

## CHAPTER 6. TECHNIQUES OF INTEGRATION

### Section 6.1 Integration by Parts (page 321)

1.  $\int x \cos x \, dx$

$$\begin{aligned} U &= x & dV &= \cos x \, dx \\ dU &= dx & V &= \sin x \\ &= x \sin x - \int \sin x \, dx \\ &= x \sin x + \cos x + C. \end{aligned}$$

2.  $\int (x+3)e^{2x} \, dx$

$$\begin{aligned} U &= x+3 & dV &= e^{2x} \, dx \\ dU &= dx & V &= \frac{1}{2}e^{2x} \\ &= \frac{1}{2}(x+3)e^{2x} - \frac{1}{2} \int e^{2x} \, dx \\ &= \frac{1}{2}(x+3)e^{2x} - \frac{1}{4}e^{2x} + C. \end{aligned}$$

3.  $\int x^2 \cos \pi x \, dx$

$$\begin{aligned} U &= x^2 & dV &= \cos \pi x \, dx \\ dU &= 2x \, dx & V &= \frac{\sin \pi x}{\pi} \\ &= \frac{x^2 \sin \pi x}{\pi} - \frac{2}{\pi} \int x \sin \pi x \, dx \\ U &= x & dV &= \sin \pi x \, dx \\ dU &= dx & V &= -\frac{\cos \pi x}{\pi} \\ &= \frac{x^2 \sin \pi x}{\pi} - \frac{2}{\pi} \left( -\frac{x \cos \pi x}{\pi} + \frac{1}{\pi} \int \cos \pi x \, dx \right) \\ &= \frac{1}{\pi} x^2 \sin \pi x + \frac{2}{\pi^2} x \cos \pi x - \frac{2}{\pi^3} \sin \pi x + C. \end{aligned}$$

4.  $\int (x^2 - 2x)e^{kx} \, dx$

$$\begin{aligned} U &= x^2 - 2x & dV &= e^{kx} \\ dU &= (2x - 2) \, dx & V &= \frac{1}{k}e^{kx} \\ &= \frac{1}{k}(x^2 - 2x)e^{kx} - \frac{1}{k} \int (2x - 2)e^{kx} \, dx \\ U &= x - 1 & dV &= e^{kx} \, dx \\ dU &= dx & V &= \frac{1}{k}e^{kx} \\ &= \frac{1}{k}(x^2 - 2x)e^{kx} - \frac{2}{k} \left[ \frac{1}{k}(x - 1)e^{kx} - \frac{1}{k} \int e^{kx} \, dx \right] \\ &= \frac{1}{k}(x^2 - 2x)e^{kx} - \frac{2}{k^2}(x - 1)e^{kx} + \frac{2}{k^3}e^{kx} + C. \end{aligned}$$

5.  $\int x^3 \ln x \, dx$

$$\begin{aligned} U &= \ln x & dV &= x^3 \, dx \\ dU &= \frac{dx}{x} & V &= \frac{x^4}{4} \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{4} \int x^3 \, dx \\ &= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C. \end{aligned}$$

6.  $\int x(\ln x)^3 \, dx = I_3$  where

$$\begin{aligned} I_n &= \int x(\ln x)^n \, dx \\ U &= (\ln x)^n & dV &= x \, dx \\ dU &= \frac{n}{x}(\ln x)^{n-1} \, dx & V &= \frac{1}{2}x^2 \\ &= \frac{1}{2}x^2(\ln x)^n - \frac{n}{2} \int x(\ln x)^{n-1} \, dx \\ &= \frac{1}{2}x^2(\ln x)^n - \frac{n}{2}I_{n-1} \\ I_3 &= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2}I_2 \\ &= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{2} \left[ \frac{1}{2}x^2(\ln x)^2 - \frac{2}{2}I_1 \right] \\ &= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{2} \left[ \frac{1}{2}x^2(\ln x) - \frac{1}{2}I_0 \right] \\ &= \frac{1}{2}x^2(\ln x)^3 - \frac{3}{4}x^2(\ln x)^2 + \frac{3}{4}x^2(\ln x) - \frac{3}{4} \int x \, dx \\ &= \frac{x^2}{2} \left[ (\ln x)^3 - \frac{3}{2}(\ln x)^2 + \frac{3}{2}(\ln x) - \frac{3}{4} \right] + C. \end{aligned}$$

7.  $\int \tan^{-1} x \, dx$

$$\begin{aligned} U &= \tan^{-1} x & dV &= dx \\ dU &= \frac{dx}{1+x^2} & V &= x \\ &= x \tan^{-1} x - \int \frac{x \, dx}{1+x^2} \\ &= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C. \end{aligned}$$

8.  $\int x^2 \tan^{-1} x \, dx$

$$\begin{aligned} U &= \tan^{-1} x & dV &= x^2 \, dx \\ dU &= \frac{dx}{1+x^2} & V &= \frac{x^3}{3} \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} \, dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \left( x - \frac{x}{1+x^2} \right) \, dx \\ &= \frac{x^3}{3} \tan^{-1} x - \frac{x^2}{6} + \frac{1}{6} \ln(1+x^2) + C. \end{aligned}$$

$$\begin{aligned}
 9. \quad & \int x \sin^{-1} x \, dx \\
 & U = \sin^{-1} x \quad dV = x \, dx \\
 & dU = \frac{dx}{\sqrt{1-x^2}} \quad V = \frac{x^2}{2} \\
 & = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \frac{x^2 dx}{\sqrt{1-x^2}} \quad \text{Let } x = \sin \theta \\
 & \quad \quad \quad dx = \cos \theta \, d\theta \\
 & = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{2} \int \sin^2 \theta \, d\theta \\
 & = \frac{1}{2} x^2 \sin^{-1} x - \frac{1}{4} (\theta - \sin \theta \cos \theta) + C \\
 & = \left( \frac{1}{2} x^2 - \frac{1}{4} \right) \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} + C.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & \int x^5 e^{-x^2} \, dx = I_2 \text{ where} \\
 & I_n = \int x^{(2n+1)} e^{-x^2} \, dx
 \end{aligned}$$

$$\begin{aligned}
 & U = x^{2n} \quad dV = x e^{-x^2} \, dx \\
 & dU = 2nx^{(2n-1)} \, dx \quad V = -\frac{1}{2} e^{-x^2} \\
 & = -\frac{1}{2} x^{2n} e^{-x^2} + n \int x^{(2n-1)} e^{-x^2} \, dx \\
 & = -\frac{1}{2} x^{2n} e^{-x^2} + n I_{n-1} \\
 I_2 & = -\frac{1}{2} x^4 e^{-x^2} + 2 \left[ -\frac{1}{2} x^2 e^{-x^2} + \int x e^{-x^2} \, dx \right] \\
 & = -\frac{1}{2} e^{-x^2} (x^4 + 2x^2 + 2) + C.
 \end{aligned}$$

$$11. \quad I_n = \int_0^{\pi/4} \sec^n x \, dx$$

$$\begin{aligned}
 & U = \sec^{n-2} x \quad dV = \sec^2 x \, dx \\
 & dU = (n-2) \sec^{n-2} x \tan x \, dx \quad V = \tan x \\
 & = \tan x \sec^{n-2} x \Big|_0^{\pi/4} - (n-2) \int_0^{\pi/4} \sec^{n-2} x \tan^2 x \, dx \\
 & = (\sqrt{2})^{n-2} - (n-2)(I_n - I_{n-2}). \\
 (n-1)I_n & = (\sqrt{2})^{n-2} + (n-2)I_{n-2}.
 \end{aligned}$$

Therefore

$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}, \quad (n \geq 2).$$

For  $n = 5$  we have

$$\begin{aligned}
 \int_0^{\pi/4} \sec^5 x \, dx & = I_5 = \frac{2\sqrt{2}}{4} + \frac{3}{4} I_3 \\
 & = \frac{\sqrt{2}}{2} + \frac{3}{4} \left( \frac{\sqrt{2}}{2} + \frac{1}{2} I_1 \right) \\
 & = \frac{7\sqrt{2}}{8} + \frac{3}{8} \ln |\sec x + \tan x| \Big|_0^{\pi/4} \\
 & = \frac{7\sqrt{2}}{8} + \frac{3}{8} \ln(1 + \sqrt{2}).
 \end{aligned}$$

$$12. \quad I = \int \tan^2 x \sec x \, dx$$

$$\begin{aligned}
 & U = \tan x \quad dV = \sec x \tan x \, dx \\
 & dU = \sec^2 x \, dx \quad V = \sec x \\
 & = \sec x \tan x - \int \sec^3 x \, dx \\
 & = \sec x \tan x - \int (1 + \tan^2 x) \sec x \, dx \\
 & = \sec x \tan x - \ln |\sec x + \tan x| - I \\
 \text{Thus, } I & = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C.
 \end{aligned}$$

$$13. \quad I = \int e^{2x} \sin 3x \, dx$$

$$\begin{aligned}
 & U = e^{2x} \quad dV = \sin 3x \, dx \\
 & dU = 2e^{2x} \, dx \quad V = -\frac{1}{3} \cos 3x \\
 & = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \int e^{2x} \cos 3x \, dx \\
 & \quad \quad \quad U = e^{2x} \quad dV = \cos 3x \, dx \\
 & \quad \quad \quad dU = 2e^{2x} \, dx \quad V = \frac{1}{3} \sin 3x \\
 & = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{3} \left( \frac{1}{3} e^{2x} \sin 3x - \frac{2}{3} I \right) \\
 \frac{13}{9} I & = -\frac{1}{3} e^{2x} \cos 3x + \frac{2}{9} e^{2x} \sin 3x + C_1 \\
 I & = \frac{1}{13} e^{2x} (2 \sin 3x - 3 \cos 3x) + C.
 \end{aligned}$$

$$14. \quad I = \int x e^{\sqrt{x}} \, dx \quad \text{Let } x = w^2 \\
 \quad \quad \quad dx = 2w \, dw$$

$$= 2 \int w^3 e^w \, dw = 2I_3 \text{ where}$$

$$I_n = \int w^n e^w \, dw$$

$$\begin{aligned}
 & U = w^n \quad dV = e^w \, dw \\
 & dU = n w^{n-1} \, dw \quad V = e^w \\
 & = w^n e^w - n I_{n-1}. \\
 I & = 2I_3 = 2w^3 e^w - 6[w^2 e^w - 2(w e^w - I_0)] \\
 & = e^{\sqrt{x}} (2x\sqrt{x} - 6x + 12\sqrt{x} - 12) + C.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \int_{1/2}^1 \frac{\sin^{-1} x}{x^2} dx \\
 & U = \sin^{-1} x \quad dV = \frac{dx}{x^2} \\
 & dU = \frac{dx}{\sqrt{1-x^2}} \quad V = -\frac{1}{x} \\
 & = -\frac{1}{x} \sin^{-1} x \Big|_{1/2}^1 + \int_{1/2}^1 \frac{dx}{x\sqrt{1-x^2}} \quad \text{Let } x = \sin \theta \\
 & \quad \quad \quad dx = \cos \theta d\theta \\
 & = -\frac{\pi}{2} + \frac{\pi}{3} + \int_{\pi/6}^{\pi/2} \csc \theta d\theta \\
 & = -\frac{\pi}{6} - \ln |\csc \theta + \cot \theta| \Big|_{\pi/6}^{\pi/2} \\
 & = -\frac{\pi}{6} - \ln 1 + \ln(2 + \sqrt{3}) = \ln(2 + \sqrt{3}) - \frac{\pi}{6}.
 \end{aligned}$$

$$\begin{aligned}
 16. \quad & \int_0^1 \sqrt{x} \sin(\pi \sqrt{x}) dx \quad \text{Let } x = w^2 \\
 & \quad \quad \quad dx = 2w dw \\
 & = 2 \int_0^1 w^2 \sin(\pi w) dw \\
 & \quad U = w^2 \quad dV = \sin(\pi w) dw \\
 & \quad dU = 2w dw \quad V = -\frac{\cos(\pi w)}{\pi} \\
 & = -\frac{2}{\pi} w^2 \cos(\pi w) \Big|_0^1 + \frac{4}{\pi} \int_0^1 w \cos(\pi w) dw \\
 & \quad U = w \quad dV = \cos(\pi w) dw \\
 & \quad dU = dw \quad V = \frac{\sin(\pi w)}{\pi} \\
 & = \frac{2}{\pi} + \frac{4}{\pi} \left[ \frac{w}{\pi} \sin(\pi w) \right] \Big|_0^1 - \frac{4}{\pi^2} \int_0^1 \sin(\pi w) dw \\
 & = \frac{2}{\pi} + \frac{4}{\pi^3} \cos(\pi w) \Big|_0^1 = \frac{2}{\pi} + \frac{4}{\pi^3} (-2) = \frac{2}{\pi} - \frac{8}{\pi^3}.
 \end{aligned}$$

$$\begin{aligned}
 17. \quad & \int x \sec^2 x dx \\
 & U = x \quad dV = \sec^2 x dx \\
 & dU = dx \quad V = \tan x \\
 & = x \tan x - \int \tan x dx \\
 & = x \tan x - \ln |\sec x| + C.
 \end{aligned}$$

$$\begin{aligned}
 18. \quad & \int x \sin^2 x dx = \frac{1}{2} \int (x - x \cos 2x) dx \\
 & = \frac{x^2}{4} - \frac{1}{2} \int x \cos 2x dx \\
 & \quad U = x \quad dV = \cos 2x dx \\
 & \quad dU = dx \quad V = \frac{1}{2} \sin 2x \\
 & = \frac{x^2}{4} - \frac{1}{2} \left[ \frac{1}{2} x \sin 2x - \frac{1}{2} \int \sin 2x dx \right] \\
 & = \frac{x^2}{4} - \frac{x}{4} \sin 2x - \frac{1}{8} \cos 2x + C.
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & I = \int \cos(\ln x) dx \\
 & \quad U = \cos(\ln x) \quad dV = dx \\
 & \quad dU = -\frac{\sin(\ln x)}{x} dx \quad V = x \\
 & = x \cos(\ln x) + \int \sin(\ln x) dx \\
 & \quad U = \sin(\ln x) \quad dV = dx \\
 & \quad dU = \frac{\cos(\ln x)}{x} dx \quad V = x \\
 & = x \cos(\ln x) + x \sin(\ln x) - I \\
 & I = \frac{1}{2} (x \cos(\ln x) + x \sin(\ln x)) + C.
 \end{aligned}$$

$$\begin{aligned}
 20. \quad & I = \int_1^e \sin(\ln x) dx \\
 & \quad U = \sin(\ln x) \quad dV = dx \\
 & \quad dU = \frac{\cos(\ln x)}{x} dx \quad V = x \\
 & = x \sin(\ln x) \Big|_1^e - \int_1^e \cos(\ln x) dx \\
 & \quad U = \cos(\ln x) \quad dV = dx \\
 & \quad dU = -\frac{\sin(\ln x)}{x} dx \quad V = x \\
 & = e \sin(1) - \left[ x \cos(\ln x) \Big|_1^e + I \right] \\
 & \text{Thus, } I = \frac{1}{2} [e \sin(1) - e \cos(1) + 1].
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & \int \frac{\ln(\ln x)}{x} dx \quad \text{Let } u = \ln x \\
 & \quad \quad \quad du = \frac{dx}{x} \\
 & = \int \ln u du \\
 & \quad U = \ln u \quad dV = du \\
 & \quad dU = \frac{du}{u} \quad V = u \\
 & = u \ln u - \int du = u \ln u - u + C \\
 & = (\ln x)(\ln(\ln x)) - \ln x + C.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad & \int_0^4 \sqrt{x} e^{\sqrt{x}} dx \quad \text{Let } x = w^2 \\
 & \quad \quad \quad dx = 2w dw \\
 & = 2 \int_0^2 w^2 e^w dw = 2I_2 \\
 & \quad \text{See solution \#16 for the formula} \\
 & \quad I_n = \int w^n e^w dw = w^n e^w - n I_{n-1}. \\
 & = 2 \left( w^2 e^w \Big|_0^2 - 2I_1 \right) = 8e^2 - 4 \left( w e^w \Big|_0^2 - I_0 \right) \\
 & = 8e^2 - 8e^2 + 4 \int_0^2 e^w dw = 4(e^2 - 1).
 \end{aligned}$$

$$23. \int \cos^{-1} x \, dx$$

$$\begin{aligned} U &= \cos^{-1} x & dV &= dx \\ dU &= -\frac{dx}{\sqrt{1-x^2}} & V &= x \\ &= x \cos^{-1} x + \int \frac{x \, dx}{\sqrt{1-x^2}} \\ &= x \cos^{-1} x - \sqrt{1-x^2} + C. \end{aligned}$$

$$24. \int x \sec^{-1} x \, dx$$

$$\begin{aligned} U &= \sec^{-1} x & dV &= x \, dx \\ dU &= \frac{dx}{|x|\sqrt{x^2-1}} & V &= \frac{1}{2}x^2 \\ &= \frac{1}{2}x^2 \sec^{-1} x - \frac{1}{2} \int \frac{|x|}{\sqrt{x^2-1}} \, dx \\ &= \frac{1}{2}x^2 \sec^{-1} x - \frac{1}{2} \operatorname{sgn}(x) \sqrt{x^2-1} + C. \end{aligned}$$

$$25. \int_1^2 \sec^{-1} x \, dx$$

$$\begin{aligned} &= \int_1^2 \cos^{-1} \frac{1}{x} \, dx \\ U &= \cos^{-1} \frac{1}{x} & dV &= dx \\ dU &= -\frac{1}{\sqrt{1-\frac{1}{x^2}}} \left(-\frac{1}{x^2}\right) dx & V &= x \\ &= x \cos^{-1} \frac{1}{x} \Big|_1^2 - \int_1^2 \frac{dx}{\sqrt{x^2-1}} \quad \text{Let } x = \sec \theta \\ &= \frac{2\pi}{3} - 0 - \int_0^{\pi/3} \sec \theta \, d\theta \\ &= \frac{2\pi}{3} - \ln |\sec \theta + \tan \theta| \Big|_0^{\pi/3} \\ &= \frac{2\pi}{3} - \ln(2 + \sqrt{3}). \end{aligned}$$

$$26. \int (\sin^{-1} x)^2 \, dx \quad \text{Let } x = \sin \theta$$

$$\begin{aligned} &dx = \cos \theta \, d\theta \\ &= \int \theta^2 \cos \theta \, d\theta \\ U &= \theta^2 & dV &= \cos \theta \, d\theta \\ dU &= 2\theta \, d\theta & V &= \sin \theta \\ &= \theta^2 \sin \theta - 2 \int \theta \sin \theta \, d\theta \\ U &= \theta & dV &= \sin \theta \, d\theta \\ dU &= d\theta & V &= -\cos \theta \\ &= \theta^2 \sin \theta - 2(-\theta \cos \theta + \int \cos \theta \, d\theta) \\ &= \theta^2 \sin \theta + 2\theta \cos \theta - 2 \sin \theta + C \\ &= x(\sin^{-1} x)^2 + 2\sqrt{1-x^2}(\sin^{-1} x) - 2x + C. \end{aligned}$$

$$27. \int x(\tan^{-1} x)^2 \, dx$$

$$\begin{aligned} U &= (\tan^{-1} x)^2 & dV &= x \, dx \\ dU &= \frac{2 \tan^{-1} x \, dx}{1+x^2} & V &= \frac{x^2}{2} \\ &= \frac{x^2}{2} (\tan^{-1} x)^2 - \int \frac{x^2 \tan^{-1} x}{1+x^2} \, dx \quad \text{Let } u = \tan^{-1} x \\ & & du &= \frac{dx}{1+x^2} \\ &= \frac{x^2}{2} (\tan^{-1} x)^2 - \int u \tan^2 u \, du \\ &= \frac{x^2}{2} (\tan^{-1} x)^2 + \int (u - u \sec^2 u) \, du \\ &= \frac{x^2}{2} (\tan^{-1} x)^2 + \frac{u^2}{2} - \int u \sec^2 u \, du \\ U &= u & dV &= \sec^2 u \, du \\ dU &= du & V &= \tan u \\ &= \frac{1}{2}(x^2+1)(\tan^{-1} x)^2 - u \tan u + \int \tan u \, du \\ &= \frac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x \tan^{-1} x + \ln |\sec u| + C \\ &= \frac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x \tan^{-1} x + \frac{1}{2} \ln(1+x^2) + C \end{aligned}$$

28. By the procedure used in Example 4 of Section 7.1,

$$\begin{aligned} \int e^x \cos x \, dx &= \frac{1}{2} e^x (\sin x + \cos x) + C; \\ \int e^x \sin x \, dx &= \frac{1}{2} e^x (\sin x - \cos x) + C. \end{aligned}$$

Now

$$\begin{aligned} &\int x e^x \cos x \, dx \\ U &= x & dV &= e^x \cos x \, dx \\ dU &= dx & V &= \frac{1}{2} e^x (\sin x + \cos x) \\ &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} \int e^x (\sin x + \cos x) \, dx \\ &= \frac{1}{2} x e^x (\sin x + \cos x) \\ &\quad - \frac{1}{4} e^x (\sin x - \cos x + \sin x + \cos x) + C \\ &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} e^x \sin x + C. \end{aligned}$$

$$\begin{aligned}
 29. \quad \text{Area} = A &= \int_0^{\pi} e^{-x} \sin x \, dx \\
 U &= e^{-x} \quad dV = \sin x \, dx \\
 dU &= -e^{-x} \, dx \quad V = -\cos x \\
 &= -e^{-x} \cos x \Big|_0^{\pi} - \int_0^{\pi} e^{-x} \cos x \, dx \\
 U &= e^{-x} \quad dV = \cos x \, dx \\
 dU &= -e^{-x} \, dx \quad V = \sin x \\
 &= e^{-\pi} + 1 - \left( e^{-x} \sin x \Big|_0^{\pi} + A \right) \\
 \text{Thus Area} = A &= \frac{1 + e^{-\pi}}{2} \text{ units}^2
 \end{aligned}$$

30. The tangent line to  $y = \ln x$  at  $x = 1$  is  $y = x - 1$ .  
Hence,

$$\begin{aligned}
 \text{Shaded area} &= \frac{1}{2}(1)(1) + (1)(e - 2) - \int_1^e \ln x \, dx \\
 &= e - \frac{3}{2} - (x \ln x - x) \Big|_1^e \\
 &= e - \frac{3}{2} - e + e + 0 - 1 = e - \frac{5}{2} \text{ sq. units.}
 \end{aligned}$$

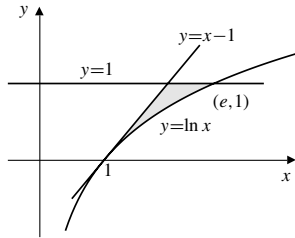


Fig. 6.1.30

$$\begin{aligned}
 31. \quad I_n &= \int (\ln x)^n \, dx \\
 U &= (\ln x)^n \quad dV = dx \\
 dU &= n(\ln x)^{n-1} \frac{dx}{x} \quad V = x \\
 I_n &= x(\ln x)^n - nI_{n-1}. \\
 I_4 &= x(\ln x)^4 - 4I_3 \\
 &= x(\ln x)^4 - 4(x(\ln x)^3 - 3I_2) \\
 &= x(\ln x)^4 - 4x(\ln x)^3 + 12(x(\ln x)^2 - 2I_1) \\
 &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 \\
 &\quad - 24(x \ln x - x) + C \\
 &= x((\ln x)^4 - 4(\ln x)^3 + 12(\ln x)^2 - 24 \ln x + 24) + C.
 \end{aligned}$$

$$\begin{aligned}
 32. \quad I_n &= \int_0^{\pi/2} x^n \sin x \, dx \\
 U &= x^n \quad dV = \sin x \, dx \\
 dU &= nx^{n-1} \, dx \quad V = -\cos x \\
 &= -x^n \cos x \Big|_0^{\pi/2} + n \int_0^{\pi/2} x^{n-1} \cos x \, dx \\
 U &= x^{n-1} \quad dV = \cos x \, dx \\
 dU &= (n-1)x^{n-2} \, dx \quad V = \sin x \\
 &= n \left[ x^{n-1} \sin x \Big|_0^{\pi/2} - (n-1) \int_0^{\pi/2} x^{n-2} \sin x \, dx \right] \\
 &= n \left( \frac{\pi}{2} \right)^{n-1} - n(n-1)I_{n-2}, \quad (n \geq 2). \\
 I_0 &= \int_0^{\pi/2} \sin x \, dx = -\cos x \Big|_0^{\pi/2} = 1. \\
 I_6 &= 6 \left( \frac{\pi}{2} \right)^5 - 6(5) \left\{ 4 \left( \frac{\pi}{2} \right)^3 - 4(3) \left[ 2 \left( \frac{\pi}{2} \right) - 2(1)I_0 \right] \right\} \\
 &= \frac{3}{16}\pi^5 - 15\pi^3 + 360\pi - 720.
 \end{aligned}$$

$$\begin{aligned}
 33. \quad I_n &= \int \sin^n x \, dx \quad (n \geq 2) \\
 U &= \sin^{n-1} x \quad dV = \sin x \, dx \\
 dU &= (n-1) \sin^{n-2} x \cos x \, dx \quad V = -\cos x \\
 &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x \, dx \\
 &= -\sin^{n-1} x \cos x + (n-1)(I_{n-2} - I_n) \\
 nI_n &= -\sin^{n-1} x \cos x + (n-1)I_{n-2} \\
 I_n &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} I_{n-2}.
 \end{aligned}$$

Note:  $I_0 = x + C$ ,  $I_1 = -\cos x + C$ . Hence

$$\begin{aligned}
 I_6 &= -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} I_4 \\
 &= -\frac{1}{6} \sin^5 x \cos x + \frac{5}{6} \left( -\frac{1}{4} \sin^3 x \cos x + \frac{3}{4} I_2 \right) \\
 &= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x \\
 &\quad + \frac{5}{8} \left( -\frac{1}{2} \sin x \cos x + \frac{1}{2} I_0 \right) \\
 &= -\frac{1}{6} \sin^5 x \cos x - \frac{5}{24} \sin^3 x \cos x - \frac{5}{16} \sin x \cos x \\
 &\quad + \frac{5}{16} x + C \\
 &= \frac{5x}{16} - \cos x \left( \frac{\sin^5 x}{6} + \frac{5 \sin^3 x}{24} + \frac{5 \sin x}{16} \right) + C.
 \end{aligned}$$

$$\begin{aligned}
 I_7 &= -\frac{1}{7} \sin^6 x \cos x + \frac{6}{7} I_5 \\
 &= -\frac{1}{7} \sin^6 x \cos x + \frac{6}{7} \left( -\frac{1}{5} \sin^4 x \cos x + \frac{4}{5} I_3 \right) \\
 &= -\frac{1}{7} \sin^6 x \cos x - \frac{6}{35} \sin^4 x \cos x \\
 &\quad + \frac{24}{35} \left( -\frac{1}{3} \sin^2 x \cos x + \frac{2}{3} I_1 \right) \\
 &= -\frac{1}{7} \sin^6 x \cos x - \frac{6}{35} \sin^4 x \cos x - \frac{8}{35} \sin^2 x \cos x \\
 &\quad - \frac{16}{35} \cos x + C \\
 &= -\cos x \left( \frac{\sin^6 x}{7} + \frac{6 \sin^4 x}{35} + \frac{8 \sin^2 x}{35} + \frac{16}{35} \right) + C.
 \end{aligned}$$

34. We have

$$\begin{aligned}
 I_n &= \int \sec^n x \, dx \quad (n \geq 3) \\
 U &= \sec^{n-2} x & dV &= \sec^2 x \, dx \\
 dU &= (n-2) \sec^{n-2} x \tan x \, dx & V &= \tan x \\
 &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx \\
 &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx \\
 &= \sec^{n-2} x \tan x - (n-2) I_n + (n-2) I_{n-2} + C \\
 I_n &= \frac{1}{n-1} (\sec^{n-2} x \tan x) + \frac{n-2}{n-1} I_{n-2} + C. \\
 I_1 &= \int \sec x \, dx = \ln |\sec x + \tan x| + C; \\
 I_2 &= \int \sec^2 x \, dx = \tan x + C. \\
 I_6 &= \frac{1}{5} (\sec^4 x \tan x) + \frac{4}{5} \left( \frac{1}{3} \sec^2 x \tan x + \frac{2}{3} I_2 \right) + C \\
 &= \frac{1}{5} \sec^4 x \tan x + \frac{4}{15} \sec^2 x \tan x + \frac{8}{15} \tan x + C. \\
 I_7 &= \frac{1}{6} (\sec^5 x \tan x) + \frac{5}{6} \left[ \frac{1}{4} \sec^3 x \tan x + \right. \\
 &\quad \left. \frac{3}{4} \left( \frac{1}{2} \sec x \tan x + \frac{1}{2} I_1 \right) \right] + C \\
 &= \frac{1}{6} \sec^5 x \tan x + \frac{5}{24} \sec^3 x \tan x + \frac{15}{48} \sec x \tan x + \\
 &\quad \frac{15}{48} \ln |\sec x + \tan x| + C.
 \end{aligned}$$

$$\begin{aligned}
 35. \quad I_n &= \int \frac{dx}{(x^2 + a^2)^n} = \frac{1}{a^2} \int \frac{x^2 + a^2 - x^2}{(x^2 + a^2)^n} \, dx \\
 &= \frac{1}{a^2} \int \frac{dx}{(x^2 + a^2)^{n-1}} - \frac{1}{a^2} \int \frac{x^2}{(x^2 + a^2)^n} \, dx
 \end{aligned}$$

$$\begin{aligned}
 U &= x & dV &= \frac{x \, dx}{(x^2 + a^2)^n} \\
 dU &= dx & V &= \frac{-1}{2(n-1)(x^2 + a^2)^{n-1}} \\
 &= \frac{1}{a^2} I_{n-1} - \frac{1}{a^2} \left( \frac{-x}{2(n-1)(x^2 + a^2)^{n-1}} \right. \\
 &\quad \left. + \frac{1}{2(n-1)} \int \frac{dx}{(x^2 + a^2)^{n-1}} \right) \\
 I_n &= \frac{x}{2(n-1)a^2(x^2 + a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}. \\
 \text{Now } I_1 &= \frac{1}{a} \tan^{-1} \frac{x}{a}, \text{ so} \\
 I_3 &= \frac{x}{4a^2(x^2 + a^2)^2} + \frac{3}{4a^2} I_2 \\
 &= \frac{x}{4a^2(x^2 + a^2)^2} + \frac{3}{4a^2} \left( \frac{x}{2a^2(x^2 + a^2)} + \frac{1}{2a^2} I_1 \right) \\
 &= \frac{x}{4a^2(x^2 + a^2)^2} + \frac{3x}{8a^4(x^2 + a^2)} + \frac{3}{8a^5} \tan^{-1} \frac{x}{a} + C.
 \end{aligned}$$

36. Given that  $f(a) = f(b) = 0$ .

$$\begin{aligned}
 &\int_a^b (x-a)(b-x) f''(x) \, dx \\
 U &= (x-a)(b-x) & dV &= f''(x) \, dx \\
 dU &= (b+a-2x) \, dx & V &= f'(x) \\
 &= (x-a)(b-x) f'(x) \Big|_a^b - \int_a^b (b+a-2x) f'(x) \, dx \\
 U &= b+a-2x & dV &= f'(x) \, dx \\
 dU &= -2 \, dx & V &= f(x) \\
 &= 0 - \left[ (b+a-2x) f(x) \Big|_a^b + 2 \int_a^b f(x) \, dx \right] \\
 &= -2 \int_a^b f(x) \, dx.
 \end{aligned}$$

37. Given:  $f''$  and  $g''$  are continuous on  $[a, b]$ , and  $f(a) = g(a) = f(b) = g(b) = 0$ . We have

$$\begin{aligned}
 &\int_a^b f(x) g''(x) \, dx \\
 U &= f(x) & dV &= g''(x) \, dx \\
 dU &= f'(x) \, dx & V &= g'(x) \\
 &= f(x) g'(x) \Big|_a^b - \int_a^b f'(x) g'(x) \, dx.
 \end{aligned}$$

Similarly,

$$\int_a^b f''(x) g(x) \, dx = f'(x) g(x) \Big|_a^b - \int_a^b f'(x) g'(x) \, dx.$$

Thus we have

$$\begin{aligned} \int_a^b f(x)g''(x) dx - \int_a^b f''(x)g(x) dx \\ = \left( f(x)g'(x) - f'(x)g(x) \right) \Big|_a^b = 0 \end{aligned}$$

by the assumptions on  $f$  and  $g$ . Thus

$$\int_a^b f(x)g''(x) dx = \int_a^b f''(x)g(x) dx.$$

This equation is also valid for any (sufficiently smooth) functions  $f$  and  $g$  for which

$$f(b)g'(b) - f'(b)g(b) = f(a)g'(a) - f'(a)g(a).$$

Examples are functions which are periodic with period  $b - a$ , or if  $f(a) = f(b) = f'(a) = f'(b) = 0$ , or if instead  $g$  satisfies such conditions. Other combinations of conditions on  $f$  and  $g$  will also do.

38.  $I_n = \int_0^{\pi/2} \cos^n x dx.$

a) For  $0 \leq x \leq \pi/2$  we have  $0 \leq \cos x \leq 1$ , and so  $0 \leq \cos^{2n+2} x \leq \cos^{2n+1} x \leq \cos^{2n} x$ . Therefore  $0 \leq I_{2n+2} \leq I_{2n+1} \leq I_{2n}$ .

b) Since  $I_n = \frac{n-1}{n} I_{n-2}$ , we have  $I_{2n+2} = \frac{2n+1}{2n+2} I_{2n}$ . Combining this with part (a), we get

$$\frac{2n+1}{2n+2} = \frac{I_{2n+2}}{I_{2n}} \leq \frac{I_{2n+1}}{I_{2n}} \leq 1.$$

The left side approaches 1 as  $n \rightarrow \infty$ , so, by the Squeeze Theorem,

$$\lim_{n \rightarrow \infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

c) By Example 6 we have, since  $2n+1$  is odd and  $2n$  is even,

$$\begin{aligned} I_{2n+1} &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{4}{5} \cdot \frac{2}{3} \\ I_{2n} &= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}. \end{aligned}$$

Multiplying the expression for  $I_{2n+1}$  by  $\pi/2$  and dividing by the expression for  $I_{2n}$ , we obtain, by part (b),

$$\lim_{n \rightarrow \infty} \frac{\frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{\pi}{2}}{\frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}} = \frac{\pi}{2} \times 1 = \frac{\pi}{2},$$

or, rearranging the factors on the left,

$$\lim_{n \rightarrow \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n-1} \cdot \frac{2n}{2n+1} = \frac{\pi}{2}.$$

## Section 6.2 Inverse Substitutions (page 328)

1.  $\int \frac{dx}{\sqrt{1-4x^2}}$  Let  $u = 2x$   
 $du = 2 dx$   
 $= \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1}(2x) + C.$

2.  $\int \frac{x^2 dx}{\sqrt{1-4x^2}}$  Let  $2x = \sin u$   
 $2 dx = \cos u du$   
 $= \frac{1}{8} \int \frac{\sin^2 u \cos u du}{\cos u}$   
 $= \frac{1}{16} \int (1 - \cos 2u) du = \frac{u}{16} - \frac{\sin 2u}{32} + C$   
 $= \frac{1}{16} \sin^{-1} 2x - \frac{1}{16} \sin u \cos u + C$   
 $= \frac{1}{16} \sin^{-1} 2x - \frac{1}{8} x \sqrt{1-4x^2} + C.$

3.  $\int \frac{x^2 dx}{\sqrt{9-x^2}}$  Let  $x = 3 \sin \theta$   
 $dx = 3 \cos \theta d\theta$   
 $= \int \frac{9 \sin^2 \theta 3 \cos \theta d\theta}{3 \cos \theta}$   
 $= \frac{9}{2} (\theta - \sin \theta \cos \theta) + C$   
 $= \frac{9}{2} \sin^{-1} \frac{x}{3} - \frac{1}{2} x \sqrt{9-x^2} + C.$

4.  $\int \frac{dx}{x\sqrt{1-4x^2}}$  Let  $x = \frac{1}{2} \sin \theta$   
 $dx = \frac{1}{2} \cos \theta d\theta$   
 $= \int \frac{\cos \theta d\theta}{\sin \theta \sqrt{1-\sin^2 \theta}} = \int \csc \theta d\theta$   
 $= \ln |\csc \theta - \cot \theta| + C = \ln \left| \frac{1}{2x} - \frac{\sqrt{1-4x^2}}{2x} \right| + C$   
 $= \ln \left| \frac{1 - \sqrt{1-4x^2}}{x} \right| + C_1.$

5.  $\int \frac{dx}{x^2 \sqrt{9-x^2}}$  Let  $x = 3 \sin \theta$   
 $dx = 3 \cos \theta d\theta$   
 $= \int \frac{3 \cos \theta d\theta}{9 \sin^2 \theta 3 \cos \theta}$   
 $= \frac{1}{9} \int \csc^2 \theta d\theta$   
 $= -\frac{1}{9} \cot \theta + C = -\frac{1}{9} \frac{\sqrt{9-x^2}}{x} + C.$

$$\begin{aligned}
 6. \quad & \int \frac{dx}{x\sqrt{9-x^2}} \quad \text{Let } x = 3 \sin \theta \\
 & \quad \quad \quad dx = 3 \cos \theta \, d\theta \\
 & = \int \frac{3 \cos \theta \, d\theta}{3 \sin \theta \cdot 3 \cos \theta} = \frac{1}{3} \int \csc \theta \, d\theta \\
 & = \frac{1}{3} \ln |\csc \theta - \cot \theta| + C = \frac{1}{3} \ln \left| \frac{3}{x} - \frac{\sqrt{9-x^2}}{x} \right| + C \\
 & = \frac{1}{3} \ln \left| \frac{3 - \sqrt{9-x^2}}{x} \right| + C.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad & \int \frac{x+1}{\sqrt{9-x^2}} \, dx = \int \frac{x \, dx}{\sqrt{9-x^2}} + \int \frac{dx}{\sqrt{9-x^2}} \\
 & = -\sqrt{9-x^2} + \sin^{-1} \frac{x}{3} + C.
 \end{aligned}$$

$$\begin{aligned}
 8. \quad & \int \frac{dx}{\sqrt{9+x^2}} \quad \text{Let } x = 3 \tan \theta \\
 & \quad \quad \quad dx = 3 \sec^2 \theta \, d\theta \\
 & = \int \frac{3 \sec^2 \theta \, d\theta}{3 \sec \theta} = \int \sec \theta \, d\theta \\
 & = \ln |\sec \theta + \tan \theta| + C = \ln(x + \sqrt{9+x^2}) + C_1.
 \end{aligned}$$

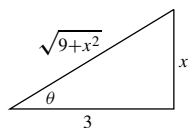


Fig. 6.2.8

$$\begin{aligned}
 9. \quad & \int \frac{x^3 \, dx}{\sqrt{9+x^2}} \quad \text{Let } u = 9+x^2 \\
 & \quad \quad \quad du = 2x \, dx \\
 & = \frac{1}{2} \int \frac{(u-9) \, du}{\sqrt{u}} = \frac{1}{2} \int (u^{1/2} - 9u^{-1/2}) \, du \\
 & = \frac{1}{3} u^{3/2} - 9u^{1/2} + C \\
 & = \frac{1}{3} (9+x^2)^{3/2} - 9\sqrt{9+x^2} + C.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad & \int \frac{\sqrt{9+x^2}}{x^4} \, dx \quad \text{Let } x = 3 \tan \theta \\
 & \quad \quad \quad dx = 3 \sec^2 \theta \, d\theta \\
 & = \int \frac{(3 \sec \theta)(3 \sec^2 \theta \, d\theta)}{81 \tan^4 \theta} \\
 & = \frac{1}{9} \int \frac{\sec^3 \theta}{\tan^4 \theta} \, d\theta = \frac{1}{9} \int \frac{\cos \theta}{\sin^4 \theta} \, d\theta \quad \text{Let } u = \sin \theta \\
 & \quad \quad \quad du = \cos \theta \, d\theta \\
 & = \frac{1}{9} \int \frac{du}{u^4} = -\frac{1}{27u^3} + C = -\frac{1}{27 \sin^3 \theta} + C \\
 & = -\frac{(9+x^2)^{3/2}}{27x^3} + C.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad & \int \frac{dx}{(a^2-x^2)^{3/2}} \quad \text{Let } x = a \sin \theta \\
 & \quad \quad \quad dx = a \cos \theta \, d\theta \\
 & = \int \frac{a \cos \theta \, d\theta}{a^3 \cos^3 \theta} = \frac{1}{a^2} \int \sec^2 \theta \, d\theta \\
 & = \frac{1}{a^2} \tan \theta + C = \frac{1}{a^2} \frac{x}{\sqrt{a^2-x^2}} + C.
 \end{aligned}$$

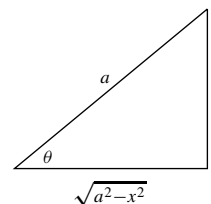


Fig. 6.2.11

$$\begin{aligned}
 12. \quad & \int \frac{dx}{(a^2+x^2)^{3/2}} \quad \text{Let } x = a \tan \theta \\
 & \quad \quad \quad dx = a \sec^2 \theta \, d\theta \\
 & = \int \frac{a \sec^2 \theta \, d\theta}{(a^2 + a^2 \tan^2 \theta)^{3/2}} = \int \frac{a \sec^2 \theta \, d\theta}{a^3 \sec^3 \theta} \\
 & = \frac{1}{a^2} \int \cos \theta \, d\theta = \frac{1}{a^2} \sin \theta + C = \frac{x}{a^2 \sqrt{a^2+x^2}} + C.
 \end{aligned}$$

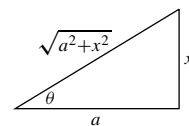


Fig. 6.2.12

$$\begin{aligned}
 13. \quad & \int \frac{x^2 \, dx}{(a^2-x^2)^{3/2}} \quad \text{Let } x = a \sin \theta \\
 & \quad \quad \quad dx = a \cos \theta \, d\theta \\
 & = \int \frac{a^2 \sin^2 \theta \cdot a \cos \theta \, d\theta}{a^3 \cos^3 \theta} \\
 & = \int \tan^2 \theta \, d\theta = \int (\sec^2 \theta - 1) \, d\theta \\
 & = \tan \theta - \theta + C \quad (\text{see Fig. s6-5-17}) \\
 & = \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1} \frac{x}{a} + C.
 \end{aligned}$$

$$\begin{aligned}
 14. \quad & \int \frac{dx}{(1+2x^2)^{5/2}} \quad \text{Let } x = \frac{1}{\sqrt{2}} \tan \theta \\
 & dx = \frac{1}{\sqrt{2}} \sec^2 \theta d\theta \\
 & = \frac{1}{\sqrt{2}} \int \frac{\sec^2 \theta d\theta}{(1+\tan^2 \theta)^{5/2}} = \frac{1}{\sqrt{2}} \int \cos^3 \theta d\theta \\
 & = \frac{1}{\sqrt{2}} \int (1 - \sin^2 \theta) \cos \theta d\theta \quad \text{Let } u = \sin \theta \\
 & \quad du = \cos \theta d\theta \\
 & = \frac{1}{\sqrt{2}} \int (1 - u^2) du = \frac{1}{\sqrt{2}} \left( u - \frac{1}{3} u^3 \right) + C \\
 & = \frac{1}{\sqrt{2}} \sin \theta - \frac{1}{3\sqrt{2}} \sin^3 \theta + C \\
 & = \frac{\sqrt{2}x}{\sqrt{2}\sqrt{1+2x^2}} - \frac{1}{3\sqrt{2}} \left( \frac{\sqrt{2}x}{\sqrt{1+2x^2}} \right)^3 + C \\
 & = \frac{4x^3 + 3x}{3(1+2x^2)^{3/2}} + C.
 \end{aligned}$$

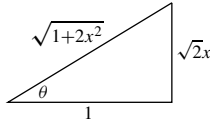


Fig. 6.2.14

$$\begin{aligned}
 15. \quad & \int \frac{dx}{x\sqrt{x^2-4}} \quad \text{Let } x = 2 \sec \theta \quad (x > 2) \\
 & dx = 2 \sec \theta \tan \theta d\theta \\
 & = \int \frac{2 \sec \theta \tan \theta d\theta}{2 \sec \theta \tan \theta} \\
 & = \frac{1}{2} \int d\theta = \frac{\theta}{2} + C = \frac{1}{2} \sec^{-1} \frac{x}{2} + C.
 \end{aligned}$$

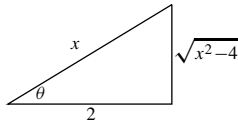


Fig. 6.2.15

$$\begin{aligned}
 16. \quad & \int \frac{dx}{x^2\sqrt{x^2-a^2}} \quad \text{Let } x = a \sec \theta \quad (a > 0) \\
 & dx = a \sec \theta \tan \theta d\theta \\
 & = \int \frac{a \sec \theta \tan \theta d\theta}{a^2 \sec^2 \theta a \tan \theta} \\
 & = \frac{1}{a^2} \int \cos \theta d\theta = \frac{1}{a^2} \sin \theta + C \\
 & = \frac{1}{a^2} \frac{\sqrt{x^2-a^2}}{x} + C.
 \end{aligned}$$

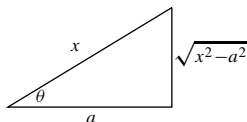


Fig. 6.2.16

$$17. \quad \int \frac{dx}{x^2+2x+10} = \int \frac{dx}{(x+1)^2+9} = \frac{1}{3} \tan^{-1} \frac{x+1}{3} + C.$$

$$\begin{aligned}
 18. \quad & \int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad \text{Let } u = x + \frac{1}{2} \\
 & \quad du = dx \\
 & = \int \frac{du}{u^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2}{\sqrt{3}} u \right) + C \\
 & = \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) + C.
 \end{aligned}$$

$$\begin{aligned}
 19. \quad & \int \frac{dx}{(4x^2+4x+5)^2} \\
 & = \int \frac{dx}{\left((2x+1)^2+4\right)^2} \quad \text{Let } 2x+1 = 2 \tan \theta \\
 & \quad 2 dx = 2 \sec^2 \theta d\theta \\
 & = \int \frac{\sec^2 \theta d\theta}{16 \sec^4 \theta} = \frac{1}{16} \int \cos^2 \theta d\theta \\
 & = \frac{1}{32} (\theta + \sin \theta \cos \theta) \\
 & = \frac{1}{32} \tan^{-1} \frac{2x+1}{2} + \frac{1}{16} \frac{2x+1}{4x^2+4x+5} + C.
 \end{aligned}$$

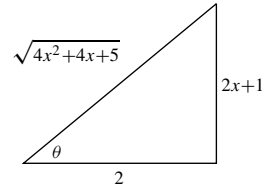


Fig. 6.2.19

$$\begin{aligned}
 20. \quad & \int \frac{x dx}{x^2-2x+3} = \int \frac{(x-1)+1}{(x-1)^2+2} dx \quad \text{Let } u = x-1 \\
 & \quad du = dx \\
 & = \int \frac{u du}{u^2+2} + \int \frac{du}{u^2+2} \\
 & = \frac{1}{2} \ln(u^2+2) + \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{u}{\sqrt{2}} \right) + C \\
 & = \frac{1}{2} \ln(x^2-2x+3) + \frac{1}{\sqrt{2}} \tan^{-1} \left( \frac{x-1}{\sqrt{2}} \right) + C.
 \end{aligned}$$

$$\begin{aligned}
 21. \quad & \int \frac{x dx}{\sqrt{2ax-x^2}} \\
 & = \int \frac{x dx}{\sqrt{a^2-(x-a)^2}} \quad \text{Let } x-a = a \sin \theta \\
 & \quad dx = a \cos \theta d\theta \\
 & = \int \frac{(a+a \sin \theta)a \cos \theta d\theta}{a \cos \theta} \\
 & = a(\theta - \cos \theta) + C \\
 & = a \sin^{-1} \frac{x-a}{a} - \sqrt{2ax-x^2} + C.
 \end{aligned}$$

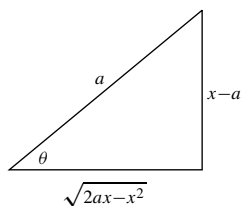


Fig. 6.2.21

$$\begin{aligned}
 22. \quad & \int \frac{dx}{(4x-x^2)^{3/2}} \\
 &= \int \frac{dx}{[4-(2-x)^2]^{3/2}} \quad \text{Let } 2-x = 2 \sin u \\
 & \quad \quad \quad -dx = 2 \cos u \, du \\
 &= -\int \frac{2 \cos u \, du}{8 \cos^3 u} = -\frac{1}{4} \int \sec^2 u \, du \\
 &= -\frac{1}{4} \tan u + C = \frac{1}{4} \frac{x-2}{\sqrt{4x-x^2}} + C.
 \end{aligned}$$

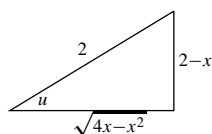


Fig. 6.2.22

$$\begin{aligned}
 23. \quad & \int \frac{x \, dx}{(3-2x-x^2)^{3/2}} \\
 &= \int \frac{x \, dx}{(4-(x+1)^2)^{3/2}} \quad \text{Let } x+1 = 2 \sin \theta \\
 & \quad \quad \quad dx = 2 \cos \theta \, d\theta \\
 &= \int \frac{(2 \sin \theta - 1) 2 \cos \theta \, d\theta}{8 \cos^3 \theta} \\
 &= \frac{1}{2} \int \sec \theta \tan \theta \, d\theta - \frac{1}{4} \int \sec^2 \theta \, d\theta \\
 &= \frac{1}{2} \sec \theta - \frac{1}{4} \tan \theta + C \\
 &= \frac{1}{\sqrt{3-2x-x^2}} - \frac{1}{4} \frac{x+1}{\sqrt{3-2x-x^2}} + C \\
 &= \frac{1}{4} \cdot \frac{3-x}{\sqrt{3-2x-x^2}} + C.
 \end{aligned}$$

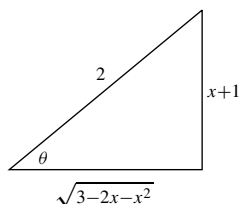


Fig. 6.2.23

$$\begin{aligned}
 24. \quad & \int \frac{dx}{(x^2+2x+2)^2} = \int \frac{dx}{[(x+1)^2+1]^2} \quad \text{Let } x+1 = \tan u \\
 & \quad \quad \quad dx = \sec^2 u \, du \\
 &= \int \frac{\sec^2 u \, du}{\sec^4 u} = \int \cos^2 u \, du \\
 &= \frac{1}{2} \int (1 + \cos 2u) \, du = \frac{u}{2} + \frac{\sin 2u}{4} + C \\
 &= \frac{1}{2} \tan^{-1}(x+1) + \frac{1}{2} \sin u \cos u + C \\
 &= \frac{1}{2} \tan^{-1}(x+1) + \frac{1}{2} \frac{x+1}{x^2+2x+2} + C.
 \end{aligned}$$

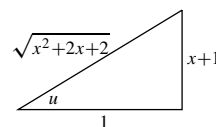


Fig. 6.2.24

$$\begin{aligned}
 25. \quad & \int \frac{dx}{(1+x^2)^3} \quad \text{Let } x = \tan \theta \\
 & \quad \quad \quad dx = \sec^2 \theta \, d\theta \\
 &= \int \frac{\sec^2 \theta}{\sec^6 \theta} \, d\theta = \int \cos^4 \theta \, d\theta \\
 &= \int \left( \frac{1 + \cos 2\theta}{2} \right)^2 \, d\theta \\
 &= \frac{1}{4} \int \left( 1 + 2 \cos 2\theta + \frac{1 + \cos 4\theta}{2} \right) \, d\theta \\
 &= \frac{3\theta}{8} + \frac{\sin 2\theta}{4} + \frac{\sin 4\theta}{32} + C \\
 &= \frac{3\theta}{8} + \frac{\sin \theta \cos \theta}{2} + \frac{\sin 2\theta \cos 2\theta}{16} + C \\
 &= \frac{3\theta}{8} + \frac{\sin \theta \cos \theta}{2} + \frac{1}{8} \sin \theta \cos \theta (2 \cos^2 \theta - 1) + C \\
 &= \frac{3}{8} \tan^{-1} x + \frac{1}{2} \cdot \frac{x}{1+x^2} + \frac{1}{8} \cdot \frac{x}{1+x^2} \left( \frac{2}{1+x^2} - 1 \right) + C \\
 &= \frac{3}{8} \tan^{-1} x + \frac{3}{8} \cdot \frac{x}{1+x^2} + \frac{1}{4} \cdot \frac{x}{(1+x^2)^2} + C \\
 &= \frac{3}{8} \tan^{-1} x + \frac{3x^3+5x}{8(1+x^2)^2} + C.
 \end{aligned}$$

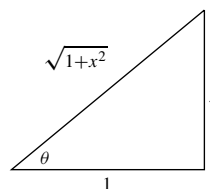


Fig. 6.2.25

$$\begin{aligned}
 26. \quad & \int \frac{x^2 dx}{(1+x^2)^2} \quad \text{Let } x = \tan u \\
 & \quad \quad \quad dx = \sec^2 u du \\
 & = \int \frac{\tan^2 u \sec^2 u du}{\sec^4 u} = \int \frac{\tan^2 u du}{\sec^2 u} \\
 & = \int \sin^2 u du = \frac{1}{2} \int (1 - \cos 2u) du \\
 & = \frac{u}{2} - \frac{\sin u \cos u}{2} + C \\
 & = \frac{1}{2} \tan^{-1} x - \frac{1}{2} \frac{x}{1+x^2} + C.
 \end{aligned}$$

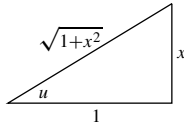


Fig. 6.2.26

$$\begin{aligned}
 27. \quad & \int \frac{\sqrt{1-x^2}}{x^3} dx \quad \text{Let } x = \sin \theta \\
 & \quad \quad \quad dx = \cos \theta d\theta \\
 & = \int \frac{\cos^2 \theta}{\sin^3 \theta} d\theta = I, \text{ where} \\
 I & = \int \cot^2 \theta \csc \theta d\theta \\
 & \quad \quad \quad U = \cot \theta \quad dV = \cot \theta \csc \theta d\theta \\
 & \quad \quad \quad dU = -\csc^2 \theta d\theta \quad V = -\csc \theta \\
 & = -\csc \theta \cot \theta - \int \csc^3 \theta d\theta \\
 & = -\csc \theta \cot \theta - \int \csc \theta d\theta - I.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 I & = -\frac{1}{2} \csc \theta \cot \theta + \frac{1}{2} \ln |\csc \theta + \cot \theta| + C \\
 & = -\frac{1}{2} \frac{\sqrt{1-x^2}}{x^2} + \frac{1}{2} \ln \left| \frac{1}{x} + \frac{\sqrt{1-x^2}}{x} \right| + C \\
 & = \frac{1}{2} \ln(1 + \sqrt{1-x^2}) - \frac{1}{2} \ln |x| - \frac{1}{2} \frac{\sqrt{1-x^2}}{x^2} + C.
 \end{aligned}$$

$$\begin{aligned}
 28. \quad & I = \int \sqrt{9+x^2} dx \quad \text{Let } x = 3 \tan \theta \\
 & \quad \quad \quad dx = 3 \sec^2 \theta d\theta \\
 & = \int 3 \sec \theta \cdot 3 \sec^2 \theta d\theta \\
 & = 9 \int \sec^3 \theta d\theta \\
 & \quad \quad \quad U = \sec \theta \quad dV = \sec^2 \theta d\theta \\
 & \quad \quad \quad dU = \sec \theta \tan \theta d\theta \quad V = \tan \theta \\
 & = 9 \sec \theta \tan \theta - 9 \int \sec \theta \tan^2 \theta d\theta \\
 & = 9 \sec \theta \tan \theta - 9 \int \sec \theta (\sec^2 \theta - 1) d\theta \\
 & = 9 \sec \theta \tan \theta + 9 \int \sec \theta d\theta - 9 \int \sec^3 \theta d\theta \\
 & = 9 \sec \theta \tan \theta + 9 \ln |\sec \theta + \tan \theta| - I \\
 I & = \frac{9}{2} \left[ \left( \frac{\sqrt{9+x^2}}{3} \right) \left( \frac{x}{3} \right) \right] + \frac{9}{2} \ln \left| \frac{\sqrt{9+x^2}}{3} + \frac{x}{3} \right| + C \\
 & = \frac{1}{2} x \sqrt{9+x^2} + \frac{9}{2} \ln (\sqrt{9+x^2} + x) + C_1. \\
 & \quad \quad \quad (\text{where } C_1 = C - \frac{9}{2} \ln 3)
 \end{aligned}$$

$$\begin{aligned}
 29. \quad & \int \frac{dx}{2+\sqrt{x}} \quad \text{Let } x = u^2 \\
 & \quad \quad \quad dx = 2u du \\
 & = \int \frac{2u du}{2+u} = 2 \int \left( 1 - \frac{2}{u+2} \right) du \\
 & = 2u - 4 \ln |u+2| + C = 2\sqrt{x} - 4 \ln(2+\sqrt{x}) + C.
 \end{aligned}$$

$$\begin{aligned}
 30. \quad & \int \frac{dx}{1+x^{1/3}} \quad \text{Let } x = u^3 \\
 & \quad \quad \quad dx = 3u^2 du \\
 & = 3 \int \frac{u^2 du}{1+u} \quad \text{Let } v = 1+u \\
 & \quad \quad \quad dv = du \\
 & = 3 \int \frac{v^2 - 2v + 1}{v} dv = 3 \int \left( v - 2 + \frac{1}{v} \right) dv \\
 & = 3 \left( \frac{v^2}{2} - 2v + \ln |v| \right) + C \\
 & = \frac{3}{2} (1+x^{1/3})^2 - 6(1+x^{1/3}) + 3 \ln |1+x^{1/3}| + C.
 \end{aligned}$$

$$\begin{aligned}
 31. \quad & I = \int \frac{1+x^{1/2}}{1+x^{1/3}} dx \quad \text{Let } x = u^6 \\
 & \quad \quad \quad dx = 6u^5 du \\
 & = \int \frac{1+u^3}{1+u^2} 6u^5 du = 6 \int \frac{u^8+u^5}{1+u^2} du.
 \end{aligned}$$

Division is required to render the last integrand as a polynomial with a remainder fraction of simpler form: observe that

$$\begin{aligned}
 u^8 & = u^8 + u^6 - u^6 - u^4 + u^4 + u^2 - u^2 - 1 + 1 \\
 & = (u^2 + 1)(u^6 - u^4 + u^2 - 1) + 1 \\
 u^5 & = u^5 + u^3 - u^3 - u + u \\
 & = (u^2 + 1)(u^3 - u) + u.
 \end{aligned}$$

Thus

$$\frac{u^8 + u^5}{u^2 + 1} = u^6 - u^4 + u^3 + u^2 - u - 1 + \frac{u + 1}{u^2 + 1}.$$

Therefore

$$\begin{aligned} I &= 6 \int \left( u^6 - u^4 + u^3 + u^2 - u - 1 + \frac{u + 1}{u^2 + 1} \right) du \\ &= 6 \left( \frac{u^7}{7} - \frac{u^5}{5} + \frac{u^4}{4} + \frac{u^3}{3} - \frac{u^2}{2} - u \right. \\ &\quad \left. + \frac{1}{2} \ln(u^2 + 1) + \tan^{-1} u \right) + C \\ &= \frac{6}{7} x^{7/6} - \frac{6}{5} x^{5/6} + \frac{3}{2} x^{2/3} + 2x^{1/2} - 3x^{1/3} - 6x^{1/6} \\ &\quad + 3 \ln(1 + x^{1/3}) + 6 \tan^{-1} x^{1/6} + C. \end{aligned}$$

$$\begin{aligned} 32. \quad & \int \frac{x\sqrt{2-x^2}}{\sqrt{x^2+1}} dx \quad \text{Let } u^2 = x^2 + 1 \\ & \quad 2u du = 2x dx \\ &= \int \frac{u\sqrt{3-u^2}}{u} du \\ &= \int \sqrt{3-u^2} du \quad \text{Let } u = \sqrt{3} \sin v \\ & \quad du = \sqrt{3} \cos v dv \\ &= \int (\sqrt{3} \cos v) \sqrt{3} \cos v dv = 3 \int \cos^2 v dv \\ &= \frac{3}{2} (v + \sin v \cos v) + C \\ &= \frac{3}{2} \sin^{-1} \left( \frac{u}{\sqrt{3}} \right) + \frac{3}{2} \frac{u\sqrt{3-u^2}}{3} + C \\ &= \frac{3}{2} \sin^{-1} \left( \sqrt{\frac{x^2+1}{3}} \right) + \frac{1}{2} \sqrt{(x^2+1)(2-x^2)} + C. \end{aligned}$$

$$\begin{aligned} 33. \quad & \int_{-\ln 2}^0 e^x \sqrt{1-e^{2x}} dx \quad \text{Let } e^x = \sin \theta \\ & \quad e^x dx = \cos \theta d\theta \\ &= \int_{\pi/6}^{\pi/2} \cos^2 \theta d\theta = \frac{1}{2} (\theta + \sin \theta \cos \theta) \Big|_{\pi/6}^{\pi/2} \\ &= \frac{1}{2} \left( \frac{\pi}{2} - \frac{\sqrt{3}}{4} \right) = \frac{\pi}{4} - \frac{\sqrt{3}}{8}. \end{aligned}$$

$$\begin{aligned} 34. \quad & \int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}} dx \quad \text{Let } u = \sin x \\ & \quad du = \cos x dx \\ &= \int_0^1 \frac{du}{\sqrt{1+u^2}} \quad \text{Let } u = \tan w \\ & \quad du = \sec^2 w dw \\ &= \int_0^{\pi/4} \frac{\sec^2 w dw}{\sec w} = \int_0^{\pi/4} \sec w dw \\ &= \ln |\sec w + \tan w| \Big|_0^{\pi/4} \\ &= \ln |\sqrt{2} + 1| - \ln |1 + 0| = \ln(\sqrt{2} + 1). \end{aligned}$$

$$\begin{aligned} 35. \quad & \int_{-1}^{\sqrt{3}-1} \frac{dx}{x^2 + 2x + 2} \\ &= \int_{-1}^{\sqrt{3}-1} \frac{dx}{(x+1)^2 + 1} \quad \text{Let } u = x + 1 \\ & \quad du = dx \\ &= \int_0^{\sqrt{3}} \frac{du}{u^2 + 1} = \tan^{-1} u \Big|_0^{\sqrt{3}} = \frac{\pi}{3}. \end{aligned}$$

$$\begin{aligned} 36. \quad & \int_1^2 \frac{dx}{x^2 \sqrt{9-x^2}} \quad \text{Let } x = 3 \sin u \\ & \quad dx = 3 \cos u du \\ &= \int_{x=1}^{x=2} \frac{3 \cos u du}{9 \sin^2 u (3 \cos u)} = \frac{1}{9} \int_{x=1}^{x=2} \csc^2 u du \\ &= \frac{1}{9} (-\cot u) \Big|_{x=1}^{x=2} = -\frac{1}{9} \left( \frac{\sqrt{9-x^2}}{x} \right) \Big|_{x=1}^{x=2} \\ &= -\frac{1}{9} \left( \frac{\sqrt{5}}{2} - \frac{\sqrt{8}}{1} \right) = \frac{2\sqrt{2}}{9} - \frac{\sqrt{5}}{18}. \end{aligned}$$

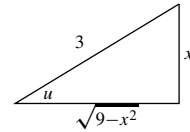


Fig. 6.2.36

$$\begin{aligned} 37. \quad & \int \frac{d\theta}{2 + \sin \theta} \quad \text{Let } x = \tan(\theta/2), \\ & \quad \sin \theta = \frac{2x}{1+x^2}, \quad d\theta = \frac{2 dx}{1+x^2} \\ &= \int \frac{\frac{2 dx}{1+x^2}}{2 + \frac{2x}{1+x^2}} = \int \frac{dx}{1+x+x^2} \\ &= \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}} + C \\ &= \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2 \tan(\theta/2) + 1}{\sqrt{3}} \right) + C. \end{aligned}$$

$$\begin{aligned} 38. \quad & \int_0^{\pi/2} \frac{d\theta}{1 + \cos \theta + \sin \theta} \quad \text{Let } x = \tan \frac{\theta}{2}, \quad d\theta = \frac{2}{1+x^2} dx, \\ & \quad \cos \theta = \frac{1-x^2}{1+x^2}, \quad \sin \theta = \frac{2x}{1+x^2}. \\ &= \int_0^1 \frac{\left( \frac{2}{1+x^2} \right) dx}{1 + \left( \frac{1-x^2}{1+x^2} \right) + \left( \frac{2x}{1+x^2} \right)} \\ &= 2 \int_0^1 \frac{dx}{2+2x} = \int_0^1 \frac{dx}{1+x} \\ &= \ln |1+x| \Big|_0^1 = \ln 2. \end{aligned}$$

$$\begin{aligned}
 39. \quad & \int \frac{d\theta}{3 + 2 \cos \theta} \quad \text{Let } x = \tan(\theta/2), \\
 & \cos \theta = \frac{1 - x^2}{1 + x^2}, \quad d\theta = \frac{2 dx}{1 + x^2} \\
 & = \int \frac{\frac{2 dx}{1 + x^2}}{3 + \frac{2 - 2x^2}{1 + x^2}} = \int \frac{2 dx}{5 + x^2} \\
 & = \frac{2}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + C = \frac{2}{\sqrt{5}} \tan^{-1} \left( \frac{\tan(\theta/2)}{\sqrt{5}} \right) + C.
 \end{aligned}$$

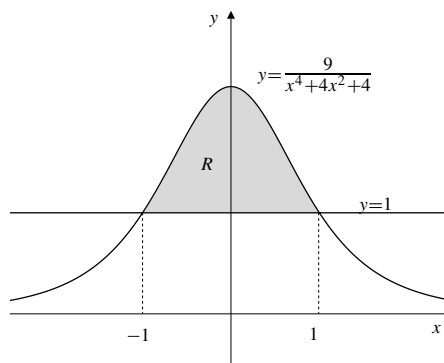


Fig. 6.2.41

$$\begin{aligned}
 40. \quad & \text{Area} = \int_{1/2}^1 \frac{dx}{\sqrt{2x - x^2}} = \int_{1/2}^1 \frac{dx}{\sqrt{1 - (x - 1)^2}} \\
 & \text{Let } u = x - 1 \\
 & du = dx \\
 & = \int_{-1/2}^0 \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u \Big|_{-1/2}^0 \\
 & = 0 - \left( -\frac{\pi}{6} \right) = \frac{\pi}{6} \text{ sq. units.}
 \end{aligned}$$

$$41. \quad \text{For intersection of } y = \frac{9}{x^4 + 4x^2 + 4} \text{ and } y = 1 \text{ we have}$$

$$\begin{aligned}
 x^4 + 4x^2 + 4 &= 9 \\
 x^4 + 4x^2 - 5 &= 0 \\
 (x^2 + 5)(x^2 - 1) &= 0,
 \end{aligned}$$

so the intersections are at  $x = \pm 1$ . The required area is

$$\begin{aligned}
 A &= 2 \int_0^1 \left( \frac{9 dx}{x^4 + 4x^2 + 4} - 1 \right) dx \\
 &= 18 \int_0^1 \frac{dx}{(x^2 + 2)^2} - 2 \quad \text{Let } x = \sqrt{2} \tan \theta \\
 & \quad \quad \quad dx = \sqrt{2} \sec^2 \theta \\
 &= 18 \int_{\theta=0}^{\theta=\pi/4} \frac{\sqrt{2} \sec^2 \theta d\theta}{4 \sec^4 \theta} - 2 \\
 &= \frac{9}{\sqrt{2}} \int_0^{\pi/4} \cos^2 \theta d\theta - 2 \\
 &= \frac{9}{2\sqrt{2}} \left( \theta + \sin \theta \cos \theta \right) \Big|_0^{\pi/4} - 2 \\
 &= \frac{9}{2\sqrt{2}} \left( \tan^{-1} \frac{x}{\sqrt{2}} + \frac{\sqrt{2}x}{x^2 + 2} \right) \Big|_0^1 - 2 \\
 &= \frac{9}{2\sqrt{2}} \left( \tan^{-1} \frac{1}{\sqrt{2}} \right) - \frac{1}{2} \text{ units}^2
 \end{aligned}$$

$$\begin{aligned}
 42. \quad & \text{Average value} = \frac{1}{4} \int_0^4 \frac{dx}{(x^2 - 4x + 8)^{3/2}} \\
 & = \frac{1}{4} \int_0^4 \frac{dx}{[(x - 2)^2 + 4]^{3/2}} \\
 & \quad \text{Let } x - 2 = 2 \tan u \\
 & \quad \quad dx = 2 \sec^2 u du \\
 & = \frac{1}{4} \int_{-\pi/4}^{\pi/4} \frac{2 \sec^2 u du}{8 \sec^3 u} \\
 & = \frac{1}{16} \int_{-\pi/4}^{\pi/4} \cos u du = \frac{1}{16} \sin u \Big|_{-\pi/4}^{\pi/4} \\
 & = \frac{1}{16} \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) = \frac{\sqrt{2}}{16}.
 \end{aligned}$$

$$\begin{aligned}
 43. \quad & \text{Area of } R \\
 & = 2 \int_0^{\sqrt{a^2 - b^2}} \left( \sqrt{a^2 - x^2} - b \right) dx \quad \text{Let } x = a \sin \theta \\
 & \quad \quad \quad dx = a \cos \theta d\theta \\
 & = 2 \int_{\theta=0}^{\theta=\sin^{-1} \frac{\sqrt{a^2 - b^2}}{a}} a^2 \cos^2 \theta d\theta - 2b \sqrt{a^2 - b^2} \\
 & = a^2 (\theta + \sin \theta \cos \theta) \Big|_0^{\sin^{-1} \frac{\sqrt{a^2 - b^2}}{a}} - 2b \sqrt{a^2 - b^2} \\
 & = \left( a^2 \sin^{-1} \frac{x}{a} + x \sqrt{a^2 - x^2} \right) \Big|_0^{\sqrt{a^2 - b^2}} - 2b \sqrt{a^2 - b^2} \\
 & = a^2 \sin^{-1} \sqrt{1 - \frac{b^2}{a^2}} + b \sqrt{a^2 - b^2} - 2b \sqrt{a^2 - b^2} \\
 & = a^2 \cos^{-1} \frac{b}{a} - b \sqrt{a^2 - b^2} \text{ units}^2
 \end{aligned}$$

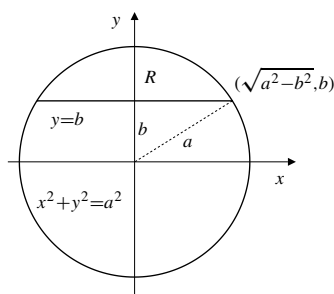


Fig. 6.2.43

44. The circles intersect at  $x = \frac{1}{4}$ , so the common area is  $A_1 + A_2$  where

$$\begin{aligned} A_1 &= 2 \int_{1/4}^1 \sqrt{1-x^2} dx \quad \text{Let } x = \sin u \\ &\quad dx = \cos u du \\ &= 2 \int_{x=1/4}^{x=1} \cos^2 u du \\ &= (u + \sin u \cos u) \Big|_{x=1/4}^{x=1} \\ &= (\sin^{-1} x + x \sqrt{1-x^2}) \Big|_{x=1/4}^{x=1} \\ &= \frac{\pi}{2} - \sin^{-1} \frac{1}{4} - \frac{\sqrt{15}}{16} \text{ sq. units.} \\ A_2 &= 2 \int_0^{1/4} \sqrt{4-(x-2)^2} dx \quad \text{Let } x-2 = 2 \sin v \\ &\quad dx = 2 \cos v dv \\ &= 8 \int_{x=0}^{x=1/4} \cos^2 v dv \\ &= 4(v + \sin v \cos v) \Big|_{x=0}^{x=1/4} \\ &= 4 \left[ \sin^{-1} \left( \frac{x-2}{2} \right) + \left( \frac{x-2}{2} \right) \frac{\sqrt{4-x^2}}{2} \right] \Big|_{x=0}^{x=1/4} \\ &= 4 \left[ \sin^{-1} \left( -\frac{7}{8} \right) - \frac{7\sqrt{15}}{64} + \frac{\pi}{2} \right] \\ &= -4 \sin^{-1} \left( \frac{7}{8} \right) - \frac{7\sqrt{15}}{16} + 2\pi \text{ sq. units.} \end{aligned}$$

Hence, the common area is

$$\begin{aligned} A_1 + A_2 &= \frac{5\pi}{2} - \frac{\sqrt{15}}{2} \\ &\quad - \sin^{-1} \left( \frac{1}{4} \right) - 4 \sin^{-1} \left( \frac{7}{8} \right) \text{ sq. units.} \end{aligned}$$

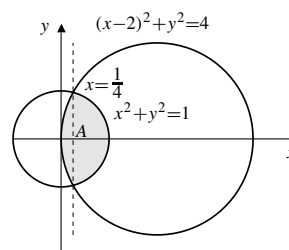


Fig. 6.2.44

45. Required area  $= \int_3^4 \left( \sqrt{25-x^2} - \frac{12}{x} \right) dx$
- $$\begin{aligned} &= \int_3^4 \sqrt{25-x^2} dx - \int_3^4 \frac{12}{x} dx \\ &\quad \text{Let } x = 5 \sin u, dx = 5 \cos u du \text{ in the first integral.} \\ &= \int_{x=3}^{x=4} 25 \cos^2 u du - 12 \ln x \Big|_3^4 \\ &= \frac{25}{2} (u + \sin u \cos u) \Big|_{x=3}^{x=4} - 12 \ln \frac{4}{3} \\ &= \frac{25}{2} \left( \sin^{-1} \frac{x}{5} \right) + \frac{1}{2} x \sqrt{25-x^2} \Big|_3^4 - 12 \ln \frac{4}{3} \\ &= \frac{25}{2} \left( \sin^{-1} \frac{4}{5} - \sin^{-1} \frac{3}{5} \right) - 12 \ln \frac{4}{3} \text{ sq. units.} \end{aligned}$$

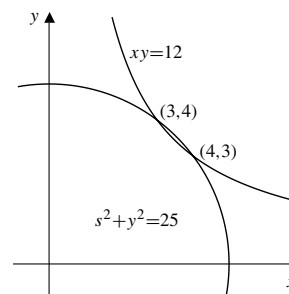


Fig. 6.2.45

46. Shaded area  $= 2 \int_c^a b \sqrt{1 - \left( \frac{x}{a} \right)^2} dx$  Let  $x = a \sin u$   
 $dx = a \cos u du$
- $$\begin{aligned} &= 2ab \int_{x=c}^{x=a} \cos^2 u du \\ &= ab(u + \sin u \cos u) \Big|_{x=c}^{x=a} \\ &= \left( ab \sin^{-1} \frac{x}{a} + \frac{b}{a} x \sqrt{a^2 - x^2} \right) \Big|_c^a \\ &= ab \left( \frac{\pi}{2} - \sin^{-1} \frac{c}{a} \right) - \frac{cb}{a} \sqrt{a^2 - c^2} \text{ sq. units.} \end{aligned}$$

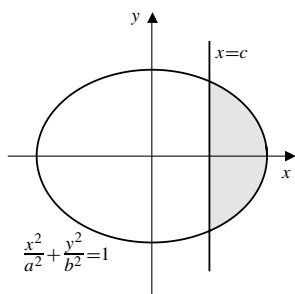


Fig. 6.2.46

47. Area of  $R$

$$\begin{aligned}
 &= \frac{Y}{2} \sqrt{1+Y^2} - \int_1^{\sqrt{1+Y^2}} \sqrt{x^2-1} \, dx \\
 &\quad \text{Let } x = \sec \theta \\
 &\quad dx = \sec \theta \tan \theta \, d\theta \\
 &= \frac{Y}{2} \sqrt{1+Y^2} - \int_0^{\tan^{-1} Y} \sec \theta \tan^2 \theta \, d\theta \\
 &= \frac{Y}{2} \sqrt{1+Y^2} - \int_0^{\tan^{-1} Y} \sec^3 \theta \, d\theta \\
 &\quad + \int_0^{\tan^{-1} Y} \sec \theta \, d\theta \\
 &= \frac{Y}{2} \sqrt{1+Y^2} + \left( -\frac{1}{2} \sec \theta \tan \theta \right. \\
 &\quad \left. - \frac{1}{2} \ln |\sec \theta + \tan \theta| + \ln |\sec \theta + \tan \theta| \right) \Big|_0^{\tan^{-1} Y} \\
 &= \frac{Y}{2} \sqrt{1+Y^2} - \frac{Y}{2} \sqrt{1+Y^2} + \frac{1}{2} \ln(Y + \sqrt{1+Y^2}) \\
 &= \frac{1}{2} \ln(Y + \sqrt{1+Y^2}) \text{ units}^2
 \end{aligned}$$

If  $Y = \sinh t$ , then we have

$$\text{Area} = \frac{1}{2} \ln(\sinh t + \cosh t) = \frac{1}{2} \ln e^t = \frac{t}{2} \text{ units}^2$$

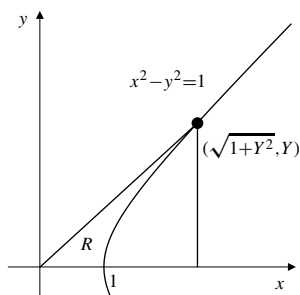


Fig. 6.2.47

$$\begin{aligned}
 48. \quad &\int \frac{dx}{\sqrt{x^2-a^2}} \quad \text{Let } x = a \cosh u \\
 &\quad dx = a \sinh u \, du \\
 &= \int \frac{a \sinh u \, du}{a \sinh u} = u + C \\
 &= \cosh^{-1} \frac{x}{a} + C = \ln(x + \sqrt{x^2-a^2}) + C, \quad (x \geq a). \\
 &\int \frac{dx}{x^2 \sqrt{x^2-a^2}} = \int \frac{a \sinh u \, du}{a^2 \cosh^2 u a \sinh u} \\
 &= \frac{1}{a^2} \int \text{sech}^2 u \, du = \frac{1}{a^2} \tanh u + C \\
 &= \frac{1}{a^2} \tanh \left( \cosh^{-1} \frac{x}{a} \right) + C \\
 &\quad \frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} - \frac{1}{\frac{x}{a} - \sqrt{\frac{x^2}{a^2} - 1}} \\
 &= \frac{1}{a^2} \cdot \frac{\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} + \frac{1}{\frac{x}{a} - \sqrt{\frac{x^2}{a^2} - 1}}}{\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} + \frac{1}{\frac{x}{a} - \sqrt{\frac{x^2}{a^2} - 1}}} + C \\
 &= \frac{\sqrt{x^2-a^2}}{a^2 x} + C_1.
 \end{aligned}$$

Section 6.3 Integrals of Rational Functions (page 336)

$$\begin{aligned}
 1. \quad &\int \frac{2 \, dx}{2x-3} = \ln |2x-3| + C. \\
 2. \quad &\int \frac{dx}{5-4x} = -\frac{1}{4} \ln |5-4x| + C. \\
 3. \quad &\int \frac{x \, dx}{\pi x + 2} = \frac{1}{\pi} \int \frac{\pi x + 2 - 2}{\pi x + 2} \, dx \\
 &= \frac{x}{\pi} - \frac{2}{\pi^2} \ln |\pi x + 2| + C. \\
 4. \quad &\int \frac{x^2}{x-4} \, dx = \int \left( x + 4 + \frac{16}{x-4} \right) \, dx \\
 &= \frac{x^2}{2} + 4x + 16 \ln |x-4| + C. \\
 5. \quad &\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3} \\
 &= \frac{Ax + 3A + Bx - 3B}{x^2-9} \\
 &\Rightarrow \begin{cases} A+B=0 \\ 3(A-B)=1 \end{cases} \Rightarrow A = \frac{1}{6}, \quad B = -\frac{1}{6}. \\
 &\int \frac{dx}{x^2-9} = \frac{1}{6} \int \frac{dx}{x-3} - \frac{1}{6} \int \frac{dx}{x+3} \\
 &= \frac{1}{6} (\ln |x-3| - \ln |x+3|) + C \\
 &= \frac{1}{6} \ln \left| \frac{x-3}{x+3} \right| + C.
 \end{aligned}$$

$$\begin{aligned}
 6. \quad \frac{1}{5-x^2} &= \frac{A}{\sqrt{5}-x} + \frac{B}{\sqrt{5}+x} \\
 &= \frac{(A+B)\sqrt{5} + (A-B)x}{5-x^2} \\
 &\Rightarrow \begin{cases} A+B = \frac{1}{\sqrt{5}} \\ A-B = 0 \end{cases} \Rightarrow A = B = \frac{1}{2\sqrt{5}}. \\
 \int \frac{1}{5-x^2} dx &= \frac{1}{2\sqrt{5}} \int \left( \frac{1}{\sqrt{5}-x} + \frac{1}{\sqrt{5}+x} \right) dx \\
 &= \frac{1}{2\sqrt{5}} \left( -\ln|\sqrt{5}-x| + \ln|\sqrt{5}+x| \right) + C \\
 &= \frac{1}{2\sqrt{5}} \ln \left| \frac{\sqrt{5}+x}{\sqrt{5}-x} \right| + C.
 \end{aligned}$$

$$\begin{aligned}
 7. \quad \frac{1}{a^2-x^2} &= \frac{A}{a-x} + \frac{B}{a+x} \\
 &= \frac{Aa + Ax + Ba - Bx}{a^2-x^2} \\
 &\Rightarrow \begin{cases} Aa + Ba = 1 \\ A - B = 0 \end{cases} \Rightarrow A = B = \frac{1}{2a}. \\
 \text{Thus} \\
 \int \frac{dx}{a^2-x^2} &= \frac{1}{2a} \int \frac{dx}{a-x} + \frac{1}{2a} \int \frac{dx}{a+x} \\
 &= \frac{1}{2a} \left( -\ln|a-x| + \ln|a+x| \right) + C \\
 &= \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C.
 \end{aligned}$$

$$\begin{aligned}
 8. \quad \frac{1}{b^2-a^2x^2} &= \frac{A}{b-ax} + \frac{B}{b+ax} \\
 &= \frac{(A+B)b + (A-B)ax}{b^2-a^2x^2} \\
 &\Rightarrow A = B = \frac{1}{2b} \\
 \int \frac{dx}{b^2-a^2x^2} &= \frac{1}{2b} \int \left( \frac{1}{b-ax} + \frac{1}{b+ax} \right) dx \\
 &= \frac{1}{2b} \left( \frac{-\ln|b-ax|}{a} + \frac{\ln|b+ax|}{a} \right) + C \\
 &= \frac{1}{2ab} \ln \left| \frac{b+ax}{b-ax} \right| + C.
 \end{aligned}$$

$$\begin{aligned}
 9. \quad \int \frac{x^2 dx}{x^2+x-2} &= \int \left( 1 - \frac{x-2}{x^2+x-2} \right) dx \\
 &= x - \int \frac{x-2}{x^2+x-2} dx. \\
 \text{If } \frac{x-2}{x^2+x-2} &= \frac{A}{x+2} + \frac{B}{x-1} = \frac{Ax - A + Bx + 2B}{x^2+x-2}, \\
 \text{then } A+B &= 1 \text{ and } -A+2B = -2, \text{ so that } A = 4/3 \\
 \text{and } B &= -1/3. \text{ Thus}
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{x^2 dx}{x^2+x-2} &= x - \frac{4}{3} \int \frac{dx}{x+2} + \frac{1}{3} \int \frac{dx}{x-1} \\
 &= x - \frac{4}{3} \ln|x+2| + \frac{1}{3} \ln|x-1| + C.
 \end{aligned}$$

$$\begin{aligned}
 10. \quad \frac{x}{3x^2+8x-3} &= \frac{A}{3x-1} + \frac{B}{x+3} \\
 &= \frac{(A+3B)x + (3A-B)}{3x^2+8x-3} \\
 &\Rightarrow \begin{cases} A+3B = 1 \\ 3A-B = 0 \end{cases} \Rightarrow A = \frac{1}{10}, B = \frac{3}{10}. \\
 \int \frac{x dx}{3x^2+8x-3} &= \frac{1}{10} \int \left( \frac{1}{3x-1} + \frac{3}{x+3} \right) dx \\
 &= \frac{1}{30} \ln|3x-1| + \frac{3}{10} \ln|x+3| + C.
 \end{aligned}$$

$$\begin{aligned}
 11. \quad \frac{x-2}{x^2+x} &= \frac{A}{x} + \frac{B}{x+1} = \frac{Ax + A + Bx}{x^2+x} \\
 &\Rightarrow \begin{cases} A+B = 1 \\ A = -2 \end{cases} \Rightarrow A = -2, B = 3. \\
 \int \frac{x-2}{x^2+x} dx &= 3 \int \frac{dx}{x+1} - 2 \int \frac{dx}{x} \\
 &= 3 \ln|x+1| - 2 \ln|x| + C.
 \end{aligned}$$

$$\begin{aligned}
 12. \quad \frac{1}{x^3+9x} &= \frac{A}{x} + \frac{Bx+C}{x^2+9} \\
 &= \frac{Ax^2+9A+Bx^2+Cx}{x^3+9x} \\
 &\Rightarrow \begin{cases} A+B = 0 \\ C = 0 \\ 9A = 1 \end{cases} \Rightarrow A = \frac{1}{9}, B = -\frac{1}{9}, C = 0. \\
 \int \frac{dx}{x^3+9x} &= \frac{1}{9} \int \left( \frac{1}{x} - \frac{x}{x^2+9} \right) dx \\
 &= \frac{1}{9} \ln|x| - \frac{1}{18} \ln(x^2+9) + K.
 \end{aligned}$$

$$13. \quad \int \frac{dx}{1-6x+9x^2} = \int \frac{dx}{(1-3x)^2} = \frac{1}{3(1-3x)} + C.$$

$$\begin{aligned}
 14. \quad \int \frac{x}{2+6x+9x^2} dx &= \int \frac{x}{(3x+1)^2+1} dx \quad \text{Let } u = 3x+1 \\
 &\quad du = 3 dx \\
 \frac{1}{9} \int \frac{u-1}{u^2+1} du &= \frac{1}{9} \int \frac{u}{u^2+1} du - \frac{1}{9} \int \frac{1}{u^2+1} du \\
 &= \frac{1}{18} \ln(u^2+1) - \frac{1}{9} \tan^{-1} u + C \\
 &= \frac{1}{18} \ln(2+6x+9x^2) - \frac{1}{9} \tan^{-1}(3x+1) + C.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad \int \frac{x^2+1}{6x-9x^2} dx &= \frac{1}{9} \int \frac{9x^2-6x+6x+9}{6x-9x^2} dx \\
 &= -\frac{x}{9} + \frac{1}{9} \int \frac{2x+3}{x(2-3x)} dx.
 \end{aligned}$$

Now

$$\begin{aligned}
 \frac{2x+3}{x(2-3x)} &= \frac{A}{x} + \frac{B}{2-3x} = \frac{2A-3Ax+Bx}{x(2-3x)} \\
 &\Rightarrow 2A = 3, \quad -3A+B = 2 \\
 &\Rightarrow A = \frac{3}{2}, \quad B = \frac{13}{2}.
 \end{aligned}$$

Therefore we have

$$\begin{aligned} \int \frac{x^2+1}{6x-9x^2} dx &= -\frac{x}{9} + \frac{1}{6} \int \frac{dx}{x} + \frac{13}{18} \int \frac{dx}{2-3x} \\ &= -\frac{x}{9} + \frac{1}{6} \ln|x| - \frac{13}{54} \ln|2-3x| + C. \end{aligned}$$

16. First divide to obtain

$$\begin{aligned} \frac{x^3+1}{x^2+7x+12} &= x-7 + \frac{37x+85}{(x+4)(x+3)} \\ \frac{37x+85}{(x+4)(x+3)} &= \frac{A}{x+4} + \frac{B}{x+3} \\ &= \frac{(A+B)x+3A+4B}{x^2+7x+12} \\ \Rightarrow \begin{cases} A+B=37 \\ 3A+4B=85 \end{cases} &\Rightarrow A=63, B=-26. \end{aligned}$$

Now we have

$$\begin{aligned} \int \frac{x^3+1}{12+7x+x^2} dx &= \int \left( x-7 + \frac{63}{x+4} - \frac{26}{x+3} \right) dx \\ &= \frac{x^2}{2} - 7x + 63 \ln|x+4| - 26 \ln|x+3| + C. \end{aligned}$$

$$\begin{aligned} 17. \quad \frac{1}{x(x^2-a^2)} &= \frac{A}{x} + \frac{B}{x-a} + \frac{C}{x+a} \\ &= \frac{Ax^2 - Aa^2 + Bx^2 + Bax + Cx^2 - Cax}{x(x^2-a^2)} \\ \Rightarrow \begin{cases} A+B+C=0 \\ B-C=0 \\ -Aa^2=1 \end{cases} &\Rightarrow \begin{cases} A=-1/a^2 \\ B=C=1/(2a^2). \end{cases} \end{aligned}$$

Thus we have

$$\begin{aligned} \int \frac{dx}{x(x^2-a^2)} &= \frac{1}{2a^2} \left( -2 \int \frac{dx}{x} + \int \frac{dx}{x-a} + \int \frac{dx}{x+a} \right) \\ &= \frac{1}{2a^2} (-2 \ln|x| + \ln|x-a| + \ln|x+a|) + K \\ &= \frac{1}{2a^2} \ln \frac{|x^2-a^2|}{x^2} + K. \end{aligned}$$

18. The partial fraction decomposition is

$$\begin{aligned} \frac{1}{x^4-a^4} &= \frac{A}{x-a} + \frac{B}{x+a} + \frac{Cx+D}{x^2+a^2} \\ &= \frac{A(x^3+ax^2+a^2x+a^3) + B(x^3-ax^2+a^2x-a^3)}{x^4-a^4} \\ &\quad + \frac{C(x^3-a^2x) + D(x^2-a^2)}{x^4-a^4} \\ \Rightarrow \begin{cases} A+B+C=0 \\ aA-aB+D=0 \\ a^2A+a^2B-a^2C=0 \\ a^3A-a^3B-a^2D=1 \end{cases} \\ \Rightarrow A &= \frac{1}{4a^3}, B = -\frac{1}{4a^3}, C=0, D = -\frac{1}{2a^2}. \end{aligned}$$

$$\begin{aligned} \int \frac{dx}{x^4-a^4} &= \frac{1}{4a^3} \int \left( \frac{1}{x-a} - \frac{1}{x+a} - \frac{2a}{x^2+a^2} \right) dx \\ &= \frac{1}{4a^3} \ln \left| \frac{x-a}{x+a} \right| - \frac{1}{2a^3} \tan^{-1} \left( \frac{x}{a} \right) + K. \end{aligned}$$

$$\begin{aligned} 19. \quad \frac{a^3}{x^3-a^3} &= \frac{A}{x-a} + \frac{Bx+C}{x^2+ax+a^2} \\ &= \frac{Ax^2 + Aax + Aa^2 + Bx^2 + Bax + Cx - Ca}{x^3-a^3} \\ \Rightarrow \begin{cases} A+B=0 \\ Aa-Ba+C=0 \\ Aa^2-Ca=a^3 \end{cases} &\Rightarrow \begin{cases} A=a/3 \\ B=-a/3 \\ C=-2a^2/3. \end{cases} \end{aligned}$$

Therefore we have

$$\begin{aligned} \int \frac{x^3}{x^3-a^3} dx &= \int \left( 1 + \frac{a^3}{x^3-a^3} \right) dx \\ &= x + \frac{a}{3} \int \frac{dx}{x-a} - \frac{a}{3} \int \frac{x+2a}{x^2+ax+a^2} \\ &= x + \frac{a}{3} \ln|x-a| - \frac{a}{6} \int \frac{2x+a+3a}{x^2+ax+a^2} \\ &= x + \frac{a}{3} \ln|x-a| - \frac{a}{6} \ln(x^2+ax+a^2) \\ &\quad - \frac{a^2}{2} \int \frac{dx}{\left(x+\frac{a}{2}\right)^2 + \frac{3}{4}a^2} \\ &= x + \frac{a}{3} \ln|x-a| - \frac{a}{6} \ln(x^2+ax+a^2) \\ &\quad - \frac{a^2}{2} \frac{2}{\sqrt{3}a} \tan^{-1} \frac{x+(a/2)}{(\sqrt{3}a)/2} + K \\ &= x + \frac{a}{3} \ln|x-a| - \frac{a}{6} \ln(x^2+ax+a^2) \\ &\quad - \frac{a}{\sqrt{3}} \tan^{-1} \frac{2x+a}{\sqrt{3}a} + K. \end{aligned}$$

20. Here the expansion is

$$\begin{aligned}\frac{1}{x^3 + 2x^2 + 2x} &= \frac{A}{x} + \frac{Bx + C}{x^2 + 2x + 2} \\ &= \frac{A(x^2 + 2x + 2) + Bx^2 + Cx}{x^3 + 2x^2 + 2x} \\ &\Rightarrow \begin{cases} A + B = 0 \\ 2A + C = 0 \\ 2A = 1 \end{cases} \Rightarrow A = -B = \frac{1}{2}, \quad C = -1,\end{aligned}$$

so we have

$$\begin{aligned}\int \frac{dx}{x^3 + 2x^2 + 2x} &= \frac{1}{2} \int \frac{dx}{x} - \frac{1}{2} \int \frac{x+2}{x^2 + 2x + 2} dx \\ \text{Let } u &= x + 1 \\ du &= dx \\ &= \frac{1}{2} \ln|x| - \frac{1}{2} \int \frac{u+1}{u^2 + 1} du \\ &= \frac{1}{2} \ln|x| - \frac{1}{4} \ln(u^2 + 1) - \frac{1}{2} \tan^{-1} u + K \\ &= \frac{1}{2} \ln|x| - \frac{1}{4} \ln(x^2 + 2x + 2) - \frac{1}{2} \tan^{-1}(x + 1) + K.\end{aligned}$$

$$\begin{aligned}21. \quad \frac{1}{x^3 - 4x^2 + 3x} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x-3} \\ &= \frac{A(x^2 - 4x + 3) + B(x^2 - 3x) + C(x^2 - x)}{x^3 - 4x^2 + 3x} \\ &\Rightarrow \begin{cases} A + B + C = 0 \\ -4A - 3B - C = 0 \\ 3A = 1 \end{cases} \\ &\Rightarrow A = \frac{1}{3}, \quad B = -\frac{1}{2}, \quad C = \frac{1}{6}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\int \frac{dx}{x^3 - 4x^2 + 3x} &= \frac{1}{3} \int \frac{dx}{x} - \frac{1}{2} \int \frac{dx}{x-1} + \frac{1}{6} \int \frac{dx}{x-3} \\ &= \frac{1}{3} \ln|x| - \frac{1}{2} \ln|x-1| + \frac{1}{6} \ln|x-3| + K.\end{aligned}$$

22. Here the expansion is

$$\begin{aligned}\frac{x^2 + 1}{x^3 + 8} &= \frac{A}{x+2} + \frac{Bx + C}{x^2 - 2x + 4} \\ &= \frac{A(x^2 - 2x + 4) + B(x^2 + 2x) + C(x + 2)}{x^3 + 8} \\ &\Rightarrow \begin{cases} A + B = 1 \\ -2A + 2B + C = 0 \\ 4A + 2C = 1 \end{cases} \Rightarrow A = \frac{5}{12}, \quad B = \frac{7}{12}, \quad C = -\frac{1}{3},\end{aligned}$$

so we have

$$\begin{aligned}\int \frac{x^2 + 1}{x^3 + 8} dx &= \frac{5}{12} \int \frac{dx}{x+2} + \frac{1}{12} \int \frac{7x-4}{(x-1)^2 + 3} dx \\ \text{Let } u &= x - 1 \\ du &= dx \\ &= \frac{5}{12} \ln|x+2| + \frac{1}{12} \int \frac{7u+3}{u^2 + 3} du \\ &= \frac{5}{12} \ln|x+2| + \frac{7}{24} \ln(x^2 - 2x + 4) \\ &\quad + \frac{1}{4\sqrt{3}} \tan^{-1} \frac{x-1}{\sqrt{3}} + K.\end{aligned}$$

$$\begin{aligned}23. \quad \frac{1}{(x^2 - 1)^2} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} \\ &= \frac{1}{(x^2 - 1)^2} (A(x-1)(x+1)^2 + B(x+1)^2 \\ &\quad + C(x+1)(x-1)^2 + D(x-1)^2) \\ &\Rightarrow \begin{cases} A + C = 0 \\ A + B - C + D = 0 \\ -A + 2B - C - 2D = 0 \\ -A + B + C + D = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{4} \\ B = C = D = \frac{1}{4} \end{cases}.\end{aligned}$$

Thus

$$\begin{aligned}\int \frac{dx}{(x^2 - 1)^2} &= \frac{1}{4} \left( -\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} \right. \\ &\quad \left. + \int \frac{dx}{x+1} + \int \frac{dx}{(x+1)^2} \right) \\ &= \frac{1}{4} \left( \ln|x+1| - \ln|x-1| - \frac{1}{x-1} - \frac{1}{x+1} \right) + K \\ &= \frac{1}{4} \ln \left| \frac{x+1}{x-1} \right| - \frac{x}{2(x^2 - 1)} + K.\end{aligned}$$

24. The expansion is

$$\begin{aligned}\frac{x^2}{(x^2 - 1)(x^2 - 4)} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2} \\ A &= \lim_{x \rightarrow 1} \frac{x^2}{(x+1)(x^2 - 4)} = \frac{1}{2(-3)} = -\frac{1}{6} \\ B &= \lim_{x \rightarrow -1} \frac{x^2}{(x-1)(x^2 - 4)} = \frac{1}{-2(-3)} = \frac{1}{6} \\ C &= \lim_{x \rightarrow 2} \frac{x^2}{(x^2 - 1)(x+2)} = \frac{4}{3(4)} = \frac{1}{3} \\ D &= \lim_{x \rightarrow -2} \frac{x^2}{(x^2 - 1)(x-2)} = \frac{4}{3(-4)} = -\frac{1}{3}.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{x^2}{(x^2 - 1)(x^2 - 4)} dx &= -\frac{1}{6} \ln|x-1| + \frac{1}{6} \ln|x+1| + \\ &\quad \frac{1}{3} \ln|x-2| - \frac{1}{3} \ln|x+2| + K.\end{aligned}$$

$$\begin{aligned}
 25. \quad \frac{1}{x^4 - 3x^3} &= \frac{1}{x^3(x-3)} \\
 &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-3} \\
 &= \frac{A(x^3 - 3x^2) + B(x^2 - 3x) + C(x-3) + Dx^3}{x^3(x-3)} \\
 &\Rightarrow \begin{cases} A + D = 0 \\ -3A + B = 0 \\ -3B + C = 0 \\ -3C = 1 \end{cases} \\
 &\Rightarrow \begin{cases} A = -1/27 \\ B = -1/9 \\ C = -1/3 \\ D = 1/27 \end{cases}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \int \frac{dx}{x^4 - 3x^3} &= -\frac{1}{27} \int \frac{dx}{x} - \frac{1}{9} \int \frac{dx}{x^2} - \frac{1}{3} \int \frac{dx}{x^3} + \frac{1}{27} \int \frac{dx}{x-3} \\
 &= \frac{1}{27} \ln \left| \frac{x-3}{x} \right| + \frac{1}{9x} + \frac{1}{6x^2} + K.
 \end{aligned}$$

26. We have

$$\begin{aligned}
 \int \frac{x dx}{(x^2 - x + 1)^2} &= \int \frac{x dx}{\left[ \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \right]^2} \quad \text{Let } u = x - \frac{1}{2} \\
 &= \int \frac{u du}{(u^2 + \frac{3}{4})^2} + \frac{1}{2} \int \frac{du}{(u^2 + \frac{3}{4})^2} \\
 \text{Let } u &= \frac{\sqrt{3}}{2} \tan v, \\
 du &= \frac{\sqrt{3}}{2} \sec^2 v dv \text{ in the second integral.} \\
 &= -\frac{1}{2} \left( \frac{1}{u^2 + \frac{3}{4}} \right) + \frac{1}{2} \int \frac{\frac{\sqrt{3}}{2} \sec^2 v dv}{\frac{9}{16} \sec^4 v} \\
 &= \frac{-1}{2(x^2 - x + 1)} + \frac{4}{3\sqrt{3}} \int \cos^2 v dv \\
 &= \frac{-1}{2(x^2 - x + 1)} + \frac{2}{3\sqrt{3}} (v + \sin v \cos v) + C \\
 &= \frac{-1}{2(x^2 - x + 1)} + \frac{2}{3\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + \frac{2}{3\sqrt{3}} \frac{2(x-\frac{1}{2})\sqrt{3}}{(2\sqrt{x^2-x+1})^2} + C \\
 &= \frac{2}{3\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}} + \frac{x-2}{3(x^2-x+1)} + C.
 \end{aligned}$$

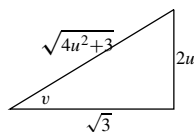


Fig. 6.3.26

$$\begin{aligned}
 27. \quad \frac{t}{(t+1)(t^2+1)^2} &= \frac{A}{t+1} + \frac{Bt+C}{t^2+1} + \frac{Dt+E}{(t^2+1)^2} \\
 &= \frac{1}{(t+1)(t^2+1)^2} \left( A(t^4+2t^2+1) + B(t^4+t^3+t^2+t) \right. \\
 &\quad \left. + C(t^3+t^2+t+1) + D(t^2+t) + E(t+1) \right) \\
 &\Rightarrow \begin{cases} A+B=0 \\ B+C=0 \\ 2A+B+C+D=0 \\ B+C+D+E=1 \\ A+C+E=0 \end{cases} \Rightarrow \begin{cases} B=-A \\ C=A \\ D=-2A \\ 2D=1 \\ E=-2A \end{cases}
 \end{aligned}$$

Thus  $A = -1/4 = C$ ,  $B = 1/4$ ,  $D = E = 1/2$ . We have

$$\begin{aligned}
 \int \frac{t dt}{(t+1)(t^2+1)^2} &= -\frac{1}{4} \int \frac{dt}{t+1} + \frac{1}{4} \int \frac{(t-1) dt}{t^2+1} + \frac{1}{2} \int \frac{(t+1) dt}{(t^2+1)^2} \\
 &= -\frac{1}{4} \ln|t+1| + \frac{1}{8} \ln(t^2+1) - \frac{1}{4} \tan^{-1} t \\
 &\quad - \frac{1}{4(t^2+1)} + \frac{1}{2} \int \frac{dt}{(t^2+1)^2} \quad \text{Let } t = \tan \theta \\
 &\quad \quad \quad dt = \sec^2 \theta d\theta \\
 &= -\frac{1}{4} \ln|t+1| + \frac{1}{8} \ln(t^2+1) - \frac{1}{4} \tan^{-1} t \\
 &\quad - \frac{1}{4(t^2+1)} + \frac{1}{2} \int \cos^2 \theta d\theta \\
 &= -\frac{1}{4} \ln|t+1| + \frac{1}{8} \ln(t^2+1) - \frac{1}{4} \tan^{-1} t \\
 &\quad - \frac{1}{4(t^2+1)} + \frac{1}{4} (\theta + \sin \theta \cos \theta) + K \\
 &= -\frac{1}{4} \ln|t+1| + \frac{1}{8} \ln(t^2+1) - \frac{1}{4} \tan^{-1} t \\
 &\quad - \frac{1}{4(t^2+1)} + \frac{1}{4} \tan^{-1} t + \frac{1}{4} \frac{t}{t^2+1} + K \\
 &= \frac{1}{4} \frac{t-1}{t^2+1} - \frac{1}{4} \ln|t+1| + \frac{1}{8} \ln(t^2+1) + K.
 \end{aligned}$$

28. We have

$$\begin{aligned}
 \int \frac{dt}{(t-1)(t^2-1)^2} &= \int \frac{dt}{(t-1)^3(t+1)^2} \quad \text{Let } u = t-1 \\
 &= \int \frac{du}{u^3(u+2)^2} \quad du = dt \\
 \frac{1}{u^3(u+2)^2} &= \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{D}{u+2} + \frac{E}{(u+2)^2} \\
 &= \frac{A(u^4+4u^3+4u^2) + B(u^3+4u^2+4u)}{u^3(u+2)^2} \\
 &\quad + \frac{C(u^2+4u+4) + D(u^4+2u^3) + Eu^3}{u^3(u+2)^2}
 \end{aligned}$$

$$\Rightarrow \begin{cases} A + D = 0 \\ 4A + B + 2D + E = 0 \\ 4A + 4B + C = 0 \\ 4B + 4C = 0 \\ 4C = 1 \end{cases}$$

$$\Rightarrow A = \frac{3}{16}, B = -\frac{1}{4}, C = \frac{1}{4}, D = -\frac{3}{16}, E = -\frac{1}{8}.$$

$$\int \frac{du}{u^3(u+2)^2}$$

$$= \frac{3}{16} \int \frac{du}{u} - \frac{1}{4} \int \frac{du}{u^2} + \frac{1}{4} \int \frac{du}{u^3}$$

$$- \frac{3}{16} \int \frac{du}{u+2} - \frac{1}{8} \int \frac{du}{(u+2)^2}$$

$$= \frac{3}{16} \ln|t-1| + \frac{1}{4(t-1)} - \frac{1}{8(t-1)^2} -$$

$$\frac{3}{16} \ln|t+1| + \frac{1}{8(t+1)} + K.$$

29.  $I = \int \frac{dx}{x(3+x^2)\sqrt{1-x^2}}$  Let  $1-x^2 = u^2$   
 $-2x dx = 2u du$

$$= - \int \frac{du}{(1-u^2)(4-u^2)}.$$

$$\frac{1}{(1-u^2)(4-u^2)}$$

$$= \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{2-u} + \frac{D}{2+u}$$

$$A = \frac{1}{(1+u)(4-u^2)} \Big|_{u=1} = \frac{1}{6}$$

$$B = \frac{1}{(1-u)(4-u^2)} \Big|_{u=-1} = \frac{1}{6}$$

$$C = \frac{1}{(1-u^2)(2+u)} \Big|_{u=2} = -\frac{1}{12}$$

$$D = \frac{1}{(1-u^2)(2-u)} \Big|_{u=-2} = -\frac{1}{12}.$$

Thus

$$I = - \left( \frac{1}{6} \int \frac{du}{1-u} + \frac{1}{6} \int \frac{du}{1+u} \right.$$

$$\left. - \frac{1}{12} \int \frac{du}{2-u} - \frac{1}{12} \int \frac{du}{2+u} \right)$$

$$= \frac{1}{6} \ln \left| \frac{1-u}{1+u} \right| + \frac{1}{12} \ln \left| \frac{2+u}{2-u} \right| + K$$

$$= \frac{1}{6} \ln \left| \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \right| + \frac{1}{12} \ln \left| \frac{2+\sqrt{1-x^2}}{2-\sqrt{1-x^2}} \right| + K$$

$$= \frac{1}{6} \ln \frac{(1-\sqrt{1-x^2})^2}{x^2} + \frac{1}{12} \ln \frac{(2+\sqrt{1-x^2})^2}{3+x^2} + K.$$

30.  $\int \frac{dx}{e^{2x} - 4e^x + 4} = \int \frac{dx}{(e^x - 2)^2}$  Let  $u = e^x$   
 $du = e^x dx$

$$= \int \frac{du}{u(u-2)^2}$$

$$\frac{1}{u(u-2)^2} = \frac{A}{u} + \frac{B}{u-2} + \frac{C}{(u-2)^2}$$

$$= \frac{A(u^2 - 4u + 4) + B(u^2 - 2u) + Cu}{u(u-2)^2}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -4A - 2B + C = 0 \\ 4A = 1 \end{cases} \Rightarrow A = \frac{1}{4}, B = -\frac{1}{4}, C = \frac{1}{2}.$$

$$\int \frac{du}{u(u-2)^2} = \frac{1}{4} \int \frac{du}{u} - \frac{1}{4} \int \frac{du}{u-2} + \frac{1}{2} \int \frac{du}{(u-2)^2}$$

$$= \frac{1}{4} \ln|u| - \frac{1}{4} \ln|u-2| - \frac{1}{2(u-2)} + K$$

$$= \frac{x}{4} - \frac{1}{4} \ln|e^x - 2| - \frac{1}{2(e^x - 2)} + K.$$

31.  $I = \int \frac{dx}{x(1+x^2)^{3/2}}$  Let  $x = \tan \theta$   
 $dx = \sec^2 \theta d\theta$

$$= \int \frac{\sec^2 \theta d\theta}{\tan \theta \sec^3 \theta} = \int \frac{\cos^2 \theta d\theta}{\sin \theta}$$

$$= \int \frac{\cos^2 \theta \sin \theta d\theta}{\sin^2 \theta}$$
 Let  $u = \cos \theta$   
 $du = -\sin \theta d\theta$ 

$$= - \int \frac{u^2 du}{1-u^2} = u + \int \frac{du}{u^2-1}.$$

We have

$$\frac{1}{u^2-1} = \frac{1}{2} \left( \frac{1}{u-1} - \frac{1}{u+1} \right).$$

Thus

$$I = u + \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| + C$$

$$= \cos \theta + \frac{1}{2} \ln \left| \frac{\cos \theta - 1}{\cos \theta + 1} \right| + C$$

$$= \frac{1}{\sqrt{1+x^2}} + \frac{1}{2} \ln \left| \frac{\frac{1}{\sqrt{1+x^2}} - 1}{\frac{1}{\sqrt{1+x^2}} + 1} \right| + C$$

$$= \frac{1}{\sqrt{1+x^2}} + \frac{1}{2} \ln \left( \frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} \right) + C.$$

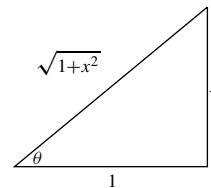


Fig. 6.3.31

32. We have

$$\begin{aligned}
 I &= \int \frac{dx}{x(1-x^2)^{3/2}} \quad \text{Let } u^2 = 1-x^2 \\
 &\quad 2u du = -2x dx \\
 &= - \int \frac{u du}{(1-u^2)u^3} = - \int \frac{du}{(1-u^2)u^2} \\
 &\quad \frac{1}{u^2(1-u^2)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{1-u} + \frac{D}{1+u} \\
 &= \frac{A(u-u^3) + B(1-u^2) + C(u^2+u^3) + D(u^2-u^3)}{u^2(1-u^2)} \\
 &\Rightarrow \begin{cases} -A+C-D=0 \\ -B+C+D=0 \\ A=0 \\ B=1 \end{cases} \\
 &\Rightarrow A=0, B=1, C=\frac{1}{2}, D=\frac{1}{2}. \\
 I &= - \int \frac{du}{(1-u^2)u^2} = - \int \frac{du}{u^2} - \frac{1}{2} \int \frac{du}{1-u} - \frac{1}{2} \int \frac{du}{1+u} \\
 &= \frac{1}{u} + \frac{1}{2} \ln|1-u| - \frac{1}{2} \ln|1+u| + K \\
 &= \frac{1}{\sqrt{1-x^2}} + \frac{1}{2} \ln \left| \frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}} \right| + K \\
 &= \frac{1}{\sqrt{1-x^2}} + \ln(1-\sqrt{1-x^2}) - \ln|x| + K.
 \end{aligned}$$

$$\begin{aligned}
 33. \quad &\int \frac{dx}{x^2(x^2-1)^{3/2}} \quad \text{Let } x = \sec \theta \\
 &\quad dx = \sec \theta \tan \theta d\theta \\
 &= \int \frac{\sec \theta \tan \theta d\theta}{\sec^2 \theta \tan^3 \theta} = \int \frac{\cos^3 \theta d\theta}{\sin^2 \theta} \\
 &= \int \frac{1-\sin^2 \theta}{\sin^2 \theta} \cos \theta d\theta \quad \text{Let } u = \sin \theta \\
 &\quad du = \cos \theta d\theta \\
 &= \int \frac{1-u^2}{u^2} du = -\frac{1}{u} - u + C \\
 &= -\left(\frac{1}{\sin \theta} + \sin \theta\right) + C \\
 &= -\left(\frac{x}{\sqrt{x^2-1}} + \frac{\sqrt{x^2-1}}{x}\right) + C.
 \end{aligned}$$

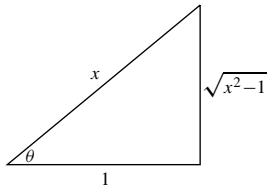


Fig. 6.3.33

$$\begin{aligned}
 34. \quad &\int \frac{d\theta}{\cos \theta (1+\sin \theta)} \quad \text{Let } u = \sin \theta \\
 &\quad du = \cos \theta d\theta \\
 &= \int \frac{du}{(1-u^2)(1+u)} = \int \frac{du}{(1-u)(1+u)^2} \\
 &\quad \frac{1}{(1-u)(1+u)^2} = \frac{A}{1-u} + \frac{B}{1+u} + \frac{C}{(1+u)^2} \\
 &= \frac{A(1+2u+u^2) + B(1-u^2) + C(1-u)}{(1-u)(1+u)^2} \\
 &\Rightarrow \begin{cases} A-B=0 \\ 2A-C=0 \\ A+B+C=1 \end{cases} \Rightarrow A=\frac{1}{4}, B=\frac{1}{4}, C=\frac{1}{2}. \\
 &\int \frac{du}{(1-u)(1+u)^2} \\
 &= \frac{1}{4} \int \frac{du}{1-u} + \frac{1}{4} \int \frac{du}{1+u} + \frac{1}{2} \int \frac{du}{(1+u)^2} \\
 &= \frac{1}{4} \ln \left| \frac{1+\sin \theta}{1-\sin \theta} \right| - \frac{1}{2(1+\sin \theta)} + C.
 \end{aligned}$$

35. Since  $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$ , we have

$$\ln Q(x) = \ln(x-a_1) + \ln(x-a_2) + \cdots + \ln(x-a_n),$$

and, differentiating both sides,

$$\begin{aligned}
 \frac{Q'(x)}{Q(x)} &= \frac{d}{dx} [\ln Q(x)] = \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_n} \\
 \frac{1}{Q(x)} &= \frac{1}{Q'(x)} \left[ \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_n} \right].
 \end{aligned}$$

Since

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n},$$

we have

$$\begin{aligned}
 &\frac{P(x)}{Q'(x)} \left[ \frac{1}{x-a_1} + \frac{1}{x-a_2} + \cdots + \frac{1}{x-a_n} \right] \\
 &= \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}.
 \end{aligned}$$

Multiply both sides by  $x-a_1$  and get

$$\begin{aligned}
 &\frac{P(x)}{Q'(x)} \left[ 1 + \frac{x-a_1}{x-a_2} + \cdots + \frac{x-a_1}{x-a_n} \right] \\
 &= A_1 + \frac{A_2(x-a_1)}{x-a_2} + \cdots + \frac{A_n(x-a_1)}{x-a_n}.
 \end{aligned}$$

Now let  $x = a_1$  and obtain  $\frac{P(a_1)}{Q'(a_1)} = A_1$ .

Similarly,  $A_j = \frac{P(a_j)}{Q'(a_j)}$  for  $1 \leq j \leq n$ .

**Section 6.4 Integration Using Computer Algebra or Tables (page 340)**

2. According to Maple

$$\begin{aligned} \int \frac{1+x+x^2}{(x^4-1)(x^4-16)^2} dx \\ = \frac{\ln(x-1)}{300} - \frac{\ln(x+1)}{900} - \frac{7}{15,360(x-2)} \\ - \frac{613}{460,800} \ln(x-2) - \frac{1}{5,120(x+2)} + \frac{79}{153,600} \ln(x+2) \\ - \frac{\ln(x^2+1)}{900} + \frac{47}{115,200} \ln(x^2+4) \\ - \frac{23}{25,600} \tan^{-1}(x/2) - \frac{6x+8}{15,360(x^2+4)} \end{aligned}$$

One suspects it has forgotten to use absolute values in some of the logarithms.

3. Neither the author's version of Maple nor his version of Mathematics would do

$$I = \int \frac{t^5}{\sqrt{3-2t^4}} dt$$

as presented. Both did an integration by parts and left an unevaluated integral. Both managed to evaluate the integral after the substitution  $u = t^2$  was made. (See Exercise 4.) However, Derive had no trouble doing the integral in its original form, giving as the answer

$$\frac{3\sqrt{2}}{16} \sin^{-1} \frac{\sqrt{6}t^2}{3} - \frac{t^2\sqrt{3-2t^4}}{8}.$$

4. Maple, Mathematica, and Derive readily gave

$$\int_0^1 \frac{1}{(x^2+1)^3} dx = \frac{3\pi}{32} + \frac{1}{4}.$$

5. Use the 6th integral in the list involving  $\sqrt{x^2 \pm a^2}$ .

$$\int \frac{x^2 dx}{\sqrt{x^2-2}} = \frac{x}{2} \sqrt{x^2-2} + \ln|x+\sqrt{x^2-2}| + C$$

6. Use the last integral in the list involving  $\sqrt{x^2 \pm a^2}$ .

$$\int \sqrt{(x^2+4)^3} dx = \frac{x}{4}(x^2+10)\sqrt{x^2+4} + 6\ln|x+\sqrt{x^2+4}| + C$$

7. Use the 8th integral in the list involving  $\sqrt{x^2 \pm a^2}$  after making the change of variable  $x = \sqrt{3}t$ .

$$\begin{aligned} \int \frac{dt}{t^2\sqrt{3t^2+5}} \quad \text{Let } x = \sqrt{3}t \\ dx = \sqrt{3}dt \\ = \frac{3}{\sqrt{3}} \int \frac{dx}{x^2\sqrt{x^2+5}} \\ = -\sqrt{3} \frac{\sqrt{x^2+5}}{5x} + C = -\frac{\sqrt{3t^2+5}}{5t} + C \end{aligned}$$

8. Use the 8th integral in the miscellaneous algebraic set.

$$\int \frac{dt}{t\sqrt{3t-5}} = \frac{2}{\sqrt{5}} \tan^{-1} \sqrt{\frac{3t-5}{5}} + C$$

9. The 5th and 4th integrals in the exponential/logarithmic set give

$$\begin{aligned} \int x^4(\ln x)^4 dx &= \frac{x^5(\ln x)^4}{5} - \frac{4}{5} \int x^4(\ln x)^3 dx \\ &= \frac{x^5(\ln x)^4}{5} - \frac{4}{5} \left( \frac{x^5(\ln x)^3}{5} - \frac{3}{2} \int x^4(\ln x)^2 dx \right) \\ &= x^5 \left( \frac{(\ln x)^4}{5} - \frac{4(\ln x)^3}{25} \right) \\ &\quad + \frac{12}{25} \left( \frac{x^5(\ln x)^2}{5} - \frac{2}{5} \int x^5 \ln x dx \right) \\ &= x^5 \left( \frac{(\ln x)^4}{5} - \frac{4(\ln x)^3}{25} + \frac{12(\ln x)^2}{125} - \frac{24 \ln x}{625} + \frac{24}{3,125} \right) + C. \end{aligned}$$

10. We make a change of variable and then use the first two integrals in the exponential/logarithmic set.

$$\begin{aligned} \int x^7 e^{x^2} dx \quad \text{Let } u = x^2 \\ du = 2x dx \\ = \frac{1}{2} \int u^3 e^u du \\ = \frac{1}{2} \left( u^3 e^u - 3 \int u^2 e^u du \right) \\ = \frac{u^3 e^u}{2} - \frac{3}{2} \left( u^2 e^u - 2 \int u e^u du \right) \\ = \left( \frac{u^3}{2} - \frac{3u^2}{2} + 3(u-1) \right) e^u + C \\ = \left( \frac{x^6}{2} - \frac{3x^4}{2} + 3x^2 - 3 \right) e^{x^2} + C \end{aligned}$$

11. Use integrals 14 and 12 in the miscellaneous algebraic set.

$$\begin{aligned} \int x\sqrt{2x-x^2} dx &= -\frac{(2x-x^2)^{3/2}}{3} + \frac{3}{2} \int \sqrt{2x-x^2} dx \\ &= -\frac{(2x-x^2)^{3/2}}{3} + \frac{x-1}{2} \sqrt{2x-x^2} + \frac{1}{2} \sin^{-1}(x-1) + C \end{aligned}$$

12. Use integrals 17 and 16 in the miscellaneous algebraic set.

$$\begin{aligned} \int \frac{\sqrt{2x-x^2}}{x^2} dx &= -\frac{(2x-x^2)^{3/2}}{x^2} - \frac{1}{1} \int \frac{\sqrt{2x-x^2}}{x} dx \\ &= -\frac{(2x-x^2)^{3/2}}{x^2} - \sqrt{2x-x^2} - \sin^{-1}(x-1) + C \end{aligned}$$

13. Use the last integral in the miscellaneous algebraic set.

$$\int \frac{dx}{(\sqrt{4x-x^2})^3} = \frac{x-2}{4} \frac{1}{\sqrt{4x-x^2}} + C$$

14. Use the last integral in the miscellaneous algebraic set. Then complete the square, change variables, and use the second last integral in the elementary list.

$$\begin{aligned} \int \frac{dx}{(\sqrt{4x-x^2})^4} &= \frac{x-2}{8} (\sqrt{4x-x^2})^{-2} + \frac{1}{8} \int \frac{dx}{4x-x^2} \\ &= \frac{x-2}{8(4x-x^2)} + \frac{1}{8} \int \frac{dx}{4-(x-2)^2} \quad \text{Let } u = x-2 \\ &\quad du = dx \\ &= \frac{x-2}{8(4x-x^2)} + \frac{1}{8} \int \frac{du}{4-u^2} \\ &= \frac{x-2}{8(4x-x^2)} + \frac{1}{32} \ln \left| \frac{u+2}{u-2} \right| + C \\ &= \frac{x-2}{8(4x-x^2)} + \frac{1}{32} \ln \left| \frac{x}{x-4} \right| + C \end{aligned}$$

## Section 6.5 Improper Integrals (page 347)

1.  $\int_1^\infty \frac{1}{(x-1)^3} dx$  Let  $u = x-1$   
 $du = dx$   
 $= \int_1^\infty \frac{du}{u^3} = \lim_{R \rightarrow \infty} \int_1^R \frac{du}{u^3}$   
 $= \lim_{R \rightarrow \infty} \left. \frac{-1}{2u^2} \right|_1^R = \lim_{R \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2R^2} \right) = \frac{1}{2}$

2.  $\int_3^\infty \frac{1}{(2x-1)^{2/3}} dx$  Let  $u = 2x-1$   
 $du = 2 dx$   
 $= \frac{1}{2} \int_5^\infty \frac{du}{u^{2/3}} = \frac{1}{2} \lim_{R \rightarrow \infty} \int_5^R u^{-2/3} du$   
 $= \frac{1}{2} \lim_{R \rightarrow \infty} 3u^{1/3} \Big|_5^R = \infty$  (diverges)

3.  $\int_0^\infty e^{-2x} dx = \lim_{R \rightarrow \infty} \left. \frac{e^{-2x}}{-2} \right|_0^R$   
 $= \lim_{R \rightarrow \infty} \left( \frac{1}{2} - \frac{1}{2e^R} \right)$   
 $= \frac{1}{2}$ . This integral converges.

4.  $\int_{-\infty}^{-1} \frac{dx}{x^2+1} = \lim_{R \rightarrow -\infty} \int_R^{-1} \frac{dx}{x^2+1}$   
 $= \lim_{R \rightarrow -\infty} \left[ \tan^{-1}(-1) - \tan^{-1}(R) \right]$   
 $= -\frac{\pi}{4} - \left( -\frac{\pi}{2} \right) = \frac{\pi}{4}$ .

This integral converges.

5.  $\int_{-1}^1 \frac{dx}{(x+1)^{2/3}} = \lim_{c \rightarrow -1+} 3(x+1)^{1/3} \Big|_c^1$   
 $= \lim_{c \rightarrow -1+} 3 \left( 2^{1/3} - (1+c)^{1/3} \right)$   
 $= 3 \sqrt[3]{2}$ . This integral converges.

6.  $\int_0^a \frac{dx}{a^2-x^2} = \lim_{C \rightarrow a-} \int_0^C \frac{dx}{a^2-x^2}$   
 $= \lim_{C \rightarrow a-} \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| \Big|_0^C$   
 $= \lim_{C \rightarrow a-} \frac{1}{2a} \ln \frac{a+C}{a-C} = \infty$ .  
 The integral diverges to infinity.

7.  $\int_0^1 \frac{dx}{(1-x)^{1/3}}$  Let  $u = 1-x$   
 $du = -dx$   
 $= \int_1^0 \frac{du}{u^{1/3}} = \lim_{c \rightarrow 0+} \int_c^1 \frac{du}{u^{1/3}}$   
 $= \lim_{c \rightarrow 0+} \left. \frac{3}{2} u^{2/3} \right|_c^1 = \frac{3}{2}$

8.  $\int_0^1 \frac{dx}{x\sqrt{1-x}}$  Let  $u^2 = 1-x$   
 $2u du = -dx$   
 $= \int_0^1 \frac{2u du}{(1-u^2)u} = 2 \lim_{c \rightarrow 1-} \int_0^c \frac{du}{1-u^2}$   
 $= 2 \lim_{c \rightarrow 1-} \frac{1}{2} \ln \left| \frac{u+1}{u-1} \right| \Big|_0^c = \infty$  (diverges)

$$\begin{aligned} 9. \quad \int_0^{\pi/2} \frac{\cos x \, dx}{(1 - \sin x)^{2/3}} \quad & \text{Let } u = 1 - \sin x \\ & du = -\cos x \, dx \\ & = \int_0^1 u^{-2/3} du = \lim_{c \rightarrow 0+} 3u^{1/3} \Big|_c^1 = 3. \end{aligned}$$

The integral converges.

$$\begin{aligned} 10. \quad \int_0^\infty x e^{-x} \, dx &= \lim_{R \rightarrow \infty} \int_0^R x e^{-x} \, dx \\ & \quad U = x \quad dV = e^{-x} \, dx \\ & \quad dU = dx \quad V = -e^{-x} \\ &= \lim_{R \rightarrow \infty} \left( -x e^{-x} \Big|_0^R + \int_0^R e^{-x} \, dx \right) \\ &= \lim_{R \rightarrow \infty} \left( -\frac{R}{e^R} - \frac{1}{e^R} + 1 \right) = 1. \end{aligned}$$

The integral converges.

$$\begin{aligned} 11. \quad \int_0^1 \frac{dx}{\sqrt{x(1-x)}} &= 2 \int_0^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}} \\ &= 2 \lim_{c \rightarrow 0+} \int_c^{1/2} \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}} \\ &= 2 \lim_{c \rightarrow 0+} \sin^{-1}(2x - 1) \Big|_c^{1/2} = \pi. \end{aligned}$$

The integral converges.

$$\begin{aligned} 12. \quad \int_0^\infty \frac{x}{1+2x^2} \, dx &= \lim_{R \rightarrow \infty} \int_0^R \frac{x}{1+2x^2} \, dx \\ &= \lim_{R \rightarrow \infty} \frac{1}{4} \ln(1+2x^2) \Big|_0^R \\ &= \lim_{R \rightarrow \infty} \left[ \frac{1}{4} \ln(1+2R^2) - \frac{1}{4} \ln 1 \right] = \infty. \end{aligned}$$

This integral diverges to infinity.

$$\begin{aligned} 13. \quad \int_0^\infty \frac{x \, dx}{(1+2x^2)^{3/2}} \quad & \text{Let } u = 1+2x^2 \\ & du = 4x \, dx \\ &= \frac{1}{4} \int_1^\infty \frac{du}{u^{3/2}} = \frac{1}{4} \lim_{R \rightarrow \infty} \left( -\frac{2}{\sqrt{u}} \right) \Big|_1^R \\ &= \frac{1}{2}. \quad \text{The integral converges.} \end{aligned}$$

$$\begin{aligned} 14. \quad \int_0^{\pi/2} \sec x \, dx &= \lim_{C \rightarrow (\pi/2)-} \ln |\sec x + \tan x| \Big|_0^C \\ &= \lim_{C \rightarrow (\pi/2)-} \ln |\sec C + \tan C| = \infty. \end{aligned}$$

This integral diverges to infinity.

$$\begin{aligned} 15. \quad \int_0^{\pi/2} \tan x \, dx &= \lim_{c \rightarrow (\pi/2)-} \ln |\sec x| \Big|_0^c \\ &= \lim_{c \rightarrow (\pi/2)-} \ln \sec c = \infty. \end{aligned}$$

This integral diverges to infinity.

$$\begin{aligned} 16. \quad \int_e^\infty \frac{dx}{x(\ln x)} \quad & \text{Let } u = \ln x \\ & du = \frac{dx}{x} \\ &= \lim_{R \rightarrow \infty} \int_1^{\ln R} \frac{du}{u} = \lim_{R \rightarrow \infty} \ln |u| \Big|_1^{\ln R} \\ &= \lim_{R \rightarrow \infty} \ln(\ln R) - \ln 1 = \infty. \end{aligned}$$

This integral diverges to infinity.

$$\begin{aligned} 17. \quad \int_1^e \frac{dx}{x\sqrt{\ln x}} \quad & \text{Let } u = \ln x \\ & du = dx/x \\ &= \int_0^1 \frac{du}{\sqrt{u}} = \lim_{c \rightarrow 0+} 2\sqrt{u} \Big|_c^1 = 2. \end{aligned}$$

This integral converges.

$$\begin{aligned} 18. \quad \int_e^\infty \frac{dx}{x(\ln x)^2} \quad & \text{Let } u = \ln x \\ & du = \frac{dx}{x} \\ &= \lim_{R \rightarrow \infty} \int_1^{\ln R} \frac{du}{u^2} = \lim_{R \rightarrow \infty} \left( -\frac{1}{\ln R} + 1 \right) = 1. \end{aligned}$$

The integral converges.

$$\begin{aligned} 19. \quad I &= \int_{-\infty}^\infty \frac{x \, dx}{1+x^2} = \int_{-\infty}^0 + \int_0^\infty = I_1 + I_2 \\ I_2 &= \int_0^\infty \frac{x \, dx}{1+x^2} \quad \text{Let } u = 1+x^2 \\ & \quad du = 2x \, dx \\ &= \lim_{R \rightarrow \infty} \frac{1}{2} \int_1^R \frac{du}{u} = \infty \quad (\text{diverges}) \end{aligned}$$

$$\begin{aligned} 20. \quad I &= \int_{-\infty}^\infty \frac{x \, dx}{1+x^4} = \int_{-\infty}^0 + \int_0^\infty = I_1 + I_2 \\ I_2 &= \int_0^\infty \frac{x \, dx}{1+x^4} \quad \text{Let } u = x^2 \\ & \quad du = 2x \, dx \\ &= \frac{1}{2} \int_0^\infty \frac{du}{1+u^2} = \frac{1}{2} \lim_{R \rightarrow \infty} \tan^{-1} u \Big|_0^R = \frac{\pi}{4} \end{aligned}$$

Similarly,  $I_1 = -\frac{\pi}{4}$ . Therefore,  $I = 0$ .

$$\begin{aligned} 21. \quad I &= \int_{-\infty}^\infty x e^{-x^2} \, dx = \int_{-\infty}^0 + \int_0^\infty = I_1 + I_2 \\ I_2 &= \int_0^\infty x e^{-x^2} \, dx \quad \text{Let } u = x^2 \\ & \quad du = 2x \, dx \\ &= \frac{1}{2} \int_0^\infty e^{-u} \, du = \frac{1}{2} \lim_{R \rightarrow \infty} -e^{-u} \Big|_0^R = \frac{1}{2} \end{aligned}$$

Similarly,  $I_1 = -\frac{1}{2}$ . Therefore,  $I = 0$ .

$$\begin{aligned} 22. \quad I &= \int_{-\infty}^{\infty} e^{-|x|} dx = \int_{-\infty}^0 e^x dx + \int_0^{\infty} e^{-x} dx = I_1 + I_2 \\ I_2 &= \int_0^{\infty} e^{-x} dx = 1 \end{aligned}$$

Similarly,  $I_1 = 1$ . Therefore,  $I = 2$ .

$$\begin{aligned} 23. \quad \text{Area of } R &= -\int_0^1 \ln x dx = -\lim_{c \rightarrow 0+} (x \ln x - x) \Big|_0^1 \\ &= -(0 - 1) + \lim_{c \rightarrow 0+} (c \ln c - c) \\ &= 1 - 0 = 1 \text{ units}^2 \end{aligned}$$

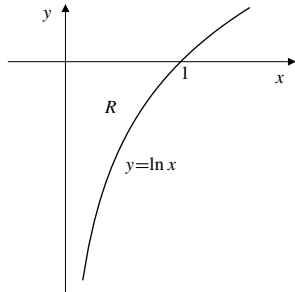


Fig. 6.5.23

$$\begin{aligned} 24. \quad \text{Area of shaded region} &= \int_0^{\infty} (e^{-x} - e^{-2x}) dx \\ &= \lim_{R \rightarrow \infty} \left( -e^{-x} + \frac{1}{2}e^{-2x} \right) \Big|_0^R \\ &= \lim_{R \rightarrow \infty} \left( -e^{-R} + \frac{1}{2}e^{-2R} + 1 - \frac{1}{2} \right) = \frac{1}{2} \text{ sq. units.} \end{aligned}$$

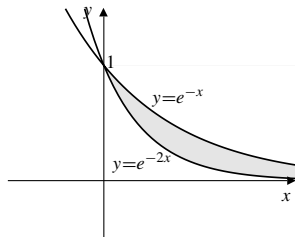


Fig. 6.5.24

$$\begin{aligned} 25. \quad \text{Area} &= \int_1^{\infty} \left( \frac{4}{2x+1} - \frac{2}{x+2} \right) dx \\ &= \lim_{R \rightarrow \infty} 2 \left( \ln(2x+1) - \ln(x+2) \right) \Big|_1^R \\ &= \lim_{R \rightarrow \infty} 2 \ln \left( \frac{2R+1}{R+2} \right) - 0 = 2 \ln 2 \text{ sq. units.} \end{aligned}$$

26. The required area is

$$\begin{aligned} \text{Area} &= \int_0^{\infty} x^{-2} e^{-1/x} dx \\ &= \int_0^1 x^{-2} e^{-1/x} dx + \int_1^{\infty} x^{-2} e^{-1/x} dx \\ &= I_1 + I_2. \end{aligned}$$

Then let  $u = -\frac{1}{x}$  and  $du = x^{-2} dx$  in both  $I_1$  and  $I_2$ :

$$\begin{aligned} I_1 &= \lim_{C \rightarrow 0+} \int_C^1 x^{-2} e^{-1/x} dx = \lim_{C \rightarrow 0+} \int_{-1/C}^{-1} e^u du \\ &= \lim_{C \rightarrow 0+} (e^{-1} - e^{-1/C}) = \frac{1}{e}. \\ I_2 &= \lim_{R \rightarrow \infty} \int_1^R x^{-2} e^{-1/x} dx = \lim_{R \rightarrow \infty} \int_{-1}^{-1/R} e^u du \\ &= \lim_{R \rightarrow \infty} (e^{-1/R} - e^{-1}) = 1 - \frac{1}{e}. \end{aligned}$$

Hence, the total area is  $I_1 + I_2 = 1$  square unit.

27. First assume that  $p \neq 1$ . Since  $a > 0$  we have

$$\begin{aligned} \int_a^{\infty} x^{-p} dx &= \lim_{R \rightarrow \infty} \frac{x^{-p+1}}{-p+1} \Big|_a^R \\ &= \frac{-a^{-p+1}}{1-p} + \lim_{R \rightarrow \infty} \frac{1}{(1-p)R^{p-1}} \\ &= \begin{cases} \frac{1}{(p-1)a^{p-1}} & \text{if } p > 1 \\ \infty & \text{if } p < 1 \end{cases} \\ \int_0^a x^{-p} dx &= \lim_{c \rightarrow 0+} \frac{x^{-p+1}}{-p+1} \Big|_c^a \\ &= \frac{a^{-p+1}}{1-p} + \lim_{c \rightarrow 0+} \frac{c^{1-p}}{p-1} \\ &= \begin{cases} \frac{a^{1-p}}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p > 1. \end{cases} \end{aligned}$$

If  $p = 1$  both integrals diverge as shown in Examples 2 and 6(a).

$$\begin{aligned} 28. \quad \int_{-1}^1 \frac{x \operatorname{sgn} x}{x+2} dx &= \int_{-1}^0 \frac{-x}{x+2} dx + \int_0^1 \frac{x}{x+2} dx \\ &= \int_{-1}^0 \left( -1 + \frac{2}{x+2} \right) dx + \int_0^1 \left( 1 - \frac{2}{x+2} \right) dx \\ &= (-x + 2 \ln |x+2|) \Big|_{-1}^0 + (x - 2 \ln |x+2|) \Big|_0^1 = \ln \frac{16}{9}. \end{aligned}$$

$$\begin{aligned} 29. \quad \int_0^2 x^2 \operatorname{sgn}(x-1) dx &= \int_0^1 -x^2 dx + \int_1^2 x^2 dx \\ &= -\frac{x^3}{3} \Big|_0^1 + \frac{x^3}{3} \Big|_1^2 = -\frac{1}{3} + \frac{8}{3} - \frac{1}{3} = 2. \end{aligned}$$

30. Since  $\frac{x^2}{x^5+1} \leq \frac{1}{x^3}$  for all  $x \geq 0$ , therefore

$$\begin{aligned} I &= \int_0^\infty \frac{x^2}{x^5+1} dx \\ &= \int_0^1 \frac{x^2}{x^5+1} dx + \int_1^\infty \frac{x^2}{x^5+1} dx \\ &\leq \int_0^1 \frac{x^2}{x^5+1} dx + \int_1^\infty \frac{dx}{x^3} \\ &= I_1 + I_2. \end{aligned}$$

Since  $I_1$  is a proper integral (finite) and  $I_2$  is a convergent improper integral, (see Theorem 2), therefore  $I$  converges.

31.  $\frac{1}{1+\sqrt{x}} \geq \frac{1}{2\sqrt{x}}$  on  $[1, \infty)$ .

Since  $\int_1^\infty \frac{dx}{\sqrt{x}}$  diverges to infinity, so must  $\int_1^\infty \frac{dx}{1+\sqrt{x}}$ .

Therefore  $\int_0^\infty \frac{dx}{1+\sqrt{x}}$  also diverges to infinity.

32. Since  $\frac{x\sqrt{x}}{x^2-1} \geq \frac{1}{\sqrt{x}}$  for all  $x > 1$ , therefore

$$I = \int_2^\infty \frac{x\sqrt{x}}{x^2-1} dx \geq \int_2^\infty \frac{dx}{\sqrt{x}} = I_1 = \infty.$$

Since  $I_1$  is a divergent improper integral,  $I$  diverges.

33.  $\int_0^\infty e^{-x^3} dx = \left( \int_0^1 + \int_1^\infty \right) e^{-x^3} dx$ .

Now  $\int_0^1 e^{-x^3} dx$  is a proper integral, and is therefore finite. Since  $x^3 \geq x$  on  $[1, \infty)$ , we have

$$\int_1^\infty e^{-x^3} dx \leq \int_1^\infty e^{-x} dx = \frac{1}{e}.$$

Thus  $\int_0^\infty e^{-x^3} dx$  converges.

34. On  $[0,1]$ ,  $\frac{1}{\sqrt{x}+x^2} \leq \frac{1}{\sqrt{x}}$ . On  $[1, \infty)$ ,  $\frac{1}{\sqrt{x}+x^2} \leq \frac{1}{x^2}$ . Thus,

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x}+x^2} &\leq \int_0^1 \frac{dx}{\sqrt{x}} \\ \int_1^\infty \frac{dx}{\sqrt{x}+x^2} &\leq \int_1^\infty \frac{dx}{x^2}. \end{aligned}$$

Since both of these integrals are convergent, therefore so is their sum  $\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$ .

35.  $\frac{e^x}{x+1} \geq \frac{e^{-1}}{x+1}$  on  $[-1, 1]$ . Thus

$$\int_{-1}^1 \frac{e^x}{x+1} dx \geq \frac{1}{e} \int_{-1}^1 \frac{dx}{x+1} = \infty.$$

The given integral diverges to infinity.

36. Since  $\sin x \leq x$  for all  $x \geq 0$ , thus  $\frac{\sin x}{x} \leq 1$ . Then

$$I = \int_0^\pi \frac{\sin x}{x} dx = \lim_{\epsilon \rightarrow 0^+} \int_\epsilon^\pi \frac{\sin x}{x} dx \leq \int_0^\pi (1) dx = \pi.$$

Hence,  $I$  converges.

37. Since  $\sin x \geq \frac{2x}{\pi}$  on  $[0, \pi/2]$ , we have

$$\begin{aligned} \int_0^\infty \frac{|\sin x|}{x^2} dx &\geq \int_0^{\pi/2} \frac{\sin x}{x^2} dx \\ &\geq \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{x} = \infty. \end{aligned}$$

The given integral diverges to infinity.

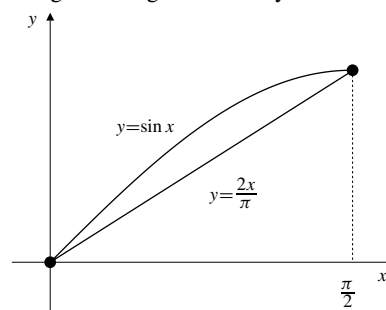


Fig. 6.5.37

38. Since  $0 \leq 1 - \cos \sqrt{x} = 2 \sin^2 \left( \frac{\sqrt{x}}{2} \right) \leq 2 \left( \frac{\sqrt{x}}{2} \right)^2 = \frac{x}{2}$ ,

for  $x \geq 0$ , therefore  $\int_0^{\pi^2} \frac{dx}{1 - \cos \sqrt{x}} \geq 2 \int_0^{\pi^2} \frac{dx}{x}$ , which diverges to infinity.

39. On  $(0, \pi/2]$ ,  $\sin x < x$ , and so  $\csc x \geq 1/x$ . Thus

$$\int_0^{\pi/2} \csc x dx > \int_0^{\pi/2} \frac{dx}{x} = \infty.$$

Therefore  $\int_{-\pi/2}^{\pi/2} \csc x dx$  must diverge. (It is of the form  $\infty - \infty$ .)

40. Since  $\ln x$  grows more slowly than any positive power of  $x$ , therefore we have  $\ln x \leq kx^{1/4}$  for some constant  $k$  and every  $x \geq 2$ . Thus,  $\frac{1}{\sqrt{x} \ln x} \geq \frac{1}{kx^{3/4}}$  for  $x \geq 2$

and  $\int_2^\infty \frac{dx}{\sqrt{x} \ln x}$  diverges to infinity by comparison with  $\frac{1}{k} \int_2^\infty \frac{dx}{x^{3/4}}$ .

$$41. \int_0^\infty \frac{dx}{xe^x} = \left( \int_0^1 + \int_1^\infty \right) \frac{dx}{xe^x}. \text{ But}$$

$$\int_0^1 \frac{dx}{xe^x} \geq \frac{1}{e} \int_0^1 \frac{dx}{x} = \infty.$$

Thus the given integral must diverge to infinity.

$$42. \text{ We are given that } \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}.$$

a) First we calculate

$$\begin{aligned} \int_0^\infty x^2 e^{-x^2} dx &= \lim_{R \rightarrow \infty} \int_0^R x^2 e^{-x^2} dx \\ U &= x \quad dV = x e^{-x^2} dx \\ dU &= dx \quad V = -\frac{1}{2} e^{-x^2} \\ &= \lim_{R \rightarrow \infty} \left[ -\frac{1}{2} x e^{-x^2} \Big|_0^R + \frac{1}{2} \int_0^R e^{-x^2} dx \right] \\ &= -\frac{1}{2} \lim_{R \rightarrow \infty} R e^{-R^2} + \frac{1}{2} \int_0^\infty e^{-x^2} dx \\ &= 0 + \frac{1}{4} \sqrt{\pi} = \frac{1}{4} \sqrt{\pi}. \end{aligned}$$

b) Similarly,

$$\begin{aligned} \int_0^\infty x^4 e^{-x^2} dx &= \lim_{R \rightarrow \infty} \int_0^R x^4 e^{-x^2} dx \\ U &= x^3 \quad dV = x e^{-x^2} dx \\ dU &= 3x^2 dx \quad V = -\frac{1}{2} e^{-x^2} \\ &= \lim_{R \rightarrow \infty} \left[ -\frac{1}{2} x^3 e^{-x^2} \Big|_0^R + \frac{3}{2} \int_0^R x^2 e^{-x^2} dx \right] \\ &= -\frac{1}{2} \lim_{R \rightarrow \infty} R^3 e^{-R^2} + \frac{3}{2} \int_0^\infty x^2 e^{-x^2} dx \\ &= 0 + \frac{3}{2} \left( \frac{1}{4} \sqrt{\pi} \right) = \frac{3}{8} \sqrt{\pi}. \end{aligned}$$

43. Since  $f$  is continuous on  $[a, b]$ , there exists a positive constant  $K$  such that  $|f(x)| \leq K$  for  $a \leq x \leq b$ . If  $a < c < b$ , then

$$\begin{aligned} \left| \int_c^b f(x) dx - \int_a^b f(x) dx \right| \\ = \left| \int_c^a f(x) dx \right| \leq K(c-a) \rightarrow 0 \text{ as } c \rightarrow a+. \end{aligned}$$

$$\text{Thus } \lim_{c \rightarrow a+} \int_c^b f(x) dx = \int_a^b f(x) dx.$$

Similarly

$$\begin{aligned} \left| \int_a^c f(x) dx - \int_a^b f(x) dx \right| \\ = \left| \int_b^c f(x) dx \right| \leq K(b-c) \rightarrow 0 \text{ as } c \rightarrow b-. \end{aligned}$$

$$\text{Thus } \lim_{c \rightarrow b-} \int_a^c f(x) dx = \int_a^b f(x) dx.$$

$$44. \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

a) Since  $\lim_{t \rightarrow \infty} t^{x-1} e^{-t/2} = 0$ , there exists  $T > 0$  such that  $t^{x-1} e^{-t/2} \leq 1$  if  $t \geq T$ . Thus

$$0 \leq \int_T^\infty t^{x-1} e^{-t} dt \leq \int_T^\infty e^{-t} dt = 2e^{-T/2}$$

and  $\int_T^\infty t^{x-1} e^{-t} dt$  converges by the comparison theorem.

If  $x > 0$ , then

$$0 \leq \int_0^T t^{x-1} e^{-t} dt < \int_0^T t^{x-1} dt$$

converges by Theorem 2(b). Thus the integral defining  $\Gamma(x)$  converges.

$$\begin{aligned} b) \Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt \\ &= \lim_{\substack{c \rightarrow 0+ \\ R \rightarrow \infty}} \int_c^R t^x e^{-t} dt \\ U &= t^x \quad dV = e^{-t} dt \\ dU &= x t^{x-1} dt \quad V = -e^{-t} \\ &= \lim_{\substack{c \rightarrow 0+ \\ R \rightarrow \infty}} \left( -t^x e^{-t} \Big|_c^R + x \int_c^R t^{x-1} e^{-t} dt \right) \\ &= 0 + x \int_0^\infty t^{x-1} e^{-t} dt = x \Gamma(x). \end{aligned}$$

$$c) \Gamma(1) = \int_0^\infty e^{-t} dt = 1 = 0!.$$

By (b),  $\Gamma(2) = 1\Gamma(1) = 1 \times 1 = 1 = 1!$ .

In general, if  $\Gamma(k+1) = k!$  for some positive integer  $k$ , then

$$\Gamma(k+2) = (k+1)\Gamma(k+1) = (k+1)k! = (k+1)!.$$

Hence  $\Gamma(n+1) = n!$  for all integers  $n \geq 0$ , by induction.

$$\begin{aligned} d) \Gamma\left(\frac{1}{2}\right) &= \int_0^\infty t^{-1/2} e^{-t} dt \quad \text{Let } t = x^2 \\ &\quad \quad \quad dt = 2x dx \\ &= \int_0^\infty \frac{1}{x} e^{-x^2} 2x dx = 2 \int_0^\infty e^{-x^2} dx = \sqrt{\pi} \\ \Gamma\left(\frac{3}{2}\right) &= \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}. \end{aligned}$$

**Section 6.6 The Trapezoid and Midpoint Rules (page 354)**

1. The exact value of  $I$  is

$$I = \int_0^2 (1+x^2) dx = \left(x + \frac{x^3}{3}\right) \Big|_0^2 = 2 + \frac{8}{3} \approx 4.6666667.$$

The approximations are

$$\begin{aligned} T_4 &= \frac{1}{2} \left[ \frac{1}{2} + \left(1 + \frac{1}{4}\right) + (1+1) + \left(1 + \frac{9}{4}\right) + \frac{5}{2} \right] = 4.75 \\ M_4 &= \frac{1}{2} \left[ \left(1 + \frac{1}{16}\right) + \left(1 + \frac{9}{16}\right) + \left(1 + \frac{25}{16}\right) + \left(1 + \frac{49}{16}\right) \right] \\ &= 4.625 \\ T_8 &= \frac{1}{2} (T_4 + M_4) = 4.6875 \\ M_8 &= \frac{1}{4} \left[ \left(1 + \frac{1}{64}\right) + \left(1 + \frac{9}{64}\right) + \left(1 + \frac{25}{64}\right) \right. \\ &\quad \left. + \left(1 + \frac{49}{64}\right) + \left(1 + \frac{81}{64}\right) + \left(1 + \frac{121}{64}\right) \right. \\ &\quad \left. + \left(1 + \frac{169}{64}\right) + \left(1 + \frac{225}{64}\right) \right] = 4.65625 \\ T_{16} &= \frac{1}{2} (T_8 + M_8) = 4.671875. \end{aligned}$$

The exact errors are

$$\begin{aligned} I - T_4 &= -0.0833333; & I - M_4 &= 0.0416667; \\ I - T_8 &= -0.0208333; & I - M_8 &= 0.0104167; \\ I - T_{16} &= -0.0052083. \end{aligned}$$

If  $f(x) = 1 + x^2$ , then  $f''(x) = 2 = K$ , and  $\frac{K(2-0)}{12} = \frac{1}{3}$ . Therefore, the error bounds are

$$\begin{aligned} \text{Trapezoid : } |I - T_n| &\leq \frac{1}{3} \left(\frac{1}{2}\right)^2 \approx 0.0833333; \\ |I - T_8| &\leq \frac{1}{3} \left(\frac{1}{4}\right)^2 \approx 0.0208333; \\ |I - T_{16}| &\leq \frac{1}{3} \left(\frac{1}{8}\right)^2 \approx 0.0052083. \\ \text{Midpoint : } |I - M_n| &\leq \frac{1}{6} \left(\frac{1}{2}\right)^2 \approx 0.0416667; \\ |I - M_8| &\leq \frac{1}{6} \left(\frac{1}{4}\right)^2 \approx 0.0104167. \end{aligned}$$

Note that the actual errors are equal to these estimates since  $f$  is a quadratic function.

2. The exact value of  $I$  is

$$\begin{aligned} I &= \int_0^1 e^{-x} dx = -e^{-x} \Big|_0^1 \\ &= 1 - \frac{1}{e} \approx 0.6321206. \end{aligned}$$

The approximations are

$$\begin{aligned} T_4 &= \frac{1}{4} \left( \frac{1}{2} e^0 + e^{-1/4} + e^{-1/2} + e^{-3/4} + \frac{1}{2} e^{-1} \right) \\ &\approx 0.6354094 \\ M_4 &= \frac{1}{4} (e^{-1/8} + e^{-3/8} + e^{-5/8} + e^{-7/8}) \\ &\approx 0.6304774 \\ T_8 &= \frac{1}{2} (T_4 + M_4) \approx 0.6329434 \\ M_8 &= \frac{1}{8} (e^{-1/16} + e^{-3/16} + e^{-5/16} + e^{-7/16} + \\ &\quad e^{-9/16} + e^{-11/16} + e^{-13/16} + e^{-15/16}) \\ &\approx 0.6317092 \\ T_{16} &= \frac{1}{2} (T_8 + M_8) \approx 0.6323263. \end{aligned}$$

The exact errors are

$$\begin{aligned} I - T_4 &= -0.0032888; & I - M_4 &= 0.0016432; \\ I - T_8 &= -0.0008228; & I - M_8 &= 0.0004114; \\ I - T_{16} &= -0.0002057. \end{aligned}$$

If  $f(x) = e^{-x}$ , then  $f^{(2)}(x) = e^{-x}$ . On  $[0,1]$ ,  $|f^{(2)}(x)| \leq 1$ . Therefore, the error bounds are:

$$\begin{aligned} \text{Trapezoid : } |I - T_n| &\leq \frac{1}{12} \left(\frac{1}{n}\right)^2 \\ |I - T_4| &\leq \frac{1}{12} \left(\frac{1}{16}\right) \approx 0.0052083; \\ |I - T_8| &\leq \frac{1}{12} \left(\frac{1}{64}\right) \approx 0.001302; \\ |I - T_{16}| &\leq \frac{1}{12} \left(\frac{1}{256}\right) \approx 0.0003255. \\ \text{Midpoint : } |I - M_n| &\leq \frac{1}{24} \left(\frac{1}{n}\right)^2 \\ |I - M_4| &\leq \frac{1}{24} \left(\frac{1}{16}\right) \approx 0.0026041; \\ |I - M_8| &\leq \frac{1}{24} \left(\frac{1}{64}\right) \approx 0.000651. \end{aligned}$$

Note that the actual errors satisfy these bounds.

3. The exact value of  $I$  is

$$I = \int_0^{\pi/2} \sin x dx = 1.$$

The approximations are

$$\begin{aligned}T_4 &= \frac{\pi}{8} \left( 0 + \sin \frac{\pi}{8} + \sin \frac{\pi}{4} + \sin \frac{3\pi}{8} + \frac{1}{2} \right) \approx 0.9871158 \\M_4 &= \frac{\pi}{8} \left( \sin \frac{\pi}{16} + \sin \frac{3\pi}{16} + \sin \frac{5\pi}{16} + \sin \frac{7\pi}{16} \right) \approx 1.0064545 \\T_8 &= \frac{1}{2}(T_4 + M_4) \approx 0.9967852 \\M_8 &= \frac{\pi}{16} \left( \sin \frac{\pi}{32} + \sin \frac{3\pi}{32} + \sin \frac{5\pi}{32} \right. \\&\quad \left. + \sin \frac{7\pi}{32} + \sin \frac{9\pi}{32} + \sin \frac{11\pi}{32} \right. \\&\quad \left. + \sin \frac{13\pi}{32} + \sin \frac{15\pi}{32} \right) \approx 1.0016082 \\T_{16} &= \frac{1}{2}(T_8 + M_8) \approx 0.9991967.\end{aligned}$$

The actual errors are

$$\begin{aligned}I - T_4 &\approx 0.0128842; & I - M_4 &\approx -0.0064545; \\I - T_8 &\approx 0.0032148; & I - M_8 &\approx -0.0016082; \\I - T_{16} &\approx 0.0008033.\end{aligned}$$

If  $f(x) = \sin x$ , then  $f''(x) = -\sin x$ , and  $|f''(x)| \leq 1 = K$ . Therefore, the error bounds are:

$$\begin{aligned}\text{Trapezoid : } |I - T_4| &\leq \frac{1}{12} \left( \frac{\pi}{2} - 0 \right) \left( \frac{\pi}{8} \right)^2 \approx 0.020186; \\|I - T_8| &\leq \frac{1}{12} \left( \frac{\pi}{2} - 0 \right) \left( \frac{\pi}{16} \right)^2 \approx 0.005047; \\|I - T_{16}| &\leq \frac{1}{12} \left( \frac{\pi}{2} - 0 \right) \left( \frac{\pi}{32} \right)^2 \approx 0.001262. \\ \text{Midpoint : } |I - M_4| &\leq \frac{1}{24} \left( \frac{\pi}{2} - 0 \right) \left( \frac{\pi}{8} \right)^2 \approx 0.010093; \\|I - M_8| &\leq \frac{1}{24} \left( \frac{\pi}{2} - 0 \right) \left( \frac{\pi}{16} \right)^2 \approx 0.002523.\end{aligned}$$

Note that the actual errors satisfy these bounds.

4. The exact value of  $I$  is

$$I = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} \approx 0.7853982.$$

The approximations are

$$\begin{aligned}T_4 &= \frac{1}{4} \left[ \frac{1}{2}(1) + \frac{16}{17} + \frac{4}{5} + \frac{16}{25} + \frac{1}{2} \left( \frac{1}{2} \right) \right] \\&\approx 0.7827941 \\M_4 &= \frac{1}{4} \left[ \frac{64}{65} + \frac{64}{73} + \frac{64}{89} + \frac{64}{113} \right] \\&\approx 0.7867001 \\T_8 &= \frac{1}{2}(T_4 + M_4) \approx 0.7847471 \\M_8 &= \frac{1}{8} \left[ \frac{256}{257} + \frac{256}{265} + \frac{256}{281} + \frac{256}{305} + \right. \\&\quad \left. \frac{256}{337} + \frac{256}{377} + \frac{256}{425} + \frac{256}{481} \right] \\&\approx 0.7857237 \\T_{16} &= \frac{1}{2}(T_8 + M_8) \approx 0.7852354.\end{aligned}$$

The exact errors are

$$\begin{aligned}I - T_4 &= 0.0026041; & I - M_4 &= -0.0013019; \\I - T_8 &= 0.0006511; & I - M_8 &= -0.0003255; \\I - T_{16} &= 0.0001628.\end{aligned}$$

Since  $f(x) = \frac{1}{1+x^2}$ , then  $f'(x) = \frac{-2x}{(1+x^2)^2}$  and  $f''(x) = \frac{6x^2-2}{(1+x^2)^3}$ . On  $[0,1]$ ,  $|f''(x)| \leq 4$ . Therefore, the error bounds are

$$\begin{aligned}\text{Trapezoid : } |I - T_n| &\leq \frac{4}{12} \left( \frac{1}{n} \right)^2 \\|I - T_4| &\leq \frac{4}{12} \left( \frac{1}{16} \right) \approx 0.0208333; \\|I - T_8| &\leq \frac{4}{12} \left( \frac{1}{64} \right) \approx 0.0052083; \\|I - T_{16}| &\leq \frac{4}{12} \left( \frac{1}{256} \right) \approx 0.001302. \\ \text{Midpoint : } |I - M_n| &\leq \frac{4}{24} \left( \frac{1}{n} \right)^2 \\|I - M_4| &\leq \frac{4}{24} \left( \frac{1}{16} \right) \approx 0.0104167; \\|I - M_8| &\leq \frac{4}{24} \left( \frac{1}{64} \right) \approx 0.0026042.\end{aligned}$$

The exact errors are much smaller than these bounds. In part, this is due to very crude estimates made for  $|f''(x)|$ .

$$\begin{aligned}5. \quad T_4 &= \frac{2}{2}[3 + 2(5 + 8 + 7) + 3] = 46 \\T_8 &= \frac{1}{2}[3 + 2(3.8 + 5 + 6.7 + 8 + 8 + 7 + 5.2) + 3] = 46.7\end{aligned}$$

6.  $M_4 = 2(3.8 + 6.7 + 8 + 5.2) = 47.4$

7.  $T_4 = 100 \times \frac{2}{2}[0 + 2(5.5 + 5 + 4.5) + 0] = 3,000 \text{ km}^2$   
 $T_8 = 100 \times \frac{1}{2}[0 + 2(4 + 5.5 + 5.5 + 5 + 5.5 + 4.5 + 4) + 0]$   
 $= 3,400 \text{ km}^2$

8.  $M_4 = 100 \times 2(4 + 5.5 + 5.5 + 4) = 3,800 \text{ km}^2$

9. We have

$$T_4 = 0.4 \left( \frac{1}{2}(1.4142) + 1.3860 + 1.3026 + 1.1772 \right. \\ \left. + \frac{1}{2}(0.9853) \right) \approx 2.02622$$

$$M_4 = (0.4)(1.4071 + 1.3510 + 1.2411 + 1.0817) \approx 2.03236$$

$$T_8 = (T_4 + M_4)/2 \approx 2.02929$$

$$M_8 = (0.2)(1.4124 + 1.3983 + 1.3702 + 1.3285 \\ + 1.2734 + 1.2057 + 1.1258 + 1.0348) \approx 2.02982$$

$$T_{16} = (T_8 + M_8)/2 \approx 2.029555.$$

10. The approximations for  $I = \int_0^1 e^{-x^2} dx$  are

$$M_8 = \frac{1}{8} \left( e^{-1/256} + e^{-9/256} + e^{-25/256} + e^{-49/256} + \right. \\ \left. e^{-81/256} + e^{-121/256} + e^{-169/256} + e^{-225/256} \right) \\ \approx 0.7473$$

$$T_{16} = \frac{1}{16} \left[ \frac{1}{2}(1) + e^{-1/256} + e^{-1/64} + e^{-9/256} + e^{-1/16} + \right. \\ \left. e^{-25/256} + e^{-9/64} + e^{-49/256} + e^{-1/4} + e^{-81/256} + \right. \\ \left. e^{-25/64} + e^{-121/256} + e^{-9/16} + e^{-169/256} + e^{-49/64} + \right. \\ \left. e^{-225/256} + \frac{1}{2}e^{-1} \right] \\ \approx 0.74658.$$

Since  $f(x) = e^{-x^2}$ , we have  $f'(x) = -2xe^{-x^2}$ ,  
 $f''(x) = 2(2x^2 - 1)e^{-x^2}$ , and  $f'''(x) = 4x(3 - 2x^2)e^{-x^2}$ .  
 Since  $f'''(x) \neq 0$  on  $(0,1)$ , therefore the maximum value  
 of  $|f'''(x)|$  on  $[0, 1]$  must occur at an endpoint of that  
 interval. We have  $f'''(0) = -2$  and  $f'''(1) = 2/e$ , so  
 $|f'''(x)| \leq 2$  on  $[0, 1]$ . The error bounds are

$$|I - M_n| \leq \frac{2}{24} \left( \frac{1}{n} \right)^2 \Rightarrow |I - M_8| \leq \frac{2}{24} \left( \frac{1}{64} \right) \\ \approx 0.00130.$$

$$|I - T_n| \leq \frac{2}{12} \left( \frac{1}{n} \right)^2 \Rightarrow |I - T_{16}| \leq \frac{2}{12} \left( \frac{1}{256} \right) \\ \approx 0.000651.$$

According to the error bounds,

$$\int_0^1 e^{-x^2} dx = 0.747,$$

accurate to two decimal places, with error no greater than  
 1 in the third decimal place.

11.  $I = \int_0^{\pi/2} \frac{\sin x}{x} dx$ . Note that  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ .

$$T_8 = \frac{\pi}{16} \left[ \frac{1}{2} + \frac{16}{\pi} \sin \frac{\pi}{16} + \frac{8}{\pi} \sin \frac{\pi}{8} + \frac{16}{3\pi} \sin \frac{3\pi}{16} + \frac{4}{\pi} \sin \frac{\pi}{4} \right. \\ \left. + \frac{16}{5\pi} \sin \frac{5\pi}{16} + \frac{8}{3\pi} \sin \frac{3\pi}{8} + \frac{16}{7\pi} \sin \frac{7\pi}{16} + \frac{1}{2} \left( \frac{2}{\pi} \right) \right] \\ \approx 1.3694596$$

$$M_8 = \frac{\pi}{16} \left[ \frac{32}{\pi} \sin \frac{\pi}{32} + \frac{32}{3\pi} \sin \frac{3\pi}{32} + \frac{32}{5\pi} \sin \frac{5\pi}{32} + \frac{32}{7\pi} \sin \frac{7\pi}{32} \right. \\ \left. + \frac{32}{9\pi} \sin \frac{9\pi}{32} + \frac{32}{11\pi} \sin \frac{11\pi}{32} + \frac{32}{13\pi} \sin \frac{13\pi}{32} \right. \\ \left. + \frac{32}{15\pi} \sin \frac{15\pi}{32} \right] \approx 1.3714136$$

$$T_{16} = (T_8 + M_8)/2 \approx 1.3704366, \quad I \approx 1.370.$$

12. The exact value of  $I$  is

$$I = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}.$$

The approximation is

$$T_1 = (1) \left[ \frac{1}{2}(0)^2 + \frac{1}{2}(1)^2 \right] = \frac{1}{2}.$$

The actual error is  $I - T_1 = -\frac{1}{6}$ . However, since  
 $f(x) = x^2$ , then  $f''(x) = 2$  on  $[0,1]$ , so the error estimate  
 here gives

$$|I - T_1| \leq \frac{2}{12}(1)^2 = \frac{1}{6}.$$

Since this is the actual size of the error in this case, the  
 constant "12" in the error estimate cannot be improved  
 (i.e., cannot be made larger).

13.  $I = \int_0^1 x^2 dx = \frac{1}{3}$ .  $M_1 = \left( \frac{1}{2} \right)^2 (1) = \frac{1}{4}$ . The actual  
 error is  $I - M_1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ .

Since the second derivative of  $x^2$  is 2, the error estimate  
 is

$$|I - M_1| \leq \frac{2}{24}(1 - 0)^2(1^2) = \frac{1}{12}.$$

Thus the constant in the error estimate for the Midpoint  
 Rule cannot be improved; no smaller constant will work  
 for  $f(x) = x^2$ .

14. Let  $y = f(x)$ . We are given that  $m_1$  is the midpoint of  $[x_0, x_1]$  where  $x_1 - x_0 = h$ . By tangent line approximate in the subinterval  $[x_0, x_1]$ ,

$$f(x) \approx f(m_1) + f'(m_1)(x - m_1).$$

The error in this approximation is

$$E(x) = f(x) - f(m_1) - f'(m_1)(x - m_1).$$

If  $f''(t)$  exists for all  $t$  in  $[x_0, x_1]$  and  $|f''(t)| \leq K$  for some constant  $K$ , then by Theorem 4 of Section 3.5,

$$|E(x)| \leq \frac{K}{2}(x - m_1)^2.$$

Hence,

$$|f(x) - f(m_1) - f'(m_1)(x - m_1)| \leq \frac{K}{2}(x - m_1)^2.$$

We integrate both sides of this inequality. Noting that  $x_1 - m_1 = m_1 - x_0 = \frac{1}{2}h$ , we obtain for the left side

$$\begin{aligned} & \left| \int_{x_0}^{x_1} f(x) dx - \int_{x_0}^{x_1} f(m_1) dx - \int_{x_0}^{x_1} f'(m_1)(x - m_1) dx \right| \\ &= \left| \int_{x_0}^{x_1} f(x) dx - f(m_1)h - f'(m_1) \frac{(x - m_1)^2}{2} \Big|_{x_0}^{x_1} \right| \\ &= \left| \int_{x_0}^{x_1} f(x) dx - f(m_1)h \right|. \end{aligned}$$

Integrating the right-hand side, we get

$$\begin{aligned} \int_{x_0}^{x_1} \frac{K}{2}(x - m_1)^2 dx &= \frac{K}{2} \frac{(x - m_1)^3}{3} \Big|_{x_0}^{x_1} \\ &= \frac{K}{6} \left( \frac{h^3}{8} + \frac{h^3}{8} \right) = \frac{K}{24}h^3. \end{aligned}$$

Hence,

$$\begin{aligned} & \left| \int_{x_0}^{x_1} f(x) dx - f(m_1)h \right| \\ &= \left| \int_{x_0}^{x_1} [f(x) - f(m_1) - f'(m_1)(x - m_1)] dx \right| \\ &\leq \frac{K}{24}h^3. \end{aligned}$$

A similar estimate holds on each subinterval  $[x_{j-1}, x_j]$  for  $1 \leq j \leq n$ . Therefore,

$$\begin{aligned} \left| \int_a^b f(x) dx - M_n \right| &= \left| \sum_{j=1}^n \left( \int_{x_{j-1}}^{x_j} f(x) dx - f(m_j)h \right) \right| \\ &\leq \sum_{j=1}^n \left| \int_{x_{j-1}}^{x_j} f(x) dx - f(m_j)h \right| \\ &\leq \sum_{j=1}^n \frac{K}{24}h^3 = \frac{K}{24}nh^3 = \frac{K(b-a)}{24}h^2 \end{aligned}$$

because  $nh = b - a$ .

## Section 6.7 Simpson's Rule (page 359)

$$1. \quad I = \int_0^2 (1 + x^2) dx = \frac{14}{3} \approx 4.6666667$$

$$S_4 = \frac{1}{6} \left[ 1 + 4 \left( 1 + \frac{1}{4} \right) + 2(1 + 1) + 4 \left( 1 + \frac{9}{4} \right) + (1 + 4) \right] = \frac{14}{3}$$

$$S_8 = \frac{1}{12} \left[ 1 + 4 \left( 1 + \frac{1}{16} \right) + 2 \left( 1 + \frac{1}{4} \right) + 4 \left( 1 + \frac{9}{16} \right) + 2(1 + 1) + 4 \left( 1 + \frac{25}{16} \right) + 2 \left( 1 + \frac{9}{4} \right) + 4 \left( 1 + \frac{49}{16} \right) + (1 + 4) \right] = \frac{14}{3}$$

The errors are zero because Simpson approximations are exact for polynomials of degree up to three.

2. The exact value of  $I$  is

$$\begin{aligned} I &= \int_0^1 e^{-x} dx = -e^{-x} \Big|_0^1 \\ &= 1 - \frac{1}{e} \approx 0.6321206. \end{aligned}$$

The approximations are

$$\begin{aligned} S_4 &= \frac{1}{12} (e^0 + 4e^{-1/4} + 2e^{-1/2} + 4e^{-3/4} + e^{-1}) \\ &\approx 0.6321342 \\ S_8 &= \frac{1}{24} (e^0 + 4e^{-1/8} + 2e^{-1/4} + 4e^{-3/8} + 2e^{-1/2} + 4e^{-5/8} + 2e^{-3/4} + 4e^{-7/8} + e^{-1}) \\ &\approx 0.6321214. \end{aligned}$$

The actual errors are

$$I - S_4 = -0.0000136; \quad I - S_8 = -0.0000008.$$

These errors are evidently much smaller than the corresponding errors for the corresponding Trapezoid Rule approximations.

3.  $I = \int_0^{\pi/2} \sin x \, dx = 1.$

$$S_4 = \frac{\pi}{24} \left( 0 + 4 \sin \frac{\pi}{8} + 2 \sin \frac{\pi}{4} + 4 \sin \frac{3\pi}{8} + \sin \frac{\pi}{2} \right) \approx 1.0001346$$

$$S_8 = \frac{\pi}{48} \left( 0 + 4 \sin \frac{\pi}{16} + 2 \sin \frac{\pi}{8} + 4 \sin \frac{3\pi}{16} + 2 \sin \frac{\pi}{4} + 4 \sin \frac{5\pi}{16} + 2 \sin \frac{3\pi}{8} + 4 \sin \frac{7\pi}{16} + \sin \frac{\pi}{2} \right) \approx 1.0000083.$$

Errors:  $I - S_4 \approx -0.0001346$ ;  $I - S_8 \approx -0.0000083$ .

4. The exact value of  $I$  is

$$I = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} \approx 0.7853982.$$

The approximations are

$$S_4 = \frac{1}{12} \left[ 1 + 4 \left( \frac{16}{17} \right) + 2 \left( \frac{4}{5} \right) + 4 \left( \frac{16}{25} \right) + \frac{1}{2} \right] \approx 0.7853922$$

$$S_8 = \frac{1}{24} \left[ 1 + 4 \left( \frac{64}{65} \right) + 2 \left( \frac{16}{17} \right) + 4 \left( \frac{64}{73} \right) + 2 \left( \frac{4}{5} \right) + 4 \left( \frac{64}{89} \right) + 2 \left( \frac{16}{25} \right) + 4 \left( \frac{64}{113} \right) + \frac{1}{2} \right] \approx 0.7853981.$$

The actual errors are

$$I - S_4 = 0.0000060; \quad I - S_8 = 0.0000001,$$

accurate to 7 decimal places. These errors are evidently much smaller than the corresponding errors for the corresponding Trapezoid Rule approximation.

5.  $S_8 = \frac{1}{3} [3 + 4(3.8 + 6.7 + 8 + 5.2) + 2(5 + 8 + 7) + 3] \approx 46.93$

6.  $S_8 = 100 \times \frac{1}{3} [0 + 4(4 + 5.5 + 5.5 + 4) + 2(5.5 + 5 + 4.5) + 0] \approx 3,533 \text{ km}^2$

7. If  $f(x) = e^{-x}$ , then  $f^{(4)}(x) = e^{-x}$ , and  $|f^{(4)}(x)| \leq 1$  on  $[0, 1]$ . Thus

$$|I - S_4| \leq \frac{1(1-0)}{180} \left( \frac{1}{4} \right)^4 \approx 0.000022$$

$$|I - S_8| \leq \frac{1(1-0)}{180} \left( \frac{1}{8} \right)^4 \approx 0.0000014.$$

If  $f(x) = \sin x$ , then  $f^{(4)}(x) = \sin x$ , and  $|f^{(4)}(x)| \leq 1$  on  $[0, \pi/2]$ . Thus

$$|I - S_4| \leq \frac{1((\pi/2)-0)}{180} \left( \frac{\pi}{8} \right)^4 \approx 0.00021$$

$$|I - S_8| \leq \frac{1((\pi/2)-0)}{180} \left( \frac{\pi}{16} \right)^4 \approx 0.000013.$$

8. Let  $I = \int_a^b f(x) \, dx$ , and the interval  $[a, b]$  be subdivided into  $2n$  subintervals of equal length  $h = (b-a)/2n$ . Let  $y_j = f(x_j)$  and  $x_j = a + jh$  for  $0 \leq j \leq 2n$ , then

$$\begin{aligned} S_{2n} &= \frac{1}{3} \left( \frac{b-a}{2n} \right) \left[ y_0 + 4y_1 + 2y_2 + \cdots \right. \\ &\quad \left. + 2y_{2n-2} + 4y_{2n-1} + y_{2n} \right] \\ &= \frac{1}{3} \left( \frac{b-a}{2n} \right) \left[ y_0 + 4 \sum_{j=1}^{2n-1} y_j - 2 \sum_{j=1}^{n-1} y_{2j} + y_{2n} \right] \end{aligned}$$

and

$$\begin{aligned} T_{2n} &= \frac{1}{2} \left( \frac{b-a}{2n} \right) \left( y_0 + 2 \sum_{j=1}^{2n-1} y_j + y_{2n} \right) \\ T_n &= \frac{1}{2} \left( \frac{b-a}{n} \right) \left( y_0 + 2 \sum_{j=1}^{n-1} y_{2j} + y_{2n} \right). \end{aligned}$$

Since  $T_{2n} = \frac{1}{2}(T_n + M_n) \Rightarrow M_n = 2T_{2n} - T_n$ , then

$$\begin{aligned} \frac{T_n + 2M_n}{3} &= \frac{T_n + 2(2T_{2n} - T_n)}{3} = \frac{4T_{2n} - T_n}{3} \\ \frac{2T_{2n} + M_n}{3} &= \frac{2T_{2n} + 2T_{2n} - T_n}{3} = \frac{4T_{2n} - T_n}{3}. \end{aligned}$$

Hence,

$$\frac{T_n + 2M_n}{3} = \frac{2T_{2n} + M_n}{3} = \frac{4T_{2n} - T_n}{3}.$$

Using the formulas of  $T_{2n}$  and  $T_n$  obtained above,

$$\begin{aligned} &\frac{4T_{2n} - T_n}{3} \\ &= \frac{1}{3} \left[ \frac{4}{2} \left( \frac{b-a}{2n} \right) \left( y_0 + 2 \sum_{j=1}^{2n-1} y_j + y_{2n} \right) \right. \\ &\quad \left. - \frac{1}{2} \left( \frac{b-a}{n} \right) \left( y_0 + 2 \sum_{j=1}^{n-1} y_{2j} + y_{2n} \right) \right] \\ &= \frac{1}{3} \left( \frac{b-a}{2n} \right) \left[ y_0 + 4 \sum_{j=1}^{2n-1} y_j - 2 \sum_{j=1}^{n-1} y_{2j} + y_{2n} \right] \\ &= S_{2n}. \end{aligned}$$

Hence,

$$S_{2n} = \frac{4T_{2n} - T_n}{3} = \frac{T_n + 2M_n}{3} = \frac{2T_{2n} + M_n}{3}.$$

9. We use the results of Exercise 9 of Section 7.6 and Exercise 8 of this section.

$$\begin{aligned} I &= \int_0^{1.6} f(x) dx \\ S_4 &= \frac{0.4}{3} (1.4142 + 4(1.3860) + 2(1.3026) + 4(1.1772) \\ &\quad + 0.9853) \approx 2.0343333 \\ S_8 &= (T_4 + 2M_4)/3 \approx 2.0303133 \\ S_{16} &= (T_8 + 2M_8)/3 \approx 2.0296433. \end{aligned}$$

10. The approximations for  $I = \int_0^1 e^{-x^2} dx$  are

$$\begin{aligned} S_8 &= \frac{1}{3} \left( \frac{1}{8} \right) \left[ 1 + 4 \left( e^{-1/64} + e^{-9/64} + e^{-25/64} + e^{-49/64} \right) + 2 \left( e^{-1/16} + e^{-1/4} + e^{-9/16} \right) + e^{-1} \right] \\ &\approx 0.7468261 \\ S_{16} &= \frac{1}{3} \left( \frac{1}{16} \right) \left[ 1 + 4 \left( e^{-1/256} + e^{-9/256} + e^{-25/256} + e^{-49/256} + e^{-81/256} + e^{-121/256} + e^{-169/256} + e^{-225/256} \right) + 2 \left( e^{-1/64} + e^{-1/16} + e^{-9/64} + e^{-1/4} + e^{-25/64} + e^{-9/16} + e^{-49/64} \right) + e^{-1} \right] \\ &\approx 0.7468243. \end{aligned}$$

If  $f(x) = e^{-x^2}$ , then  $f^{(4)}(x) = 4e^{-x^2}(4x^4 - 12x^2 + 3)$ . On  $[0, 1]$ ,  $|f^{(4)}(x)| \leq 12$ , and the error bounds are

$$\begin{aligned} |I - S_n| &\leq \frac{12(1)}{180} \left( \frac{1}{n} \right)^4 \\ |I - S_8| &\leq \frac{12}{180} \left( \frac{1}{8} \right)^4 \approx 0.0000163 \\ |I - S_{16}| &\leq \frac{12}{180} \left( \frac{1}{16} \right)^4 \approx 0.0000010. \end{aligned}$$

Comparing the two approximations,

$$I = \int_0^1 e^{-x^2} dx = 0.7468,$$

accurate to 4 decimal places.

$$11. \quad I = \int_0^1 x^4 dx = \frac{1}{5}. \quad S_2 = \frac{1}{6} \left[ 0^4 + 4 \left( \frac{1}{2} \right)^4 + 1^4 \right] = \frac{5}{24}.$$

If  $f(x) = x^4$ , then  $f^{(4)}(x) = 24$ .

$$\text{Error estimate: } |I - S_2| \leq \frac{24(1-0)}{180} \left( \frac{1}{2} \right)^4 = \frac{1}{120}.$$

$$\text{Actual error: } |I - S_2| = \left| \frac{1}{5} - \frac{5}{24} \right| = \frac{1}{120}.$$

Thus the error estimate cannot be improved.

12. The exact value of  $I$  is

$$I = \int_0^1 x^3 dx = \frac{x^4}{4} \Big|_0^1 = \frac{1}{4}.$$

The approximation is

$$S_2 = \frac{1}{3} \left( \frac{1}{2} \right) \left[ 0^3 + 4 \left( \frac{1}{2} \right)^3 + 1^3 \right] = \frac{1}{4}.$$

The actual error is zero. Hence, Simpson's Rule is exact for the cubic function  $f(x) = x^3$ . Since it is evidently exact for quadratic functions  $f(x) = Bx^2 + Cx + D$ , it must also be exact for arbitrary cubics  $f(x) = Ax^3 + Bx^2 + Cx + D$ .

### Section 6.8 Other Aspects of Approximate Integration (page 364)

- $\int_0^1 \frac{dx}{x^{1/3}(1+x)}$  Let  $x = u^3$   
 $= 3 \int_0^1 \frac{u^2 du}{u(1+u^3)} = 3 \int_0^1 \frac{u du}{1+u^3}.$
- $\int_0^1 \frac{e^x}{\sqrt{1-x}} dx$  Let  $t^2 = 1-x$   
 $2t dt = -dx$   
 $= - \int_1^0 \frac{e^{1-t^2}}{t} 2t dt = 2 \int_0^1 e^{1-t^2} dt.$
- One possibility: let  $x = \sin \theta$  and get

$$I = \int_{-1}^1 \frac{e^x dx}{\sqrt{1-x^2}} = \int_{-\pi/2}^{\pi/2} e^{\sin \theta} d\theta.$$

Another possibility:

$$I = \int_{-1}^0 \frac{e^x dx}{\sqrt{1-x^2}} + \int_0^1 \frac{e^x dx}{\sqrt{1-x^2}} = I_1 + I_2.$$

In  $I_1$  put  $1+x = u^2$ ; in  $I_2$  put  $1-x = u^2$ :

$$\begin{aligned} I_1 &= \int_0^1 \frac{2e^{u^2-1} u du}{u\sqrt{2-u^2}} = 2 \int_0^1 \frac{e^{u^2-1} du}{\sqrt{2-u^2}} \\ I_2 &= \int_0^1 \frac{2e^{1-u^2} u du}{u\sqrt{2-u^2}} = 2 \int_0^1 \frac{e^{1-u^2} du}{\sqrt{2-u^2}} \end{aligned}$$

so  $I = 2 \int_0^1 \frac{e^{u^2-1} + e^{1-u^2}}{\sqrt{2-u^2}} du.$

4.  $\int_1^\infty \frac{dx}{x^2 + \sqrt{x} + 1}$  Let  $x = \frac{1}{t^2}$   
 $dx = -\frac{2dt}{t^3}$   
 $= \int_1^0 \frac{1}{\left(\frac{1}{t^2}\right)^2 + \sqrt{\frac{1}{t^2}} + 1} \left(-\frac{2dt}{t^3}\right)$   
 $= 2 \int_0^1 \frac{t dt}{t^4 + t^3 + 1}.$

5.  $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$  Let  $\sin x = u^2$   
 $2u du = \cos x dx = \sqrt{1-u^4} dx$   
 $= 2 \int_0^1 \frac{u du}{u\sqrt{1-u^4}}$   
 $= 2 \int_0^1 \frac{du}{\sqrt{(1-u)(1+u)(1+u^2)}}$  Let  $1-u = v^2$   
 $-du = 2v dv$   
 $= 4 \int_0^1 \frac{v dv}{v\sqrt{(1+1-v^2)(1+(1-v^2)^2)}}$   
 $= 4 \int_0^1 \frac{dv}{\sqrt{(2-v^2)(2-2v^2+v^4)}}.$

6. Let

$$\int_0^\infty \frac{dx}{x^4 + 1} = \int_0^1 \frac{dx}{x^4 + 1} + \int_1^\infty \frac{dx}{x^4 + 1} = I_1 + I_2.$$

Let  $x = \frac{1}{t}$  and  $dx = -\frac{dt}{t^2}$  in  $I_2$ , then

$$I_2 = \int_1^0 \frac{1}{\left(\frac{1}{t}\right)^4 + 1} \left(-\frac{dt}{t^2}\right) = \int_0^1 \frac{t^2}{1+t^4} dt.$$

Hence,

$$\begin{aligned} \int_0^\infty \frac{dx}{x^4 + 1} &= \int_0^1 \left( \frac{1}{x^4 + 1} + \frac{x^2}{1 + x^4} \right) dx \\ &= \int_0^1 \frac{x^2 + 1}{x^4 + 1} dx. \end{aligned}$$

7.  $I = \int_0^1 \sqrt{x} dx = \frac{2}{3} \approx 0.666667.$

$$T_2 = \frac{1}{2} \left( 0 + \sqrt{\frac{1}{2}} + \frac{1}{2} \right) \approx 0.603553$$

$$T_4 = \frac{1}{4} \left( 2T_2 + \sqrt{\frac{1}{4}} + \sqrt{\frac{3}{4}} \right) \approx 0.643283$$

$$T_8 = \frac{1}{8} \left( 4T_4 + \sqrt{\frac{1}{8}} + \sqrt{\frac{3}{8}} + \sqrt{\frac{5}{8}} + \sqrt{\frac{7}{8}} \right) \approx 0.658130$$

$$\begin{aligned} T_{16} &= \frac{1}{16} \left( 8T_8 + \sqrt{\frac{1}{16}} + \sqrt{\frac{3}{16}} + \sqrt{\frac{5}{16}} + \sqrt{\frac{7}{16}} \right. \\ &\quad \left. + \sqrt{\frac{9}{16}} + \sqrt{\frac{11}{16}} + \sqrt{\frac{13}{16}} + \sqrt{\frac{15}{16}} \right) \approx 0.663581. \end{aligned}$$

The errors are

$$I - T_2 \approx 0.0631$$

$$I - T_4 \approx 0.0234$$

$$I - T_8 \approx 0.0085$$

$$I - T_{16} \approx 0.0031.$$

Observe that, although these errors are decreasing, they are not decreasing like  $1/n^2$ ; that is,

$$|I - T_{2n}| > \frac{1}{4} |I - T_n|.$$

This is because the second derivative of  $f(x) = \sqrt{x}$  is  $f''(x) = -1/(4x^{3/2})$ , which is *not bounded* on  $[0, 1]$ .

8. Let

$$\begin{aligned} I &= \int_1^\infty e^{-x^2} dx \quad \text{Let } x = \frac{1}{t} \\ dx &= -\frac{dt}{t^2} \\ &= \int_1^0 e^{-(1/t)^2} \left(-\frac{1}{t^2}\right) dt = \int_0^1 \frac{e^{-1/t^2}}{t^2} dt. \end{aligned}$$

Observe that

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{e^{-1/t^2}}{t^2} &= \lim_{t \rightarrow 0^+} \frac{t^{-2}}{e^{1/t^2}} \left[ \frac{\infty}{\infty} \right] \\ &= \lim_{t \rightarrow 0^+} \frac{-2t^{-3}}{e^{1/t^2}(-2t^{-3})} \\ &= \lim_{t \rightarrow 0^+} \frac{1}{e^{1/t^2}} = 0. \end{aligned}$$

Hence,

$$\begin{aligned} S_2 &= \frac{1}{3} \left( \frac{1}{2} \right) \left[ 0 + 4(4e^{-4}) + e^{-1} \right] \\ &\approx 0.1101549 \\ S_4 &= \frac{1}{3} \left( \frac{1}{4} \right) \left[ 0 + 4(16e^{-16}) + 2(4e^{-4}) \right. \\ &\quad \left. + 4 \left( \frac{16}{9} e^{-16/9} \right) + e^{-1} \right] \\ &\approx 0.1430237 \\ S_8 &= \frac{1}{3} \left( \frac{1}{8} \right) \left[ 0 + 4 \left( 64e^{-64} + \frac{64}{9} e^{-64/9} + \frac{64}{25} e^{-64/25} + \right. \right. \\ &\quad \left. \left. \frac{64}{49} e^{-64/49} \right) + 2 \left( 16e^{-16} + 4e^{-4} + \frac{16}{9} e^{-16/9} \right) + e^{-1} \right] \\ &\approx 0.1393877. \end{aligned}$$

Hence,  $I \approx 0.14$ , accurate to 2 decimal places. These approximations do not converge very quickly, because the fourth derivative of  $e^{-1/t^2}$  has very large values for some values of  $t$  near 0. In fact, higher and higher derivatives behave more and more badly near 0, so higher order methods cannot be expected to work well either.

9. Referring to Example 5, we have

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(f; 0, x),$$

where  $R_n(f; 0, x) = \frac{e^X x^{n+1}}{(n+1)!}$ , for some  $X$  between 0 and  $x$ . Now

$$|R_n(f; 0, -x^2)| \leq \frac{x^{2n+2}}{(n+1)!}$$

if  $0 \leq x \leq 1$  for any  $x$ , since  $-x^2 \leq X \leq 0$ . Therefore

$$\begin{aligned} \left| \int_0^1 R_n(f; 0, -x^2) dx \right| &\leq \frac{1}{(n+1)!} \int_0^1 x^{2n+2} dx \\ &= \frac{1}{(2n+3)(n+1)!}. \end{aligned}$$

This error

will be less than  $10^{-4}$  if  $(2n+3)(n+1)! > 10,000$ . Since  $15 \times 7! > 10,000$ ,  $n = 6$  will do. Thus we use seven terms of the series ( $0 \leq n \leq 6$ ):

$$\begin{aligned} \int_0^1 e^{-x^2} dx &\approx \int_0^1 \left( 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \frac{x^{12}}{6!} \right) dx \\ &= 1 - \frac{1}{3} + \frac{1}{5 \times 2!} - \frac{1}{7 \times 3!} + \frac{1}{9 \times 4!} - \frac{1}{11 \times 5!} + \frac{1}{13 \times 6!} \\ &\approx 0.74684 \quad \text{with error less than } 10^{-4}. \end{aligned}$$

10. We are given that  $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$  and from the previous exercise  $\int_0^1 e^{-x^2} dx = 0.74684$ . Therefore,

$$\begin{aligned} \int_1^\infty e^{-x^2} dx &= \int_0^\infty e^{-x^2} dx - \int_0^1 e^{-x^2} dx \\ &= \frac{1}{2}\sqrt{\pi} - 0.74684 \\ &= 0.139 \quad (\text{to 3 decimal places}). \end{aligned}$$

11. If  $f(x) = ax^3 + bx^2 + cx + d$ , then, by symmetry,

$$\begin{aligned} \int_{-1}^1 f(x) dx &= 2 \int_0^1 (bx^2 + d) dx = 2 \left( \frac{b}{3} + d \right) \\ Af(-u) + Af(u) &= 2A(bu^2 + d). \end{aligned}$$

These two expressions are identical provided  $A = 1$  and  $u^2 = 1/3$ , so  $u = 1/\sqrt{3}$ .

12. For any function  $f$  we use the approximation

$$\int_{-1}^1 f(x) dx \approx f(-1/\sqrt{3}) + f(1/\sqrt{3}).$$

We have

$$\begin{aligned} \int_{-1}^1 x^4 dx &\approx \left( -\frac{1}{\sqrt{3}} \right)^4 + \left( \frac{1}{\sqrt{3}} \right)^4 = \frac{2}{9} \\ \text{Error} &= \int_{-1}^1 x^4 dx - \frac{2}{9} = \frac{2}{5} - \frac{2}{9} \approx 0.17778 \\ \int_{-1}^1 \cos x dx &\approx \cos \left( -\frac{1}{\sqrt{3}} \right) + \cos \left( \frac{1}{\sqrt{3}} \right) \approx 1.67582 \\ \text{Error} &= \int_{-1}^1 \cos x dx - 1.67582 \approx 0.00712 \\ \int_{-1}^1 e^x dx &\approx e^{-1/\sqrt{3}} + e^{1/\sqrt{3}} \approx 2.34270 \\ \text{Error} &= \int_{-1}^1 e^x dx - 2.34270 \approx 0.00771. \end{aligned}$$

13. If  $F(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ , then, by symmetry,

$$\begin{aligned} \int_{-1}^1 F(x) dx &= 2 \int_0^1 (bx^4 + dx^2 + f) dx = 2 \left( \frac{b}{5} + \frac{d}{3} + f \right) \\ Af(-u) + BF(0) + AF(u) &= 2A(bu^4 + du^2 + f) + Bf. \end{aligned}$$

These two expressions are identical provided

$$Au^4 = \frac{1}{5}, \quad Au^2 = \frac{1}{3}, \quad A + \frac{B}{2} = 1.$$

Dividing the first two equations gives  $u^2 = 3/5$ , so  $u = \sqrt{3/5}$ . Then  $3A/5 = 1/3$ , so  $A = 5/9$ , and finally,  $B = 8/9$ .

14. For any function  $f$  we use the approximation

$$\int_{-1}^1 f(x) dx \approx \frac{5}{9} [f(-\sqrt{3/5}) + f(\sqrt{3/5})] + \frac{8}{9} f(0).$$

We have

$$\int_{-1}^1 x^6 dx \approx \frac{5}{9} \left[ \left(-\sqrt{\frac{3}{5}}\right)^6 + \left(\sqrt{\frac{3}{5}}\right)^6 \right] + 0 = 0.24000$$

$$\text{Error} = \int_{-1}^1 x^6 dx - 0.24000 \approx 0.04571$$

$$\int_{-1}^1 \cos x dx \approx \frac{5}{9} \left[ \cos \left(-\sqrt{\frac{3}{5}}\right) + \cos \left(\sqrt{\frac{3}{5}}\right) \right] + \frac{8}{9}$$

$$\approx 1.68300$$

$$\text{Error} = \int_{-1}^1 \cos x dx - 1.68300 \approx 0.00006$$

$$\int_{-1}^1 e^x dx \approx e^{-\sqrt{3/5}} + e^{\sqrt{3/5}} \approx 2.35034$$

$$\text{Error} = \int_{-1}^1 e^x dx - 2.35034 \approx 0.00006.$$

15.  $I = \int_0^1 e^{-x^2} dx$

$$T_0^0 = T_1 = R_0 = (1) \left( \frac{1}{2} e^0 + \frac{1}{2} e^{-1} \right) \approx 0.6839397$$

$$T_1^0 = T_2 = \frac{1}{2} \left( \frac{1}{2} e^0 + e^{-1/4} + \frac{1}{2} e^{-1} \right) \approx 0.7313703$$

$$T_2^0 = T_4 = \frac{1}{4} \left( 2T_2 + e^{-1/16} + e^{-9/16} \right) \approx 0.7429841$$

$$T_3^0 = T_8 = \frac{1}{8} \left( 4T_4 + e^{-1/64} + e^{-9/64} + e^{-25/64} + e^{-49/64} \right)$$

$$\approx 0.7458656$$

$$T_1^1 = S_2 = R_1 = \frac{4T_1^0 - T_0^0}{3} \approx 0.7471805$$

$$T_2^1 = S_4 = \frac{4T_2^0 - T_1^0}{3} \approx 0.7468554$$

$$T_3^1 = S_8 = \frac{4T_3^0 - T_2^0}{3} \approx 0.7468261$$

$$T_2^2 = R_2 = \frac{16T_2^1 - T_1^1}{15} \approx 0.7468337$$

$$T_3^2 = \frac{16T_3^1 - T_2^1}{15} \approx 0.7468242$$

$$T_3^3 = R_3 = \frac{64T_3^2 - T_2^2}{63} \approx 0.7468241$$

$$I \approx 0.746824 \text{ to 6 decimal places.}$$

16. From Exercise 9 in Section 7.6, for  $I = \int_0^{1.6} f(x) dx$ ,

$$T_0^0 = T_1 = 1.9196$$

$$T_1^0 = T_2 = 2.00188$$

$$T_2^0 = T_4 = 2.02622$$

$$T_3^0 = T_8 = 2.02929.$$

Hence,

$$R_1 = T_1^1 = \frac{4T_1^0 - T_0^0}{3} = 2.0346684$$

$$T_2^1 = \frac{4T_2^0 - T_1^0}{3} = 2.0343333 = S_4$$

$$R_2 = T_2^2 = \frac{16T_2^1 - T_1^1}{15} = 2.0346684$$

$$T_3^1 = \frac{4T_3^0 - T_2^0}{3} = 2.0303133 = S_8$$

$$T_3^2 = \frac{16T_3^1 - T_2^1}{15} = 2.0300453$$

$$R_3 = T_3^3 = \frac{64T_3^2 - T_2^2}{63} = 2.0299719.$$

17.  $T_1^1 = S_2 = \frac{2h}{3} (y_0 + 4y_2 + y_4)$

$$T_2^1 = S_4 = \frac{h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4)$$

$$R_2 = T_2^2 = \frac{16T_2^1 - T_1^1}{15}$$

$$= \frac{\frac{16h}{3} (y_0 + 4y_1 + 2y_2 + 4y_3 + y_4) - \frac{2h}{3} (y_0 + 4y_2 + y_4)}{15}$$

$$= \frac{h}{45} (14y_0 + 64y_1 + 24y_2 + 64y_3 + 14y_4)$$

$$= \frac{2h}{45} (7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4)$$

18. Let

$$I = \int_{\pi}^{\infty} \frac{\sin x}{1+x^2} dx \quad \text{Let } x = \frac{1}{t}$$

$$dx = -\frac{dt}{t^2}$$

$$= \int_{1/\pi}^0 \frac{\sin \left(\frac{1}{t}\right)}{1 + \left(\frac{1}{t^2}\right)} \left(-\frac{1}{t^2}\right) dt$$

$$= \int_0^{1/\pi} \frac{\sin \left(\frac{1}{t}\right)}{1 + t^2} dt.$$

The transformation is not suitable because the derivative of  $\sin\left(\frac{1}{t}\right)$  is  $-\frac{1}{t^2}\cos\left(\frac{1}{t}\right)$ , which has very large values at some points close to 0.

In order to approximate the integral  $I$  to an desired degree of accuracy, say with error less than  $\epsilon$  in absolute value, we have to divide the integral into two parts:

$$\begin{aligned} I &= \int_{\pi}^{\infty} \frac{\sin x}{1+x^2} dx \\ &= \int_{\pi}^t \frac{\sin x}{1+x^2} dx + \int_t^{\infty} \frac{\sin x}{1+x^2} dx \\ &= I_1 + I_2. \end{aligned}$$

If  $t \geq \tan \frac{\pi - \epsilon}{2}$ , then

$$\begin{aligned} \int_t^{\infty} \frac{\sin x}{1+x^2} dx &< \int_t^{\infty} \frac{dx}{1+x^2} \\ &= \tan^{-1}(x) \Big|_t^{\infty} = \frac{\pi}{2} - \tan^{-1}(t) \leq \frac{\epsilon}{2}. \end{aligned}$$

Now let  $A$  be a numerical approximation to the proper integral  $\int_{\pi}^t \frac{\sin x}{1+x^2} dx$ , having error less than  $\epsilon/2$  in absolute value. Then

$$\begin{aligned} |I - A| &= |I_1 + I_2 - A| \\ &\leq |I_1 - A| + |I_2| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence,  $A$  is an approximation to the integral  $I$  with the desired accuracy.

$$\begin{aligned} 19. \quad f(x) &= \frac{\sin x}{x}, \quad f'(x) = \frac{x \cos x - \sin x}{x^2}, \\ f''(x) &= \frac{x^2(\cos x - x \sin x - \cos x) - (x \cos x - \sin x)2x}{x^4} \\ &= \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3}. \end{aligned}$$

Now use l'Hôpital's Rule to get

$$\begin{aligned} \lim_{x \rightarrow 0} f''(x) &= \lim_{x \rightarrow 0} \frac{-2x \sin x - x^2 \cos x - 2 \cos x + 2x \sin x + 2 \cos x}{3x^2} \\ &= \lim_{x \rightarrow 0} -\frac{\cos x}{3} = -\frac{1}{3}. \end{aligned}$$

### Review Exercises on Techniques of Integration (page 365)

$$\begin{aligned} 1. \quad \frac{x}{2x^2 + 5x + 2} &= \frac{A}{2x+1} + \frac{B}{x+2} \\ &= \frac{Ax + 2A + 2Bx + B}{2x^2 + 5x + 2} \\ &\Rightarrow \begin{cases} A + 2B = 1 \\ 2A + B = 0 \end{cases} \end{aligned}$$

Thus  $A = -1/3$  and  $B = 2/3$ . We have

$$\begin{aligned} \int \frac{x dx}{2x^2 + 5x + 2} &= -\frac{1}{3} \int \frac{dx}{2x+1} + \frac{2}{3} \int \frac{dx}{x+2} \\ &= \frac{2}{3} \ln|x+2| - \frac{1}{6} \ln|2x+1| + C. \end{aligned}$$

$$\begin{aligned} 2. \quad \int \frac{x}{(x-1)^3} dx &\quad \text{Let } u = x-1 \\ &\quad du = dx \\ &= \int \frac{u+1}{u^3} du = \int \left( \frac{1}{u^2} + \frac{1}{u^3} \right) du \\ &= -\frac{1}{u} - \frac{1}{2u^2} + C = -\frac{1}{x-1} - \frac{1}{2(x-1)^2} + C. \end{aligned}$$

$$\begin{aligned} 3. \quad \int \sin^3 x \cos^3 x dx &= \int \sin^2 x (1 - \sin^2 x) \cos x dx \quad \text{Let } u = \sin x \\ &\quad du = \cos x dx \\ &= \int (u^2 - u^4) du = \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C. \end{aligned}$$

$$\begin{aligned} 4. \quad \int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}} dx &\quad \text{Let } u = 1 + \sqrt{x} \\ &\quad du = \frac{dx}{2\sqrt{x}} \\ &= 2 \int u^{1/3} du = 2 \left( \frac{3}{4} \right) u^{4/3} + C \\ &= \frac{3}{2} (1 + \sqrt{x})^{4/3} + C. \end{aligned}$$

$$\begin{aligned} 5. \quad \frac{3}{4x^2 - 1} &= \frac{A}{2x-1} + \frac{B}{2x+1} \\ &= \frac{2Ax + A + 2Bx + B}{4x^2 - 1} \\ &\Rightarrow \begin{cases} 2A + 2B = 0 \\ A - B = 3 \end{cases} \Rightarrow A = -B = \frac{3}{2} \end{aligned}$$

$$\begin{aligned}\int \frac{3 dx}{4x^2 - 1} &= \frac{3}{2} \left( \int \frac{dx}{2x - 1} - \int \frac{dx}{2x + 1} \right) \\ &= \frac{3}{4} \ln \left| \frac{2x - 1}{2x + 1} \right| + C.\end{aligned}$$

$$\begin{aligned}6. \quad \int (x^2 + x - 2) \sin 3x \, dx \\ U = x^2 + x - 2 \quad dV = \sin 3x \\ dU = (2x + 1) dx \quad V = -\frac{1}{3} \cos 3x \\ = -\frac{1}{3}(x^2 + x - 2) \cos 3x + \frac{1}{3} \int (2x + 1) \cos 3x \, dx\end{aligned}$$

$$\begin{aligned}U = 2x + 1 \quad dV = \cos 3x \, dx \\ dU = 2 \, dx \quad V = \frac{1}{3} \sin 3x \\ = -\frac{1}{3}(x^2 + x - 2) \cos 3x + \frac{1}{9}(2x + 1) \sin 3x \\ - \frac{2}{9} \int \sin 3x \, dx \\ = -\frac{1}{3}(x^2 + x - 2) \cos 3x + \frac{1}{9}(2x + 1) \sin 3x \\ + \frac{2}{27} \cos 3x + C.\end{aligned}$$

$$\begin{aligned}7. \quad \int \frac{\sqrt{1-x^2}}{x^4} \, dx \quad \text{Let } x = \sin \theta \\ dx = \cos \theta \, d\theta \\ = \int \frac{\cos^2 \theta}{\sin^4 \theta} \, d\theta \\ = \int \csc^2 \theta \cot^2 \theta \, d\theta \quad \text{Let } v = \cot \theta \\ dv = -\csc^2 \theta \, d\theta \\ = -\int v^2 \, dv = -\frac{v^3}{3} + C \\ = -\frac{\cot^3 \theta}{3} + C = -\frac{1}{3} \left( \frac{\sqrt{1-x^2}}{x} \right)^3 + C.\end{aligned}$$

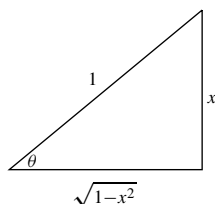


Fig. RT.7

$$\begin{aligned}8. \quad \int x^3 \cos(x^2) \, dx \quad \text{Let } w = x^2 \\ dw = 2x \, dx \\ = \frac{1}{2} \int w \cos w \, dw \\ U = w \quad dV = \cos w \, dw \\ dU = dw \quad V = \sin w \\ = \frac{1}{2} w \sin w - \frac{1}{2} \int \sin w \, dw \\ = \frac{1}{2} x^2 \sin(x^2) + \frac{1}{2} \cos(x^2) + C.\end{aligned}$$

$$\begin{aligned}9. \quad \int \frac{x^2 \, dx}{(5x^3 - 2)^{2/3}} \quad \text{Let } u = 5x^3 - 2 \\ du = 15x^2 \, dx \\ = \frac{1}{15} \int u^{-2/3} \, du = \frac{1}{5} u^{1/3} + C \\ = \frac{1}{5} (5x^3 - 2)^{1/3} + C.\end{aligned}$$

$$\begin{aligned}10. \quad \frac{1}{x^2 + 2x - 15} = \frac{A}{x - 3} + \frac{B}{x + 5} = \frac{(A + B)x + (5A - 3B)}{x^2 + 2x - 15} \\ \Rightarrow \begin{cases} A + B = 0 \\ 5A - 3B = 1 \end{cases} \Rightarrow A = \frac{1}{8}, \quad B = -\frac{1}{8}. \\ \int \frac{dx}{x^2 + 2x - 15} = \frac{1}{8} \int \frac{dx}{x - 3} - \frac{1}{8} \int \frac{dx}{x + 5} \\ = \frac{1}{8} \ln \left| \frac{x - 3}{x + 5} \right| + C.\end{aligned}$$

$$\begin{aligned}11. \quad \int \frac{dx}{(4 + x^2)^2} \quad \text{Let } x = 2 \tan \theta \\ dx = 2 \sec^2 \theta \, d\theta \\ = \int \frac{2 \sec^2 \theta \, d\theta}{16 \sec^4 \theta} = \frac{1}{8} \int \cos^2 \theta \, d\theta \\ = \frac{1}{16} (\theta + \sin \theta \cos \theta) + C \\ = \frac{1}{16} \tan^{-1} \frac{x}{2} + \frac{1}{8} \left( \frac{x}{4 + x^2} \right) + C.\end{aligned}$$

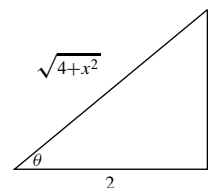


Fig. RT.11

$$\begin{aligned}12. \quad \int (\sin x + \cos x)^2 \, dx = \int (1 + \sin 2x) \, dx \\ = x - \frac{1}{2} \cos 2x + C.\end{aligned}$$

$$\begin{aligned}
 13. \quad & \int 2^x \sqrt{1+4^x} dx \quad \text{Let } 2^x = \tan \theta \\
 & \quad \quad \quad 2^x \ln 2 dx = \sec^2 \theta d\theta \\
 & = \frac{1}{\ln 2} \int \sec^3 \theta d\theta \\
 & = \frac{1}{2 \ln 2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C \\
 & = \frac{1}{2 \ln 2} (2^x \sqrt{1+4^x} + \ln(2^x + \sqrt{1+4^x})) + C.
 \end{aligned}$$

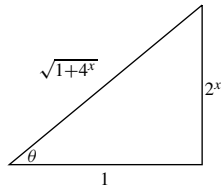


Fig. RT.13

$$\begin{aligned}
 14. \quad & \int \frac{\cos x}{1 + \sin^2 x} dx \quad \text{Let } u = \sin x \\
 & \quad \quad \quad du = \cos x dx \\
 & = \int \frac{du}{1+u^2} = \tan^{-1} u + C \\
 & = \tan^{-1}(\sin x) + C.
 \end{aligned}$$

$$\begin{aligned}
 15. \quad & \int \frac{\sin^3 x}{\cos^7 x} dx = \int \tan^3 x \sec^4 x dx \\
 & = \int \tan^3 x (1 + \tan^2 x) \sec^2 x dx \quad \text{Let } u = \tan x \\
 & \quad \quad \quad du = \sec^2 x dx \\
 & = \int (u^3 + u^5) du = \frac{u^4}{4} + \frac{u^6}{6} + C \\
 & = \frac{1}{4} \tan^4 x + \frac{1}{6} \tan^6 x + C.
 \end{aligned}$$

16. We have

$$\begin{aligned}
 & \int \frac{x^2 dx}{(3+5x^2)^{3/2}} \quad \text{Let } x = \sqrt{\frac{3}{5}} \tan u \\
 & \quad \quad \quad dx = \sqrt{\frac{3}{5}} \sec^2 u du \\
 & = \int \frac{(\frac{3}{5} \tan^2 u)(\sqrt{\frac{3}{5}} \sec^2 u du)}{(3)^{3/2} \sec^3 u} \\
 & = \frac{1}{5\sqrt{5}} \int (\sec u - \cos u) du \\
 & = \frac{1}{5\sqrt{5}} (\ln |\sec u + \tan u| - \sin u) + C \\
 & = \frac{1}{5\sqrt{5}} \left( \ln \left| \frac{\sqrt{5x^2+3}}{\sqrt{3}} + \frac{\sqrt{5x}}{\sqrt{3}} \right| - \frac{\sqrt{5x}}{\sqrt{5x^2+3}} \right) + C \\
 & = \frac{1}{5\sqrt{5}} \ln(\sqrt{5x^2+3} + \sqrt{5x}) - \frac{x}{5\sqrt{5x^2+3}} + C_0, \\
 & \quad \text{where } C_0 = C - \frac{1}{5\sqrt{5}} \ln \sqrt{3}.
 \end{aligned}$$

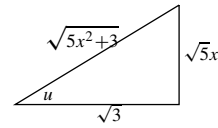


Fig. RT.16

$$\begin{aligned}
 17. \quad I &= \int e^{-x} \sin 2x dx \\
 U &= e^{-x} & dV &= \sin 2x dx \\
 dU &= -e^{-x} dx & V &= -\frac{1}{2} \cos 2x \\
 &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{2} \int e^{-x} \cos 2x dx
 \end{aligned}$$

$$\begin{aligned}
 U &= e^{-x} & dV &= \cos 2x dx \\
 dU &= -e^{-x} dx & V &= \frac{1}{2} \sin 2x \\
 &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{2} \left( \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} I \right) \\
 &= -\frac{1}{2} e^{-x} \cos 2x - \frac{1}{4} e^{-x} \sin 2x - \frac{1}{4} I \\
 I &= -e^{-x} \left( \frac{2}{5} \cos 2x + \frac{1}{5} \sin 2x \right) + C.
 \end{aligned}$$

$$\begin{aligned}
 18. \quad I &= \int \frac{2x^2 + 4x - 3}{x^2 + 5x} dx = \int \frac{2x^2 + 10x - 6x - 3}{x^2 + 5x} dx \\
 &= \int \left[ 2 - \frac{6x+3}{x(x+5)} \right] dx \\
 \frac{6x+3}{x(x+5)} &= \frac{A}{x} + \frac{B}{x+5} = \frac{(A+B)x + 5A}{x(x+5)} \\
 \Rightarrow \begin{cases} A+B=6 \\ 5A=3 \end{cases} &\Rightarrow A = \frac{3}{5}, B = \frac{27}{5}. \\
 I &= \int 2 dx - \frac{3}{5} \int \frac{dx}{x} - \frac{27}{5} \int \frac{dx}{x+5} \\
 &= 2x - \frac{3}{5} \ln |x| - \frac{27}{5} \ln |x+5| + C.
 \end{aligned}$$

$$\begin{aligned}
 19. \quad I &= \int \cos(3 \ln x) dx \\
 U &= \cos(3 \ln x) & dV &= dx \\
 dU &= -\frac{3 \sin(3 \ln x) dx}{x} & V &= x \\
 &= x \cos(3 \ln x) + 3 \int \sin(3 \ln x) dx \\
 U &= \sin(3 \ln x) & dV &= dx \\
 dU &= \frac{3 \cos(3 \ln x) dx}{x} & V &= x \\
 &= x \cos(3 \ln x) + 3 \left( x \sin(3 \ln x) - 3I \right) \\
 I &= \frac{1}{10} x \cos(3 \ln x) + \frac{3}{10} x \sin(3 \ln x) + C.
 \end{aligned}$$

$$\begin{aligned}
 20. \quad \frac{1}{4x^3 + x} &= \frac{A}{x} + \frac{Bx + C}{4x^2 + 1} \\
 &= \frac{A(4x^2 + 1) + Bx^2 + Cx}{4x^3 + x} \\
 \Rightarrow \begin{cases} 4A + B = 0 \\ C = 0, A = 1 \end{cases} &\Rightarrow B = -4. \\
 \int \frac{1}{4x^3 + x} dx &= \int \frac{dx}{x} - 4 \int \frac{x dx}{4x^2 + 1} \\
 &= \ln|x| - \frac{1}{2} \ln(4x^2 + 1) + C.
 \end{aligned}$$

$$\begin{aligned}
 21. \quad \int \frac{x \ln(1 + x^2)}{1 + x^2} dx \quad &\text{Let } u = \ln(1 + x^2) \\
 &du = \frac{2x dx}{1 + x^2} \\
 &= \frac{1}{2} \int u du = \frac{u^2}{4} + C \\
 &= \frac{1}{4} (\ln(1 + x^2))^2 + C.
 \end{aligned}$$

$$\begin{aligned}
 22. \quad \int \sin^2 x \cos^4 x dx \\
 &= \int \frac{1}{2} (1 - \cos 2x) \left[ \frac{1}{2} (1 + \cos 2x) \right]^2 dx \\
 &= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) dx \\
 &= \frac{1}{8} x + \frac{1}{16} \sin 2x - \frac{1}{16} \int (1 + \cos 4x) dx \\
 &\quad - \frac{1}{8} \int (1 - \sin^2 2x) \cos 2x dx \\
 &= \frac{x}{8} + \frac{1}{16} \sin 2x - \frac{x}{16} - \frac{1}{64} \sin 4x - \frac{1}{16} \sin 2x \\
 &\quad + \frac{1}{48} \sin^3 2x + C \\
 &= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.
 \end{aligned}$$

$$\begin{aligned}
 23. \quad \int \frac{x^2 dx}{\sqrt{2 - x^2}} \quad &\text{Let } x = \sqrt{2} \sin \theta \\
 &dx = \sqrt{2} \cos \theta d\theta \\
 &= 2 \int \sin^2 \theta d\theta = \theta - \sin \theta \cos \theta + C \\
 &= \sin^{-1} \frac{x}{\sqrt{2}} - \frac{x\sqrt{2 - x^2}}{2} + C.
 \end{aligned}$$

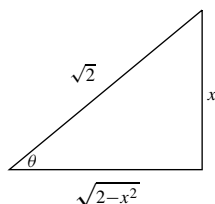


Fig. RT.23

24. We have

$$\begin{aligned}
 I &= \int \tan^4 x \sec x dx \\
 U &= \tan^3 x \quad dV = \tan x \sec x dx \\
 dU &= 3 \tan^2 x \sec^2 x dx \quad V = \sec x \\
 &= \tan^3 x \sec x - 3 \int \tan^2 x \sec^3 x dx \\
 &= \tan^3 x \sec x - 3 \int \tan^2 x (\tan^2 x + 1) \sec x dx \\
 &= \tan^3 x \sec x - 3I - 3J \quad \text{where} \\
 J &= \int \tan^2 x \sec x dx \\
 U &= \tan x \quad dV = \tan x \sec x dx \\
 dU &= \sec^2 x dx \quad V = \sec x \\
 &= \tan x \sec x - \int \sec^3 x dx \\
 &= \tan x \sec x - \int (\tan^2 x + 1) \sec x dx \\
 &= \tan x \sec x - J - \ln |\sec x + \tan x| + C \\
 J &= \frac{1}{2} \tan x \sec x - \frac{1}{2} \ln |\sec x + \tan x| + C. \\
 I &= \frac{1}{4} \tan^3 x \sec x - \frac{3}{8} \tan x \sec x \\
 &\quad + \frac{3}{8} \ln |\sec x + \tan x| + C.
 \end{aligned}$$

$$\begin{aligned}
 25. \quad \int \frac{x^2 dx}{(4x + 1)^{10}} \quad &\text{Let } u = 4x + 1 \\
 &du = 4 dx \\
 &= \frac{1}{4} \int \left( \frac{u - 1}{4} \right)^2 \frac{1}{u^{10}} du \\
 &= \frac{1}{64} \int (u^{-8} - 2u^{-9} + u^{-10}) du \\
 &= -\frac{1}{448} u^{-7} + \frac{1}{256} u^{-8} - \frac{1}{576} u^{-9} + C \\
 &= \frac{1}{64} \left( -\frac{1}{7(4x + 1)^7} + \frac{1}{4(4x + 1)^8} - \frac{1}{9(4x + 1)^9} \right) + C.
 \end{aligned}$$

26. We have

$$\begin{aligned}
 \int x \sin^{-1} \left( \frac{x}{2} \right) dx \\
 U &= \sin^{-1} \left( \frac{x}{2} \right) \quad dV = x dx \\
 dU &= \frac{dx}{\sqrt{4 - x^2}} \quad V = \frac{x^2}{2} \\
 &= \frac{x^2}{2} \sin^{-1} \left( \frac{x}{2} \right) - \frac{1}{2} \int \frac{x^2 dx}{\sqrt{4 - x^2}} \quad \text{Let } x = 2 \sin u \\
 &\quad dx = 2 \cos u du \\
 &= \frac{x^2}{2} \sin^{-1} \left( \frac{x}{2} \right) - 2 \int \sin^2 u du \\
 &= \frac{x^2}{2} \sin^{-1} \left( \frac{x}{2} \right) - \int (1 - \cos 2u) du \\
 &= \frac{x^2}{2} \sin^{-1} \left( \frac{x}{2} \right) - u + \sin u \cos u + C
 \end{aligned}$$

$$= \left( \frac{x^2}{2} - 1 \right) \sin^{-1} \left( \frac{x}{2} \right) + \frac{1}{4} x \sqrt{4 - x^2} + C.$$

27.  $\int \sin^5(4x) dx$

$$= \int (1 - \cos^2 4x)^2 \sin 4x dx \quad \begin{array}{l} \text{Let } u = \cos 4x \\ du = -4 \sin 4x dx \end{array}$$

$$= -\frac{1}{4} \int (1 - 2u^2 + u^4) du$$

$$= -\frac{1}{4} \left( u - \frac{2}{3} u^3 + \frac{1}{5} u^5 \right) + C$$

$$= -\frac{1}{4} \cos 4x + \frac{1}{6} \cos^3 4x - \frac{1}{20} \cos^5 4x + C.$$

28. We have

$$I = \int \frac{dx}{x^5 - 2x^3 + x} = \int \frac{x dx}{x^6 - 2x^4 + x^2} \quad \begin{array}{l} \text{Let } u = x^2 \\ du = 2x dx \end{array}$$

$$= \frac{1}{2} \int \frac{du}{u^3 - 2u^2 + u} = \frac{1}{2} \int \frac{du}{u(u-1)^2}$$

$$\frac{1}{u(u-1)^2} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{(u-1)^2}$$

$$= \frac{A(u^2 - 2u + 1) + B(u^2 - u) + Cu}{u^3 - 2u^2 + u}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ -2A - B + C = 0 \\ A = 1 \end{cases} \Rightarrow A = 1, B = -1, C = 1.$$

$$\frac{1}{2} \int \frac{du}{u^3 - 2u^2 + u} = \frac{1}{2} \int \frac{du}{u} - \frac{1}{2} \int \frac{du}{u-1} + \frac{1}{2} \int \frac{du}{(u-1)^2}$$

$$= \frac{1}{2} \ln |u| - \frac{1}{2} \ln |u-1| - \frac{1}{2} \frac{1}{u-1} + K$$

$$= \frac{1}{2} \ln \frac{x^2}{|x^2 - 1|} - \frac{1}{2(x^2 - 1)} + K.$$

29.  $\int \frac{dx}{2 + e^x}$

$$= \int \frac{e^{-x} dx}{2e^{-x} + 1} \quad \begin{array}{l} \text{Let } u = 2e^{-x} + 1 \\ du = -2e^{-x} dx \end{array}$$

$$= -\frac{1}{2} \int \frac{du}{u} = -\frac{1}{2} \ln(2e^{-x} + 1) + C.$$

30. Let

$$I_n = \int x^n 3^x dx$$

$$\begin{array}{ll} U = x^n & dV = 3^x dx \\ dU = nx^{n-1} dx & V = \frac{3^x}{\ln 3} \end{array}$$

$$= \frac{x^n 3^x}{\ln 3} - \frac{n}{\ln 3} I_{n-1}.$$

$$I_0 = \int 3^x dx = \frac{3^x}{\ln 3} + C.$$

Hence,

$$I_3 = \int x^3 3^x dx$$

$$= \frac{x^3 3^x}{\ln 3} - \frac{3}{\ln 3} \left[ \frac{x^2 3^x}{\ln 3} - \frac{2}{\ln 3} \left( \frac{x 3^x}{\ln 3} - \frac{1}{\ln 3} I_0 \right) \right] + C_1$$

$$= 3^x \left[ \frac{x^3}{\ln 3} - \frac{3x^2}{(\ln 3)^2} + \frac{6x}{(\ln 3)^3} - \frac{6}{(\ln 3)^4} \right] + C_1.$$

31.  $\int \frac{\sin^2 x \cos x}{2 - \sin x} dx$  Let  $u = \sin x$   
 $du = \cos x dx$

$$= \int \frac{u^2 du}{2 - u} \quad \begin{array}{l} \text{Let } 2 - u = v \\ du = -dv \end{array}$$

$$= - \int \frac{4 - 4v + v^2}{v} dv = \int \left( -\frac{4}{v} + 4 - v \right) dv$$

$$= -4 \ln |v| + 4v - \frac{v^2}{2} + C$$

$$= -4 \ln |2 - u| + 4(2 - u) - \frac{1}{2} (2 - u)^2 + C$$

$$= -4 \ln(2 - \sin x) - 2 \sin x - \frac{1}{2} \sin^2 x + C_1.$$

32. We have

$$\int \frac{x^2 + 1}{x^2 + 2x + 2} dx = \int \left( 1 - \frac{2x + 1}{x^2 + 2x + 2} \right) dx$$

$$= x - \int \frac{2x + 1}{(x + 1)^2 + 1} dx \quad \begin{array}{l} \text{Let } u = x + 1 \\ du = dx \end{array}$$

$$= x - \int \frac{2u - 1}{u^2 + 1} du$$

$$= x - \ln |u^2 + 1| + \tan^{-1} u + C$$

$$= x - \ln(x^2 + 2x + 2) + \tan^{-1}(x + 1) + C.$$

33.  $\int \frac{dx}{x^2 \sqrt{1 - x^2}}$  Let  $x = \sin \theta$   
 $dx = \cos \theta d\theta$

$$= \int \frac{\cos \theta d\theta}{\sin^2 \theta \cos \theta} = \int \csc^2 \theta d\theta$$

$$= -\cot \theta + C = -\frac{\sqrt{1 - x^2}}{x} + C.$$

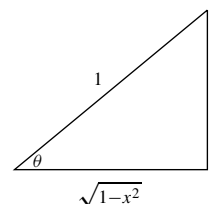


Fig. RT.33

34. We have

$$\begin{aligned} \int x^3 (\ln x)^2 dx \\ U = (\ln x)^2 \quad dV = x^3 dx \\ dU = \frac{2}{x} \ln x dx \quad V = \frac{1}{4} x^4 \\ = \frac{1}{4} x^4 (\ln x)^2 - \frac{1}{2} \int x^3 \ln x dx \\ U = \ln x \quad dV = x^3 dx \\ dU = \frac{1}{x} dx \quad V = \frac{1}{4} x^4 \\ = \frac{1}{4} x^4 (\ln x)^2 - \frac{1}{8} x^4 \ln x + \frac{1}{8} \int x^3 dx \\ = \frac{x^4}{4} \left[ (\ln x)^2 - \frac{1}{2} \ln x + \frac{1}{8} \right] + C. \end{aligned}$$

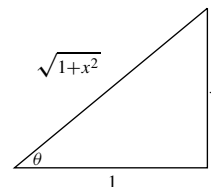


Fig. RT.37

$$\begin{aligned} 35. \int \frac{x^3 dx}{\sqrt{1-4x^2}} \quad \text{Let } 2x = \sin \theta \\ 2dx = \cos \theta d\theta \\ = \frac{1}{16} \int \frac{\sin^3 \theta \cos \theta d\theta}{\cos \theta} = \frac{1}{16} \int (1 - \cos^2 \theta) \sin \theta d\theta \\ = \frac{1}{16} \left( -\cos \theta + \frac{1}{3} \cos^3 \theta \right) + C \\ = \frac{1}{48} (1 - 4x^2)^{3/2} - \frac{1}{16} \sqrt{1-4x^2} + C. \end{aligned}$$

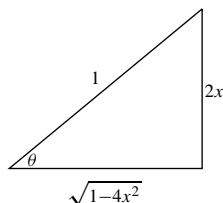


Fig. RT.35

$$\begin{aligned} 36. \int \frac{e^{1/x}}{x^2} dx \quad \text{Let } u = \frac{1}{x} \\ du = -\frac{1}{x^2} dx \\ = - \int e^u du = -e^u + C = -e^{1/x} + C. \end{aligned}$$

$$\begin{aligned} 37. \int \frac{x+1}{\sqrt{x^2+1}} dx \\ = \sqrt{x^2+1} + \int \frac{dx}{\sqrt{x^2+1}} \quad \text{Let } x = \tan \theta \\ dx = \sec^2 \theta d\theta \\ = \sqrt{x^2+1} + \int \sec \theta d\theta \\ = \sqrt{x^2+1} + \ln |\sec \theta + \tan \theta| + C \\ = \sqrt{x^2+1} + \ln(x + \sqrt{x^2+1}) + C. \end{aligned}$$

$$\begin{aligned} 38. \int e^{(x^{1/3})} \quad \text{Let } x = u^3 \\ dx = 3u^2 du \\ = 3 \int u^2 e^u du = 3I_2 \end{aligned}$$

See solution to #16 of Section 6.6 for

$$\begin{aligned} I_n = \int u^n e^u dx = u^n e^u - n I_{n-1}. \\ = 3[u^2 e^u - 2(u e^u - e^u)] + C \\ = e^{(x^{1/3})} (3x^{2/3} - 6x^{1/3} + 6) + C. \end{aligned}$$

$$39. I = \int \frac{x^3 - 3}{x^3 - 9x} dx = \int \left( 1 + \frac{9x - 3}{x^3 - 9x} \right) dx.$$

$$\begin{aligned} \frac{9x - 3}{x^3 - 9x} = \frac{A}{x} + \frac{B}{x - 3} + \frac{C}{x + 3} \\ = \frac{Ax^2 - 9A + Bx^2 + 3Bx + Cx^2 - 3Cx}{x^3 - 9x} \\ \Rightarrow \begin{cases} A + B + C = 0 \\ 3B - 3C = 9 \\ -9A = -3 \end{cases} \Rightarrow \begin{cases} A = 1/3 \\ B = 4/3 \\ C = -5/3. \end{cases} \end{aligned}$$

Thus we have

$$\begin{aligned} I = x + \frac{1}{3} \int \frac{dx}{x} + \frac{4}{3} \int \frac{dx}{x - 3} - \frac{5}{3} \int \frac{dx}{x + 3} \\ = x + \frac{1}{3} \ln |x| + \frac{4}{3} \ln |x - 3| - \frac{5}{3} \ln |x + 3| + K. \end{aligned}$$

$$\begin{aligned} 40. \int \frac{10^{\sqrt{x+2}}}{\sqrt{x+2}} dx \quad \text{Let } u = \sqrt{x+2} \\ du = \frac{dx}{2\sqrt{x+2}} \\ = 2 \int 10^u du = \frac{2}{\ln 10} 10^u + C = \frac{2}{\ln 10} 10^{\sqrt{x+2}} + C. \end{aligned}$$

$$\begin{aligned} 41. \int \sin^5 x \cos^9 x dx \\ = \int (1 - \cos^2 x)^2 \cos^9 x \sin x dx \quad \text{Let } u = \cos x \\ du = -\sin x dx \\ = - \int (1 - 2u^2 + u^4) u^9 du \\ = -\frac{u^{10}}{10} + \frac{u^{12}}{6} - \frac{u^{14}}{14} + C \\ = \frac{\cos^{12} x}{6} - \frac{\cos^{10} x}{10} - \frac{\cos^{14} x}{14} + C. \end{aligned}$$

42. Assume that  $x \geq 1$  and let  $x = \sec u$  and  $dx = \sec u \tan u \, du$ . Then

$$\begin{aligned} \int \frac{x^2 dx}{\sqrt{x^2 - 1}} &= \int \frac{\sec^3 u \tan u \, du}{\tan u} = \int \sec^3 u \, du \\ &= \frac{1}{2} \sec u \tan u + \frac{1}{2} \ln |\sec u + \tan u| + C \\ &= \frac{1}{2} x \sqrt{x^2 - 1} + \frac{1}{2} \ln |x + \sqrt{x^2 - 1}| + C. \end{aligned}$$

Differentiation shows that this solution is valid for  $x \leq -1$  also.

43.  $I = \int \frac{x \, dx}{x^2 + 2x - 1} = \int \frac{(x + 1 - 1) \, dx}{(x + 1)^2 - 2}$  Let  $u = x + 1$   
 $du = dx$   
 $= \int \frac{u - 1}{u^2 - 2} \, du = \frac{1}{2} \ln |u^2 - 2| - \int \frac{du}{u^2 - 2}.$

$$\begin{aligned} \frac{1}{u^2 - 2} &= \frac{A}{u - \sqrt{2}} + \frac{B}{u + \sqrt{2}} \\ &= \frac{Au + \sqrt{2}A + Bu - \sqrt{2}B}{u^2 - 2} \\ \Rightarrow \begin{cases} A + B = 0 \\ \sqrt{2}(A - B) = 1 \end{cases} \\ \Rightarrow A = -B &= \frac{1}{2\sqrt{2}}. \end{aligned}$$

Thus we have

$$\begin{aligned} I &= \frac{1}{2} \ln |u^2 - 2| - \frac{1}{2\sqrt{2}} \ln \left| \frac{u - \sqrt{2}}{u + \sqrt{2}} \right| + K \\ &= \frac{1}{2} \ln |x^2 + 2x - 1| - \frac{1}{2\sqrt{2}} \ln \left| \frac{x + 1 - \sqrt{2}}{x + 1 + \sqrt{2}} \right| + K. \end{aligned}$$

44.  $\int \frac{2x - 3}{\sqrt{4 - 3x + x^2}} \, dx$  Let  $u = 4 - 3x + x^2$   
 $du = (-3 + 2x) \, dx$   
 $= \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{4 - 3x + x^2} + C.$

45.  $\int x^2 \sin^{-1} 2x \, dx$   
 $U = \sin^{-1} 2x \quad dV = x^2 \, dx$   
 $dU = \frac{2 \, dx}{\sqrt{1 - 4x^2}} \quad V = \frac{x^3}{3}$   
 $= \frac{x^3}{3} \sin^{-1} 2x - \frac{2}{3} \int \frac{x^3 \, dx}{\sqrt{1 - 4x^2}}$  Let  $v = 1 - 4x^2$   
 $dv = -8x \, dx$   
 $= \frac{x^3}{3} \sin^{-1} 2x - \frac{2}{3} \int \frac{1 - v}{4v^{1/2}} \left(-\frac{1}{8} dv\right)$   
 $= \frac{x^3}{3} \sin^{-1} 2x + \frac{1}{48} \int (v^{-1/2} - v^{1/2}) \, dv$   
 $= \frac{x^3}{3} \sin^{-1} 2x + \frac{1}{24} \sqrt{v} - \frac{1}{72} v^{3/2} + C$   
 $= \frac{x^3}{3} \sin^{-1} 2x + \frac{1}{24} \sqrt{1 - 4x^2} - \frac{1}{72} (1 - 4x^2)^{3/2} + C.$

46. Let  $\sqrt{3}x = \sec u$  and  $\sqrt{3} \, dx = \sec u \tan u \, du$ . Then

$$\begin{aligned} \int \frac{\sqrt{3x^2 - 1}}{x} \, dx &= \int \frac{\tan u \frac{1}{\sqrt{3}} \sec u \tan u \, du}{\frac{1}{\sqrt{3}} \sec u} \\ &= \int \tan^2 u \, du = \int (\sec^2 u - 1) \, du \\ &= \tan u - u + C = \sqrt{3x^2 - 1} - \sec^{-1}(\sqrt{3}x) + C \\ &= \sqrt{3x^2 - 1} + \sin^{-1} \left( \frac{1}{\sqrt{3}x} \right) + C_1. \end{aligned}$$

47.  $\int \cos^4 x \sin^4 x \, dx = \frac{1}{16} \int \sin^4 2x \, dx$   
 $= \frac{1}{64} \int (1 - \cos 4x)^2 \, dx$   
 $= \frac{1}{64} \int \left( 1 - 2 \cos 4x + \frac{1 + \cos 8x}{2} \right) \, dx$   
 $= \frac{1}{64} \left( \frac{3x}{2} - \frac{\sin 4x}{2} + \frac{\sin 8x}{16} \right) + C$   
 $= \frac{1}{128} \left( 3x - \sin 4x + \frac{\sin 8x}{8} \right) + C.$

48.  $\int \sqrt{x - x^2} \, dx$   
 $= \int \sqrt{\frac{1}{4} - (x - \frac{1}{2})^2} \, dx$  Let  $x - \frac{1}{2} = \frac{1}{2} \sin u$   
 $dx = \frac{1}{2} \cos u \, du$   
 $= \frac{1}{4} \int \cos^2 u \, du = \frac{1}{8} u + \frac{1}{8} \sin u \cos u + C$   
 $= \frac{1}{8} \sin^{-1}(2x - 1) + \frac{1}{4} (2x - 1) \sqrt{x - x^2} + C.$

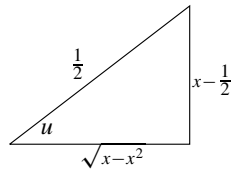


Fig. RT.48

$$\begin{aligned}
 49. \quad & \int \frac{dx}{(4+x)\sqrt{x}} \quad \text{Let } x = u^2 \\
 & \quad \quad \quad dx = 2u \, du \\
 & = \int \frac{2u \, du}{(4+u^2)u} = 2 \int \frac{du}{4+u^2} \\
 & = \frac{2}{2} \tan^{-1} \frac{u}{2} + C = \tan^{-1} \frac{\sqrt{x}}{2} + C.
 \end{aligned}$$

$$\begin{aligned}
 50. \quad & \int x \tan^{-1} \left( \frac{x}{3} \right) dx \\
 & \quad \quad U = \tan^{-1} \left( \frac{x}{3} \right) \quad dV = x \, dx \\
 & \quad \quad dU = \frac{3 \, dx}{9+x^2} \quad V = \frac{x^2}{2} \\
 & = \frac{x^2}{2} \tan^{-1} \left( \frac{x}{3} \right) - \frac{3}{2} \int \frac{x^2}{9+x^2} dx \\
 & = \frac{x^2}{2} \tan^{-1} \left( \frac{x}{3} \right) - \frac{3}{2} \int \left( 1 - \frac{9}{9+x^2} \right) dx \\
 & = \frac{x^2}{2} \tan^{-1} \left( \frac{x}{3} \right) - \frac{3x}{2} + \frac{9}{2} \tan^{-1} \left( \frac{x}{3} \right) + C.
 \end{aligned}$$

$$\begin{aligned}
 51. \quad & I = \int \frac{x^4 - 1}{x^3 + 2x^2} dx \\
 & = \int \frac{x^4 + 2x^3 - 2x^3 - 4x^2 + 4x^2 - 1}{x^3 + 2x^2} dx \\
 & = \int \left( x - 2 + \frac{4x^2 - 1}{x^3 + 2x^2} \right) dx. \\
 & \quad \quad \frac{4x^2 - 1}{x^3 + 2x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} \\
 & \quad \quad = \frac{Ax^2 + 2Ax + Bx + 2B + Cx^2}{x^3 + 2x^2} \\
 & \quad \quad \Rightarrow \begin{cases} A + C = 4 \\ 2A + B = 0 \\ 2B = -1 \end{cases} \Rightarrow \begin{cases} A = 1/4 \\ B = -1/2 \\ C = 15/4 \end{cases}
 \end{aligned}$$

Thus

$$\begin{aligned}
 I &= \frac{x^2}{2} - 2x + \frac{1}{4} \int \frac{dx}{x} - \frac{1}{2} \int \frac{dx}{x^2} + \frac{15}{4} \int \frac{dx}{x+2} \\
 &= \frac{x^2}{2} - 2x + \frac{1}{4} \ln |x| + \frac{1}{2x} + \frac{15}{4} \ln |x+2| + K.
 \end{aligned}$$

52. Let  $u = x^2$  and  $du = 2x \, dx$ ; then we have

$$I = \int \frac{dx}{x(x^2 + 4)^2} = \int \frac{x \, dx}{x^2(x^2 + 4)^2} = \frac{1}{2} \int \frac{du}{u(u+4)^2}.$$

Since

$$\begin{aligned}
 \frac{1}{u(u+4)^2} &= \frac{A}{u} + \frac{B}{u+4} + \frac{C}{(u+4)^2} \\
 &= \frac{A(u^2 + 8u + 16) + B(u^2 + 4u) + Cu}{u(u+4)^2}
 \end{aligned}$$

$$\Rightarrow \begin{cases} A + B = 0 \\ 8A + 4B + C = 0 \\ 16A = 1 \end{cases} \Rightarrow A = \frac{1}{16}, \quad B = -\frac{1}{16}, \quad C = -\frac{1}{4},$$

therefore

$$\begin{aligned}
 I &= \frac{1}{32} \int \frac{du}{u} - \frac{1}{32} \int \frac{du}{u+4} - \frac{1}{8} \int \frac{du}{(u+4)^2} \\
 &= \frac{1}{32} \ln \left| \frac{u}{u+4} \right| + \frac{1}{8} \frac{1}{u+4} + C \\
 &= \frac{1}{32} \ln \left| \frac{x^2}{x^2+4} \right| + \frac{1}{8(x^2+4)} + C.
 \end{aligned}$$

$$\begin{aligned}
 53. \quad & \int \frac{\sin(2 \ln x)}{x} dx \quad \text{Let } u = 2 \ln x \\
 & \quad \quad du = \frac{2}{x} dx
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \int \sin u \, du = -\frac{1}{2} \cos u + C \\
 &= -\frac{1}{2} \cos(2 \ln x) + C.
 \end{aligned}$$

54. Since

$$\begin{aligned}
 I &= \int \frac{\sin(\ln x)}{x^2} dx \\
 & \quad \quad U = \sin(\ln x) \quad dV = \frac{dx}{x^2} \\
 & \quad \quad dU = \frac{\cos(\ln x)}{x} dx \quad V = -\frac{1}{x} \\
 &= -\frac{\sin(\ln x)}{x} + \int \frac{\cos(\ln x)}{x^2} dx \\
 & \quad \quad U = \cos(\ln x) \quad dV = \frac{dx}{x^2} \\
 & \quad \quad dU = -\frac{\sin(\ln x)}{x} dx \quad V = -\frac{1}{x} \\
 &= -\frac{\sin(\ln x)}{x} - \frac{\cos(\ln x)}{x} - I,
 \end{aligned}$$

therefore

$$I = -\frac{1}{2x} [\sin(\ln x) + \cos(\ln x)] + C.$$

$$\begin{aligned}
 55. \quad & \int \frac{e^{2 \tan^{-1} x}}{1+x^2} dx \quad \text{Let } u = 2 \tan^{-1} x \\
 & \quad \quad du = \frac{2 \, dx}{1+x^2} \\
 &= \frac{1}{2} \int e^u \, du = \frac{1}{2} e^u + C = \frac{1}{2} e^{2 \tan^{-1} x} + C.
 \end{aligned}$$

56. We have

$$I = \int \frac{x^3 + x - 2}{x^2 - 7} dx = \int \frac{x^3 - 7x + 8x - 2}{x^2 - 7} dx \\ = \int \left( x + \frac{8x - 2}{x^2 - 7} \right) dx.$$

Since

$$\frac{8x - 2}{x^2 - 7} = \frac{A}{x + \sqrt{7}} + \frac{B}{x - \sqrt{7}} = \frac{(A + B)x + (B - A)\sqrt{7}}{x^2 - 7} \\ \Rightarrow \begin{cases} A + B = 8 \\ B - A = -\frac{2}{\sqrt{7}} \end{cases} \Rightarrow A = 4 + \frac{1}{\sqrt{7}}, \quad B = 4 - \frac{1}{\sqrt{7}},$$

therefore

$$I = \int \left( x + \frac{8x - 2}{x^2 - 7} \right) dx \\ = \frac{x^2}{2} + \left( 4 + \frac{1}{\sqrt{7}} \right) \int \frac{dx}{x + \sqrt{7}} + \left( 4 - \frac{1}{\sqrt{7}} \right) \int \frac{dx}{x - \sqrt{7}} \\ = \frac{x^2}{2} + \left( 4 + \frac{1}{\sqrt{7}} \right) \ln |x + \sqrt{7}| + \left( 4 - \frac{1}{\sqrt{7}} \right) \ln |x - \sqrt{7}| + C.$$

$$57. \int \frac{\ln(3 + x^2)}{3 + x^2} x dx \quad \text{Let } u = \ln(3 + x^2) \\ du = \frac{2x dx}{3 + x^2} \\ = \frac{1}{2} \int u du = \frac{u^2}{4} + C = \frac{1}{4} (\ln(3 + x^2))^2 + C.$$

$$58. \int \cos^7 x dx = \int (1 - \sin^2 x)^3 \cos x dx \quad \text{Let } u = \sin x \\ du = \cos x dx \\ = \int (1 - u^2)^3 du = \int (1 - 3u^2 + 3u^4 - u^6) du \\ = u - u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7 + C \\ = \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C.$$

$$59. \int \frac{\sin^{-1}(x/2)}{(4 - x^2)^{1/2}} dx \quad \text{Let } u = \sin^{-1}(x/2) \\ du = \frac{dx}{2\sqrt{1 - (x^2/4)}} = \frac{dx}{\sqrt{4 - x^2}} \\ = \int u du = \frac{u^2}{2} + C = \frac{1}{2} (\sin^{-1}(x/2))^2 + C.$$

60. We have

$$\int \tan^4(\pi x) dx = \int \tan^2(\pi x) [\sec^2(\pi x) - 1] dx \\ = \int \tan^2(\pi x) \sec^2(\pi x) dx - \int [\sec^2(\pi x) - 1] dx \\ = \frac{1}{3\pi} \tan^3(\pi x) - \frac{1}{\pi} \tan(\pi x) + x + C.$$

$$61. \int \frac{(x + 1) dx}{\sqrt{x^2 + 6x + 10}} \\ = \int \frac{(x + 3 - 2) dx}{\sqrt{(x + 3)^2 + 1}} \quad \text{Let } u = x + 3 \\ du = dx \\ = \int \frac{(u - 2) du}{\sqrt{u^2 + 1}} \\ = \sqrt{u^2 + 1} - 2 \int \frac{du}{\sqrt{u^2 + 1}} \quad \text{Let } u = \tan \theta \\ du = \sec^2 \theta d\theta \\ = \sqrt{x^2 + 6x + 10} - 2 \int \sec \theta d\theta \\ = \sqrt{x^2 + 6x + 10} - 2 \ln |\sec \theta + \tan \theta| + C \\ = \sqrt{x^2 + 6x + 10} - 2 \ln \left( x + 3 + \sqrt{x^2 + 6x + 10} \right) + C.$$

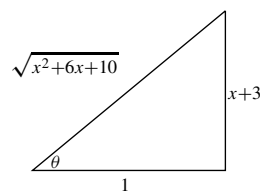


Fig. RT.61

$$62. \int e^x (1 - e^{2x})^{5/2} dx \quad \text{Let } e^x = \sin u \\ e^x dx = \cos u du \\ = \int \cos^6 u du = \left( \frac{1}{2} \right)^3 \int (1 + \cos 2u)^3 du \\ = \frac{1}{8} \int (1 + 3 \cos 2u + 3 \cos^2 2u + \cos^3 2u) du \\ = \frac{u}{8} + \frac{3}{16} \sin 2u + \frac{3}{16} \int (1 + \cos 4u) du + \\ \frac{1}{8} \int (1 - \sin^2 2u) \cos 2u du \\ = \frac{5u}{16} + \frac{3}{16} \sin 2u + \frac{3}{64} \sin 4u + \frac{\sin 2u}{16} \\ - \frac{1}{48} \sin^3 2u + C \\ = \frac{5}{16} \sin^{-1}(e^x) + \frac{1}{4} \sin[2 \sin^{-1}(e^x)] + \\ \frac{3}{64} \sin[4 \sin^{-1}(e^x)] - \frac{1}{48} \sin^3[2 \sin^{-1}(e^x)] + C \\ = \frac{5}{16} \sin^{-1}(e^x) + \frac{1}{2} e^x \sqrt{1 - e^{2x}} \\ + \frac{3}{16} e^x \sqrt{1 - e^{2x}} (1 - 2e^{2x}) \\ - \frac{1}{6} e^{3x} (1 - e^{2x})^{3/2} + C.$$

$$\begin{aligned}
 63. \quad & \int \frac{x^3 dx}{(x^2 + 2)^{7/2}} \quad \text{Let } x = \sqrt{2} \tan \theta \\
 & \quad \quad \quad dx = \sqrt{2} \sec^2 \theta d\theta \\
 & = \int \frac{2\sqrt{2} \tan^3 \theta \sqrt{2} \sec^2 \theta d\theta}{8\sqrt{2} \sec^7 \theta} \\
 & = \frac{1}{2\sqrt{2}} \int \sin^3 \theta \cos^2 \theta d\theta \\
 & = \frac{1}{2\sqrt{2}} \int (1 - \cos^2 \theta) \cos^2 \theta \sin \theta d\theta \quad \text{Let } u = \cos \theta \\
 & \quad \quad \quad du = -\sin \theta d\theta \\
 & = \frac{1}{2\sqrt{2}} \int (u^4 - u^2) du = \frac{1}{2\sqrt{2}} \left( \frac{u^5}{5} - \frac{u^3}{3} \right) + C \\
 & = \frac{1}{2\sqrt{2}} \left( \frac{1}{5} \left( \frac{\sqrt{2}}{\sqrt{2+x^2}} \right)^5 - \frac{1}{3} \left( \frac{\sqrt{2}}{\sqrt{2+x^2}} \right)^3 \right) + C \\
 & = \frac{2}{5(2+x^2)^{5/2}} - \frac{1}{3(2+x^2)^{3/2}} + C.
 \end{aligned}$$

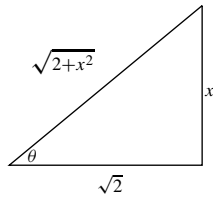


Fig. RT.63

$$\begin{aligned}
 64. \quad & \int \frac{x^2}{2x^2 - 3} dx = \frac{1}{2} \int \left( 1 + \frac{3}{2x^2 - 3} \right) dx \\
 & = \frac{x}{2} + \frac{\sqrt{3}}{4} \int \left( \frac{1}{\sqrt{2}x - \sqrt{3}} - \frac{1}{\sqrt{2}x + \sqrt{3}} \right) dx \\
 & = \frac{x}{2} + \frac{\sqrt{3}}{4\sqrt{2}} \ln \left| \frac{\sqrt{2}x - \sqrt{3}}{\sqrt{2}x + \sqrt{3}} \right| + C.
 \end{aligned}$$

$$\begin{aligned}
 65. \quad & \int \frac{x^{1/2}}{1+x^{1/3}} dx \quad \text{Let } x = u^6 \\
 & \quad \quad \quad dx = 6u^5 du \\
 & = 6 \int \frac{u^8}{u^2 + 1} du \\
 & = 6 \int \frac{u^8 + u^6 - u^6 - u^4 + u^4 + u^2 - u^2 - 1 + 1}{u^2 + 1} du \\
 & = 6 \int \left( u^6 - u^4 + u^2 - 1 + \frac{1}{u^2 + 1} \right) du \\
 & = 6 \left( \frac{u^7}{7} - \frac{u^5}{5} + \frac{u^3}{3} - u + \tan^{-1} u \right) + C \\
 & = \frac{6}{7} x^{7/6} - \frac{6}{5} x^{5/6} + 2\sqrt{x} - 6x^{1/6} + 6 \tan^{-1} x^{1/6} + C.
 \end{aligned}$$

66. We have

$$\int \frac{dx}{x(x^2 + x + 1)^{1/2}}$$

$$\begin{aligned}
 & = \int \frac{dx}{x[(x + \frac{1}{2})^2 + \frac{3}{4}]^{1/2}} \quad \text{Let } x + \frac{1}{2} = \frac{\sqrt{3}}{2} \tan \theta \\
 & \quad \quad \quad dx = \frac{\sqrt{3}}{2} \sec^2 \theta d\theta \\
 & = \int \frac{\frac{\sqrt{3}}{2} \sec^2 \theta d\theta}{\left( \frac{\sqrt{3}}{2} \tan \theta - \frac{1}{2} \right) \left( \frac{\sqrt{3}}{2} \sec \theta \right)} \\
 & = \int \frac{2 \sec \theta d\theta}{\sqrt{3} \tan \theta - 1} = 2 \int \frac{d\theta}{\sqrt{3} \sin \theta - \cos \theta} \\
 & = 2 \int \frac{\sqrt{3} \sin \theta + \cos \theta}{3 \sin^2 \theta - \cos^2 \theta} d\theta \\
 & = 2\sqrt{3} \int \frac{\sin \theta d\theta}{3 \sin^2 \theta - \cos^2 \theta} + 2 \int \frac{\cos \theta d\theta}{3 \sin^2 \theta - \cos^2 \theta} \\
 & = 2\sqrt{3} \int \frac{\sin \theta d\theta}{3 - 4 \cos^2 \theta} + 2 \int \frac{\cos \theta d\theta}{4 \sin^2 \theta - 1} \\
 & \quad \text{Let } u = \cos \theta, du = -\sin \theta d\theta \text{ in the first integral;} \\
 & \quad \text{let } v = \sin \theta, dv = \cos \theta d\theta \text{ in the second integral.} \\
 & = -2\sqrt{3} \int \frac{du}{3 - 4u^2} + 2 \int \frac{dv}{4v^2 - 1} \\
 & = -\frac{\sqrt{3}}{2} \int \frac{du}{\frac{3}{4} - u^2} - \frac{1}{2} \int \frac{dv}{\frac{1}{4} - v^2} \\
 & = -\frac{\sqrt{3}}{2} \left( \frac{1}{2} \right) \left( \frac{2}{\sqrt{3}} \right) \ln \left| \frac{\cos \theta + \frac{\sqrt{3}}{2}}{\cos \theta - \frac{\sqrt{3}}{2}} \right| \\
 & \quad - \frac{1}{2} \left( \frac{1}{2} \right) (2) \ln \left| \frac{\sin \theta + \frac{1}{2}}{\sin \theta - \frac{1}{2}} \right| + C \\
 & = \frac{1}{2} \ln \left| \frac{\left( \cos \theta - \frac{\sqrt{3}}{2} \right) \left( \sin \theta - \frac{1}{2} \right)}{\left( \cos \theta + \frac{\sqrt{3}}{2} \right) \left( \sin \theta + \frac{1}{2} \right)} \right| + C.
 \end{aligned}$$

Since  $\sin \theta = \frac{2x+1}{2\sqrt{x^2+x+1}}$  and  $\cos \theta = \frac{\sqrt{3}}{2\sqrt{x^2+x+1}}$ ,  
therefore

$$\int \frac{dx}{x(x^2 + x + 1)^{1/2}} = \frac{1}{2} \ln \left| \frac{(x+2) - 2\sqrt{x^2+x+1}}{(x+2) + 2\sqrt{x^2+x+1}} \right| + C.$$

$$\begin{aligned}
 67. \quad & \int \frac{1+x}{1+\sqrt{x}} dx \quad \text{Let } x = u^2 \\
 & \quad \quad \quad dx = 2u du \\
 & = 2 \int \frac{u(1+u^2)}{1+u} du \\
 & = 2 \int \frac{u^3 + u^2 - u^2 - u + 2u + 2 - 2}{1+u} du \\
 & = 2 \int \left( u^2 - u + 2 - \frac{2}{1+u} \right) du \\
 & = 2 \left( \frac{u^3}{3} - \frac{u^2}{2} + 2u - 2 \ln|1+u| \right) + C \\
 & = \frac{2}{3} x^{3/2} - x + 4\sqrt{x} - 4 \ln(1+\sqrt{x}) + C.
 \end{aligned}$$

$$\begin{aligned}
 68. \quad & \int \frac{x dx}{4x^4 + 4x^2 + 5} \quad \text{Let } u = x^2 \\
 & \quad \quad \quad du = 2x dx \\
 & = \frac{1}{2} \int \frac{du}{4u^2 + 4u + 5} \\
 & = \frac{1}{2} \int \frac{du}{(2u+1)^2 + 4} \quad \text{Let } w = 2u+1 \\
 & \quad \quad \quad dw = 2du \\
 & = \frac{1}{4} \int \frac{dw}{w^2 + 4} = \frac{1}{8} \tan^{-1} \left( \frac{w}{2} \right) + C \\
 & = \frac{1}{8} \tan^{-1} \left( x^2 + \frac{1}{2} \right) + C.
 \end{aligned}$$

$$\begin{aligned}
 69. \quad & \int \frac{x dx}{(x^2 - 4)^2} \quad \text{Let } u = x^2 - 4 \\
 & \quad \quad \quad du = 2x dx \\
 & = \frac{1}{2} \int \frac{du}{u^2} = -\frac{1}{2u} + C \\
 & = -\frac{1}{2(x^2 - 4)} + C = -\frac{1}{2x^2 - 8} + C.
 \end{aligned}$$

70. Use the partial fraction decomposition

$$\begin{aligned}
 & \frac{1}{x^3 + x^2 + x} = \frac{A}{x} + \frac{Bx + C}{x^2 + x + 1} \\
 & = \frac{A(x^2 + x + 1) + Bx^2 + Cx}{x^3 + x^2 + x} \\
 & \Rightarrow \begin{cases} A + B = 0 \\ A + C = 0 \\ A = 1 \end{cases} \Rightarrow A = 1, B = -1, C = -1.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \int \frac{dx}{x^3 + x^2 + x} \\
 & = \int \frac{dx}{x} - \int \frac{x+1}{x^2 + x + 1} dx \quad \text{Let } u = x + \frac{1}{2} \\
 & \quad \quad \quad du = dx \\
 & = \ln|x| - \int \frac{u + \frac{1}{2}}{u^2 + \frac{3}{4}} du \\
 & = \ln|x| - \frac{1}{2} \ln(x^2 + x + 1) - \frac{1}{\sqrt{3}} \tan^{-1} \left( \frac{2x+1}{\sqrt{3}} \right) + C.
 \end{aligned}$$

$$\begin{aligned}
 71. \quad & \int x^2 \tan^{-1} x dx \\
 & \quad \quad \quad U = \tan^{-1} x \quad dV = x^2 dx \\
 & \quad \quad \quad dU = \frac{dx}{1+x^2} \quad V = \frac{x^3}{3} \\
 & = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3 dx}{1+x^2} \\
 & = \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3 + x - x}{x^2 + 1} dx \\
 & = \frac{x^3}{3} \tan^{-1} x - \frac{1}{6} x^2 + \frac{1}{6} \ln(1+x^2) + C.
 \end{aligned}$$

$$\begin{aligned}
 72. \quad & \int e^x \sec(e^x) dx \quad \text{Let } u = e^x \\
 & \quad \quad \quad du = e^x dx \\
 & = \int \sec u du = \ln|\sec u + \tan u| + C \\
 & = \ln|\sec(e^x) + \tan(e^x)| + C.
 \end{aligned}$$

$$\begin{aligned}
 73. \quad I &= \int \frac{dx}{4 \sin x - 3 \cos x} \quad \text{Let } z = \tan \frac{x}{2}, \quad dx = \frac{2 dz}{1+z^2} \\
 & \quad \quad \quad \cos x = \frac{1-z^2}{1+z^2}, \quad \sin x = \frac{2z}{1+z^2} \\
 & = \int \frac{\frac{2 dz}{1+z^2}}{\frac{8z}{1+z^2} - \frac{3-3z^2}{1+z^2}} \\
 & = 2 \int \frac{dz}{3z^2 + 8z - 3} = 2 \int \frac{dz}{(3z-1)(z+3)}.
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{(3z-1)(z+3)} = \frac{A}{3z-1} + \frac{B}{z+3} \\
 & = \frac{Az + 3A + 3Bz - B}{(3z-1)(z+3)} \\
 & \Rightarrow \begin{cases} A + 3B = 0 \\ 3A - B = 1 \end{cases} \Rightarrow \begin{cases} A = 3/10 \\ B = -1/10. \end{cases}
 \end{aligned}$$

Thus

$$\begin{aligned}
 I &= \frac{3}{5} \int \frac{dz}{3z-1} - \frac{1}{5} \int \frac{dz}{z+3} \\
 & = \frac{1}{5} \ln|3z-1| - \frac{1}{5} \ln|z+3| + C \\
 & = \frac{1}{5} \ln \left| \frac{3 \tan^{-1}(x/2) - 1}{\tan^{-1}(x/2) + 3} \right| + C.
 \end{aligned}$$

$$\begin{aligned}
 74. \quad & \int \frac{dx}{x^{1/3} - 1} \quad \text{Let } x = (u+1)^3 \\
 & \quad \quad \quad dx = 3(u+1)^2 du \\
 & = 3 \int \frac{(u+1)^2}{u} du = 3 \int \left( u + 2 + \frac{1}{u} \right) du \\
 & = 3 \left( \frac{u^2}{2} + 2u + \ln|u| \right) + C \\
 & = \frac{3}{2} (x^{1/3} - 1)^2 + 6(x^{1/3} - 1) + 3 \ln|x^{1/3} - 1| + C.
 \end{aligned}$$

$$\begin{aligned}
 75. \quad & \int \frac{dx}{\tan x + \sin x} \\
 &= \int \frac{\cos x \, dx}{\sin x(1 + \cos x)} \quad \text{Let } z = \tan(x/2), \quad dx = \frac{2 \, dz}{1 + z^2} \\
 & \quad \cos x = \frac{1 - z^2}{1 + z^2}, \quad \sin x = \frac{2z}{1 + z^2} \\
 &= \int \frac{\frac{1 - z^2}{1 + z^2} \frac{2 \, dz}{1 + z^2}}{\frac{2z}{1 + z^2} \left(1 + \frac{1 - z^2}{1 + z^2}\right)} \\
 &= \int \frac{(1 - z^2) \, dz}{z(1 + z^2 + 1 - z^2)} = \frac{1}{2} \int \frac{1 - z^2}{z} \, dz \\
 &= \frac{1}{2} \ln |z| - \frac{z^2}{4} + C \\
 &= \frac{1}{2} \ln \left| \tan \frac{x}{2} \right| - \frac{1}{4} \left( \tan \frac{x}{2} \right)^2 + C.
 \end{aligned}$$

Remark: Since

$$\tan^2 \frac{x}{2} = \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}} = \frac{1 - \cos x}{1 + \cos x},$$

the answer can also be written as

$$\frac{1}{4} \ln \left| \frac{1 - \cos x}{1 + \cos x} \right| - \frac{1}{4} \cdot \frac{1 - \cos x}{1 + \cos x} + C.$$

$$\begin{aligned}
 76. \quad & \int \frac{x \, dx}{\sqrt{3 - 4x - 4x^2}} = \int \frac{x \, dx}{\sqrt{4 - (2x + 1)^2}} \quad \text{Let } u = 2x + 1 \\
 & \quad du = 2 \, dx \\
 &= \frac{1}{4} \int \frac{u - 1}{\sqrt{4 - u^2}} \, du \\
 &= -\frac{1}{4} \sqrt{4 - u^2} - \frac{1}{4} \sin^{-1} \left( \frac{u}{2} \right) + C \\
 &= -\frac{1}{4} \sqrt{3 - 4x - 4x^2} - \frac{1}{4} \sin^{-1} \left( x + \frac{1}{2} \right) + C.
 \end{aligned}$$

$$\begin{aligned}
 77. \quad & \int \frac{\sqrt{x}}{1 + x} \, dx \quad \text{Let } x = u^2 \\
 & \quad dx = 2u \, du \\
 &= 2 \int \frac{u^2 \, du}{1 + u^2} = 2 \int \left( 1 - \frac{1}{1 + u^2} \right) \, du \\
 &= 2 \left( u - \tan^{-1} u \right) + C = 2\sqrt{x} - 2 \tan^{-1} \sqrt{x} + C.
 \end{aligned}$$

$$\begin{aligned}
 78. \quad & \int \sqrt{1 + e^x} \, dx \quad \text{Let } u^2 = 1 + e^x \\
 & \quad 2u \, du = e^x \, dx \\
 &= \int \frac{2u^2 \, du}{u^2 - 1} = \int \left( 2 + \frac{2}{u^2 - 1} \right) \, du \\
 &= \int \left( 2 + \frac{1}{u - 1} - \frac{1}{u + 1} \right) \, du \\
 &= 2u + \ln \left| \frac{u - 1}{u + 1} \right| + C \\
 &= 2\sqrt{1 + e^x} + \ln \left| \frac{\sqrt{1 + e^x} - 1}{\sqrt{1 + e^x} + 1} \right| + C.
 \end{aligned}$$

$$79. \quad I = \int \frac{x^4 \, dx}{x^3 - 8} = \int \left( x + \frac{8x}{x^3 - 8} \right) \, dx.$$

$$\begin{aligned}
 \frac{8x}{x^3 - 8} &= \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 2x + 4} \\
 &= \frac{Ax^2 + 2Ax + 4A + Bx^2 - 2Bx + Cx - 2C}{x^3 - 8} \\
 &\Rightarrow \begin{cases} A + B = 0 \\ 2A - 2B + C = 8 \\ 4A - 2C = 0 \end{cases} \Rightarrow \begin{cases} B = -A \\ C = 2A \\ 6A = 8 \end{cases}
 \end{aligned}$$

Thus  $A = 4/3$ ,  $B = -4/3$ ,  $C = 8/3$ . We have

$$\begin{aligned}
 I &= \frac{x^2}{2} + \frac{4}{3} \int \frac{dx}{x - 2} - \frac{4}{3} \int \frac{x - 2}{x^2 + 2x + 4} \, dx \\
 &= \frac{x^2}{2} + \frac{4}{3} \ln |x - 2| - \frac{4}{3} \int \frac{x + 1 - 3}{(x + 1)^2 + 3} \, dx \\
 &= \frac{x^2}{2} + \frac{4}{3} \ln |x - 2| - \frac{2}{3} \ln(x^2 + 2x + 4) \\
 &\quad + \frac{4}{\sqrt{3}} \tan^{-1} \frac{x + 1}{\sqrt{3}} + K.
 \end{aligned}$$

80. By the procedure used in Example 4 of Section 7.1,

$$\begin{aligned}
 \int e^x \cos x \, dx &= \frac{1}{2} e^x (\sin x + \cos x) + C, \\
 \int e^x \sin x \, dx &= \frac{1}{2} e^x (\sin x - \cos x) + C.
 \end{aligned}$$

Now

$$\begin{aligned}
 & \int x e^x \cos x \, dx \\
 & \quad U = x \quad dV = e^x \cos x \, dx \\
 & \quad dU = dx \quad V = \frac{1}{2} e^x (\sin x + \cos x) \\
 &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} \int e^x (\sin x + \cos x) \, dx \\
 &= \frac{1}{2} x e^x (\sin x + \cos x) \\
 & \quad - \frac{1}{4} e^x (\sin x - \cos x + \sin x + \cos x) + C \\
 &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} e^x \sin x + C.
 \end{aligned}$$

**Other Review Exercises 6 (page 366)**

$$\begin{aligned} 1. \quad & \frac{d}{dx} e^x [(ax+b)\cos x + (cx+d)\sin x] \\ &= e^x [(ax+b)\cos x + (cx+d)\sin x + a\cos x + c\sin x \\ &\quad - (ax+b)\sin x + (cx+d)\cos x] \\ &= e^x [(a+c)x + b + a + d)\cos x \\ &\quad + ((c-a)x + d + c - b)\sin x] \end{aligned}$$

If  $a+c=1$ ,  $b+a+d=0$ ,  $c-a=0$ , and  $d+c-b=0$ , then  $a=c=-d=1/2$  and  $b=0$ . Thus

$$I = \int x e^x \cos x \, dx = \frac{e^x}{2} [x \cos x + (x-1) \sin x] + C.$$

If  $a+c=0$ ,  $b+a+d=0$ ,  $c-a=1$ , and  $d+c-b=0$ , then  $b=c=-a=1/2$  and  $d=0$ . Thus

$$J = \int x e^x \sin x \, dx = \frac{e^x}{2} [x \sin x - (x-1) \cos x] + C.$$

$$\begin{aligned} 2. \quad & \int_0^\infty x^r e^{-x} \, dx \\ &= \lim_{\substack{c \rightarrow 0+ \\ R \rightarrow \infty}} \int_c^R x^r e^{-x} \, dx \\ &\quad U = x^r \quad dV = e^{-x} \, dx \\ &\quad dU = r x^{r-1} \, dx \quad V = -e^{-x} \\ &= \lim_{\substack{c \rightarrow 0+ \\ R \rightarrow \infty}} -x^r e^{-x} \Big|_c^R + r \int_0^\infty x^{r-1} e^{-x} \, dx \\ &= \lim_{c \rightarrow 0+} c^r e^{-c} + r \int_0^\infty x^{r-1} e^{-x} \, dx \end{aligned}$$

because  $\lim_{R \rightarrow \infty} R^r e^{-R} = 0$  for any  $r$ . In order to ensure that  $\lim_{c \rightarrow 0+} c^r e^{-c} = 0$  we must have  $\lim_{c \rightarrow 0+} c^r = 0$ , so we need  $r > 0$ .

$$\begin{aligned} 3. \quad & \int_0^{\pi/2} \csc x \, dx = \lim_{c \rightarrow 0+} -\ln |\csc x + \cot x| \Big|_c^{\pi/2} \\ &= \lim_{c \rightarrow 0+} \ln |\csc c + \cot c| = \infty \text{ (diverges)} \end{aligned}$$

$$\begin{aligned} 4. \quad & \int_1^\infty \frac{dx}{x+x^3} = \lim_{R \rightarrow \infty} \int_1^R \left( \frac{1}{x} - \frac{x}{1+x^2} \right) dx \\ &= \lim_{R \rightarrow \infty} \left( \ln|x| - \frac{1}{2} \ln(1+x^2) \right) \Big|_1^R \\ &= \lim_{R \rightarrow \infty} \frac{1}{2} \left( \ln \frac{R^2}{1+R^2} + \ln 2 \right) = \frac{\ln 2}{2} \end{aligned}$$

$$\begin{aligned} 5. \quad & \int_0^1 \sqrt{x} \ln x \, dx \quad \text{Let } x = u^2 \\ &\quad dx = 2u \, du \\ &= \int_0^1 u(2 \ln u) 2u \, du \\ &= 4 \int_0^1 u^2 \ln u \, du \\ &\quad U = \ln u \quad dV = u^2 \, du \\ &\quad dU = \frac{du}{u} \quad V = \frac{u^3}{3} \\ &= 4 \lim_{c \rightarrow 0+} \left( \frac{u^3}{3} \ln u \Big|_c^1 - \frac{1}{3} \int_c^1 u^2 \, du \right) \\ &= -\frac{4}{3} \lim_{c \rightarrow 0+} c^3 \ln c - \frac{4}{9} (1 - c^3) = -\frac{4}{9} \end{aligned}$$

$$\begin{aligned} 6. \quad & \int_0^1 \frac{dx}{x\sqrt{1-x^2}} > \int_0^1 \frac{dx}{x} = \infty \text{ (diverges)} \\ & \text{Therefore } \int_{-1}^1 \frac{dx}{x\sqrt{1-x^2}} \text{ diverges.} \end{aligned}$$

$$\begin{aligned} 7. \quad & I = \int_0^\infty \frac{dx}{\sqrt{x}e^x} = \int_0^1 + \int_1^\infty = I_1 + I_2 \\ & I_1 = \int_0^1 \frac{dx}{\sqrt{x}e^x} < \int_0^1 \frac{dx}{\sqrt{x}} = 2 \\ & I_2 = \int_1^\infty \frac{dx}{\sqrt{x}e^x} < \int_1^\infty e^{-x} \, dx = \frac{1}{e} \\ & \text{Thus } I \text{ converges, and } I < 2 + (1/e). \end{aligned}$$

$$\begin{aligned} 8. \quad & \text{Volume} = \int_0^{60} A(x) \, dx. \text{ The approximation is} \\ & T_6 = \frac{10}{2} [10,200 + 2(9,200 + 8,000 + 7,100 \\ &\quad + 4,500 + 2,400) + 100] \\ &\quad \approx 364,000 \text{ m}^3. \end{aligned}$$

$$\begin{aligned} 9. \quad & S_6 = \frac{10}{3} [10,200 + 4(9,200 + 7,100 + 2,400) \\ &\quad + 2(8,000 + 4,500) + 100] \\ &\quad \approx 367,000 \text{ m}^3 \end{aligned}$$

$$\begin{aligned} 10. \quad & I = \int_0^1 \sqrt{2 + \sin(\pi x)} \, dx \\ & T_4 = \frac{1}{8} [\sqrt{2} + 2(\sqrt{2 + \sin(\pi/4)} + \sqrt{2 + \sin(\pi/2)} \\ &\quad + \sqrt{2 + \sin(3\pi/4)} + \sqrt{2})] \\ &\quad \approx 1.609230 \\ & M_4 = \frac{1}{4} [\sqrt{2 + \sin(\pi/8)} + \sqrt{2 + \sin(3\pi/8)} \\ &\quad + \sqrt{2 + \sin(5\pi/8)} + \sqrt{2 + \sin(7\pi/8)}] \\ &\quad \approx 1.626765 \\ & I \approx 1.6 \end{aligned}$$

$$\begin{aligned} 11. \quad T_8 &= \frac{1}{2}(T_4 + M_4) \approx 1.617996 \\ S_8 &= \frac{1}{3}(T_4 + 2M_4) \approx 1.62092 \\ I &\approx 1.62 \end{aligned}$$

$$\begin{aligned} 12. \quad I &= \int_{1/2}^{\infty} \frac{x^2}{x^5 + x^3 + 1} dx \quad \text{Let } x = 1/t \\ &\quad dx = -(1/t^2) dt \\ &= \int_0^2 \frac{(1/t^4) dt}{(1/t^5) + (1/t^3) + 1} = \int_0^2 \frac{t dt}{t^5 + t^2 + 1} \\ T_4 &\approx 0.4444 \quad M_4 \approx 0.4799 \\ T_8 &\approx 0.4622 \quad M_8 \approx 0.4708 \\ S_8 &\approx 0.4681 \quad S_{16} \approx 0.4680 \\ I &\approx 0.468 \text{ to 3 decimal places} \end{aligned}$$

$$\begin{aligned} 13. \quad a) \quad T_4 &= 1 \left( \frac{0.730}{2} + 1.001 + 1.332 + 1.729 + \frac{2.198}{2} \right) \\ &= 5.526 \\ S_4 &= \frac{1}{3} \left( 0.730 + 2.198 + 4(1.001 + 1.729) + 2(1.332) \right) \\ &= 5.504. \end{aligned}$$

$$b) \text{ If } T_8 = 5.5095, \text{ then } S_8 = \frac{4T_8 - T_4}{3} = 5.504.$$

c) Yes,  $S_4 = S_8$  suggests that  $S_n$  may be independent of  $n$ , which is consistent with a polynomial of degree not exceeding 3.

### Challenging Problems 6 (page 367)

$$\begin{aligned} 1. \quad a) \quad &\text{Long division of } x^2 + 1 \text{ into } \\ &x^4(1-x)^4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 \text{ yields} \\ &\frac{x^4(1-x)^4}{x^2+1} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}. \end{aligned}$$

Integrating both sides over  $[0, 1]$  leads at once to

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx = \frac{22}{7} - 4 \tan^{-1} 1 = \frac{22}{7} - \pi.$$

$$\begin{aligned} \text{Since } \frac{x^4(1-x)^4}{x^2+1} &> 0 \text{ on } (0, 1), \quad \frac{22}{7} - \pi > 0, \text{ and so} \\ \pi &< \frac{22}{7}. \end{aligned}$$

b) If  $I = \int_0^1 x^4(1-x)^4 dx$ , then since  $1 < x^2 + 1 < 2$  on  $(0, 1)$ , we have

$$I > \int_0^1 \frac{x^4(1-x)^4}{x^2+1} dx > \frac{I}{2}.$$

Thus  $I > (22/7) - \pi > I/2$ , or

$$\frac{22}{7} - I < \pi < \frac{22}{7} - \frac{I}{2}.$$

$$c) \quad I = \int_0^1 (x^8 - 4x^7 + 6x^6 - 4x^5 + x^4) dx = \frac{1}{630}. \text{ Thus}$$

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

$$\begin{aligned} 2. \quad a) \quad I_n &= \int (1-x^2)^n dx \\ U &= (1-x^2)^n \quad dV = dx \\ dU &= -2nx(1-x^2)^{n-1} dx \quad V = x \\ &= x(1-x^2)^n + 2n \int x^2(1-x^2)^{n-1} dx \\ &= x(1-x^2)^n - 2n \int (1-x^2-1)(1-x^2)^{n-1} dx \\ &= x(1-x^2)^n - 2nI_n + 2nI_{n-1}, \quad \text{so} \\ I_n &= \frac{1}{2n+1} x(1-x^2)^n + \frac{2n}{2n+1} I_{n-1}. \end{aligned}$$

b) Let  $J_n = \int_0^1 (1-x^2)^n dx$ . Observe that  $J_0 = 1$ . By (a), if  $n > 0$ , then we have

$$J_n = \frac{x(1-x^2)^n}{2n+1} \Big|_0^1 + \frac{2n}{2n+1} J_{n-1} = \frac{2n}{2n+1} J_{n-1}.$$

Therefore,

$$\begin{aligned} J_n &= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{4}{5} \cdot \frac{2}{3} J_0 \\ &= \frac{[(2n)(2n-2) \cdots (4)(2)]^2}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}. \end{aligned}$$

c) From (a):

$$I_{n-1} = \frac{2n+1}{2n} I_n - \frac{1}{2n} x(1-x^2)^n.$$

Thus

$$\begin{aligned} \int (1-x^2)^{-3/2} dx &= I_{-3/2} \\ &= \frac{2(-1/2)+1}{-1} I_{-1/2} - \frac{1}{-1} x(1-x^2)^{-1/2} \\ &= \frac{x}{\sqrt{1-x^2}}. \end{aligned}$$

3. a)  $x^4 + x^2 + 1 = (x^2 + 1)^2 - x^2 = (x^2 - x + 1)(x^2 + x + 1)$ .  
Thus

$$\begin{aligned} \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx &= \int \frac{x^2 + 1}{(x^2 - x + 1)(x^2 + x + 1)} dx \\ &= \frac{1}{2} \int \left( \frac{1}{x^2 - x + 1} + \frac{1}{x^2 + x + 1} \right) dx \\ &= \frac{1}{2} \int \left( \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}} + \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \right) dx \\ &= \frac{1}{\sqrt{3}} \left( \tan^{-1} \frac{2x - 1}{\sqrt{3}} + \tan^{-1} \frac{2x + 1}{\sqrt{3}} \right) + C. \end{aligned}$$

- b)  $x^4 + 1 = (x^2 + 1)^2 - 2x^2 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)$ .  
Thus

$$\begin{aligned} \int \frac{x^2 + 1}{x^4 + 1} dx &= \int \frac{x^2 + 1}{(x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1)} dx \\ &= \frac{1}{2} \int \left( \frac{1}{x^2 - \sqrt{2}x + 1} + \frac{1}{x^2 + \sqrt{2}x + 1} \right) dx \\ &= \frac{1}{2} \int \left( \frac{1}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{3}{4}} + \frac{1}{\left(x + \frac{\sqrt{2}}{2}\right)^2 + \frac{3}{4}} \right) dx \\ &= \frac{1}{\sqrt{3}} \left( \tan^{-1} \frac{2x - \sqrt{2}}{\sqrt{3}} + \tan^{-1} \frac{2x + \sqrt{2}}{\sqrt{3}} \right) + C. \end{aligned}$$

4.  $I_{m,n} = \int_0^1 x^m (\ln x)^n dx$  Let  $x = e^{-t}$   
 $dx = -e^{-t} dt$

$$\begin{aligned} &= \int_0^\infty e^{-mt} (-t)^n e^{-t} dt \\ &= (-1)^n \int_0^\infty t^n e^{-(m+1)t} dt \quad \text{Let } u = (m+1)t \\ &\quad du = (m+1) dt \\ &= \frac{(-1)^n}{(m+1)^n} \int_0^\infty u^n e^{-u} du \\ &= \frac{(-1)^n}{(m+1)^n} \Gamma(n+1) \quad (\text{see \#50 in Section 7.5}) \\ &= \frac{(-1)^n n!}{(m+1)^n}. \end{aligned}$$

5. a)  $0 < I_n = \int_0^1 x^n e^{-x} dx < \int_0^1 x^n dx = \frac{1}{n+1}$ ,  
because  $0 < e^{-x} < 1$  on  $(0, 1)$ . Thus  $\lim_{n \rightarrow \infty} I_n = 0$   
by the Squeeze Theorem.

b)  $I_0 = \int_0^1 e^{-x} dx = -e^{-x} \Big|_0^1 = 1 - \frac{1}{e}$

$$\begin{aligned} I_n &= \int_0^1 x^n e^{-x} dx \\ U &= x^n \quad dV = e^{-x} dx \\ dU &= nx^{n-1} dx \quad V = -e^{-x} \\ &= -x^n e^{-x} \Big|_0^1 + n \int_0^1 x^{n-1} e^{-x} dx \\ &= nI_{n-1} - \frac{1}{e} \quad \text{if } n \geq 1 \end{aligned}$$

- c) The formula

$$I_n = n! \left( 1 - \frac{1}{e} \sum_{j=0}^n \frac{1}{j!} \right)$$

holds for  $n = 0$  by part (b). Assume that it holds for some integer  $n = k \geq 0$ . Then by (b),

$$\begin{aligned} I_{k+1} &= (k+1)I_k - \frac{1}{e} = (k+1)k! \left( 1 - \frac{1}{e} \sum_{j=0}^k \frac{1}{j!} \right) - \frac{1}{e} \\ &= (k+1)! \left( 1 - \frac{1}{e} \sum_{j=0}^k \frac{1}{j!} - \frac{1}{e(k+1)!} \right) \\ &= (k+1)! \left( 1 - \frac{1}{e} \sum_{j=0}^{k+1} \frac{1}{j!} \right). \end{aligned}$$

Thus the formula holds for all  $n \geq 0$ , by induction.

- d) Since  $\lim_{n \rightarrow \infty} I_n = 0$ , we must have

$$\lim_{n \rightarrow \infty} \left( 1 - \frac{1}{e} \sum_{j=0}^n \frac{1}{j!} \right) = 0.$$

Thus  $e = \lim_{n \rightarrow \infty} \sum_{j=0}^n \frac{1}{j!}$ .

6.  $I = \int_0^1 e^{-Kx} dx = \frac{e^{-Kx}}{-K} \Big|_0^1 = \frac{1}{K} \left( 1 - \frac{1}{e^K} \right)$ .

For very large  $K$ , the value of  $I$  is very small ( $I < 1/K$ ). However,

$$\begin{aligned} T_{100} &= \frac{1}{100} (1 + \cdots) > \frac{1}{100} \\ S_{100} &= \frac{1}{300} (1 + \cdots) > \frac{1}{300} \\ M_{100} &= \frac{1}{100} (e^{-K/200} + \cdots) < \frac{1}{100}. \end{aligned}$$

In each case the  $\dots$  represent terms much less than the first term (shown) in the sum. Evidently  $M_{100}$  is smallest if  $k$  is much greater than 100, and is therefore the best approximation.  $T_{100}$  appears to be the worst.

7. a) Let  $f(x) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$ . Then

$$\int_{-h}^h f(x) dx = 2 \left( \frac{Bh^5}{5} + \frac{Dh^3}{3} + Fh \right).$$

Also

$$\begin{aligned} & 2h \left[ af(-h) + bf(-h/2) + cf(0) + bf(h/2) + af(h) \right] \\ &= 2 \left[ a \left( 2Bh^5 + 2Dh^3 + 2F \right) \right. \\ & \quad \left. + b \left( \frac{2Bh^5}{16} + \frac{2Dh^3}{4} + 2F \right) + cFh \right]. \end{aligned}$$

These expressions will be identical if the coefficients of like powers of  $h$  on the two sides are identical.

Thus

$$2a + \frac{2b}{16} = \frac{1}{5}, \quad 2a + \frac{2b}{4} = \frac{1}{3}, \quad 2a + 2b + c = 1.$$

Solving these equations, we get  $a = 7/90$ ,  $b = 16/45$ , and  $c = 2/15$ . The approximation for the integral of any function  $f$  on  $[m-h, m+h]$  is

$$\begin{aligned} \int_{m-h}^{m+h} f(x) dx &\approx 2h \left[ \frac{7}{90} f(m-h) + \frac{16}{45} f(m - \frac{1}{2}h) \right. \\ & \quad \left. + \frac{2}{15} f(m) + \frac{16}{45} f(m + \frac{1}{2}h) + \frac{7}{90} f(m+h) \right]. \end{aligned}$$

- b) If  $m = h = 1/2$ , we obtain

$$\begin{aligned} \int_0^1 e^{-x} dx &\approx 1 \left[ \frac{7}{90} e^0 + \frac{16}{45} e^{-1/4} + \frac{2}{15} e^{-1/2} \right. \\ & \quad \left. + \frac{16}{45} e^{-3/4} + \frac{7}{90} e^{-1} \right] \\ &\approx 0.63212087501. \end{aligned}$$

With two intervals having  $h = 1/4$  and  $m = 1/4$  and  $m = 3/4$ , we get

$$\begin{aligned} \int_0^1 e^{-x} dx &\approx \frac{1}{2} \left[ \frac{7}{90} e^0 + \frac{16}{45} e^{-1/8} + \frac{2}{15} e^{-1/4} \right. \\ & \quad \left. + \frac{16}{45} e^{-3/8} + \frac{7}{45} e^{-1/2} \right. \\ & \quad \left. + \frac{16}{45} e^{-5/8} + \frac{2}{15} e^{-3/4} + \frac{16}{45} e^{-7/8} + \frac{7}{90} e^{-1} \right] \\ &\approx 0.63212055883. \end{aligned}$$

8. a)  $f'(x) < 0$  on  $[1, \infty)$ , and  $\lim_{x \rightarrow \infty} f(x) = 0$ . Therefore

$$\begin{aligned} \int_1^\infty |f'(x)| dx &= - \int_1^\infty f'(x) dx \\ &= - \lim_{R \rightarrow \infty} \int_1^R f'(x) dx \\ &= \lim_{R \rightarrow \infty} (f(1) - f(R)) = f(1). \end{aligned}$$

Thus

$$\left| \int_R^\infty f'(x) \cos x dx \right| \leq \int_R^\infty |f'(x)| dx \rightarrow 0 \text{ as } R \rightarrow \infty.$$

Thus  $\lim_{R \rightarrow \infty} \int_1^R f'(x) \cos x dx$  exists.

- b)  $\int_1^\infty f(x) \sin x dx$

$$\begin{aligned} U &= f(x) & dV &= \sin x dx \\ dU &= f'(x) dx & V &= -\cos x \\ &= \lim_{R \rightarrow \infty} f(x) \cos x \Big|_1^R + \int_1^\infty f'(x) \cos x dx \\ &= -f(1) \cos(1) + \int_1^\infty f'(x) \cos x dx; \\ &\text{the integral converges.} \end{aligned}$$

- c)  $f(x) = 1/x$  satisfies the conditions of part (a), so

$$\int_1^\infty \frac{\sin x}{x} dx \text{ converges}$$

by part (b). Similarly, it can be shown that

$$\int_1^\infty \frac{\cos(2x)}{x} dx \text{ converges.}$$

But since  $|\sin x| \geq \sin^2 x = \frac{1}{2}(1 - \cos(2x))$ , we have

$$\int_1^\infty \frac{|\sin x|}{x} dx \geq \int_1^\infty \frac{1 - \cos(2x)}{2x} dx.$$

The latter integral diverges because  $\int_1^\infty (1/x) dx$  diverges to infinity while  $\int_1^\infty (\cos(2x))/(2x) dx$  converges. Therefore

$$\int_1^\infty \frac{|\sin x|}{x} dx \text{ diverges to infinity.}$$