CHAPTER 5. INTEGRATION

Section 5.1 Sums and Sigma Notation (page 278)

1. $\displaystyle\sum_{i=1}^{4} i^3 = 1^3 + 2^3 + 3^3 + 4^3$

2. $\displaystyle\sum_{j=1}^{100} \frac{j}{j+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{100}{101}$

3. $\displaystyle\sum_{i=1}^{n} 3^i = 3 + 3^2 + 3^3 + \cdots + 3^n$

4. $\displaystyle\sum_{i=0}^{n-1} \frac{(-1)^i}{i+1} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n-1}}{n}$

5. $\displaystyle\sum_{j=3}^{n} \frac{(-2)^j}{(j-2)^2} = -\frac{2^3}{1^2} + \frac{2^4}{2^2} - \frac{2^5}{3^2} + \cdots + \frac{(-1)^n 2^n}{(n-2)^2}$

6. $\displaystyle\sum_{j=1}^{n} \frac{j^2}{n^3} = \frac{1}{n^3} + \frac{4}{n^3} + \frac{9}{n^3} + \cdots + \frac{n^2}{n^3}$

7. $\displaystyle 5 + 6 + 7 + 8 + 9 = \sum_{i=5}^{9} i$

8. $2 + 2 + 2 + \cdots + 2$ (200 terms) equals $\displaystyle\sum_{i=1}^{200} 2$

9. $\displaystyle 2^2 - 3^2 + 4^2 - 5^2 + \cdots - 99^2 = \sum_{i=2}^{99} (-1)^i i^2$

10. $\displaystyle 1 + 2x + 3x^2 + 4x^3 + \cdots + 100x^{99} = \sum_{i=1}^{100} i x^{i-1}$

11. $\displaystyle 1 + x + x^2 + x^3 + \cdots + x^n = \sum_{i=0}^{n} x^i$

12. $\displaystyle 1 - x + x^2 - x^3 + \cdots + x^{2n} = \sum_{i=0}^{2n} (-1)^i x^i$

13. $\displaystyle 1 - \frac{1}{4} + \frac{1}{9} - \cdots + \frac{(-1)^{n-1}}{n^2} = \sum_{i=1}^{n} \frac{(-1)^{i-1}}{i^2}$

14. $\displaystyle \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \cdots + \frac{n}{2^n} = \sum_{i=1}^{n} \frac{i}{2^i}$

15. $\displaystyle\sum_{j=0}^{99} \sin j = \sum_{i=1}^{100} \sin(i-1)$

16. $\displaystyle\sum_{k=-5}^{m} \frac{1}{k^2+1} = \sum_{i=1}^{m+6} \frac{1}{((i-6)^2+1}$

17. $\displaystyle\sum_{i=1}^{n} (i^2+2i) = \frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} = \frac{n(n+1)(2n+7)}{6}$

18. $\displaystyle\sum_{j=1}^{1,000} (2j+3) = \frac{2(1,000)(1,001)}{2} + 3,000 = 1,004,000$

19. $\displaystyle\sum_{k=1}^{n} (\pi^k - 3) = \frac{\pi(\pi^n - 1)}{\pi - 1} - 3n$

20. $\displaystyle\sum_{i=1}^{n} (2^i - i^2) = 2^{n+1} - 2 - \tfrac{1}{6}n(n+1)(2n+1)$

21. $\displaystyle\sum_{m=1}^{n} \ln m = \ln 1 + \ln 2 + \cdots + \ln n = \ln(n!)$

22. $\displaystyle\sum_{i=0}^{n} e^{i/n} = \frac{e^{(n+1)/n} - 1}{e^{1/n} - 1}$

23. $2 + 2 + \cdots + 2$ (200 terms) equals 400

24. $1 + x + x^2 + \cdots + x^n = \begin{cases} \dfrac{1 - x^{n+1}}{1-x} & \text{if } x \neq 1 \\ n+1 & \text{if } x = 1 \end{cases}$

25. $1 - x + x^2 - x^3 + \cdots + x^{2n} = \begin{cases} \dfrac{1 + x^{2n+1}}{1+x} & \text{if } x \neq -1 \\ 2n+1 & \text{if } x = -1 \end{cases}$

26. Let $f(x) = 1 + x + x^2 + \cdots + x^{100} = \dfrac{x^{101} - 1}{x - 1}$ if $x \neq 1$. Then

$$\begin{aligned} f'(x) &= 1 + 2x + 3x^2 + \cdots + 100x^{99} \\ &= \frac{d}{dx}\frac{x^{101} - 1}{x - 1} = \frac{100x^{101} - 101x^{100} + 1}{(x-1)^2}. \end{aligned}$$

27. $2^2 - 3^2 + 4^2 - 5^2 + \cdots + 98^2 - 99^2$

$$\begin{aligned} &= \sum_{k=1}^{49} [(2k)^2 - (2k+1)^2] = \sum_{k=1}^{49} [4k^2 - 4k^2 - 4k - 1] \\ &= -\sum_{k=1}^{49} [4k+1] = -4\frac{49 \times 50}{2} - 49 = -4,949 \end{aligned}$$

28. Let $s = \dfrac{1}{2} + \dfrac{2}{4} + \dfrac{3}{8} + \cdots + \dfrac{n}{2^n}$. Then

$$\frac{s}{2} = \frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \cdots + \frac{n}{2^{n+1}}.$$

Subtracting these two sums, we get

$$\begin{aligned}\frac{s}{2} &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} - \frac{n}{2^{n+1}} \\ &= \frac{1}{2}\,\frac{1-(1/2^n)}{1-(1/2)} - \frac{n}{2^{n+1}} \\ &= 1 - \frac{n+2}{2^{n+1}}.\end{aligned}$$

Thus $s = 2 + (n+2)/2^n$.

29. $$\begin{aligned}\sum_{i=m}^{n}\Big(f(i+1)-f(i)\Big) &= \sum_{i=m}^{n} f(i+1) - \sum_{i=m}^{n} f(i) \\ &= \sum_{j=m+1}^{n+1} f(j) - \sum_{i=m}^{n} f(i) \\ &= f(n+1) - f(m),\end{aligned}$$
because each sum has only one term that is not cancelled by a term in the other sum. It is called "telescoping" because the sum "folds up" to a sum involving only part of the first and last terms.

30. $\displaystyle\sum_{n=1}^{10}(n^4 - (n-1)^4 = 10^4 - 0^4 = 10,000$

31. $\displaystyle\sum_{j=1}^{m}(2^j - 2^{j-1}) = 2^m - 2^0 = 2^m - 1$

32. $\displaystyle\sum_{i=m}^{2m}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{m} - \frac{1}{2m+1} = \frac{m+1}{m(2m+1)}$

33. $\displaystyle\sum_{j=1}^{m}\frac{1}{j(j+1)} = \sum_{j=1}^{m}\left(\frac{1}{j} - \frac{1}{j+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

34. The number of small shaded squares is $1 + 2 + \cdots + n$. Since each has area 1, the total area shaded is $\sum_{i=1}^{n} i$. But this area consists of a large right-angled triangle of area $n^2/2$ (below the diagonal), and n small triangles (above the diagonal) each of area 1/2. Equating these areas, we get

$$\sum_{i=1}^{n} i = \frac{n^2}{2} + n\frac{1}{2} = \frac{n(n+1)}{2}.$$

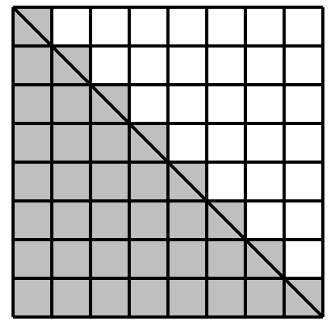

Fig. 5.1.34

35. To show that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$

we write n copies of the identity

$$(k+1)^2 - k^2 = 2k + 1,$$

one for each k from 1 to n:

$$\begin{aligned}2^2 - 1^2 &= 2(1) + 1 \\ 3^2 - 2^2 &= 2(2) + 1 \\ 4^2 - 3^2 &= 2(3) + 1 \\ &\ \vdots \\ (n+1)^2 - n^2 &= 2(n) + 1.\end{aligned}$$

Adding the left and right sides of these formulas we get

$$(n+1)^2 - 1^2 = 2\sum_{i=1}^{n} i + n.$$

Hence, $\sum_{i=1}^{n} i = \dfrac{1}{2}(n^2 + 2n + 1 - 1 - n) = \dfrac{n(n+1)}{2}$.

36. The formula $\sum_{i=1}^{n} i = n(n+1)/2$ holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n =$ some number $k \geq 1$; that is, $\sum_{i=1}^{k} i = k(k+1)/2$. Then for $n = k+1$, we have

$$\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}.$$

Thus the formula also holds for $n = k+1$. By induction, it holds for all positive integers n.

37. The formula $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$ holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n =$ some number $k \ge 1$; that is, $\sum_{i=1}^{k} i^2 = k(k+1)(2k+1)/6$. Then for $n = k+1$, we have

$$\begin{aligned}\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k+1}{6}[2k^2 + k + 6k + 6] \\ &= \frac{k+1}{6}(k+2)(2k+3) \\ &= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}.\end{aligned}$$

Thus the formula also holds for $n = k+1$. By induction, it holds for all positive integers n.

38. The formula $\sum_{i=1}^{n} r^{i-1} = (r^n - 1)/(r-1)$ (for $r \ne 1$) holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n =$ some number $k \ge 1$; that is, $\sum_{i=1}^{k} r^{i-1} = (r^k - 1)/(r-1)$. Then for $n = k+1$, we have

$$\sum_{i=1}^{k+1} r^{i-1} = \sum_{i=1}^{k} r^{i-1} + r^k = \frac{r^k - 1}{r-1} + r^k = \frac{r^{k+1} - 1}{r-1}.$$

Thus the formula also holds for $n = k+1$. By induction, it holds for all positive integers n.

39.

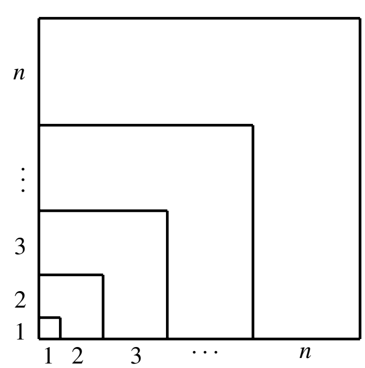

Fig. 5.1.39

The L-shaped region with short side i is a square of side $i(i+1)/2$ with a square of side $(i-1)i/2$ cut out. Since

$$\left(\frac{i(i+1)}{2}\right)^2 - \left(\frac{(i-1)i}{2}\right)^2 = \frac{i^4 + 2i^3 + i^2 - (i^4 - 2i^3 + i^2)}{4} = i^3,$$

that L-shaped region has area i^3. The sum of the areas of the n L-shaped regions is the area of the large square of side $n(n+1)/2$, so

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

40. To show that

$$\sum_{j=1}^{n} j^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4},$$

we write n copies of the identity

$$(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1,$$

one for each k from 1 to n:

$$\begin{aligned}2^4 - 1^4 &= 4(1)^3 + 6(1)^2 + 4(1) + 1 \\ 3^4 - 2^4 &= 4(2)^3 + 6(2)^2 + 4(2) + 1 \\ 4^4 - 3^4 &= 4(3)^3 + 6(3)^2 + 4(3) + 1 \\ &\vdots \\ (n+1)^4 - n^4 &= 4(n)^3 + 6(n)^2 + 4(n) + 1.\end{aligned}$$

Adding the left and right sides of these formulas we get

$$\begin{aligned}(n+1)^4 - 1^4 &= 4\sum_{j=1}^{n} j^3 + 6\sum_{j=1}^{n} j^2 + 4\sum_{j=1}^{n} j + n \\ &= 4\sum_{j=1}^{n} j^3 + \frac{6n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2} + n.\end{aligned}$$

Hence,

$$\begin{aligned}4\sum_{j=1}^{n} j^3 &= (n+1)^4 - 1 - n(n+1)(2n+1) - 2n(n+1) - n \\ &= n^2(n+1)^2\end{aligned}$$

so $\displaystyle\sum_{j=1}^{n} j^3 = \frac{n^2(n+1)^2}{4}$.

41. The formula $\sum_{i=1}^{n} i^3 = n^2(n+1)^2/4$ holds for $n = 1$, since it says $1 = 1$ in this case. Now assume that it holds for $n =$ some number $k \ge 1$; that is, $\sum_{i=1}^{k} i^3 = k^2(k+1)^2/4$. Then for $n = k+1$, we have

$$\begin{aligned}\sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2}{4}[k^2 + 4(k+1)] \\ &= \frac{(k+1)^2}{4}(k+2)^2.\end{aligned}$$

Thus the formula also holds for $n = k + 1$. By induction, it holds for all positive integers n.

42. To find $\sum_{j=1}^{n} j^4 = 1^4 + 2^4 + 3^4 + \cdots + n^4$, we write n copies of the identity

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1,$$

one for each k from 1 to n:

$$\begin{aligned} 2^5 - 1^5 &= 5(1)^4 + 10(1)^3 + 10(1)^2 + 5(1) + 1 \\ 3^5 - 2^5 &= 5(2)^4 + 10(2)^3 + 10(2)^2 + 5(2) + 1 \\ 4^5 - 3^5 &= 5(3)^4 + 10(3)^3 + 10(3)^2 + 5(3) + 1 \\ &\vdots \\ (n+1)^5 - n^5 &= 5(n)^4 + 10(n)^3 + 10(n)^2 + 5(n) + 1. \end{aligned}$$

Adding the left and right sides of these formulas we get

$$(n+1)^5 - 1^5 = 5\sum_{j=1}^{n} j^4 + 10\sum_{j=1}^{n} j^3 + 10\sum_{j=1}^{n} j^2 + 5\sum_{j=1}^{n} j + n.$$

Substituting the known formulas for all the sums except $\sum_{j=1}^{n} j^4$, and solving for this quantity, gives

$$\sum_{j=1}^{n} j^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}.$$

Of course we got Maple to do the donkey work!

43.
$$\sum_{i=1}^{n} i^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$
$$\sum_{i=1}^{n} i^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$
$$\sum_{i=1}^{n} i^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$
$$\sum_{i=1}^{n} i^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \cdots$$

We would guess (correctly) that

$$\sum_{i=1}^{n} i^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \cdots.$$

Section 5.2 Areas as Limits of Sums (page 284)

1. The area is the limit of the sum of the areas of the rectangles shown in the figure. It is

$$\begin{aligned} A &= \lim_{n\to\infty} \frac{1}{n}\left[\frac{3}{n} + \frac{3\times 2}{n} + \frac{3\times 3}{n} + \cdots + \frac{3n}{n}\right] \\ &= \lim_{n\to\infty} \frac{3}{n^2}(1+2+3+\cdots+n) \\ &= \lim_{n\to\infty} \frac{3}{n^2}\cdot\frac{n(n+1)}{2} = \frac{3}{2}\text{sq. units.} \end{aligned}$$

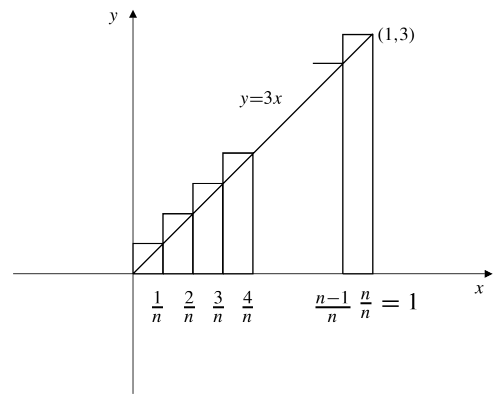

Fig. 5.2.1

2. This is similar to #1; the rectangles now have width $3/n$ and the ith has height $2(3i/n)+1$, the value of $2x+1$ at $x = 3i/n$. The area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(2\frac{3i}{n} + 1\right) \\ &= \lim_{n\to\infty} \frac{18}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}n \\ &= \lim_{n\to\infty} \frac{18}{n^2}\frac{n(n+1)}{2} + 3 = 9 + 3 = 12\text{sq. units.} \end{aligned}$$

3. This is similar to #1; the rectangles have width $(3-1)/n = 2/n$ and the ith has height the value of $2x-1$ at $x = 1 + (2i/n)$. The area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2 + 2\frac{2i}{n} - 1\right) \\ &= \lim_{n\to\infty} \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}n \\ &= \lim_{n\to\infty} \frac{8}{n^2}\frac{n(n+1)}{2} + 2 = 4 + 2 = 6\text{sq. units.} \end{aligned}$$

4. This is similar to #1; the rectangles have width $(2-(-1))/n = 3/n$ and the ith has height the value of $3x+4$ at $x = -1 + (3i/n)$. The area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\left(-3 + 3\frac{3i}{n} + 4\right) \\ &= \lim_{n\to\infty} \frac{27}{n^2}\sum_{i=1}^{n} i + \frac{3}{n}n \\ &= \lim_{n\to\infty} \frac{27}{n^2}\frac{n(n+1)}{2} + 3 = \frac{27}{2} + 3 = \frac{33}{2}\text{sq. units.} \end{aligned}$$

5. The area is the limit of the sum of the areas of the rectangles shown in the figure. It is

$$\begin{aligned} A &= \lim_{n\to\infty} \frac{2}{n}\left[\left(1+\frac{2}{n}\right)^2 + \left(1+\frac{4}{n}\right)^2 + \cdots + \left(1+\frac{2n}{n}\right)^2\right] \\ &= \lim_{n\to\infty} \frac{2}{n}\Big[1 + \frac{4}{n} + \frac{4}{n^2} + 1 + \frac{8}{n} + \frac{16}{n^2} \\ &\qquad + \cdots + 1 + \frac{4n}{n} + \frac{4n^2}{n^2}\Big] \\ &= \lim_{n\to\infty}\left(2 + \frac{8}{n^2}\cdot\frac{n(n+1)}{2} + \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\ &= 2 + 4 + \frac{8}{3} = \frac{26}{3}\text{sq. units.} \end{aligned}$$

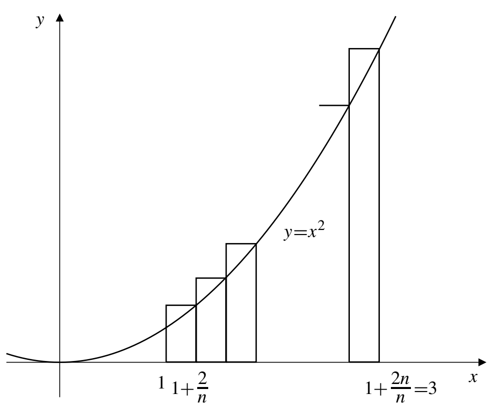

Fig. 5.2.5

6. Divide $[0, a]$ into n equal subintervals of length $\Delta x = \dfrac{a}{n}$ by points $x_i = \dfrac{ia}{n}$, $(0 \le i \le n)$. Then

$$\begin{aligned} S_n &= \sum_{i=1}^{n}\left(\frac{a}{n}\right)\left[\left(\frac{ia}{n}\right)^2 + 1\right] \\ &= \left(\frac{a}{n}\right)^3 \sum_{i=1}^{n} i^2 + \frac{a}{n}\sum_{i=1}^{n}(1) \end{aligned}$$

(Use Theorem 1(a) and 1(c).)

$$\begin{aligned} &= \left(\frac{a}{n}\right)^3 \frac{n(n+1)(2n+1)}{6} + \frac{a}{n}(n) \\ &= \frac{a^3}{6}\frac{(n+1)(2n+1)}{n^2} + a. \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} S_n = \frac{a^3}{3} + a\text{sq. units.}$$

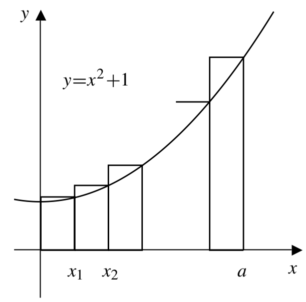

Fig. 5.2.6

7. The required area is (see the figure)

$$\begin{aligned} A &= \lim_{n\to\infty}\frac{3}{n}\Big[\left(-1+\frac{3}{n}\right)^2 + 2\left(-1+\frac{3}{n}\right) + 3 \\ &\qquad + \left(-1+\frac{6}{n}\right)^2 + 2\left(-1+\frac{6}{n}\right) + 3 \\ &\qquad + \cdots + \left(-1+\frac{3n}{n}\right)^2 + 2\left(-1+\frac{3n}{n}\right) + 3\Big] \\ &= \lim_{n\to\infty}\frac{3}{n}\Big[\left(1 - \frac{6}{n} + \frac{3^2}{n^2} - 2 + \frac{6}{n} + 3\right) \\ &\qquad + \left(1 - \frac{12}{n} + \frac{6^2}{n^2} - 2 + \frac{12}{n} + 3\right) \\ &\qquad + \cdots + \left(1 - \frac{6n}{n} + \frac{9n^2}{n^2} - 2 + \frac{6n}{n} + 3\right)\Big] \\ &= \lim_{n\to\infty}\left(6 + \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right) \\ &= 6 + 9 = 15\text{sq. units.} \end{aligned}$$

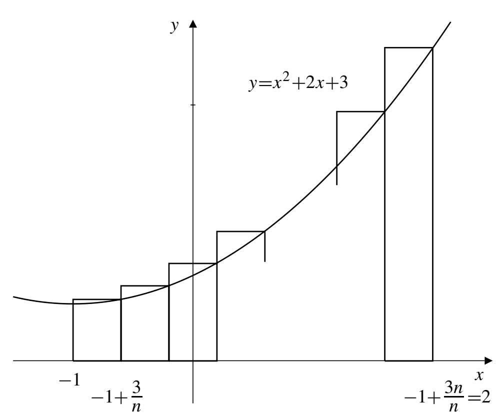

Fig. 5.2.7

8.

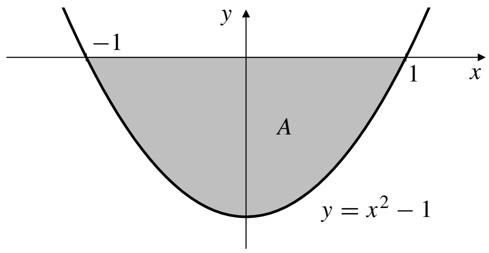

Fig. 5.2.8

The region in question lies between $x = -1$ and $x = 1$ and is symmetric about the y-axis. We can therefore double the area between $x = 0$ and $x = 1$. If we divide this interval into n equal subintervals of width $1/n$ and use the distance $0 - (x^2 - 1) = 1 - x^2$ between $y = 0$ and $y = x^2 - 1$ for the heights of rectangles, we find that the required area is

$$\begin{aligned} A &= 2 \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n}\left(1 - \frac{i^2}{n^2}\right) \\ &= 2 \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{i^2}{n^3}\right) \\ &= 2 \lim_{n\to\infty} \left(\frac{n}{n} - \frac{n(n+1)(2n+1)}{6n^3}\right) = 2 - \frac{4}{6} = \frac{4}{3} \text{ sq. units.} \end{aligned}$$

9.

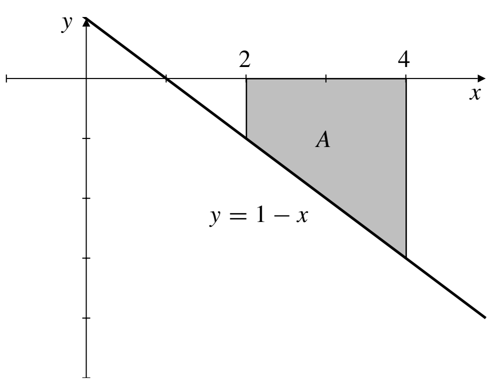

Fig. 5.2.9

The height of the region at position x is $0 - (1 - x) = x - 1$. The "base" is an interval of length 2, so we approximate using n rectangles of width $2/n$. The shaded area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2 + \frac{2i}{n} - 1\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2}{n} + \frac{4i}{n^2}\right) \\ &= \lim_{n\to\infty} \left(\frac{2n}{n} + 4\frac{n(n+1)}{2n^2}\right) = 2 + 2 = 4 \text{ sq. units.} \end{aligned}$$

10.

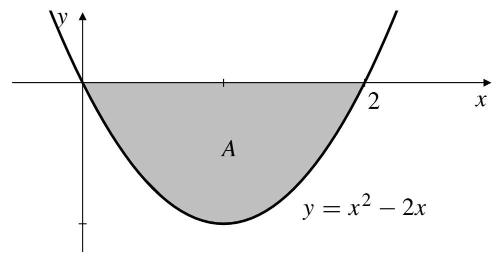

Fig. 5.2.10

The height of the region at position x is $0 - (x^2 - 2x) = 2x - x^2$. The "base" is an interval of length 2, so we approximate using n rectangles of width $2/n$. The shaded area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}\left(2\frac{2i}{n} - \frac{4i^2}{n^2}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8i}{n^2} - \frac{8i^2}{n^3}\right) \\ &= \lim_{n\to\infty} \left(\frac{8}{n^2}\,\frac{n(n+1)}{2} - \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right) \\ &= 4 - \frac{8}{3} = \frac{4}{3} \text{ sq. units.} \end{aligned}$$

11.

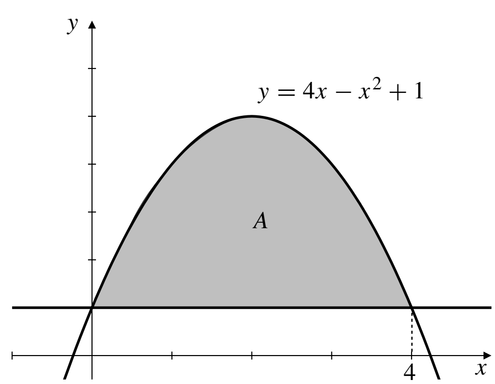

Fig. 5.2.11

The height of the region at position x is $4x - x^2 + 1 - 1 = 4x - x^2$. The "base" is an interval of length 4, so we approximate using n rectangles of width $4/n$. The shaded area is

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n}\left(4\frac{4i}{n} - \frac{16i^2}{n^2}\right) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{64i}{n^2} - \frac{64i^2}{n^3}\right) \\ &= \lim_{n\to\infty} \left(\frac{64}{n^2}\frac{n(n+1)}{2} - \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}\right) \\ &= 32 - \frac{64}{3} = \frac{32}{3} \text{ sq. units.} \end{aligned}$$

12. Divide $[0, b]$ into n equal subintervals of length $\Delta x = \frac{b}{n}$ by points $x_i = \frac{ib}{n}$, $(0 \le i \le n)$. Then

$$\begin{aligned} S_n &= \sum_{i=1}^{n} \frac{b}{n}\left(e^{(ib/n)}\right) = \frac{b}{n}\sum_{i=1}^{n}\left(e^{(b/n)}\right)^i \\ &= \frac{b}{n}e^{(b/n)}\sum_{i=1}^{n}\left(e^{(b/n)}\right)^{i-1} \qquad \text{(Use Thm. 6.1.2(d).)} \\ &= \frac{b}{n}e^{(b/n)}\frac{e^{(b/n)n} - 1}{e^{(b/n)} - 1} \\ &= \frac{b}{n}e^{(b/n)}\frac{e^b - 1}{e^{(b/n)} - 1}. \end{aligned}$$

Let $r = \frac{b}{n}$.

$$\begin{aligned} \text{Area} &= \lim_{n\to\infty} S_n = (e^b - 1)\lim_{r\to 0+} e^r \lim_{r\to 0+}\frac{r}{e^r - 1} \qquad \left[\frac{0}{0}\right] \\ &= (e^b - 1)(1)\lim_{r\to 0+}\frac{1}{e^r} = e^b - 1 \text{sq. units.} \end{aligned}$$

13. The required area is

$$\begin{aligned} A &= \lim_{n\to\infty}\frac{2}{n}\left[2^{-1+(2/n)} + 2^{-1+(4/n)} + \cdots + 2^{-1+(2n/n)}\right] \\ &= \lim_{n\to\infty}\frac{2^{2/n}}{n}\left[1 + \left(2^{2/n}\right) + \left(2^{2/n}\right)^2 + \cdots + \left(2^{2/n}\right)^{n-1}\right] \\ &= \lim_{n\to\infty}\frac{2^{2/n}}{n}\cdot\frac{\left(2^{2/n}\right)^n - 1}{2^{2/n} - 1} \\ &= \lim_{n\to\infty} 2^{2/n} \times 3 \times \frac{1}{n(2^{2/n} - 1)} \\ &= 3\lim_{n\to\infty}\frac{1}{n(2^{2/n} - 1)}. \end{aligned}$$

Now we can use l'Hôpital's rule to evaluate

$$\begin{aligned} \lim_{n\to\infty} n(2^{2/n} - 1) &= \lim_{n\to\infty}\frac{2^{2/n} - 1}{\frac{1}{n}} \qquad \left[\frac{0}{0}\right] \\ &= \lim_{n\to\infty}\frac{2^{2/n}\ln 2\left(\frac{-2}{n^2}\right)}{\frac{-1}{n^2}} \\ &= \lim_{n\to\infty} 2^{(2/n)+1}\ln 2 = 2\ln 2. \end{aligned}$$

Thus the area is $\frac{3}{2\ln 2}$ square units.

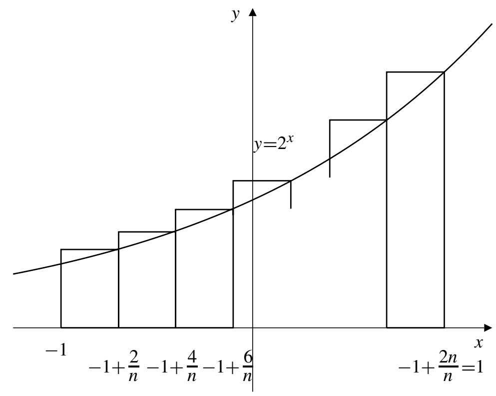

Fig. 5.2.13

14.
$$\begin{aligned} \text{Area} &= \lim_{n\to\infty}\frac{b}{n}\left[\left(\frac{b}{n}\right)^3 + \left(\frac{2b}{n}\right)^3 + \cdots + \left(\frac{nb}{n}\right)^3\right] \\ &= \lim_{n\to\infty}\frac{b^4}{n^4}(1^3 + 2^3 + 3^3 + \cdots + n^3) \\ &= \lim_{n\to\infty}\frac{b^4}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \frac{b^4}{4}\text{sq. units.} \end{aligned}$$

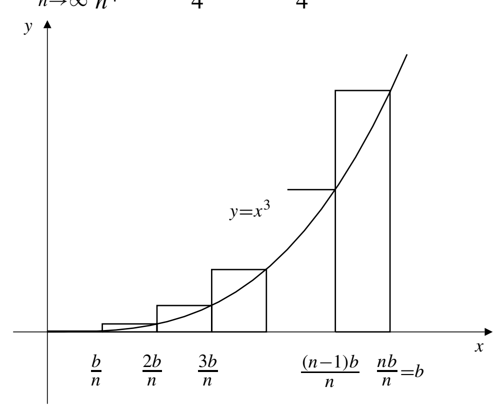

Fig. 5.2.14

15. Let $t = \left(\frac{b}{a}\right)^{1/n}$ and let

$$x_0 = a,\ x_1 = at,\ x_2 = at^2,\ \ldots,\ x_n = at^n = b.$$

The i th subinterval $[x_{i-1}, x_i]$ has length $\Delta x_i = at^{i-1}(t-1)$. Since $f(x_{i-1}) = \dfrac{1}{at^{i-1}}$, we form the sum

$$\begin{aligned} S_n &= \sum_{i=1}^{n} at^{i-1}(t-1)\left(\frac{1}{at^{i-1}}\right) \\ &= n(t-1) = n\left[\left(\frac{b}{a}\right)^{1/n} - 1\right]. \end{aligned}$$

Let $r = \dfrac{1}{n}$ and $c = \dfrac{b}{a}$. The area under the curve is

$$\begin{aligned} A &= \lim_{n\to\infty} S_n = \lim_{r\to 0+} \frac{c^r - 1}{r} \qquad \left[\frac{0}{0}\right] \\ &= \lim_{r\to 0+} \frac{c^r \ln c}{1} = \ln c = \ln\left(\frac{b}{a}\right) \text{ square units.} \end{aligned}$$

This is not surprising because it follows from the *definition* of ln.

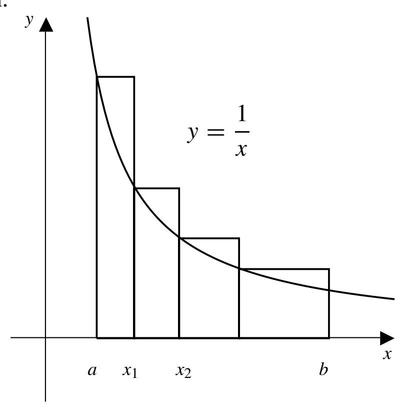

Fig. 5.2.15

16.

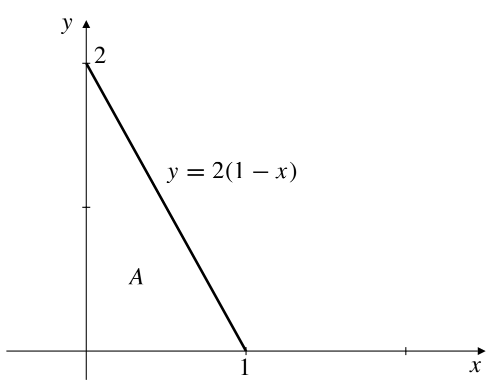

Fig. 5.2.16

$s_n = \displaystyle\sum_{i=1}^{n} \frac{2}{n}\left(1 - \frac{i}{n}\right)$ represents a sum of areas of n rectangles each of width $1/n$ and having heights equal to the height to the graph $y = 2(1-x)$ at the points $x = i/n$. Thus $\lim_{n\to\infty} S_n$ is the area A of the triangle in the figure above, and therefore has the value 1.

17.

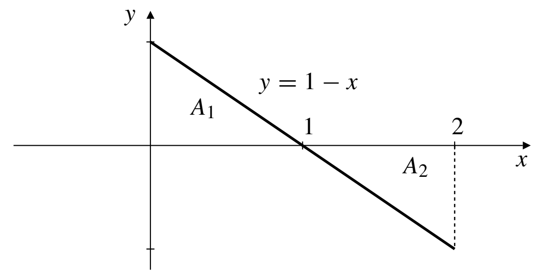

Fig. 5.2.17

$s_n = \displaystyle\sum_{i=1}^{n} \frac{2}{n}\left(1 - \frac{2i}{n}\right)$ represents a sum of areas of n rectangles each of width $2/n$ and having heights equal to the height to the graph $y = 1 - x$ at the points $x = 2i/n$. Half of these rectangles have negative height, and $\lim_{n\to\infty} S_n$ is the difference $A_1 - A_2$ of the areas of the two triangles in the figure above. It has the value 0 since the two triangles have the same area.

18.

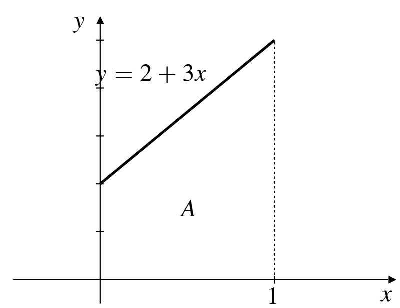

Fig. 5.2.18

$s_n = \displaystyle\sum_{i=1}^{n} \frac{2n+3i}{n^2} = \sum_{i=1}^{n} \frac{1}{n}\left(2 + \frac{3i}{n}\right)$ represents a sum of areas of n rectangles each of width $1/n$ and having heights equal to the height to the graph $y = 2 + 3x$ at the points $x = i/n$. Thus $\lim_{n\to\infty} S_n$ is the area of the trapezoid in the figure above, and has the value $1(2+5)/2 = 7/2$.

19. $$\begin{aligned} S_n &= \sum_{j=1}^{n} \frac{1}{n}\sqrt{1 - \left(\frac{j}{n}\right)^2} \\ &= \text{sum of areas of rectangles in the figure.} \end{aligned}$$

Thus the limit of S_n is the area of a quarter circle of unit radius:

$$\lim_{n\to\infty} S_n = \frac{\pi}{4}.$$

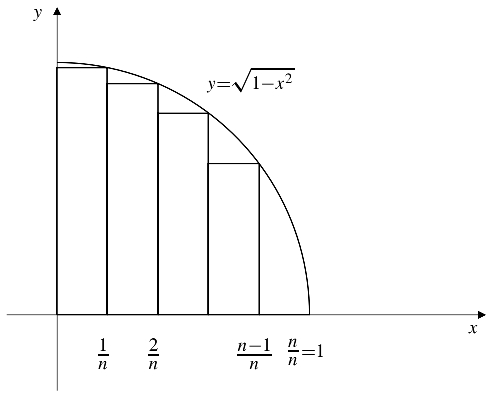

Fig. 5.2.19

Section 5.3 The Definite Integral (page 290)

1. $f(x) = x$ on $[0, 2]$, $n = 8$.

$$P_8 = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 1\right\}$$

$$L(f, P_8) = \frac{2-0}{8}\left[0 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4}\right] = \frac{7}{4}$$

$$U(f, P_8) = \frac{2-0}{8}\left[\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 + \frac{5}{4} + \frac{3}{2} + \frac{7}{4} + 2\right] = \frac{9}{4}$$

2. $f(x) = x^2$ on $[0, 4]$, $n = 4$.

$$L(f, P_4) = \left(\frac{4-0}{4}\right)[0 + (1)^2 + (2)^2 + (3)^2] = 14.$$

$$U(f, P_4) = \left(\frac{4-0}{4}\right)[(1)^2 + (2)^2 + (3)^2 + (4)^2] = 30.$$

3. $f(x) = e^x$ on $[-2, 2]$, $n = 4$.

$$L(f, P_4) = 1(e^{-2} + e^{-1} + e^0 + e^1) = \frac{e^4 - 1}{e^2(e-1)} \approx 4.22$$

$$U(f, P_4) = 1(e^{-1} + e^0 + e^1 + e^2) = \frac{e^4 - 1}{e(e-1)} \approx 11.48.$$

4. $f(x) = \ln x$ on $[1, 2]$, $n = 5$.

$$L(f, P_5) = \left(\frac{2-1}{5}\right)\left[\ln 1 + \ln\frac{6}{5} + \ln\frac{7}{5} + \ln\frac{8}{5} + \ln\frac{9}{5}\right]$$
$$\approx 0.3153168.$$

$$U(f, P_5) = \left(\frac{2-1}{5}\right)\left[\ln\frac{6}{5} + \ln\frac{7}{5} + \ln\frac{8}{5} + \ln\frac{9}{5} + \ln 2\right]$$
$$\approx 0.4539462.$$

5. $f(x) = \sin x$ on $[0, \pi]$, $n = 6$.

$$P_6 = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi\right\}$$

$$L(f, P_6) = \frac{\pi}{6}\left[0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0\right]$$
$$= \frac{\pi}{6}(1 + \sqrt{3}) \approx 1.43,$$

$$U(f, P_6) = \frac{\pi}{6}\left[\frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2}\right]$$
$$= \frac{\pi}{6}(3 + \sqrt{3}) \approx 2.48.$$

6. $f(x) = \cos x$ on $[0, 2\pi]$, $n = 4$.

$$L(f, P_4) = \left(\frac{2\pi}{4}\right)\left[\cos\frac{\pi}{2} + \cos\pi + \cos\pi + \cos\frac{3\pi}{2}\right] = -\pi.$$

$$U(f, P_4) = \left(\frac{2\pi}{4}\right)\left[\cos 0 + \cos\frac{\pi}{2} + \cos\frac{3\pi}{2} + \cos 2\pi\right] = \pi.$$

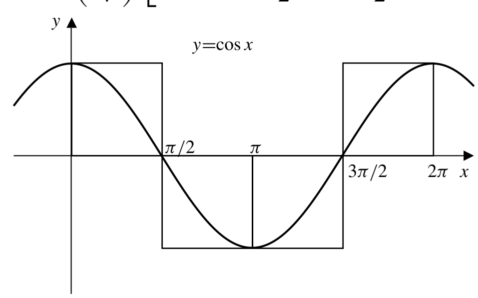

Fig. 5.3.6

7. $f(x) = x$ on $[0, 1]$. $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$. We have

$$L(f, P_n) = \frac{1}{n}\left(0 + \frac{1}{n} + \frac{2}{n} + \cdots + \frac{n-1}{n}\right)$$
$$= \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{n-1}{2n},$$

$$U(f, P_n) = \frac{1}{n}\left(\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \cdots + \frac{n}{n}\right)$$
$$= \frac{1}{n^2} \cdot \frac{n(n+1)n}{2} = \frac{n+1}{2n}.$$

Thus $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = 1/2$.
If P is any partition of $[0, 1]$, then

$$L(f, P) \le U(f, P_n) = \frac{n+1}{2n}$$

for every n, so $L(f, P) \le \lim_{n\to\infty} U(f, P_n) = 1/2$. Similarly, $U(f, P) \ge 1/2$. If there exists any number I such that $L(f, P) \le I \le U(f, P)$ for all P, then I cannot be less than 1/2 (or there would exist a P_n such that $L(f, P_n) > I$), and, similarly, I cannot be greater than 1/2 (or there would exist a P_n such that $U(f, P_n) < I$). Thus $I = 1/2$ and $\int_0^1 x\,dx = 1/2$.

8. $f(x) = 1 - x$ on $[0, 2]$. $P_n = \left\{0, \frac{2}{n}, \frac{4}{n}, \ldots, \frac{2n-2}{n}, \frac{2n}{n}\right\}$. We have

$$\begin{aligned}
L(f, P_n) &= \frac{2}{n}\left(\left(1 - \frac{2}{n}\right) + \left(1 - \frac{4}{n}\right) + \cdots + \left(1 - \frac{2n}{n}\right)\right) \\
&= \frac{2}{n}n - \frac{4}{n^2}\sum_{i=1}^{n} i \\
&= 2 - \frac{4}{n^2}\frac{n(n+1)}{2} = -\frac{2}{n} \to 0 \text{ as } n \to \infty, \\
U(f, P_n) &= \frac{2}{n}\left(\left(1 - \frac{0}{n}\right) + \left(1 - \frac{2}{n}\right) + \cdots + \left(1 - \frac{2n-2}{n}\right)\right) \\
&= \frac{2}{n}n - \frac{4}{n^2}\sum_{i=0}^{n-1} i \\
&= 2 - \frac{4}{n^2}\frac{(n-1)n}{2} = \frac{2}{n} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus $\int_0^2 (1 - x)\,dx = 0$.

9. $f(x) = x^3$ on $[0, 1]$. $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n}\right\}$. We have (using the result of Exercise 51 (or 52) of Section 6.1)

$$\begin{aligned}
L(f, P_n) &= \frac{1}{n}\left(\left(\frac{0}{n}\right)^3 + \left(\frac{1}{n}\right)^3 + \cdots + \left(\frac{n-1}{n}\right)^3\right) \\
&= \frac{1}{n^4}\sum_{i=0}^{n-1} i^3 = \frac{1}{n^4}\frac{(n-1)^2 n^2}{4} \\
&= \frac{1}{4}\left(\frac{n-1}{n}\right)^2 \to \frac{1}{4} \text{ as } n \to \infty, \\
U(f, P_n) &= \frac{1}{n}\left(\left(\frac{1}{n}\right)^3 + \left(\frac{2}{n}\right)^3 + \cdots + \left(\frac{n}{n}\right)^3\right) \\
&= \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{1}{n^4}\frac{n^2(n+1)^2}{4} \\
&= \frac{1}{4}\left(\frac{n+1}{n}\right)^2 \to \frac{1}{4} \text{ as } n \to \infty.
\end{aligned}$$

Thus $\int_0^1 x^3\,dx = \frac{1}{4}$.

10. $f(x) = e^x$ on $[0, 3]$. $P_n = \left\{0, \frac{3}{n}, \frac{6}{n}, \ldots, \frac{3n-3}{n}, \frac{3n}{n}\right\}$. We have (using the result of Exercise 51 (or 52) of Section 6.1)

$$\begin{aligned}
L(f, P_n) &= \frac{3}{n}\left(e^{0/n} + e^{3/n} + e^{6/n} + \cdots + e^{3(n-1)/n}\right) \\
&= \frac{3}{n}\frac{e^{3n/n} - 1}{e^{3/n} - 1} = \frac{3(e^3 - 1)}{n(e^{3/n} - 1)}, \\
U(f, P_n) &= \frac{3}{n}\left(e^{3/n} + e^{6/n} + e^{9/n} + \cdots + e^{3n/n}\right) = e^{3/n}L(f, P_n).
\end{aligned}$$

By l'Hôpital's Rule,

$$\begin{aligned}
\lim_{n\to\infty} n(e^{3/n} - 1) &= \lim_{n\to\infty}\frac{e^{3/n} - 1}{1/n} \\
&= \lim_{n\to\infty}\frac{e^{3/n}(-3/n^2)}{-1/n^2} = \lim_{n\to\infty}\frac{3e^{3/n}}{1} = 3.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n) = e^3 - 1 = \int_0^3 e^x\,dx.$$

11. $\displaystyle \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i}{n}} = \int_0^1 \sqrt{x}\,dx$

12. $\displaystyle \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\sqrt{\frac{i-1}{n}} = \int_0^1 \sqrt{x}\,dx$

13. $\displaystyle \lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\sin\frac{\pi i}{n} = \int_0^{\pi} \sin x\,dx$

14. $\displaystyle \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}\ln\left(1 + \frac{2i}{n}\right) = \int_0^2 \ln(1 + x)\,dx$

15. $\displaystyle \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\tan^{-1}\left(\frac{2i-1}{2n}\right) = \int_0^1 \tan^{-1} x\,dx$

Note that $\dfrac{2i-1}{2n}$ is the midpoint of $\left[\dfrac{i-1}{n}, \dfrac{i}{n}\right]$.

16. $\displaystyle \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2 + i^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{1}{1 + (i/n)^2} = \int_0^1 \frac{dx}{1 + x^2}$

17. Let $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$ where $1 \le i \le n - 1$. Since f is continuous and nondecreasing,

$$\begin{aligned}
L(f, P_n) &= f(a)\Delta x + f(x_1)\Delta x + \\
&\qquad f(x_2)\Delta x + \cdots + f(x_{n-1})\Delta x \\
&= \frac{b-a}{n}\left[f(a) + \sum_{i=1}^{n-1} f(x_i)\right], \\
U(f, P_n) &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + \\
&\qquad f(x_{n-1})\Delta x + f(b)\Delta x \\
&= \frac{b-a}{n}\left[\sum_{i=1}^{n-1} f(x_i) + f(b)\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
&U(f, P_n) - L(f, P_n) \\
&= \frac{b-a}{n}\left[\sum_{i=1}^{n-1} f(x_i) + f(b) - f(a) - \sum_{i=1}^{n-1} f(x_i)\right] \\
&= \frac{(b-a)(f(b) - f(a))}{n}.
\end{aligned}$$

Since

$$\lim_{n\to\infty}[U(f,P_n) - L(f,P-n)] = 0,$$

therefore f must be integrable on $[a,b]$.

18. $P = \{x_0 < x_1 < \cdots < x_n\}$,
$P' = \{x_0 < x_1 < \cdots < x_{j-1} < x' < x_j < \cdots < x_n\}$.
Let m_i and M_i be, respectively, the minimum and maximum values of $f(x)$ on the interval $[x_{i-1}, x_i]$, for $1 \le i \le n$. Then

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$
$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

If m_j' and M_j' are the minimum and maximum values of $f(x)$ on $[x_{j-1}, x']$, and if m_j'' and M_j'' are the corresponding values for $[x', x_j]$, then

$$m_j' \ge m_j, \quad m_j'' \ge m_j, \quad M_j' \le M_j, \quad M_j'' \le M_j.$$

Therefore we have

$$m_j(x_j - x_{j-1}) \le m_j'(x' - x_{j-1}) + m_j''(x_j - x'),$$
$$M_j(x_j - x_{j-1}) \ge M_j'(x' - x_{j-1}) + M_j''(x_j - x').$$

Hence $L(f,P) \le L(f,P')$ and $U(f,P) \ge U(f,P')$.

If P'' is any refinement of P we can add the new points in P'' to those in P one at a time, and thus obtain

$$L(f,P) \le L(f,P''), \qquad U(f,P'') \le U(f,P).$$

Section 5.4 Properties of the Definite Integral (page 296)

1. $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx + \int_c^a f(x)\,dx = \int_a^c f(x)\,dx - \int_a^c f(x)\,dx = 0$$

2. $$\int_0^2 3f(x)\,dx + \int_1^3 3f(x)\,dx - \int_0^3 2f(x)\,dx - \int_1^2 3f(x)\,dx$$
$$= \int_0^1 (3-2)f(x)\,dx + \int_1^2 (3+3-2-3)f(x)\,dx + \int_2^3 (3-2)f(x)\,dx$$
$$= \int_0^3 f(x)\,dx$$

3. $\int_{-2}^{2}(x+2)\,dx = \frac{1}{2}(4)(4) = 8$

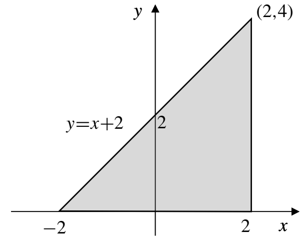

Fig. 5.4.3

4. $\int_0^2 (3x+1)\,dx = \text{shaded area} = \frac{1}{2}(1+7)(2) = 8$

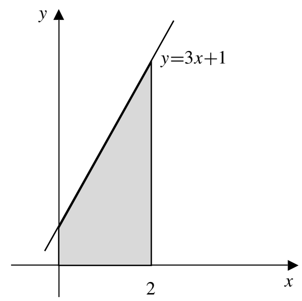

Fig. 5.4.4

5. $\int_a^b x\,dx = \dfrac{b^2}{2} - \dfrac{a^2}{2}$

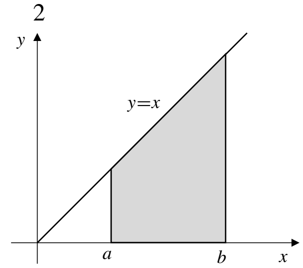

Fig. 5.4.5

6. $\int_{-1}^{2}(1-2x)\,dx = A_1 - A_2 = 0$

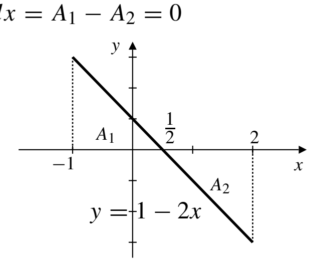

Fig. 5.4.6

7. $\int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{2-t^2}\,dt = \frac{1}{2}\pi(\sqrt{2})^2 = \pi$

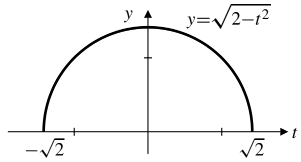

Fig. 5.4.7

8. $\displaystyle\int_{-\sqrt{2}}^{0} \sqrt{2-x^2}\,dx = \text{quarter disk} = \frac{1}{4}\pi(\sqrt{2})^2 = \frac{\pi}{2}$

9. $\displaystyle\int_{-\pi}^{\pi} \sin(x^3)\,dx = 0$. (The integrand is an odd function and the interval of integration is symmetric about $x = 0$.)

10. $\int_{-a}^{a}(a-|s|)\,ds = \text{shaded area} = 2(\frac{1}{2}a^2) = a^2$

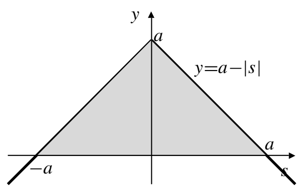

Fig. 5.4.10

11. $\int_{-1}^{1}(u^5 - 3u^3 + \pi)\,du = \pi\int_{-1}^{1} du = 2\pi$

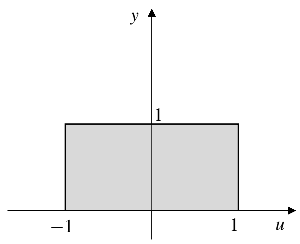

Fig. 5.4.11

12. Let $y = \sqrt{2x-x^2} \Rightarrow y^2 + (x-1)^2 = 1$.

$$\int_0^2 \sqrt{2x-x^2}\,dx = \text{shaded area} = \frac{1}{2}\pi(1)^2 = \frac{\pi}{2}.$$

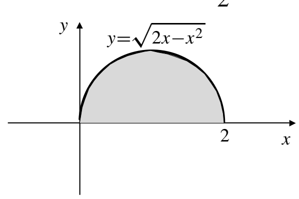

Fig. 5.4.12

13. $\displaystyle\int_{-4}^{4}(e^x - e^{-x})\,dx = 0$ (odd function, symmetric interval)

14. $$\int_{-3}^{3}(2+t)\sqrt{9-t^2}\,dt = 2\int_{-3}^{3}\sqrt{9-t^2}\,dt + \int_{-3}^{3} t\sqrt{9-t^2}\,dt$$
$$= 2\left(\frac{1}{2}\pi 3^2\right) + 0 = 9\pi$$

15. $$\int_0^1 \sqrt{4-x^2}\,dx = \text{area } A_1 \text{ in figure below}$$
$$= \frac{1}{4}\text{area of circle} - \text{area } A_2$$
(see #14 below)
$$= \frac{1}{4}(\pi 2^2) - \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$$

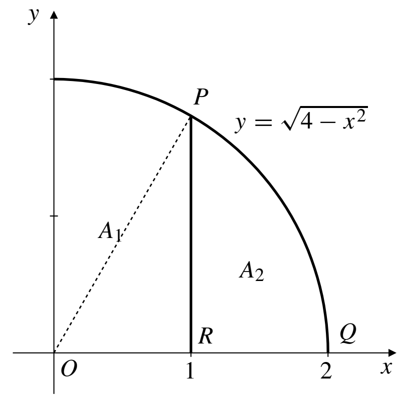

Fig. 5.4.15

16. $$\int_1^2 \sqrt{4-x^2}\,dx = \text{area } A_2 \text{ in figure above}$$
$$= \text{area sector POQ} - \text{area triangle POR}$$
$$= \frac{1}{6}(\pi 2^2) - \frac{1}{2}(1)\sqrt{3}$$
$$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$

17. $\displaystyle\int_0^2 6x^2\,dx = 6\int_0^2 x^2\,dx = 6\frac{3^3}{3} = 16$

18. $$\int_2^3 (x^2-4)\,dx = \int_0^3 x^2\,dx - \int_0^2 x^2\,dx - 4(3-2)$$
$$= \frac{3^3}{3} - \frac{2^3}{3} - 4 = \frac{7}{3}$$

19. $$\int_{-2}^{2}(4-t^2)\,dt = 2\int_0^2 (4-t^2)\,dt$$
$$= 2\left(2(4) - \frac{2^3}{3}\right) = \frac{32}{3}$$

20. $\displaystyle\int_0^2 (v^2 - v)\,dv = \frac{2^3}{3} - \frac{2^2}{2} = \frac{2}{3}$

21. $$\int_0^1 (x^2 + \sqrt{1-x^2})\,dx = \frac{1^3}{3} + \frac{1}{4}(\pi 1^2)$$
$$= \frac{1}{3} + \frac{\pi}{4}$$

22. $$\int_{-6}^{6} x^2(2+\sin x)\,dx = \int_{-6}^{6} 2x^2\,dx + \int_{-6}^{6} x^2 \sin x\,dx$$
$$= 4\int_0^6 x^2\,dx + 0 = \frac{4}{3}(6^3) = 288$$

23. $$\int_1^2 \frac{1}{x}\,dx = \ln 2$$

24. $$\int_2^4 \frac{1}{t}\,dt = \int_1^4 \frac{1}{t}\,dt - \int_1^2 \frac{1}{t}\,dt$$
$$= \ln 4 - \ln 2 = \ln(4/2) = \ln 2$$

25. $$\int_{1/3}^1 \frac{1}{t}\,dt = -\int_1^{1/3} \frac{1}{t}\,dt = -\ln\frac{1}{3} = \ln 3$$

26. $$\int_{1/4}^3 \frac{1}{s}\,ds = \int_1^3 \frac{1}{s}\,ds - \int_1^{1/4} \frac{1}{s}\,ds$$
$$= \ln 3 - \ln\frac{1}{4} = \ln 3 + \ln 4 = \ln 12$$

27. Average $$= \frac{1}{4-0}\int_0^4 (x+2)\,dx$$
$$= \frac{1}{4}\left[\frac{1}{2}(4^2) + 2(4)\right] = 4$$

28. Average $$= \frac{1}{b-a}\int_a^b (x+2)\,dx$$
$$= \frac{1}{b-a}\left[\frac{1}{2}(b^2-a^2) + 2(b-a)\right]$$
$$= \frac{1}{2}(b+a) + 2 = \frac{4+a+b}{2}$$

29. Average $$= \frac{1}{\pi-(-\pi)}\int_{-\pi}^{\pi} (1+\sin t)\,dt$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} 1\,dt + \int_{-\pi}^{\pi} \sin t\,dt\right]$$
$$= \frac{1}{2\pi}[2\pi + 0] = 1$$

30. Average $$= \frac{1}{3-0}\int_0^3 x^2\,dx = \frac{1}{3}\,\frac{3^3}{3} = 3$$

31. Average value $$= \frac{1}{2-0}\int_0^2 (4-x^2)^{1/2}\,dx$$
$$= \frac{1}{2}(\text{shaded area})$$
$$= \frac{1}{2}\left(\frac{1}{4}\pi(2)^2\right) = \frac{\pi}{2}$$

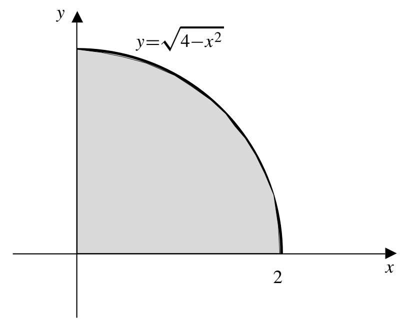

Fig. 5.4.31

32. Average value $$= \frac{1}{2-(1/2)}\int_{1/2}^2 \frac{1}{s}\,ds$$
$$= \frac{2}{3}\left(\ln 2 - \ln\frac{1}{2}\right) = \frac{4}{3}\ln 2$$

33. $\int_{-1}^2 \operatorname{sgn} x\,dx = 2 - 1 = 1$

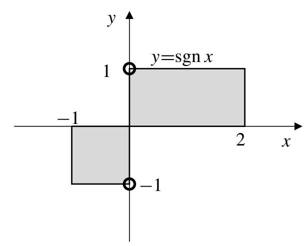

Fig. 5.4.33

34. Let
$$f(x) = \begin{cases} 1+x & \text{if } x < 0 \\ 2 & \text{if } x \ge 0. \end{cases}$$
Then
$$\int_{-3}^2 f(x)\,dx = \text{area}(1) + \text{area}(2) - \text{area}(3)$$
$$= (2\times 2) + \tfrac{1}{2}(1)(1) - \tfrac{1}{2}(2)(2) = 2\tfrac{1}{2}.$$

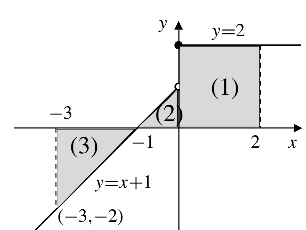

Fig. 5.4.34

35. $$\int_0^2 g(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 x\,dx$$
$$= \frac{1^3}{3} + \frac{2^2-1^2}{2} = \frac{11}{6}$$

36. $\displaystyle\int_0^3 |2-x|\,dx = \int_0^2 (2-x)\,dx + \int_2^3 (x-2)\,dx$

$$= \left(2x - \frac{x^2}{2}\right)\Bigg|_0^2 + \left(\frac{x^2}{2} - 2x\right)\Bigg|_2^3$$

$$= 4 - 2 - 0 + \frac{9}{2} - 6 - 2 + 4 = \frac{5}{2}$$

37. $\displaystyle I = \int_0^2 \sqrt{4-x^2}\,\mathrm{sgn}(x-1)\,dx$

$= \text{area } A_1 \ - \text{ area } A_2.$

Area $A_1 = \frac{1}{6}\pi 2^2 - \frac{1}{2}(1)(\sqrt{3}) = \frac{2}{3}\pi - \frac{1}{2}\sqrt{3}.$

Area $A_2 = \frac{1}{4}\pi 2^2 -$ area $A_1 = \frac{1}{3}\pi + \frac{1}{2}\sqrt{3}.$

Therefore $I = (\pi/3) - \sqrt{3}.$

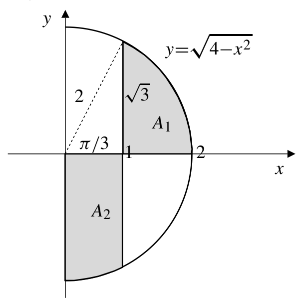

Fig. 5.4.37

38. $\int_0^{3.5} [x]\,dx =$ shaded area $= 1 + 2 + 1.5 = 4.5.$

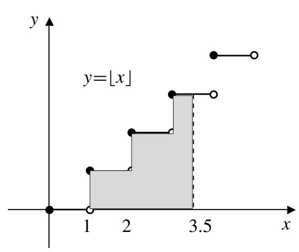

Fig. 5.4.38

39.

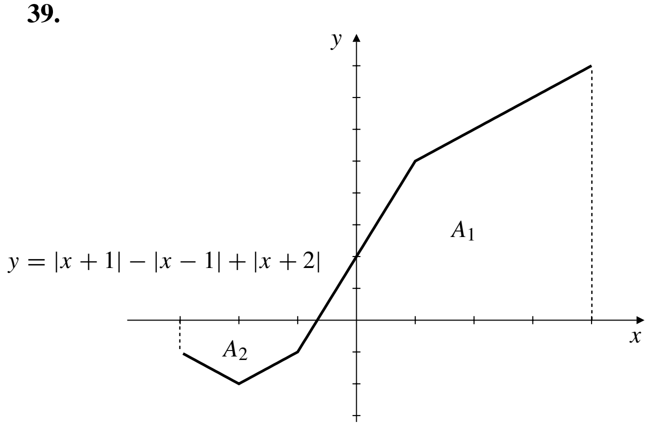

Fig. 5.4.39

$$\int_{-3}^4 (|x+1| - |x-1| + |x+2|)\,dx$$

$= \text{area } A_1 \ - \text{area } A_2$

$$= \frac{1}{2}\frac{5}{3}(5) + \frac{5+8}{2}(3) - \frac{1+2}{2}(1) - \frac{1+2}{2}(1) - \frac{1}{2}\frac{1}{3}(1) = \frac{41}{2}$$

40.

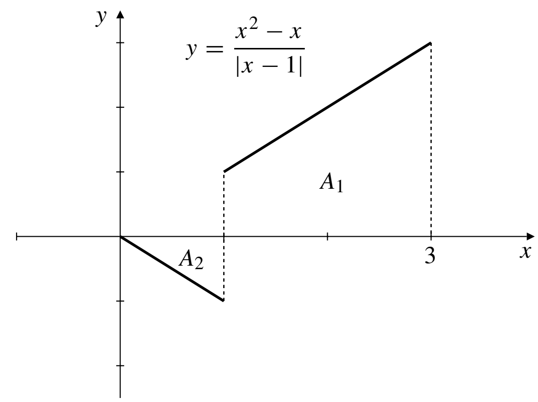

Fig. 5.4.40

$$\int_0^3 \frac{x^2-x}{|x-1|}\,dx$$

$= \text{area } A_1 \ - \text{area } A_2$

$$= \frac{1+3}{2}(2) - \frac{1}{2}(1)(1) = \frac{7}{2}$$

41. Average $\displaystyle = \frac{1}{4}\int_{-2}^2 |x+1|\,\mathrm{sgn}\,x\,dx$

$$= \frac{1}{4}\left(\int_0^2 (x+1)\,dx - \int_{-2}^0 |x+1|\,dx\right)$$

$$= \frac{1}{4}\left(\frac{1+3}{2}\times 2 - 2\times\frac{1}{2}\times 1\times 1\right)$$

$$= 1 - \frac{1}{4} = \frac{3}{4}.$$

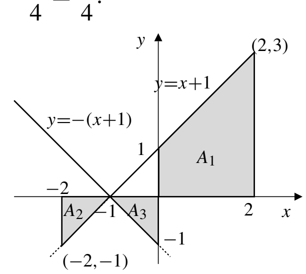

Fig. 5.4.41

42. $$\int_a^b \left(f(x) - \bar{f}\right) dx = \int_a^b f(x)\,dx - \int_a^b \bar{f}\,dx = (b-a)\bar{f} - \bar{f}\int_a^b dx = (b-a)\bar{f} - (b-a)\bar{f} = 0$$

43. $$\int_a^b \left(f(x) - k\right)^2 dx = \int_a^b \left(f(x)\right)^2 dx - 2k\int_a^b f(x)\,dx + k^2\int_a^b dx = \int_a^b \left(f(x)\right)^2 dx - 2k(b-a)\bar{f} + k^2(b-a) = (b-a)(k-\bar{f})^2 + \int_a^b \left(f(x)\right)^2 dx - (b-a)\bar{f}^2$$

This is minimum if $k = \bar{f}$.

Section 5.5 The Fundamental Theorem of Calculus (page 301)

1. $$\int_0^2 x^3\,dx = \frac{x^4}{4}\bigg|_0^2 = \frac{16-0}{4} = 4$$

2. $$\int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\bigg|_0^4 = \frac{16}{3}$$

3. $$\int_{1/2}^1 \frac{1}{x^2}\,dx = \frac{-1}{x}\bigg|_{1/2}^1 = -1-(-2) = 1$$

4. $$\int_{-2}^{-1}\left(\frac{1}{x^2} - \frac{1}{x^3}\right) dx = \left(-\frac{1}{x} + \frac{1}{2x^2}\right)\bigg|_{-2}^{-1} = 1 + \frac{1}{2} - \left(\frac{1}{2} + \frac{1}{8}\right) = \frac{7}{8}$$

5. $$\int_{-1}^2 (3x^2 - 4x + 2)\,dx = (x^3 - 2x^2 + 2x)\bigg|_{-1}^2 = 9$$

6. $$\int_1^2 \left(\frac{2}{x^3} - \frac{x^3}{2}\right) dx = \left(-\frac{1}{x^2} - \frac{x^4}{8}\right)\bigg|_1^2 = -9/8$$

7. $$\int_{-2}^2 (x^2+3)^2\,dx = 2\int_0^2 (x^4 + 6x^2 + 9)\,dx = 2\left(\frac{x^5}{5} + 2x^3 + 9x\right)\bigg|_0^2 = 2\left(\frac{32}{5} + 16 + 18\right) = \frac{404}{5}$$

8. $$\int_4^9 \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx = \frac{2}{3}x^{3/2} - 2\sqrt{x}\bigg|_4^9 = \left[\frac{2}{3}(9)^{3/2} - 2\sqrt{9}\right] - \left[\frac{2}{3}(4)^{3/2} - 2\sqrt{4}\right] = \frac{32}{3}$$

9. $$\int_{-\pi/4}^{-\pi/6} \cos x\,dx = \sin x\bigg|_{-\pi/4}^{-\pi/6} = -\frac{1}{2} + \frac{1}{\sqrt{2}} = \frac{2-\sqrt{2}}{2\sqrt{2}}$$

10. $$\int_0^{\pi/3} \sec^2\theta\,d\theta = \tan\theta\bigg|_0^{\pi/3} = \tan\frac{\pi}{3} = \sqrt{3}$$

11. $$\int_{\pi/4}^{\pi/3} \sin\theta\,d\theta = -\cos\theta\bigg|_{\pi/4}^{\pi/3} = \frac{\sqrt{2}-1}{2}$$

12. $$\int_0^{2\pi} (1 + \sin u)\,du = (u - \cos u)\bigg|_0^{2\pi} = 2\pi$$

13. $$\int_{-\pi}^{\pi} e^x\,dx = e^x\bigg|_{-\pi}^{\pi} = e^{\pi} - e^{-\pi}$$

14. $\displaystyle\int_{-2}^2 (e^x - e^{-x})\,dx = 0$ (odd function, symmetric interval)

15. $$\int_0^e a^x\,dx = \frac{a^x}{\ln a}\bigg|_0^e = \frac{a^e - 1}{\ln a}$$

16. $$\int_{-1}^1 2^x\,dx = \frac{2^x}{\ln 2}\bigg|_{-1}^1 = \frac{2}{\ln 2} - \frac{1}{2\ln 2} = \frac{3}{2\ln 2}$$

17. $$\int_{-1}^1 \frac{dx}{1+x^2} = \tan^{-1}x\bigg|_{-1}^1 = \frac{\pi}{2}$$

18. $$\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\bigg|_0^{1/2} = \frac{\pi}{6}$$

19. $$\int_{-1}^1 \frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}\bigg|_{-1}^1 = \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right) = \frac{\pi}{3}$$

20. $$\int_{-2}^0 \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2}\bigg|_{-2}^0 = 0 - \frac{1}{2}\tan^{-1}(-1) = \frac{\pi}{8}$$

21. Area $R = \int_0^1 x^4\,dx = \dfrac{x^5}{5}\bigg|_0^1 = \dfrac{1}{5}$ sq. units.

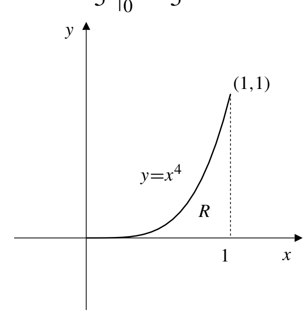

Fig. 5.5.21

22. $$\text{Area} = \int_e^{e^2} \frac{1}{x}\,dx = \ln x \Big|_e^{e^2}$$
$$= \ln e^2 - \ln e = 2 - 1 = 1 \text{ sq. units.}$$

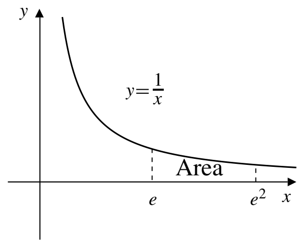

Fig. 5.5.22

23. $$\text{Area } R = -\int_0^4 (x^2 - 4x)\,dx$$
$$= -\left(\frac{x^3}{3} - 2x^2\right)\Big|_0^4$$
$$= -\left(\frac{64}{3} - 32\right) = \frac{32}{3} \text{sq. units.}$$

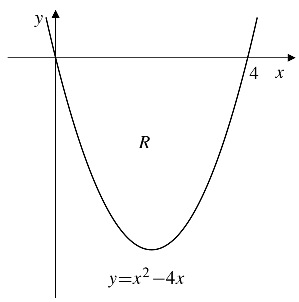

Fig. 5.5.23

24. Since $y = 5 - 2x - 3x^2 = (5 + 3x)(1 - x)$, therefore $y = 0$ at $x = -\frac{5}{3}$ and 1, and $y > 0$ if $-\frac{5}{3} < x < 1$. Thus, the area is

$$\int_{-1}^{1} (5 - 2x - 3x^2)\,dx = 2\int_0^1 (5 - 3x^2)\,dx$$
$$= 2(5x - x^3)\Big|_0^1$$
$$= 2(5 - 1) = 8 \text{ sq. units.}$$

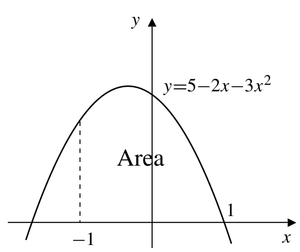

Fig. 5.5.24

25. For intersection of $y = x^2 - 3x + 3$ and $y = 1$, we have

$$x^2 - 3x + 3 = 1$$
$$x^2 - 3x + 2 = 0$$
$$(x - 2)(x - 1) = 0.$$

Thus $x = 1$ or $x = 2$. The indicated region has area

$$\text{Area } R = 1 - \int_1^2 (x^2 - 3x + 3)\,dx$$
$$= 1 - \left(\frac{x^3}{3} - \frac{3x^2}{2} + 3x\right)\Big|_1^2$$
$$= 1 - \left(\frac{8}{3} - 6 + 6 - \left[\frac{1}{3} - \frac{3}{2} + 3\right]\right) = \frac{1}{6}\text{sq. units.}$$

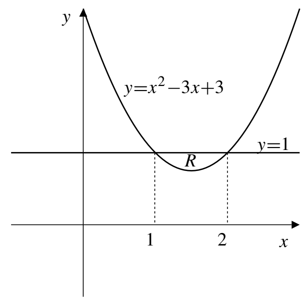

Fig. 5.5.25

26. Since $y = \sqrt{x}$ and $y = \dfrac{x}{2}$ intersect where $\sqrt{x} = \dfrac{x}{2}$, that is, at $x = 0$ and $x = 4$, thus,

$$\text{Area} = \int_0^4 \sqrt{x}\,dx - \int_0^4 \frac{x}{2}\,dx$$
$$= \frac{2}{3}x^{3/2}\Big|_0^4 - \frac{x^2}{4}\Big|_0^4$$
$$= \frac{16}{3} - \frac{16}{4} = \frac{4}{3} \text{ sq. units.}$$

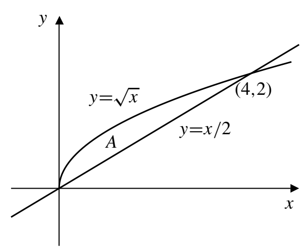

Fig. 5.5.26

27. $$\text{Area } R = 2 \times \text{shaded area}$$
$$= 2\left(\frac{1}{2} - \int_0^1 x^2\,dx\right)$$
$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}\text{sq. units.}$$

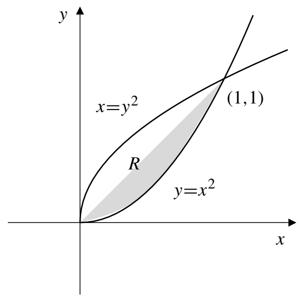

Fig. 5.5.27

28. The two graphs intersect at $(\pm 3, 3)$, thus

$$\begin{aligned}\text{Area} &= 2\int_0^3 (12 - x^2)\,dx - 2\int_0^3 x\,dx \\ &= 2\left(12x - \frac{1}{3}x^3\right)\bigg|_0^3 - 2\left(\frac{1}{2}x^2\right)\bigg|_0^3 \\ &= 2(36 - 9) - 9 = 45 \text{ sq. units.}\end{aligned}$$

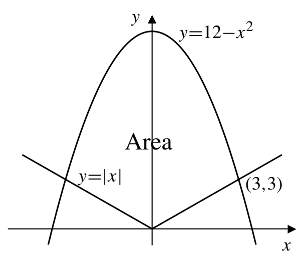

Fig. 5.5.28

29. $$\begin{aligned}\text{Area } R &= \int_0^1 x^{1/3}\,dx - \int_0^1 x^{1/2}\,dx \\ &= \frac{3}{4}x^{4/3}\bigg|_0^1 - \frac{2}{3}x^{3/2}\bigg|_0^1 = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}\text{sq. units.}\end{aligned}$$

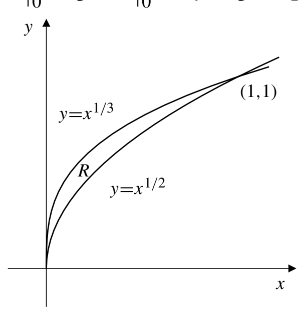

Fig. 5.5.29

30. Area $= \int_{-a}^0 e^{-x}\,dx = -e^{-x}\bigg|_{-a}^0 = e^a - 1$ sq. units.

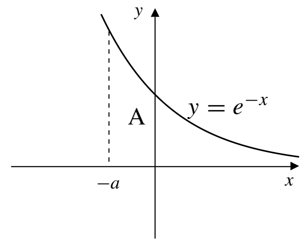

Fig. 5.5.30

31. $$\begin{aligned}\text{Area R} &= \int_0^{2\pi} (1 - \cos x)\,dx \\ &= (x - \sin x)\bigg|_0^{2\pi} = 2\pi\,\text{sq. units.}\end{aligned}$$

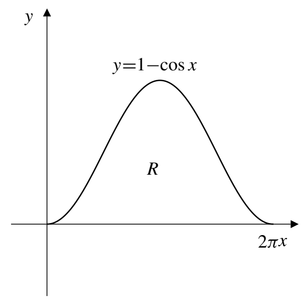

Fig. 5.5.31

32. $$\begin{aligned}\text{Area} &= \int_1^{27} x^{-1/3}\,dx = \frac{3}{2}x^{2/3}\bigg|_1^{27} \\ &= \frac{3}{2}(27)^{2/3} - \frac{3}{2} = 12 \text{ sq. units.}\end{aligned}$$

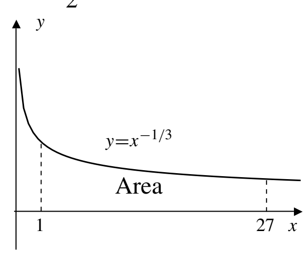

Fig. 5.5.32

33. $$\begin{aligned}\int_0^{3\pi/2} |\cos x|\,dx &= \int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{3\pi/2} \cos x\,dx \\ &= \sin x\bigg|_0^{\pi/2} - \sin x\bigg|_{\pi/2}^{3\pi/2} \\ &= 1 + 1 + 1 = 3\end{aligned}$$

34. $$\begin{aligned}\int_1^3 \frac{\operatorname{sgn}(x-2)}{x^2}\,dx &= -\int_1^2 \frac{dx}{x^2} + \int_2^3 \frac{dx}{x^2} \\ &= \frac{1}{x}\bigg|_1^2 - \frac{1}{x}\bigg|_2^3 = -\frac{1}{3}\end{aligned}$$

35. Average value

$$= \frac{1}{2}\int_0^2 (1+x+x^2+x^3)\,dx$$
$$= \frac{1}{2}\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}\right)\Bigg|_0^2$$
$$= \frac{1}{2}\left(2+2+\frac{8}{3}+4\right) = \frac{16}{3}.$$

36. Average value $= \dfrac{1}{2-(-2)}\displaystyle\int_{-2}^2 e^{3x}\,dx$

$$= \frac{1}{4}\left(\frac{1}{3}e^{3x}\right)\Bigg|_{-2}^2$$
$$= \frac{1}{12}(e^6 - e^{-6}).$$

37. Avg. $= \dfrac{1}{1/\ln 2}\displaystyle\int_0^{1/\ln 2} 2^x\,dx = (\ln 2)\frac{2^x}{\ln 2}\Bigg|_0^{1/\ln 2} = e-1$

38. Since

$$g(t) = \begin{cases} 0, & \text{if } 0 \le t \le 1, \\ 1, & \text{if } 1 < t \le 3, \end{cases}$$

the average value of $g(t)$ over [0,3] is

$$\frac{1}{3}\left[\int_0^1 (0)\,dt + \int_1^3 1\,dt\right] = \frac{1}{3}\left[0 + t\Big|_1^3\right]$$
$$= \frac{1}{3}(3-1) = \frac{2}{3}.$$

39. $\dfrac{d}{dx}\displaystyle\int_2^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x}$

40. $\dfrac{d}{dt}\displaystyle\int_t^3 \frac{\sin x}{x}\,dx = \frac{d}{dt}\left[-\int_3^t \frac{\sin x}{x}\,dx\right] = -\frac{\sin t}{t}$

41. $\dfrac{d}{dx}\displaystyle\int_{x^2}^0 \frac{\sin t}{t}\,dt = -\frac{d}{dx}\int_0^{x^2} \frac{\sin t}{t}\,dt$

$$= -2x\frac{\sin x^2}{x^2} = -2\frac{\sin x^2}{x}$$

42. $\dfrac{d}{dx}x^2\displaystyle\int_0^{x^2} \frac{\sin u}{u}\,du$

$$= 2x\int_0^{x^2} \frac{\sin u}{u}\,du + x^2\frac{d}{dx}\int_0^{x^2}\frac{\sin u}{u}\,du$$
$$= 2x\int_0^{x^2} \frac{\sin u}{u}\,du + x^2\left[\frac{2x\sin x^2}{x^2}\right]$$
$$= 2x\int_0^{x^2} \frac{\sin u}{u}\,du + 2x\sin(x^2)$$

43. $\dfrac{d}{dt}\displaystyle\int_{-\pi}^t \frac{\cos y}{1+y^2}\,dy = \frac{\cos t}{1+t^2}$

44. $\dfrac{d}{d\theta}\displaystyle\int_{\sin\theta}^{\cos\theta} \frac{1}{1-x^2}\,dx$

$$= \frac{d}{d\theta}\left[\int_a^{\cos\theta}\frac{1}{1-x^2}\,dx - \int_a^{\sin\theta}\frac{1}{1-x^2}\,dx\right]$$
$$= \frac{-\sin\theta}{1-\cos^2\theta} - \frac{\cos\theta}{1-\sin^2\theta}$$
$$= \frac{-1}{\sin\theta} - \frac{1}{\cos\theta} = -\csc\theta - \sec\theta$$

45.

$$F(t) = \int_0^t \cos(x^2)\,dx$$
$$F(\sqrt{x}) = \int_0^{\sqrt{x}} \cos(u^2)\,du$$
$$\frac{d}{dx}F(\sqrt{x}) = \cos x\,\frac{1}{2\sqrt{x}} = \frac{\cos x}{2\sqrt{x}}$$

46. $H(x) = 3x\displaystyle\int_4^{x^2} e^{-\sqrt{t}}\,dt$

$$H'(x) = 3\int_4^{x^2} e^{-\sqrt{t}}\,dt + 3x(2xe^{-|x|})$$
$$H'(2) = 3\int_4^4 e^{-\sqrt{t}}\,dt + 3(2)(4e^{-2})$$
$$= 3(0) + 24e^{-2} = \frac{24}{e^2}$$

47. $f(x) = \pi + \pi\displaystyle\int_1^x f(t)\,dt$

$f'(x) = \pi f(x) \quad\Longrightarrow\quad f(x) = Ce^{\pi x}$

$\pi = f(1) = Ce^{\pi} \quad\Longrightarrow\quad C = \pi e^{-\pi}$

$f(x) = \pi\, e^{\pi(x-1)}$.

48. $f(x) = 1 - \displaystyle\int_0^x f(t)\,dt$

$f'(x) = -f(x) \quad\Longrightarrow\quad f(x) = Ce^{-x}$

$1 = f(0) = C$

$f(x) = e^{-x}$.

49. The function $1/x^2$ is not *defined* (and therefore not continuous) at $x = 0$, so the Fundamental Theorem of Calculus cannot be applied to it on the interval $[-1, 1]$. Since $1/x^2 > 0$ wherever it is defined, we would expect $\displaystyle\int_{-1}^1 \frac{dx}{x^2}$ to be *positive* if it exists at all (which it doesn't).

50. If $F(x) = \displaystyle\int_{17}^x \frac{\sin t}{1+t^2}\,dt$, then $F'(x) = \dfrac{\sin x}{1+x^2}$ and $F(17) = 0$.

51. $F(x) = \int_0^{2x-x^2} \cos\left(\dfrac{1}{1+t^2}\right)dt$.

Note that $0 < \dfrac{1}{1+t^2} \le 1$ for all t, and hence

$$0 < \cos(1) \le \cos\left(\frac{1}{1+t^2}\right) \le 1.$$

The integrand is continuous for all t, so $F(x)$ is defined and differentiable for all x. Since $\lim_{x\to\pm\infty}(2x-x^2)=-\infty$, therefore $\lim_{x\to\pm\infty}F(x)=-\infty$. Now

$$F'(x)=(2-2x)\cos\left(\frac{1}{1+(2x-x^2)^2}\right)=0$$

only at $x=1$. Therefore F must have a maximum value at $x=1$, and no minimum value.

52. $$\lim_{n\to\infty}\frac{1}{n}\left[\left(1+\frac{1}{n}\right)^5+\left(1+\frac{2}{n}\right)^5+\cdots+\left(1+\frac{n}{n}\right)^5\right]$$
$=$ area below $y=x^5$, above $y=0$, between $x=1$ and $x=2$
$$=\int_1^2 x^5\,dx=\frac{1}{6}x^6\bigg|_1^2=\frac{1}{6}(2^6-1)=\frac{21}{2}$$

53. $$\lim_{n\to\infty}\frac{\pi}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\cdots+\sin\frac{n\pi}{n}\right)$$
$=\lim_{n\to\infty}$ sum of areas of rectangles shown in figure
$$=\int_0^\pi \sin x\,dx=-\cos x\bigg|_0^\pi=2$$

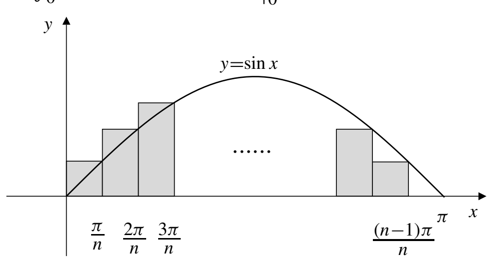

Fig. 5.5.53

54. $$\lim_{n\to\infty}\left(\frac{n}{n^2+1}+\frac{n}{n^2+4}+\frac{n}{n^2+9}+\cdots+\frac{n}{2n^2}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{n^2}{n^2+1}+\frac{n^2}{n^2+4}+\frac{n^2}{n^2+9}+\cdots+\frac{n^2}{2n^2}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\cdots+\frac{1}{1+\left(\frac{n}{n}\right)^2}\right)$$
$=$ area below $y=\dfrac{1}{1+x^2}$, above $y=0$, between $x=0$ and $x=1$
$$=\int_0^1\frac{1}{1+x^2}\,dx=\tan^{-1}x\bigg|_0^1=\frac{\pi}{4}$$

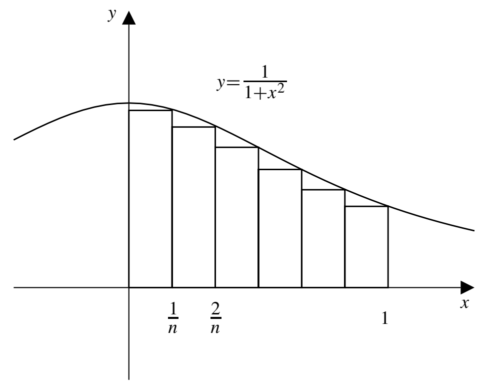

Fig. 5.5.54

Section 5.6 The Method of Substitution (page 308)

1. $\displaystyle\int e^{5-2x}\,dx$ Let $u=5-2x$
$du=-2\,dx$
$$=-\frac{1}{2}\int e^u\,du=-\frac{1}{2}e^u+C=-\frac{1}{2}e^{5-2x}+C.$$

2. $\displaystyle\int\cos(ax+b)\,dx$ Let $u=ax+b$
$du=a\,dx$
$$=\frac{1}{a}\int\cos u\,du=\frac{1}{a}\sin u+C$$
$$=\frac{1}{a}\sin(ax+b)+C.$$

3. $\displaystyle\int\sqrt{3x+4}\,dx$ Let $u=3x+4$
$du=3\,dx$
$$=\frac{1}{3}\int u^{1/2}\,du=\frac{2}{9}u^{3/2}+C=\frac{2}{9}(3x+4)^{3/2}+C.$$

4. $\displaystyle\int e^{2x}\sin(e^{2x})\,dx$ Let $u=e^{2x}$
$du=2e^{2x}\,dx$
$$=\frac{1}{2}\int\sin u\,du=-\frac{1}{2}\cos u+C$$
$$=-\frac{1}{2}\cos(e^{2x})+C.$$

5. $\displaystyle\int\frac{x\,dx}{(4x^2+1)^5}$ Let $u=4x^2+1$
$du=8x\,dx$
$$=\frac{1}{8}\int u^{-5}\,du=-\frac{1}{32}u^{-4}+C=\frac{-1}{32(4x^2+1)^4}+C.$$

6. $\displaystyle\int\frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ Let $u=\sqrt{x}$
$du=\dfrac{dx}{2\sqrt{x}}$
$$=2\int\sin u\,du=-2\cos u+C$$
$$=-2\cos\sqrt{x}+C.$$

7. $\displaystyle\int xe^{x^2}\,dx$ Let $u = x^2$, $du = 2x\,dx$

$$= \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

8. $\displaystyle\int x^2 2^{x^3+1}\,dx$ Let $u = x^3 + 1$, $du = 3x^2\,dx$

$$= \frac{1}{3}\int 2^u\,du = \frac{1}{3}\frac{2^u}{\ln 2} + C$$
$$= \frac{2^{x^3+1}}{3\ln 2} + C.$$

9. $\displaystyle\int \frac{\cos x}{4+\sin^2 x}\,dx$ Let $u = \sin x$, $du = \cos x\,dx$

$$= \int \frac{du}{4+u^2}$$
$$= \frac{1}{2}\tan^{-1}\frac{u}{2} + C = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\sin x\right) + C.$$

10. $\displaystyle\int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx$ Let $u = \tan x$, $du = \sec^2 x\,dx$

$$= \frac{du}{\sqrt{1-u^2}}$$
$$= \sin^{-1} u + C$$
$$= \sin^{-1}(\tan x) + C.$$

11. $\displaystyle\int \frac{e^x+1}{e^x-1}\,dx$

$$= \int \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{x/2}}\,dx \quad \text{Let } u = e^{x/2} - e^{-x/2},\ du = \tfrac{1}{2}\left(e^{x/2}+e^{-x/2}\right)dx$$
$$= 2\int \frac{du}{u} = 2\ln|u| + C$$
$$= 2\ln\left|e^{x/2} - e^{-x/2}\right| + C = \ln\left|e^x + e^{-x} - 2\right| + C.$$

12. $\displaystyle\int \frac{\ln t}{t}\,dt$ Let $u = \ln t$, $du = \dfrac{dt}{t}$

$$= \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln t)^2 + C.$$

13. $\displaystyle\int \frac{ds}{\sqrt{4-5s}}$ Let $u = 4 - 5s$, $du = -5\,ds$

$$= -\frac{1}{5}\int \frac{du}{\sqrt{u}}$$
$$= -\frac{2}{5}u^{1/2} + C = -\frac{2}{5}\sqrt{4-5s} + C.$$

14. $\displaystyle\int \frac{x+1}{\sqrt{x^2+2x+3}}\,dx$ Let $u = x^2 + 2x + 3$, $du = 2(x+1)\,dx$

$$= \frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = \sqrt{u} + C = \sqrt{x^2+2x+3} + C$$

15. $\displaystyle\int \frac{t}{\sqrt{4-t^4}}\,dt$ Let $u = t^2$, $du = 2t\,dt$

$$= \frac{1}{2}\int \frac{du}{\sqrt{4-u^2}}$$
$$= \frac{1}{2}\sin^{-1}\frac{u}{2} + C = \frac{1}{2}\sin^{-1}\left(\frac{t^2}{2}\right) + C.$$

16. $\displaystyle\int \frac{x^2}{2+x^6}\,dx$ Let $u = x^3$, $du = 3x^2\,dx$

$$= \frac{1}{3}\int \frac{du}{2+u^2} = \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$$
$$= \frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{x^3}{\sqrt{2}}\right) + C.$$

17. $\displaystyle\int \frac{dx}{e^x+1} = \int \frac{e^{-x}\,dx}{1+e^{-x}}$ Let $u = 1 + e^{-x}$, $du = -e^{-x}\,dx$

$$= -\int \frac{du}{u} = -\ln|u| + C = -\ln(1+e^{-x}) + C.$$

18. $\displaystyle\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1}$ Let $u = e^x$, $du = e^x\,dx$

$$= \int \frac{du}{u^2+1} = \tan^{-1} u + C$$
$$= \tan^{-1} e^x + C.$$

19. $\displaystyle\int \tan x \ln\cos x\,dx$ Let $u = \ln\cos x$, $du = -\tan x\,dx$

$$= -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\left(\ln\cos x\right)^2 + C.$$

20. $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{x\,dx}{\sqrt{1-x^2}} + \int \frac{dx}{\sqrt{1-x^2}} \quad \text{Let } u = 1 - x^2,\ du = -2x\,dx$$

in the first integral only

$$= -\frac{1}{2}\int \frac{du}{\sqrt{u}} + \sin^{-1} x = -\sqrt{u} + \sin^{-1} x + C$$
$$= -\sqrt{1-x^2} + \sin^{-1} x + C.$$

21. $\displaystyle\int \frac{dx}{x^2+6x+13} = \int \frac{dx}{(x+3)^2+4}$ Let $u = x + 3$, $du = dx$

$$= \int \frac{du}{u^2+4} = \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$
$$= \frac{1}{2}\tan^{-1}\frac{x+3}{2} + C.$$

22. $\displaystyle\int \frac{dx}{\sqrt{4+2x-x^2}} = \frac{dx}{\sqrt{5-(1-x)^2}}$ Let $u = 1 - x$, $du = -dx$

$$= -\int \frac{du}{\sqrt{5-u^2}} = -\sin^{-1}\left(\frac{u}{\sqrt{5}}\right) + C$$
$$= -\sin^{-1}\left(\frac{1-x}{\sqrt{5}}\right) + C = \sin^{-1}\left(\frac{x-1}{\sqrt{5}}\right) + C.$$

23. $\int \sin^3 x \cos^5 x\,dx$

$= \int \sin x(\cos^5 x - \cos^7 x)\,dx$ Let $u = \cos x$, $du = -\sin x\,dx$

$= \int (u^7 - u^5)\,du$

$= \frac{u^8}{8} - \frac{u^6}{6} + C = \frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C.$

24. $\int \sin^4 t \cos^5 t\,dt$

$= \int \sin^4 t(1 - \sin^2 t)^2 \cos t\,dt$ Let $u = \sin t$, $du = \cos t\,dt$

$= \int (u^4 - 2u^6 + u^8)\,du = \frac{u^5}{5} - \frac{2u^7}{7} + \frac{u^9}{9} + C$

$= \frac{1}{5}\sin^5 t - \frac{2}{7}\sin^7 t + \frac{1}{9}\sin^9 t + C.$

25. $\int \sin ax \cos^2 ax\,dx$ Let $u = \cos ax$, $du = -a \sin ax\,dx$

$= -\frac{1}{a}\int u^2\,du$

$= -\frac{u^3}{3a} + C = -\frac{1}{3a}\cos^3 ax + C.$

26. $\int \sin^2 x \cos^2 x\,dx = \int \left(\frac{\sin 2x}{2}\right)^2 dx$

$= \frac{1}{4}\int \frac{1 - \cos 4x}{2}\,dx = \frac{x}{8} - \frac{\sin 4x}{32} + C.$

27. $\int \sin^6 x\,dx = \int \left(\frac{1 - \cos 2x}{2}\right)^3 dx$

$= \frac{1}{8}\int (1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x)\,dx$

$= \frac{x}{8} - \frac{3\sin 2x}{16} + \frac{3}{16}\int (1 + \cos 4x)\,dx$

$\quad - \frac{1}{8}\int \cos 2x(1 - \sin^2 2x)\,dx$ Let $u = \sin 2x$, $du = 2\cos 2x\,dx$

$= \frac{5x}{16} - \frac{3\sin 2x}{16} + \frac{3\sin 4x}{64} - \frac{1}{16}\int (1 - u^2)\,du$

$= \frac{5x}{16} - \frac{3\sin 2x}{16} + \frac{3\sin 4x}{64} - \frac{\sin 2x}{16} + \frac{\sin^3 2x}{48} + C$

$= \frac{5x}{16} - \frac{\sin 2x}{4} + \frac{3\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$

28. $\int \cos^4 x\,dx = \int \frac{[1 + \cos(2x)]^2}{4}\,dx$

$= \frac{1}{4}\int [1 + 2\cos(2x) + \cos^2(2x)]\,dx$

$= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{1}{8}\int 1 + \cos(4x)\,dx$

$= \frac{x}{4} + \frac{\sin(2x)}{4} + \frac{x}{8} + \frac{\sin(4x)}{32} + C$

$= \frac{3x}{8} + \frac{\sin(2x)}{4} + \frac{\sin(4x)}{32} + C.$

29. $\int \sec^5 x \tan x\,dx$ Let $u = \sec x$, $du = \sec x\,\tan x\,dx$

$= \int u^4\,du = \frac{u^5}{5} + C = \frac{\sec^5 x}{5} + C.$

30. $\int \sec^6 x \tan^2 x\,dx$

$= \int \sec^2 x \tan^2 x(1 + \tan^2 x)^2\,dx$ Let $u = \tan x$, $du = \sec^2 x\,dx$

$= \int (u^2 + 2u^4 + u^6)\,du = \frac{1}{3}u^3 + \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$

$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$

31. $\int \sqrt{\tan x}\sec^4 x\,dx$

$= \int \sqrt{\tan x}(1 + \tan^2 x)\sec^2 x\,dx$ Let $u = \tan x$, $du = \sec^2 x\,dx$

$= \int \left(u^{1/2} + u^{5/2}\right) du$

$= \frac{2u^{3/2}}{3} + \frac{2u^{7/2}}{7} + C$

$= \frac{2}{3}(\tan x)^{3/2} + \frac{2}{7}(\tan x)^{7/2} + C.$

32. $\int \sin^{-2/3} x \cos^3 x\,dx$ Let $u = \sin x$, $du = \cos x\,dx$

$= \int \frac{1 - u^2}{u^{2/3}}\,du = 3u^{1/3} - \frac{3}{7}u^{7/3} + C$

$= 3\sin^{1/3} x - \frac{3}{7}\sin^{7/3} x + C.$

33. $\int \cos x \sin^4(\sin x)\,dx$ Let $u = \sin x$, $du = \cos x\,dx$

$= \int \sin^4 u\,du = \int \left(\frac{1 - \cos 2u}{2}\right)^2 du$

$= \frac{1}{4}\int \left(1 - 2\cos 2u + \frac{1 + \cos 4u}{2}\right) du$

$= \frac{3u}{8} - \frac{\sin 2u}{4} + \frac{\sin 4u}{32} + C$

$= \frac{3}{8}\sin x - \frac{1}{4}\sin(2\sin x) + \frac{1}{32}\sin(4\sin x) + C.$

34. $\displaystyle\int \frac{\sin^3(\ln x)\cos^3(\ln x)}{x}\,dx$ Let $u = \sin(\ln x)$, $du = \dfrac{\cos(\ln x)}{x}\,dx$

$$= \int u^3(1-u^2)\,du = \frac{1}{4}u^4 - \frac{1}{6}u^6 + C$$
$$= \frac{1}{4}\sin^4(\ln x) - \frac{1}{6}\sin^6(\ln x) + C.$$

35. $\displaystyle\int \frac{\sin^2 x}{\cos^4 x}\,dx$

$= \displaystyle\int \tan^2 x\sec^2 x\,dx$ Let $u = \tan x$, $du = \sec^2 x\,dx$

$$= \int u^2\,du = \frac{u^3}{3} + C = \frac{1}{3}\tan^3 x + C.$$

36. $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\,dx = \int \tan^3 x\sec x\,dx$

$= \displaystyle\int (\sec^2 x - 1)\sec x\tan x\,dx$ Let $u = \sec x$, $du = \sec x\tan x\,dx$

$$= \int (u^2-1)\,du = \tfrac{1}{3}u^3 - u + C$$
$$= \tfrac{1}{3}\sec^3 x - \sec x + C.$$

37. $\displaystyle\int \csc^5 x\cot^5 x\,dx$

$= \displaystyle\int \csc x\cot x\csc^4 x(\csc^2 x - 1)^2\,dx$

Let $u = \csc x$, $du = -\csc x\cot x\,dx$

$$= -\int (u^8 - 2u^6 + u^4)\,du$$
$$= -\frac{u^9}{9} + \frac{2u^7}{7} - \frac{u^5}{5} + C$$
$$= -\frac{1}{9}\csc^9 x + \frac{2}{7}\csc^7 x - \frac{1}{5}\csc^5 x + C.$$

38. $\displaystyle\int \frac{\cos^4 x}{\sin^8 x}\,dx = \int \cot^4 x\csc^4 x\,dx$

$= \displaystyle\int \cot^4 x(1+\cot^2 x)\csc^2 x\,dx$ Let $u = \cot x$, $du = -\csc^2 x\,dx$

$$= -\int u^4(1+u^2)\,du = -\frac{u^5}{5} - \frac{u^7}{7} + C$$
$$= -\frac{1}{5}\cot^5 x - \frac{1}{7}\cot^7 x + C.$$

39. $\displaystyle\int_0^4 x^3(x^2+1)^{-1/2}\,dx$ Let $u = x^2+1$, $x^2 = u - 1$, $du = 2x\,dx$

$$= \frac{1}{2}\int_1^{17}(u-1)u^{-1/2}\,du$$
$$= \frac{1}{2}\left(\frac{2}{3}u^{3/2} - 2u^{1/2}\right)\Bigg|_1^{17}$$
$$= \frac{17\sqrt{17}-1}{3} - (\sqrt{17}-1) = \frac{14\sqrt{17}}{3} + \frac{2}{3}.$$

40. $\displaystyle\int_1^{\sqrt{e}} \frac{\sin(\pi\ln x)}{x}\,dx$ Let $u = \pi\ln x$, $du = \dfrac{\pi}{x}\,dx$

$$= \frac{1}{\pi}\int_0^{\pi/2}\sin u\,du = -\frac{1}{\pi}\cos u\Big|_0^{\pi/2}$$
$$= -\frac{1}{\pi}(0-1) = \frac{1}{\pi}.$$

41. $\displaystyle\int_0^{\pi/2}\sin^4 x\,dx = \int_0^{\pi/2}\left(\frac{1-\cos 2x}{2}\right)^2$

$$= \frac{1}{4}\int_0^{\pi/2}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)dx$$
$$= \frac{3x}{8}\Big|_0^{\pi/2} - \frac{\sin 2x}{4}\Big|_0^{\pi/2} + \frac{\sin 4x}{32}\Big|_0^{\pi/2} = \frac{3\pi}{16}.$$

42. $\displaystyle\int_{\pi/4}^{\pi}\sin^5 x\,dx$

$= \displaystyle\int_{\pi/4}^{\pi}(1-\cos^2 x)^2\sin x\,dx$ Let $u = \cos x$, $du = -\sin x\,dx$

$$= -\int_{1/\sqrt{2}}^{-1}(1-2u^2+u^4)\,du = u - \frac{2}{3}u^3 + \frac{1}{5}u^5\Big|_{-1}^{1/\sqrt{2}}$$
$$= \frac{1}{\sqrt{2}} - \frac{1}{3\sqrt{2}} + \frac{1}{20\sqrt{2}} - \left(-1 + \frac{2}{3} - \frac{1}{5}\right) = \frac{43}{60\sqrt{2}} + \frac{8}{15}.$$

43. $\displaystyle\int_e^{e^2}\frac{dt}{t\ln t}$ Let $u = \ln t$, $du = \dfrac{dt}{t}$

$$= \int_1^2 \frac{du}{u} = \ln u\Big|_1^2 = \ln 2 - \ln 1 = \ln 2.$$

44. $\displaystyle\int_{\pi^2/16}^{\pi^2/9}\frac{2^{\sin\sqrt{x}}\cos\sqrt{x}}{\sqrt{x}}\,dx$ Let $u = \sin\sqrt{x}$, $du = \dfrac{\cos\sqrt{x}}{2\sqrt{x}}\,dx$

$$= 2\int_{1/\sqrt{2}}^{\sqrt{3}/2}2^u\,du = \frac{2(2^u)}{\ln 2}\Bigg|_{1/\sqrt{2}}^{\sqrt{3}/2}$$
$$= \frac{2}{\ln 2}(2^{\sqrt{3}/2} - 2^{1/\sqrt{2}}).$$

45. $$\int_0^{\pi/2} \sqrt{1+\cos x}\,dx = \int_0^{\pi/2} \sqrt{2\cos^2\frac{x}{2}}\,dx$$
$$= \sqrt{2}\int_0^{\pi/2} \cos\frac{x}{2}\,dx = 2\sqrt{2}\sin\frac{x}{2}\Big|_0^{\pi/2} = 2.$$
$$\int_0^{\pi/2} \sqrt{1-\sin x}\,dx$$
$$= \int_0^{\pi/2} \sqrt{1-\cos\left(\tfrac{\pi}{2}-x\right)}\,dx \quad \text{Let } u = \tfrac{\pi}{2}-x, \quad du = -dx$$
$$= -\int_{\pi/2}^{0} \sqrt{1-\cos u}\,du$$
$$= \int_0^{\pi/2} \sqrt{2\sin^2\frac{u}{2}}\,du = \sqrt{2}\left(-2\cos\frac{u}{2}\right)\Big|_0^{\pi/2}$$
$$= -2+2\sqrt{2} = 2(\sqrt{2}-1).$$

46. $$\text{Area} = \int_0^2 \frac{x}{x^2+16}\,dx \quad \text{Let } u = x^2+16, \quad du = 2x\,dx$$
$$= \frac{1}{2}\int_{16}^{20} \frac{du}{u} = \frac{1}{2}\ln u\Big|_{16}^{20}$$
$$= \frac{1}{2}(\ln 20 - \ln 16) = \frac{1}{2}\ln\left(\frac{5}{4}\right) \text{ sq. units.}$$

47. $$\text{Area } R = \int_0^2 \frac{x\,dx}{x^4+16} \quad \text{Let } u = x^2, \quad du = 2x\,dx$$
$$= \frac{1}{2}\int_0^4 \frac{du}{u^2+16} = \frac{1}{8}\tan^{-1}\frac{u}{4}\Big|_0^4 = \frac{\pi}{32}\text{sq. units.}$$

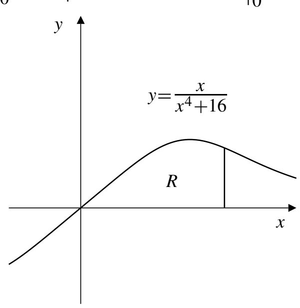

Fig. 5.6.47

48. The area bounded by the ellipse $(x^2/a^2)+(y^2/b^2)=1$ is

$$4\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,dx \quad \text{Let } x = au, \quad dx = a\,du$$
$$= 4ab\int_0^1 \sqrt{1-u^2}\,du.$$

The integral is the area of a quarter circle of radius 1. Hence

$$\text{Area} = 4ab\left(\frac{\pi(1)^2}{4}\right) = \pi ab \text{ sq. units.}$$

49. We start with the addition formulas

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

and take half their sum and half their difference to obtain

$$\cos x\cos y = \frac{1}{2}\Big(\cos(x+y)+\cos(x-y)\Big)$$
$$\sin x\sin y = \frac{1}{2}\Big(\cos(x-y)-\cos(x+y)\Big).$$

Similarly, taking half the sum of the formulas

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$\sin(x-y) = \sin x\cos y - \cos x\sin y,$$

we obtain

$$\sin x\cos y = \frac{1}{2}\Big(\sin(x+y)+\sin(x-y)\Big).$$

50. We have

$$\int \cos ax\cos bx\,dx$$
$$= \frac{1}{2}\int [\cos(ax-bx)+\cos(ax+bx)]\,dx$$
$$= \frac{1}{2}\int \cos[(a-b)x]\,dx + \frac{1}{2}\int \cos[(a+b)x]\,dx$$

Let $u=(a-b)x$, $du=(a-b)\,dx$ in the first integral; let $v=(a+b)x$, $dv=(a+b)\,dx$ in the second integral.

$$= \frac{1}{2(a-b)}\int \cos u\,du + \frac{1}{2(a+b)}\int \cos v\,dv$$
$$= \frac{1}{2}\left[\frac{\sin[(a-b)x]}{(a-b)} + \frac{\sin[(a+b)x]}{(a+b)}\right] + C.$$
$$\int \sin ax\sin bx\,dx$$
$$= \frac{1}{2}\int [\cos(ax-bx)-\cos(ax+bx)]\,dx$$
$$= \frac{1}{2}\left[\frac{\sin[(a-b)x]}{(a-b)} - \frac{\sin[(a+b)x]}{(a+b)}\right] + C.$$
$$\int \sin ax\cos bx\,dx$$
$$= \frac{1}{2}\int [\sin(ax+bx)+\sin(ax-bx)]\,dx$$
$$= \frac{1}{2}\Big[\int \sin[(a+b)x]\,dx + \int \sin[(a-b)x]\,dx\Big]$$
$$= -\frac{1}{2}\left[\frac{\cos[(a+b)x]}{(a+b)} + \frac{\cos[(a-b)x]}{(a-b)}\right] + C.$$

51. If m and n are integers, and $m \neq n$, then

$$\begin{aligned}
&\int_{-\pi}^{\pi} \begin{Bmatrix} \cos mx \cos nx \\ \sin mx \sin nx \end{Bmatrix} dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \Big(\cos(m-n)x \pm \cos(m+n)x\Big)\, dx \\
&= \frac{1}{2}\left(\frac{\sin(m-n)x}{m-n} \pm \frac{\sin(m+n)x}{m+n}\right)\Bigg|_{-\pi}^{\pi} \\
&= 0 \pm 0 = 0.
\end{aligned}$$

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin mx \cos nx\, dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \Big(\sin(m+n)x + \sin(m-n)x\Big)\, dx \\
&= -\frac{1}{2}\left(\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right)\Bigg|_{-\pi}^{\pi} \\
&= 0 \text{ (by periodicity).}
\end{aligned}$$

If $m = n \neq 0$ then

$$\begin{aligned}
&\int_{-\pi}^{\pi} \sin mx \cos mx\, dx \\
&= \frac{1}{2}\int_{-\pi}^{\pi} \sin 2mx\, dx \\
&= -\frac{1}{4m}\cos 2mx\Bigg|_{-\pi}^{\pi} = 0 \text{ (by periodicity).}
\end{aligned}$$

52. If $1 \le m \le k$, we have

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos mx\, dx = \frac{a_0}{2}&\int_{-\pi}^{\pi} \cos mx\, dx \\
&+ \sum_{n=1}^{k} a_n \int_{-\pi}^{\pi} \cos nx\ \cos mx\, dx \\
&+ \sum_{n=1}^{k} b_n \int_{-\pi}^{\pi} \sin nx\ \cos mx\, dx.
\end{aligned}$$

By the previous exercise, all the integrals on the right side are zero except the one in the first sum having $n = m$. Thus the whole right side reduces to

$$\begin{aligned}
a_m \int_{-\pi}^{\pi} \cos^2(mx)\, dx &= a_m \int_{-\pi}^{\pi} \frac{1+\cos(2mx)}{2}\, dx \\
&= \frac{a_m}{2}(2\pi + 0) = \pi a_m.
\end{aligned}$$

Thus

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx.$$

A similar argument shows that

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx.$$

For $m = 0$ we have

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos mx\, dx &= \int_{-\pi}^{\pi} f(x)\, dx \\
&= \frac{a_0}{2}\int_{-\pi}^{\pi} dx \\
&\quad + \sum_{n=1}^{k} (a_n \cos(nx) + b_n \sin(nx))\, dx \\
&= \frac{a_0}{2}(2\pi) + 0 + 0 = a_0 \pi,
\end{aligned}$$

so the formula for a_m holds for $m = 0$ also.

Section 5.7 Areas of Plane Regions (page 313)

1. Area of $R = \displaystyle\int_0^1 (x - x^2)\, dx$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \text{ sq. units.}$$

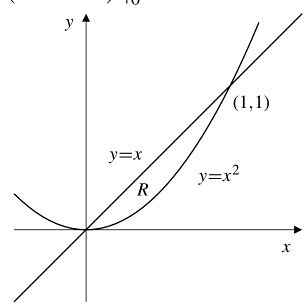

Fig. 5.7.1

2. Area of $R = \displaystyle\int_0^1 (\sqrt{x} - x^2)\, dx$

$$= \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Bigg|_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units.}$$

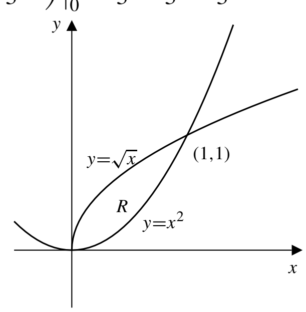

Fig. 5.7.2

3. Area of $R = 2\displaystyle\int_0^2 (8 - 2x^2)\, dx$

$$= \left(16x - \frac{4}{3}x^3\right)\Bigg|_0^2 = \frac{64}{3} \text{ sq. units.}$$

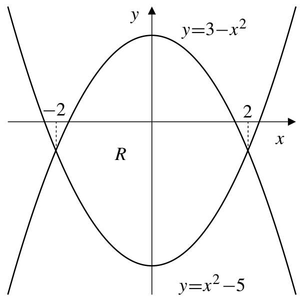

Fig. 5.7.3

4. For intersections:

$x^2 - 2x = 6x - x^2 \Rightarrow 2x^2 - 8x = 0$

i.e., $x = 0$ or 4.

$$\text{Area of } R = \int_0^4 \Big[6x - x^2 - (x^2 - 2x)\Big]\,dx$$

$$= \int_0^4 (8x - 2x^2)\,dx$$

$$= \left(4x^2 - \frac{2}{3}x^3\right)\Bigg|_0^4 = \frac{64}{3} \text{ sq. units.}$$

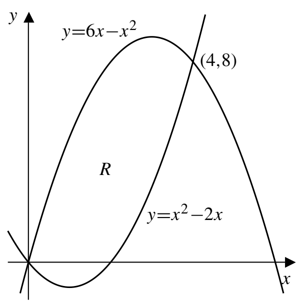

Fig. 5.7.4

5. For intersections:

$$\left.\begin{array}{r} 2y = 4x - x^2 \\ 2y + 3x = 6 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{c} 4x - x^2 = 6 - 3x \\ x^2 - 7x + 6 = 0 \\ (x-1)(x-6) = 0 \end{array}$$

Thus intersections of the curves occur at $x = 1$ and $x = 6$. We have

$$\text{Area of } R = \int_1^6 \left(2x - \frac{x^2}{2} - 3 + \frac{3x}{2}\right) dx$$

$$= \left(\frac{7x^2}{4} - \frac{x^3}{6} - 3x\right)\Bigg|_1^6$$

$$= \frac{245}{4} - 36 + \frac{1}{6} - 15 = \frac{125}{12} \text{ sq. units.}$$

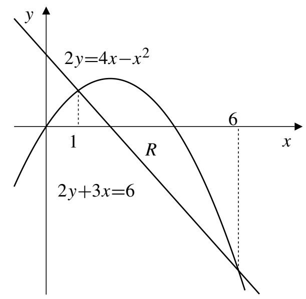

Fig. 5.7.5

6. For intersections:

$7 + y = 2y^2 - y + 3 \Rightarrow 2y^2 - 2y - 4 = 0$

$2(y-2)(y+1) = 0 \Rightarrow$ i.e., $y = -1$ or 2.

$$\text{Area of } R = \int_{-1}^2 [(7+y) - (2y^2 - y + 3)]\,dy$$

$$= 2\int_{-1}^2 (2 + y - y^2)\,dy$$

$$= 2\left(2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\Bigg|_{-1}^2 = 9 \text{ sq. units.}$$

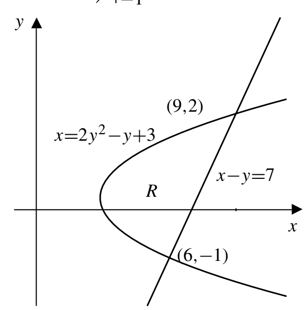

Fig. 5.7.6

7. $\text{Area of } R = 2\int_0^1 (x - x^3)\,dx$

$$= 2\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{1}{2} \text{ sq. units.}$$

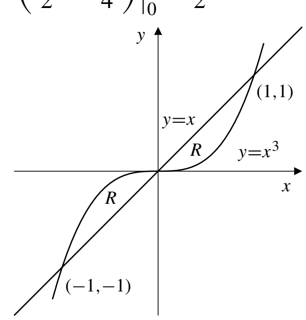

Fig. 5.7.7

8. 4 Shaded area $= \int_0^1 (x^2 - x^3)\,dx$

$$= \left(\frac{1}{3}x^3 - \frac{1}{4}x^4\right)\bigg|_0^1 = \frac{1}{12} \text{ sq. units.}$$

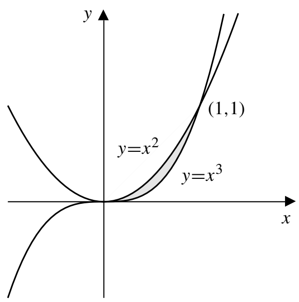

Fig. 5.7.8

9. Area of $R = \int_0^1 (\sqrt{x} - x^3)\,dx$

$$= \left(\frac{2}{3}x^{3/2} - \frac{x^4}{4}\right)\bigg|_0^1 = \frac{5}{12} \text{ sq. units.}$$

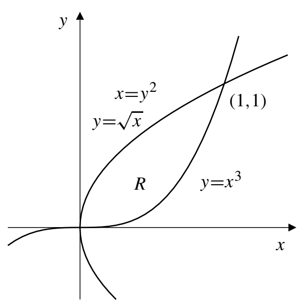

Fig. 5.7.9

10. For intersections:

$y^2 = 2y^2 - y - 2 \Rightarrow y^2 - y - 2 = 0$

$(y-2)(y+1) = 0 \Rightarrow$ i.e., $y = -1$ or 2.

Area of $R = \int_{-1}^2 [y^2 - (2y^2 - y - 2)]\,dy$

$$= \int_{-1}^2 [2 + y - y^2]\,dy = \left(2y + \frac{1}{2}y^2 - \frac{1}{3}y^3\right)\bigg|_{-1}^2$$

$$= \frac{9}{2} \text{ sq. units.}$$

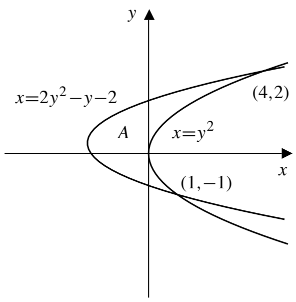

Fig. 5.7.10

11. For intersections: $\frac{1}{x} = y = \frac{5-2x}{2}$.

Thus $2x^2 - 5x + 2 = 0$, i.e., $(2x-1)(x-2) = 0$. The graphs intersect at $x = 1/2$ and $x = 2$. Thus

$$\begin{aligned}\text{Area of } R &= \int_{1/2}^2 \left(\frac{5-2x}{2} - \frac{1}{x}\right) dx \\ &= \left(\frac{5x}{2} - \frac{x^2}{2} - \ln x\right)\bigg|_{1/2}^2 \\ &= \frac{15}{8} - 2\ln 2 \text{ sq. units.}\end{aligned}$$

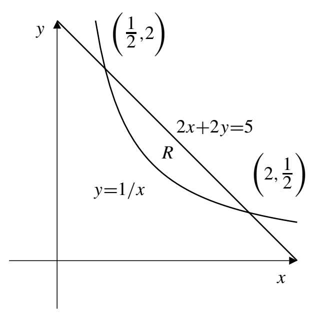

Fig. 5.7.11

12. Area of shaded region $= 2\int_0^1 [(1-x^2) - (x^2-1)^2]\,dx$

$$= 2\int_0^1 (x^2 - x^4)\,dx = 2\left(\frac{1}{3}x^3 - \frac{1}{5}x^5\right)\bigg|_0^1 = \frac{4}{15} \text{ sq. units.}$$

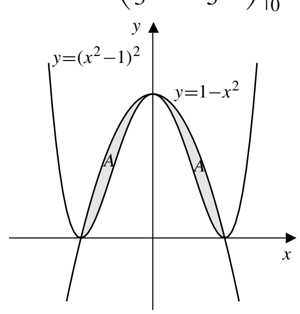

Fig. 5.7.12

13. The curves $y = \dfrac{x^2}{2}$ and $y = \dfrac{1}{1+x^2}$ intersect at $x = \pm 1$. Thus

$$\begin{aligned}\text{Area of } R &= 2\int_0^1 \left(\frac{1}{1+x^2} - \frac{x^2}{2}\right) dx \\ &= 2\left(\tan^{-1} x - \frac{x^3}{6}\right)\Bigg|_0^1 = \frac{\pi}{2} - \frac{1}{3} \text{ sq. units.}\end{aligned}$$

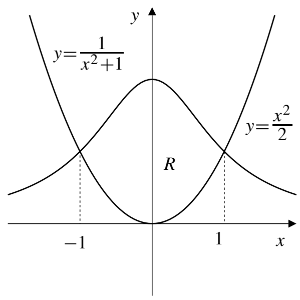

Fig. 5.7.13

14. For intersections:
$\dfrac{4x}{3+x^2} = 1 \Rightarrow x^2 - 4x + 3 = 0$
i.e., $x = 1$ or 3.

$$\text{Shaded area } = \int_1^3 \left[\frac{4x}{3+x^2} - 1\right] dx$$
$$= [2\ln(3+x^2) - x]\Big|_1^3 = 2\ln 3 - 2 \text{ sq. units.}$$

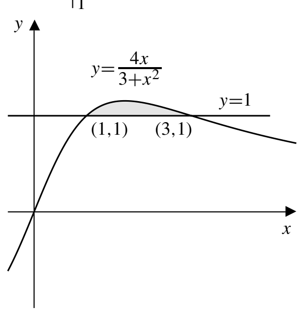

Fig. 5.7.14

15. The curves $y = \dfrac{4}{x^2}$ and $y = 5 - x^2$ intersect where $x^4 - 5x^2 + 4 = 0$, i.e., where $(x^2-4)(x^2-1) = 0$. Thus the intersections are at $x = \pm 1$ and $x = \pm 2$. We have

$$\begin{aligned}\text{Area of } R &= 2\int_1^2 \left(5 - x^2 - \frac{4}{x^2}\right) dx \\ &= 2\left(5x - \frac{x^3}{3} + \frac{4}{x}\right)\Bigg|_1^2 = \frac{4}{3} \text{ sq. units.}\end{aligned}$$

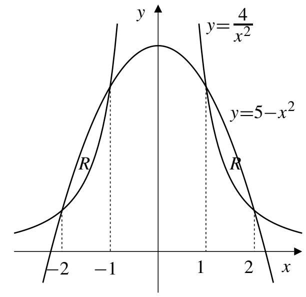

Fig. 5.7.15

16.
$$\begin{aligned}\text{Area } A &= \int_{-\pi}^{\pi} (\sin y - (y^2 - \pi^2))\, dy \\ &= \left(-\cos y + \pi^2 y - \frac{y^3}{3}\right)\Bigg|_{-\pi}^{\pi} = \frac{4\pi^3}{3} \text{ sq. units.}\end{aligned}$$

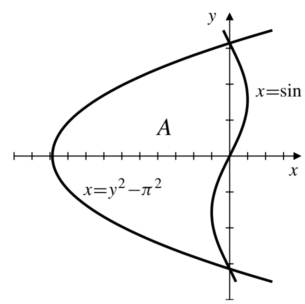

Fig. 5.7.16

17.
$$\begin{aligned}\text{Area of } R &= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx \\ &= -(\cos x + \sin x)\Big|_{\pi/4}^{5\pi/4} \\ &= \sqrt{2} + \sqrt{2} = 2\sqrt{2} \text{ sq. units.}\end{aligned}$$

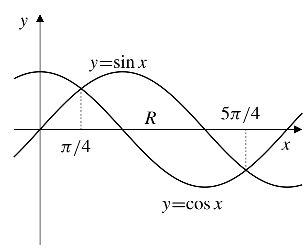

Fig. 5.7.17

18.
$$\begin{aligned}\text{Area } &= \int_{-\pi/2}^{\pi/2} (1 - \sin^2 x)\, dx \\ &= 2\int_0^{\pi/2} \frac{1+\cos(2x)}{2}\, dx \\ &= \left(x + \frac{\sin(2x)}{2}\right)\Bigg|_0^{\pi/2} = \frac{\pi}{2} \text{ sq. units.}\end{aligned}$$

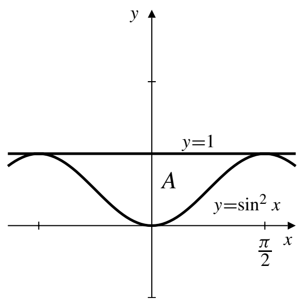

Fig. 5.7.18

19. Area $A = \int_0^{\pi/2} (\sin x - \sin^2 x)\, dx$

$$= \left(-\cos x + \frac{\sin x \cos x - x}{2}\right)\Bigg|_0^{\pi/2} = 1 - \frac{\pi}{4} \text{ sq. units.}$$

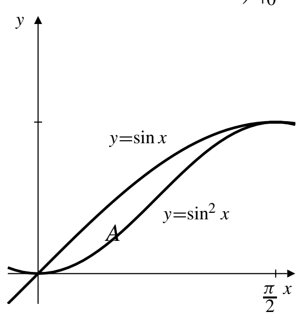

Fig. 5.7.19

20. Area $A = 2\int_0^{\pi/4} (\cos^2 x - \sin^2 x)\, dx$

$$= 2\int_0^{\pi/4} \cos(2x)\, dx = \sin(2x)\Big|_0^{\pi/4} = 1 \text{ sq. units.}$$

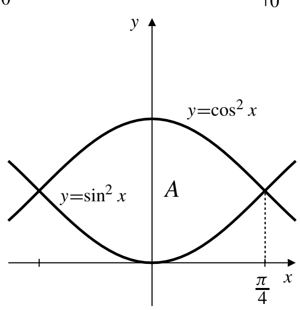

Fig. 5.7.20

21. For intersections: $\dfrac{4x}{\pi} = \tan x \;\Rightarrow\; x = 0$ or $\dfrac{\pi}{4}$.

$$\text{Area } = \int_0^{\pi/4} \left(\frac{4x}{\pi} - \tan x\right) dx$$

$$= \left(\frac{2}{\pi}x^2 - \ln|\sec x|\right)\Bigg|_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{2}\ln 2 \text{ sq. units.}$$

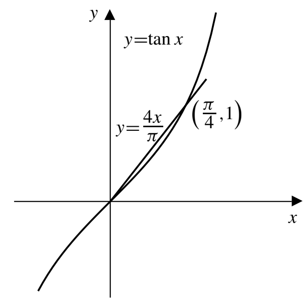

Fig. 5.7.21

22. For intersections: $x^{1/3} = \tan(\pi x/4)$. Thus $x = \pm 1$.

$$\text{Area } A = 2\int_0^1 \left(x^{1/3} - \tan\frac{\pi x}{4}\right) dx$$

$$= 2\left(\frac{3}{4}x^{4/3} - \frac{4}{\pi}\ln\left|\sec\frac{\pi x}{4}\right|\right)\Bigg|_0^1$$

$$= \frac{3}{2} - \frac{8}{\pi}\ln\sqrt{2} = \frac{3}{2} - \frac{4}{\pi}\ln 2 \text{ sq. units.}$$

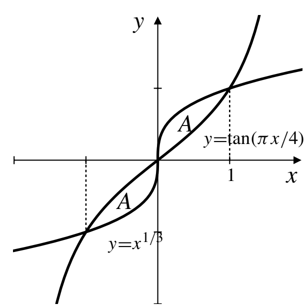

Fig. 5.7.22

23. For intersections: $\sec x = 2$. Thus $x = \pm\pi/3$.

$$\text{Area } A = 2\int_0^{\pi/3} (2 - \sec x)\, dx$$

$$= (4x - 2\ln|\sec x + \tan x|)\Big|_0^{\pi/3}$$

$$= \frac{4\pi}{3} - 2\ln(2 + \sqrt{3}) \text{ sq. units.}$$

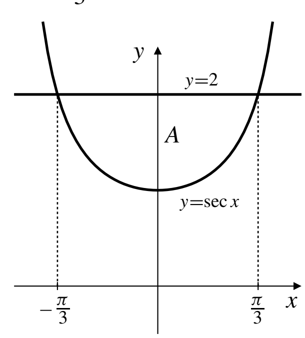

Fig. 5.7.23

24. For intersections: $|x| = \sqrt{2}\cos(\pi x/4)$. Thus $x = \pm 1$.

$$\begin{aligned}\text{Area } A &= 2\int_0^1 \left(\sqrt{2}\cos\frac{\pi x}{4} - x\right) dx \\ &= \left(\frac{8\sqrt{2}}{\pi}\sin\frac{\pi x}{4} - x^2\right)\Bigg|_0^1 \\ &= \frac{8}{\pi} - 1 \text{ sq. units.}\end{aligned}$$

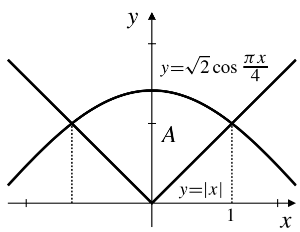

Fig. 5.7.24

25. For intersections: $x = \sin(\pi x/2)$. Thus $x = \pm 1$.

$$\begin{aligned}\text{Area } A &= 2\int_0^1 \left(\sin\frac{\pi x}{2} - x\right) dx \\ &= \left(-\frac{4}{\pi}\cos\frac{\pi x}{2} - x^2\right)\Bigg|_0^1 \\ &= \frac{4}{\pi} - 1 \text{ sq. units.}\end{aligned}$$

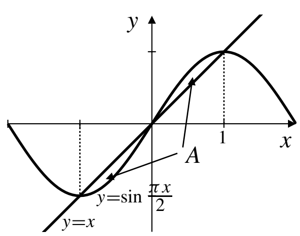

Fig. 5.7.25

26. For intersections: $e^x = x + 2$. There are two roots, both of which must be found numerically. We used a TI-85 solve routine to get $x_1 \approx -1.841406$ and $x_2 \approx 1.146193$. Thus

$$\begin{aligned}\text{Area } A &= \int_{x_1}^{x_2} \left(x + 2 - e^x\right) dx \\ &= \left(\frac{x^2}{2} + 2x - e^x\right)\Bigg|_{x_1}^{x_2} \\ &\approx 1.949091 \text{ sq. units.}\end{aligned}$$

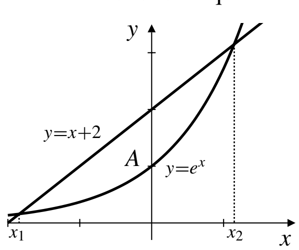

Fig. 5.7.26

27.
$$\begin{aligned}\text{Area of } R &= 4\int_0^1 \sqrt{x^2 - x^4}\, dx \\ &= 4\int_0^1 x\sqrt{1 - x^2}\, dx \qquad \text{Let } u = 1 - x^2 \\ &\qquad\qquad\qquad\qquad\quad\ du = -2x\, dx \\ &= 2\int_0^1 u^{1/2}\, du = \frac{4}{3}u^{3/2}\Bigg|_0^1 = \frac{4}{3} \text{ sq. units.}\end{aligned}$$

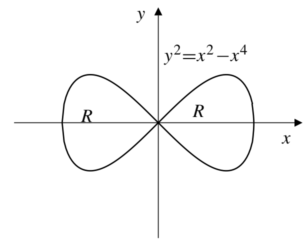

Fig. 5.7.27

28.
$$\text{Loop area } = 2\int_{-2}^0 x^2\sqrt{2 + x}\, dx \quad \text{Let } u^2 = 2 + x, \quad 2u\, du = dx$$

$$\begin{aligned}&= 2\int_0^{\sqrt{2}} (u^2 - 2)^2 u(2u)\, du = 4\int_0^{\sqrt{2}} (u^6 - 4u^4 + 4u^2)\, du \\ &= 4\left(\frac{1}{7}u^7 - \frac{4}{5}u^5 + \frac{4}{3}u^3\right)\Bigg|_0^{\sqrt{2}} = \frac{256\sqrt{2}}{105} \text{ sq. units.}\end{aligned}$$

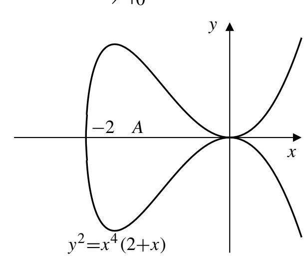

Fig. 5.7.28

29. The tangent line to $y = e^x$ at $x = 1$ is $y - e = e(x - 1)$, or $y = ex$. Thus

$$\begin{aligned}\text{Area of } R &= \int_0^1 (e^x - ex)\, dx \\ &= \left(e^x - \frac{ex^2}{2}\right)\Bigg|_0^1 = \frac{e}{2} - 1 \text{ sq. units.}\end{aligned}$$

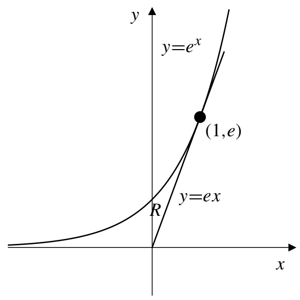

Fig. 5.7.29

30. The tangent line to $y = x^3$ at $(1, 1)$ is $y-1 = 3(x-1)$, or $y = 3x - 2$. The intersections of $y = x^3$ and this tangent line occur where $x^3 - 3x + 2 = 0$. Of course $x = 1$ is a (double) root of this cubic equation, which therefore factors to $(x-1)^2(x+2) = 0$. The other intersection is at $x = -2$. Thus

$$\begin{aligned}\text{Area of } R &= \int_{-2}^{1}(x^3 - 3x + 2)\,dx \\ &= \left(\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right)\bigg|_{-2}^{1} \\ &= -\frac{15}{4} - \frac{3}{2} + 6 + 2 + 4 = \frac{27}{4} \text{ sq. units.}\end{aligned}$$

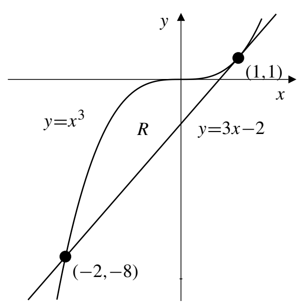

Fig. 5.7.30

Review Exercises 5 (page 314)

1. $\dfrac{1}{j^2} - \dfrac{1}{(j+1)^2} = \dfrac{j^2 + 2j + 1 - j^2}{j^2(j+1)^2} = \dfrac{2j+1}{j^2(j+1)^2}$

$$\begin{aligned}\sum_{j=1}^{n}\frac{2j+1}{j^2(j+1)^2} &= \sum_{j=1}^{n}\left(\frac{1}{j^2} - \frac{1}{(j+1)^2}\right) \\ &= \frac{1}{1^2} - \frac{1}{(n+1)^2} = \frac{n^2+2n}{(n+1)^2}\end{aligned}$$

2. The number of balls is

$$\begin{aligned}&40 \times 30 + 39 \times 29 + \cdots + 12 \times 2 + 11 \times 1 \\ &= \sum_{i=1}^{30} i(i+10) = \frac{(30)(31)(61)}{6} + 10\,\frac{(30)(31)}{2} = 14{,}105.\end{aligned}$$

3. $x_i = 1 + (2i/n)$, $(i = 0, 1, 2, \ldots, n)$, $\Delta x_i = 2/n$.

$$\begin{aligned}\int_1^3 f(x)\,dx &= \lim_{n\to\infty}\sum_{i=1}^{n}(x_i^2 - 2x_i + 3)\frac{2}{n} \\ &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[\left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(2 + \frac{4i}{n}\right) + 3\right] \\ &= \lim_{n\to\infty}\frac{2}{n}\sum_{i=1}^{n}\left[2 + \frac{4}{n^2}i^2\right] \\ &= \lim_{n\to\infty}\left(\frac{4}{n}n + \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6}\right) \\ &= 4 + \frac{8}{3} = \frac{20}{3}\end{aligned}$$

4. $R_n = \sum_{i=1}^{n}(1/n)\sqrt{1 + (i/n)}$ is a Riemann sum for $f(x) = \sqrt{1+x}$ on the interval $[0, 1]$. Thus

$$\begin{aligned}\lim_{n\to\infty} R_n &= \int_0^1 \sqrt{1+x}\,dx \\ &= \frac{2}{3}(1+x)^{3/2}\bigg|_0^1 = \frac{4\sqrt{2}-2}{3}.\end{aligned}$$

5. $\displaystyle\int_{-\pi}^{\pi}(2 - \sin x)\,dx = 2(2\pi) - \int_{-\pi}^{\pi}\sin x\,dx = 4\pi - 0 = 4\pi$

6. $\displaystyle\int_0^{\sqrt{5}}\sqrt{5 - x^2}\,dx = 1/4$ of the area of a circle of radius $\sqrt{5}$

$$= \frac{1}{4}\pi(\sqrt{5})^2 = \frac{5\pi}{4}$$

7. $\int_1^3\left(1 - \dfrac{x}{2}\right)\,dx = \text{area } A_1 - \text{area } A_2 = 0$

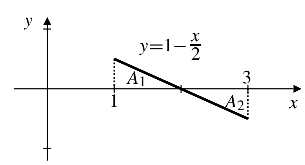

Fig. R-5.7

8. $\int_0^{\pi}\cos x\,dx = \text{area } A_1 - \text{area } A_2 = 0$

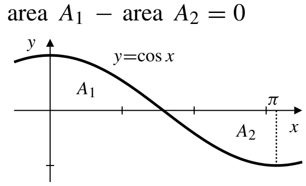

Fig. R-5.8

9. $\bar{f} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}(2 - \sin(x^3))\,dx = \dfrac{1}{2\pi}[2(2\pi) - 0] = 2$

10. $\bar{h} = \dfrac{1}{3}\displaystyle\int_0^3|x - 2|\,dx = \dfrac{1}{3}\,\dfrac{5}{2} = \dfrac{5}{6}$ (via #9)

11. $f(t) = \int_{13}^{t} \sin(x^2)\,dx, \quad f'(t) = \sin(t^2)$

12. $f(x) = \int_{-13}^{\sin x} \sqrt{1+t^2}\,dt, \quad f'(x) = \sqrt{1+\sin^2 x}(\cos x)$

13. $g(s) = \int_{4s}^{1} e^{\sin u}\,du, \quad g'(s) = -4e^{\sin(4s)}$

14. $g(\theta) = \int_{e^{\sin\theta}}^{e^{\cos\theta}} \ln x\,dx$

$$\begin{aligned} g'(\theta) &= (\ln(e^{\cos\theta}))e^{\cos\theta}(-\sin\theta) - (\ln(e^{\sin\theta}))e^{\sin\theta}\cos\theta \\ &= -\sin\theta\cos\theta(e^{\cos\theta} + e^{\sin\theta}) \end{aligned}$$

15. $2f(x) + 1 = 3\int_{x}^{1} f(t)\,dt$

$$\begin{aligned} &2f'(x) = -3f(x) \quad \Longrightarrow \quad f(x) = Ce^{-3x/2} \\ &2f(1) + 1 = 0 \\ &-\frac{1}{2} = f(1) = Ce^{-3/2} \quad \Longrightarrow \quad C = -\frac{1}{2}e^{3/2} \\ &f(x) = -\frac{1}{2}e^{(3/2)(1-x)}. \end{aligned}$$

16. $I = \int_0^{\pi} xf(\sin x)\,dx \quad$ Let $x = \pi - u$, $dx = -du$

$$\begin{aligned} &= -\int_{\pi}^{0} (\pi - u)f(\sin(\pi - u))\,du \quad (\text{but } \sin(\pi - u) = \sin u) \\ &= \pi\int_0^{\pi} f(\sin u)\,du - \int_0^{\pi} uf(\sin u)\,du \\ &= \pi\int_0^{\pi} f(\sin x)\,dx - I. \end{aligned}$$

Now, solving for I, we get

$$\int_0^{\pi} xf(\sin x)\,dx = I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$$

17. $y = 2 + x - x^2$ and $y = 0$ intersect where $2 + x - x^2 = 0$, that is, where $(2-x)(1+x) = 0$, namely at $x = -1$ and $x = 2$. Since $2 + x - x^2 \ge 0$ on $[-1, 2]$, the required area is

$$\int_{-1}^{2} (2+x-x^2)\,dx = \left(2x + \frac{x^2}{2} - \frac{x^3}{3}\right)\bigg|_{-1}^{2} = \frac{9}{2} \text{ sq. units..}$$

18. The area bounded by $y = (x-1)^2$, $y = 0$, and $x = 0$ is

$$\int_0^1 (x-1)^2\,dx = \frac{(x-1)^3}{3}\bigg|_0^1 = \frac{1}{3} \text{ sq. units..}$$

19. $x = y - y^4$ and $x = 0$ intersect where $y - y^4 = 0$, that is, at $y = 0$ and $y = 1$. Since $y - y^4 \ge 0$ on $[0, 1]$, the required area is

$$\int_0^1 (y - y^4 - 0)\,dy = \left(\frac{y^2}{2} - \frac{y^5}{5}\right)\bigg|_0^1 = \frac{3}{10} \text{ sq. units.}$$

20. $y = 4x - x^2$ and $y = 3$ meet where $x^2 - 4x + 3 = 0$, that is, at $x = 1$ and $x = 3$. Since $4x - x^2 \ge 3$ on $[1, 3]$, the required area is

$$\int_1^3 (4x - x^2 - 3)\,dx = \left(2x^2 - \frac{x^3}{3} - 3x\right)\bigg|_1^3 = \frac{4}{3} \text{ sq. units.}$$

21. $y = \sin x$ and $y = \cos(2x)$ intersect at $x = \pi/6$, but nowhere else in the interval $[0, \pi/6]$. The area between the curves in that interval is

$$\begin{aligned} \int_0^{\pi/6} (\cos(2x) - \sin x)\,dx &= \left(\tfrac{1}{2}\sin(2x) + \cos x\right)\bigg|_0^{\pi/6} \\ &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} - 1 = \frac{3\sqrt{3}}{4} - 1 \text{ sq. units..} \end{aligned}$$

22. $y = 5 - x^2$ and $y = 4/x^2$ meet where $5 - x^2 = 4/x^2$, that is, where

$$\begin{aligned} x^4 - 5x^2 + 4 &= 0 \\ (x^2 - 1)(x^2 - 4) &= 0. \end{aligned}$$

There are four intersections: $x = \pm 1$ and $x = \pm 2$. By symmetry (see the figure) the total area bounded by the curves is

$$2\int_1^2 \left(5 - x^2 - \frac{4}{x^2}\right)dx = 2\left(5x - \frac{x^3}{3} + \frac{4}{x}\right)\bigg|_1^2 = \frac{4}{3} \text{ sq. units.}$$

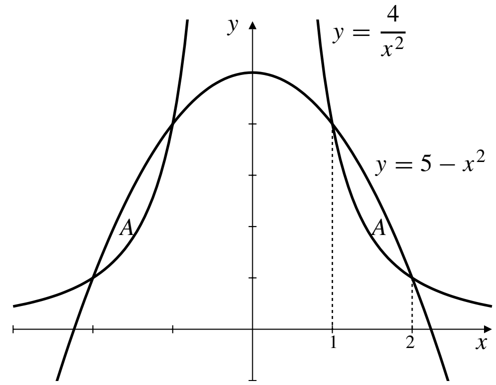

Fig. R-5.22

23. $\int x^2\cos(2x^3 + 1)\,dx \quad$ Let $u = 2x^3 + 1$, $du = 6x^2\,dx$

$$= \frac{1}{6}\int \cos u\,du = \frac{\sin u}{6} + C = \frac{\sin(2x^3 + 1)}{6} + C$$

24. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$ Let $u = \ln x$, $du = dx/x$

$$= \int_0^1 u\,du = \frac{u^2}{2}\bigg|_0^1 = \frac{1}{2}$$

25. $\displaystyle\int_0^4 \sqrt{9t^2 + t^4}\,dt$

$$= \int_0^4 t\sqrt{9+t^2}\,dt \quad \text{Let } u = 9 + t^2,\ du = 2t\,dt$$

$$= \frac{1}{2}\int_9^{25}\sqrt{u}\,du = \frac{1}{3}u^{3/2}\bigg|_9^{25} = \frac{98}{3}$$

26. $\displaystyle\int \sin^3(\pi x)\,dx$

$$= \int \sin(\pi x)\Big(1 - \cos^2(\pi x)\Big)\,dx \quad \text{Let } u = \cos(\pi x),\ du = -\pi\sin(\pi x)\,dx$$

$$= -\frac{1}{\pi}\int (1 - u^2)\,du$$

$$= \frac{1}{\pi}\left(\frac{u^3}{3} - u\right) + C = \frac{1}{3\pi}\cos^3(\pi x) - \frac{1}{\pi}\cos(\pi x) + C$$

27. $\displaystyle\int_0^{\ln 2} \frac{e^u}{4 + e^{2u}}\,du$ Let $v = e^u$, $dv = e^u\,du$

$$= \int_1^2 \frac{dv}{4 + v^2}$$

$$= \frac{1}{2}\tan^{-1}\frac{v}{2}\bigg|_1^2 = \frac{\pi}{8} - \frac{1}{2}\tan^{-1}\frac{1}{2}$$

28. $\displaystyle\int_1^{\sqrt[4]{e}} \frac{\tan^2(\pi\ln x)}{x}\,dx$ Let $u = \pi\ln x$, $du = (\pi/x)\,dx$

$$= \frac{1}{\pi}\int_0^{\pi/4}\tan^2 u\,du = \frac{1}{\pi}\int_0^{\pi/4}(\sec^2 u - 1)\,du$$

$$= \frac{1}{\pi}(\tan u - u)\bigg|_0^{\pi/4} = \frac{1}{\pi} - \frac{1}{4}$$

29. $\displaystyle\int \frac{\sin\sqrt{2s+1}}{\sqrt{2s+1}}\,ds$ Let $u = \sqrt{2s+1}$, $du = ds/\sqrt{2s+1}$

$$= \int \sin u\,du = -\cos u + C = -\cos\sqrt{2s+1} + C$$

30.
$$\int \cos^2\frac{t}{5}\sin^2\frac{t}{5}\,dt = \frac{1}{4}\int \sin^2\frac{2t}{5}\,dt$$
$$= \frac{1}{8}\int\left(1 - \cos\frac{4t}{5}\right)dt$$
$$= \frac{1}{8}\left(t - \frac{5}{4}\sin\frac{4t}{5}\right) + C$$

31. $\displaystyle F(x) = \int_0^{x^2-2x}\frac{1}{1+t^2}\,dt.$

Since $1/(1+t^2) > 0$ for all t, $F(x)$ will be minimum when

$$x^2 - 2x = (x-1)^2 - 1$$

is minimum, that is, when $x = 1$. The minimum value is

$$F(1) = \int_0^{-1}\frac{dt}{1+t^2} = \tan^{-1}t\bigg|_0^{-1} = -\frac{\pi}{4}.$$

F has no maximum value; $F(x) < \pi/2$ for all x, but $F(x) \to \pi/2$ if $x^2 - 2x \to \infty$, which happens as $x \to \pm\infty$.

32. $f(x) = 4x - x^2 \ge 0$ if $0 \le x \le 4$, and $f(x) < 0$ otherwise. If $a < b$, then $\int_a^b f(x)\,dx$ will be maximum if $[a, b] = [0, 4]$; extending the interval to the left of 0 or to the right of 4 will introduce negative contributions to the integral. The maximum value is

$$\int_0^4 (4x - x^2)\,dx = \left(2x^2 - \frac{x^3}{3}\right)\bigg|_0^4 = \frac{32}{3}.$$

33. The average value of $v(t) = dx/dt$ over $[t_0, t_1]$ is

$$\frac{1}{t_1 - t_0}\int_{t_0}^{t_1}\frac{dx}{dt}\,dt = \frac{1}{t_1 - t_0}\,x(t)\bigg|_{t_0}^{t_1} = \frac{x(t_1) - x(t_0)}{t_1 - t_0} = v_{\text{av}}.$$

34. If $y(t)$ is the distance the object falls in t seconds from its release time, then

$$y''(t) = g, \quad y(0) = 0, \quad \text{and } y'(0) = 0.$$

Antidifferentiating twice and using the initial conditions leads to

$$y(t) = \frac{1}{2}gt^2.$$

The average height during the time interval $[0, T]$ is

$$\frac{1}{T}\int_0^T \frac{1}{2}gt^2\,dt = \frac{g}{2T}\,\frac{T^3}{3} = \frac{gT^2}{6} = y\left(\frac{T}{\sqrt{3}}\right).$$

35. Let $f(x) = ax^3 + bx^2 + cx + d$ so that

$$\int_0^1 f(x)\,dx = \frac{a}{4} + \frac{b}{3} + \frac{c}{2} + d.$$

We want this integral to be $\Big(f(x_1) + f(x_2)\Big)/2$ for all choices of a, b, c, and d. Thus we require that

$$a(x_1^3 + x_2^3) + b(x_1^2 + x_2^2) + c(x_1 + x^2) + 2d$$
$$= 2\int_0^1 f(x)\,dx = \frac{a}{2} + \frac{2b}{3} + c + 2d.$$

It follows that x_1 and x_2 must satisfy

$$x_1^3 + x_2^3 = \frac{1}{2} \qquad (1)$$
$$x_1^2 + x_2^2 = \frac{2}{3} \qquad (2)$$
$$x_2 + x_2 = 1. \qquad (3)$$

At first glance this system may seem overdetermined; there are three equations in only two unknowns. However, they do admit a solution as we now show. Squaring equation (3) and subtracting equation (2) we get $2x_1x_2 = 1/3$. Subtracting this latter equation from equation (2) then gives $(x_2 - x_1)^2 = 1/3$, so that $x_2 - x_1 = 1/\sqrt{3}$ (the positive square root since we want $x_1 < x_2$). Adding and subtracting this equation and equation (3) then produces the values $x_2 = (\sqrt{3}+1)/(2\sqrt{3})$ and $x_1 = (\sqrt{3}-1)/(2\sqrt{3})$. These values also satisfy equation (1) since

$$x_1^3 + x_2^3 = (x_2 + x_2)(x_1^2 - x_1x_2 + x_2^2) = 1 \times \left(\frac{2}{3} - \frac{1}{6}\right) = \frac{1}{2}.$$

Challenging Problems 5 (page 315)

1. $x_i = 2^{i/n}$, $0 \le i \le n$, $f(x) = 1/x$ on $[1, 2]$. Since f is decreasing, f is largest at the left endpoint and smallest at the right endpoint of any interval $[2^{(i-1)/n}, 2^{i/n}]$ of the partition. Thus

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} \frac{1}{2^{(i-1)/n}}(2^{i/n} - 2^{(i-1)/n}) \\ &= \sum_{i=1}^{n}(2^{1/n} - 1) = n(2^{1/n} - 1) \\ L(f, P_n) &= \sum_{i=1}^{n} \frac{1}{2^{i/n}}(2^{i/n} - 2^{(i-1)/n}) \\ &= \sum_{i=1}^{n}(1 - 2^{-1/n}) = n(1 - 2^{-1/n}) = \frac{U(f, P_n)}{2^{1/n}}. \end{aligned}$$

Now, by l'Hôpital's rule,

$$\begin{aligned} \lim_{n\to\infty} n(2^{1/n} - 1) &= \lim_{x\to\infty} \frac{2^{1/x} - 1}{1/x} \qquad \left[\frac{0}{0}\right] \\ &= \lim_{x\to\infty} \frac{2^{1/x}\ln 2(-1/x^2)}{-1/x^2} = \ln 2. \end{aligned}$$

Thus $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \ln s$.

2. a)
$$\begin{aligned} &\cos\Big((j+\tfrac{1}{2})t\Big) - \cos\Big((j-\tfrac{1}{2})t\Big) \\ &= \cos(jt)\cos(\tfrac{1}{2}t) - \sin(jt)\sin(\tfrac{1}{2}t) \\ &\qquad - \cos(jt)\cos(\tfrac{1}{2}t) - \sin(jt)\sin(\tfrac{1}{2}t) \\ &= -2\sin(jt)\sin(\tfrac{1}{2}t). \end{aligned}$$
Therefore, we obtain a telescoping sum:

$$\begin{aligned} &\sum_{j=1}^{n}\sin(jt) \\ &= -\frac{1}{2\sin(\frac{1}{2}t)}\sum_{j=1}^{n}\Big[\cos\Big((j+\tfrac{1}{2})t\Big) - \cos\Big((j-\tfrac{1}{2})t\Big)\Big] \\ &= -\frac{1}{2\sin(\frac{1}{2}t)}\Big[\cos\Big((n+\tfrac{1}{2})t\Big) - \cos(\tfrac{1}{2}t)\Big] \\ &= \frac{1}{2\sin(\frac{1}{2}t)}\Big[\cos(\tfrac{1}{2}t) - \cos\Big((n+\tfrac{1}{2})t\Big)\Big]. \end{aligned}$$

b) Let $P_n = \{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \ldots \frac{n\pi}{2n}\}$ be the partition of $[0, \pi/2]$ into n subintervals of equal length $\Delta x = \pi/2n$. Using $t = \pi/2n$ in the formula obtained in part (a), we get

$$\begin{aligned} &\int_0^{\pi/2}\sin x\,dx \\ &= \lim_{n\to\infty}\sum_{j=1}^{n}\sin\left(\frac{j\pi}{2n}\right)\frac{\pi}{2n} \\ &= \lim_{n\to\infty}\frac{\pi}{2n}\,\frac{1}{2\sin(\pi/(4n))}\left(\cos\frac{\pi}{4n} - \cos\frac{(2n+1)\pi}{4n}\right) \\ &= \lim_{n\to\infty}\frac{\pi/(4n)}{\sin(\pi/(4n))}\lim_{n\to\infty}\left(\cos\frac{\pi}{4n} - \cos\frac{(2n+1)\pi}{4n}\right) \\ &= 1 \times \left(\cos 0 - \cos\frac{\pi}{2}\right) = 1. \end{aligned}$$

3. a)
$$\begin{aligned} &\sin\Big((j+\tfrac{1}{2})t\Big) - \sin\Big((j-\tfrac{1}{2})t\Big) \\ &= \sin(jt)\cos(\tfrac{1}{2}t) + \cos(jt)\sin(\tfrac{1}{2}t) \\ &\qquad - \sin(jt)\cos(\tfrac{1}{2}t) + \cos(jt)\sin(\tfrac{1}{2}t) \\ &= 2\cos(jt)\sin(\tfrac{1}{2}t). \end{aligned}$$
Therefore, we obtain a telescoping sum:

$$\begin{aligned} &\sum_{j=1}^{n}\cos(jt) \\ &= \frac{1}{2\sin(\frac{1}{2}t)}\sum_{j=1}^{n}\Big[\sin\Big((j+\tfrac{1}{2})t\Big) - \sin\Big((j-\tfrac{1}{2})t\Big)\Big] \\ &= \frac{1}{2\sin(\frac{1}{2}t)}\Big[\sin\Big((n+\tfrac{1}{2})t\Big) - \sin(\tfrac{1}{2}t)\Big]. \end{aligned}$$

b) Let $P_n = \{0, \frac{\pi}{3n}, \frac{2\pi}{3n}, \frac{3\pi}{3n}, \ldots \frac{n\pi}{3n}\}$ be the partition of $[0, \pi/3]$ into n subintervals of equal length $\Delta x = \pi/3n$. Using $t = \pi/3n$ in the formula obtained in part (a), we get

$$\begin{aligned}
&\int_0^{\pi/2} \cos x\,dx \\
&= \lim_{n\to\infty} \sum_{j=1}^{n} \cos\left(\frac{j\pi}{3n}\right)\frac{\pi}{3n} \\
&= \lim_{n\to\infty} \frac{\pi}{3n}\,\frac{1}{2\sin(\pi/(6n))}\left(\sin\frac{(2n+1)\pi}{6n} - \sin\frac{\pi}{6n}\right) \\
&= \lim_{n\to\infty} \frac{\pi/(6n)}{\sin(\pi/(6n))}\lim_{n\to\infty}\left(\sin\frac{(2n+1)\pi}{6n} - \sin\frac{\pi}{6n}\right) \\
&= 1 \times \left(\sin\frac{\pi}{3} - \sin 0\right) = \frac{\sqrt{3}}{2}.
\end{aligned}$$

4. $f(x) = 1/x^2$, $1 = x_0 < x_1 < x_2 < \cdots < x_n = 2$. If $c_i = \sqrt{x_{i-1}x_i}$, then

$$x_{i-1}^2 < x_{i=1}x_i = c_i^2 < x_i^2,$$

so $x_{i-1} < c_i < x_i$. We have

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \sum_{i=1}^{n} \frac{1}{x_{i-1}x_i}(x_i - x_{i-1}) \\
&= \sum_{i=1}^{n}\left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right) \qquad \text{(telescoping)} \\
&= \frac{1}{x_0} - \frac{1}{x_n} = 1 - \frac{1}{2} = \frac{1}{2}.
\end{aligned}$$

Thus $\displaystyle\int_1^2 \frac{dx}{x^2} = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \frac{1}{2}$.

5. We want to prove that for each positive integer k,

$$\sum_{j=1}^{n} j^k = \frac{n^{k+1}}{k+1} + \frac{n^k}{2} + P_{k-1}(n),$$

where P_{k-1} is a polynomial of degree at most $k-1$. First check the case $k = 1$:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} = \frac{n^{1+1}}{1+1} + \frac{n}{2} + P_0(n),$$

where $P_0(n) = 0$ certainly has degree ≤ 0. Now assume that the formula above holds for $k = 1, 2, 3, \ldots, m$. We will show that it also holds for $k = m+1$. To this end, sum the the formula

$$(j+1)^{m+2} - j^{m+2} = (m+2)j^{m+1} + \frac{(m+2)(m+1)}{2}j^m + \cdots + 1$$

(obtained by the Binomial Theorem) for $j = 1, 2, \ldots, n$. The left side telescopes, and we get

$$\begin{aligned}
(n+1)^{m+2} - 1^{m+2} &= (m+2)\sum_{j=1}^{n} j^{m+1} \\
&\quad + \frac{(m+2)(m+1)}{2}\sum_{j=1}^{n} j^m + \cdots + \sum_{j=1}^{n} 1.
\end{aligned}$$

Expanding the binomial power on the left and using the induction hypothesis on the other terms we get

$$\begin{aligned}
n^{m+2} + (m+2)n^{m+1} + \cdots &= (m+2)\sum_{j=1}^{n} j^{m+1} \\
&\quad + \frac{(m+2)(m+1)}{2}\,\frac{n^{m+1}}{m+1} + \cdots,
\end{aligned}$$

where the $\cdots$ represent terms of degree m or lower in the variable n. Solving for the remaining sum, we get

$$\begin{aligned}
&\sum_{j=1}^{n} j^{m+1} \\
&= \frac{1}{m+2}\left(n^{m+2} + (m+2)n^{m+1} + \cdots - \frac{m+2}{2}n^{m+1} - \cdots\right) \\
&= \frac{n^{m+2}}{m+2} + \frac{n^{m+1}}{2} + \cdots
\end{aligned}$$

so that the formula is also correct for $k = m+1$. Hence it is true for all positive integers k by induction.

b) Using the technique of Example 2 in Section 6.2 and the result above,

$$\begin{aligned}
\int_0^a x^k\,dx &= \lim_{n\to\infty}\frac{a}{n}\sum_{j=1}^{n}\left(\frac{a}{n}\right)^j \\
&= a^{k+1}\lim_{n\to\infty}\frac{1}{n^{k+1}}\sum_{j=1}^{n} j^k \\
&= a^{k+1}\lim_{n\to\infty}\left(\frac{1}{k+1} + \frac{1}{2n} + \frac{P_{k-1}(n)}{n^{k+1}}\right) \\
&= \frac{a^{k+1}}{k+1}.
\end{aligned}$$

6. Let $f(x) = ax^3 + bx^2 + cx + d$. We used Maple to calculate the following:

The tangent to $y = f(x)$ at $P = (p, f(p))$ has equation

$$y = g(x) = ap^3 + bp^2 + cp + d + (3ap^2 + 2bp + c)(x - p).$$

This line intersects $y = f(x)$ at $x = p$ (double root) and at $x = q$, where

$$q = -\frac{2ap + b}{a}.$$

Similarly, the tangent to $y = f(x)$ at $x = q$ has equation

$$y = h(x) = aq^3 + bq^2 + cq + d + (3aq^2 + 2bq + c)(x - q),$$

and intersects $y = f(x)$ at $x = q$ (double root) and $x = r$, where

$$r = -\frac{2aq + b}{a} = \frac{4ap + b}{a}.$$

The area between $y = f(x)$ and the tangent line at P is the absolute value of

$$\int_p^q (f(x) - g(x)\,dx$$
$$= -\frac{1}{12}\left(\frac{81a^4p^4 + 108a^3bp^3 + 54a^2b^2p^2 + 12ab^3p + b^4}{a^3}\right).$$

The area between $y = f(x)$ and the tangent line at $Q = (q, f(q))$ is the absolute value of

$$\int_q^r (f(x) - h(x)\,dx$$
$$= -\frac{4}{3}\left(\frac{81a^4p^4 + 108a^3bp^3 + 54a^2b^2p^2 + 12ab^3p + b^4}{a^3}\right),$$

which is 16 times the area between $y = f(x)$ and the tangent at P.

7. We continue with the calculations begun in the previous problem. P and Q are as they were in that problem, but $R = (r, f(r))$ is now the inflection point of $y = f(x)$, given by $f''(r) = 0$. Maple gives

$$r = -\frac{b}{3a}.$$

Since

$$p - r = \frac{b + 3ap}{a} \text{ and } r - q = \frac{2(b + 3ap)}{a}$$

have the same sign, R must lie between Q and P on the curve $y = f(x)$. The line QR has a rather complicated equation $y = k(x)$, which we won't reproduce here, but the area between this line and the curve $y = f(x)$ is the absolute value of $\int_r^q (f(x) - k(x))\,dx$, which Maple evaluates to be

$$-\frac{4}{81}\left(\frac{81a^4p^4 + 108a^3bp^3 + 54a^2b^2p^2 + 12ab^3p + b^4}{a^3}\right),$$

which is 16/27 of the area between the curve and its tangent at P. This leaves 11/27 of that area to lie between the curve, QR, and the tangent, so QR divides the area between $y = f(x)$ and its tangent at P in the ratio 16/11.

8. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. The tangent to $y = f(x)$ at $P = (p, f(p))$ has equation

$$y = g(x) = ap^4 + bp^3 + cp^2 + dp + e + (4ap^3 + 3bp^2 + 2cp + d)(x - p),$$

and intersects $y = f(x)$ at $x = p$ (double root) and at the two points

$$x = \frac{-2ap - b \pm \sqrt{b^2 - 4ac - 4abp - 8a^2p^2}}{2a}.$$

If these latter two points coincide, then the tangent is a "double tangent." This happens if

$$8a^2p^2 + 4abp + 4ac - b^2 = 0,$$

which has two solutions, which we take to be p and q:

$$p = \frac{-b + \sqrt{3b^2 - 8ac}}{4a}$$
$$q = \frac{-b - \sqrt{3b^2 - 8ac}}{4a} = -p - \frac{b}{2a}.$$

(Both roots exist and are distinct provided $3b^2 > 8ac$.) The point T corresponds to $x = t = (p + q)/2 = -b/4a$. The tangent to $y = f(x)$ at $x = t$ has equation

$$y = h(x) = -\frac{3b^4}{256a^3} + \frac{b^2c}{16a^2} - \frac{bd}{4a} + e + \left(\frac{b^3}{8a^2} - \frac{bc}{2a} + d\right)\left(x + \frac{b}{4a}\right)$$

and it intersects $y = f(x)$ at the points U and V with x-coordinates

$$u = \frac{-b - \sqrt{2}\sqrt{3b^2 - 8ac}}{4a},$$
$$v = \frac{-b + \sqrt{2}\sqrt{3b^2 - 8ac}}{4a}.$$

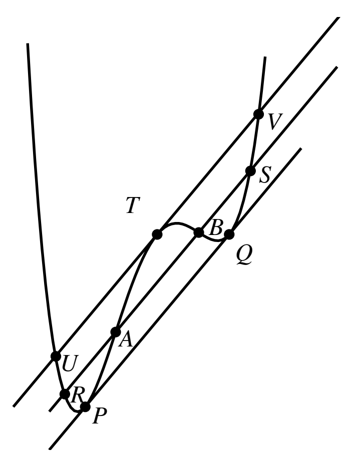

Fig. C-5.8

a) The areas between the curve $y = f(x)$ and the lines PQ and UV are, respectively, the absolute values of

$$A_1 = \int_p^q (f(x)-g(x))\,dx \text{ and } A_2 = \int_u^v (h(x)-f(x))\,dx.$$

Maple calculates these two integrals and simplifies the ratio A_1/A_2 to be $1/\sqrt{2}$.

b) The two inflection points A and B of f have x-coordinates shown by Maple to be

$$\alpha = \frac{-3b - \sqrt{3(3b^2 - 8ac)}}{12a} \quad \text{and}$$
$$\beta = \frac{-3b + \sqrt{3(3b^2 - 8ac)}}{12a}.$$

It then determines the four points of intersection of the line $y = k(x)$ through these inflection points and the curve. The other two points have x-coordinates

$$r = \frac{-3b - \sqrt{15(3b^2 - 8ac)}}{12a} \quad \text{and}$$
$$s = \frac{-3b + \sqrt{15(3b^2 - 8ac)}}{12a}.$$

The region bounded by RS and the curve $y = f(x)$ is divided into three parts by A and B. The areas of these three regions are the absolute values of

$$A_1 = \int_r^\alpha (k(x) - f(x))\,dx$$
$$A_2 = \int_\alpha^\beta (f(x) - k(x))\,dx$$
$$A_3 = \int_\beta^s (k(x) - f(x))\,dx.$$

The expressions calculated by Maple for $k(x)$ and for these three areas are very complicated, but Maple simplifies the rations A_3/A_1 and A_2/A_1 to 1 and 2 respectively, as was to be shown.