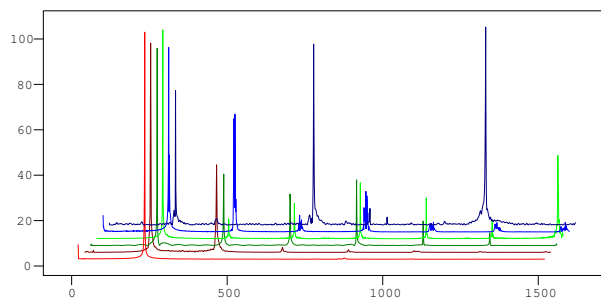


FIGURE 14. Fourier series for G5 performed on different musical instruments (same order and colour as on the [previous Figure](#))

The Fourier analysis tells that:

- (i) All sound have the same base (i.e. the lowest) frequencies which corresponds to the G5 tone, i.e. 788 Gz.

- (ii) The higher frequencies, which are necessarily are multiples of 788 Gz to avoid dissonance, appears with different weights for different instruments.

The Fourier analysis is very useful in the signal processing and is indeed the fundamental tool. However it is not universal and has very serious limitations. Consider the simple case of the signals plotted on the Figure 15(a) and (b). They are both made out of same two pure harmonics:

- (i) On the first signal the two harmonics (drawn in blue and green) follow one after another in time on Figure 15(a);  
(ii) They just blended in equal proportions over the whole interval on Figure 15(b).

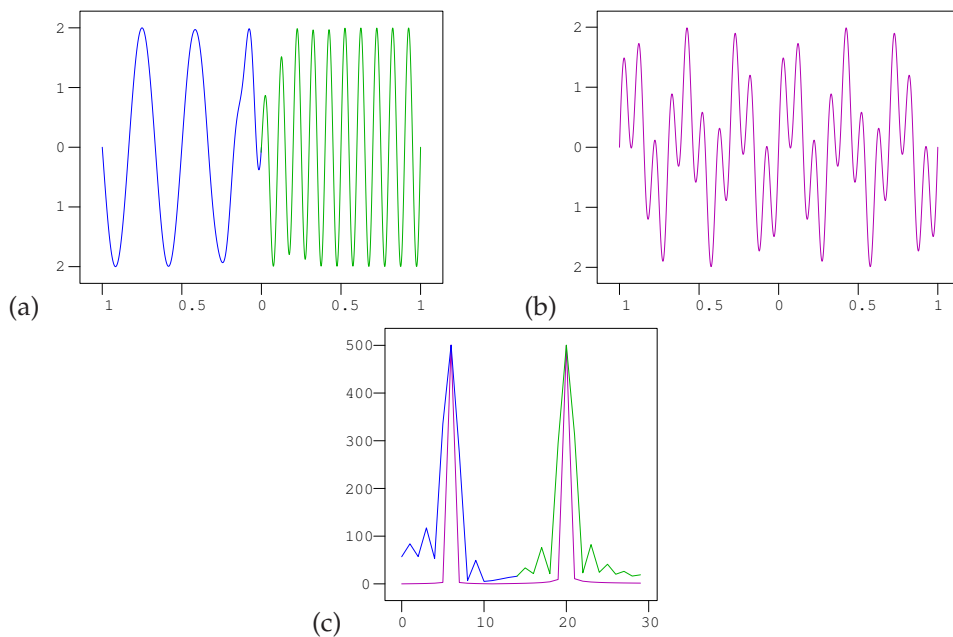


FIGURE 15. Limits of the Fourier analysis: different frequencies separated in time

This appear to be two very different signals. However the Fourier performed over the whole interval does not seems to be very different, see Figure 15(c). Both transforms (drawn in blue-green and pink) have two major pikes corresponding to the pure frequencies. It is not very easy to extract differences between signals from their Fourier transform (yet this should be possible according to our study).

Even a better picture could be obtained if we use *windowed Fourier transform*, namely use a sliding “window” of the constant width instead of the entire interval for the Fourier transform. Yet even better analysis could be obtained by means of *wavelets* already mentioned in Remark 4.12 in connection with Plancherel’s formula. Roughly, wavelets correspond to a sliding window of a variable size—narrow for high frequencies and wide for low.

## 5. DUALITY OF LINEAR SPACES

Everything has another side

Orthonormal basis allows to reduce any question on Hilbert space to a question on sequence of numbers. This is powerful but sometimes heavy technique. Sometime we need a smaller and faster tool to study questions which are represented by a single number, for example to demonstrate that two vectors are different it is enough to show that there is a unequal values of a single coordinate. In such cases *linear functionals* are just what we needed.

-Is it functional?  
-Yes, it works!

**5.1. Dual space of a normed space.**

**Definition 5.1.** A *linear functional* on a vector space  $V$  is a linear mapping  $\alpha : V \rightarrow \mathbb{C}$  (or  $\alpha : V \rightarrow \mathbb{R}$  in the real case), i.e.

$$\alpha(ax + by) = a\alpha(x) + b\alpha(y), \quad \text{for all } x, y \in V \text{ and } a, b \in \mathbb{C}.$$

**Exercise 5.2.** Show that  $\alpha(0)$  is necessarily 0.

We will not consider any functionals but linear, thus bellow *functional* always means *linear functional*.

**Example 5.3.** (i) Let  $V = \mathbb{C}^n$  and  $c_k, k = 1, \dots, n$  be complex numbers. Then  $\alpha((x_1, \dots, x_n)) = c_1x_1 + \dots + c_nx_n$  is a linear functional.

(ii) On  $C[0, 1]$  a functional is given by  $\alpha(f) = \int_0^1 f(t) dt$ .

(iii) On a Hilbert space  $H$  for any  $x \in H$  a functional  $\alpha_x$  is given by  $\alpha_x(y) = \langle y, x \rangle$ .

**Theorem 5.4.** Let  $V$  be a normed space and  $\alpha$  is a linear functional. The following are equivalent:

- (i)  $\alpha$  is continuous (at any point of  $V$ ).
- (ii)  $\alpha$  is continuous at point 0.
- (iii)  $\sup\{|\alpha(x)| : \|x\| \leq 1\} < \infty$ , i.e.  $\alpha$  is a bounded linear functional.

*Proof.* Implication 5.4(i)  $\Rightarrow$  5.4(ii) is trivial.

Show 5.4(ii)  $\Rightarrow$  5.4(iii). By the **definition of continuity**: for any  $\epsilon > 0$  there exists  $\delta > 0$  such that  $\|v\| < \delta$  implies  $|\alpha(v) - \alpha(0)| < \epsilon$ . Take  $\epsilon = 1$  then  $|\alpha(\delta x)| < 1$  for all  $x$  with norm less than 1 because  $\|\delta x\| < \delta$ . But from linearity of  $\alpha$  the inequality  $|\alpha(\delta x)| < 1$  implies  $|\alpha(x)| < 1/\delta < \infty$  for all  $\|x\| \leq 1$ .

5.4(iii)  $\Rightarrow$  5.4(i). Let mentioned supremum be  $M$ . For any  $x, y \in V$  such that  $x \neq y$  vector  $(x - y)/\|x - y\|$  has norm 1. Thus  $|\alpha((x - y)/\|x - y\|)| < M$ . By the linearity of  $\alpha$  this implies that  $|\alpha(x) - \alpha(y)| < M\|x - y\|$ . Thus  $\alpha$  is continuous.  $\square$

**Definition 5.5.** The *dual space*  $X^*$  of a normed space  $X$  is the set of continuous linear functionals on  $X$ . Define a norm on it by

$$(5.1) \quad \|\alpha\| = \sup_{\|x\| \leq 1} |\alpha(x)|.$$

- Exercise 5.6.**
- (i) Show that  $X^*$  is a linear space with natural operations.
  - (ii) Show that (5.1) defines a norm on  $X^*$ .
  - (iii) Show that  $|\alpha(x)| \leq \|\alpha\| \cdot \|x\|$  for all  $x \in X, \alpha \in X^*$ .

**Theorem 5.7.**  $X^*$  is a Banach space with the defined norm (even if  $X$  was incomplete).

*Proof.* Due to Exercise 5.6 we only need to show that  $X^*$  is complete. Let  $(\alpha_n)$  be a Cauchy sequence in  $X^*$ , then for any  $x \in X$  scalars  $\alpha_n(x)$  form a Cauchy sequence, since  $|\alpha_m(x) - \alpha_n(x)| \leq \|\alpha_m - \alpha_n\| \cdot \|x\|$ . Thus the sequence has a limit

and we define  $\alpha$  by  $\alpha(x) = \lim_{n \rightarrow \infty} \alpha_n(x)$ . Clearly  $\alpha$  is a linear functional on  $X$ . We should show that it is bounded and  $\alpha_n \rightarrow \alpha$ . Given  $\epsilon > 0$  there exists  $N$  such that  $\|\alpha_n - \alpha_m\| < \epsilon$  for all  $n, m \geq N$ . If  $\|x\| \leq 1$  then  $|\alpha_n(x) - \alpha_m(x)| \leq \epsilon$ , let  $m \rightarrow \infty$  then  $|\alpha_n(x) - \alpha(x)| \leq \epsilon$ , so

$$|\alpha(x)| \leq |\alpha_n(x)| + \epsilon \leq \|\alpha_n\| + \epsilon,$$

i.e.  $\|\alpha\|$  is finite and  $\|\alpha_n - \alpha\| \leq \epsilon$ , thus  $\alpha_n \rightarrow \alpha$ .  $\square$

**Definition 5.8.** The kernel of linear functional  $\alpha$ , write  $\ker \alpha$ , is the set all vectors  $x \in X$  such that  $\alpha(x) = 0$ .

**Exercise 5.9.** Show that

- (i)  $\ker \alpha$  is a subspace of  $X$ .
- (ii) If  $\alpha \neq 0$  then  $\ker \alpha$  is a proper subspace of  $X$ .
- (iii) If  $\alpha$  is continuous then  $\ker \alpha$  is closed.

Study one and get any other for free!  
Hilbert spaces sale

## 5.2. Self-duality of Hilbert space.

**Lemma 5.10 (Riesz–Fréchet).** Let  $H$  be a Hilbert space and  $\alpha$  a continuous linear functional on  $H$ , then there exists the unique  $y \in H$  such that  $\alpha(x) = \langle x, y \rangle$  for all  $x \in H$ . Also  $\|\alpha\|_{H^*} = \|y\|_H$ .

*Proof.* Uniqueness: if  $\langle x, y \rangle = \langle x, y' \rangle \Leftrightarrow \langle x, y - y' \rangle = 0$  for all  $x \in H$  then  $y - y'$  is self-orthogonal and thus is zero (Exercise 3.2(i)).

Existence: we may assume that  $\alpha \neq 0$  (otherwise take  $y = 0$ ), then  $M = \ker \alpha$  is a closed proper subspace of  $H$ . Since  $H = M \oplus M^\perp$ , there exists a non-zero  $z \in M^\perp$ , by scaling we could get  $\alpha(z) = 1$ . Then for any  $x \in H$ :

$$x = (x - \alpha(x)z) + \alpha(x)z, \quad \text{with } x - \alpha(x)z \in M, \alpha(x)z \in M^\perp.$$

Because  $\langle x, z \rangle = \alpha(x) \langle z, z \rangle = \alpha(x) \|z\|^2$  for any  $x \in H$  we set  $y = z / \|z\|^2$ .

Equality of the norms  $\|\alpha\|_{H^*} = \|y\|_H$  follows from the **Cauchy–Bunyakovskii–Schwarz inequality** in the form  $\alpha(x) \leq \|x\| \cdot \|y\|$  and the identity  $\alpha(y / \|y\|) = \|y\|$ .  $\square$

**Example 5.11.** On  $L_2[0, 1]$  let  $\alpha(f) = \langle f, t^2 \rangle = \int_0^1 f(t)t^2 dt$ . Then

$$\|\alpha\| = \|t^2\| = \left( \int_0^1 (t^2)^2 dt \right)^{1/2} = \frac{1}{\sqrt{5}}.$$

## 6. OPERATORS

All the space's a stage,  
and all functionals and operators merely players!

All our previous considerations were only a preparation of the stage and now the main actors come forward to perform a play. The vectors spaces are not so interesting while we consider them in statics, what really make them exciting is the their transformations. The natural first steps is to consider transformations which respect both linear structure and the norm.

### 6.1. Linear operators.

**Definition 6.1.** A linear operator  $T$  between two normed spaces  $X$  and  $Y$  is a mapping  $T : X \rightarrow Y$  such that  $T(\lambda v + \mu u) = \lambda T(v) + \mu T(u)$ . The kernel of linear operator  $\ker T$  and image are defined by

$$\ker T = \{x \in X : Tx = 0\} \quad \text{Im } T = \{y \in Y : y = Tx, \text{ for some } x \in X\}.$$

**Exercise 6.2.** Show that kernel of  $T$  is a linear subspace of  $X$  and image of  $T$  is a linear subspace of  $Y$ .

As usual we are interested also in connections with the second (topological) structure:

**Definition 6.3.** A norm of linear operator is defined:

$$(6.1) \quad \|T\| = \sup\{\|Tx\|_Y : \|x\|_X \leq 1\}.$$

$T$  is a bounded linear operator if  $\|T\| = \sup\{\|Tx\| : \|x\|\} < \infty$ .

**Exercise 6.4.** Show that  $\|Tx\| \leq \|T\| \cdot \|x\|$  for all  $x \in X$ .

**Example 6.5.** Consider the following examples and determine kernel and images of the mentioned operators.

- (i) On a normed space  $X$  define the zero operator to a space  $Y$  by  $Z : x \rightarrow 0$  for all  $x \in X$ . Its norm is 0.
- (ii) On a normed space  $X$  define the identity operator by  $I_X : x \rightarrow x$  for all  $x \in X$ . Its norm is 1.
- (iii) On a normed space  $X$  any linear functional define a linear operator from  $X$  to  $\mathbb{C}$ , its norm as operator is the same as functional.
- (iv) The set of operators from  $\mathbb{C}^n$  to  $\mathbb{C}^m$  is given by  $n \times m$  matrices which acts on vector by the matrix multiplication. All linear operators on finite-dimensional spaces are bounded.
- (v) On  $\ell_2$ , let  $S(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$  be the right shift operator. Clearly  $\|Sx\| = \|x\|$  for all  $x$ , so  $\|S\| = 1$ .
- (vi) On  $L_2[a, b]$ , let  $w(t) \in C[a, b]$  and define multiplication operator  $M_w f$  by  $(M_w f)(t) = w(t)f(t)$ . Now:

$$\begin{aligned} \|M_w f\|^2 &= \int_a^b |w(t)|^2 |f(t)|^2 dt \\ &\leq K^2 \int_a^b |f(t)|^2 dt, \quad \text{where } K = \|w\|_\infty = \sup_{[a,b]} |w(t)|, \end{aligned}$$

so  $\|M_w\| \leq K$ .

**Exercise 6.6.** Show that for multiplication operator in fact there is the equality of norms  $\|M_w\|_2 = \|w(t)\|_\infty$ .

**Theorem 6.7.** Let  $T : X \rightarrow Y$  be a linear operator. The following conditions are equivalent:

- (i)  $T$  is continuous on  $X$ ;
- (ii)  $T$  is continuous at the point 0.
- (iii)  $T$  is a bounded linear operator.

*Proof.* Proof essentially follows the proof of similar Theorem 5.4. □

## 6.2. $B(H)$ as a Banach space (and even algebra).

**Theorem 6.8.** Let  $B(X, Y)$  be the space of bounded linear operators from  $X$  and  $Y$  with the norm *defined above*. If  $Y$  is complete, then  $B(X, Y)$  is a Banach space.

*Proof.* The proof repeat proof of the Theorem 5.7, which is a particular case of the present theorem for  $Y = \mathbb{C}$ , see Example 6.5(iii).  $\square$

**Theorem 6.9.** Let  $T \in B(X, Y)$  and  $S \in B(Y, Z)$ , where  $X, Y$ , and  $Z$  are normed spaces. Then  $ST \in B(X, Z)$  and  $\|ST\| \leq \|S\| \|T\|$ .

*Proof.* Clearly  $(ST)x = S(Tx) \in Z$ , and

$$\|STx\| \leq \|S\| \|Tx\| \leq \|S\| \|T\| \|x\|,$$

which implies norm estimation if  $\|x\| \leq 1$ .  $\square$

**Corollary 6.10.** Let  $T \in B(X, X) = B(X)$ , where  $X$  is a normed space. Then for any  $n \geq 1$ ,  $T^n \in B(X)$  and  $\|T^n\| \leq \|T\|^n$ .

*Proof.* It is induction by  $n$  with the trivial base  $n = 1$  and the step following from the previous theorem.  $\square$

*Remark 6.11.* Some texts use notations  $L(X, Y)$  and  $L(X)$  instead of ours  $B(X, Y)$  and  $B(X)$ .

**Definition 6.12.** Let  $T \in B(X, Y)$ . We say  $T$  is an *invertible operator* if there exists  $S \in B(Y, X)$  such that

$$ST = I_X \quad \text{and} \quad TS = I_Y.$$

Such an  $S$  is called the *inverse operator* of  $T$ .

**Exercise 6.13.** Show that

- (i) for an invertible operator  $T : X \rightarrow Y$  we have  $\ker T = \{0\}$  and  $\mathcal{R}T = Y$ .
- (ii) the inverse operator is unique (if exists at all). (Assume existence of  $S$  and  $S'$ , then consider operator  $STS'$ .)

**Example 6.14.** We consider inverses to operators from Exercise 6.5.

- (i) The zero operator is never invertible unless the pathological spaces  $X = Y = \{0\}$ .
- (ii) The **identity operator**  $I_X$  is the inverse of itself.
- (iii) A linear functional is not invertible unless it is non-zero and  $X$  is one dimensional.
- (iv) An operator  $\mathbb{C}^n \rightarrow \mathbb{C}^m$  is invertible if and only if  $m = n$  and corresponding square matrix is non-singular, i.e. has non-zero determinant.
- (v) The **right shift**  $S$  is not invertible on  $\ell_2$  (it is one-to-one but is not onto). But the *left shift operator*  $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$  is its *left inverse*, i.e.  $TS = I$  but  $TS \neq I$  since  $ST(1, 0, 0, \dots) = (0, 0, \dots)$ .  $T$  is not invertible either (it is onto but not one-to-one), however  $S$  is its *right inverse*.
- (vi) **Operator of multiplication**  $M_w$  is invertible if and only if  $w^{-1} \in C[a, b]$  and inverse is  $M_{w^{-1}}$ . For example  $M_{1+t}$  is invertible  $L_2[0, 1]$  and  $M_t$  is not.

## 6.3. Adjoint.

**Theorem 6.15.** Let  $H$  and  $K$  be Hilbert Spaces and  $T \in B(H, K)$ . Then there exists operator  $T^* \in B(K, H)$  such that

$$\langle Th, k \rangle_K = \langle h, T^*k \rangle_H \quad \text{for all } h \in H, k \in K.$$

Such  $T^*$  is called the *adjoint operator* of  $T$ . Also  $T^{**} = T$  and  $\|T^*\| = \|T\|$ .

*Proof.* For any fixed  $k \in K$  the expression  $h \mapsto \langle Th, k \rangle_K$  defines a bounded linear functional on  $H$ . By the **Riesz–Fréchet lemma** there is a *unique*  $y \in H$  such that  $\langle Th, k \rangle_K = \langle h, y \rangle_H$  for all  $h \in H$ . Define  $T^*k = y$  then  $T^*$  is linear:

$$\begin{aligned} \langle h, T^*(\lambda_1 k_1 + \lambda_2 k_2) \rangle_H &= \langle Th, \lambda_1 k_1 + \lambda_2 k_2 \rangle_K \\ &= \bar{\lambda}_1 \langle Th, k_1 \rangle_K + \bar{\lambda}_2 \langle Th, k_2 \rangle_K \\ &= \bar{\lambda}_1 \langle h, T^*k_1 \rangle_H + \bar{\lambda}_2 \langle h, T^*k_2 \rangle_H \\ &= \langle h, \lambda_1 T^*k_1 + \lambda_2 T^*k_2 \rangle_H \end{aligned}$$

So  $T^*(\lambda_1 k_1 + \lambda_2 k_2) = \lambda_1 T^*k_1 + \lambda_2 T^*k_2$ .  $T^{**}$  is defined by  $\langle k, T^{**}h \rangle = \langle T^*k, h \rangle$  and the identity  $\langle T^{**}h, k \rangle = \langle h, T^*k \rangle = \langle Th, k \rangle$  for all  $h$  and  $k$  shows  $T^{**} = T$ . Also:

$$\begin{aligned} \|T^*k\|^2 &= \langle T^*k, T^*k \rangle = \langle k, TT^*k \rangle \\ &\leq \|k\| \cdot \|TT^*k\| \leq \|k\| \cdot \|T\| \cdot \|T^*k\|, \end{aligned}$$

which implies  $\|T^*k\| \leq \|T\| \cdot \|k\|$ , consequently  $\|T^*\| \leq \|T\|$ . The opposite inequality follows from the identity  $\|T\| = \|T^{**}\|$ .  $\square$

**Exercise 6.16.** (i) For operators  $T_1$  and  $T_2$  show that

$$(T_1 T_2)^* = T_2^* T_1^*, \quad (T_1 + T_2)^* = T_1^* + T_2^* \quad (\lambda T)^* = \bar{\lambda} T^*.$$

(ii) If  $A$  is an operator on a Hilbert space  $H$  then  $(\ker A)^\perp = \text{Im } A^*$ .

#### 6.4. Hermitian, unitary and normal operators.

**Definition 6.17.** An operator  $T : H \rightarrow H$  is a *Hermitian operator* or *self-adjoint operator* if  $T = T^*$ , i.e.  $\langle Tx, y \rangle = \langle x, Ty \rangle$  for all  $x, y \in H$ .

**Example 6.18.** (i) On  $\ell_2$  the adjoint  $S^*$  to the **right shift operator**  $S$  is given by the **left shift**  $S^* = T$ , indeed:

$$\begin{aligned} \langle Sx, y \rangle &= \langle (0, x_1, x_2, \dots), (y_1, y_2, \dots) \rangle \\ &= x_1 \bar{y}_2 + x_2 \bar{y}_3 + \dots = \langle (x_1, x_2, \dots), (y_2, y_3, \dots) \rangle \\ &= \langle x, Ty \rangle. \end{aligned}$$

Thus  $S$  is *not* Hermitian.

(ii) Let  $D$  be *diagonal operator* on  $\ell_2$  given by

$$D(x_1, x_2, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \dots).$$

where  $(\lambda_k)$  is any bounded complex sequence. It is easy to check that  $\|D\| = \|(\lambda_n)\|_\infty = \sup_k |\lambda_k|$  and

$$D^*(x_1, x_2, \dots) = (\bar{\lambda}_1 x_1, \bar{\lambda}_2 x_2, \dots),$$

thus  $D$  is Hermitian if and only if  $\lambda_k \in \mathbb{R}$  for all  $k$ .

(iii) If  $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$  is represented by multiplication of a column vector by a matrix  $A$ , then  $T^*$  is multiplication by the matrix  $A^*$ —transpose and conjugate to  $A$ .

**Exercise 6.19.** Show that for any bounded operator  $T$  operators  $T_1 = \frac{1}{2}(T \pm T^*)$ ,  $T^*T$  and  $TT^*$  are Hermitians.

**Theorem 6.20.** Let  $T$  be a Hermitian operator on a Hilbert space. Then

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x \rangle|.$$

*Proof.* If  $Tx = 0$  for all  $x \in H$ , both sides of the identity are 0. So we suppose that  $\exists x \in H$  for which  $Tx \neq 0$ .

We see that  $|\langle Tx, x \rangle| \leq \|Tx\| \|x\| \leq \|T\| \|x^2\|$ , so  $\sup_{\|x\|=1} |\langle Tx, x \rangle| \leq \|T\|$ . To get the inequality the other way around, we first write  $s := \sup_{\|x\|=1} |\langle Tx, x \rangle|$ . Then for any  $x \in H$ , we have  $|\langle Tx, x \rangle| \leq s \|x^2\|$ .

We now consider

$$\begin{aligned} \langle T(x+y), x+y \rangle &= \langle Tx, x \rangle + \langle Tx, y \rangle + \langle Ty, x \rangle + \langle Ty, y \rangle = \langle Tx, x \rangle + 2\Re \langle Tx, y \rangle + \langle Ty, y \rangle \\ (\text{because } T \text{ being Hermitian gives } \langle Ty, x \rangle &= \langle y, Tx \rangle = \overline{\langle Tx, y \rangle}) \text{ and, similarly,} \\ \langle T(x-y), x-y \rangle &= \langle Tx, x \rangle - 2\Re \langle Tx, y \rangle + \langle Ty, y \rangle. \end{aligned}$$

Subtracting gives

$$4\Re \langle Tx, y \rangle = \langle T(x+y), x+y \rangle - \langle T(x-y), x-y \rangle \leq s(\|x+y\|^2 + \|x-y\|^2) = 2s(\|x\|^2 + \|y\|^2),$$

by the parallelogram identity.

Now, for  $x \in H$  such that  $Tx \neq 0$ , we put  $y = \|Tx\|^{-1} \|x\| Tx$ . Then  $\|y\| = \|x\|$  and when we substitute into the previous inequality, we get

$$4 \|Tx\| \|x\| = 4\Re \langle Tx, y \rangle \leq 4s \|x^2\|,$$

So  $\|Tx\| \leq s \|x\|$  and it follows that  $\|T\| \leq s$ , as required.  $\square$

**Definition 6.21.** We say that  $U : H \rightarrow H$  is a *unitary operator* on a Hilbert space  $H$  if  $U^* = U^{-1}$ , i.e.  $U^*U = UU^* = I$ .

**Example 6.22.** (i) If  $D : \ell_2 \rightarrow \ell_2$  is a **diagonal operator** such that  $De_k = \lambda_k e_k$ , then  $D^*e_k = \bar{\lambda}_k e_k$  and  $D$  is unitary if and only if  $|\lambda_k| = 1$  for all  $k$ .  
(ii) The **shift operator**  $S$  satisfies  $S^*S = I$  but  $SS^* \neq I$  thus  $S$  is **not** unitary.

**Theorem 6.23.** For an operator  $U$  on a complex Hilbert space  $H$  the following are equivalent:

- (i)  $U$  is unitary;
- (ii)  $U$  is surjection and an isometry, i.e.  $\|Ux\| = \|x\|$  for all  $x \in H$ ;
- (iii)  $U$  is a surjection and preserves the inner product, i.e.  $\langle Ux, Uy \rangle = \langle x, y \rangle$  for all  $x, y \in H$ .

*Proof.* **6.23(i)  $\Rightarrow$  6.23(ii).** Clearly unitarity of operator implies its invertibility and hence surjectivity. Also

$$\|Ux\|^2 = \langle Ux, Ux \rangle = \langle x, U^*Ux \rangle = \langle x, x \rangle = \|x\|^2.$$

**6.23(ii)  $\Rightarrow$  6.23(iii).** Using the polarisation identity (cf. polarisation in equation (2.6)):

$$\begin{aligned} 4 \langle Tx, y \rangle &= \langle T(x+y), x+y \rangle + i \langle T(x+iy), x+iy \rangle \\ &\quad - \langle T(x-y), x-y \rangle - i \langle T(x-iy), x-iy \rangle. \\ &= \sum_{k=0}^3 i^k \langle T(x+i^k y), x+i^k y \rangle \end{aligned}$$



Take  $T = U^*U$  and  $T = I$ , then

$$\begin{aligned} 4 \langle U^*Ux, y \rangle &= \sum_{k=0}^3 i^k \langle U^*U(x + i^k y), x + i^k y \rangle \\ &= \sum_{k=0}^3 i^k \langle U(x + i^k y), U(x + i^k y) \rangle \\ &= \sum_{k=0}^3 i^k \langle (x + i^k y), (x + i^k y) \rangle \\ &= 4 \langle x, y \rangle. \end{aligned}$$

6.23(iii)  $\Rightarrow$  6.23(i). Indeed  $\langle U^*Ux, y \rangle = \langle x, y \rangle$  implies  $\langle (U^*U - I)x, y \rangle = 0$  for all  $x, y \in H$ , then  $U^*U = I$ . Since  $U$  should be invertible by surjectivity we see that  $U^* = U^{-1}$ .  $\square$

**Definition 6.24.** A normal operator  $T$  is one for which  $T^*T = TT^*$ .

- Example 6.25.**
- (i) Any self-adjoint operator  $T$  is normal, since  $T^* = T$ .
  - (ii) Any unitary operator  $U$  is normal, since  $U^*U = I = UU^*$ .
  - (iii) Any **diagonal operator**  $D$  is normal, since  $De_k = \lambda_k e_k, D^*e_k = \bar{\lambda}_k e_k$ , and  $DD^*e_k = D^*De_k = |\lambda_k|^2 e_k$ .
  - (iv) The **shift operator**  $S$  is **not** normal.
  - (v) A finite matrix is normal (as an operator on  $\ell_2^n$ ) if and only if it has an orthonormal basis in which it is diagonal.

*Remark 6.26.* Theorems 6.20 and 6.23(ii) draw similarity between those types of operators and multiplications by complex numbers. Indeed Theorem 6.20 said that an operator which significantly change direction of vectors (“rotates”) cannot be Hermitian, just like a multiplication by a real number scales but do not rotate. On the other hand Theorem 6.23(ii) says that unitary operator just rotate vectors but do not scale, as a multiplication by an unimodular complex number. We will see further such connections in Theorem 7.17.

## 7. SPECTRAL THEORY

Beware of ghosts<sup>2</sup> in this area!

As we saw operators could be added and multiplied each other, in some sense they behave like numbers, but are much more complicated. In this lecture we will associate to each operator a set of complex numbers which reflects certain (unfortunately not all) properties of this operator.

The analogy between operators and numbers become even more deeper since we could construct *functions of operators* (called *functional calculus*) in a way we build numeric functions. The most important functions of this sort is called *resolvent* (see Definition 7.5). The methods of analytical functions are very powerful in operator theory and students may wish to refresh their knowledge of complex analysis before this part.

**7.1. The spectrum of an operator on a Hilbert space.** An *eigenvalue* of operator  $T \in B(H)$  is a complex number  $\lambda$  such that there exists a nonzero  $x \in H$ , called *eigenvector* with property  $Tx = \lambda x$ , in other words  $x \in \ker(T - \lambda I)$ .

In finite dimensions  $T - \lambda I$  is invertible if and only if  $\lambda$  is **not** an eigenvalue. In infinite dimensions it is not the same: the **right shift operator**  $S$  is not invertible but 0 is not its eigenvalue because  $Sx = 0$  implies  $x = 0$  (check!).

**Definition 7.1.** The *resolvent set*  $\rho(T)$  of an operator  $T$  is the set

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}.$$

The *spectrum of operator*  $T \in B(H)$ , denoted  $\sigma(T)$ , is the complement of the resolvent set  $\rho(T)$ :

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is **not** invertible}\}.$$

**Example 7.2.** If  $H$  is finite dimensional the from previous discussion follows that  $\sigma(T)$  is the set of eigenvalues of  $T$  for any  $T$ .

Even this example demonstrates that spectrum does not provide a complete description for operator even in finite-dimensional case. For example, both operators in  $\mathbb{C}^2$  given by matrices  $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  have a single point spectrum  $\{0\}$ , however are rather different. The situation became even worse in the infinite dimensional spaces.

**Theorem 7.3.** The spectrum  $\sigma(T)$  of a bounded operator  $T$  is a nonempty compact (i.e. closed and bounded) subset of  $\mathbb{C}$ .

For the proof we will need several Lemmas.

**Lemma 7.4.** Let  $A \in B(H)$ . If  $\|A\| < 1$  then  $I - A$  is invertible in  $B(H)$  and inverse is given by the Neumann series (C. Neumann, 1877):

$$(7.1) \quad (I - A)^{-1} = I + A + A^2 + A^3 + \dots = \sum_{k=0}^{\infty} A^k.$$

*Proof.* Define the sequence of operators  $B_n = I + A + \dots + A^n$ —the partial sums of the infinite series (7.1). It is a Cauchy sequence, indeed:

$$\begin{aligned} \|B_n - B_m\| &= \|A^{m+1} + A^{m+2} + \dots + A^n\| && \text{(if } n < m\text{)} \\ &\leq \|A^{m+1}\| + \|A^{m+2}\| + \dots + \|A^n\| \\ &\leq \|A\|^{m+1} + \|A\|^{m+2} + \dots + \|A\|^n \\ &\leq \frac{\|A\|^{m+1}}{1 - \|A\|} < \epsilon \end{aligned}$$

for a large  $m$ . By the **completeness** of  $B(H)$  there is a limit, say  $B$ , of the sequence  $B_n$ . It is a simple algebra to check that  $(I - A)B_n = B_n(I - A) = I - A^{n+1}$ , passing to the limit in the norm topology, where  $A^{n+1} \rightarrow 0$  and  $B_n \rightarrow B$  we get:

$$(I - A)B = B(I - A) = I \quad \Leftrightarrow \quad B = (I - A)^{-1}.$$

□

**Definition 7.5.** The *resolvent* of an operator  $T$  is the operator valued function defined on the resolvent set by the formula:

$$(7.2) \quad R(\lambda, T) = (T - \lambda I)^{-1}.$$

**Corollary 7.6.** (i) If  $|\lambda| > \|T\|$  then  $\lambda \in \rho(T)$ , hence the spectrum is bounded.

(ii) The resolvent set  $\rho(T)$  is open, i.e for any  $\lambda \in \rho(T)$  then there exist  $\epsilon > 0$  such that all  $\mu$  with  $|\lambda - \mu| < \epsilon$  are also in  $\rho(T)$ , i.e. the resolvent set is open and the spectrum is closed.

Both statements together imply that the spectrum is compact.

*Proof.* (i) If  $|\lambda| > \|T\|$  then  $\|\lambda^{-1}T\| < 1$  and the operator  $T - \lambda I = -\lambda(I - \lambda^{-1}T)$  has the inverse

$$(7.3) \quad R(\lambda, T) = (T - \lambda I)^{-1} = - \sum_{k=0}^{\infty} \lambda^{-k-1} T^k.$$

by the **previous Lemma**.

(ii) Indeed:

$$\begin{aligned} T - \mu I &= T - \lambda I + (\lambda - \mu)I \\ &= (T - \lambda I)(I + (\lambda - \mu)(T - \lambda I)^{-1}). \end{aligned}$$

The last line is an invertible operator because  $T - \lambda I$  is invertible by the assumption and  $I + (\lambda - \mu)(T - \lambda I)^{-1}$  is invertible by the **previous Lemma**, since  $\|(\lambda - \mu)(T - \lambda I)^{-1}\| < 1$  if  $\epsilon < \|(T - \lambda I)^{-1}\|$ . □

**Exercise 7.7.** (i) Prove the *first resolvent identity*:

$$(7.4) \quad R(\lambda, T) - R(\mu, T) = (\lambda - \mu)R(\lambda, T)R(\mu, T)$$

(ii) Use the identity (7.4) to show that  $(T - \mu I)^{-1} \rightarrow (T - \lambda I)^{-1}$  as  $\mu \rightarrow \lambda$ .

(iii) Use the identity (7.4) to show that for  $z \in \rho(T)$  the complex derivative  $\frac{d}{dz}R(z, T)$  of the resolvent  $R(z, T)$  is well defined, i.e. the resolvent is an analytic function operator valued function of  $z$ .

**Lemma 7.8.** *The spectrum is non-empty.*

*Proof.* Let us assume the opposite,  $\sigma(T) = \emptyset$  then the resolvent function  $R(\lambda, T)$  is well defined for all  $\lambda \in \mathbb{C}$ . As could be seen from the von Neumann series (7.3)  $\|R(\lambda, T)\| \rightarrow 0$  as  $\lambda \rightarrow \infty$ . Thus for any vectors  $x, y \in H$  the function  $f(\lambda) = \langle R(\lambda, T)x, y \rangle$  is analytic (see Exercise 7.7(iii)) function tending to zero at infinity. Then by the Liouville theorem from complex analysis  $R(\lambda, T) = 0$ , which is impossible. Thus the spectrum is not empty. □

*Proof of Theorem 7.3.* Spectrum is nonempty by Lemma 7.8 and compact by Corollary 7.6. □

*Remark 7.9.* Theorem 7.3 gives the maximal possible description of the spectrum, indeed any non-empty compact set could be a spectrum for some bounded operator, see Problem A.23.

**7.2. The spectral radius formula.** The following definition is of interest.

**Definition 7.10.** The *spectral radius* of  $T$  is

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

From the Lemma 7.6(i) immediately follows that  $r(T) \leq \|T\|$ . The more accurate estimation is given by the following theorem.

**Theorem 7.11.** *For a bounded operator  $T$  we have*

$$(7.5) \quad r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}.$$

We start from the following general lemma:

**Lemma 7.12.** *Let a sequence  $(a_n)$  of positive real numbers satisfies inequalities:  $0 \leq a_{m+n} \leq a_m + a_n$  for all  $m$  and  $n$ . Then there is a limit  $\lim_{n \rightarrow \infty} (a_n/n)$  and its equal to  $\inf_n (a_n/n)$ .*

*Proof.* The statements follows from the observation that for any  $n$  and  $m = nk + l$  with  $0 \leq l \leq n$  we have  $a_m \leq ka_n + la_1$  thus, for big  $m$  we got  $a_m/m \leq a_n/n + la_1/m \leq a_n/n + \epsilon$ . □

*Proof of Theorem 7.11.* The existence of the limit  $\lim_{n \rightarrow \infty} \|T^n\|^{1/n}$  in (7.5) follows from the previous Lemma since by the Lemma 6.9  $\log \|T^{n+m}\| \leq \log \|T^n\| + \log \|T^m\|$ . Now we are using some results from the complex analysis. The Laurent series for the resolvent  $R(\lambda, T)$  in the neighbourhood of infinity is given by the von Neumann series (7.3). The radius of its convergence (which is equal, obviously, to  $r(T)$ ) by the Hadamard theorem is exactly  $\lim_{n \rightarrow \infty} \|T^n\|^{1/n}$ .  $\square$

**Corollary 7.13.** *There exists  $\lambda \in \sigma(T)$  such that  $|\lambda| = r(T)$ .*

*Proof.* Indeed, as it is known from the complex analysis the boundary of the convergence circle of a Laurent (or Taylor) series contain a singular point, the singular point of the resolvent is obviously belongs to the spectrum.  $\square$

**Example 7.14.** Let us consider the **left shift operator**  $S^*$ , for any  $\lambda \in \mathbb{C}$  such that  $|\lambda| < 1$  the vector  $(1, \lambda, \lambda^2, \lambda^3, \dots)$  is in  $\ell_2$  and is an eigenvector of  $S^*$  with eigenvalue  $\lambda$ , so the open unit disk  $|\lambda| < 1$  belongs to  $\sigma(S^*)$ . On the other hand spectrum of  $S^*$  belongs to the closed unit disk  $|\lambda| \leq 1$  since  $r(S^*) \leq \|S^*\| = 1$ . Because spectrum is closed it should coincide with the closed unit disk, since the open unit disk is dense in it. Particularly  $1 \in \sigma(S^*)$ , but it is easy to see that 1 is not an eigenvalue of  $S^*$ .

**Proposition 7.15.** *For any  $T \in B(H)$  the spectrum of the adjoint operator is  $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$ .*

*Proof.* If  $(T - \lambda I)V = 0$  then  $V(T - \lambda I) = 0$  by taking adjoints  $V^*(T^* - \bar{\lambda}I) = (T^* - \bar{\lambda}I)V^* = 0$ . So  $\lambda \in \rho(T)$  implies  $\bar{\lambda} \in \rho(T^*)$ , using the property  $T^{**} = T$  we could invert the implication and get the statement of proposition.  $\square$

**Example 7.16.** In continuation of Example 7.14 using the previous Proposition we conclude that  $\sigma(S)$  is also the closed unit disk, but  $S$  does not have eigenvalues at all!

### 7.3. Spectrum of Special Operators.

**Theorem 7.17.** (i) *If  $U$  is a unitary operator then  $\sigma(U) \subseteq \{z : |z| = 1\}$ .*

(ii) *If  $T$  is Hermitian then  $\sigma(T) \subseteq \mathbb{R}$ .*

*Proof.* (i) If  $|\lambda| > 1$  then  $\|\lambda^{-1}U\| < 1$  and then  $\lambda I - U = \lambda(I - \lambda^{-1}U)$  is invertible, thus  $\lambda \notin \sigma(U)$ . If  $|\lambda| < 1$  then  $\|\lambda U^*\| < 1$  and then  $\lambda I - U = U(\lambda U^* - I)$  is invertible, thus  $\lambda \notin \sigma(U)$ . The remaining set is exactly  $\{z : |z| = 1\}$ .

(ii) Without loss of generality we could assume that  $\|T\| < 1$ , otherwise we could multiply  $T$  by a small real scalar. Let us consider the *Cayley transform* which maps real axis to the unit circle:

$$U = (T - iI)(T + iI)^{-1}.$$

Straightforward calculations show that  $U$  is unitary if  $T$  is Hermitian. Let us take  $\lambda \notin \mathbb{R}$  and  $\lambda \neq -i$  (this case could be checked directly by Lemma 7.4). Then the Cayley transform  $\mu = (\lambda - i)(\lambda + i)^{-1}$  of  $\lambda$  is not on the unit circle and thus the operator

$$U - \mu I = (T - iI)(T + iI)^{-1} - (\lambda - i)(\lambda + i)^{-1}I = 2i(\lambda + i)^{-1}(T - \lambda I)(T + iI)^{-1},$$

is invertible, which implies invertibility of  $T - \lambda I$ . So  $\lambda \notin \mathbb{R}$ .  $\square$

The above reduction of a self-adjoint operator to a unitary one (it can be done on the opposite direction as well!) is an important tool which can be applied in other questions as well, e.g. in the following exercise.

- Exercise 7.18.** (i) Show that an operator  $U : f(t) \mapsto e^{it}f(t)$  on  $L_2[0, 2\pi]$  is unitary and has the entire unit circle  $\{|z| = 1\}$  as its spectrum .  
 (ii) Find a self-adjoint operator  $T$  with the entire real line as its spectrum.

8. COMPACTNESS

It is not easy to study linear operators “in general” and there are many questions about operators in Hilbert spaces raised many decades ago which are still unanswered. Therefore it is reasonable to single out classes of operators which have (relatively) simple properties. Such a class of operators more closed to finite dimensional ones will be studied here.

These operators are so compact that we even can fit them in our course

**8.1. Compact operators.** Let us recall some topological definition and results.

**Definition 8.1.** A *compact set* in a metric space is defined by the property that any its covering by a family of open sets contains a subcovering by a finite subfamily.

In the finite dimensional vector spaces  $\mathbb{R}^n$  or  $\mathbb{C}^n$  there is the following equivalent definition of compactness (equivalence of 8.2(i) and 8.2(ii) is known as *Heine–Borel theorem*):

**Theorem 8.2.** *If a set  $E$  in  $\mathbb{R}^n$  or  $\mathbb{C}^n$  has any of the following properties then it has other two as well:*

- (i)  $E$  is bounded and closed;
- (ii)  $E$  is compact;
- (iii) Any infinite subset of  $E$  has a limiting point belonging to  $E$ .

**Exercise\* 8.3.** Which equivalences from above are not true any more in the infinite dimensional spaces?

**Definition 8.4.** Let  $X$  and  $Y$  be normed spaces,  $T \in B(X, Y)$  is a *finite rank operator* if  $\text{Im } T$  is a finite dimensional subspace of  $Y$ .  $T$  is a *compact operator* if whenever  $(x_i)_1^\infty$  is a bounded sequence in  $X$  then its image  $(Tx_i)_1^\infty$  has a convergent subsequence in  $Y$ .

The set of finite rank operators is denote by  $F(X, Y)$  and the set of compact operators—by  $K(X, Y)$

**Exercise 8.5.** Show that both  $F(X, Y)$  and  $K(X, Y)$  are linear subspaces of  $B(X, Y)$ .

We intend to show that  $F(X, Y) \subset K(X, Y)$ .

**Lemma 8.6.** *Let  $Z$  be a finite-dimensional normed space. Then there is a number  $N$  and a mapping  $S : \ell_2^N \rightarrow Z$  which is invertible and such that  $S$  and  $S^{-1}$  are bounded.*

*Proof.* The proof is given by an explicit construction. Let  $N = \dim Z$  and  $z_1, z_2, \dots, z_N$  be a basis in  $Z$ . Let us define

$$S : \ell_2^N \rightarrow Z \quad \text{by} \quad S(a_1, a_2, \dots, a_N) = \sum_{k=1}^N a_k z_k,$$

then we have an estimation of norm:

$$\begin{aligned} \|Sa\| &= \left\| \sum_{k=1}^N a_k z_k \right\| \leq \sum_{k=1}^N |a_k| \|z_k\| \\ &\leq \left( \sum_{k=1}^N |a_k|^2 \right)^{1/2} \left( \sum_{k=1}^N \|z_k\|^2 \right)^{1/2}. \end{aligned}$$

So  $\|S\| \leq \left(\sum_1^N \|z_k\|^2\right)^{1/2}$  and  $S$  is continuous.

Clearly  $S$  has the trivial kernel, particularly  $\|Sa\| > 0$  if  $\|a\| = 1$ . By the **Heine–Borel theorem** the unit sphere in  $\ell_2^N$  is compact, consequently the continuous function  $a \mapsto \left\|\sum_1^N a_k z_k\right\|$  attains its lower bound, which has to be positive. This means there exists  $\delta > 0$  such that  $\|a\| = 1$  implies  $\|Sa\| > \delta$ , or, equivalently if  $\|z\| < \delta$  then  $\|S^{-1}z\| < 1$ . The later means that  $\|S^{-1}\| \leq \delta^{-1}$  and boundedness of  $S^{-1}$ .  $\square$

**Corollary 8.7.** For any two metric spaces  $X$  and  $Y$  we have  $F(X, Y) \subset K(X, Y)$ .

*Proof.* Let  $T \in F(X, Y)$ , if  $(x_n)_1^\infty$  is a bounded sequence in  $X$  then  $((Tx_n)_1^\infty \subset Z = \text{Im } T$  is also bounded. Let  $S : \ell_2^N \rightarrow Z$  be a map constructed in the **above Lemma**. The sequence  $(S^{-1}Tx_n)_1^\infty$  is bounded in  $\ell_2^N$  and thus has a limiting point, say  $a_0$ . Then  $Sa_0$  is a limiting point of  $(Tx_n)_1^\infty$ .  $\square$

There is a simple condition which allows to determine which diagonal operators are compact (particularly the **identity operator**  $I_X$  is *not* compact if  $\dim X = \infty$ ):

**Proposition 8.8.** Let  $T$  is a **diagonal operator** and given by identities  $Te_n = \lambda_n e_n$  for all  $n$  in a basis  $e_n$ .  $T$  is compact if and only if  $\lambda_n \rightarrow 0$ .

*Proof.* If  $\lambda_n \not\rightarrow 0$  then there exists a subsequence  $\lambda_{n_k}$  and  $\delta > 0$  such that  $|\lambda_{n_k}| > \delta$  for all  $k$ . Now the sequence  $(e_{n_k})$  is bounded but its image  $Te_{n_k} = \lambda_{n_k} e_{n_k}$  has no convergent subsequence because for any  $k \neq l$ :

$$\|\lambda_{n_k} e_{n_k} - \lambda_{n_l} e_{n_l}\| = (|\lambda_{n_k}|^2 + |\lambda_{n_l}|^2)^{1/2} \geq \sqrt{2}\delta,$$

i.e.  $Te_{n_k}$  is not a Cauchy sequence, see Figure 16.

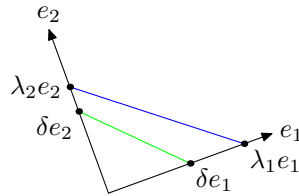


FIGURE 16. Distance between scales of orthonormal vectors

For the converse, note that if  $\lambda_n \rightarrow 0$  then we can define a finite rank operator  $T_m$ ,  $m \geq 1$ —*m*-“truncation” of  $T$  by:

$$(8.1) \quad T_m e_n = \begin{cases} T e_n = \lambda_n e_n, & 1 \leq n \leq m; \\ 0, & n > m. \end{cases}$$

Then obviously

$$(T - T_m)e_n = \begin{cases} 0, & 1 \leq n \leq m; \\ \lambda_n e_n, & n > m, \end{cases}$$

and  $\|T - T_m\| = \sup_{n > m} |\lambda_n| \rightarrow 0$  if  $m \rightarrow \infty$ . All  $T_m$  are finite rank operators (so are compact) and  $T$  is also compact as their limit—by the **next Theorem**.  $\square$

**Theorem 8.9.** Let  $T_m$  be a sequence of compact operators convergent to an operator  $T$  in the norm topology (i.e.  $\|T - T_m\| \rightarrow 0$ ) then  $T$  is compact itself. Equivalently  $K(X, Y)$  is a closed subspace of  $B(X, Y)$ .

*Proof.* Take a bounded sequence  $(x_n)_1^\infty$ . From compactness  
of  $T_1 \Rightarrow \exists$  subsequence  $(x_n^{(1)})_1^\infty$  of  $(x_n)_1^\infty$  s.t.  $(T_1 x_n^{(1)})_1^\infty$  is convergent.  
of  $T_2 \Rightarrow \exists$  subsequence  $(x_n^{(2)})_1^\infty$  of  $(x_n^{(1)})_1^\infty$  s.t.  $(T_2 x_n^{(2)})_1^\infty$  is convergent.  
of  $T_3 \Rightarrow \exists$  subsequence  $(x_n^{(3)})_1^\infty$  of  $(x_n^{(2)})_1^\infty$  s.t.  $(T_3 x_n^{(3)})_1^\infty$  is convergent.  
... ..

Could we find a subsequence which converges for all  $T_m$  simultaneously? The first guess “take the intersection of all above sequences  $(x_n^{(k)})_1^\infty$ ” does not work because the intersection could be empty. The way out is provided by the *diagonal argument* (see Table 2): a subsequence  $(T_m x_k^{(k)})_1^\infty$  is convergent for all  $m$ , because at latest after the term  $x_m^{(m)}$  it is a subsequence of  $(x_k^{(m)})_1^\infty$ .

$T_1 x_1^{(1)}$	$T_1 x_2^{(1)}$	$T_1 x_3^{(1)}$	...	$T_1 x_n^{(1)}$	...	$\rightarrow$	$a_1$
$T_2 x_1^{(2)}$	$T_2 x_2^{(2)}$	$T_2 x_3^{(2)}$	...	$T_2 x_n^{(2)}$	...	$\rightarrow$	$a_2$
$T_3 x_1^{(3)}$	$T_3 x_2^{(3)}$	$T_3 x_3^{(3)}$	...	$T_3 x_n^{(3)}$	...	$\rightarrow$	$a_3$
...	...	...	...	...	...		
$T_n x_1^{(n)}$	$T_n x_2^{(n)}$	$T_n x_3^{(n)}$	...	$T_n x_n^{(n)}$	...	$\rightarrow$	$a_n$
...	...	...	...	...	...		$\downarrow$
							$\searrow$
							<b>a</b>

TABLE 2. The “diagonal argument”.

We are claiming that a subsequence  $(T x_k^{(k)})_1^\infty$  of  $(T x_n)_1^\infty$  is convergent as well. We use here  $\epsilon/3$  argument (see Figure 17): for a given  $\epsilon > 0$  choose  $p \in \mathbb{N}$  such that  $\|T - T_p\| < \epsilon/3$ . Because  $(T_p x_k^{(k)}) \rightarrow 0$  it is a Cauchy sequence, thus there exists

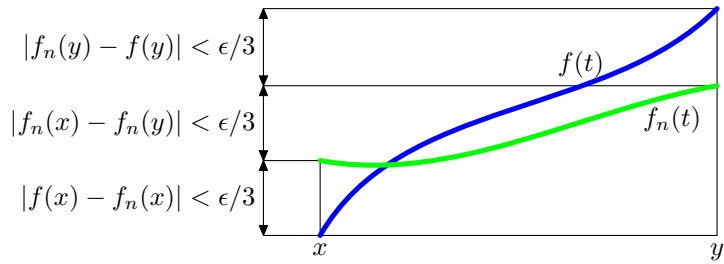


FIGURE 17. The  $\epsilon/3$  argument to estimate  $|f(x) - f(y)|$ .

$n_0 > p$  such that  $\|T_p x_k^{(k)} - T_p x_l^{(l)}\| < \epsilon/3$  for all  $k, l > n_0$ . Then:

$$\begin{aligned} \|T x_k^{(k)} - T x_l^{(l)}\| &= \|(T x_k^{(k)} - T_p x_k^{(k)}) + (T_p x_k^{(k)} - T_p x_l^{(l)}) + (T_p x_l^{(l)} - T x_l^{(l)})\| \\ &\leq \|T x_k^{(k)} - T_p x_k^{(k)}\| + \|T_p x_k^{(k)} - T_p x_l^{(l)}\| + \|T_p x_l^{(l)} - T x_l^{(l)}\| \\ &\leq \epsilon \end{aligned}$$

Thus  $T$  is compact. □

## 8.2. Hilbert–Schmidt operators.

**Definition 8.10.** Let  $T : H \rightarrow K$  be a bounded linear map between two Hilbert spaces. Then  $T$  is said to be *Hilbert–Schmidt operator* if there exists an orthonormal basis in  $H$  such that the series  $\sum_{k=1}^{\infty} \|Te_k\|^2$  is convergent.

- Example 8.11.** (i) Let  $T : \ell_2 \rightarrow \ell_2$  be a **diagonal operator** defined by  $Te_n = e_n/n$ , for all  $n \geq 1$ . Then  $\sum \|Te_n\|^2 = \sum n^{-2} = \pi^2/6$  (see Example 4.14) is finite.
- (ii) The **identity operator**  $I_H$  is **not** a Hilbert–Schmidt operator, unless  $H$  is finite dimensional.

A relation to compact operator is as follows.

**Theorem 8.12.** *All Hilbert–Schmidt operators are compact. (The opposite inclusion is false, give a counterexample!)*

*Proof.* Let  $T \in B(H, K)$  have a convergent series  $\sum \|Te_n\|^2$  in an orthonormal basis  $(e_n)_1^{\infty}$  of  $H$ . We again (see (8.1)) define the  $m$ -truncation of  $T$  by the formula

$$(8.2) \quad T_m e_n = \begin{cases} Te_n, & 1 \leq n \leq m; \\ 0, & n > m. \end{cases}$$

Then  $T_m(\sum_1^{\infty} a_k e_k) = \sum_1^m a_k e_k$  and each  $T_m$  is a **finite rank operator** because its image is spanned by the finite set of vectors  $Te_1, \dots, Te_m$ . We claim that  $\|T - T_m\| \rightarrow 0$ . Indeed by linearity and definition of  $T_m$ :

$$(T - T_m) \left( \sum_{n=1}^{\infty} a_n e_n \right) = \sum_{n=m+1}^{\infty} a_n (Te_n).$$

Thus:

$$(8.3) \quad \left\| (T - T_m) \left( \sum_{n=1}^{\infty} a_n e_n \right) \right\| = \left\| \sum_{n=m+1}^{\infty} a_n (Te_n) \right\|$$

$$\leq \sum_{n=m+1}^{\infty} |a_n| \|Te_n\|$$

$$\leq \left( \sum_{n=m+1}^{\infty} |a_n|^2 \right)^{1/2} \left( \sum_{n=m+1}^{\infty} \|Te_n\|^2 \right)^{1/2}$$

$$(8.4) \quad \leq \left\| \sum_{n=1}^{\infty} a_n e_n \right\| \left( \sum_{n=m+1}^{\infty} \|Te_n\|^2 \right)^{1/2}$$

so  $\|T - T_m\| \rightarrow 0$  and by the **previous Theorem**  $T$  is compact as a limit of compact operators.  $\square$

**Corollary 8.13** (from the above proof). *For a Hilbert–Schmidt operator*

$$\|T\| \leq \left( \sum_{n=m+1}^{\infty} \|Te_n\|^2 \right)^{1/2}.$$

*Proof.* Just consider difference of  $T$  and  $T_0 = 0$  in (8.3)–(8.4).  $\square$

**Example 8.14.** An *integral operator*  $T$  on  $L_2[0, 1]$  is defined by the formula:

$$(8.5) \quad (Tf)(x) = \int_0^1 K(x, y) f(y) dy, \quad f(y) \in L_2[0, 1],$$



where the continuous on  $[0, 1] \times [0, 1]$  function  $K$  is called the *kernel of integral operator*.

**Theorem 8.15.** *Integral operator (8.5) is Hilbert–Schmidt.*

*Proof.* Let  $(e_n)_{n \in \mathbb{Z}}$  be an orthonormal basis of  $L_2[0, 1]$ , e.g.  $(e^{2\pi i n t})_{n \in \mathbb{Z}}$ . Let us consider the kernel  $K_x(y) = K(x, y)$  as a function of the argument  $y$  depending from the parameter  $x$ . Then:

$$(Te_n)(x) = \int_0^1 K(x, y)e_n(y) dy = \int_0^1 K_x(y)e_n(y) dy = \langle K_x, \bar{e}_n \rangle.$$

So  $\|Te_n\|^2 = \int_0^1 |\langle K_x, \bar{e}_n \rangle|^2 dx$ . Consequently:

$$\begin{aligned} \sum_{-\infty}^{\infty} \|Te_n\|^2 &= \sum_{-\infty}^{\infty} \int_0^1 |\langle K_x, \bar{e}_n \rangle|^2 dx \\ (8.6) \qquad &= \int_0^1 \sum_1^{\infty} |\langle K_x, \bar{e}_n \rangle|^2 dx \\ &= \int_0^1 \|K_x\|^2 dx \\ &= \int_0^1 \int_0^1 |K(x, y)|^2 dx dy < \infty \end{aligned}$$

**Exercise 8.16.** Justify the exchange of summation and integration in (8.6). □

*Remark 8.17.* The definition 8.14 and Theorem 8.15 work also for any  $T : L_2[a, b] \rightarrow L_2[c, d]$  with a continuous kernel  $K(x, y)$  on  $[c, d] \times [a, b]$ .

**Definition 8.18.** Define *Hilbert–Schmidt norm* of a Hilbert–Schmidt operator  $A$  by  $\|A\|_{HS}^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2$  (it is independent of the choice of orthonormal basis  $(e_n)_{n=1}^{\infty}$ , see Question A.27).

**Exercise\* 8.19.** Show that set of Hilbert–Schmidt operators with the above norm is a Hilbert space and find the an expression for the inner product.

**Example 8.20.** Let  $K(x, y) = x - y$ , then

$$(Tf)(x) = \int_0^1 (x - y)f(y) dy = x \int_0^1 f(y) dy - \int_0^1 yf(y) dy$$

is a rank 2 operator. Furthermore:

$$\begin{aligned} \|T\|_{HS}^2 &= \int_0^1 \int_0^1 (x - y)^2 dx dy = \int_0^1 \left[ \frac{(x - y)^3}{3} \right]_{x=0}^1 dy \\ &= \int_0^1 \frac{(1 - y)^3}{3} + \frac{y^3}{3} dy = \left[ -\frac{(1 - y)^4}{12} + \frac{y^4}{12} \right]_0^1 = \frac{1}{6}. \end{aligned}$$

On the other hand there is an orthonormal basis such that

$$Tf = \frac{1}{\sqrt{12}} \langle f, e_1 \rangle e_1 - \frac{1}{\sqrt{12}} \langle f, e_2 \rangle e_2,$$

and  $\|T\| = \frac{1}{\sqrt{12}}$  and  $\sum_1^2 \|Te_k\|^2 = \frac{1}{6}$  and we get  $\|T\| \leq \|T\|_{\text{HS}}$  in agreement with Corollary 8.13.

## 9. THE SPECTRAL THEOREM FOR COMPACT NORMAL OPERATORS

Recall from Section 6.4 that an operator  $T$  is **normal** if  $TT^* = T^*T$ ; **Hermitian** ( $T^* = T$ ) and **unitary** ( $T^* = T^{-1}$ ) operators are normal.

### 9.1. Spectrum of normal operators.

**Theorem 9.1.** *Let  $T \in B(H)$  be a normal operator then*

- (i)  $\ker T = \ker T^*$ , so  $\ker(T - \lambda I) = \ker(T^* - \bar{\lambda}I)$  for all  $\lambda \in \mathbb{C}$
- (ii) *Eigenvectors corresponding to distinct eigenvalues are orthogonal.*
- (iii)  $\|T\| = r(T)$ .

*Proof.* (i) Obviously:

$$\begin{aligned} x \in \ker T &\Leftrightarrow \langle Tx, Tx \rangle = 0 \Leftrightarrow \langle T^*Tx, x \rangle = 0 \\ &\Leftrightarrow \langle TT^*x, x \rangle = 0 \Leftrightarrow \langle T^*x, T^*x \rangle = 0 \\ &\Leftrightarrow x \in \ker T^*. \end{aligned}$$

The second part holds because normalities of  $T$  and  $T - \lambda I$  are equivalent.

- (ii) If  $Tx = \lambda x$ ,  $Ty = \mu y$  then from the previous statement  $T^*y = \bar{\mu}y$ . If  $\lambda \neq \mu$  then the identity

$$\lambda \langle x, y \rangle = \langle Tx, y \rangle = \langle x, T^*y \rangle = \mu \langle x, y \rangle$$

implies  $\langle x, y \rangle = 0$ .

- (iii) Let  $S = T^*T$  then normality of  $T$  implies that  $S$  is Hermitian (check!). Consequently inequality

$$\|Sx\|^2 = \langle Sx, Sx \rangle = \langle S^2x, x \rangle \leq \|S^2\| \|x\|^2$$

implies  $\|S\|^2 \leq \|S^2\|$ . But the opposite inequality follows from the Theorem 6.9, thus we have the equality  $\|S^2\| = \|S\|^2$  and more generally by induction:  $\|S^{2^m}\| = \|S\|^{2^m}$  for all  $m$ .

Now we claim  $\|S\| = \|T\|^2$ . From Theorem 6.9 and 6.15 we get  $\|S\| = \|T^*T\| \leq \|T\|^2$ . On the other hand if  $\|x\| = 1$  then

$$\|T^*T\| \geq |\langle T^*Tx, x \rangle| = \langle Tx, Tx \rangle = \|Tx\|^2$$

implies the opposite inequality  $\|S\| \geq \|T\|^2$ . And because  $(T^{2^m})^*T^{2^m} = (T^*T)^{2^m}$  we get the equality

$$\|T^{2^m}\|^2 = \|(T^*T)^{2^m}\| = \|T^*T\|^{2^m} = \|T\|^{2^{m+1}}.$$

Thus:

$$r(T) = \lim_{m \rightarrow \infty} \|T^{2^m}\|^{1/2^m} = \lim_{m \rightarrow \infty} \|T\|^{2^{m+1}/2^{m+1}} = \|T\|.$$

by the spectral radius formula (7.5). □

**Example 9.2.** It is easy to see that normality is important in 9.1(iii), indeed the non-normal operator  $T$  given by the matrix  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  in  $\mathbb{C}$  has one-point spectrum  $\{0\}$ , consequently  $r(T) = 0$  but  $\|T\| = 1$ .

**Lemma 9.3.** *Let  $T$  be a compact normal operator then*

- (i) *The set of eigenvalues of  $T$  is either finite or a countable sequence tending to zero.*
- (ii) *All the eigenspaces, i.e.  $\ker(T - \lambda I)$ , are finite-dimensional for all  $\lambda \neq 0$ .*

*Remark 9.4.* This Lemma is true for any compact operator, but we will not use that in our course.

*Proof.* (i) Let  $H_0$  be the closed linear span of eigenvectors of  $T$ . Then  $T$  restricted to  $H_0$  is a diagonal compact operator with the same set of eigenvalues  $\lambda_n$  as in  $H$ . Then  $\lambda_n \rightarrow 0$  from Proposition 8.8.

**Exercise 9.5.** Use the proof of Proposition 8.8 to give a direct demonstration.

*Solution.* Or straightforwardly assume opposite: there exist an  $\delta > 0$  and infinitely many eigenvalues  $\lambda_n$  such that  $|\lambda_n| > \delta$ . By the **previous Theorem** there is an orthonormal sequence  $v_n$  of corresponding eigenvectors  $Tv_n = \lambda_n v_n$ . Now the sequence  $(v_n)$  is bounded but its image  $Tv_n = \lambda_n e_n$  has no convergent subsequence because for any  $k \neq l$ :

$$\|\lambda_k v_k - \lambda_l v_l\| = (|\lambda_k|^2 + |\lambda_l|^2)^{1/2} \geq \sqrt{2}\delta,$$

i.e.  $Te_{n_k}$  is not a Cauchy sequence, see Figure 16. □

- (ii) Similarly if  $H_0 = \ker(T - \lambda I)$  is infinite dimensional, then restriction of  $T$  on  $H_0$  is  $\lambda I$ —which is non-compact by Proposition 8.8. Alternatively consider the infinite orthonormal sequence  $(v_n)$ ,  $Tv_n = \lambda v_n$  as in Exercise 9.5. □

**Lemma 9.6.** *Let  $T$  be a compact normal operator. Then all non-zero points  $\lambda \in \sigma(T)$  are eigenvalues and there exists an eigenvalue of modulus  $\|T\|$ .*

*Proof.* Assume without loss of generality that  $T \neq 0$ . Let  $\lambda \in \sigma(T)$ , without loss of generality (multiplying by a scalar)  $\lambda = 1$ .

We claim that if 1 is not an eigenvalue then there exist  $\delta > 0$  such that

$$(9.1) \quad \|(I - T)x\| \geq \delta \|x\|.$$

Otherwise there exists a sequence of vectors  $(x_n)$  with unit norm such that  $(I - T)x_n \rightarrow 0$ . Then from the compactness of  $T$  for a subsequence  $(x_{n_k})$  there is  $y \in H$  such that  $Tx_{n_k} \rightarrow y$ , then  $x_n \rightarrow y$  implying  $Ty = y$  and  $y \neq 0$ —i.e.  $y$  is eigenvector with eigenvalue 1.

Now we claim  $\text{Im}(I - T)$  is closed, i.e.  $y \in \overline{\text{Im}(I - T)}$  implies  $y \in \text{Im}(I - T)$ . Indeed, if  $(I - T)x_n \rightarrow y$ , then there is a subsequence  $(x_{n_k})$  such that  $Tx_{n_k} \rightarrow z$  implying  $x_{n_k} \rightarrow y + z$ , then  $(I - T)(z + y) = y$ .

Finally  $I - T$  is injective, i.e.  $\ker(I - T) = \{0\}$ , by (9.1). By the property 9.1(i),  $\ker(I - T^*) = \{0\}$  as well. But because always  $\ker(I - T^*) = \text{Im}(I - T)^\perp$  (check!) we got surjectivity, i.e.  $\text{Im}(I - T)^\perp = \{0\}$ , of  $I - T$ . Thus  $(I - T)^{-1}$  exists and is bounded because (9.1) implies  $\|y\| > \delta \|(I - T)^{-1}y\|$ . Thus  $1 \notin \sigma(T)$ .

The existence of eigenvalue  $\lambda$  such that  $|\lambda| = \|T\|$  follows from combination of Lemma 7.13 and Theorem 9.1(iii). □

## 9.2. Compact normal operators.

**Theorem 9.7** (The spectral theorem for compact normal operators). *Let  $T$  be a compact normal operator on a Hilbert space  $H$ . Then there exists an orthonormal sequence  $(e_n)$  of eigenvectors of  $T$  and corresponding eigenvalues  $(\lambda_n)$  such that:*

$$(9.2) \quad Tx = \sum_n \lambda_n \langle x, e_n \rangle e_n, \quad \text{for all } x \in H.$$

If  $(\lambda_n)$  is an infinite sequence it tends to zero.

Conversely, if  $T$  is given by a formula (9.2) then it is compact and normal.

*Proof.* Suppose  $T \neq 0$ . Then by the **previous Theorem** there exists an eigenvalue  $\lambda_1$  such that  $|\lambda_1| = \|T\|$  with corresponding eigenvector  $e_1$  of the unit norm. Let  $H_1 = \text{Lin}(e_1)^\perp$ . If  $x \in H_1$  then

$$(9.3) \quad \langle Tx, e_1 \rangle = \langle x, T^*e_1 \rangle = \langle x, \bar{\lambda}_1 e_1 \rangle = \lambda_1 \langle x, e_1 \rangle = 0,$$

thus  $Tx \in H_1$  and similarly  $T^*x \in H_1$ . Write  $T_1 = T|_{H_1}$  which is again a normal compact operator with a norm does not exceeding  $\|T\|$ . We could inductively repeat this procedure for  $T_1$  obtaining sequence of eigenvalues  $\lambda_2, \lambda_3, \dots$  with eigenvectors  $e_2, e_3, \dots$ . If  $T_n = 0$  for a finite  $n$  then theorem is already proved. Otherwise we have an infinite sequence  $\lambda_n \rightarrow 0$ . Let

$$x = \sum_1^n \langle x, e_k \rangle e_k + y_n \quad \Rightarrow \quad \|x\|^2 = \sum_1^n |\langle x, e_k \rangle|^2 + \|y_n\|^2, \quad y_n \in H_n,$$

from **Pythagoras's theorem**. Then  $\|y_n\| \leq \|x\|$  and  $\|Ty_n\| \leq \|T_n\| \|y_n\| \leq |\lambda_n| \|x\| \rightarrow 0$  by Lemma 9.3. Thus

$$Tx = \lim_{n \rightarrow \infty} \left( \sum_1^n \langle x, e_n \rangle Te_n + Ty_n \right) = \sum_1^\infty \lambda_n \langle x, e_n \rangle e_n$$

Conversely, if  $Tx = \sum_1^\infty \lambda_n \langle x, e_n \rangle e_n$  then

$$\langle Tx, y \rangle = \sum_1^\infty \lambda_n \langle x, e_n \rangle \langle e_n, y \rangle = \sum_1^\infty \langle x, e_n \rangle \lambda_n \overline{\langle y, e_n \rangle},$$

thus  $T^*y = \sum_1^\infty \bar{\lambda}_n \langle y, e_n \rangle e_n$ . Then we got the normality of  $T$ :  $T^*Tx = TT^*x = \sum_1^\infty |\lambda_n|^2 \langle x, e_n \rangle e_n$ . Also  $T$  is compact because it is a uniform limit of the finite rank operators  $T_n x = \sum_1^n \lambda_n \langle x, e_n \rangle e_n$ .  $\square$

**Corollary 9.8.** Let  $T$  be a compact normal operator on a separable Hilbert space  $H$ , then there exists a orthonormal basis  $g_k$  such that

$$Tx = \sum_1^\infty \lambda_n \langle x, g_n \rangle g_n,$$

and  $\lambda_n$  are eigenvalues of  $T$  including zeros.

*Proof.* Let  $(e_n)$  be the orthonormal sequence constructed in the proof of the **previous Theorem**. Then  $x$  is perpendicular to all  $e_n$  if and only if its in the kernel of  $T$ . Let  $(f_n)$  be any orthonormal basis of  $\ker T$ . Then the union of  $(e_n)$  and  $(f_n)$  is the orthonormal basis  $(g_n)$  we have looked for.  $\square$

**Exercise 9.9.** Finish all details in the above proof.

**Corollary 9.10** (Singular value decomposition). If  $T$  is any compact operator on a separable Hilbert space then there exists orthonormal sequences  $(e_k)$  and  $(f_k)$  such that  $Tx = \sum_k \mu_k \langle x, e_k \rangle f_k$  where  $(\mu_k)$  is a sequence of positive numbers such that  $\mu_k \rightarrow 0$  if it is an infinite sequence.

*Proof.* Operator  $T^*T$  is compact and Hermitian (hence normal). From the **previous Corollary** there is an orthonormal basis  $(e_k)$  such that  $T^*Tx = \sum_n \lambda_n \langle x, e_n \rangle e_n$  for some positive  $\lambda_n = \|Te_n\|^2$ . Let  $\mu_n = \|Te_n\|$  and  $f_n = Te_n/\mu_n$ . Then  $f_n$  is an orthonormal sequence (check!) and

$$Tx = \sum_n \langle x, e_n \rangle Te_n = \sum_n \langle x, e_n \rangle \mu_n f_n.$$

$\square$

**Corollary 9.11.** *A bounded operator in a Hilber space is compact if and only if it is a uniform limit of the finite rank operators.*

*Proof.* Sufficiency follows from 8.9. Necessity: by the previous Corollary  $Tx = \sum_n \langle x, e_n \rangle \mu_n f_n$  thus  $T$  is a uniform limit of operators  $T_n x = \sum_{n=1}^m \langle x, e_n \rangle \mu_n f_n$  which are of finite rank.  $\square$

10. APPLICATIONS TO INTEGRAL EQUATIONS

In this lecture we will study the *Fredholm equation* defined as follows. Let the *integral operator* with a *kernel*  $K(x, y)$  defined on  $[a, b] \times [a, b]$  be defined as before:

$$(10.1) \quad (T\phi)(x) = \int_a^b K(x, y)\phi(y) dy.$$

The Fredholm equation of the *first* and *second* kinds correspondingly are:

$$(10.2) \quad T\phi = f \quad \text{and} \quad \phi - \lambda T\phi = f,$$

for a function  $f$  on  $[a, b]$ . A special case is given by *Volterra equation* by an operator integral operator (10.1)  $T$  with a kernel  $K(x, y) = 0$  for all  $y > x$  which could be written as:

$$(10.3) \quad (T\phi)(x) = \int_a^x K(x, y)\phi(y) dy.$$

We will consider integral operators with kernels  $K$  such that  $\int_a^b \int_a^b K(x, y) dx dy < \infty$ , then by Theorem 8.15  $T$  is a **Hilbert–Schmidt operator** and in particular bounded.

As a reason to study Fredholm operators we will mention that solutions of differential equations in mathematical physics (notably heat and wave equations) requires a decomposition of a function  $f$  as a linear combination of functions  $K(x, y)$  with “coefficients”  $\phi$ . This is an continuous analog of a discrete decomposition into **Fourier series**.

Using ideas from the proof of Lemma 7.4 we define *Neumann series* for the resolvent:

$$(10.4) \quad (I - \lambda T)^{-1} = I + \lambda T + \lambda^2 T^2 + \dots,$$

which is valid for all  $\lambda < \|T\|^{-1}$ .

**Example 10.1.** Solve the Volterra equation

$$\phi(x) - \lambda \int_0^x y\phi(y) dy = x^2, \quad \text{on } L_2[0, 1].$$

In this case  $I - \lambda T\phi = f$ , with  $f(x) = x^2$  and:

$$K(x, y) = \begin{cases} y, & 0 \leq y \leq x; \\ 0, & x < y \leq 1. \end{cases}$$

Straightforward calculations shows:

$$(Tf)(x) = \int_0^x y \cdot y^2 dy = \frac{x^4}{4},$$

$$(T^2f)(x) = \int_0^x y \frac{y^4}{4} dy = \frac{x^6}{24}, \dots$$

and generally by induction:

$$(\mathbb{T}^n f)(x) = \int_0^x y \frac{y^{2n}}{2^{n-1}n!} dy = \frac{x^{2n+2}}{2^n(n+1)!}.$$

Hence:

$$\begin{aligned} \phi(x) &= \sum_0^\infty \lambda^n \mathbb{T}^n f = \sum_0^\infty \frac{\lambda^n x^{2n+2}}{2^n(n+1)!} \\ &= \frac{2}{\lambda} \sum_0^\infty \frac{\lambda^{n+1} x^{2n+2}}{2^{n+1}(n+1)!} \\ &= \frac{2}{\lambda} (e^{\lambda x^2/2} - 1) \quad \text{for all } \lambda \in \mathbb{C} \setminus \{0\}, \end{aligned}$$

because in this case  $r(\mathbb{T}) = 0$ . For the Fredholm equations this is not always the case, see Tutorial problem A.29.

Among other integral operators there is an important subclass with *separable kernel*, namely a kernel which has a form:

$$(10.5) \quad K(x, y) = \sum_{j=1}^n g_j(x) h_j(y).$$

In such a case:

$$\begin{aligned} (\mathbb{T}\phi)(x) &= \int_a^b \sum_{j=1}^n g_j(x) h_j(y) \phi(y) dy \\ &= \sum_{j=1}^n g_j(x) \int_a^b h_j(y) \phi(y) dy, \end{aligned}$$

i.e. the image of  $\mathbb{T}$  is spanned by  $g_1(x), \dots, g_n(x)$  and is finite dimensional, consequently the solution of such equation reduces to linear algebra.

**Example 10.2.** Solve the Fredholm equation (actually find eigenvectors of  $\mathbb{T}$ ):

$$\begin{aligned} \phi(x) &= \lambda \int_0^{2\pi} \cos(x+y) \phi(y) dy \\ &= \lambda \int_0^{2\pi} (\cos x \cos y - \sin x \sin y) \phi(y) dy. \end{aligned}$$

Clearly  $\phi(x)$  should be a linear combination  $\phi(x) = A \cos x + B \sin x$  with coefficients  $A$  and  $B$  satisfying to:

$$\begin{aligned} A &= \lambda \int_0^{2\pi} \cos y (A \cos y + B \sin y) dy, \\ B &= -\lambda \int_0^{2\pi} \sin y (A \cos y + B \sin y) dy. \end{aligned}$$

Basic calculus implies  $A = \lambda\pi A$  and  $B = -\lambda\pi B$  and the only nonzero solutions are:

$$\begin{aligned} \lambda = \pi^{-1} \quad A \neq 0 \quad B = 0 \\ \lambda = -\pi^{-1} \quad A = 0 \quad B \neq 0 \end{aligned}$$

We develop some Hilbert–Schmidt theory for integral operators.

**Theorem 10.3.** *Suppose that  $K(x, y)$  is a continuous function on  $[a, b] \times [a, b]$  and  $K(x, y) = \overline{K(y, x)}$  and operator  $T$  is defined by (10.1). Then*

- (i)  $T$  is a self-adjoint Hilbert–Schmidt operator.
- (ii) All eigenvalues of  $T$  are real and satisfy  $\sum_n \lambda_n^2 < \infty$ .
- (iii) The eigenvectors  $v_n$  of  $T$  can be chosen as an orthonormal basis of  $L_2[a, b]$ , are continuous for nonzero  $\lambda_n$  and

$$T\phi = \sum_{n=1}^{\infty} \lambda_n \langle \phi, v_n \rangle v_n \quad \text{where} \quad \phi = \sum_{n=1}^{\infty} \langle \phi, v_n \rangle v_n$$

*Proof.* (i) The condition  $K(x, y) = \overline{K(y, x)}$  implies the Hermitian property of  $T$ :

$$\begin{aligned} \langle T\phi, \psi \rangle &= \int_a^b \left( \int_a^b K(x, y) \phi(y) dy \right) \bar{\psi}(x) dx \\ &= \int_a^b \int_a^b K(x, y) \phi(y) \bar{\psi}(x) dx dy \\ &= \int_a^b \phi(y) \left( \int_a^b \overline{K(y, x) \psi(x)} dx \right) dy \\ &= \langle \phi, T\psi \rangle. \end{aligned}$$

The Hilbert–Schmidt property (and hence compactness) was proved in Theorem 8.15.

- (ii) Spectrum of  $T$  is real as for any Hermitian operator, see Theorem 7.17(ii) and finiteness of  $\sum_n \lambda_n^2$  follows from Hilbert–Schmidt property
- (iii) The existence of orthonormal basis consisting from eigenvectors ( $v_n$ ) of  $T$  was proved in Corollary 9.8. If  $\lambda_n \neq 0$  then:

$$\begin{aligned} v_n(x_1) - v_n(x_2) &= \lambda_n^{-1} ((Tv_n)(x_1) - (Tv_n)(x_2)) \\ &= \frac{1}{\lambda_n} \int_a^b (K(x_1, y) - K(x_2, y)) v_n(y) dy \end{aligned}$$

and by Cauchy–Schwarz–Bunyakovskii inequality:

$$|v_n(x_1) - v_n(x_2)| \leq \frac{1}{|\lambda_n|} \|v_n\|_2 \int_a^b |K(x_1, y) - K(x_2, y)| dy$$

which tends to 0 due to (uniform) continuity of  $K(x, y)$ . □

**Theorem 10.4.** *Let  $T$  be as in the previous Theorem. Then if  $\lambda \neq 0$  and  $\lambda^{-1} \notin \sigma(T)$ , the unique solution  $\phi$  of the Fredholm equation of the second kind  $\phi - \lambda T\phi = f$  is*

$$(10.6) \quad \phi = \sum_1^{\infty} \frac{\langle f, v_n \rangle}{1 - \lambda \lambda_n} v_n.$$

*Proof.* Let  $\phi = \sum_1^\infty a_n v_n$  where  $a_n = \langle \phi, v_n \rangle$ , then

$$\phi - \lambda T\phi = \sum_1^\infty a_n(1 - \lambda\lambda_n)v_n = f = \sum_1^\infty \langle f, v_n \rangle v_n$$

if and only if  $a_n = \langle f, v_n \rangle / (1 - \lambda\lambda_n)$  for all  $n$ . Note  $1 - \lambda\lambda_n \neq 0$  since  $\lambda^{-1} \notin \sigma(T)$ .

Because  $\lambda_n \rightarrow 0$  we got  $\sum_1^\infty |a_n|^2$  by its comparison with  $\sum_1^\infty |\langle f, v_n \rangle|^2 = \|f\|^2$ , thus the solution exists and is unique by the **Riesz-Fisher Theorem**.  $\square$

See Exercise A.30 for an example.

**Theorem 10.5** (Fredholm alternative). *Let  $T \in K(H)$  be compact normal and  $\lambda \in \mathbb{C} \setminus \{0\}$ . Consider the equations:*

$$(10.7) \quad \phi - \lambda T\phi = 0$$

$$(10.8) \quad \phi - \lambda T\phi = f$$

then *either*

- (A) *the only solution to (10.7) is  $\phi = 0$  and (10.8) has a unique solution for any  $f \in H$ ; or*
- (B) *there exists a nonzero solution to (10.7) and (10.8) can be solved if and only if  $f$  is orthogonal all solutions to (10.7).*

*Proof.* (A) If  $\phi = 0$  is the only solution of (10.7), then  $\lambda^{-1}$  is not an eigenvalue of  $T$  and then by Lemma 9.6 is neither in spectrum of  $T$ . Thus  $I - \lambda T$  is invertible and the unique solution of (10.8) is given by  $\phi = (I - \lambda T)^{-1}f$ .

- (B) A nonzero solution to (10.7) means that  $\lambda^{-1} \in \sigma(T)$ . Let  $(v_n)$  be an orthonormal basis of eigenvectors of  $T$  for eigenvalues  $(\lambda_n)$ . By Lemma 9.3(ii) only a finite number of  $\lambda_n$  is equal to  $\lambda^{-1}$ , say they are  $\lambda_1, \dots, \lambda_N$ , then

$$(I - \lambda T)\phi = \sum_{n=1}^\infty (1 - \lambda\lambda_n) \langle \phi, v_n \rangle v_n = \sum_{n=N+1}^\infty (1 - \lambda\lambda_n) \langle \phi, v_n \rangle v_n.$$

If  $f = \sum_1^\infty \langle f, v_n \rangle v_n$  then the identity  $(I - \lambda T)\phi = f$  is only possible if  $\langle f, v_n \rangle = 0$  for  $1 \leq n \leq N$ . Conversely from that condition we could give a solution

$$\phi = \sum_{n=N+1}^\infty \frac{\langle f, v_n \rangle}{1 - \lambda\lambda_n} v_n + \phi_0, \quad \text{for any } \phi_0 \in \text{Lin}(v_1, \dots, v_N),$$

which is again in  $H$  because  $f \in H$  and  $\lambda_n \rightarrow 0$ .  $\square$

**Example 10.6.** Let us consider

$$(T\phi)(x) = \int_0^1 (2xy - x - y + 1)\phi(y) dy.$$

Because the kernel of  $T$  is real and symmetric  $T = T^*$ , the kernel is also separable:

$$(T\phi)(x) = x \int_0^1 (2y - 1)\phi(y) dy + \int_0^1 (-y + 1)\phi(y) dy,$$

and  $T$  of the rank 2 with image of  $T$  spanned by  $1$  and  $x$ . By direct calculations:

$$\begin{aligned} T: 1 &\mapsto \frac{1}{2} \\ T: x &\mapsto \frac{1}{6}x + \frac{1}{6}, \end{aligned} \quad \text{or } T \text{ is given by the matrix } \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} \end{pmatrix}$$



According to linear algebra decomposition over eigenvectors is:

$$\lambda_1 = \frac{1}{2} \quad \text{with vector} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\lambda_2 = \frac{1}{6} \quad \text{with vector} \quad \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$$

with normalisation  $v_1(y) = 1$ ,  $v_2(y) = \sqrt{12}(y - 1/2)$  and we complete it to an orthonormal basis  $(v_n)$  of  $L_2[0, 1]$ . Then

- If  $\lambda \neq 2$  or  $6$  then  $(I - \lambda T)\phi = f$  has a unique solution (cf. equation (10.6)):

$$\begin{aligned} \phi &= \sum_{n=1}^2 \frac{\langle f, v_n \rangle}{1 - \lambda \lambda_n} v_n + \sum_{n=3}^{\infty} \langle f, v_n \rangle v_n \\ &= \sum_{n=1}^2 \frac{\langle f, v_n \rangle}{1 - \lambda \lambda_n} v_n + \left( f - \sum_{n=1}^2 \langle f, v_n \rangle v_n \right) \\ &= f + \sum_{n=1}^2 \frac{\lambda \lambda_n}{1 - \lambda \lambda_n} \langle f, v_n \rangle v_n. \end{aligned}$$

- If  $\lambda = 2$  then the solutions exist provided  $\langle f, v_1 \rangle = 0$  and are:

$$\phi = f + \frac{\lambda \lambda_2}{1 - \lambda \lambda_2} \langle f, v_2 \rangle v_2 + C v_1 = f + \frac{1}{2} \langle f, v_2 \rangle v_2 + C v_1, \quad C \in \mathbb{C}.$$

- If  $\lambda = 6$  then the solutions exist provided  $\langle f, v_2 \rangle = 0$  and are:

$$\phi = f + \frac{\lambda \lambda_1}{1 - \lambda \lambda_1} \langle f, v_1 \rangle v_1 + C v_2 = f - \frac{3}{2} \langle f, v_2 \rangle v_2 + C v_2, \quad C \in \mathbb{C}.$$

## 11. BANACH AND NORMED SPACES

We will work with either the field of real numbers  $\mathbb{R}$  or the complex numbers  $\mathbb{C}$ . To avoid repetition, we use  $\mathbb{K}$  to denote either  $\mathbb{R}$  or  $\mathbb{C}$ .

**11.1. Normed spaces.** Recall, see Defn. 2.3, a *norm* on a vector space  $V$  is a map  $\|\cdot\| : V \rightarrow [0, \infty)$  such that

- (i)  $\|u\| = 0$  only when  $u = 0$ ;
- (ii)  $\|\lambda u\| = |\lambda| \|u\|$  for  $\lambda \in \mathbb{K}$  and  $u \in V$ ;
- (iii)  $\|u + v\| \leq \|u\| + \|v\|$  for  $u, v \in V$ .

A norm induces a *metric*, see Defn. 2.1, on  $V$  by setting  $d(u, v) = \|u - v\|$ . When  $V$  is *complete*, see Defn. 2.6, for this metric, we say that  $V$  is a *Banach space*.

**Theorem 11.1.** *Every finite-dimensional normed vector space is a Banach space.*

We will use the following simple inequality:

**Lemma 11.2.** *Let two real numbers  $1 < p, q < \infty$  are related through  $\frac{1}{p} + \frac{1}{q} = 1$  then*

$$(11.1) \quad |ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q},$$

for any complex  $a$  and  $b$ .

*Proof.* Consider the function  $\phi(t) = t^m - mt$  for an  $1 < m < \infty$ . From its derivative  $\phi'(t) = m(t^{m-1} - 1)$  we find the only critical point on  $[0, \infty)$ , which is its maximum. Thus write the inequality  $\phi(t) \leq \phi(1)$  for  $t = a^p/b^q$  and  $m = 1/p$ . After transformation we get  $ab^{-q/p} - 1 \leq \frac{1}{p}(a^p b^{-q} - 1)$  and multiplication by  $b^q$  with rearrangements lead to the desired result.  $\square$

**Proposition 11.3 (Hölder's Inequality).** *For  $1 < p < \infty$ , let  $q \in (1, \infty)$  be such that  $1/p + 1/q = 1$ . For  $n \geq 1$  and  $u, v \in \mathbb{K}^n$ , we have that*

$$\sum_{j=1}^n |u_j v_j| \leq \left( \sum_{j=1}^n |u_j|^p \right)^{\frac{1}{p}} \left( \sum_{j=1}^n |v_j|^q \right)^{\frac{1}{q}}.$$

*Proof.* For reasons become clear soon we use the notation  $\|u\| = \left( \sum_{j=1}^n |u_j|^p \right)^{\frac{1}{p}}$  and  $\|v\| = \left( \sum_{j=1}^n |v_j|^q \right)^{\frac{1}{q}}$  and define for  $1 \leq i \leq n$ :

$$a_i = \frac{u_i}{\|u\|} \quad \text{and} \quad b_i = \frac{v_i}{\|v\|}.$$

Summing up for  $1 \leq i \leq n$  all inequalities obtained from (11.1):

$$|a_i b_i| \leq \frac{|a_i|^p}{p} + \frac{|b_i|^q}{q},$$

we get the result.  $\square$

Using Hölder inequality we can derive the following one:

**Proposition 11.4 (Minkowski's Inequality).** *For  $1 < p < \infty$ , and  $n \geq 1$ , let  $u, v \in \mathbb{K}^n$ . Then*

$$\left( \sum_{j=1}^n |u_j + v_j|^p \right)^{1/p} \leq \left( \sum_{j=1}^n |u_j|^p \right)^{1/p} + \left( \sum_{j=1}^n |v_j|^p \right)^{1/p}.$$

*Proof.* For  $p > 1$  we have:

$$(11.2) \quad \sum_1^n |x_k + y_k|^p = \sum_1^n |x_k| |x_k + y_k|^{p-1} + \sum_1^n |y_k| |x_k + y_k|^{p-1}.$$

By Hölder inequality

$$\sum_1^n |x_k| |x_k + y_k|^{p-1} \leq \left( \sum_1^n |x_k|^p \right)^{\frac{1}{p}} \left( \sum_1^n |x_k + y_k|^{q(p-1)} \right)^{\frac{1}{q}}.$$

Adding a similar inequality for the second term in the right hand side of (11.2) and division by  $\left( \sum_1^n |x_k + y_k|^{q(p-1)} \right)^{\frac{1}{q}}$  yields the result.  $\square$

Minkowski's inequality shows that for  $1 \leq p < \infty$  (the case  $p = 1$  is easy) we can define a norm  $\|\cdot\|_p$  on  $\mathbb{K}^n$  by

$$\|u\|_p = \left( \sum_{j=1}^n |u_j|^p \right)^{1/p} \quad (u = (u_1, \dots, u_n) \in \mathbb{K}^n).$$

We can define an infinite analogue of this. Let  $1 \leq p < \infty$ , let  $\ell^p$  be the space of all scalar sequences  $(x_n)$  with  $\sum_n |x_n|^p < \infty$ . A careful use of Minkowski's inequality shows that  $\ell^p$  is a vector space. Then  $\ell^p$  becomes a normed space for the  $\|\cdot\|_p$  norm.

Recall that a Cauchy sequence, see Defn. 2.5, in a normed space is bounded: if  $(x_n)$  is Cauchy then we can find  $N$  with  $\|x_n - x_m\| < 1$  for all  $n, m \geq N$ . Then  $\|x_n\| \leq \|x_n - x_N\| + \|x_N\| < \|x_N\| + 1$  for  $n \geq N$ , so in particular,  $\|x_n\| \leq \max(\|x_1\|, \|x_2\|, \dots, \|x_{N-1}\|, \|x_N\| + 1)$ .

**Theorem 11.5.** For  $1 \leq p < \infty$ , the space  $\ell^p$  is a Banach space.

*Proof.* Most completeness proofs are similar to this, see Thm. 2.24. So we shall prove this result in detail. Let  $(x^{(n)})$  be a Cauchy-sequence in  $\ell^p$ ; we wish to show this converges to some vector in  $\ell^p$ .

For each  $n$ ,  $x^{(n)} \in \ell^p$  so is a sequence of scalars, say  $(x_k^{(n)})_{k=1}^\infty$ . As  $(x^{(n)})$  is Cauchy, for each  $\epsilon > 0$  there exists  $N_\epsilon$  so that  $\|x^{(n)} - x^{(m)}\|_p \leq \epsilon$  for  $n, m \geq N_\epsilon$ .

For  $k$  fixed,

$$|x_k^{(n)} - x_k^{(m)}| \leq \left( \sum_j |x_j^{(n)} - x_j^{(m)}|^p \right)^{1/p} = \|x^{(n)} - x^{(m)}\|_p \leq \epsilon,$$

when  $n, m \geq N_\epsilon$ . Thus the scalar sequence  $(x_k^{(n)})_{n=1}^\infty$  is Cauchy in  $\mathbb{K}$  and hence converges, to  $y_k$  say.

Let  $y = (y_k)$ , so that  $y$  is a candidate for the limit of  $(x^{(n)})$ . Firstly, we check that  $y \in \ell^p$ . We calculate,

$$\begin{aligned} \|y\|_p &= \lim_{K \rightarrow \infty} \left( \sum_{k=1}^K |y_k|^p \right)^{1/p} = \lim_{K \rightarrow \infty} \left( \sum_{k=1}^K \lim_{n \rightarrow \infty} |x_k^{(n)}|^p \right)^{1/p} \\ &= \lim_{K \rightarrow \infty} \lim_{n \rightarrow \infty} \left( \sum_{k=1}^K |x_k^{(n)}|^p \right)^{1/p} \leq \lim_{K \rightarrow \infty} \lim_{n \rightarrow \infty} \left( \sum_{k=1}^\infty |x_k^{(n)}|^p \right)^{1/p} \\ &= \lim_{n \rightarrow \infty} \|x^{(n)}\|_p < \infty, \end{aligned}$$

as  $(x^{(n)})$  is Cauchy, and hence bounded.

Finally, we check that  $x^{(n)} \rightarrow y$  in  $\ell^p$ . For  $\epsilon > 0$ , let  $n \geq N_\epsilon$ , so that

$$\begin{aligned} \|x^{(n)} - y\|_p &= \lim_{K \rightarrow \infty} \left( \sum_{k=1}^K |x_k^{(n)} - y_k|^p \right)^{1/p} = \lim_{K \rightarrow \infty} \left( \sum_{k=1}^K \lim_{m \rightarrow \infty} |x_k^{(n)} - x_k^{(m)}|^p \right)^{1/p} \\ &= \lim_{K \rightarrow \infty} \lim_{m \rightarrow \infty} \left( \sum_{k=1}^K |x_k^{(n)} - x_k^{(m)}|^p \right)^{1/p} \leq \lim_{K \rightarrow \infty} \lim_{m \rightarrow \infty} \left( \sum_{k=1}^{\infty} |x_k^{(n)} - x_k^{(m)}|^p \right)^{1/p} \\ &= \lim_{m \rightarrow \infty} \|x^{(n)} - x^{(m)}\|_p \leq \epsilon, \end{aligned}$$

as  $n \geq N_\epsilon$ . Hence  $\|x^{(n)} - y\|_p \rightarrow 0$ .  $\square$

For  $p = \infty$ , there are two analogies to the  $\ell^p$  spaces. The first is arguably more natural, but we write  $c_0$  for it.  $c_0$  is the space of all scalar sequences  $(x_n)$  which converge to 0. We equip  $c_0$  with the sup norm,

$$\|(x_n)\|_\infty = \sup_{n \in \mathbb{N}} |x_n| \quad ((x_n) \in c_0).$$

This is defined, as if  $x_n \rightarrow 0$ , then  $(x_n)$  is bounded. Similarly, we define  $\ell^\infty$  to be the vector space of all bounded scalar sequences, with the  $\|\cdot\|_\infty$  norm. Hence  $c_0$  is a subspace of  $\ell^\infty$ , and we can check that  $c_0$  is closed.

**Theorem 11.6.** *The spaces  $c_0$  and  $\ell^\infty$  are Banach spaces.*

*Proof.* This will be a variant of the previous proof: it's shorter, but the "trick" is maybe harder to remember. We do the  $\ell^\infty$  case. Again, let  $(x^{(n)})$  be a Cauchy sequence in  $\ell^\infty$ , and for each  $n$ , let  $x^{(n)} = (x_k^{(n)})_{k=1}^\infty$ . For  $\epsilon > 0$  we can find  $N$  such that  $\|x^{(n)} - x^{(m)}\|_\infty < \epsilon$  for  $n, m \geq N$ . Thus, for any  $k$ , we see that  $|x_k^{(n)} - x_k^{(m)}| < \epsilon$  when  $n, m \geq N$ . So  $(x_k^{(n)})_{n=1}^\infty$  is Cauchy, and hence converges, say to  $x_k \in \mathbb{K}$ . Let  $x = (x_k)$ .

Let  $m \geq N$ , so that for any  $k$ , we have that

$$|x_k - x_k^{(m)}| = \lim_{n \rightarrow \infty} |x_k^{(n)} - x_k^{(m)}| \leq \epsilon.$$

As  $k$  was arbitrary, we see that  $\sup_k |x_k - x_k^{(m)}| \leq \epsilon$ . So, firstly, this shows that  $(x - x^{(m)}) \in \ell^\infty$ , and so also  $x = (x - x^{(m)}) + x^{(m)} \in \ell^\infty$ . Secondly, we have shown that  $\|x - x^{(m)}\|_\infty \leq \epsilon$  when  $m \geq N$ , so  $x^{(m)} \rightarrow x$  in norm.  $\square$

**Example 11.7.** We can also consider a Banach space of functions  $L_p[a, b]$  with the norm

$$\|f\|_p = \left( \int_a^b |f(t)|^p dt \right)^{1/p}.$$

See the discussion after Defn. 2.22 for a realisation of such spaces.

**11.2. Bounded linear operators.** Recall what a *linear* map is, see Defn. 6.1. A linear map is often called an *operator*. A linear map  $T : E \rightarrow F$  between normed spaces is *bounded* if there exists  $M > 0$  such that  $\|T(x)\| \leq M \|x\|$  for  $x \in E$ , see Defn. 6.3. We write  $B(E, F)$  for the set of operators from  $E$  to  $F$ . For the natural operations,  $B(E, F)$  is a vector space. We norm  $B(E, F)$  by setting

$$(11.3) \quad \|T\| = \sup \left\{ \frac{\|T(x)\|}{\|x\|} : x \in E, x \neq 0 \right\}.$$

**Exercise 11.8.** Show that

- (i) The expression (11.3) is a norm in the sense of Defn. 2.3.

(ii) We equivalently have

$$\|T\| = \sup\{\|T(x)\| : x \in E, \|x\| \leq 1\} = \sup\{\|T(x)\| : x \in E, \|x\| = 1\}.$$

**Proposition 11.9.** *For a linear map  $T : E \rightarrow F$  between normed spaces, the following are equivalent:*

- (i)  $T$  is continuous (for the metrics induced by the norms on  $E$  and  $F$ );
- (ii)  $T$  is continuous at 0;
- (iii)  $T$  is bounded.

*Proof.* Proof essentially follows the proof of similar Theorem 5.4. See also discussion about usefulness of this theorem there.  $\square$

**Theorem 11.10.** *Let  $E$  be a normed space, and let  $F$  be a Banach space. Then  $B(E, F)$  is a Banach space.*

*Proof.* Let  $(T_n)$  be a Cauchy sequence in  $B(E, F)$ . For  $x \in E$ , check that  $(T_n(x))$  is Cauchy in  $F$ , and hence converges to, say,  $T(x)$ , as  $F$  is complete. Then check that  $T : E \rightarrow F$  is linear, bounded, and that  $\|T_n - T\| \rightarrow 0$ .  $\square$

We write  $B(E)$  for  $B(E, E)$ . For normed spaces  $E, F$  and  $G$ , and for  $T \in B(E, F)$  and  $S \in B(F, G)$ , we have that  $ST = S \circ T \in B(E, G)$  with  $\|ST\| \leq \|S\| \|T\|$ .

For  $T \in B(E, F)$ , if there exists  $S \in B(F, E)$  with  $ST = I_E$ , the identity of  $E$ , and  $TS = I_F$ , then  $T$  is said to be *invertible*, and write  $T = S^{-1}$ . In this case, we say that  $E$  and  $F$  are *isomorphic spaces*, and that  $T$  is an *isomorphism*.

If  $\|T(x)\| = \|x\|$  for each  $x \in E$ , we say that  $T$  is an *isometry*. If additionally  $T$  is an isomorphism, then  $T$  is an *isometric isomorphism*, and we say that  $E$  and  $F$  are *isometrically isomorphic*.

**11.3. Dual Spaces.** Let  $E$  be a normed vector space, and let  $E^*$  (also written  $E'$ ) be  $B(E, \mathbb{K})$ , the space of bounded linear maps from  $E$  to  $\mathbb{K}$ , which we call *functionals*, or more correctly, *bounded linear functionals*, see Defn. 5.1. Notice that as  $\mathbb{K}$  is complete, the above theorem shows that  $E^*$  is always a Banach space.

**Theorem 11.11.** *Let  $1 < p < \infty$ , and again let  $q$  be such that  $1/p + 1/q = 1$ . Then the map  $\ell^q \rightarrow (\ell^p)^*$ ;  $u \mapsto \phi_u$ , is an isometric isomorphism, where  $\phi_u$  is defined, for  $u = (u_j) \in \ell^q$ , by*

$$\phi_u(x) = \sum_{j=1}^{\infty} u_j x_j \quad (x = (x_j) \in \ell^p).$$

*Proof.* By Holder's inequality, we see that

$$|\phi_u(x)| \leq \sum_{j=1}^{\infty} |u_j| |x_j| \leq \left( \sum_{j=1}^{\infty} |u_j|^q \right)^{1/q} \left( \sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} = \|u\|_q \|x\|_p.$$

So the sum converges, and hence  $\phi_u$  is defined. Clearly  $\phi_u$  is linear, and the above estimate also shows that  $\|\phi_u\| \leq \|u\|_q$ . The map  $u \mapsto \phi_u$  is also clearly linear, and we've just shown that it is norm-decreasing.

Now let  $\phi \in (\ell^p)^*$ . For each  $n$ , let  $e_n = (0, \dots, 0, 1, 0, \dots)$  with the 1 in the  $n$ th position. Then, for  $x = (x_n) \in \ell^p$ ,

$$\left\| x - \sum_{k=1}^n x_k e_k \right\|_p = \left( \sum_{k=n+1}^{\infty} |x_k|^p \right)^{1/p} \rightarrow 0,$$

as  $n \rightarrow \infty$ . As  $\phi$  is continuous, we see that

$$\phi(x) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \phi(x_k e_k) = \sum_{k=1}^{\infty} x_k \phi(e_k).$$

Let  $u_k = \phi(e_k)$  for each  $k$ . If  $u = (u_k) \in \ell^q$  then we would have that  $\phi = \phi_u$ .

Let us fix  $N \in \mathbb{N}$ , and define

$$x_k = \begin{cases} 0 & : u_k = 0 \text{ or } k > N, \\ \overline{u_k} |u_k|^{q-2} & : u_k \neq 0. \end{cases}$$

Then we see that

$$\sum_{k=1}^{\infty} |x_k|^p = \sum_{k=1}^N |u_k|^{p(q-1)} = \sum_{k=1}^N |u_k|^q,$$

as  $p(q-1) = q$ . Then, by the previous paragraph,

$$\phi(x) = \sum_{k=1}^{\infty} x_k u_k = \sum_{k=1}^N |u_k|^q.$$

Hence

$$\|\phi\| \geq \frac{|\phi(x)|}{\|x\|_p} = \left( \sum_{k=1}^N |u_k|^q \right)^{1-1/p} = \left( \sum_{k=1}^N |u_k|^q \right)^{1/q}.$$

By letting  $N \rightarrow \infty$ , it follows that  $u \in \ell^q$  with  $\|u\|_q \leq \|\phi\|$ . So  $\phi = \phi_u$  and  $\|\phi\| = \|\phi_u\| \leq \|u\|_q$ . Hence every element of  $(\ell^p)^*$  arises as  $\phi_u$  for some  $u$ , and also  $\|\phi_u\| = \|u\|_q$ .  $\square$

Loosely speaking, we say that  $\ell^q = (\ell^p)^*$ , although we should always be careful to keep in mind the exact map which gives this.

Similarly, we can show that  $c_0^* = \ell^1$  and that  $(\ell^1)^* = \ell^\infty$  (the implementing isometric isomorphism is giving by the same summation formula).

**11.4. Hahn–Banach Theorem.** Mathematical induction is a well known method to prove statements depending from a natural number. The mathematical induction is based on the following property of natural numbers: any subset of  $\mathbb{N}$  has the least element. This observation can be generalised to the transfinite induction described as follows.

A *poset* is a set  $X$  with a relation  $\preceq$  such that  $a \preceq a$  for all  $a \in X$ , if  $a \preceq b$  and  $b \preceq a$  then  $a = b$ , and if  $a \preceq b$  and  $b \preceq c$ , then  $a \preceq c$ . We say that  $(X, \preceq)$  is *total* if for every  $a, b \in X$ , either  $a \preceq b$  or  $b \preceq a$ . For a subset  $S \subseteq X$ , an element  $a \in X$  is an *upper bound* for  $S$  if  $s \preceq a$  for every  $s \in S$ . An element  $a \in X$  is *maximal* if whenever  $b \in X$  is such that  $a \preceq b$ , then also  $b \preceq a$ .

Then **Zorn's Lemma** tells us that if  $X$  is a non-empty poset such that every total subset has an upper bound, then  $X$  has a maximal element. Really this is an *axiom* which we have to assume, in addition to the usual axioms of set-theory. Zorn's Lemma is equivalent to the *axiom of choice* and *Zermelo's theorem*.

**Theorem 11.12** (Hahn–Banach Theorem). *Let  $E$  be a normed vector space, and let  $F \subseteq E$  be a subspace. Let  $\phi \in F^*$ . Then there exists  $\psi \in E^*$  with  $\|\psi\| \leq \|\phi\|$  and  $\psi(x) = \phi(x)$  for each  $x \in F$ .*

*Proof.* We do the real case. An "extension" of  $\phi$  is a bounded linear map  $\phi_G : G \rightarrow \mathbb{R}$  such that  $F \subseteq G \subseteq E$ ,  $\phi_G(x) = \phi(x)$  for  $x \in F$ , and  $\|\phi_G\| \leq \|\phi\|$ . We introduce a partial order on the pairs  $(G, \phi_G)$  of subspaces and functionals as follows:  $(G_1, \phi_{G_1}) \preceq (G_2, \phi_{G_2})$  if and only if  $G_1 \subseteq G_2$  and  $\phi_{G_1}(x) = \phi_{G_2}(x)$  for all  $x \in G_1$ . A Zorn's Lemma argument shows that a maximal extension  $\phi_G : G \rightarrow \mathbb{R}$  exists. We shall show that if  $G \neq E$ , then we can extend  $\phi_G$ , a contradiction.

Let  $x \notin G$ , so an extension  $\phi_1$  of  $\phi$  to the linear span of  $G$  and  $x$  must have the form

$$\phi_1(\tilde{x} + \alpha x) = \phi(x) + \alpha \alpha \quad (\tilde{x} \in G, \alpha \in \mathbb{R}),$$

for some  $\alpha \in \mathbb{R}$ . Under this,  $\phi_1$  is linear and extends  $\phi$ , but we also need to ensure that  $\|\phi_1\| \leq \|\phi\|$ . That is, we need

$$(11.4) \quad |\phi(\tilde{x}) + \alpha\alpha| \leq \|\phi\| \|\tilde{x} + \alpha x\| \quad (\tilde{x} \in G, \alpha \in \mathbb{R}).$$

To simplify proof put  $-\alpha y = \tilde{x}$  in (11.4) and divide both sides of the identity by  $\alpha$ . Thus we need to show that there exist such  $\alpha$  that

$$|\alpha - \phi(y)| \leq \|\phi\| \|x - y\| \quad \text{for all } y \in G, \alpha \in \mathbb{R},$$

or

$$\phi(y) - \|\phi\| \|x - y\| \leq \alpha \leq \phi(y) + \|\phi\| \|x - y\|.$$

For any  $y_1$  and  $y_2$  in  $G$  we have:

$$\phi(y_1) - \phi(y_2) \leq \|\phi\| \|y_1 - y_2\| \leq \|\phi\| (\|x - y_2\| + \|x - y_1\|).$$

Thus

$$\phi(y_1) - \|\phi\| \|x - y_1\| \leq \phi(y_2) + \|\phi\| \|x - y_2\|.$$

As  $y_1$  and  $y_2$  were arbitrary,

$$\sup_{y \in G} (-\phi(y) - \|\phi\| \|y + x\|) \leq \inf_{y \in G} (-\phi(y) + \|\phi\| \|y + x\|).$$

Hence we can choose  $\alpha$  between the inf and the sup.

The complex case follows by “complexification”. □

The Hahn-Banach theorem tells us that a functional from a subspace can be extended to the whole space without increasing the norm. In particular, extending a functional on a one-dimensional subspace yields the following.

**Corollary 11.13.** *Let  $E$  be a normed vector space, and let  $x \in E$ . Then there exists  $\phi \in E^*$  with  $\|\phi\| = 1$  and  $\phi(x) = \|x\|$ .*

Another useful result which can be proved by Hahn-Banach is the following.

**Corollary 11.14.** *Let  $E$  be a normed vector space, and let  $F$  be a subspace of  $E$ . For  $x \in E$ , the following are equivalent:*

- (i)  $x \in \bar{F}$  the closure of  $F$ ;
- (ii) for each  $\phi \in E^*$  with  $\phi(y) = 0$  for each  $y \in F$ , we have that  $\phi(x) = 0$ .

*Proof.* **11.14(i)  $\Rightarrow$  11.14(ii)** follows because we can find a sequence  $(y_n)$  in  $F$  with  $y_n \rightarrow x$ ; then it's immediate that  $\phi(x) = 0$ , because  $\phi$  is continuous. Conversely, we show that if **11.14(i)** doesn't hold then **11.14(ii)** doesn't hold (that is, the contrapositive to **11.14(ii)  $\Rightarrow$  11.14(i)**).

So,  $x \notin \bar{F}$ . Define  $\psi : \text{lin}\{F, x\} \rightarrow \mathbb{K}$  by

$$\psi(y + tx) = t \quad (y \in F, t \in \mathbb{K}).$$

This is well-defined, for if  $y + tx = y' + t'x$  then either  $t = t'$ , or otherwise  $x = (t - t')^{-1}(y' - y) \in F$  which is a contradiction. The map  $\psi$  is obviously linear, so we need to show that it is bounded. Towards a contradiction, suppose that  $\psi$  is not bounded, so we can find a sequence  $(y_n + t_n x)$  with  $\|y_n + t_n x\| \leq 1$  for each  $n$ , and yet  $|\psi(y_n + t_n x)| = |t_n| \rightarrow \infty$ . Then  $\|t_n^{-1}y_n + x\| \leq 1/|t_n| \rightarrow 0$ , so that the sequence  $(-t_n^{-1}y_n)$ , which is in  $F$ , converges to  $x$ . So  $x$  is in the closure of  $F$ , a contradiction. So  $\psi$  is bounded. By Hahn-Banach theorem, we can find some  $\phi \in E^*$  extending  $\psi$ . For  $y \in F$ , we have  $\phi(y) = \psi(y) = 0$ , while  $\phi(x) = \psi(x) = 1$ , so **11.14(ii)** doesn't hold, as required. □

We define  $E^{**} = (E^*)^*$  to be the bidual of  $E$ , and define  $J : E \rightarrow E^{**}$  as follows. For  $x \in E$ ,  $J(x)$  should be in  $E^{**}$ , that is, a map  $E^* \rightarrow \mathbb{K}$ . We define this to be the map  $\phi \mapsto \phi(x)$  for  $\phi \in E^*$ . We write this as

$$J(x)(\phi) = \phi(x) \quad (x \in E, \phi \in E^*).$$

The Corollary 11.13 shows that  $J$  is an isometry; when  $J$  is surjective (that is, when  $J$  is an isomorphism), we say that  $E$  is *reflexive*. For example,  $\ell^p$  is reflexive for  $1 < p < \infty$ .

**11.5.  $C(X)$  Spaces.** This section is not examinable, *BUT* a pre-requisite for the course is the Topology course, and so I assume that you know the basics about compact spaces, and so forth. Standard facts about topology will be used in later sections of the course.

All our topological spaces are assumed *Hausdorff*. Let  $X$  be a compact space, and let  $C_{\mathbb{K}}(X)$  be the space of continuous functions from  $X$  to  $\mathbb{K}$ , with pointwise operations, so that  $C_{\mathbb{K}}(X)$  is a vector space. We norm  $C_{\mathbb{K}}(X)$  by setting

$$\|f\|_{\infty} = \sup_{x \in X} |f(x)| \quad (f \in C_{\mathbb{K}}(X)).$$

**Theorem 11.15.** *Let  $X$  be a compact space. Then  $C_{\mathbb{K}}(X)$  is a Banach space.*

Let  $E$  be a vector space, and let  $\|\cdot\|_{(1)}$  and  $\|\cdot\|_{(2)}$  be norms on  $E$ . These norms are *equivalent* if there exists  $m > 0$  with

$$m^{-1} \|x\|_{(2)} \leq \|x\|_{(1)} \leq m \|x\|_{(2)} \quad (x \in E).$$

**Theorem 11.16.** *Let  $E$  be a finite-dimensional vector space with basis  $\{e_1, \dots, e_n\}$ , so we can identify  $E$  with  $\mathbb{K}^n$  as vector spaces, and hence talk about the norm  $\|\cdot\|_2$  on  $E$ . If  $\|\cdot\|$  is any norm on  $E$ , then  $\|\cdot\|$  and  $\|\cdot\|_2$  are equivalent.*

**Corollary 11.17.** *Let  $E$  be a finite-dimensional normed space. Then a subset  $X \subseteq E$  is compact if and only if it is closed and bounded.*

**Lemma 11.18.** *Let  $E$  be a normed vector space, and let  $F$  be a closed subspace of  $E$  with  $E \neq F$ . For  $0 < \theta < 1$ , we can find  $x_0 \in E$  with  $\|x_0\| \leq 1$  and  $\|x_0 - y\| > \theta$  for  $y \in F$ .*

**Theorem 11.19.** *Let  $E$  be an infinite-dimensional normed vector space. Then the closed unit ball of  $E$ , the set  $\{x \in E : \|x\| \leq 1\}$ , is not compact.*

*Proof.* Use the above lemma to construct a sequence  $(x_n)$  in the closed unit ball of  $E$  with, say,  $\|x_n - x_m\| \geq 1/2$  for each  $n \neq m$ . Then  $(x_n)$  can have no convergent subsequence, and so the closed unit ball cannot be compact.  $\square$

## 12. MEASURE THEORY

The presentation in this section is close to [2-4].

### 12.1. Basic Measure Theory.

**Definition 12.1.** Let  $X$  be a set. A  $\sigma$ -algebra on  $X$  is a collection of subsets of  $X$ , say  $\mathcal{R} \subseteq 2^X$ , such that

- (i)  $X \in \mathcal{R}$ ;
- (ii) if  $A, B \in \mathcal{R}$ , then  $A \setminus B \in \mathcal{R}$ ;
- (iii) if  $(A_n)$  is any sequence in  $\mathcal{R}$ , then  $\cup_n A_n \in \mathcal{R}$ .

If we drop the first condition from the definition we got a *ring*.

**Exercise 12.2.** Show that the empty set belongs to any  $\sigma$ -algebra.

**Definition 12.3.** A *semiring*  $S$  of sets is the collection such that



- (i) it is closed under intersection;
- (ii) for  $A, B \in S$  we have  $A \setminus B = C_1 \sqcup \dots \sqcup C_N$  with  $C_k \in S$ .

**Example 12.4.** The following are semirings but not rings:

- (i) The collection of intervals  $[a, b)$  on the real line;
- (ii) The collection of all rectangles  $\{a \leq x < b, c \leq y < d\}$  on the plane.

**Exercise 12.5.** Let  $S$  be a semiring. Show that

- (i) The collection of all disjoint unions  $\sqcup_{k=1}^n A_k$ , where  $A_k \in S$ , is a ring. We call it the ring  $R(S)$  generated by the semiring  $S$ .
- (ii) Any ring containing  $S$  contains  $R(S)$  as well.

For a  $\sigma$ -algebra  $R$  and  $A, B \in R$ , we have

$$A \cap B = X \setminus (X \setminus (A \cap B)) = X \setminus ((X \setminus A) \cup (X \setminus B)) \in R;$$

similarly,  $R$  is closed under taking (countably) infinite intersections.

As the intersection of a family of  $\sigma$ -algebras is again a  $\sigma$ -algebra, and the power set  $2^X$  is a  $\sigma$ -algebra, it follows that given any collection  $D \subseteq 2^X$ , there is a  $\sigma$ -algebra  $R$  such that  $D \subseteq R$ , such that if  $S$  is any other  $\sigma$ -algebra, with  $D \subseteq S$ , then  $R \subseteq S$ . We call  $R$  the  $\sigma$ -algebra generated by  $D$ . In a similar way we can

**Definition 12.6.** A *measure* is a map  $\mu : R \rightarrow [0, \infty]$  defined on a  $\sigma$ -algebra  $R$ , such that

- (i)  $\mu(\emptyset) = 0$ .
- (ii) if  $(A_n)$  is a finite subset of  $R$ , such that  $A_j$  are pairwise disjoint (that is,  $A_n \cap A_m = \emptyset$  for  $n \neq m$ ), then  $\mu(\cup_n A_n) = \sum_n \mu(A_n)$ . This property is called *additivity* of a measure.

In analysis we are interested in infinities and limits, thus the following extension of additivity is very important.

**Definition 12.7.** In terms of the previous definition we say that  $\mu$  is *countably additive* (or  $\sigma$ -additive) if for any countable family  $(A_n)$  of pairwise disjoint sets from  $R$  we have  $\mu(\cup_n A_n) = \sum_n \mu(A_n)$ . If the sum diverges, then as it will be the sum of positive numbers, we can, without problem, define it to be  $+\infty$ .

**Example 12.8.** (i) Fix a point  $a \in \mathbb{R}$  and define a measure  $\mu$  by the condition  $\mu(A) = 1$  if  $a \in A$  and  $\mu(A) = 0$  otherwise.

(ii) For the semiring from Example 12.4(i) define  $\mu([a, b)) = b - a$ . This is a measure, and we will show its  $\sigma$ -additivity.

(iii) For the semiring from Example 12.4(ii), define  $\mu(V) = (b - a)(d - c)$  for the rectangle  $V = \{a \leq x < b, c \leq y < d\}$ . It is again a  $\sigma$ -additive measure.

We will see how to define a measure which is not  $\sigma$ -additive in Section 12.4.

**Definition 12.9.** A measure  $\mu$  is *finite* if  $\mu(A) < \infty$  for all  $A \in R$ .

A measure  $\mu$  is  *$\sigma$ -finite* if  $X$  is a union of countable number of sets  $X_k$ , such that the restriction of  $\mu$  to each  $X_k$  is finite.

**Proposition 12.10.** Let  $\mu$  be a measure on a  $\sigma$ -algebra  $R$ . Then:

- (i) If  $A, B \in R$  with  $A \subseteq B$ , then  $\mu(A) \leq \mu(B)$ ;
- (ii) If  $A, B \in R$  with  $A \subseteq B$  and  $\mu(B) < \infty$ , then  $\mu(B \setminus A) = \mu(B) - \mu(A)$ ;
- (iii) If  $(A_n)$  is a sequence in  $R$ , with  $A_1 \subseteq A_2 \subseteq A_3 \subseteq \dots$ . Then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(\cup A_n).$$

- (iv) If  $(A_n)$  is a sequence in  $R$ , with  $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ . If  $\mu(A_m) < \infty$  for some  $m$ , then

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu(\cap A_n).$$

**12.2. Extension of Measures.** From now on we consider only finite measures, an extension to  $\sigma$ -finite measures will be done later.

**Proposition 12.11.** *Any measure  $\mu'$  on a semiring  $S$  is uniquely extended to a measure  $\mu$  on the generated ring  $R(S)$ , see Ex. 12.5. If the initial measure was  $\sigma$ -additive, then the extension is  $\sigma$ -additive as well.*

*Proof.* If an extension exists it shall satisfy  $\mu(A) = \sqcup_{k=1}^n \mu'(A_k)$ , where  $A_k \in S$ . We need to show for this definition two elements:

- (i) Consistency, i.e. independence of the value from a presentation of  $A \in R(S)$  as  $A = \sqcup_{k=1}^n A_k$ , where  $A_k \in S$ . For two different presentation  $A = \sqcup_{k=1}^n A_k$  and  $A = \sqcup_{k=1}^m B_k$  define  $C_{jk} = A_j \cap B_k$  and express the value of  $\mu(A)$  through them.
- (ii) Additivity. For  $A = \sqcup_{k=1}^n A_k$ , where  $A_k \in R(S)$  present  $A = \sqcup_{j=1}^{n(k)} C_{jk}$  and obtain additivity from this sub-presentation.

Finally show the  $\sigma$ -additivity. For a set  $A = \sqcup_{k=1}^{\infty} A_k$ , where  $A_k \in R(S)$ , find presentations  $A = \sqcup_{k=1}^n B_k$  and  $B_k = \sqcup_{j=1}^{n(k)} B_{jk}$ . Then use  $\sigma$ -additivity of  $\mu'$  and the possibility to change summation signs in series with non-negative terms.  $\square$

In a similar way we can extend a measure from a semiring to corresponding  $\sigma$ -ring, however it can be done even for a larger family. To do that we will use the following notion.

**Definition 12.12.** Let  $S$  be a semi-ring of subsets in  $X$ , and  $\mu$  be a measure defined on  $S$ . An *outer measure*  $\mu^*$  on  $X$  is a map  $\mu^* : 2^X \rightarrow [0, \infty]$  defined by:

$$\mu^*(A) = \inf \left\{ \sum_k \mu(A_k), \text{ such that } A \subseteq \cup_k A_k, \quad A_k \in S \right\}.$$

**Proposition 12.13.** *An outer measure has the following properties:*

- (i)  $\mu^*(\emptyset) = 0$ ;
- (ii) if  $A \subseteq B$  then  $\mu^*(A) \leq \mu^*(B)$ ;
- (iii) if  $(A_n)$  is any sequence in  $2^X$ , then  $\mu^*(\cup_n A_n) \leq \sum_n \mu^*(A_n)$ .

The final condition says that an outer measure is *countably sub-additive*.

**Example 12.14.** The *Lebesgue outer measure* on  $\mathbb{R}$  is defined out of the measure from Example 12.8(ii), that is for  $A \subseteq \mathbb{R}$ , as

$$\mu^*(A) = \inf \left\{ \sum_{j=1}^{\infty} (b_j - a_j) : A \subseteq \cup_{j=1}^{\infty} [a_j, b_j] \right\}.$$

We make this definition, as intuitively, the "length", or measure, of the interval  $[a, b]$  is  $(b - a)$ .

For example, for outer Lebesgue measure we have  $\mu^*(A) = 0$  for any countable set, which follows, as clearly  $\mu^*({x}) = 0$  for any  $x \in \mathbb{R}$ .

**Lemma 12.15.** *Let  $a < b$ . Then  $\mu^*([a, b]) = b - a$ .*

*Proof.* For  $\epsilon > 0$ , as  $[a, b] \subseteq [a, b + \epsilon]$ , we have that  $\mu^*([a, b]) \leq (b - a) + \epsilon$ . As  $\epsilon > 0$ , was arbitrary,  $\mu^*([a, b]) \leq b - a$ .

To show the opposite inequality we observe that  $[a, b] \subset [a, b]$  and  $\mu^*[a, b] = b - a$  (because  $[a, b]$  is in the semi-ring) so  $\mu^*[a, b] \geq b - a$  by 12.13(ii).  $\square$

Our next aim is to construct measures from outer measures. We use the notation  $A \Delta B = (A \cup B) \setminus (A \cap B)$  for *symmetric difference of sets*.

**Definition 12.16.** Given an outer measure  $\mu^*$  defined by a semiring  $S$ , we define  $A \subseteq X$  to be *Lebesgue measurable* if for any  $\varepsilon > 0$  there is a countable union  $B$  of elements in  $S$  such that  $\mu^*(A \Delta B) < \varepsilon$ .

Obviously all elements of  $S$  are measurable. An alternative definition of a measurable set is due to Carathéodory.

**Definition 12.17.** Given an outer measure  $\mu^*$ , we define  $E \subseteq X$  to be *Carathéodory measurable* if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E),$$

for any  $A \subseteq X$ .

As  $\mu^*$  is sub-additive, this is equivalent to

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \setminus E) \quad (A \subseteq X),$$

as the other inequality is automatic.

**Exercise\* 12.18.** Show that measurability by Lebesgue and Carathéodory are equivalent.

**Theorem 12.19 (Lebesgue).** Let  $\mu^*$  be an outer measure on  $X$  defined by a semiring  $S$ , and let  $L$  be the collection of all Lebesgue measurable sets for  $\mu^*$ . Then  $L$  is a  $\sigma$ -algebra, and if  $\mu$  is the restriction of  $\mu^*$  to  $L$ , then  $\mu$  is a measure.

*Sketch of proof.* First we show that  $\mu^*(A) = \mu(A)$  for a set  $A \in R(S)$ . If  $A \subset \cup_k A_k$  then  $\mu(A) \leq \sum_k \mu(A_k)$ , taking infimum we get  $\mu(A) < \mu^*(A)$ . The opposite inequality follows because  $A \in R(S)$  implies a disjoint representation  $A = \sqcup_k B_k$ .

Now we will show that  $R(S)$  is an incomplete metric space, with the measure  $\mu$  being uniformly continuous functions. Measurable sets make the completion of  $R(S)$  with  $\mu$  being continuation of  $\mu^*$  to the completion by continuity.

Define a distance between elements  $A, B \in L$  as the outer measure of the symmetric difference of  $A$  and  $B$ :  $d(A, B) = \mu^*(A \Delta B)$ . Introduce equivalence relation  $A \sim B$  if  $d(A, B) = 0$  and use the inclusion for the triangle inequality:

$$A \Delta B \subseteq (A \Delta C) \cup (C \Delta B)$$

Then Lebesgue measurable sets make the closure of  $S$  with respect to this distance.

We can check that measurable sets form an algebra. To this end we need to make estimations, say, of  $\mu^*((A_1 \cap A_2) \Delta (B_1 \cap B_2))$  in terms of  $\mu^*(A_i \Delta B_i)$ .

Now we show that  $L$  is  $\sigma$ -algebra. Let  $A_k \in L$  and  $A = \cup_k A_k$ . Then for any  $\varepsilon > 0$  there exist  $B_k \in S$  such that  $\mu^*(A_k \Delta B_k) < \frac{\varepsilon}{2^k}$ . Define  $B = \cup_k B_k$ . Then

$$(\cup_k A_k) \Delta (\cup_k B_k) \subset \cup_k (A_k \Delta B_k) \text{ implies } \mu^*(A \Delta B) < \varepsilon.$$

Now define  $B'_1 = B_1$  and  $B'_k = B_k \setminus \cup_{i=1}^{k-1} B_i$ . Then  $B = \cup_k B'_k$  and  $B'_k \in R(S)$ . From the convergence of the series there is  $N$  such that  $\sum_{k=N}^{\infty} \mu(B_k) < \varepsilon$ . Let  $B' = \cup_{k=1}^N B'_k$ . Then  $\mu^*(B \Delta B') \leq \varepsilon$  and  $\mu^*(A \Delta B') \leq 2\varepsilon$ .

To check that  $\mu^*$  is measure on  $L$  we use the following

**Lemma 12.20.**  $|\mu^*(A) - \mu^*(B)| \leq \mu^*(A \Delta B)$ , that is  $\mu^*$  is uniformly continuous in the metric  $d(A, B)$ .

*Proof of the Lemma.* Use inclusions  $A \subset B \cup (A \Delta B)$  □

To show additivity take  $A_{1,2} \in L$ ,  $A = A_1 \sqcup A_2$ ,  $B_{1,2} \in R$  and  $\mu^*(A_i \Delta B_i) < \varepsilon$ . Then  $\mu^*(A \Delta (B_1 \cup B_2)) < 2\varepsilon$  and  $|\mu^*(A) - \mu^*(B_1 \cup B_2)| < 2\varepsilon$ . Thus  $\mu^*(B_1 \cup B_2) = \mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) - \mu(B_1 \cap B_2)$ , but  $\mu(B_1 \cup B_2) = d(B_1 \cup B_2, \emptyset) = d(B_1 \cup B_2, A_1 \cap A_2) < 2\varepsilon$ . Therefore

$$|\mu^*(B_1 \cup B_2) - \mu(B_1) - \mu(B_2)| < 2\varepsilon.$$

Combining everything together we get:

$$|\mu^*(A) - \mu^*(A_1) - \mu^*(A_2)| < 6\varepsilon.$$

Thus  $\mu^*$  is additive.

Check the countable additivity for  $A = \sqcup_k A_k$ . The Inequality  $\mu^*(A) \leq \sum_k \mu^*(A_k)$  follows from countable sub-additivity. The opposite inequality is the limiting case of the finite inequality  $\mu^*(A) \geq \sum_{k=1}^N \mu^*(A_k)$  following from additivity and monotonicity of  $\mu$ .  $\square$

**Corollary 12.21.** *Let  $E \subseteq \mathbb{R}$  be open or closed. Then  $E$  is Lebesgue measurable.*

*Proof.* This is a common trick, using the density and the countability of the rationals. As  $\sigma$ -algebras are closed under taking complements, we need only show that open sets are Lebesgue measurable.

Intervals  $(a, b)$  are Lebesgue measurable by the very definition. Now let  $U \subseteq \mathbb{R}$  be open. For each  $x \in U$ , there exists  $a_x < b_x$  with  $x \in (a_x, b_x) \subseteq U$ . By making  $a_x$  slightly larger, and  $b_x$  slightly smaller, we can ensure that  $a_x, b_x \in \mathbb{Q}$ . Thus  $U = \cup_x (a_x, b_x)$ . Each interval is measurable, and there are at most a countable number of them (endpoints make a countable set) thus  $U$  is the countable (or finite) union of Lebesgue measurable sets, and hence  $U$  is Lebesgue measurable itself.  $\square$

We perform now an extension of finite measure to  $\sigma$ -finite one. Let there is  $\sigma$ -additive and  $\sigma$ -finite measure  $\mu$  defined on a semiring in  $X = \sqcup_k X_k$ , where restriction of  $\mu$  to every  $X_k$  is finite. Consider the Lebesgue extension  $\mu_k$  of  $\mu$  defined within  $X_k$ . A set  $A \subset X$  is measurable if every intersection  $A \cap X_k$  is  $\mu_k$  measurable. For a such measurable set  $A$  we define its measure by the identity:

$$\mu(A) = \sum_k \mu_k(A \cap X_k).$$

We call a measure  $\mu$  defined on  $L$  *complete* if whenever  $E \subseteq X$  is such that there exists  $F \in L$  with  $\mu(F) = 0$  and  $E \subseteq F$ , we have that  $E \in L$ . Measures constructed from outer measures by the above theorem are always complete. On the example sheet, we saw how to form a complete measure from a given measure. We call sets like  $E$  *null sets*: complete measures are useful, because it is helpful to be able to say that null sets are in our  $\sigma$ -algebra. Null sets can be quite complicated. For the Lebesgue measure, all countable subsets of  $\mathbb{R}$  are null, but then so is the Cantor set, which is uncountable.

**Definition 12.22.** If we have a property  $P(x)$  which is true except possibly  $x \in A$  and  $\mu(A) = 0$ , we say  $P(x)$  is *almost everywhere* or *a.e.*.

**12.3. Complex-Valued Measures and Charges.** We oftenly need expand the measures to be more than only positive-valued.

**Definition 12.23.** Let  $X$  be a set, and  $R$  be a  $\sigma$ -ring. A real- (complex-) valued function  $\nu$  on  $R$  is called a *charge* (or *signed measure*) if it is countably additive as follows: for any  $A_k \in R$  the identity  $A = \sqcup_k A_k$  implies the series  $\sum_k \nu(A_k)$  is absolute convergent and has the sum  $\nu(A)$ .

**Example 12.24.** Any linear combination of  $\sigma$ -additive measures on  $\mathbb{R}$  with real (complex) coefficients is real (complex) charge.

The opposite statement is also true:

**Theorem 12.25.** *Any real (complex) charge  $\nu$  has a representation  $\nu = \mu_1 - \mu_2$  ( $\nu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ ), where  $\mu_k$  are  $\sigma$ -additive measures.*

To prove the theorem we need the following definition.

**Definition 12.26.** The *variation of a charge* on a set  $A$  is  $|\nu|(A) = \sup \sum_k |\nu(A_k)|$  for all disjoint splitting  $A = \sqcup_k A_k$ .

**Example 12.27.** If  $\nu = \mu_1 - \mu_2$ , then  $|\nu|(A) \leq \mu_1(A) + \mu_2(A)$ . The inequality becomes an identity for *disjunctive measures* on  $A$  (that is there is a splitting  $A = A_1 \sqcup A_2$  such that  $\mu_2(A_1) = \mu_1(A_2) = 0$ ).

The relation of variation to charge is as follows:

**Theorem 12.28.** For any charge  $\nu$  the function  $|\nu|$  is a  $\sigma$ -additive measure.

Finally to prove the Thm. 12.25 we use the following

**Proposition 12.29.** For any charge  $\nu$  the function  $|\nu| - \nu$  is a  $\sigma$ -additive measure as well.

From the Thm. 12.25 we can deduce

**Corollary 12.30.** The collection of all charges on a  $\sigma$ -algebra  $\mathcal{R}$  is a linear space which is complete with respect to the distance:

$$d(\nu_1, \nu_2) = \sup_{A \in \mathcal{R}} |\nu_1(A) - \nu_2(A)|.$$

The following result is also important:

**Theorem 12.31 (Hahn Decomposition).** Let  $\nu$  be a charge. There exist  $A, B \in \mathcal{L}$ , called a Hahn decomposition of  $(X, \nu)$ , with  $A \cap B = \emptyset$ ,  $A \cup B = X$  and such that for any  $E \in \mathcal{L}$ ,

$$\nu(A \cap E) \geq 0, \quad \nu(B \cap E) \leq 0.$$

This need not be unique.

*Proof.* We only sketch this. We say that  $A \in \mathcal{L}$  is *positive* if

$$\nu(E \cap A) \geq 0 \quad (E \in \mathcal{L}),$$

and similarly define what it means for a measurable set to be *negative*. Suppose that  $\nu$  never takes the value  $-\infty$  (the other case follows by considering the charge  $-\nu$ ).

Let  $\beta = \inf \nu(B_0)$  where we take the infimum over all negative sets  $B_0$ . If  $\beta = -\infty$  then for each  $n$ , we can find a negative  $B_n$  with  $\nu(B_n) \leq -n$ . But then  $B = \cup_n B_n$  would be negative with  $\nu(B) \leq -n$  for any  $n$ , so that  $\nu(B) = -\infty$  a contradiction.

So  $\beta > -\infty$  and so for each  $n$  we can find a negative  $B_n$   $\nu(B_n) < \beta + 1/n$ . Then we can show that  $B = \cup_n B_n$  is negative, and argue that  $\nu(B) \leq \beta$ . As  $B$  is negative, actually  $\nu(B) = \beta$ .

There then follows a very tedious argument, by contradiction, to show that  $A = X \setminus B$  is a positive set. Then  $(A, B)$  is the required decomposition.  $\square$

**12.4. Constructing Measures, Products.** Consider the semiring  $S$  of intervals  $[a, b)$ . There is a simple description of all measures on it. For a measure  $\mu$  define

$$F_\mu(t) = \begin{cases} \mu([0, t)) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -\mu([t, 0)) & \text{if } t < 0, \end{cases}$$

$F_\mu$  is monotonic and any monotonic function  $F$  defines a measure  $\mu$  on  $S$  by the by  $\mu([a, b)) = F(b) - F(a)$ . The correspondence is one-to-one with the additional assumption  $F(0) = 0$ .

**Theorem 12.32.** The above measure  $\mu$  is  $\sigma$ -additive on  $S$  if and only if  $F$  is continuous from the left:  $F(t - 0) = F(t)$  for all  $t \in \mathbb{R}$ .

*Proof.* The necessity:  $F(t) - F(t - 0) = \lim_{\varepsilon \rightarrow 0} \mu([t - \varepsilon, t]) = 0$ .

For sufficiency assume  $[a, b] = \sqcup_k [a_k, b_k]$ . The inequality  $\mu([a, b]) \geq \sum_k \mu([a_k, b_k])$  follows from additivity and monotonicity. For the opposite inequality take  $\delta_k$  s.t.  $F(b) - F(b - \delta) < \varepsilon$  and  $F(a_k) - F(a_k - \delta_k) < \varepsilon/2^k$  (use left continuity of  $F$ ). Then the interval  $[a, b - \delta]$  is covered by  $(a_k - \delta_k, b_k)$ , there is finite subcovering. Thus  $\mu([a, b - \delta]) \leq \sum_{k=1}^N \mu([a_k - \delta_k, b_k])$ .  $\square$

**Example 12.33.** (i) Take  $F(t) = t$ , then the corresponding measure is the Lebesgue measure on  $\mathbb{R}$ .

(ii) Take  $F(t)$  be the integer part of  $t$ , then  $\mu$  counts the number of integer within the set.

(iii) Define the *Cantor function* as follows  $\alpha(x) = 1/2$  on  $(1/3, 2/3)$ ;  $\alpha(x) = 1/4$  on  $(1/9, 2/9)$ ;  $\alpha(x) = 3/4$  on  $(7/9, 8/9)$ . This function is monotonic and can be continued to  $[0, 1]$  by continuity. The resulting measure is as follows:

- The measure of the entire interval is 1.
- Measure of every point is zero.
- The measure of the Cantor set is 1, while its Lebesgue measure is 0.

Another possibility to build measures is their product.

**Definition 12.34.** Let  $X$  and  $Y$  be spaces, and let  $S$  and  $T$  be semirings on  $X$  and  $Y$  respectively. Then  $S \times T$  is the semiring consisting of  $\{A \times B : A \in S, B \in T\}$  ("generalised rectangles"). Let  $\mu$  and  $\nu$  be measures on  $S$  and  $T$  respectively. Define the *product measure*  $\mu \times \nu$  on  $S \times T$  by the rule  $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ .

**Example 12.35.** The measure from Example 12.8(iii) is the product of two copies of pre-Lebesgue measures from Example 12.8(ii).

### 13. INTEGRATION

We now come to the main use of measure theory: to define a general theory of integration.

**13.1. Measurable functions.** From now on, by a *measure space* we shall mean a triple  $(X, L, \mu)$ , where  $X$  is a set,  $L$  is a  $\sigma$ -algebra on  $X$ , and  $\mu$  is a measure defined on  $L$ . We say that the members of  $L$  are *measurable*, or  $L$ -measurable, if necessary to avoid confusion.

**Definition 13.1.** A function  $f : X \rightarrow \mathbb{R}$  is *measurable* if

$$E_c(f) = \{x \in X : f(x) < c\}$$

is in  $L$  for any  $c \in \mathbb{R}$ .

A complex-valued function is measurable if its real and imaginary parts are measurable.

**Lemma 13.2.** *The following are equivalent:*

- (i) A function  $f$  is measurable;
- (ii) For any  $a < b$  the set  $f^{-1}((a, b))$  is measurable;
- (iii) For any open set  $U \in \mathbb{R}$  the set  $f^{-1}(U)$  is measurable.

*Proof.* Use that any open set  $U \in \mathbb{R}$  is a union of countable set of intervals  $(a, b)$ , cf. proof of Cor. 12.21.  $\square$

**Corollary 13.3.** *Let  $f$  be measurable and  $g$  be continuous, then the composition  $g(f(x))$  is measurable.*

*Proof.* The preimage of  $(-\infty, c)$  under a continuous  $g$  is an open set, and its preimage under  $f$  is measurable.  $\square$

**Theorem 13.4.** *Let  $f, g : X \rightarrow \mathbb{R}$  be measurable. Then  $af, f + g, fg, \max(f, g)$  and  $\min(f, g)$  are all measurable. That is measurable functions form an algebra and this algebra is closed under convergence a.e.*

*Proof.* Use Cor. 13.3 to show measurability of  $\lambda f, |f|$  and  $f^2$ .

Next use the following identities:

$$\begin{aligned} E_c(f_1 + f_2) &= \cup_{r \in \mathbb{Q}} (E_r(f_1) \cap E_{c-r}(f_2)), \\ f_1 f_2 &= \frac{(f_1 + f_2)^2 - (f_1 - f_2)^2}{4}, \\ \max(f_1, f_2) &= \frac{(f_1 + f_2) + |f_1 - f_2|}{2}. \end{aligned}$$

If  $(f_n)$  is a non-increasing sequence of measurable functions converging to  $f$ . Then  $E_c(f) = \cup_n E_c(f_n)$ .

Moreover any limit can be replaced by two monotonic limits:

$$\lim_{n \rightarrow \infty} f_n(x) = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} \max(f_n(x), f_{n+1}(x), \dots, f_{n+k}(x)).$$

Finally if  $f_1$  is measurable and  $f_2 = f_1$  almost everywhere, then  $f_2$  is measurable as well. □

We can define several types of convergence for measurable functions

**Definition 13.5.** We say that sequence  $(f_n)$  of functions converges

(i) *uniformly* to  $f$  (notated  $f_n \Rightarrow f$ ) if

$$\sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0;$$

(ii) *almost everywhere* to  $f$  (notated  $f_n \xrightarrow{a.e.} f$ ) if

$$f_n(x) \rightarrow f(x) \quad \text{for all } x \in X \setminus A, \mu(A) = 0;$$

(iii) *in measure*  $\mu$  to  $f$  (notated  $f_n \xrightarrow{\mu} f$ ) if for all  $\varepsilon > 0$

$$\mu(\{x \in X : |f_n(x) - f(x)| > \varepsilon\}) \rightarrow 0.$$

Clearly uniform convergence implies both convergences a.e and in measure.

**Theorem 13.6.** *On finite measures convergence a.e. implies convergence in measure.*

*Proof.* Define  $A_n(\varepsilon) = \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$ . Let  $B_n(\varepsilon) = \cup_{k \geq n} A_k(\varepsilon)$ . Clearly  $B_n(\varepsilon) \supset B_{n+1}(\varepsilon)$ , let  $B(\varepsilon) = \cap_1^\infty B_n(\varepsilon)$ . If  $x \in B(\varepsilon)$  then  $f_n(x) \not\rightarrow f(x)$ . Thus  $\mu(B(\varepsilon)) = 0$ , but  $\mu(B(\varepsilon)) = \lim_{n \rightarrow \infty} \mu(B_n(\varepsilon))$ . Since  $A_n(\varepsilon) \subset B_n(\varepsilon)$  we see that  $\mu(A_n(\varepsilon)) \rightarrow 0$ . □

**Exercise 13.7.** Present examples of sequences  $(f_n)$  and functions  $f$  such that:

- (i)  $f_n \xrightarrow{\mu} f$  but not  $f_n \xrightarrow{a.e.} f$ .
- (ii)  $f_n \xrightarrow{a.e.} f$  but not  $f_n \Rightarrow f$ .

However we can slightly “fix” either the set or the sequence to “upgrade” the convergence as shown in the following two theorems.

**Theorem 13.8 (Egorov).** *If  $f_n \xrightarrow{a.e.} f$  on a finite measure set  $X$  then for any  $\sigma > 0$  there is  $E_\sigma \subset X$  with  $\mu(E_\sigma) < \sigma$  and  $f_n \Rightarrow f$  on  $X \setminus E_\sigma$ .*

*Proof.* We use  $A_n(\varepsilon)$  and  $B_n(\varepsilon)$  from the proof of Thm. 13.6. For every  $\varepsilon > 0$  we seen  $\mu(B_n(\varepsilon)) \rightarrow 0$ , thus for each  $k$  there is  $N(k)$  such that  $\mu(B_{N(k)}(1/k)) < \sigma/2^k$ . Put  $E_\sigma = \cup_k B_{N(k)}(1/k)$ . □

**Theorem 13.9.** *If  $f_n \xrightarrow{\mu} f$  then there is a subsequence  $(n_k)$  such that  $f_{n_k} \xrightarrow{a.e.} f$  for  $k \rightarrow \infty$ .*

*Proof.* In the notations of two previous proofs: for every natural  $k$  take  $n_k$  such that  $\mu(A_{n_k}(1/k)) < 1/2^k$ . Define  $C_m = \cup_{k=m}^{\infty} A_{n_k}(1/k)$  and  $C = \cap C_m$ . Then  $\mu(C) = 0$  and if  $x \notin C$  then there is such  $N$  that  $x \notin A_{n_k}(1/k)$  for all  $k > N$ . That means that  $|f_{n_k}(x) - f(x)| < 1/k$  for all such  $k$ , i.e.  $f_{n_k}(x) \rightarrow f(x)$ .  $\square$

**Exercise 13.10.** Read about Luzin's C-property.

**13.2. Lebsgue Integral.** First we define a sort of "basis" for the space of integral functions.

**Definition 13.11.** For  $A \subseteq X$ , we define  $\chi_A$  to be the *indicator function* of  $A$ , by

$$\chi_A(x) = \begin{cases} 1 & : x \in A, \\ 0 & : x \notin A. \end{cases}$$

Then, if  $\chi_A$  is measurable, then  $\chi_A^{-1}((1/2, 3/2)) = A \in L$ ; conversely, if  $A \in L$ , then  $X \setminus A \in L$ , and we see that for any  $U \subseteq \mathbb{R}$  open,  $\chi_A^{-1}(U)$  is either  $\emptyset$ ,  $A$ ,  $X \setminus A$ , or  $X$ , all of which are in  $L$ . So  $\chi_A$  is measurable if and only if  $A \in L$ .

**Definition 13.12.** A measurable function  $f : X \rightarrow \mathbb{R}$  is *simple* if it attain only a countable number of values.

**Lemma 13.13.** A function  $f : X \rightarrow \mathbb{R}$  is simple if and only if

$$(13.1) \quad f = \sum_{k=1}^{\infty} t_k \chi_{A_k}$$

for some  $(t_k)_{k=1}^{\infty} \subseteq \mathbb{R}$  and  $A_k \in L$ . That is, simple functions are linear combinations of indicator functions of measurable sets.

Moreover in the above representation the sets  $A_k$  can be pair-wise disjoint and all  $t_k$  pair-wise different. In this case the representation is unique.

Notice that it is now obvious that

**Corollary 13.14.** The collection of simple functions forms a vector space: this wasn't clear from the original definition.

**Definition 13.15.** A simple function in the form (13.1) with disjoint  $A_k$  is called *summable* if the following series converges:

$$\sum_{k=1}^{\infty} |t_k| \mu(A_k) \quad \text{if } f \text{ has the above unique representation} \quad f = \sum_{k=1}^{\infty} t_k \chi_{A_k}$$

It is another combinatorial exercise to show that this definition is independent of the way we write  $f$ .

**Definition 13.16.** For any summable function  $f$  from the previous Definition we define the *integral* of a simple function  $f : X \rightarrow \mathbb{R}$  over a measurable set  $A$  by setting

$$\int_A f \, d\mu = \sum_{k=1}^{\infty} t_k \mu(A_k \cap A).$$

Clearly the series converges for any summable function  $f$ . Moreover

**Lemma 13.17.** The value of integral of a summable function is independent from its representation by the sum of indicators over pair-wise disjoint sets.

*Proof.* This is another slightly tedious combinatorial exercise. You need to prove that the integral of a simple function is well-defined, in the sense that it is independent of the way we choose to write the simple function.  $\square$



We will denote by  $S(X)$  the collection of all simple summable functions on  $X$ .

**Proposition 13.18.** *Let  $f, g : X \rightarrow \mathbb{R}$  be in  $S(X)$  (that is simple summable), let  $a, b \in \mathbb{R}$  and  $A$  is a measurable. Then:*

- (i)  $\int_A af + bg \, d\mu = a \int_A f \, d\mu + b \int_A g \, d\mu$ , that is  $S(X)$  is a linear space;
- (ii) The correspondence  $f \rightarrow \int_A f \, d\mu$  is a linear functional on  $S(X)$ ;
- (iii) The correspondence  $A \rightarrow \int_A f \, d\mu$  is a charge;
- (iv) The function

$$(13.2) \quad d_1(f, g) = \int_X |f(x) - g(x)| \, d\mu(x)$$

has all properties of the distance on  $S(X)$  probably except separation.

- (v) For all  $A \subset X$ :

$$\left| \int_A f(x) \, d\mu(x) - \int_A g(x) \, d\mu(x) \right| \leq d_1(f, g).$$

- (vi) If  $f \leq g$  then  $\int_X f \, d\mu \leq \int_X g \, d\mu$ , that is integral is monotonic;
- (vii) For  $f \geq 0$  we have  $\int_X f \, d\mu = 0$  if and only if  $\mu(\{x \in X : f(x) \neq 0\}) = 0$ .

*Proof.* The proof is almost obvious, for example the Property 13.18(i) easily follows from Lem. 13.17.

We will outline 13.18(iii) only. Let  $f$  is an indicator function of a set  $B$ , then  $A \rightarrow \int_A f \, d\mu = \mu(A \cap B)$  is a  $\sigma$ -additive measure (and thus—a charge). By the Cor. 12.30 the same is true for finite linear combinations of indicator functions and their limits in the sense of distance  $d_1$ .  $\square$

We can identify functions which has the same values a.e. Then  $S(X)$  becomes a metric space with the distance  $d_1$  (13.2). It may be incomplete and the completion will be realised as a the following space of functions:

**Definition 13.19.** A function  $f$  is *summable* by a measure  $\mu$  if there is sequence  $(f_n) \subset S(X)$  such that

- (i) the sequence  $(f_n)$  is a Cauchy sequence in  $S(X)$ ;
- (ii)  $f_n \xrightarrow{a.e.} f$ .

Clear if a function is summable, then any equivalent function is summable as well. Set of equivalent classes will be denoted by  $L_1(X)$ .

**Lemma 13.20.** *If the measure  $\mu$  is finite than any bounded measurable function is summable.*

*Proof.* Define  $E_{kn}(f) = \{x \in X : k/n \leq f(x) < (k+1)/n\}$  and  $f_n = \sum_k \frac{k}{n} \chi_{E_{kn}}$  (note that the sum is finite due to boundedness of  $f$ ).

Since  $|f_n(x) - f(x)| < 1/n$  we have uniform convergence (thus convergence a.e.) and  $(f_n)$  is the Cauchy sequence:  $d_1(f_n, f_m) = \int_X |f_n - f_m| \, d\mu \leq (\frac{1}{n} + \frac{1}{m})\mu(X)$ .  $\square$

*Remark 13.21.* This Lemma can be extended to the space of *essentially bounded* functions  $L_\infty(X)$ , in other words  $L_\infty(X) \subset L_1(X)$  for finite measures.

**Exercise 13.22.** Convergence in the metric  $d_1$  and a.e. do not imply each other:

- (i) Give an example of  $f_n \xrightarrow{a.e.} f$  such that  $d_1(f_n, f) \not\rightarrow 0$ .
- (ii) Give an example of the sequence  $(f_n)$  and function  $f$  in  $L^1(X)$  such that  $d_1(f_n, f) \not\rightarrow 0$  but  $f_n$  does not converge to  $f$  a.e.

To build integral we need the following

**Lemma 13.23.** *Let  $(f_n)$  and  $(g_n)$  be two Cauchy sequences in  $S(X)$  with the same limit a.e., then  $d_1(f_n, g_n) \rightarrow 0$ .*

*Proof.* Let  $\phi_n = f_n - g_n$ , then this is a Cauchy sequence with zero limit a.e. Assume the opposite to the statement: there exist  $\delta > 0$  and sequence  $(n_k)$  such that  $\int_X |\phi_{n_k}| d\mu > \delta$ . Rescaling-renumbering we can obtain  $\int_X |\phi_n| d\mu > 1$ .

Take *quickly convergent subsequence* using the Cauchy property:

$$d_1(\phi_{n_k}, \phi_{n_{k+1}}) \leq 1/2^{k+2}.$$

Renumbering again assume  $d_1(\phi_k, \phi_{k+1}) \leq 1/2^{k+2}$

Since  $\phi_1$  is a simple, that is  $\phi = \sum_k t_k \chi_{A_k}$  and  $\sum_k |t_k| \chi_{A_k} = \int_X |\phi_1| d\mu \geq 1$ . Thus there exists  $N$ , such that  $\sum_{k=1}^N |t_k| \chi_{A_k} \geq 3/4$ . Put  $A = \sqcup_{k=1}^N A_k$  and  $C = \max_{1 \leq k \leq N} |t_k| = \max_{x \in A} |\phi_1(x)|$ .

By the Egorov's Theorem 13.8 there is  $E \subset A$  such that  $\mu(E) < 1/(4C)$  and  $\phi_n \Rightarrow 0$  on  $B = A \setminus E$ . Then

$$\int_B |\phi_1| d\mu = \int_A |\phi_1| d\mu - \int_E |\phi_1| d\mu \geq \frac{3}{4} - \frac{1}{4C} \cdot C = \frac{1}{2}.$$

Since

$$\int_B |\phi_n| d\mu - \int_B |\phi_{n+1}| d\mu \leq d_1(\phi_n, \phi_{n+1}) \leq \frac{1}{2^{n+2}}$$

we get

$$\int_B |\phi_n| d\mu \geq \int_B |\phi_1| d\mu - \sum_{k=1}^{n-1} \left| \int_B |\phi_k| d\mu - \int_B |\phi_{k+1}| d\mu \right| \geq \frac{1}{2} - \sum_{k=1}^{n-1} \frac{1}{2^{k+2}} > \frac{1}{4}.$$

But this contradicts to uniform convergence  $\phi_n \Rightarrow 0$  on  $B$ .  $\square$

**Corollary 13.24.** *The functional  $I_A(f) = \int_A f(x) d\mu(x)$ , defined on any  $A \in \mathcal{L}$  on the space of simple functions  $S(X)$  can be extended by continuity to the functional on  $L_1(X, \mu)$ .*

**Definition 13.25.** For an arbitrary summable  $f \in L_1(X)$ , we define the *Lebesgue integral*

$$\int_A f d\mu = \lim_{n \rightarrow \infty} \int_A f_n d\mu,$$

where the Cauchy sequence  $f_n$  of summable simple functions converges to  $f$  a.e.

**Theorem 13.26.** (i)  $L_1(X)$  is a linear space.

(ii) For any set  $A \subset X$  the correspondence  $f \mapsto \int_A f d\mu$  is a linear functional on  $L_1(X)$ .

(iii) For any  $f \in L_1(X)$  the value  $\nu(A) = \int_A f d\mu$  is a charge.

(iv)  $d_1(f, g) = \int_A |f - g| d\mu$  is a distance on  $L_1(X)$ .

*Proof.* The proof is follows from Prop. 13.18 and continuity of extension.  $\square$

*Remark 13.27.* Note that we build  $L_1(X)$  as a completion of  $S(X)$  with respect to the distance  $d_1$ . Its realisation as equivalence classes of measurable functions on  $X$  is somehow secondary to this.

**13.3. Properties of the Lebesgue Integral.** We will state some results on integral and limit, in particular we state that  $L_1(X)$  is closed in  $d_1$  metric.

**Theorem 13.28** (Lebesgue on dominated convergence). *Let  $(f_n)$  be a sequence of  $\mu$ -summable functions on  $X$ , and there is  $\phi \in L_1(X)$  such that  $|f_n(x)| \leq \phi(x)$  for all  $x \in X$ ,  $n \in \mathbb{N}$ .*

*If  $f_n \xrightarrow{a.e.} f$ , then  $f \in L_1(X)$  and for any measurable  $A$ :*

$$\lim_{n \rightarrow \infty} \int_A f_n d\mu = \int_A f d\mu.$$

*Proof.* For any measurable  $A$  the expression  $\nu(A) = \int_A \phi d\mu$  defines a finite measure on  $X$  due to non-negativeness of  $\phi$  and Thm. 13.26.

**Lemma 13.29.** *If  $g$  is measurable and bounded then  $f = \phi g$  is  $\mu$ -summable and for any  $\mu$ -measurable set  $A$  we have*

$$\int_A f \, d\mu = \int_A g \, d\nu.$$

*Proof of the Lemma.* Let  $M$  be the set of all  $g$  such that the Lemma is true.  $M$  includes any indicator functions  $g = \chi_B$  of a measurable  $B$ :

$$\int_A f \, d\mu = \int_A \phi \chi_B \, d\mu = \int_{A \cap B} \phi \, d\mu = \nu(A \cap B) = \int_A g \, d\nu.$$

Thus  $M$  contains also finite linear combinations of indicators. For any  $n \in \mathbb{N}$  and a bounded  $g$  two functions  $g_-(x) = \frac{1}{n}[\lfloor ng(x) \rfloor]$  and  $g_+(x) = g_- + \frac{1}{n}$  are finite linear combinations of indicators and are in  $M$ . Since  $g_-(x) \leq g(x) \leq g_+(x)$  we have

$$\int_A g_-(x) \, d\nu = \int_A \phi g_- \, d\mu \leq \int_A \phi g(x) \, d\mu \leq \int_A \phi g_+(x) \, d\mu = \int_A g_+(x) \, d\nu.$$

By squeeze rule for  $n \rightarrow \infty$  we have the middle term tends to  $\int_A g \, d\nu$ , that is  $g \in M$ .  $\square$

For the proof of the theorem define:

$$g_n(x) = \begin{cases} f_n(x)/\phi(x), & \text{if } \phi(x) \neq 0, \\ 0, & \text{if } \phi(x) = 0, \end{cases}$$

$$g(x) = \begin{cases} f(x)/\phi(x), & \text{if } \phi(x) \neq 0, \\ 0, & \text{if } \phi(x) = 0. \end{cases}$$

Then  $g_n$  is bounded by 1 and  $g_n \xrightarrow{a.e.} g$ . To show the theorem it will be enough to show  $\lim_{n \rightarrow \infty} \int_A g_n \, d\nu = \int_A g \, d\nu$ . For the uniformly bounded functions on the finite measure set this can be derived from the Egorov's Thm. 13.8, see an example of this in the proof of Lemma 13.23.  $\square$

**Exercise 13.30.** Give an example of  $f_n \xrightarrow{a.e.} f$  such that  $\int_X f_n \, d\mu \neq \int_X f \, d\mu$ .

**Exercise 13.31** (Chebyshev's inequality). Show that: if  $f$  is non-negative and summable, then

$$(13.3) \quad \mu\{x \in X : f(x) > c\} < \frac{1}{c} \int_X f \, d\mu.$$

**Theorem 13.32** (B. Levi's, on monotone convergence). *Let  $(f_n)$  be monotonically increasing sequence  $\mu$ -summable functions on  $X$ . Define  $f(x) = \lim_{n \rightarrow \infty} f_n(x)$  (allowing the value  $+\infty$ ).*

- (i) *If all integrals  $\int_X f_n \, d\mu$  are bounded by the same value then  $f$  is summable and  $\int_X f \, d\mu = \lim_{n \rightarrow \infty} \int_X f_n \, d\mu$ .*
- (ii) *If  $\lim_{n \rightarrow \infty} \int_X f_n \, d\mu = +\infty$  then function  $f$  is not summable.*

*Proof.* Replacing  $f_n$  by  $f_n - f_1$  and  $f$  by  $f - f_1$  we can assume  $f_n \geq 0$  and  $f \geq 0$ . Let  $E$  be the set where  $f$  is infinite, then  $E = \bigcap_N \bigcup_n E_{Nn}$ , where  $E_{Nn} = \{x \in X : f_n(x) \geq N\}$ . We have  $\int_{E_{Nn}} f_n \, d\mu > N\mu(E_{Nn})$ , but  $\int_{E_{Nn}} f_n \, d\mu \leq \int_C f_n \, d\mu \leq C$  then  $\mu(E_{Nn}) \leq C/N$  by Chebyshev's inequality. Thus  $\mu(E) = \lim_{N \rightarrow \infty} \lim_{n \rightarrow \infty} \mu(E_{Nn}) = 0$ .

Thus  $f$  is finite a.e.

**Lemma 13.33.** *Let  $f$  be a measurable non-negative function attaining only finite values.  $f$  is summable if and only if  $\sup \int_A f \, d\mu < \infty$ , where the supremum is taken over all finite-measure set  $A$  such that  $f$  is bounded on  $A$ .*

*Proof of the Lemma.* If  $f$  is summable then for any set  $A \subset X$  we have  $\int_A f \, d\mu \leq \int_X f \, d\mu < \infty$ , thus the supremum is finite.

Let  $\sup \int_A f \, d\mu = M < \infty$ , define  $B = \{x \in X : f(x) = 0\}$  and  $A_k = \{x \in X : 2^k \leq f(x) < 2^{k+1}, k \in \mathbb{Z}\}$  we have  $\mu(A_k) < M/2^k$  and  $X = B \sqcup (\bigsqcup_{k=0}^{\infty} A_k)$ . Define

$$g(x) = \begin{cases} 2^k, & \text{if } x \in A_k, \\ 0, & \text{if } x \in B, \end{cases}$$

$$f_n(x) = \begin{cases} f(x), & \text{if } x \in \bigsqcup_{k=0}^n A_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then  $g(x) \leq f(x) < 2g(x)$  and  $f_n(x) < 2g(x)$ . Function  $g$  is a simple function, its summability follows from the estimation  $\int_{\bigsqcup_{k=0}^n A_k} g \, d\mu \leq \int_{\bigsqcup_{k=0}^n A_k} f \, d\mu \leq M$ . Then  $f_n \xrightarrow{\text{a.e.}} f$  and we use the Lebesgue Thm. 13.28 on dominated convergence.  $\square$

Let  $A$  be a finite measure set such that  $f$  is bounded, then

$$\int_A f \, d\mu = \lim_{n \rightarrow \infty} \int_A f_n \, d\mu \leq \lim_{n \rightarrow \infty} \int_X f_n \, d\mu \leq C.$$

This show summability of  $f$  by the previous Lemma. The rest of statement and the second part follows from the Lebesgue Thm. 13.28 on dominated convergence.  $\square$

Now we can extend this result dropping the monotonicity assumption.

**Lemma 13.34 (Fatou).** *If a sequence  $(f_n)$  of  $\mu$ -summable non-negative functions is such that:*

- $\int_X f_n \, d\mu \leq C$  for all  $n$ ;
- $f_n \xrightarrow{\text{a.e.}} f$ ,

*then  $f$  is  $\mu$ -summable and  $\int_X f \, d\mu \leq C$ .*

*Proof of Fatou's Lemma.* Let us replace the limit  $f_n \rightarrow f$  by two monotonic limits:

$$g_{kn}(x) = \min(f_n(x), \dots, f_{n+k}(x))$$

$$g_n(x) = \lim_{k \rightarrow \infty} g_{kn}(x).$$

Then  $\lim_{n \rightarrow \infty} g_n(x) = f(x)$  a.e. From monotonicity of integral we get  $\int_X g_n \, d\mu \leq C$ . Then Levi's Thm. 13.32 implies that  $f$  is summable and  $\int_X f \, d\mu \leq C$ .  $\square$

*Remark 13.35.* Note that the price for dropping monotonicity from Thm. 13.32 to Lem. 13.34 is that the limit  $\int_X f_n \, d\mu \rightarrow \int_X f \, d\mu$  may not hold any more.

**Exercise 13.36.** Give an example such that under the Fatou's lemma condition we get  $\lim_{n \rightarrow \infty} \int_X f_n \, d\mu \neq \int_X f \, d\mu$ .

Now we can show that  $L_1(X)$  is complete:

**Theorem 13.37.**  $L_1(X)$  is a Banach space.

*Proof.* It is clear that the distance function  $d_1$  indeed define a norm  $\|f\|_1 = d_1(f, 0)$ . We only need to demonstrate the completeness. Take a Cauchy sequence  $(f_n)$  and building a subsequence if necessary, assume that its *quickly convergent* that is  $d_1(f_n, f_{n+1}) \leq 1/2^k$ . Put  $\phi_1 = f_1$  and  $\phi_n = f_n - f_{n-1}$  for  $n > 1$ . The sequence  $\psi_n(x) = \sum_{k=1}^n |\phi_k(x)|$  is monotonic, integrals  $\int_X \psi_n \, d\mu$  are bounded by the same constant. Thus by the B. Levi's Thm. 13.32 and its proof  $\psi_n \rightarrow \psi$  for a summable essentially bounded function  $\psi$ . Therefore the series  $\sum \phi_k(x)$  converges as well to a function  $f$ . But this means that  $f_n \xrightarrow{\text{a.e.}} f$ . We also notice  $|f_n(x)| \leq |\psi(x)|$ . Thus by the Lebesgue Thm. 13.28 on dominated convergence  $\lim_{n \rightarrow \infty} \int_X |f_n - f| \, d\mu = 0$ . That is  $f_n \rightarrow f$  in the norm of  $L_1(X)$ .  $\square$

The next important property of the Lebesgue integral is its *absolute continuity*.

**Theorem 13.38** (Absolute continuity of Lebesgue integral). *Let  $f \in L_1(X)$ . Then for any  $\varepsilon > 0$  there is a  $\delta > 0$  such that  $|\int_A f \, d\mu| < \varepsilon$  if  $\mu(A) < \delta$ .*

*Proof.* If  $f$  is essentially bounded by  $M$ , then it is enough to set  $\delta = \varepsilon/M$ . In general let:

$$\begin{aligned} A_n &= \{x \in X : n \leq |f(x)| < n + 1\}, \\ B_n &= \sqcup_0^n A_k, \\ C_n &= X \setminus B_n. \end{aligned}$$

Then  $\int_X |f| \, d\mu = \sum_0^\infty \int_{A_k} |f| \, d\mu$ , thus there is an  $N$  such that  $\sum_N^\infty \int_{A_k} |f| \, d\mu = \int_{C_N} |f| \, d\mu < \varepsilon/2$ . Now put  $\delta = \frac{\varepsilon}{2N+2}$ , then for any  $A \subset X$  with  $\mu(A) < \delta$ :

$$\left| \int_A f \, d\mu \right| \leq \int_A |f| \, d\mu = \int_{A \cap B_N} |f| \, d\mu + \int_{A \cap C_N} |f| \, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

**13.4. Integration on Product Measures.** We will study in some details product measures introduced in Defn. 12.34. We start from the following

**Theorem 13.39.** *Let  $X$  and  $Y$  be spaces, and let  $S$  and  $T$  be semirings on  $X$  and  $Y$  respectively and  $\mu$  and  $\nu$  be measures on  $S$  and  $T$  respectively. If  $\mu$  and  $\nu$  are  $\sigma$ -additive, then the product measure  $\nu \times \mu$  from Defn. 12.34 is  $\sigma$ -additive as well.*

*Proof.* For any  $C = A \times B \in S \times T$  let us define  $f_C(x) = \chi_A(x)\nu(B)$ . Then

$$(\mu \times \nu)(C) = \mu(A)\nu(B) = \int_X f_C \, d\mu.$$

If the same set  $C$  has a representation  $C = \sqcup_k C_k$  for  $C_k \in S \times T$ , then  $\sigma$ -additivity of  $\nu$  implies  $f_C = \sum_k f_{C_k}$ . By the Lebesgue theorem 13.28 on dominated convergence:

$$\int_X f_C \, d\mu = \sum_k \int_X f_{C_k} \, d\mu.$$

Thus

$$(\mu \times \nu)(C) = \sum_k (\mu \times \nu)(C_k).$$

□

The above correspondence  $C \mapsto f_C$  can be extended to the ring  $R(S \times T)$  generated by  $S \times T$  by the formula:

$$f_C = \sum_k f_{C_k}, \quad \text{for } C = \sqcup_k C_k \in R(S \times T).$$

We obtain:

$$\|f_{C_1} - f_{C_2}\|_{L_1} \leq (\mu \times \nu)(C_1 \Delta C_2)$$

because from the representation  $C_1 = A_1 \sqcup B$  and  $C_2 = A_2 \sqcup B$ , where  $B = C_1 \cap C_2$  one can see that  $f_{C_1} - f_{C_2} = f_{A_1} - f_{A_2}$ ,  $f_{C_1 \Delta C_2} = f_{A_1} + f_{A_2}$  together with  $|f_{A_1} - f_{A_2}| \leq f_{A_1} + f_{A_2}$  for non-negative functions..

Thus the map  $C \mapsto f_C$  can be extended to the map of  $\sigma$ -algebra  $L(X \times Y)$  of  $\mu \times \nu$ -measurable set to  $L_1(X)$  by the formula  $f_{\lim_n C_n} = \lim_n f_{C_n}$ .

**Exercise 13.40.** Describe topologies where two limits from the last formula are taken.

**Lemma 13.41.** *Let  $C \in L(X \times Y)$ . For almost every  $x \in X$  the set  $C_x = \{y \in Y : (x, y) \in C\}$  is  $\nu$ -measurable and  $\nu(C_x) = f_C(x)$ .*

*Proof.* For sets from the ring  $R(S \times T)$  it is true by the definition. If  $C^{(n)}$  is a monotonic sequence of sets, then  $\nu(\lim_n C_x^{(n)}) = \lim_n \nu(C_x^{(n)})$  by  $\sigma$ -additivity of measures. Thus the property  $\nu(C_x) = f_x(C)$  is preserved by monotonic limits. The following result of the separate interest:

**Lemma 13.42.** *Any measurable set is up to a set of zero measure can be received from elementary sets by two monotonic limits.*

*Proof of Lem. 13.42.* Let  $C$  be a measurable set, put  $C_n \in R(S \times T)$  to approximate  $C$  up to  $2^{-n}$  in  $\mu \times \nu$ . Let  $\tilde{C} = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} C_{n+k}$ , then

$$(\mu \times \nu)(C \setminus \bigcup_{k=1}^{\infty} C_{n+k}) = 0 \quad \text{and} \quad (\mu \times \nu)(\bigcup_{k=1}^{\infty} C_{n+k} \setminus C) = 2^{1-n}.$$

Then  $(\mu \times \nu)(\tilde{C} \Delta C) \leq 2^{1-n}$  for any  $n \in \mathbb{N}$ .  $\square$

Coming back to Lem. 13.41 we notice that (in the above notations)  $f_C = f_{\tilde{C}}$  almost everywhere. Then:

$$f_C(x) = f_{\tilde{C}}(x) = \nu(\tilde{C}_x) = \nu(C_x).$$

$\square$

**Theorem 13.43.** *Let  $\mu$  and  $\nu$  are  $\sigma$ -finite measures and  $C$  be a  $\mu \times \nu$  measurable set  $X \times Y$ . We define  $C_x = \{y \in Y : (x, y) \in C\}$ . Then for  $\mu$ -almost every  $x \in X$  the set  $C_x$  is  $\nu$ -measurable, function  $f_C(x) = \nu(C_x)$  is  $\mu$ -measurable and*

$$(13.4) \quad (\mu \times \nu)(C) = \int_X f_C d\mu,$$

where both parts may have the value  $+\infty$ .

*Proof.* If  $C$  has a finite measure, then the statement is reduced to Lem. 13.41 and a passage to limit in (13.4).

If  $C$  has an infinite measure, then there exists a sequence of  $C_n \subset C$ , such that  $\bigcup_n C_n = C$  and  $(\mu \times \nu)(C_n) \rightarrow \infty$ . Then  $f_C(x) = \lim_n f_{C_n}(x)$  and

$$\int_X f_{C_n} d\mu = (\mu \times \nu)(C_n) \rightarrow +\infty.$$

Thus  $f_C$  is measurable and non-summable.  $\square$

This theorem justify the well-known technique to calculation of areas (volumes) as integrals of length (areas) of the sections.

*Remark 13.44.* (i) The role of spaces  $X$  and  $Y$  in Theorem 13.43 is symmetric, thus we can swap them in the conclusion.

(ii) The Theorem 13.43 can be extended to any finite number of measure spaces. For the case of three spaces  $(X, \mu)$ ,  $(Y, \nu)$ ,  $(Z, \lambda)$  we have:

$$(13.5) \quad (\mu \times \nu \times \lambda)(C) = \int_{X \times Y} \lambda(C_{xy}) d(\mu \times \nu)(x, y) = \int_Z (\mu \times \nu)(C_z) d\lambda(z),$$

where

$$\begin{aligned} C_{xy} &= \{z \in Z : (x, y, z) \in C\}, \\ C_z &= \{(x, y) \in X \times Y : (x, y, z) \in C\}. \end{aligned}$$

**Theorem 13.45 (Fubini).** *Let  $f(x, y)$  be a summable function on the product of spaces  $(X, \mu)$  and  $(Y, \nu)$ . Then:*

- (i) *For  $\mu$ -almost every  $x \in X$  the function  $f(x, y)$  is summable on  $Y$  and  $f_Y(x) = \int_Y f(x, y) d\nu(y)$  is a  $\mu$ -summable on  $X$ .*
- (ii) *For  $\nu$ -almost every  $y \in Y$  the function  $f(x, y)$  is summable on  $X$  and  $f_X(y) = \int_X f(x, y) d\mu(x)$  is a  $\nu$ -summable on  $Y$ .*

(iii) *There are the identities:*

$$(13.6) \quad \begin{aligned} \int_{X \times Y} f(x, y) \, d(\mu \times \nu)(x, y) &= \int_X \left( \int_Y f(x, y) \, d\nu(y) \right) d\mu(x) \\ &= \int_Y \left( \int_X f(x, y) \, d\mu(x) \right) d\nu(y). \end{aligned}$$

(iv) *For a non-negative functions the existence of any repeated integral in (13.6) implies summability of  $f$  on  $X \times Y$ .*

*Proof.* From the decomposition  $f = f_+ - f_-$  we can reduce our consideration to non-negative functions. Let us consider the product of three spaces  $(X, \mu)$ ,  $(Y, \nu)$ ,  $(\mathbb{R}, \lambda)$ , with  $\lambda = dz$  being the Lebesgue measure on  $\mathbb{R}$ . Define

$$C = \{(x, y, z) \in X \times Y \times \mathbb{R} : 0 \leq z \leq f(x, y)\}.$$

Using the relation (13.5) we get:

$$\begin{aligned} C_{xy} &= \{z \in \mathbb{R} : 0 \leq z \leq f(x, y)\}, & \lambda(C_{xy}) &= f(x, y) \\ C_x &= \{(y, z) \in Y \times \mathbb{R} : 0 \leq z \leq f(x, y)\}, & (\nu \times \lambda)(C_x) &= \int_Y f(x, y) \, d\nu(y). \end{aligned}$$

the theorem follows from those relations. □

- Exercise 13.46.**
- Show that the first three conclusions of the Fubini Theorem may fail if  $f$  is not summable.
  - Show that the fourth conclusion of the Fubini Theorem may fail if  $f$  has values of different signs.

### 13.5. Absolute Continuity of Measures.

**Definition 13.47.** Let  $X$  be a set with  $\sigma$ -algebra  $\mathcal{R}$  and  $\sigma$ -finite measure  $\mu$  and finite charge  $\nu$  on  $\mathcal{R}$ . The charge  $\nu$  is *absolutely continuous* with respect to  $\mu$  if  $\mu(A) = 0$  for  $A \in \mathcal{R}$  implies  $\nu(A) = 0$ . Two charges  $\nu_1$  and  $\nu_2$  are *equivalent* if two conditions  $|\nu_1|(A) = 0$  and  $|\nu_2|(A) = 0$  are equivalent.

**Theorem 13.48 (Radon–Nikodym).** *Any charge  $\nu$  which absolutely continuous with respect to a measure  $\mu$  have the form*

$$\nu(A) = \int_A f \, d\mu,$$

where  $f$  is a function from  $L_1$ . The function  $f \in L_1$  is uniquely defined by the charge  $\nu$ .

*Proof.* Again, this is a sketch. First we will assume that  $\nu$  is a measure. Let  $D$  be the collection of measurable functions  $g : X \rightarrow [0, \infty)$  such that

$$\int_E g \, d\mu \leq \nu(E) \quad (E \in \mathcal{L}).$$

Let  $\alpha = \sup_{g \in D} \int_X g \, d\mu \leq \nu(X) < \infty$ . So we can find a sequence  $(g_n)$  in  $D$  with  $\int_X g_n \, d\mu \rightarrow \alpha$ .

We define  $f_0(x) = \sup_n g_n(x)$ . We can show that  $f_0 = \infty$  only on a set of  $\mu$ -measure zero, so if we adjust  $f_0$  on this set, we get a measurable function  $f : X \rightarrow [0, \infty)$ . There is now a long argument to show that  $f$  is as required.

If  $\nu$  is a charge, we can find  $f$  by applying the previous operation to the measures  $\nu_+$  and  $\nu_-$  (as it is easy to verify that  $\nu_+, \nu_- \ll \mu$ ).

We show that  $f$  is essentially unique. If  $g$  is another function inducing  $\nu$ , then

$$\int_E f - g \, d\mu = \nu(E) - \nu(E) = 0 \quad (E \in \mathcal{L}).$$

Let  $E = \{x \in X : f(x) - g(x) \geq 0\}$ , so as  $f - g$  is measurable,  $E \in \mathcal{L}$ . Then  $\int_E f - g \, d\mu = 0$  and  $f - g \geq 0$  on  $E$ , so by our result from integration theory, we have that  $f - g = 0$  almost everywhere on  $E$ . Similarly, if  $F = \{x \in X : f(x) - g(x) \leq 0\}$ , then  $F \in \mathcal{L}$  and  $f - g = 0$  almost everywhere on  $F$ . As  $E \cup F = X$ , we conclude that  $f = g$  almost everywhere.  $\square$

**Corollary 13.49.** *Let  $\mu$  be a measure on  $X$ ,  $\nu$  be a finite charge, which is absolutely continuous with respect to  $\mu$ . For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that  $\mu(A) < \delta$  implies  $|\nu|(A) < \varepsilon$ .*

*Proof.* By the Radon–Nikodym theorem there is a function  $f \in L_1(X, \mu)$  such that  $\nu(A) = \int_A f \, d\mu$ . Then  $|\nu|(A) = \int_A |f| \, d\mu$  and we get the statement from Theorem 13.38 on absolute continuity of the Lebesgue integral.  $\square$

## 14. FUNCTIONAL SPACES

In this section we describe various Banach spaces of functions on sets with measure.

**14.1. Integrable Functions.** Let  $(X, \mathcal{L}, \mu)$  be a measure space. For  $1 \leq p < \infty$ , we define  $\mathcal{L}_p(\mu)$  to be the space of measurable functions  $f : X \rightarrow \mathbb{K}$  such that

$$\int_X |f|^p \, d\mu < \infty.$$

We define  $\|\cdot\|_p : \mathcal{L}_p(\mu) \rightarrow [0, \infty)$  by

$$\|f\|_p = \left( \int_X |f|^p \, d\mu \right)^{1/p} \quad (f \in \mathcal{L}_p(\mu)).$$

Notice that if  $f = 0$  almost everywhere, then  $|f|^p = 0$  almost everywhere, and so  $\|f\|_p = 0$ . However, there can be non-zero functions such that  $f = 0$  almost everywhere. So  $\|\cdot\|_p$  is *not* a norm on  $\mathcal{L}_p(\mu)$ .

**Lemma 14.1.** *Let  $1 < p < \infty$ , let  $q \in (1, \infty)$  be such that  $1/p + 1/q = 1$ . For  $f \in \mathcal{L}_p(\mu)$  and  $g \in L^q(\mu)$ , we have that  $fg$  is integrable, and*

$$\int_X |fg| \, d\mu \leq \|f\|_p \|g\|_q.$$

*Proof.* Recall that we know from Lem. 11.2 that

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q} \quad (a, b \in \mathbb{K}).$$

Now we follow the steps in proof of Prop. 11.3. Define measurable functions  $a, b : X \rightarrow \mathbb{K}$  by setting

$$a(x) = \frac{f(x)}{\|f\|_p}, \quad b(x) = \frac{g(x)}{\|g\|_q} \quad (x \in X).$$

So we have that

$$|a(x)b(x)| \leq \frac{|f(x)|^p}{p \|f\|_p^p} + \frac{|g(x)|^q}{q \|g\|_q^q} \quad (x \in X).$$

By integrating, we see that

$$\int_X |ab| \, d\mu \leq \frac{1}{p \|f\|_p^p} \int_X |f|^p \, d\mu + \frac{1}{q \|g\|_q^q} \int_X |g|^q \, d\mu = \frac{1}{p} + \frac{1}{q} = 1.$$

Hence, by the definition of  $a$  and  $b$ ,

$$\int_X |fg| \leq \|f\|_p \|g\|_q,$$



as required. □

**Lemma 14.2.** *Let  $f, g \in \mathcal{L}_p(\mu)$  and let  $\alpha \in \mathbb{K}$ . Then:*

- (i)  $\|\alpha f\|_p = |\alpha| \|f\|_p$ ;
- (ii)  $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ .

*In particular,  $\mathcal{L}_p$  is a vector space.*

*Proof.* Part 14.2(i) is easy. For 14.2(ii), we need a version of Minkowski's Inequality, which will follow from the previous lemma. We essentially repeat the proof of Prop. 11.4.

Notice that the  $p = 1$  case is easy, so suppose that  $1 < p < \infty$ . We have that

$$\begin{aligned} \int_X |f + g|^p \, d\mu &= \int_X |f + g|^{p-1} |f + g| \, d\mu \\ &\leq \int_X |f + g|^{p-1} (|f| + |g|) \, d\mu \\ &= \int_X |f + g|^{p-1} |f| \, d\mu + \int_X |f + g|^{p-1} |g| \, d\mu. \end{aligned}$$

Applying the lemma, this is

$$\leq \|f\|_p \left( \int_X |f + g|^{q(p-1)} \, d\mu \right)^{1/q} + \|g\|_p \left( \int_X |f + g|^{q(p-1)} \, d\mu \right)^{1/q}.$$

As  $q(p - 1) = p$ , we see that

$$\|f + g\|_p^p \leq (\|f\|_p + \|g\|_p) \|f + g\|_p^{p/q}.$$

As  $p - p/q = 1$ , we conclude that

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p,$$

as required.

In particular, if  $f, g \in \mathcal{L}_p(\mu)$  then  $\alpha f + g \in \mathcal{L}_p(\mu)$ , showing that  $\mathcal{L}_p(\mu)$  is a vector space. □

We define an equivalence relation  $\sim$  on the space of measurable functions by setting  $f \sim g$  if and only if  $f = g$  almost everywhere. We can check that  $\sim$  is an equivalence relation (the slightly non-trivial part is that  $\sim$  is transitive).

**Proposition 14.3.** *For  $1 \leq p < \infty$ , the collection of equivalence classes  $\mathcal{L}_p(\mu)/\sim$  is a vector space, and  $\|\cdot\|_p$  is a well-defined norm on  $\mathcal{L}_p(\mu)/\sim$ .*

*Proof.* We need to show that addition, and scalar multiplication, are well-defined on  $\mathcal{L}_p(\mu)/\sim$ . Let  $\alpha \in \mathbb{K}$  and  $f_1, f_2, g_1, g_2 \in \mathcal{L}_p(\mu)$  with  $f_1 \sim f_2$  and  $g_1 \sim g_2$ . Then it's easy to see that  $\alpha f_1 + g_1 \sim \alpha f_2 + g_2$ ; but this is all that's required!

If  $f \sim g$  then  $|f|^p = |g|^p$  almost everywhere, and so  $\|f\|_p = \|g\|_p$ . So  $\|\cdot\|_p$  is well-defined on equivalence classes. In particular, if  $f \sim 0$  then  $\|f\|_p = 0$ . Conversely, if  $\|f\|_p = 0$  then  $\int_X |f|^p \, d\mu = 0$ , so as  $|f|^p$  is a positive function, we must have that  $|f|^p = 0$  almost everywhere. Hence  $f = 0$  almost everywhere, so  $f \sim 0$ . That is,

$$\{f \in \mathcal{L}_p(\mu) : f \sim 0\} = \left\{ f \in \mathcal{L}_p(\mu) : \|f\|_p = 0 \right\}.$$

It follows from the above lemma that this is a subspace of  $\mathcal{L}_p(\mu)$ .

The above lemma now immediately shows that  $\|\cdot\|_p$  is a norm on  $\mathcal{L}_p(\mu)/\sim$ . □

**Definition 14.4.** We write  $L_p(\mu)$  for the normed space  $(\mathcal{L}_p(\mu)/\sim, \|\cdot\|_p)$ .

We will abuse notation and continue to write members of  $L_p(\mu)$  as functions. Really they are equivalence classes, and so care must be taken when dealing with  $L_p(\mu)$ . For example, if  $f \in L_p(\mu)$ , it does *not* make sense to talk about the value of  $f$  at a point.

**Theorem 14.5.** *Let  $(f_n)$  be a Cauchy sequence in  $L_p(\mu)$ . There exists  $f \in L_p(\mu)$  with  $\|f_n - f\|_p \rightarrow 0$ . In fact, we can find a subsequence  $(n_k)$  such that  $f_{n_k} \rightarrow f$  pointwise, almost everywhere.*

*Proof.* We shall show this carefully for  $L_\infty(\mu)$  below. This case is similar.  $\square$

**Corollary 14.6.**  $L_p(\mu)$  is a Banach space.

**Proposition 14.7.** *Let  $(X, L, \mu)$  be a measure space, and let  $1 \leq p < \infty$ . We can define a map  $\Phi : L_q(\mu) \rightarrow L_p(\mu)^*$  by setting  $\Phi(f) = F$ , for  $f \in L_q(\mu)$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , where*

$$F : L_p(\mu) \rightarrow \mathbb{K}, \quad g \mapsto \int_X fg \, d\mu \quad (g \in L_p(\mu)).$$

*Proof.* This proof very similar to proof of Thm. 11.11. For  $f \in L_p(\mu)$  and  $g \in L_q(\mu)$ , it follows by the above lemma (Holder's Inequality), that  $fg$  is integrable, and

$$\left| \int_X fg \, d\mu \right| \leq \int_X |fg| \, d\mu \leq \|f\|_q \|g\|_p.$$

Let  $f_1, f_2 \in L_p(\mu)$  and  $g_1, g_2 \in L_q(\mu)$  with  $f_1 \sim f_2$  and  $g_1 \sim g_2$ . Then  $f_1 g_1 = f_2 g_1$  almost everywhere and  $f_2 g_1 = f_2 g_2$  almost everywhere, so  $f_1 g_1 = f_2 g_2$  almost everywhere, and hence

$$\int_X f_1 g_1 \, d\mu = \int_X f_2 g_2 \, d\mu.$$

So  $\Phi$  is well-defined.

Clearly  $\Phi$  is linear, and we have shown that  $\|\Phi(f)\| \leq \|f\|_q$ .

Let  $f \in L_q(\mu)$  and define  $g : X \rightarrow \mathbb{K}$  by

$$g(x) = \begin{cases} \overline{f(x)} |f(x)|^{q-2} & : f(x) \neq 0, \\ 0 & : f(x) = 0. \end{cases}$$

Then  $|g(x)| = |f(x)|^{q-1}$  for all  $x \in X$ , and so

$$\int_X |g|^p \, d\mu = \int_X |f|^{p(q-1)} \, d\mu = \int_X |f|^q \, d\mu,$$

so  $\|g\|_p = \|f\|_q^{q/p}$ , and so, in particular,  $g \in L_p(\mu)$ . Let  $F = \Phi(f)$ , so that

$$F(g) = \int_X fg \, d\mu = \int_X |f|^q \, d\mu = \|f\|_q^q.$$

Thus  $\|F\| \geq \|f\|_q^q / \|g\|_p = \|f\|_q$ . So we conclude that  $\|F\| = \|f\|_q$ , showing that  $\Phi$  is an isometry.  $\square$

**Proposition 14.8.** *Let  $(X, L, \mu)$  be a finite measure space, let  $1 \leq p < \infty$ , and let  $F \in L_p(\mu)^*$ . Then there exists  $g \in L_q(\mu)$ ,  $\frac{1}{p} + \frac{1}{q} = 1$  such that*

$$F(f) = \int_X fg \, d\mu \quad (f \in L_p(\mu)).$$

*Proof.* As  $\mu(X) < \infty$ , for  $E \in L$ , we have that  $\|\chi_E\|_p = \mu(E)^{1/p} < \infty$ . So  $\chi_E \in L_p(\mu)$ , and hence we can define

$$\nu(E) = F(\chi_E) \quad (E \in L).$$

We proceed to show that  $\nu$  is a signed (or complex) measure. Then we can apply the Radon-Nikodym Theorem 13.48 to find a function  $g : X \rightarrow \mathbb{K}$  such that

$$F(\chi_E) = \nu(E) = \int_E g \, d\mu \quad (E \in L).$$

There is then a long argument to show that  $g \in L^q(\mu)$  and that

$$\int_X fg \, d\mu = F(f)$$

for all  $f \in L_p(\mu)$ , and not just for  $f = \chi_E$ . □

**Proposition 14.9.** *For  $1 < p < \infty$ , we have that  $L_p(\mu)^* = L_q(\mu)$  isometrically, under the identification of the above results.*

*Remark 14.10.* Note that  $L_\infty^*$  is not isomorphic to  $L_1$ , except finite-dimensional situation. Moreover if  $\mu$  is not a point measure  $L_1$  is not a dual to any Banach space.

### 14.2. Dense Subspaces in $L_p$ .

**Proposition 14.11.** *Let  $(X, L, \mu)$  be a finite measure space, and let  $1 \leq p < \infty$ . Then the collection of simple functions is dense in  $L_p(\mu)$ .*

*Proof.* Let  $f \in L_p(\mu)$ , and suppose for now that  $f \geq 0$ . For each  $n \in \mathbb{N}$ , let

$$f_n = \max(n, 2^{-n} \lfloor 2^n f \rfloor).$$

Then each  $f_n$  is simple,  $f_n \uparrow f$ , and  $|f_n - f|^p \rightarrow 0$  pointwise. For each  $n$ , we have that

$$0 \leq f_n \leq f \implies 0 \leq f - f_n \leq f,$$

so that  $|f - f_n|^p \leq |f|^p$  for all  $n$ . As  $\int |f|^p \, d\mu < \infty$ , we can apply the Dominated Convergence Theorem to see that

$$\lim_n \int_X |f_n - f|^p \, d\mu = 0,$$

that is,  $\|f_n - f\|_p \rightarrow 0$ .

The general case follows by taking positive and negative parts, and if  $\mathbb{K} = \mathbb{C}$ , by taking real and imaginary parts first. □

**Proposition 14.12.** *Let  $([0, 1], L, \mu)$  be the restriction of Lebesgue measure to  $[0, 1]$ . We often write  $L_p([0, 1])$  instead of  $L_p(\mu)$ . For  $1 < p < \infty$ , we have that  $C_{\mathbb{K}}([0, 1])$  is dense in  $L_p([0, 1])$ .*

*Proof.* As  $[0, 1]$  is a finite measure space, and each member of  $C_{\mathbb{K}}([0, 1])$  is bounded, it is easy to see that each  $f \in C_{\mathbb{K}}([0, 1])$  is such that  $\|f\|_p < \infty$ . So it makes sense to regard  $C_{\mathbb{K}}([0, 1])$  as a subspace of  $L_p(\mu)$ . If  $C_{\mathbb{K}}([0, 1])$  is not dense in  $L_p(\mu)$ , then we can find a non-zero  $F \in L_p([0, 1])^*$  with  $F(f) = 0$  for each  $f \in C_{\mathbb{K}}([0, 1])$ . This was a corollary of the Hahn-Banach theorem 11.12.

So there exists a non-zero  $g \in L_q([0, 1])$  with

$$\int_{[0,1]} fg \, d\mu = 0 \quad (f \in C_{\mathbb{K}}([0, 1])).$$

Let  $a < b$  in  $[0, 1]$ . By approximating  $\chi_{(a,b)}$  by a continuous function, we can show that  $\int_{(a,b)} g \, d\mu = \int g\chi_{(a,b)} \, d\mu = 0$ .

Suppose for now that  $\mathbb{K} = \mathbb{R}$ . Let  $A = \{x \in [0, 1] : g(x) \geq 0\} \in L$ . By the definition of the Lebesgue (outer) measure, for  $\epsilon > 0$ , there exist sequences  $(a_n)$  and  $(b_n)$  with  $A \subseteq \cup_n (a_n, b_n)$ , and  $\sum_n (b_n - a_n) \leq \mu(A) + \epsilon$ .

For each  $N$ , consider  $\cup_{n=1}^N (a_n, b_n)$ . If some  $(a_i, b_i)$  overlaps  $(a_j, b_j)$ , then we could just consider the larger interval  $(\min(a_i, a_j), \max(b_i, b_j))$ . Formally by an induction argument, we see that we can write  $\cup_{n=1}^N (a_n, b_n)$  as a finite union of *disjoint* open intervals. By linearity, it hence follows that for  $N \in \mathbb{N}$ , if we set  $B_N = \cup_{n=1}^N (a_n, b_n)$ , then

$$\int g \chi_{B_N} d\mu = \int g \chi_{(a_1, b_1) \cup \dots \cup (a_N, b_N)} d\mu = 0.$$

Let  $B = \cup_n (a_n, b_n)$ , so  $A \subseteq B$  and  $\mu(B) \leq \sum_n (b_n - a_n) \leq \mu(A) + \epsilon$ . We then have that

$$\left| \int g \chi_{B_N} d\mu - \int g \chi_B d\mu \right| = \left| \int g \chi_{B \setminus (a_1, b_1) \cup \dots \cup (a_N, b_N)} d\mu \right|.$$

We now apply the Holder inequality to get

$$\begin{aligned} \left( \int \chi_{B \setminus (a_1, b_1) \cup \dots \cup (a_N, b_N)} d\mu \right)^{1/p} \|g\|_q &= \mu(B \setminus (a_1, b_1) \cup \dots \cup (a_N, b_N))^{1/p} \|g\|_q \\ &\leq \left( \sum_{n=N+1}^{\infty} (b_n - a_n) \right)^{1/p} \|g\|_q. \end{aligned}$$

We can make this arbitrarily small by making  $N$  large. Hence we conclude that

$$\int g \chi_B d\mu = 0.$$

Then we apply Holder again to see that

$$\left| \int g \chi_A d\mu \right| = \left| \int g \chi_A d\mu - \int g \chi_B d\mu \right| = \left| \int g \chi_{B \setminus A} d\mu \right| \leq \|g\|_q \mu(B \setminus A)^{1/p} \leq \|g\|_q \epsilon^{1/p}.$$

As  $\epsilon > 0$  was arbitrary, we see that  $\int_A g d\mu = 0$ . As  $g$  is positive on  $A$ , we conclude that  $g = 0$  almost everywhere on  $A$ .

A similar argument applied to the set  $\{x \in [0, 1] : g(x) \leq 0\}$  allows us to conclude that  $g = 0$  almost everywhere. If  $\mathbb{K} = \mathbb{C}$ , then take real and imaginary parts.  $\square$

**14.3. Continuous functions.** Let  $K$  be a compact (always assumed Hausdorff) topological space.

**Definition 14.13.** The *Borel  $\sigma$ -algebra*,  $\mathcal{B}(K)$ , on  $K$ , is the  $\sigma$ -algebra generated by the open sets in  $K$  (recall what this means from Section 11.5). A member of  $\mathcal{B}(K)$  is a *Borel set*.

Notice that if  $f : K \rightarrow \mathbb{K}$  is a continuous function, then clearly  $f$  is  $\mathcal{B}(K)$ -measurable (the inverse image of an open set will be open, and hence certainly Borel). So if  $\mu : \mathcal{B}(K) \rightarrow \mathbb{K}$  is a finite charge or complex measure (for  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$  respectively), then  $f$  will be  $\mu$ -integrable (as  $f$  is bounded) and so we can define

$$\phi_\mu : C_{\mathbb{K}}(K) \rightarrow \mathbb{K}, \quad \phi_\mu(f) = \int_K f d\mu \quad (f \in C_{\mathbb{K}}(K)).$$

Clearly  $\phi_\mu$  is linear. Suppose for now that  $\mu$  is positive, so that

$$|\phi_\mu(f)| \leq \int_K |f| d\mu \leq \|f\|_\infty \mu(K) \quad (f \in C_{\mathbb{K}}(K)).$$

So  $\phi_\mu \in C_{\mathbb{K}}(K)^*$  with  $\|\phi_\mu\| \leq \mu(K)$ .

The aim of this section is to show that all of  $C_{\mathbb{K}}(K)^*$  arises in this way.

**Definition 14.14.** A measure  $\mu : B(K) \rightarrow [0, \infty)$  is *regular* if for each  $A \in B(K)$ , we have

$$\begin{aligned} \mu(A) &= \sup\{\mu(E) : E \subseteq A \text{ and } E \text{ is compact}\} \\ &= \inf\{\mu(U) : A \subseteq U \text{ and } U \text{ is open}\}. \end{aligned}$$

A charge  $\nu = \nu_+ - \nu_-$  is *regular* if  $\nu_+$  and  $\nu_-$  are regular measures. A complex measure is *regular* if its real and imaginary parts are regular.

**Example 14.15.** (i) Many common measures on the real line, e.g. the Lebesgue measure, point measures, etc., are regular.

(ii) An example of the measure  $\mu$  on  $[0, 1]$  which is not regular:

$$\mu(\emptyset) = 0, \quad \mu(\{\frac{1}{2}\}) = 1, \quad \mu(A) = +\infty,$$

for any other subset  $A \subset [0, 1]$ .

(iii) Another example of a  $\sigma$ -additive measure  $\mu$  on  $[0, 1]$  which is not regular:

$$\mu(A) = \begin{cases} 0, & \text{if } A \text{ is at most countable;} \\ +\infty & \text{otherwise.} \end{cases}$$

As we are working only with compact spaces, for us, “compact” is the same as “closed”. Regular measures somehow interact “well” with the underlying topology on  $K$ .

We let  $M_{\mathbb{R}}(K)$  and  $M_{\mathbb{C}}(K)$  be the collection of all finite, regular, signed or complex (respectively) measures on  $B(K)$ . These are real or complex, respectively, vector spaces for the obvious definition of addition and scalar multiplication.

Recall, Defn. 12.26, that for  $\mu \in M_{\mathbb{K}}(K)$  we define the *variation* of  $\mu$

$$\|\mu\| = \sup \left\{ \sum_{n=1}^{\infty} |\mu(A_n)| \right\},$$

where the supremum is taken over all sequences  $(A_n)$  of pairwise disjoint members of  $B(K)$ , with  $\sqcup_n A_n = K$ . Such  $(A_n)$  are called *partitions*.

**Proposition 14.16.** *The variation  $\|\cdot\|$  is a norm on  $M_{\mathbb{K}}(K)$ .*

*Proof.* If  $\mu = 0$  then clearly  $\|\mu\| = 0$ . If  $\|\mu\| = 0$ , then for  $A \in B(K)$ , let  $A_1 = A, A_2 = K \setminus A$  and  $A_3 = A_4 = \dots = \emptyset$ . Then  $(A_n)$  is a partition, and so

$$0 = \sum_{n=1}^{\infty} |\mu(A_n)| = |\mu(A)| + |\mu(K \setminus A)|.$$

Hence  $\mu(A) = 0$ , and so as  $A$  was arbitrary, we have that  $\mu = 0$ .

Clearly  $\|a\mu\| = |a| \|\mu\|$  for  $a \in \mathbb{K}$  and  $\mu \in M_{\mathbb{K}}(K)$ .

For  $\mu, \lambda \in M_{\mathbb{K}}(K)$  and a partition  $(A_n)$ , we have that

$$\sum_n |(\mu + \lambda)(A_n)| = \sum_n |\mu(A_n) + \lambda(A_n)| \leq \sum_n |\mu(A_n)| + |\lambda(A_n)| \leq \|\mu\| + \|\lambda\|.$$

As  $(A_n)$  was arbitrary, we see that  $\|\mu + \lambda\| \leq \|\mu\| + \|\lambda\|$ . □

To get a handle on the “regular” condition, we need to know a little more about  $C_{\mathbb{K}}(K)$ .

**Theorem 14.17 (Urysohn’s Lemma).** *Let  $K$  be a compact space, and let  $E, F$  be closed subsets of  $K$  with  $E \cap F = \emptyset$ . There exists  $f : K \rightarrow [0, 1]$  continuous with  $f(x) = 1$  for  $x \in E$  and  $f(x) = 0$  for  $x \in F$  (written  $f(E) = \{1\}$  and  $f(F) = \{0\}$ ).*

*Proof.* See a book on (point set) topology. □

**Lemma 14.18.** *Let  $\mu : B(K) \rightarrow [0, \infty)$  be a regular measure. Then for  $U \subseteq K$  open, we have*

$$\mu(U) = \sup \left\{ \int_K f \, d\mu : f \in C_{\mathbb{R}}(K), 0 \leq f \leq \chi_U \right\}.$$

*Proof.* If  $0 \leq f \leq \chi_U$ , then

$$0 = \int_K 0 \, d\mu \leq \int_K f \, d\mu \leq \int_K \chi_U \, d\mu = \mu(U).$$

Conversely, let  $F = K \setminus U$ , a closed set. Let  $E \subseteq U$  be closed. By Urysohn Lemma 14.17, there exists  $f : K \rightarrow [0, 1]$  continuous with  $f(E) = \{1\}$  and  $f(F) = \{0\}$ . So  $\chi_E \leq f \leq \chi_U$ , and hence

$$\mu(E) \leq \int_K f \, d\mu \leq \mu(U).$$

As  $\mu$  is regular,

$$\mu(U) = \sup \{ \mu(E) : E \subseteq U \text{ closed} \} \leq \sup \left\{ \int_K f \, d\mu : 0 \leq f \leq \chi_U \right\} \leq \mu(U).$$

Hence we have equality throughout.  $\square$

**Lemma 14.19.** *Let  $\mu \in M_{\mathbb{R}}(K)$ . Then*

$$\|\mu\| = \|\Phi_{\mu}\| := \sup \left\{ \left| \int_K f \, d\mu \right| : f \in C_{\mathbb{R}}(K), \|f\|_{\infty} \leq 1 \right\}.$$

*Proof.* Let  $(A, B)$  be a Hahn decomposition (Thm. 12.31) for  $\mu$ . For  $f \in C_{\mathbb{R}}(K)$  with  $\|f\|_{\infty} \leq 1$ , we have that

$$\begin{aligned} \left| \int_K f \, d\mu \right| &\leq \left| \int_A f \, d\mu \right| + \left| \int_B f \, d\mu \right| = \left| \int_A f \, d\mu_+ \right| + \left| \int_B f \, d\mu_- \right| \\ &\leq \int_A |f| \, d\mu_+ + \int_B |f| \, d\mu_- \leq \|f\|_{\infty} (\mu(A) - \mu(B)) \leq \|f\|_{\infty} \|\mu\|, \end{aligned}$$

using the fact that  $\mu(B) \leq 0$  and that  $(A, B)$  is a partition of  $K$ .

Conversely, as  $\mu$  is regular, for  $\epsilon > 0$ , there exist closed sets  $E$  and  $F$  with  $E \subseteq A$ ,  $F \subseteq B$ , and with  $\mu_+(E) > \mu_+(A) - \epsilon$  and  $\mu_-(F) > \mu_-(B) - \epsilon$ . By Urysohn Lemma 14.17, there exists  $f : K \rightarrow [0, 1]$  continuous with  $f(E) = \{1\}$  and  $f(F) = \{0\}$ . Let  $g = 2f - 1$ , so  $g$  is continuous,  $g$  takes values in  $[-1, 1]$ , and  $g(E) = \{1\}$ ,  $g(F) = \{-1\}$ . Then

$$\begin{aligned} \int_K g \, d\mu &= \int_E 1 \, d\mu + \int_F -1 \, d\mu + \int_{K \setminus (E \cup F)} g \, d\mu \\ &= \mu(E) - \mu(F) + \int_{A \setminus E} g \, d\mu + \int_{B \setminus F} g \, d\mu \end{aligned}$$

As  $E \subseteq A$ , we have  $\mu(E) = \mu_+(E)$ , and as  $F \subseteq B$ , we have  $-\mu(F) = \mu_-(F)$ . So

$$\begin{aligned} \int_K g \, d\mu &> \mu_+(A) - \epsilon + \mu_-(B) - \epsilon + \int_{A \setminus E} g \, d\mu + \int_{B \setminus F} g \, d\mu \\ &\geq |\mu(A)| + |\mu(B)| - 2\epsilon - |\mu(A \setminus E)| - |\mu(B \setminus F)| \\ &\geq |\mu(A)| + |\mu(B)| - 4\epsilon. \end{aligned}$$

As  $\epsilon > 0$  was arbitrary, we see that  $\|\Phi_{\mu}\| \geq |\mu(A)| + |\mu(B)| = \|\mu\|$ .  $\square$

**14.4. Riesz Representation Theorem.** To facilitate an approach to the key point of this Subsection we will require some more definitions.

**Definition 14.20.** A functional  $F$  is *positive* if for any non-negative function  $f$  we have  $F(f) > 0$ .

**Lemma 14.21.** Any positive linear functional  $F$  on  $C(X)$  is continuous and  $\|F\| = F(\mathbf{1})$ , where  $\mathbf{1}$  is the function identically equal to 1 on  $X$ .

*Proof.* For any function  $f$  such that  $\|f\|_\infty \leq 1$  the function  $1 - f$  is non negative thus:  $F(\mathbf{1}) - F(f) = F(1 - f) > 0$ , Thus  $F(\mathbf{1}) > F(f)$ , that is  $F$  is bounded and its norm is  $F(\mathbf{1})$ .  $\square$

So for a positive functional you know the exact place where to spot its norm, while a linear functional can attain its norm in an generic point of the unit ball in  $C(X)$ . It is also remarkable that any bounded linear functional can be represented by a pair of positive ones.

**Lemma 14.22.** Let  $\lambda$  be a continuous linear functional on  $C(X)$ . Then there are positive functionals  $\lambda_+$  and  $\lambda_-$  on  $C(X)$ , such that  $\lambda = \lambda_+ - \lambda_-$ .

*Proof.* First, for  $f \in C_{\mathbb{R}}(K)$  with  $f \geq 0$ , we define

$$\begin{aligned} \lambda_+(f) &= \sup \{ \lambda(g) : g \in C_{\mathbb{R}}(K), 0 \leq g \leq f \} \geq 0, \\ \lambda_-(f) &= \lambda_+(f) - \lambda(f) = \sup \{ \lambda(g) - \lambda(f) : g \in C_{\mathbb{R}}(K), 0 \leq g \leq f \} \\ &= \sup \{ \lambda(h) : h \in C_{\mathbb{R}}(K), 0 \leq h + f \leq f \} \\ &= \sup \{ \lambda(h) : h \in C_{\mathbb{R}}(K), -f \leq h \leq 0 \} \geq 0. \end{aligned}$$

In sense this is similar to the Hahn decomposition (Thm. 12.31).

We can check that

$$\lambda_+(tf) = t\lambda_+(f), \quad \lambda_-(tf) = t\lambda_-(f) \quad (t \geq 0, f \geq 0).$$

For  $f_1, f_2 \geq 0$ , we have that

$$\begin{aligned} \lambda_+(f_1 + f_2) &= \sup \{ \lambda(g) : 0 \leq g \leq f_1 + f_2 \} \\ &= \sup \{ \lambda(g_1 + g_2) : 0 \leq g_1 + g_2 \leq f_1 + f_2 \} \\ &\geq \sup \{ \lambda(g_1) + \lambda(g_2) : 0 \leq g_1 \leq f_1, 0 \leq g_2 \leq f_2 \} \\ &= \lambda_+(f_1) + \lambda_+(f_2). \end{aligned}$$

Conversely, if  $0 \leq g \leq f_1 + f_2$ , then set  $g_1 = \min(g, f_1)$ , so  $0 \leq g_1 \leq f_1$ . Let  $g_2 = g - g_1$  so  $g_1 \leq g$  implies that  $0 \leq g_2$ . For  $x \in K$ , if  $g_1(x) = g(x)$  then  $g_2(x) = 0 \leq f_2(x)$ ; if  $g_1(x) = f_1(x)$  then  $f_1(x) \leq g(x)$  and so  $g_2(x) = g(x) - f_1(x) \leq f_2(x)$ . So  $0 \leq g_2 \leq f_2$ , and  $g = g_1 + g_2$ . So in the above displayed equation, we really have equality throughout, and so  $\lambda_+(f_1 + f_2) = \lambda_+(f_1) + \lambda_+(f_2)$ . As  $\lambda$  is additive, it is now immediate that  $\lambda_-(f_1 + f_2) = \lambda_-(f_1) + \lambda_-(f_2)$

For  $f \in C_{\mathbb{R}}(K)$  we put  $f_+(x) = \max(f(x), 0)$  and  $f_-(x) = -\min(f(x), 0)$ . Then  $f_{\pm} \geq 0$  and  $f = f_+ - f_-$ . We define:

$$\lambda_+(f) = \lambda_+(f_+) - \lambda_+(f_-), \quad \lambda_-(f) = \lambda_-(f_+) - \lambda_-(f_-).$$

As when we were dealing with integration, we can check that  $\lambda_+$  and  $\lambda_-$  become linear functionals; by the previous Lemma they are bounded.  $\square$

Finally, we need a technical definition.

**Definition 14.23.** For  $f \in C_{\mathbb{R}}(K)$ , we define the *support* of  $f$ , written  $\text{supp}(f)$ , to be the closure of the set  $\{x \in K : f(x) \neq 0\}$ .

**Theorem 14.24 (Riesz Representation).** *Let  $K$  be a compact (Hausdorff) space, and let  $\lambda \in C_{\mathbb{K}}(K)^*$ . There exists a unique  $\mu \in M_{\mathbb{K}}(K)$  such that*

$$\lambda(f) = \int_K f \, d\mu \quad (f \in C_{\mathbb{K}}(K)).$$

Furthermore,  $\|\lambda\| = \|\mu\|$ .

*Proof.* Let us show *uniqueness*. If  $\mu_1, \mu_2 \in M_{\mathbb{K}}(K)$  both induce  $\lambda$  then  $\mu = \mu_1 - \mu_2$  induces the zero functional on  $C_{\mathbb{K}}(K)$ . So for  $f \in C_{\mathbb{R}}(K)$ ,

$$\begin{aligned} 0 &= \Re \int_K f \, d\mu = \int_K f \, d\mu_r \\ &= \Im \int_K f \, d\mu = \int_K f \, d\mu_i. \end{aligned}$$

So  $\mu_r$  and  $\mu_i$  both induce the zero functional on  $C_{\mathbb{R}}(K)$ . By Lemma 14.19, this means that  $\|\mu_r\| = \|\mu_i\| = 0$ , showing that  $\mu = \mu_r + i\mu_i = 0$ , as required.

*Existence* is harder, and we shall only sketch it here. Firstly, we shall suppose that  $\mathbb{K} = \mathbb{R}$  and that  $\lambda$  is *positive*.

Motivated by the above Lemmas 14.18 and 14.19, for  $U \subseteq K$  open, we define

$$\mu^*(U) = \sup \{ \lambda(f) : f \in C_{\mathbb{R}}(K), 0 \leq f \leq \chi_U, \text{supp}(f) \subseteq U \}.$$

For  $A \subseteq K$  general, we define

$$\mu^*(A) = \inf \{ \mu^*(U) : U \subseteq K \text{ is open, } A \subseteq U \}.$$

We then proceed to show that

- $\mu^*$  is an outer measure: this requires a technical topological lemma, where we make use of the support condition in the definition.
- We then check that every open set is  $\mu^*$ -measurable.
- As  $B(K)$  is generated by open sets, and the collection of  $\mu^*$ -measurable sets is a  $\sigma$ -algebra, it follows that every member of  $B(K)$  is  $\mu^*$ -measurable.
- By using results from Section 12, it follows that if we let  $\mu$  be the restriction of  $\mu^*$  to  $B(K)$ , then  $\mu$  is a measure on  $B(K)$ .
- We then check that this measure is regular.
- Finally, we show that  $\mu$  does induce the functional  $\lambda$ . Arguably, it is this last step which is the hardest (or least natural to prove).

If  $\lambda$  is not positive, then by Lemma 14.22 represent it as  $\lambda = \lambda_+ - \lambda_-$  for positive  $\lambda_{\pm}$ . As  $\lambda_+$  and  $\lambda_-$  are positive functionals, we can find  $\mu_+$  and  $\mu_-$  positive measures in  $M_{\mathbb{R}}(K)$  such that

$$\lambda_+(f) = \int_K f \, d\mu_+, \quad \lambda_-(f) = \int_K f \, d\mu_- \quad (f \in C_{\mathbb{R}}(K)).$$

Then if  $\mu = \mu_+ - \mu_-$ , we see that

$$\lambda(f) = \lambda_+(f) - \lambda_-(f) = \int_K f \, d\mu \quad (f \in C_{\mathbb{R}}(K)).$$

Finally, if  $\mathbb{K} = \mathbb{C}$ , then we use the same “complexification” trick from the proof of the Hahn-Banach Theorem 11.12. Namely, let  $\lambda \in C_{\mathbb{C}}(K)^*$ , and define  $\lambda_r, \lambda_i \in C_{\mathbb{R}}(K)^*$  by

$$\lambda_r(f) = \Re \lambda(f), \quad \lambda_i(f) = \Im \lambda(f) \quad (f \in C_{\mathbb{R}}(K)).$$

These are both clearly  $\mathbb{R}$ -linear. Notice also that  $|\lambda_r(f)| = |\Re \lambda(f)| \leq |\lambda(f)| \leq \|\lambda\| \|f\|_{\infty}$ , so  $\lambda_r$  is bounded; similarly  $\lambda_i$ .



By the real version of the Riesz Representation Theorem, there exist charges  $\mu_r$  and  $\mu_i$  such that

$$\Re\lambda(f) = \lambda_r(f) = \int_{\mathbb{K}} f \, d\mu_r, \quad \Im\lambda(f) = \lambda_i(f) = \int_{\mathbb{K}} f \, d\mu_i \quad (f \in C_{\mathbb{R}}(\mathbb{K})).$$

Then let  $\mu = \mu_r + i\mu_i$ , so for  $f \in C_{\mathbb{C}}(\mathbb{K})$ ,

$$\begin{aligned} \int_{\mathbb{K}} f \, d\mu &= \int_{\mathbb{K}} f \, d\mu_r + i \int_{\mathbb{K}} f \, d\mu_i \\ &= \int_{\mathbb{K}} \Re(f) \, d\mu_r + i \int_{\mathbb{K}} \Im(f) \, d\mu_r + i \int_{\mathbb{K}} \Re(f) \, d\mu_i - \int_{\mathbb{K}} \Im(f) \, d\mu_i \\ &= \lambda_r(\Re(f)) + i\lambda_r(\Im(f)) + i\lambda_i(\Re(f)) - \lambda_i(\Im(f)) \\ &= \Re\lambda(\Re(f)) + i\Re\lambda(\Im(f)) + i\Im\lambda(\Re(f)) - \Im\lambda(\Im(f)) \\ &= \lambda(\Re(f) + i\Im(f)) = \lambda(f), \end{aligned}$$

as required. □

Notice that we have not currently proved that  $\|\mu\| = \|\lambda\|$  in the case  $\mathbb{K} = \mathbb{C}$ . See a textbook for this.

## 15. FOURIER TRANSFORM

In this section we will briefly present a theory of Fourier transform focusing on commutative group approach. We mainly follow footsteps of [2, Ch. IV].

**15.1. Convolutions on Commutative Groups.** Let  $G$  be a commutative group, we will use  $+$  sign to denote group operation, respectively the inverse elements of  $g \in G$  will be denoted  $-g$ . We assume that  $G$  has a Hausdorff topology such that operations  $(g_1, g_2) \mapsto g_1 + g_2$  and  $g \mapsto -g$  are continuous maps. We also assume that the topology is *locally compact*, that is the group neutral element has a neighbourhood with a compact closure.

**Example 15.1.** Our main examples will be as follows:

- (i)  $G = \mathbb{Z}$  the group of integers with operation of addition and the discrete topology (each point is an open set).
- (ii)  $G = \mathbb{R}$  the group of real numbers with addition and the topology defined by open intervals.
- (iii)  $G = \mathbb{T}$  the group of Euclidean rotations the unit circle in  $\mathbb{R}^2$  with the natural topology. Another realisations of the same group:
  - Unimodular complex numbers under multiplication.
  - Factor group  $\mathbb{R}/\mathbb{Z}$ , that is addition of real numbers modulus 1.

There is a homomorphism between two realisations given by  $z = e^{2\pi it}$ ,  $t \in [0, 1)$ ,  $|z| = 1$ .

We assume that  $G$  has a regular Borel measure which is invariant in the following sense.

**Definition 15.2.** Let  $\mu$  be a measure on a commutative group  $G$ ,  $\mu$  is called *invariant* (or *Haar measure*) if for any measurable  $X$  and any  $g \in G$  the sets  $g + X$  and  $-X$  are also measurable and  $\mu(X) = \mu(g + X) = \mu(-X)$ .

Such an invariant measure exists if and only if the group is locally compact, in this case the measure is uniquely defined up to the constant factor.

**Exercise 15.3.** Check that in the above three cases invariant measures are:

- $G = \mathbb{Z}$ , the invariant measure of  $X$  is equal to number of elements in  $X$ .
- $G = \mathbb{R}$  the invariant measure is the Lebesgue measure.

- $G = \mathbb{T}$  the invariant measure coincides with the Lebesgue measure.

**Definition 15.4.** A *convolution* of two functions on a commutative group  $G$  with an invariant measure  $\mu$  is defined by:

$$(15.1) \quad (f_1 * f_2)(x) = \int_G f_1(x-y) f_2(y) d\mu(y) = \int_G f_1(y) f_2(x-y) d\mu(y).$$

**Theorem 15.5.** If  $f_1, f_2 \in L_1(G, \mu)$ , then the integrals in (15.1) exist for almost every  $x \in G$ , the function  $f_1 * f_2$  is in  $L_1(G, \mu)$  and  $\|f_1 * f_2\| \leq \|f_1\| \cdot \|f_2\|$ .

*Proof.* If  $f_1, f_2 \in L_1(G, \mu)$  then by Fubini's Thm. 13.45 the function  $\phi(x, y) = f_1(x) * f_2(y)$  is in  $L_1(G \times G, \mu \times \mu)$  and  $\|\phi\| = \|f_1\| \cdot \|f_2\|$ .

Let us define a map  $\tau : G \times G \rightarrow G \times G$  such that  $\tau(x, y) = (x + y, y)$ . It is measurable (send Borel sets to Borel sets) and preserves the measure  $\mu \times \mu$ . Indeed, for an elementary set  $X = A \times B \subset G \times G$  we have:

$$\begin{aligned} (\mu \times \mu)(\tau(X)) &= \int_{G \times G} \chi_{\tau(X)}(x, y) d\mu(x) d\mu(y) \\ &= \int_{G \times G} \chi_X(x - y, y) d\mu(x) d\mu(y) \\ &= \int_G \left( \int_G \chi_X(x - y, y) d\mu(x) \right) d\mu(y) \\ &= \int_B \mu(A + y) d\mu(y) = \mu(A) \times \mu(B) = (\mu \times \mu)(X). \end{aligned}$$

We used invariance of  $\mu$  and Fubini's Thm. 13.45. Therefore we have an isometric isomorphism of  $L_1(G \times G, \mu \times \mu)$  into itself by the formula:

$$T\phi(x, y) = \phi(\tau(x, y)) = \phi(x - y, y).$$

If we apply this isomorphism to the above function  $\phi(x, y) = f_1(x) * f_2(y)$  we shall obtain the statement.  $\square$

**Definition 15.6.** Denote by  $S(k)$  the map  $S(k) : f \mapsto k * f$  which we will call *convolution operator with the kernel  $k$* .

**Corollary 15.7.** If  $k \in L_1(G)$  then the convolution  $S(k)$  is a bounded linear operator on  $L_1(G)$ .

**Theorem 15.8.** Convolution is a commutative, associative and distributive operation. In particular  $S(f_1)S(f_2) = S(f_2)S(f_1) = S(f_1 * f_2)$ .

*Proof.* Direct calculation using change of variables.  $\square$

It follows from Thm. 15.5 that convolution is a closed operation on  $L_1(G)$  and has nice properties due to Thm. 15.8. We fix this in the following definition.

**Definition 15.9.**  $L_1(G)$  equipped with the operation of convolution is called *convolution algebra*  $L_1(G)$ .

The following operators of special interest.

**Definition 15.10.** An operator of *shift*  $T(a)$  acts on functions by  $T(a) : f(x) \mapsto f(x + a)$ .

**Lemma 15.11.** An operator of shift is an isometry of  $L_1(G)$ ,  $1 \leq p \leq \infty$ .

**Theorem 15.12.** Operators of shifts and convolutions commute:

$$T(a)(f_1 * f_2) = T(a)f_1 * f_2 = f_1 * T(a)f_2,$$

or

$$T(a)S(f) = S(f)T(a) = S(T(a)f).$$

*Proof.* Just another calculation with a change of variables. □

*Remark 15.13.* Note that operator of shifts  $T(a)$  provide a *representation* of the group  $G$  by linear isometric operators in  $L_p(G)$ ,  $1 \leq p \leq \infty$ . A map  $f \mapsto S(f)$  is a *representation* of the convolution algebra

There is a useful relation between support of functions and their convolutions.

**Lemma 15.14.** *For any  $f_1, f_2 \in L_1(G)$  we have:*

$$\text{supp}(f_1 * f_2) \subset \text{supp}(f_1) + \text{supp}(f_2).$$

*Proof.* If  $x \notin \text{supp}(f_1) + \text{supp}(f_2)$  then for any  $y \in \text{supp}(f_2)$  we have  $x - y \notin \text{supp}(f_1)$ . Thus for such  $x$  convolution is the integral of the identical zero. □

**15.2. Characters of Commutative Groups.** Our purpose is to map the commutative algebra of convolutions to a commutative algebra of functions with point-wise multiplication. To this end we first represent elements of the group as operators of multiplication.

**Definition 15.15.** A *character*  $\chi : G \rightarrow \mathbb{T}$  is a continuous homomorphism of an abelian topological group  $G$  to the group  $\mathbb{T}$  of unimodular complex numbers under multiplications:

$$\chi(x + y) = \chi(x)\chi(y).$$

**Lemma 15.16.** *The product of two characters of a group is again a character of the group. If  $\chi$  is a character of  $G$  then  $\chi^{-1} = \bar{\chi}$  is a character as well.*

*Proof.* Let  $\chi_1$  and  $\chi_2$  be characters of  $G$ . Then:

$$\begin{aligned} \chi_1(gh)\chi_2(gh) &= \chi_1(g)\chi_1(h)\chi_2(g)\chi_2(h) \\ &= (\chi_1(g)\chi_2(g))(\chi_1(h)\chi_2(h)) \in \mathbb{T}. \end{aligned}$$

□

**Definition 15.17.** The *dual group* is collection of all characters of  $G$  with operation of multiplication.

The dual group becomes a topological group with the *uniform convergence on compacts*.

**Exercise 15.18.** Check that the sequence  $f_n(x) = x^n$  does not converge uniformly on compacts if considered on  $[0, 1]$ . However it does converges uniformly on compacts if considered on  $(0, 1)$ .

**Example 15.19.** If  $G = \mathbb{Z}$  then any character  $\chi$  is defined by its values  $\chi(1)$  since

$$(15.2) \quad \chi(n) = [\chi(1)]^n.$$

Since  $\chi(1)$  can be any number on  $\mathbb{T}$  we see that  $\hat{\mathbb{Z}}$  is parametrised by  $\mathbb{T}$ .

**Theorem 15.20.** *The group  $\hat{\mathbb{Z}}$  is isomorphic to  $\mathbb{T}$ .*

*Proof.* The correspondence from the above example is a group homomorphism. Indeed if  $\chi_z$  is the character with  $\chi_z(1) = z$ , then  $\chi_{z_1}\chi_{z_2} = \chi_{z_1z_2}$ . Since  $\mathbb{Z}$  is discrete, every compact consists of a finite number of points, thus uniform convergence on compacts means point-wise convergence. The equation (15.2) shows that  $\chi_{z_n} \rightarrow \chi_z$  if and only if  $\chi_{z_n}(1) \rightarrow \chi_z(1)$ , that is  $z_n \rightarrow z$ . □

**Theorem 15.21.** *The group  $\hat{\mathbb{T}}$  is isomorphic to  $\mathbb{Z}$ .*

*Proof.* For every  $n \in \mathbb{Z}$  define a character of  $\mathbb{T}$  defined by

$$(15.3) \quad \chi_n(z) = z^n, \quad z \in \mathbb{T}.$$

We will show that these are the only characters in Cor. 15.25. The isomorphism property is easy to establish. The topological isomorphism follows from discreteness of  $\hat{\mathbb{T}}$ . Indeed due to compactness of  $\mathbb{T}$  for  $n \neq m$ :

$$\max_{z \in \mathbb{T}} |\chi_n(z) - \chi_m(z)|^2 = \max_{z \in \mathbb{T}} |2 - 2\Re z^{m-n}|^2 = 4.$$

□

**Principle 15.22** (Pontryagin's duality). For any locally compact commutative topological group  $G$  the natural map  $G \rightarrow \hat{\hat{G}}$ , such that it maps  $g \in G$  to a character  $f_g$  on  $\hat{G}$  by the formula:

$$(15.4) \quad f_g(\chi) = \chi(g), \quad \chi \in \hat{G},$$

is an isomorphism of topological groups.

**Remark 15.23.** (i) The principle is not true for commutative group which are not locally compact.

(ii) Note the similarity with an embedding of a vector space into the second dual.

The Pontryagin's duality tells that the collection of all characters contains enough information to rebuild the initial group.

**Theorem 15.24.** *The group  $\hat{\mathbb{R}}$  is isomorphic to  $\mathbb{R}$ .*

*Proof.* For  $\lambda \in \mathbb{R}$  define a character  $\chi_\lambda \in \hat{\mathbb{R}}$  by

$$(15.5) \quad \chi_\lambda(x) = e^{2\pi i \lambda x}, \quad x \in \mathbb{R}.$$

Moreover any smooth character of the group  $G = (\mathbb{R}, +)$  has the form (15.5). Indeed, let  $\chi$  be a smooth character of  $\mathbb{R}$ . Put  $c = \chi'(t)|_{t=0} \in \mathbb{C}$ . Then  $\chi'(t) = c\chi(t)$  and  $\chi(t) = e^{ct}$ . We also get  $c \in i\mathbb{R}$  and any such  $c$  define a character. Then the multiplication of characters is:  $\chi_1(t)\chi_2(t) = e^{c_1 t} e^{c_2 t} = e^{(c_2+c_1)t}$ . So we have a group isomorphism.

For a generic character we can apply first the *smoothing technique* and reduce to the above case.

Let us show topological homeomorphism. If  $\lambda_n \rightarrow \lambda$  then  $\chi_{\lambda_n} \rightarrow \chi_\lambda$  uniformly on any compact in  $\mathbb{R}$  from the explicit formula of the character. Reverse, let  $\chi_{\lambda_n} \rightarrow \chi_\lambda$  uniformly on any interval. Then  $\chi_{\lambda_n - \lambda}(x) \rightarrow 1$  uniformly on  $[0, 1]$ . But

$$\begin{aligned} \sup_{[0,1]} |\chi_{\lambda_n - \lambda}(x) - 1| &= \sup_{[0,1]} |\sin \pi(\lambda_n - \lambda)x| \\ &= \begin{cases} 1, & \text{if } |\lambda_n - \lambda| \geq 1/2, \\ \sin \pi |\lambda_n - \lambda|, & \text{if } |\lambda_n - \lambda| \leq 1/2. \end{cases} \end{aligned}$$

Thus  $\lambda_n \rightarrow \lambda$ . □

**Corollary 15.25.** *Any character of the group  $\mathbb{T}$  has the form (15.3).*

*Proof.* Let  $\chi \in \hat{\mathbb{T}}$ , consider  $\chi_1(t) = \chi(e^{2\pi i t})$  which is a character of  $\mathbb{R}$ . Thus  $\chi_1(t) = e^{2\pi i \lambda t}$  for some  $\lambda \in \mathbb{R}$ . Since  $\chi_1(1) = 1$  then  $\lambda = n \in \mathbb{Z}$ . Thus  $\chi_1(t) = e^{2\pi i n t}$ , that is  $\chi(z) = z^n$  for  $z = e^{2\pi i t}$ . □

**Remark 15.26.** Although  $\hat{\mathbb{R}}$  is isomorphic to  $\mathbb{R}$  there is no a canonical form for this isomorphism (unlike for  $\mathbb{R} \rightarrow \hat{\mathbb{R}}$ ). Our choice is convenient for the Poisson formula below, however some other popular definitions are  $\lambda \rightarrow e^{i\lambda x}$  or  $\lambda \rightarrow e^{-i\lambda x}$ .

We can unify the previous three Theorem into the following statement.

**Theorem 15.27.** *Let  $G = \mathbb{R}^n \times \mathbb{Z}^k \times \mathbb{T}^l$  be the direct product of groups. Then the dual group is  $\hat{G} = \mathbb{R}^n \times \mathbb{T}^k \times \mathbb{Z}^l$ .*

**15.3. Fourier Transform on Commutative Groups.**

**Definition 15.28.** Let  $G$  be a locally compact commutative group with an invariant measure  $\mu$ . For any  $f \in L_1(G)$  define the *Fourier transform*  $\hat{f}$  by

$$(15.6) \quad \hat{f}(\chi) = \int_G f(x) \bar{\chi}(x) \, d\mu(x), \quad \chi \in \hat{G}.$$

That is the Fourier transform  $\hat{f}$  is a function on the dual group  $\hat{G}$ .

**Example 15.29.** (i) If  $G = \mathbb{Z}$ , then  $f \in L_1(\mathbb{Z})$  is a two-sided summable sequence  $(c_n)_{n \in \mathbb{Z}}$ . Its Fourier transform is the function  $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$  on  $\mathbb{T}$ . Sometimes  $f(z)$  is called *generating function* of the sequence  $(c_n)$ .

(ii) If  $G = \mathbb{T}$ , then the Fourier transform of  $f \in L^1(\mathbb{T})$  is its *Fourier coefficients*, see Section 4.1.

(iii) If  $G = \mathbb{R}$ , the Fourier transform is also the function on  $\mathbb{R}$  given by the *Fourier integral*:

$$(15.7) \quad \hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{-2\pi i \lambda x} \, dx.$$

The important properties of the Fourier transform are captured in the following statement.

**Theorem 15.30.** *Let  $G$  be a locally compact commutative group with an invariant measure  $\mu$ . The Fourier transform maps functions from  $L_1(G)$  to continuous bounded functions on  $\hat{G}$ . Moreover, a convolution is transformed to point-wise multiplication:*

$$(15.8) \quad (f_1 * f_2)^\wedge(\chi) = \hat{f}_1(\chi) \cdot \hat{f}_2(\chi),$$

a shift operator  $T(a)$ ,  $a \in G$  is transformed in multiplication by the character  $f_a \in \hat{G}$ :

$$(15.9) \quad (T(a)f)^\wedge(\chi) = f_a(\chi) \cdot \hat{f}(\chi), \quad f_a(\chi) = \chi(a)$$

and multiplication by a character  $\chi \in \hat{G}$  is transformed to the shift  $T(\chi^{-1})$ :

$$(15.10) \quad (\chi \cdot f)^\wedge(\chi_1) = T(\chi^{-1})\hat{f}(\chi_1) = \hat{f}(\chi^{-1}\chi_1).$$

*Proof.* Let  $f \in L_1(G)$ . For any  $\varepsilon > 0$  there is a compact  $K \subset G$  such that  $\int_{G \setminus K} |f| \, d\mu < \varepsilon$ . If  $\chi_n \rightarrow \chi$  in  $\hat{G}$ , then we have the uniform convergence of  $\chi_n \rightarrow \chi$  on  $K$ , so there is  $n(\varepsilon)$  such that for  $k > n(\varepsilon)$  we have  $|\chi_k(x) - \chi(x)| < \varepsilon$  for all  $x \in K$ . Then

$$\begin{aligned} \left| \hat{f}(\chi_n) - \hat{f}(\chi) \right| &\leq \int_K |f(x)| |\chi_n(x) - \chi(x)| \, d\mu(x) + \int_{G \setminus K} |f(x)| |\chi_n(x) - \chi(x)| \, d\mu(x) \\ &\leq \varepsilon \|f\| + 2\varepsilon. \end{aligned}$$

Thus  $\hat{f}$  is continuous. Its boundedness follows from the integral estimations. Algebraic maps (15.8)–(15.10) can be obtained by changes of variables under integration. For example, using Fubini's Thm. 13.45 and invariance of the measure:

$$\begin{aligned} (f_1 * f_2)^\wedge(\chi) &= \int_G \int_G f_1(s) f_2(t-s) \, ds \chi(t) \, dt \\ &= \int_G \int_G f_1(s) \chi(s) f_2(t-s) \chi(t-s) \, ds \, dt \\ &= \hat{f}_1(\chi) \hat{f}_2(\chi). \end{aligned}$$

□

**15.4. Fourier Integral.** We recall the formula (15.7):

**Definition 15.31.** We define the *Fourier integral* of a function  $f \in L_1(\mathbb{R})$  by

$$(15.11) \quad \hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{-2\pi i \lambda x} dx.$$

We already know that  $\hat{f}$  is a bounded continuous function on  $\mathbb{R}$ , a further property is:

**Lemma 15.32.** *If a sequence of functions  $(f_n) \subset L_1(\mathbb{R})$  converges in the metric  $L_1(\mathbb{R})$ , then the sequence  $(\hat{f}_n)$  converges uniformly on the real line.*

*Proof.* This follows from the estimation:

$$\left| \hat{f}_n(\lambda) - \hat{f}_m(\lambda) \right| \leq \int_{\mathbb{R}} |f_n(x) - f_m(x)| dx.$$

□

**Lemma 15.33.** *The Fourier integral  $\hat{f}$  of  $f \in L_1(\mathbb{R})$  has zero limits at  $-\infty$  and  $+\infty$ .*

*Proof.* Take  $f$  the indicator function of  $[a, b]$ . Then  $\hat{f} = \frac{1}{-2\pi i \lambda} (e^{-2\pi i a} - e^{-2\pi i b})$ ,  $\lambda \neq 0$ . Thus  $\lim_{\lambda \rightarrow \pm\infty} \hat{f}(\lambda) = 0$ . By continuity from the previous Lemma this can be extended to the closure of simple functions, the space  $L_1(\mathbb{R})$ . □

**Lemma 15.34.** *If  $f$  is absolutely continuous on every interval and  $f' \in L_1(\mathbb{R})$ , then*

$$(f')^\wedge = 2\pi i \lambda \hat{f}.$$

*More generally:*

$$(15.12) \quad (f^{(k)})^\wedge = (2\pi i \lambda)^k \hat{f}.$$

*Proof.* This is follows from integration by parts, which is possible because assumption in the Lemma. □

**Corollary 15.35.** *If  $f^{(k)} \in L_1(\mathbb{R})$  then*

$$\left| \hat{f} \right| = \frac{|\hat{f}^{(k)}|}{|2\pi \lambda|^k} \rightarrow 0,$$

*that is  $\hat{f}$  decrease at infinity faster than  $|\lambda|^{-k}$ .*

**Definition 15.36.** We say that a function  $f$  is *rapidly decreasing* if  $\lim_{x \rightarrow \pm\infty} |x^k f(x)| = 0$  for any  $k \in \mathbb{N}$ .

The collection of all smooth rapidly decreasing functions is called *Schwartz space* and is denoted by  $S$ .

**Example 15.37.** An example of a rapidly decreasing function is  $e^{-\pi x^2}$ . Find its Fourier transform.

**Lemma 15.38.** *Let  $f(x)$  and  $xf(x)$  are both in  $L_1(\mathbb{R})$ , then  $\hat{f}$  is differentiable and*

$$\hat{f}' = (-2\pi i x \hat{f}).$$

*More generally*

$$(15.13) \quad \hat{f}^{(k)} = ((-2\pi i x)^k \hat{f}).$$

*Proof.* It is follows from the differentiation under the integration sign. □

**Corollary 15.39.** *The Fourier transform of a smooth rapidly decreasing function is a smooth rapidly decreasing function.*

The relation (15.12) and (15.13) allows to reduce many partial differential equations to algebraic one, to finish the solution we need the inverse of the Fourier transform.

**Definition 15.40.** We define the *inverse Fourier transform* on  $L_1(\mathbb{R})$ :

$$(15.14) \quad \check{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{2\pi i \lambda x} dx.$$

**Theorem 15.41.** *The Fourier integral and the inverse Fourier transform are inverse maps. That is, if  $g = \hat{f}$  then  $f = \check{g}$ .*

*Proof.* The exact meaning of the statement depends from the spaces which we consider as the domain and the range. Various variants and their proofs can be found in the literature. For example, in [2, § IV.2.3], it is proven for the Schwartz space  $S$  of smooth rapidly decreasing functions using the intertwining relations (15.12) and (15.13). Thus the composition of Fourier integral and the inverse Fourier transform commutes both with operator of multiplication by  $x$  and differentiation. Then we need a result, that any operator commuting with multiplication by  $x$  is operator of multiplication by a function  $f$ , then commutation with differentiation implies  $f' = 0$ , that is  $f = \text{const}$ . The value of this constant can be evaluated by a Fourier transform on a single function, say  $e^{-\pi x^2}$ .  $\square$

**Theorem 15.42** (Plancherel identity). *The Fourier transform extends uniquely to a unitary map  $L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$ :*

$$\int_{\mathbb{R}} |f|^2 dx = \int_{\mathbb{R}} |\hat{f}|^2 d\lambda.$$

*Proof.* The proof will be done in three steps: first we establish the identity for smooth rapidly decreasing functions, then for  $L_2$  functions with compact support and finally for any  $L_2$  function.

- (i) Take  $f_1$  and  $f_2 \in S$  be smooth rapidly decreasing functions and  $g_1$  and  $g_2$  be their Fourier transform. Then (using Fubini's Thm. 13.45):

$$\begin{aligned} \int_{\mathbb{R}} f_1(t) \bar{f}_2(t) dt &= \int_{\mathbb{R}} \int_{\mathbb{R}} g_1(\lambda) e^{2\pi i \lambda t} d\lambda \bar{f}_2(t) dt \\ &= \int_{\mathbb{R}} g_1(\lambda) \int_{\mathbb{R}} e^{2\pi i \lambda t} \bar{f}_2(t) dt d\lambda \\ &= \int_{\mathbb{R}} g_1(\lambda) \bar{g}_2(\lambda) d\lambda \end{aligned}$$

Put  $f_1 = f_2 = f$  and  $g_1 = g_2 = g$  we get the identity.

- (ii) Next let  $f \in L_2(\mathbb{R})$  with a support in  $(-a, a)$  then  $f \in L_1(\mathbb{R})$  as well, thus the Fourier transform is well-defined. Let  $f_n \in S$  be a sequence with support on  $(-a, a)$  which converges to  $f$  in  $L_2$  and thus in  $L_1$ . The Fourier transform  $g_n$  converges to  $g$  uniformly and is a Cauchy sequence in  $L_2$  due to the above identity. Thus  $g_n \rightarrow g$  in  $L_2$  and we can extend the Plancherel identity by continuity to  $L_2$  functions with compact support.
- (iii) The final bit is done for a general  $f \in L_2$  the sequence

$$f_n(x) = \begin{cases} f(x), & \text{if } |x| < n, \\ 0, & \text{otherwise;} \end{cases}$$

of truncations to the interval  $(-n, n)$ . For  $f_n$  the Plancherel identity is established above, and  $f_n \rightarrow f$  in  $L_2(\mathbb{R})$ . We also build their Fourier images  $g_n$  and see that this is a Cauchy sequence in  $L_2(\mathbb{R})$ , so  $g_n \rightarrow g$ .

If  $f \in L_1 \cap L_2$  then the above  $g$  coincides with the ordinary Fourier transform on  $L_1$ .  $\square$

Proofs of the following statements are not examinable Thms. [12.19](#), [12.31](#), [13.48](#), [14.24](#), Props. [14.11](#), [14.12](#).

#### APPENDIX A. TUTORIAL PROBLEMS

These are tutorial problems intended for self-assessment of the course understanding.

**A.1. Tutorial problems I.** *All spaces are complex, unless otherwise specified.*

**A.1.** Show that  $\|f\| = |f(0)| + \sup |f'(t)|$  defines a norm on  $C^1[0, 1]$ , which is the space of (real) functions on  $[0, 1]$  with continuous derivative.

**A.2.** Show that the formula  $\langle (x_n), (y_n) \rangle = \sum_{n=1}^{\infty} x_n \bar{y}_n / n^2$  defines an inner product on  $\ell_{\infty}$ , the space of bounded (complex) sequences. What norm does it produce?

**A.3.** Use the Cauchy–Schwarz inequality for a suitable inner product to prove that for all  $f \in C[0, 1]$  the inequality

$$\left| \int_0^1 f(x)x \, dx \right| \leq C \left( \int_0^1 |f(x)|^2 \, dx \right)^{1/2}$$

holds for some constant  $C > 0$  (independent of  $f$ ) and find the smallest possible  $C$  that holds for all functions  $f$  (hint: consider the cases of equality).

**A.4.** We define the following norm on  $\ell_{\infty}$ , the space of bounded complex sequences:

$$\|(x_n)\|_{\infty} = \sup_{n \geq 1} |x_n|.$$

Show that this norm makes  $\ell_{\infty}$  into a Banach space (i.e., a complete normed space).

**A.5.** Fix a vector  $(w_1, \dots, w_n)$  whose components are strictly positive real numbers, and define an inner product on  $\mathbb{C}^n$  by

$$\langle x, y \rangle = \sum_{k=1}^n w_k x_k \bar{y}_k.$$

Show that this makes  $\mathbb{C}^n$  into a Hilbert space (i.e., a complete inner-product space).

**A.2. Tutorial problems II.**

**A.6.** Show that the supremum norm on  $C[0, 1]$  isn't given by an inner product, by finding a counterexample to the parallelogram law.

**A.7.** In  $\ell_2$  let  $e_1 = (1, 0, 0, \dots)$ ,  $e_2 = (0, 1, 0, \dots)$ ,  $e_3 = (0, 0, 1, 0, \dots)$ , and so on. Show that  $\text{Lin}(e_1, e_2, \dots) = c_{00}$ , and that  $\text{CLin}(e_1, e_2, \dots) = \ell_2$ . What is  $\text{CLin}(e_2, e_3, \dots)$ ?

**A.8.** Let  $C[-1, 1]$  have the standard  $L_2$  inner product, defined by

$$\langle f, g \rangle = \int_{-1}^1 f(t) \overline{g(t)} \, dt.$$

Show that the functions  $1$ ,  $t$  and  $t^2 - 1/3$  form an orthogonal (not orthonormal!) basis for the subspace  $P_2$  of polynomials of degree at most 2 and hence calculate the best  $L_2$ -approximation of the function  $t^4$  by polynomials in  $P_2$ .



**A.9.** Define an inner product on  $C[0, 1]$  by

$$\langle f, g \rangle = \int_0^1 \sqrt{t} f(t) \overline{g(t)} dt.$$

Use the Gram–Schmidt process to find the first 2 terms of an orthonormal sequence formed by orthonormalising the sequence  $1, t, t^2, \dots$ .

**A.10.** Consider the plane  $P$  in  $\mathbb{C}^4$  (usual inner product) spanned by the vectors  $(1, 1, 0, 0)$  and  $(1, 0, 0, -1)$ . Find orthonormal bases for  $P$  and  $P^\perp$ , and verify directly that  $(P^\perp)^\perp = P$ .

### A.3. Tutorial Problems III.

**A.11.** Let  $a$  and  $b$  be arbitrary real numbers with  $a < b$ . By using the fact that the functions  $\frac{1}{\sqrt{2\pi}} e^{inx}$ ,  $n \in \mathbb{Z}$ , are orthonormal in  $L_2[0, 2\pi]$ , together with the change of variable  $x = 2\pi(t - a)/(b - a)$ , find an orthonormal basis in  $L_2[a, b]$  of the form  $e_n(t) = \alpha e^{in\lambda t}$ ,  $n \in \mathbb{Z}$ , for suitable real constants  $\alpha$  and  $\lambda$ .

**A.12.** For which real values of  $\alpha$  is

$$\sum_{n=1}^{\infty} n^\alpha e^{int}$$

the Fourier series of a function in  $L_2[-\pi, \pi]$ ?

**A.13.** Calculate the Fourier series of  $f(t) = e^t$  on  $[-\pi, \pi]$  and use Parseval's identity to deduce that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1} = \frac{\pi}{\tanh \pi}.$$

**A.14.** Using the fact that  $(e_n)$  is a complete orthonormal system in  $L_2[-\pi, \pi]$ , where  $e_n(t) = \exp(int)/\sqrt{2\pi}$ , show that  $e_0, s_1, c_1, s_2, c_2, \dots$  is a complete orthonormal system, where  $s_n(t) = \sin nt/\sqrt{\pi}$  and  $c_n(t) = \cos nt/\sqrt{\pi}$ . Show that every  $L_2[-\pi, \pi]$  function  $f$  has a Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt,$$

converging in the  $L_2$  sense, and give a formula for the coefficients.

**A.15.** Let  $C(\mathbb{T})$  be the space of continuous (complex) functions on the circle  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  with the supremum norm. Show that, for any polynomial  $f(z)$  in  $C(\mathbb{T})$

$$\int_{|z|=1} f(z) dz = 0.$$

Deduce that the function  $f(z) = \bar{z}$  is *not* the uniform limit of polynomials on the circle (i.e., Weierstrass's approximation theorem doesn't hold in this form).

### A.4. Tutorial Problems IV.

**A.16.** Define a linear functional on  $C[0, 1]$  (continuous functions on  $[0, 1]$ ) by  $\alpha(f) = f(1/2)$ . Show that  $\alpha$  is bounded if we give  $C[0, 1]$  the supremum norm. Show that  $\alpha$  is not bounded if we use the  $L_2$  norm, because we can find a sequence  $(f_n)$  of continuous functions on  $[0, 1]$  such that  $\|f_n\|_2 \leq 1$ , but  $f_n(1/2) \rightarrow \infty$ .

**A.17.** The *Hardy space*  $H_2$  is the Hilbert space of all power series  $f(z) = \sum_{n=0}^{\infty} a_n z^n$ , such that  $\sum_{n=0}^{\infty} |a_n|^2 < \infty$ , where the inner product is given by

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}.$$

Show that the sequence  $1, z, z^2, z^3, \dots$  is an orthonormal basis for  $H_2$ .

Fix  $w$  with  $|w| < 1$  and define a linear functional on  $H_2$  by  $\alpha(f) = f(w)$ . Write down a formula for the function  $g(z) \in H_2$  such that  $\alpha(f) = \langle f, g \rangle$ . What is  $\|\alpha\|$ ?

**A.18.** The *Volterra operator*  $V : L_2[0, 1] \rightarrow L_2[0, 1]$  is defined by

$$(Vf)(x) = \int_0^x f(t) dt.$$

Use the Cauchy–Schwarz inequality to show that  $|(Vf)(x)| \leq \sqrt{x} \|f\|_2$  (hint: write  $(Vf)(x) = \langle f, J_x \rangle$  where  $J_x$  is a function that you can write down explicitly).

Deduce that  $\|Vf\|_2^2 \leq \frac{1}{2} \|f\|_2^2$ , and hence  $\|V\| \leq 1/\sqrt{2}$ .

**A.19.** Find the adjoints of the following operators:

- (i)  $A : \ell_2 \rightarrow \ell_2$ , defined by  $A(x_1, x_2, \dots) = (0, \frac{x_1}{1}, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$ ; and, on a general Hilbert space  $H$ :
- (ii) The rank-one operator  $R$ , defined by  $Rx = \langle x, y \rangle z$ , where  $y$  and  $z$  are fixed elements of  $H$ ;
- (iii) The projection operator  $P_M$ , defined by  $P_M(m + n) = m$ , where  $m \in M$  and  $n \in M^\perp$ , and  $H = M \oplus M^\perp$  as usual.

**A.20.** Let  $U \in B(H)$  be a unitary operator. Show that  $(Ue_n)$  is an orthonormal basis of  $H$  whenever  $(e_n)$  is.

Let  $\ell_2(\mathbb{Z})$  denote the Hilbert space of two-sided sequences  $(a_n)_{n=-\infty}^{\infty}$  with

$$\|(a_n)\|^2 = \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty.$$

Show that the *bilateral right shift*,  $V : \ell_2(\mathbb{Z}) \rightarrow \ell_2(\mathbb{Z})$  defined by  $V((a_n)) = (b_n)$ , where  $b_n = a_{n-1}$  for all  $n \in \mathbb{Z}$ , is unitary, whereas the usual right shift  $S$  on  $\ell_2 = \ell_2(\mathbb{N})$  is not unitary.

#### A.5. Tutorial Problems V.

**A.21.** Let  $f \in C[-\pi, \pi]$  and let  $M_f$  be the multiplication operator on  $L_2(-\pi, \pi)$ , given by  $(M_f g)(t) = f(t)g(t)$ , for  $g \in L_2(-\pi, \pi)$ . Find a function  $\tilde{f} \in C[-\pi, \pi]$  such that  $M_f^* = M_{\tilde{f}}$ .

Show that  $M_f$  is always a normal operator. When is it Hermitian? When is it unitary?

**A.22.** Let  $T$  be any operator such that  $T^n = 0$  for some integer  $n$  (such operators are called *nilpotent*). Show that  $I - T$  is invertible (hint: consider  $I + T + T^2 + \dots + T^{n-1}$ ). Deduce that  $I - T/\lambda$  is invertible for any  $\lambda \neq 0$ .

What is  $\sigma(T)$ ? What is  $r(T)$ ?

**A.23.** Let  $(\lambda_n)$  be a fixed bounded sequence of complex numbers, and define an operator on  $\ell_2$  by  $T((x_n)) = ((y_n))$ , where  $y_n = \lambda_n x_n$  for each  $n$ . Recall that  $T$  is a bounded operator and  $\|T\| = \|(\lambda_n)\|_\infty$ . Let  $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ . Prove the following:

- (i) Each  $\overline{\lambda_k}$  is an eigenvalue of  $T$ , and hence is in  $\sigma(T)$ .
- (ii) If  $\lambda \notin \overline{\Lambda}$ , then the inverse of  $T - \lambda I$  exists (and is bounded).

Deduce that  $\sigma(T) = \overline{\lambda}$ . Note, that then *any non-empty compact set could be a spectrum of some bounded operator.*

**A.24.** Let  $S$  be an *isomorphism* between Hilbert spaces  $H$  and  $K$ , that is,  $S : H \rightarrow K$  is a linear bijection such that  $S$  and  $S^{-1}$  are bounded operators. Suppose that  $T \in B(H)$ . Show that  $T$  and  $STS^{-1}$  have the same spectrum and the same eigenvalues (if any).

**A.25.** Define an operator  $U : \ell_2(\mathbb{Z}) \rightarrow L_2(-\pi, \pi)$  by  $U((a_n)) = \sum_{n=-\infty}^{\infty} a_n e^{int} / \sqrt{2\pi}$ . Show that  $U$  is a bijection and an isometry, i.e., that  $\|Ux\| = \|x\|$  for all  $x \in \ell_2(\mathbb{Z})$ .

Let  $V$  be the bilateral right shift on  $\ell_2(\mathbb{Z})$ , the unitary operator defined on Question A.20. Let  $f \in L_2(-\pi, \pi)$ . Show that  $(UVU^{-1}f)(t) = e^{it}f(t)$ , and hence, using Question A.24, show that  $\sigma(V) = \mathbb{T}$ , the unit circle, but that  $V$  has no eigenvalues.

**A.6. Tutorial Problems VI.**

**A.26.** Show that  $K(X)$  is a closed linear subspace of  $B(X)$ , and that  $AT$  and  $TA$  are compact whenever  $T \in K(X)$  and  $A \in B(X)$ . (This means that  $K(X)$  is a closed ideal of  $B(X)$ .)

**A.27.** Let  $A$  be a Hilbert–Schmidt operator, and let  $(e_n)_{n \geq 1}$  and  $(f_m)_{m \geq 1}$  be orthonormal bases of  $A$ . By writing each  $Ae_n$  as  $Ae_n = \sum_{m=1}^{\infty} \langle Ae_n, f_m \rangle f_m$ , show that

$$\sum_{n=1}^{\infty} \|Ae_n\|^2 = \sum_{m=1}^{\infty} \|A^*f_m\|^2.$$

Deduce that the quantity  $\|A\|_{HS}^2 = \sum_{n=1}^{\infty} \|Ae_n\|^2$  is independent of the choice of orthonormal basis, and that  $\|A\|_{HS} = \|A^*\|_{HS}$ . ( $\|A\|_{HS}$  is called the *Hilbert–Schmidt norm* of  $A$ .)

- A.28.** (i) Let  $T \in K(H)$  be a compact operator. Using Question A.26, show that  $T^*T$  and  $TT^*$  are compact Hermitian operators.  
 (ii) Let  $(e_n)_{n \geq 1}$  and  $(f_n)_{n \geq 1}$  be orthonormal bases of a Hilbert space  $H$ , let  $(\alpha_n)_{n \geq 1}$  be any bounded complex sequence, and let  $T \in B(H)$  be an operator defined by

$$Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, e_n \rangle f_n.$$

Prove that  $T$  is Hilbert–Schmidt precisely when  $(\alpha_n) \in \ell_2$ . Show that  $T$  is a compact operator if and only if  $\alpha_n \rightarrow 0$ , and in this case write down spectral decompositions for the compact Hermitian operators  $T^*T$  and  $TT^*$ .

**A.29.** Solve the Fredholm integral equation  $\phi - \lambda T\phi = f$ , where  $f(x) = x$  and

$$(T\phi)(x) = \int_0^1 xy^2\phi(y) dy \quad (\phi \in L_2(0, 1)),$$

for small values of  $\lambda$  by means of the Neumann series.

For what values of  $\lambda$  does the series converge? Write down a solution which is valid for all  $\lambda$  apart from one exception. What is the exception?

**A.30.** Suppose that  $h$  is a  $2\pi$ -periodic  $L_2(-\pi, \pi)$  function with Fourier series  $\sum_{n=-\infty}^{\infty} a_n e^{int}$ . Show that each of the functions  $\phi_k(y) = e^{iky}$ ,  $k \in \mathbb{Z}$ , is an eigenvector of the integral operator  $T$  on  $L_2(-\pi, \pi)$  defined by

$$(T\phi)(x) = \int_{-\pi}^{\pi} h(x-y)\phi(y) dy,$$

and calculate the corresponding eigenvalues.

Now let  $h(t) = -\log(2(1 - \cos t))$ . Assuming, without proof, that  $h(t)$  has the Fourier series  $\sum_{n \in \mathbb{Z}, n \neq 0} e^{int}/|n|$ , use the Hilbert–Schmidt method to solve the Fredholm equation  $\phi - \lambda T\phi = f$ , where  $f(t)$  has Fourier series  $\sum_{n=-\infty}^{\infty} c_n e^{int}$  and  $1/\lambda \notin \sigma(T)$ .

#### A.7. Tutorial Problems VII.

**A.31.** Use the Gram–Schmidt algorithm to find an orthonormal basis for the subspace  $X$  of  $L_2(-1, 1)$  spanned by the functions  $t$ ,  $t^2$  and  $t^4$ .

Hence find the best  $L_2(-1, 1)$  approximation of the constant function  $f(t) = 1$  by functions from  $X$ .

**A.32.** For  $n = 1, 2, \dots$  let  $\phi_n$  denote the linear functional on  $\ell_2$  defined by

$$\phi_n(x) = x_1 + x_2 + \dots + x_n,$$

where  $x = (x_1, x_2, \dots) \in \ell_2$ . Use the Riesz–Fréchet theorem to calculate  $\|\phi_n\|$ .

**A.33.** Let  $T$  be a bounded linear operator on a Hilbert space, and suppose that  $T = A + iB$ , where  $A$  and  $B$  are self-adjoint operators. Express  $T^*$  in terms of  $A$  and  $B$ , and hence solve for  $A$  and  $B$  in terms of  $T$  and  $T^*$ .

Deduce that every operator  $T$  can be written  $T = A + iB$ , where  $A$  and  $B$  are self-adjoint, in a unique way.

Show that  $T$  is normal if and only if  $AB = BA$ .

**A.34.** Let  $P_n$  be the subspace of  $L_2(-\pi, \pi)$  consisting of all polynomials of degree at most  $n$ , and let  $T_n$  be the subspace consisting of all trigonometric polynomials of the form  $f(t) = \sum_{k=-n}^n a_k e^{ikt}$ . Calculate the spectrum of the differentiation operator  $D$ , defined by  $(Df)(t) = f'(t)$ , when

- (i)  $D$  is regarded as an operator on  $P_n$ , and
- (ii)  $D$  is regarded as an operator on  $T_n$ .

*Note that both  $P_n$  and  $T_n$  are finite-dimensional Hilbert spaces.*

Show that  $T_n$  has an orthonormal basis of eigenvectors of  $D$ , whereas  $P_n$  does not.

**A.35.** Use the Neumann series to solve the Volterra integral equation  $\phi - \lambda T\phi = f$  in  $L_2[0, 1]$ , where  $\lambda \in \mathbb{C}$ ,  $f(t) = 1$  for all  $t$ , and  $(T\phi)(x) = \int_0^x t^2 \phi(t) dt$ . (You should be able to sum the infinite series.)

## APPENDIX B. SOLUTIONS OF TUTORIAL PROBLEMS

Solutions of the tutorial problems will be distributed due in time on the paper.

## APPENDIX C. COURSE IN THE NUTSHELL

**C.1. Some useful results and formulae (1).**

**C.1.** A *norm* on a vector space,  $\|x\|$ , satisfies  $\|x\| \geq 0$ ,  $\|x\| = 0$  if and only if  $x = 0$ ,  $\|\lambda x\| = |\lambda| \|x\|$ , and  $\|x + y\| \leq \|x\| + \|y\|$  (*triangle inequality*). A norm defines a metric and a complete normed space is called a *Banach space*.

**C.2.** An *inner-product space* is a vector space (usually complex) with a scalar product on it,  $\langle x, y \rangle \in \mathbb{C}$  such that  $\langle x, y \rangle = \overline{\langle y, x \rangle}$ ,  $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$ ,  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ ,  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0$  if and only if  $x = 0$ . This defines a norm by  $\|x\|^2 = \langle x, x \rangle$ . A complete inner-product space is called a *Hilbert space*. A Hilbert space is automatically a Banach space.

**C.3.** The *Cauchy–Schwarz inequality*.  $|\langle x, y \rangle| \leq \|x\| \|y\|$  with equality if and only if  $x$  and  $y$  are linearly dependent.

**C.4.** Some *examples of Hilbert spaces*. (i) Euclidean  $\mathbb{C}^n$ . (ii)  $\ell_2$ , sequences  $(a_k)$  with  $\|(a_k)\|_2^2 = \sum |a_k|^2 < \infty$ . In both cases  $\langle (a_k), (b_k) \rangle = \sum a_k \overline{b_k}$ . (iii)  $L_2[a, b]$ , functions on  $[a, b]$  with  $\|f\|_2^2 = \int_a^b |f(t)|^2 dt < \infty$ . Here  $\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$ . (iv) Any closed subspace of a Hilbert space.

**C.5.** Other *examples of Banach spaces*. (i)  $C_b(X)$ , continuous bounded functions on a topological space  $X$ . (ii)  $\ell_\infty(X)$ , all bounded functions on a set  $X$ . The supremum norms on  $C_b(X)$  and  $\ell_\infty(X)$  make them into Banach spaces. (iii) Any closed subspace of a Banach space.

**C.6.** On *incomplete spaces*. The inner-product ( $L_2$ ) norm on  $C[0, 1]$  is incomplete.  $c_{00}$  (sequences eventually zero), with the  $\ell_2$  norm, is another incomplete i.p.s.

**C.7.** The *parallelogram identity*.  $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$  in an inner-product space. Not in general normed spaces.

**C.8.** On *subspaces*. Complete  $\implies$  closed. The closure of a linear subspace is still a linear subspace.  $\text{Lin}(A)$  is the smallest subspace containing  $A$  and  $\text{CLin}(A)$  is its closure, the smallest closed subspace containing  $A$ .

**C.9.** From now on we work in inner-product spaces.

**C.10.** The *orthogonality*.  $x \perp y$  if  $\langle x, y \rangle = 0$ . An *orthogonal sequence* has  $\langle e_n, e_m \rangle = 0$  for  $n \neq m$ . If all the vectors have norm 1 it is an *orthonormal sequence (o.n.s.)*, e.g.  $e_n = (0, \dots, 0, 1, 0, 0, \dots) \in \ell_2$  and  $e_n(t) = (1/\sqrt{2\pi})e^{int}$  in  $L_2(-\pi, \pi)$ .

**C.11.** *Pythagoras's theorem*: if  $x \perp y$  then  $\|x + y\|^2 = \|x\|^2 + \|y\|^2$ .

**C.12.** The *best approximation* to  $x$  by a linear combination  $\sum_{k=1}^n \lambda_k e_k$  is  $\sum_{k=1}^n \langle x, e_k \rangle e_k$  if the  $e_k$  are orthonormal. Note that  $\langle x, e_k \rangle$  is the Fourier coefficient of  $x$  w.r.t.  $e_k$ .

**C.13.** *Bessel's inequality*.  $\|x\|^2 \geq \sum_{k=1}^n |\langle x, e_k \rangle|^2$  if  $e_1, \dots, e_n$  is an o.n.s.

**C.14.** *Riesz–Fischer theorem*. For an o.n.s.  $(e_n)$  in a Hilbert space,  $\sum \lambda_n e_n$  converges if and only if  $\sum |\lambda_n|^2 < \infty$ ; then  $\|\sum \lambda_n e_n\|^2 = \sum |\lambda_n|^2$ .

**C.15.** A *complete o.n.s.* or *orthonormal basis (o.n.b.)* is an o.n.s.  $(e_n)$  such that if  $\langle y, e_n \rangle = 0$  for all  $n$  then  $y = 0$ . In that case every vector is of the form  $\sum \lambda_n e_n$  as in the R-F theorem. Equivalently: the closed linear span of the  $(e_n)$  is the whole space.

**C.16.** *Gram–Schmidt orthonormalization process*. Start with  $x_1, x_2, \dots$  linearly independent. Construct  $e_1, e_2, \dots$  an o.n.s. by inductively setting  $y_{n+1} = x_{n+1} - \sum_{k=1}^n \langle x_{n+1}, e_k \rangle e_k$  and then normalizing  $e_{n+1} = y_{n+1} / \|y_{n+1}\|$ .

**C.17. On orthogonal complements.**  $M^\perp$  is the set of all vectors orthogonal to everything in  $M$ . If  $M$  is a closed linear subspace of a Hilbert space  $H$  then  $H = M \oplus M^\perp$ . There is also a linear map,  $P_M$  the projection from  $H$  onto  $M$  with kernel  $M^\perp$ .

**C.18. Fourier series.** Work in  $L_2(-\pi, \pi)$  with o.n.s.  $e_n(t) = (1/\sqrt{2\pi})e^{int}$ . Let  $CP(-\pi, \pi)$  be the continuous periodic functions, which are dense in  $L_2$ . For  $f \in CP(-\pi, \pi)$  write  $f_m = \sum_{n=-m}^m \langle f, e_n \rangle e_n$ ,  $m \geq 0$ . We wish to show that  $\|f_m - f\|_2 \rightarrow 0$ , i.e., that  $(e_n)$  is an o.n.b.

**C.19. The Fejér kernel.** For  $f \in CP(-\pi, \pi)$  write  $F_m = (f_0 + \dots + f_m)/(m+1)$ . Then  $F_m(x) = (1/2\pi) \int_{-\pi}^{\pi} f(t)K_m(x-t) dt$  where  $K_m(t) = (1/(m+1)) \sum_{k=0}^m \sum_{n=-k}^k e^{int}$  is the Fejér kernel. Also  $K_m(t) = (1/(m+1))[\sin^2(m+1)t/2]/[\sin^2 t/2]$ .

**C.20. Fejér's theorem.** If  $f \in CP(-\pi, \pi)$  then its Fejér sums tend uniformly to  $f$  on  $[-\pi, \pi]$  and hence in  $L_2$  norm also. Hence  $CLin((e_n)) \supseteq CP(-\pi, \pi)$  so must be all of  $L_2(-\pi, \pi)$ . Thus  $(e_n)$  is an o.n.b.

**C.21. Corollary.** If  $f \in L_2(-\pi, \pi)$  then  $f(t) = \sum c_n e^{int}$  with convergence in  $L_2$ , where  $c_n = (1/2\pi) \int_{-\pi}^{\pi} f(t)e^{-int} dt$ .

**C.22. Parseval's formula.** If  $f, g \in L_2(-\pi, \pi)$  have Fourier series  $\sum c_n e^{int}$  and  $\sum d_n e^{int}$  then  $(1/2\pi)\langle f, g \rangle = \sum c_n \bar{d}_n$ .

**C.23. Weierstrass approximation theorem.** The polynomials are dense in  $C[a, b]$  for any  $a < b$  (in the supremum norm).

## C.2. Some useful results and formulae (2).

**C.24. On dual spaces.** A linear functional on a vector space  $X$  is a linear mapping  $\alpha : X \rightarrow \mathbb{C}$  (or to  $\mathbb{R}$  in the real case), i.e.,  $\alpha(ax + by) = a\alpha(x) + b\alpha(y)$ . When  $X$  is a normed space,  $\alpha$  is continuous if and only if it is bounded, i.e.,  $\sup\{|\alpha(x)| : \|x\| \leq 1\} < \infty$ . Then we define  $\|\alpha\|$  to be this sup, and it is a norm on the space  $X^*$  of bounded linear functionals, making  $X^*$  into a Banach space.

**C.25. Riesz-Fréchet theorem.** If  $\alpha : H \rightarrow \mathbb{C}$  is a bounded linear functional on a Hilbert space  $H$ , then there is a unique  $y \in H$  such that  $\alpha(x) = \langle x, y \rangle$  for all  $x \in H$ ; also  $\|\alpha\| = \|y\|$ .

**C.26. On linear operator.** These are linear mappings  $T : X \rightarrow Y$ , between normed spaces. Defining  $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$ , finite, makes the bounded (i.e., continuous) operators into a normed space,  $B(X, Y)$ . When  $Y$  is complete, so is  $B(X, Y)$ . We get  $\|Tx\| \leq \|T\| \|x\|$ , and, when we can compose operators,  $\|ST\| \leq \|S\| \|T\|$ . Write  $B(X)$  for  $B(X, X)$ , and for  $T \in B(X)$ ,  $\|T^n\| \leq \|T\|^n$ . Inverse  $S = T^{-1}$  when  $ST = TS = I$ .

**C.27. On adjoints.**  $T \in B(H, K)$  determines  $T^* \in B(K, H)$  such that  $\langle Th, k \rangle_K = \langle h, T^*k \rangle_H$  for all  $h \in H, k \in K$ . Also  $\|T^*\| = \|T\|$  and  $T^{**} = T$ .

**C.28. On unitary operator.** Those  $U \in B(H)$  for which  $UU^* = U^*U = I$ . Equivalently,  $U$  is surjective and an isometry (and hence preserves the inner product).

*Hermitian operator or self-adjoint operator.* Those  $T \in B(H)$  such that  $T = T^*$ .

*On normal operator.* Those  $T \in B(H)$  such that  $TT^* = T^*T$  (so including Hermitian and unitary operators).

**C.29. On spectrum.**  $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ is not invertible in } B(X)\}$ . Includes all eigenvalues  $\lambda$  where  $Tx = \lambda x$  for some  $x \neq 0$ , and often other things as well. On spectral radius:  $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ . Properties:  $\sigma(T)$  is closed, bounded and

nonempty. Proof: based on the fact that  $(I - A)$  is invertible for  $\|A\| < 1$ . This implies that  $r(T) \leq \|T\|$ .

**C.30.** The *spectral radius formula*.  $r(T) = \inf_{n \geq 1} \|T^n\|^{1/n} = \lim_{n \rightarrow \infty} \|T^n\|^{1/n}$ .

Note that  $\sigma(T^n) = \{\lambda^n : \lambda \in \sigma(T)\}$  and  $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$ . The spectrum of a unitary operator is contained in  $\{|z| = 1\}$ , and the spectrum of a self-adjoint operator is real (proof by *Cayley transform*:  $U = (T - iI)(T + iI)^{-1}$  is unitary).

**C.31.** On *finite rank operator*.  $T \in F(X, Y)$  if  $\text{Im } T$  is finite-dimensional.

On *compact operator*.  $T \in K(X, Y)$  if: whenever  $(x_n)$  is bounded, then  $(Tx_n)$  has a convergent subsequence. Now  $F(X, Y) \subseteq K(X, Y)$  since bounded sequences in a finite-dimensional space have convergent subsequences (because when  $Z$  is f.d.,  $Z$  is isomorphic to  $\ell_2^n$ , i.e.,  $\exists S : \ell_2^n \rightarrow Z$  with  $S, S^{-1}$  bounded). Also limits of compact operators are compact, which shows that a diagonal operator  $Tx = \sum \lambda_n \langle x, e_n \rangle e_n$  is compact iff  $\lambda_n \rightarrow 0$ .

**C.32.** *Hilbert–Schmidt operators*.  $T$  is H–S when  $\sum \|Te_n\|^2 < \infty$  for some o.n.b.  $(e_n)$ . All such operators are compact—write them as a limit of finite rank operators  $T_k$  with  $T_k \sum_{n=1}^{\infty} a_n e_n = \sum_{n=1}^k a_n (Te_n)$ . This class includes integral operators  $T : L_2(a, b) \rightarrow L_2(a, b)$  of the form

$$(Tf)(x) = \int_a^b K(x, y)f(y) dy,$$

where  $K$  is continuous on  $[a, b] \times [a, b]$ .

**C.33.** On *spectral properties of normal operators*. If  $T$  is normal, then (i)  $\ker T = \ker T^*$ , so  $Tx = \lambda x \implies T^*x = \bar{\lambda}x$ ; (ii) eigenvectors corresponding to distinct eigenvalues are orthogonal; (iii)  $\|T\| = r(T)$ .

If  $T \in B(H)$  is compact normal, then its set of eigenvalues is either finite or a sequence tending to zero. The eigenspaces are finite-dimensional, except possibly for  $\lambda = 0$ . All nonzero points of the spectrum are eigenvalues.

**C.34.** On *spectral theorem for compact normal operators*. There is an orthonormal sequence  $(e_k)$  of eigenvectors of  $T$ , and eigenvalues  $(\lambda_k)$ , such that  $Tx = \sum_k \lambda_k \langle x, e_k \rangle e_k$ . If  $(\lambda_k)$  is an infinite sequence, then it tends to 0. All operators of the above form are compact and normal.

*Corollary.* In the spectral theorem we can have the same formula with an orthonormal *basis*, adding in vectors from  $\ker T$ .

**C.35.** On *general compact operators*. We can write  $Tx = \sum \mu_k \langle x, e_k \rangle f_k$ , where  $(e_k)$  and  $(f_k)$  are orthonormal sequences and  $(\mu_k)$  is either a finite sequence or an infinite sequence tending to 0. Hence  $T \in B(H)$  is compact if and only if it is the norm limit of a sequence of finite-rank operators.

**C.36.** On *integral equations*. Fredholm equations on  $L_2(a, b)$  are  $T\phi = f$  or  $\phi - \lambda T\phi = f$ , where  $(T\phi)(x) = \int_a^b K(x, y)\phi(y) dy$ . Volterra equations similar, except that  $T$  is now defined by  $(T\phi)(x) = \int_a^x K(x, y)\phi(y) dy$ .

**C.37.** *Neumann series*.  $(I - \lambda T)^{-1} = 1 + \lambda T + \lambda^2 T^2 + \dots$ , for  $\|\lambda T\| < 1$ .

On *separable kernel*.  $K(x, y) = \sum_{j=1}^n g_j(x)h_j(y)$ . The image of  $T$  (and hence its eigenvectors for  $\lambda \neq 0$ ) lies in the space spanned by  $g_1, \dots, g_n$ .

**C.38.** *Hilbert–Schmidt theory*. Suppose that  $K \in C([a, b] \times [a, b])$  and  $K(y, x) = \overline{K(x, y)}$ . Then (in the Fredholm case)  $T$  is a self-adjoint Hilbert–Schmidt operator



and eigenvectors corresponding to nonzero eigenvalues are continuous functions. If  $\lambda \neq 0$  and  $1/\lambda \notin \sigma(T)$ , the the solution of  $\phi - \lambda T\phi = f$  is

$$\phi = \sum_{k=1}^{\infty} \frac{\langle f, v_k \rangle}{1 - \lambda \lambda_k} v_k.$$

**C.39.** *Fredholm alternative.* Let  $T$  be compact and normal and  $\lambda \neq 0$ . Consider the equations (i)  $\phi - \lambda T\phi = 0$  and (ii)  $\phi - \lambda T\phi = f$ . Then EITHER (A) The only solution of (i) is  $\phi = 0$  and (ii) has a unique solution for all  $f$  OR (B) (i) has nonzero solutions  $\phi$  and (ii) can be solved if and only if  $f$  is orthogonal to every solution of (i).

## APPENDIX D. SUPPLEMENTARY SECTIONS

**D.1. Reminder from Complex Analysis.** The analytic function theory is the most powerful tool in the operator theory. Here we briefly recall few facts of complex analysis used in this course. Use any decent textbook on complex variables for a concise exposition. The only difference with our version that we consider function  $f(z)$  of a complex variable  $z$  taking value in an arbitrary normed space  $V$  over the field  $\mathbb{C}$ . By the direct inspection we could check that all standard proofs of the listed results work as well in this more general case.

**Definition D.1.** A function  $f(z)$  of a complex variable  $z$  taking value in a normed vector space  $V$  is called *differentiable* at a point  $z_0$  if the following limit (called *derivative* of  $f(z)$  at  $z_0$ ) exists:

$$(D.1) \quad f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

**Definition D.2.** A function  $f(z)$  is called *holomorphic* (or *analytic*) in an open set  $\Omega \subset \mathbb{C}$  if it is differentiable at any point of  $\Omega$ .

**Theorem D.3** (Laurent Series). *Let a function  $f(z)$  be analytical in the annulus  $r < z < R$  for some real  $r < R$ , then it could be uniquely represented by the Laurent series:*

$$(D.2) \quad f(z) = \sum_{k=-\infty}^{\infty} c_k z^k, \quad \text{for some } c_k \in V.$$

**Theorem D.4** (Cauchy–Hadamard). *The radii  $r'$  and  $R'$ , ( $r' < R'$ ) of convergence of the Laurent series (D.2) are given by*

$$(D.3) \quad r' = \liminf_{n \rightarrow \infty} \|c_n\|^{1/n} \quad \text{and} \quad \frac{1}{R'} = \limsup_{n \rightarrow \infty} \|c_n\|^{1/n}.$$

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