CHAPTER 3. TRANSCENDENTAL FUNCTIONS

Section 3.1 Inverse Functions (page 167)

1. $f(x) = x - 1$
$f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1 \Rightarrow x_1 = x_2$.
Thus f is one-to-one. Let $y = f^{-1}(x)$.
Then $x = f(y) = y - 1$ and $y = x + 1$. Thus
$f^{-1}(x) = x + 1$.
$\mathcal{D}(f) = \mathcal{D}(f^{-1}) = \mathbb{R} = \mathcal{R}(f) = \mathcal{R}(f^{-1})$.

2. $f(x) = 2x - 1$. If $f(x_1) = f(x_2)$, then $2x_1 - 1 = 2x_2 - 1$. Thus $2(x_1 - x_2) = 0$ and $x_1 = x_2$. Hence, f is one-to-one.
Let $y = f^{-1}(x)$. Thus $x = f(y) = 2y - 1$, so
$y = \frac{1}{2}(x + 1)$. Thus $f^{-1}(x) = \frac{1}{2}(x + 1)$.
$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = (-\infty, \infty)$.
$\mathcal{R}(f) = \mathcal{D}(f^{-1}) = (-\infty, \infty)$.

3. $f(x) = \sqrt{x - 1}$
$$\begin{aligned} f(x_1) = f(x_2) &\Leftrightarrow \sqrt{x_1 - 1} = \sqrt{x_2 - 1}, \quad (x_1, x_2 \ge 1) \\ &\Leftrightarrow x_1 - 1 = x_2 - 1 = 0 \\ &\Leftrightarrow x_1 = x_2 \end{aligned}$$
Thus f is one-to-one. Let $y = f^{-1}(x)$.
Then $x = f(y) = \sqrt{y - 1}$, and $y = 1 + x^2$. Thus
$f^{-1}(x) = 1 + x^2$, $(x \ge 0)$.
$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = [1, \infty)$, $\mathcal{R}(f) = \mathcal{D}(f^{-1}) = [0, \infty)$.

4. $f(x) = -\sqrt{x - 1}$ for $x \ge 1$.
If $f(x_1) = f(x_2)$, then $-\sqrt{x_1 - 1} = -\sqrt{x_2 - 1}$ and $x_1 - 1 = x_2 - 1$. Thus $x_1 = x_2$ and f is one-to-one.
Let $y = f^{-1}(x)$. Then $x = f(y) = -\sqrt{y - 1}$ so $x^2 = y - 1$ and $y = x^2 + 1$. Thus, $f^{-1}(x) = x^2 + 1$.
$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = [1, \infty)$. $\mathcal{R}(f) = \mathcal{D}(f^{-1}) = (-\infty, 0]$.

5. $f(x) = x^3$
$$\begin{aligned} f(x_1) = f(x_2) &\Leftrightarrow x_1^3 = x_2^3 \\ &\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \\ &\Rightarrow x_1 = x_2 \end{aligned}$$
Thus f is one-to-one. Let $y = f^{-1}(x)$.
Then $x = f(y) = y^3$ so $y = x^{1/3}$.
Thus $f^{-1}(x) = x^{1/3}$.
$\mathcal{D}(f) = \mathcal{D}(f^{-1}) = \mathbb{R} = \mathcal{R}(f) = \mathcal{R}(f^{-1})$.

6. $f(x) = 1 + \sqrt[3]{x}$. If $f(x_1) = f(x_2)$, then $1 + \sqrt[3]{x_1} = 1 + \sqrt[3]{x_2}$ so $x_1 = x_2$. Thus, f is one-to-one.
Let $y = f^{-1}(x)$ so that $x = f(y) = 1 + \sqrt[3]{y}$. Thus $y = (x - 1)^3$ and $f^{-1}(x) = (x - 1)^3$.
$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = (-\infty, \infty)$.
$\mathcal{R}(f) = \mathcal{D}(f^{-1}) = (-\infty, \infty)$.

7.
$$\begin{aligned} f(x) &= x^2, \quad (x \le 0) \\ f(x_1) = f(x_2) &\Leftrightarrow x_1^2 = x_2^2, \quad (x_1 \le 0,\ x_2 \le 0) \\ &\Leftrightarrow x_1 = x_2 \end{aligned}$$
Thus f is one-to-one. Let $y = f^{-1}(x)$.
Then $x = f(y) = y^2$ $(y \le 0)$.
therefore $y = -\sqrt{x}$ and $f^{-1}(x) = -\sqrt{x}$.
$\mathcal{D}(f) = (-\infty, 0] = \mathcal{R}(f^{-1})$,
$\mathcal{D}(f^{-1}) = [0, \infty) = \mathcal{R}(f)$.

8. $f(x) = (1 - 2x)^3$. If $f(x_1) = f(x_2)$, then $(1 - 2x_1)^3 = (1 - 2x_2)^3$ and $x_1 = x_2$. Thus, f is one-to-one.
Let $y = f^{-1}(x)$. Then $x = f(y) = (1 - 2y)^3$ so
$y = \frac{1}{2}(1 - \sqrt[3]{x})$. Thus, $f^{-1}(x) = \frac{1}{2}(1 - \sqrt[3]{x})$.
$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = (-\infty, \infty)$.
$\mathcal{R}(f) = \mathcal{D}(f^{-1}) = (-\infty, \infty)$.

9. $f(x) = \dfrac{1}{x + 1}$. $\mathcal{D}(f) = \{x : x \ne -1\} = \mathcal{R}(f^{-1})$.
$$\begin{aligned} f(x_1) = f(x_2) &\Leftrightarrow \frac{1}{x_1 + 1} = \frac{1}{x_2 + 1} \\ &\Leftrightarrow x_2 + 1 = x_1 + 1 \\ &\Leftrightarrow x_2 = x_1 \end{aligned}$$
Thus f is one-to-one; Let $y = f^{-1}(x)$.
Then $x = f(y) = \dfrac{1}{y + 1}$
so $y + 1 = \dfrac{1}{x}$ and $y = f^{-1}(x) = \dfrac{1}{x} - 1$.
$\mathcal{D}(f^{-1}) = \{x : x \ne 0\} = \mathcal{R}(f)$.

10. $f(x) = \dfrac{x}{1 + x}$. If $f(x_1) = f(x_2)$, then $\dfrac{x_1}{1 + x_1} = \dfrac{x_2}{1 + x_2}$.
Hence $x_1(1 + x_2) = x_2(1 + x_1)$ and, on simplification, $x_1 = x_2$. Thus, f is one-to-one.
Let $y = f^{-1}(x)$. Then $x = f(y) = \dfrac{y}{1 + y}$ and
$x(1 + y) = y$. Thus $y = \dfrac{x}{1 - x} = f^{-1}(x)$.
$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = (-\infty, -1) \cup (-1, \infty)$.
$\mathcal{R}(f) = \mathcal{D}(f^{-1}) = (-\infty, 1) \cup (1, \infty)$.

11. $f(x) = \dfrac{1 - 2x}{1 + x}$. $\mathcal{D}(f) = \{x : x \ne -1\} = \mathcal{R}(f^{-1})$
$$\begin{aligned} f(x_1) = f(x_2) &\Leftrightarrow \frac{1 - 2x_1}{1 + x_1} = \frac{1 - 2x_2}{1 + x_2} \\ &\Leftrightarrow 1 + x_2 - 2x_1 - 2x_1x_2 = 1 + x_1 - 2x_2 - 2x_1x_2 \\ &\Leftrightarrow 3x_2 = 3x_1 \Leftrightarrow x_1 = x_2 \end{aligned}$$
Thus f is one-to-one. Let $y = f^{-1}(x)$.
Then $x = f(y) = \dfrac{1 - 2y}{1 + y}$
so $x + xy = 1 - 2y$
and $f^{-1}(x) = y = \dfrac{1 - x}{2 + x}$.
$\mathcal{D}(f^{-1}) = \{x : x \ne -2\} = \mathcal{R}(f)$.

12. $f(x) = \dfrac{x}{\sqrt{x^2+1}}$. If $f(x_1) = f(x_2)$, then
$$\frac{x_1}{\sqrt{x_1^2+1}} = \frac{x_2}{\sqrt{x_2^2+1}}. \qquad (*)$$
Thus $x_1^2(x_2^2+1) = x_2^2(x_1^2+1)$ and $x_1^2 = x_2^2$.
From (*), x_1 and x_2 must have the same sign. Hence, $x_1 = x_2$ and f is one-to-one.
Let $y = f^{-1}(x)$. Then $x = f(y) = \dfrac{y}{\sqrt{y^2+1}}$, and $x^2(y^2+1) = y^2$. Hence $y^2 = \dfrac{x^2}{1-x^2}$. Since $f(y)$ and y have the same sign, we must have $y = \dfrac{x}{\sqrt{1-x^2}}$, so
$f^{-1}(x) = \dfrac{x}{\sqrt{1-x^2}}$.
$\mathcal{D}(f) = \mathcal{R}(f^{-1}) = (-\infty, \infty)$.
$\mathcal{R}(f) = \mathcal{D}(f^{-1}) = (-1, 1)$.

13. $g(x) = f(x) - 2$
Let $y = g^{-1}(x)$. Then $x = g(y) = f(y) - 2$, so $f(y) = x + 2$ and $g^{-1}(x) = y = f^{-1}(x+2)$.

14. $h(x) = f(2x)$. Let $y = h^{-1}(x)$. Then $x = h(y) = f(2y)$ and $2y = f^{-1}(x)$. Thus $h^{-1}(x) = y = \frac{1}{2}f^{-1}(x)$.

15. $k(x) = -3f(x)$. Let $y = k^{-1}(x)$. Then $x = k(y) = -3f(y)$, so $f(y) = -\dfrac{x}{3}$ and $k^{-1}(x) = y = f^{-1}\left(-\dfrac{x}{3}\right)$.

16. $m(x) = f(x-2)$. Let $y = m^{-1}(x)$. Then $x = m(y) = f(y-2)$, and $y - 2 = f^{-1}(x)$.
Hence $m^{-1}(x) = y = f^{-1}(x) + 2$.

17. $p(x) = \dfrac{1}{1+f(x)}$. Let $y = p^{-1}(x)$.
Then $x = p(y) = \dfrac{1}{1+f(y)}$ so $f(y) = \dfrac{1}{x} - 1$,
and $p^{-1}(x) = y = f^{-1}\left(\dfrac{1}{x} - 1\right)$.

18. $q(x) = \dfrac{f(x)-3}{2}$ Let $y = q^{-1}(x)$. Then
$x = q(y) = \dfrac{f(y)-3}{2}$ and $f(y) = 2x+3$. Hence
$q^{-1}(x) = y = f^{-1}(2x+3)$.

19. $r(x) = 1 - 2f(3-4x)$
Let $y = r^{-1}(x)$. Then $x = r(y) = 1 - 2f(3-4y)$.
$$f(3-4y) = \frac{1-x}{2}$$
$$3 - 4y = f^{-1}\left(\frac{1-x}{2}\right)$$
and $r^{-1}(x) = y = \dfrac{1}{4}\left(3 - f^{-1}\left(\dfrac{1-x}{2}\right)\right)$.

20. $s(x) = \dfrac{1+f(x)}{1-f(x)}$. Let $y = s^{-1}(x)$.
Then $x = s(y) = \dfrac{1+f(y)}{1-f(y)}$. Solving for $f(y)$ we obtain
$f(y) = \dfrac{x-1}{x+1}$. Hence $s^{-1}(x) = y = f^{-1}\left(\dfrac{x-1}{x+1}\right)$.

21. $f(x) = x^2 + 1$ if $x \ge 0$, and $f(x) = x + 1$ if $x < 0$.
If $f(x_1) = f(x_2)$ then if $x_1 \ge 0$ and $x_2 \ge 0$ then $x_1^2 + 1 = x_2^2 + 1$ so $x_1 = x_2$;
if $x_1 \ge 0$ and $x_2 < 0$ then $x_1^2 + 1 = x_2 + 1$ so $x_2 = x_1^2$ (not possible);
if $x_1 < 0$ and $x_2 \ge 0$ then $x_1 = x_2^2$ (not possible);
if $x_1 < 0$ and $x_2 < 0$ then $x_1 + 1 = x_2 + 1$ so $x_1 = x_2$.
Therefore f is one-to-one. Let $y = f^{-1}(x)$. Then
$$x = f(y) = \begin{cases} y^2+1 & \text{if } y \ge 0 \\ y+1 & \text{if } y < 0. \end{cases}$$
Thus $f^{-1}(x) = y = \begin{cases} \sqrt{x-1} & \text{if } x \ge 1 \\ x-1 & \text{if } x < 1. \end{cases}$

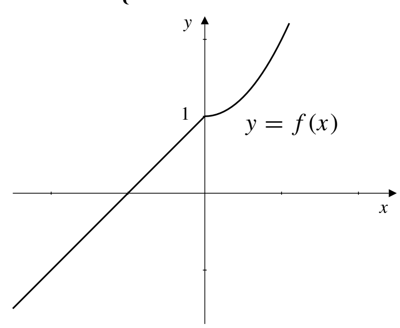

Fig. 3.1.21

22. $g(x) = x^3$ if $x \ge 0$, and $g(x) = x^{1/3}$ if $x < 0$.
Suppose $f(x_1) = f(x_2)$. If $x_1 \ge 0$ and $x_2 \ge 0$ then $x_1^3 = x_2^3$ so $x_1 = x_2$.
Similarly, $x_1 = x_2$ if both are negative. If x_1 and x_2 have opposite sign, then so do $g(x_1)$ and $g(x_2)$.
Therefore g is one-to-one. Let $y = g^{-1}(x)$. Then
$$x = g(y) = \begin{cases} y^3 & \text{if } y \ge 0 \\ y^{1/3} & \text{if } y < 0. \end{cases}$$
Thus $g^{-1}(x) = y = \begin{cases} x^{1/3} & \text{if } x \ge 0 \\ x^3 & \text{if } x < 0. \end{cases}$

23. If x_1 and x_2 are both positive or both negative, and $h(x_1) = h(x_2)$, then $x_1^2 = x_2^2$ so $x_1 = x_2$. If x_1 and x_2 have opposite sign, then $h(x_1)$ and $h(x_2)$ are on opposite sides of 1, so cannot be equal. Hence h is one-to-one.
If $y = h^{-1}(x)$, then $x = h(y) = \begin{cases} y^2+1 & \text{if } y \ge 0 \\ -y^2+1 & \text{if } y < 0 \end{cases}$. If $y \ge 0$, then $y = \sqrt{x-1}$. If $y < 0$, then $y = \sqrt{1-x}$.
Thus $h^{-1}(x) = \begin{cases} \sqrt{x-1} & \text{if } x \ge 1 \\ \sqrt{1-x} & \text{if } x < 1 \end{cases}$

24. $y = f^{-1}(x) \Leftrightarrow x = f(y) = y^3 + y$. To find $y = f^{-1}(2)$ we solve $y^3 + y = 2$ for y. Evidently $y = 1$ is the only solution, so $f^{-1}(2) = 1$.

25. $g(x) = 1$ if $x^3 + x = 10$, that is, if $x = 2$. Thus $g^{-1}(1) = 2$.

26. $h(x) = -3$ if $x|x| = -4$, that is, if $x = -2$. Thus $h^{-1}(-3) = -2$.

27. If $y = f^{-1}(x)$ then $x = f(y)$.
Thus $1 = f'(y)\dfrac{dy}{dx}$ so $\dfrac{dy}{dx} = \dfrac{1}{f'(y)} = \dfrac{1}{\frac{1}{y}} = y$
(since $f'(x) = 1/x$).

28. $f(x) = 1 + 2x^3$
Let $y = f^{-1}(x)$.
Thus $x = f(y) = 1 + 2y^3$.
$1 = 6y^2\dfrac{dy}{dx}$ so $(f^{-1})'(x) = \dfrac{dy}{dx} = \dfrac{1}{6y^2} = \dfrac{1}{6[f^{-1}(x)]^2}$

29. If $f(x) = \dfrac{4x^3}{x^2+1}$, then

$$f'(x) = \frac{(x^2+1)(12x^2) - 4x^3(2x)}{(x^2+1)^2} = \frac{4x^2(x^2+3)}{(x^2+1)^2}.$$

Since $f'(x) > 0$ for all x, except $x = 0$, f must be one-to-one and so it has an inverse.
If $y = f^{-1}(x)$, then $x = f(y) = \dfrac{4y^3}{y^2+1}$, and

$$1 = f'(y) = \frac{(y^2+1)(12y^2y') - 4y^3(2yy')}{(y^2+1)^2}.$$

Thus $y' = \dfrac{(y^2+1)^2}{4y^4+12y^2}$. Since $f(1) = 2$, therefore $f^{-1}(2) = 1$ and

$$\left(f^{-1}\right)'(2) = \left.\frac{(y^2+1)^2}{4y^4+12y^2}\right|_{y=1} = \frac{1}{4}.$$

30. If $f(x) = x\sqrt{3+x^2}$ and $y = f^{-1}(x)$, then $x = f(y) = y\sqrt{3+y^2}$, so,

$$1 = y'\sqrt{3+y^2} + y\frac{2yy'}{2\sqrt{3+y^2}} \quad\Rightarrow\quad y' = \frac{\sqrt{3+y^2}}{3+2y^2}.$$

Since $f(-1) = -2$ implies that $f^{-1}(-2) = -1$, we have

$$\left(f^{-1}\right)'(-2) = \left.\frac{\sqrt{3+y^2}}{3+2y^2}\right|_{y=-1} = \frac{2}{5}.$$

Note: $f(x) = x\sqrt{3+x^2} = -2 \Rightarrow x^2(3+x^2) = 4$
$\Rightarrow x^4 + 3x^2 - 4 = 0 \Rightarrow (x^2+4)(x^2-1) = 0$.
Since $(x^2+4) = 0$ has no real solution, therefore $x^2 - 1 = 0$ and $x = 1$ or -1. Since it is given that $f(x) = -2$, therefore x must be -1.

31. $y = f^{-1}(2) \Leftrightarrow 2 = f(y) = y^2/(1+\sqrt{y})$. We must solve $2 + 2\sqrt{y} = y^2$ for y. There is a root between 2 and 3: $f^{-1}(2) \approx 2.23362$ to 5 decimal places.

32. $g(x) = 2x + \sin x \Rightarrow g'(x) = 2 + \cos x \geq 1$ for all x. Therefore g is increasing, and so one-to-one and invertible on the whole real line.

$y = g^{-1}(x) \Leftrightarrow x = g(y) = 2y + \sin y$. For $y = g^{-1}(2)$, we need to solve $2y + \sin y - 2 = 0$. The root is between 0 and 1; to five decimal places $g^{-1}(2) = y \approx 0.68404$. Also

$$1 = \frac{dx}{dx} = (2 + \cos y)\frac{dy}{dx}$$
$$(g^{-1})'(2) = \left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2+\cos y} \approx 0.36036.$$

33. If $f(x) = x\sec x$, then $f'(x) = \sec x + x\sec x\tan x \geq 1$ for x in $(-\pi/2, \pi/2)$. Thus f is increasing, and so one-to-one on that interval. Moreover,
$\lim_{x\to-(\pi/2)+} f(x) = -\infty$ and $\lim_{x\to(\pi/2)+} f(x) = \infty$, so, being continuous, f has range $(-\infty, \infty)$, and so f^{-1} has domain $(-\infty, \infty)$.
Since $f(0) = 0$, we have $f^{-1}(0) = 0$, and

$$(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = 1.$$

34. If $y = (f \circ g)^{-1}(x)$, then $x = f \circ g(y) = f(g(y))$. Thus $g(y) = f^{-1}(x)$ and $y = g^{-1}(f^{-1}(x)) = g^{-1} \circ f^{-1}(x)$. That is, $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

35. $f(x) = \dfrac{x-a}{bx-c}$
Let $y = f^{-1}(x)$. Then $x = f(y) = \dfrac{y-a}{by-c}$ and
$bxy - cx = y - a$ so $y = \dfrac{cx-a}{bx-1}$. We have
$f^{-1}(x) = f(x)$ if $\dfrac{x-a}{bx-c} = \dfrac{cx-a}{bx-1}$. Evidently it is necessary and sufficient that $c = 1$. a and b may have any values.

36. Let $f(x)$ be an even function. Then $f(x) = f(-x)$.
Hence, f is not one-to-one and it is not invertible.
Therefore, it cannot be self-inverse.
An odd function $g(x)$ may be self-inverse if its graph is symmetric about the line $x = y$. Examples are $g(x) = x$ and $g(x) = 1/x$.

37. No. A function that is one-to-one on a single interval need not be either increasing or decreasing. For example, consider the function defined on $[0, 2]$ by

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1 \\ -x & \text{if } 1 < x \leq 2. \end{cases}$$

It is one-to-one but neither increasing nor decreasing on all of $[0, 2]$.

38. First we consider the case where the domain of f is a closed interval. Suppose that f is one-to-one and continuous on $[a, b]$, and that $f(a) < f(b)$. We show that f must be increasing on $[a, b]$. Suppose not. Then there are numbers x_1 and x_2 with $a \le x_1 < x_2 \le b$ and $f(x_1) > f(x_2)$. If $f(x_1) > f(a)$, let u be a number such that $u < f(x_1)$, $f(x_2) < u$, and $f(a) < u$. By the Intermediate-Value Theorem there exist numbers c_1 in (a, x_1) and c_2 in (x_1, x_2) such that $f(c_1) = u = f(c_2)$, contradicting the one-to-oneness of f. A similar contradiction arises if $f(x_1) \le f(a)$ because, in this case, $f(x_2) < f(b)$ and we can find c_1 in (x_1, x_2) and c_2 in (x_2, b) such that $f(c_1) = f(c_2)$. Thus f must be increasing on $[a, b]$.

A similar argument shows that if $f(a) > f(b)$, then f must be decreasing on $[a, b]$.

Finally, if the interval I where f is defined is not necessarily closed, the same argument shows that if $[a, b]$ is a subinterval of I on which f is increasing (or decreasing), then f must also be increasing (or decreasing) on any intervals of either of the forms $[x_1, b]$ or $[a, x_2]$, where x_1 and x_2 are in I and $x_1 \le a < b \le x_2$. So f must be increasing (or decreasing) on the whole of I.

Section 3.2 Exponential and Logarithmic Functions (page 171)

1. $\dfrac{3^3}{\sqrt{3^5}} = 3^{3-5/2} = 3^{1/2} = \sqrt{3}$

2. $2^{1/2}8^{1/2} = 2^{1/2}2^{3/2} = 2^2 = 4$

3. $(x^{-3})^{-2} = x^6$

4. $(\frac{1}{2})^x 4^{x/2} = \dfrac{2^x}{2^x} = 1$

5. $\log_5 125 = \log_5 5^3 = 3$

6. If $\log_4(\frac{1}{8}) = y$ then $4^y = \frac{1}{8}$, or $2^{2y} = 2^{-3}$. Thus $2y = -3$ and $\log_4(\frac{1}{8}) = y = -\frac{3}{2}$.

7. $\log_{1/3} 3^{2x} = \log_{1/3}\left(\dfrac{1}{3}\right)^{-2x} = -2x$

8. $4^{3/2} = 8 \quad \Rightarrow \quad \log_4 8 = \frac{3}{2} \quad \Rightarrow \quad 2^{\log_4 8} = 2^{3/2} = 2\sqrt{2}$

9. $10^{-\log_{10}(1/x)} = \dfrac{1}{1/x} = x$

10. Since $\log_a\left(x^{1/(\log_a x)}\right) = \dfrac{1}{\log_a x}\log_a x = 1$, therefore $x^{1/(\log_a x)} = a^1 = a$.

11. $(\log_a b)(\log_b a) = \log_a a = 1$

12. $$\log_x\left(x(\log_y y^2)\right) = \log_x(2x) = \log_x x + \log_x 2$$
$$= 1 + \log_x 2 = 1 + \frac{1}{\log_2 x}$$

13. $(\log_4 16)(\log_4 2) = 2 \times \dfrac{1}{2} = 1$

14. $\log_{15} 75 + \log_{15} 3 = \log_{15} 225 = 2$
(since $15^2 = 225$)

15. $\log_6 9 + \log_6 4 = \log_6 36 = 2$

16. $$2\log_3 12 - 4\log_3 6 = \log_3\left(\frac{4^2 \cdot 3^2}{2^4 \cdot 3^4}\right)$$
$$= \log_3(3^{-2}) = -2$$

17. $$\begin{aligned}&\log_a(x^4 + 3x^2 + 2) + \log_a(x^4 + 5x^2 + 6)\\ &\quad - 4\log_a\sqrt{x^2+2}\\ &= \log_a\left((x^2+2)(x^2+1)\right) + \log_a\left((x^2+2)(x^2+3)\right)\\ &\quad - 2\log_1(x^2+2)\\ &= \log_a(x^2+1) + \log_a(x^2+3)\\ &= \log_a(x^4 + 4x^2 + 3)\end{aligned}$$

18. $$\begin{aligned}&\log_\pi(1-\cos x) + \log_\pi(1+\cos x) - 2\log_\pi \sin x\\ &= \log_\pi\left[\frac{(1-\cos x)(1+\cos x)}{\sin^2 x}\right] = \log_\pi \frac{\sin^2 x}{\sin^2 x}\\ &= \log_\pi 1 = 0\end{aligned}$$

19. $y = 3^{\sqrt{2}}$, $\log_{10} y = \sqrt{2}\log_{10} 3$,
$y = 10^{\sqrt{2}\log_{10} 3} \approx 4.72880$

20. $\log_3 5 = (\log_{10} 5)/(\log_{10} 3 \approx 1.46497$

21. $2^{2x} = 5^{x+1}$, $2x\log_{10} 2 = (x+1)\log_{10} 5$,
$x = (\log_{10} 5)/(2\log_{10} 2 - \log_{10} 5) \approx -7.21257$

22. $x^{\sqrt{2}} = 3$, $\sqrt{2}\log_{10} x = \log_{10} 3$,
$x = 10^{(\log_{10} 3)/\sqrt{2}} \approx 2.17458$

23. $\log_x 3 = 5$, $(\log_{10} 3)/(\log_{10} x) = 5$,
$\log_{10} x = (\log_{10} 3)/5$, $x = 10^{(\log_{10} 3)/5} \approx 1.24573$

24. $\log_3 x = 5$, $(\log_{10} x)/(\log_{10} 3) = 5$,
$\log_{10} x = 5\log_{10} 3$, $x = 10^{5\log_{10} 3} = 3^5 = 243$

25. Let $u = \log_a\left(\dfrac{1}{x}\right)$ then $a^u = \dfrac{1}{x} = x^{-1}$. Hence, $a^{-u} = x$ and $u = -\log_a x$.
Thus, $\log_a\left(\dfrac{1}{x}\right) = -\log_a x$.

26. Let $\log_a x = u$, $\log_a y = v$.
Then $x = a^u$, $y = a^v$.
Thus $\dfrac{x}{y} = \dfrac{a^u}{a^v} = a^{u-v}$
and $\log_a\left(\dfrac{x}{y}\right) = u - v = \log_a x - \log_a y$.

27. Let $u = \log_a(x^y)$, then $a^u = x^y$ and $a^{u/y} = x$.
Therefore $\dfrac{u}{y} = \log_a x$, or $u = y \log_a x$.
Thus, $\log_a(x^y) = y \log_a x$.

28. Let $\log_b x = u$, $\log_b a = v$.
Thus $b^u = x$ and $b^v = a$.
Therefore $x = b^u = b^{v(u/v)} = a^{u/v}$
and $\log_a x = \dfrac{u}{v} = \dfrac{\log_b x}{\log_b a}$.

29.
$$\log_4(x+4) - 2\log_4(x+1) = \frac{1}{2}$$
$$\log_4 \frac{x+4}{(x+1)^2} = \frac{1}{2}$$
$$\frac{x+4}{(x+1)^2} = 4^{1/2} = 2$$
$2x^2 + 3x - 2 = 0$ but we need $x + 1 > 0$, so $x = 1/2$.

30. First observe that $\log_9 x = \log_3 x / \log_3 9 = \frac{1}{2}\log_3 x$. Now
$$2\log_3 x + \log_9 x = 10$$
$$\log_3 x^2 + \log_3 x^{1/2} = 10$$
$$\log_3 x^{5/2} = 10$$
$x^{5/2} = 3^{10}$, so $x = (3^{10})^{2/5} = 3^4 = 81$

31. Note that $\log_x 2 = 1/\log_2 x$.
Since $\lim_{x\to\infty} \log_2 x = \infty$, therefore $\lim_{x\to\infty} \log_x 2 = 0$.

32. Note that $\log_x(1/2) = -\log_x 2 = -1/\log_2 x$.
Since $\lim_{x\to 0+} \log_2 x = -\infty$, therefore
$\lim_{x\to 0+} \log_x(1/2) = 0$.

33. Note that $\log_x 2 = 1/\log_2 x$.
Since $\lim_{x\to 1+} \log_2 x = 0+$, therefore
$\lim_{x\to 1+} \log_x 2 = \infty$.

34. Note that $\log_x 2 = 1/\log_2 x$.
Since $\lim_{x\to 1-} \log_2 x = 0-$, therefore
$\lim_{x\to 1-} \log_x 2 = -\infty$.

35. $f(x) = a^x$ and $f'(0) = \lim\limits_{h\to 0} \dfrac{a^h - 1}{h} = k$. Thus
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x a^h - a^x}{h} \\ &= a^x \lim_{h\to 0} \frac{a^h - 1}{h} = a^x f'(0) = a^x k = k f(x). \end{aligned}$$

36.
$$\begin{aligned} y = f^{-1}(x) &\Rightarrow x = f(y) = a^y \\ &\Rightarrow 1 = \frac{dx}{dx} = k a^y \frac{dy}{dx} \\ &\Rightarrow \frac{dy}{dx} = \frac{1}{k a^y} = \frac{1}{kx}. \end{aligned}$$
Thus $(f^{-1})'(x) = 1/(kx)$.

Section 3.3 The Natural Logarithm and Exponential (page 179)

1. $\dfrac{e^3}{\sqrt{e^5}} = e^{3-5/2} = e^{1/2} = \sqrt{e}$

2. $\ln(e^{1/2}e^{2/3}) = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

3. $e^{5\ln x} = x^5$

4. $e^{(3\ln 9)/2} = 9^{3/2} = 27$

5. $\ln \dfrac{1}{e^{3x}} = \ln e^{-3x} = -3x$

6. $e^{2\ln\cos x} + \left(\ln e^{\sin x}\right)^2 = \cos^2 x + \sin^2 x = 1$

7. $3\ln 4 - 4\ln 3 = \ln \dfrac{4^3}{3^4} = \ln \dfrac{64}{81}$

8. $4\ln\sqrt{x} + 6\ln(x^{1/3}) = 2\ln x + 2\ln x = 4\ln x$

9. $2\ln x + 5\ln(x-2) = \ln\left(x^2(x-2)^5\right)$

10. $\ln(x^2 + 6x + 9) = \ln[(x+3)^2] = 2\ln(x+3)$

11.
$$2^{x+1} = 3^x$$
$$(x+1)\ln 2 = x\ln 3$$
$$x = \frac{\ln 2}{\ln 3 - \ln 2} = \frac{\ln 2}{\ln(3/2)}$$

12. $3^x = 9^{1-x} \Rightarrow 3^x = 3^{2(1-x)}$
$\Rightarrow \quad x = 2(1-x) \quad \Rightarrow \quad x = \frac{2}{3}$

13.
$$\begin{aligned} \frac{1}{2^x} &= \frac{5}{8^{x+3}} \\ -x\ln 2 &= \ln 5 - (x+3)\ln 8 \\ &= \ln 5 - (3x+9)\ln 2 \\ 2x\ln 2 &= \ln 5 - 9\ln 2 \\ x &= \frac{\ln 5 - 9\ln 2}{2\ln 2} \end{aligned}$$

14. $2^{x^2-3} = 4^x = 2^{2x} \Rightarrow x^2 - 3 = 2x$
$x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0$
Hence, $x = -1$ or 3.

15. $\ln(x/(2-x))$ is defined if $x/(2-x) > 0$, that is, if $0 < x < 2$. The domain is the interval $(0, 2)$.

16. $\ln(x^2 - x - 2) = \ln[(x-2)(x+1)]$ is defined if $(x-2)(x+1) > 0$, that is, if $x < -1$ or $x > 2$. The domain is the union $(-\infty, -1) \cup (2, \infty)$.

17. $\ln(2x - 5) > \ln(7 - 2x)$ holds if $2x - 5 > 0$, $7 - 2x > 0$, and $2x - 5 > 7 - 2x$, that is, if $x > 5/2$, $x < 7/2$, and $4x > 12$ (i.e., $x > 3$). The solution set is the interval $(3, 7/2)$.

18. $\ln(x^2-2) \le \ln x$ holds if $x^2 > 2$, $x > 0$, and $x^2 - 2 \le x$. Thus we need $x > \sqrt{2}$ and $x^2 - x - 2 \le 0$. This latter inequality says that $(x-2)(x+1) \le 0$, so it holds for $-1 \le x \le 2$. The solution set of the given inequality is $(\sqrt{2}, 2]$.

19. $y = e^{5x}, \quad y' = 5e^{5x}$

20. $y = xe^x - x, \qquad y' = e^x + xe^x - 1$

21.
$$\begin{aligned} y &= \frac{x}{e^{2x}} = xe^{-2x} \\ y' &= e^{-2x} - 2xe^{-2x} \\ &= (1-2x)e^{-2x} \end{aligned}$$

22. $y = x^2 e^{x/2}, \qquad y' = 2xe^{x/2} + \frac{1}{2}x^2 e^{x/2}$

23. $y = \ln(3x-2) \qquad y' = \dfrac{3}{3x-2}$

24. $y = \ln|3x-2|, \qquad y' = \dfrac{3}{3x-2}$

25. $y = \ln(1+e^x) \qquad y' = \dfrac{e^x}{1+e^x}$

26. $f(x) = e^{x^2}, \qquad f'(x) = (2x)e^{x^2}$

27. $y = \dfrac{e^x + e^{-x}}{2}, \quad y' = \dfrac{e^x - e^{-x}}{2}$

28. $x = e^{3t}\ \ln t, \qquad \dfrac{dx}{dt} = 3e^{3t}\ \ln t + \dfrac{1}{t}e^{3t}$

29. $y = e^{(e^x)}, \quad y' = e^x e^{(e^x)} = e^{x+e^x}$

30. $y = \dfrac{e^x}{1+e^x} = 1 - \dfrac{1}{1+e^x}, \quad y' = \dfrac{e^x}{(1+e^x)^2}$

31. $y = e^x \sin x, \quad y' = e^x(\sin x + \cos x)$

32. $y = e^{-x}\ \cos x, \qquad y' = -e^{-x}\cos x - e^{-x}\sin x$

33. $y = \ln\ln x \qquad y' = \dfrac{1}{x\ln x}$

34.
$$\begin{aligned} y &= x\ln x - x \\ y' &= \ln x + x\left(\frac{1}{x}\right) - 1 = \ln x \end{aligned}$$

35.
$$\begin{aligned} y &= x^2 \ln x - \frac{x^2}{2} \\ y' &= 2x\ln x + \frac{x^2}{x} - \frac{2x}{2} = 2x\ln x \end{aligned}$$

36. $y = \ln|\sin x|, \qquad y' = \dfrac{\cos x}{\sin x} = \cot x$

37.
$$\begin{aligned} y &= 5^{2x+1} \\ y' &= 2(5^{2x+1})\ln 5 = (2\ln 5)5^{2x+1} \end{aligned}$$

38. $y = 2^{(x^2-3x+8)}, \qquad y' = (2x-3)(\ln 2)2^{(x^2-3x+8)}$

39. $g(x) = t^x x^t, \quad g'(x) = t^x x^t \ln t + t^{x+1}x^{t-1}$

40. $h(t) = t^x - x^t, \qquad h'(t) = xt^{x-1} - x^t \ln x$

41.
$$\begin{aligned} f(s) &= \log_a(bs+c) = \frac{\ln(bs+c)}{\ln a} \\ f'(s) &= \frac{b}{(bs+c)\ln a} \end{aligned}$$

42.
$$\begin{aligned} g(x) &= \log_x(2x+3) = \frac{\ln(2x+3)}{\ln x} \\ g'(x) &= \frac{\ln x\left(\dfrac{2}{2x+3}\right) - [\ln(2x+3)]\left(\dfrac{1}{x}\right)}{(\ln x)^2} \\ &= \frac{2x\ln x - (2x+3)\ln(2x+3)}{x(2x+3)(\ln x)^2} \end{aligned}$$

43.
$$\begin{aligned} y &= x^{\sqrt{x}} = e^{\sqrt{x}\ln x} \\ y' &= e^{\sqrt{x}\ln x}\left(\frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x}\right) \\ &= x^{\sqrt{x}}\left(\frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln x + 1\right)\right) \end{aligned}$$

44. Given that $y = \left(\dfrac{1}{x}\right)^{\ln x}$, let $u = \ln x$. Then $x = e^u$ and $y = \left(\dfrac{1}{e^u}\right)^u = (e^{-u})^u = e^{-u^2}$. Hence,

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = (-2ue^{-u^2})\left(\frac{1}{x}\right) = -\frac{2\ln x}{x}\left(\frac{1}{x}\right)^{\ln x}.$$

45.
$$\begin{aligned} y &= \ln|\sec x + \tan x| \\ y' &= \frac{\sec x\tan x + \sec^2 x}{\sec x + \tan x} \\ &= \sec x \end{aligned}$$

46.
$$\begin{aligned} y &= \ln|x + \sqrt{x^2-a^2}| \\ y' &= \frac{1 + \dfrac{2x}{2\sqrt{x^2-a^2}}}{x + \sqrt{x^2-a^2}} = \frac{1}{\sqrt{x^2-a^2}} \end{aligned}$$

47.
$$\begin{aligned} y &= \ln(\sqrt{x^2+a^2} - x) \\ y' &= \frac{\dfrac{x}{\sqrt{x^2+a^2}} - 1}{\sqrt{x^2+a^2} - x} \\ &= -\frac{1}{\sqrt{x^2+a^2}} \end{aligned}$$

48.
$$\begin{aligned} y &= (\cos x)^x - x^{\cos x} = e^{x\ln\cos x} - e^{(\cos x)(\ln x)} \\ y' &= e^{x\ln\cos x}\left[\ln\cos x + x\left(\frac{1}{\cos x}\right)(-\sin x)\right] \\ &\quad - e^{(\cos x)(\ln x)}\left[-\sin x\ln x + \frac{1}{x}\cos x\right] \\ &= (\cos x)^x(\ln\cos x - x\tan x) \\ &\quad - x^{\cos x}\left(-\sin x\ln x + \frac{1}{x}\cos x\right) \end{aligned}$$

49.
$$\begin{aligned} f(x) &= xe^{ax}\\ f'(x) &= e^{ax}(1+ax)\\ f''(x) &= e^{ax}(2a+a^2x)\\ f'''(x) &= e^{ax}(3a^2+a^3x)\\ &\ \vdots\\ f^{(n)}(x) &= e^{ax}(na^{n-1}+a^nx) \end{aligned}$$

50. Since

$$\begin{aligned} \frac{d}{dx}(ax^2+bx+c)e^x &= (2ax+b)e^x+(ax^2+bx+c)e^x\\ &= [ax^2+(2a+b)x+(b+c)]e^x\\ &= [Ax^2+Bx+C]e^x. \end{aligned}$$

Thus, differentiating $(ax^2+bx+c)e^x$ produces another function of the same type with different constants. Any number of differentiations will do likewise.

51.
$$\begin{aligned} y &= e^{x^2}\\ y' &= 2xe^{x^2}\\ y'' &= 2e^{x^2}+4x^2e^{x^2}=2(1+2x^2)e^{x^2}\\ y''' &= 2(4x)e^{x^2}+2(1+2x^2)2xe^{x^2}=4(3x+2x^3)e^{x^2}\\ y^{(4)} &= 4(3+6x^2)e^{x^2}+4(3x+2x^3)2xe^{x^2}\\ &= 4(3+12x^2+4x^4)e^{x^2} \end{aligned}$$

52.
$$\begin{aligned} &f(x)=\ln(2x+1) && f'(x)=2(2x+1)^{-1}\\ &f''(x)=(-1)2^2(2x+1)^{-2} && f'''(x)=(2)2^3(2x+1)^{-3}\\ &f^{(4)}(x)=-(3!)2^4(2x+1)^{-4} \end{aligned}$$

Thus, if $n=1,2,3,\ldots$ we have
$f^{(n)}(x)=(-1)^{n-1}(n-1)!2^n(2x+1)^{-n}$.

53. a)
$$\begin{aligned} f(x) &= (x^x)^x = x^{(x^2)}\\ \ln f(x) &= x^2\ln x\\ \frac{1}{f}f' &= 2x\ln x + x\\ f' &= x^{x^2+1}(2\ln x+1) \end{aligned}$$

b)
$$\begin{aligned} g(x) &= x^{x^x}\\ \ln g &= x^x\ln x\\ \frac{1}{g'}g' &= x^x(1+\ln x)\ln x+\frac{x^x}{x}\\ g' &= x^{x^x}x^x\left(\frac{1}{x}+\ln x+(\ln x)^2\right) \end{aligned}$$

Evidently g grows more rapidly than does f as x grows large.

54. Given that $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}=a$ where $a>0$, then

$$\ln a = x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}\ln x = a\ln x.$$

Thus $\ln x=\dfrac{1}{a}\ln a=\ln a^{1/a}$, so $x=a^{1/a}$.

55.
$$\begin{aligned} f(x) &= (x-1)(x-2)(x-3)(x-4)\\ \ln f(x) &= \ln(x-1)+\ln(x-2)+\ln(x-3)+\ln(x-4)\\ \frac{1}{f(x)}f'(x) &= \frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4}\\ f'(x) &= f(x)\left(\frac{1}{x-1}+\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4}\right) \end{aligned}$$

56.
$$\begin{aligned} F(x) &= \frac{\sqrt{1+x}(1-x)^{1/3}}{(1+5x)^{4/5}}\\ \ln F(x) &= \tfrac12\ln(1+x)+\tfrac13\ln(1-x)-\tfrac45\ln(1+5x)\\ \frac{F'(x)}{F(x)} &= \frac{1}{2(1+x)}-\frac{1}{3(1-x)}-\frac{4}{(1+5x)}\\ F'(0) &= F(0)\left[\frac12-\frac13-\frac41\right]=(1)\left[\frac12-\frac13-4\right]=-\frac{23}{6} \end{aligned}$$

57.
$$\begin{aligned} f(x) &= \frac{(x^2-1)(x^2-2)(x^2-3)}{(x^2+1)(x^2+2)(x^2+3)}\\ f(2) &= \frac{3\times2\times1}{5\times6\times7}=\frac{1}{35}, \qquad f(1)=0\\ \ln f(x) &= \ln(x^2-1)+\ln(x^2-2)+\ln(x^2-3)\\ &\quad -\ln(x^2+1)-\ln(x^2+2)-\ln(x^2+3)\\ \frac{1}{f(x)}f'(x) &= \frac{2x}{x^2-1}+\frac{2x}{x^2-2}+\frac{2x}{x^2-3}\\ &\quad -\frac{2x}{x^2+1}-\frac{2x}{x^2+2}-\frac{2x}{x^2+3}\\ f'(x) &= 2xf(x)\left(\frac{1}{x^2-1}+\frac{1}{x^2-2}+\frac{1}{x^2-3}\right.\\ &\quad \left.-\frac{1}{x^2+1}-\frac{1}{x^2+2}-\frac{1}{x^2+3}\right)\\ f'(2) &= \frac{4}{35}\left(\frac13+\frac12+\frac11-\frac15-\frac16-\frac17\right)\\ &= \frac{4}{35}\times\frac{139}{105}=\frac{556}{3675} \end{aligned}$$

Since $f(x)=(x^2-1)g(x)$ where $g(1)\neq0$, then
$f'(x)=2xg(x)+(x^2-1)g'(x)$ and
$f'(1)=2g(1)+0=2\times\dfrac{(-1)(-2)}{2\times3\times4}=\dfrac16$.

58. Since $y=x^2e^{-x^2}$, then

$$y'=2xe^{-x^2}-2x^3e^{-x^2}=2x(1-x)(1+x)e^{-x^2}.$$

The tangent is horizontal at $(0,0)$ and $\left(\pm1,\dfrac1e\right)$.

59.
$f(x)=xe^{-x}$
$f'(x)=e^{-x}(1-x)$, C.P. $x=1$, $f(1)=\dfrac1e$
$f'(x)>0$ if $x<1$ (f increasing)
$f'(x)<0$ if $x>1$ (f decreasing)

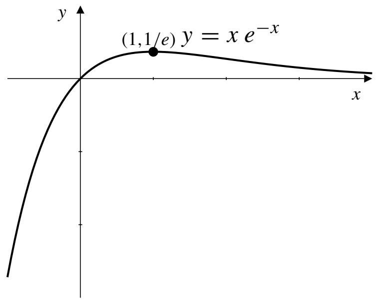

Fig. 3.3.59

60. Since $y = \ln x$ and $y' = \dfrac{1}{x} = 4$ then $x = \frac{1}{4}$ and $y = \ln \frac{1}{4} = -\ln 4$. The tangent line of slope 4 is $y = -\ln 4 + 4(x - \frac{1}{4})$, i.e., $y = 4x - 1 - \ln 4$.

61. Let the point of tangency be (a, e^a).
Tangent line has slope

$$\frac{e^a - 0}{a - 0} = \frac{d}{dx}e^x\bigg|_{x=a} = e^a.$$

Therefore, $a = 1$ and line has slope e.
The line has equation $y = ex$.

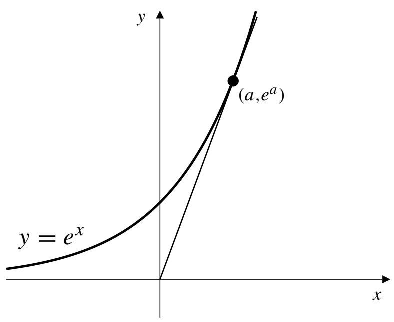

Fig. 3.3.61

62. The slope of $y = \ln x$ at $x = a$ is $y' = \dfrac{1}{x}\bigg|_{x=a} = \dfrac{1}{a}$. The line from $(0, 0)$ to $(a, \ln a)$ is tangent to $y = \ln x$ if

$$\frac{\ln a - 0}{a - 0} = \frac{1}{a}$$

i.e., if $\ln a = 1$, or $a = e$. Thus, the line is $y = \dfrac{x}{e}$.

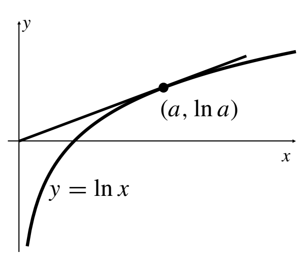

Fig. 3.3.62

63. Let the point of tangency be $(a, 2^a)$. Slope of the tangent is

$$\frac{2^a - 0}{a - 1} = \frac{d}{dx}2^x\bigg|_{x=a} = 2^a \ln 2.$$

Thus $a - 1 = \dfrac{1}{\ln 2}$, $\quad a = 1 + \dfrac{1}{\ln 2}$.
So the slope is $2^a \ln 2 = 2^{1+(1/\ln 2)} \ln 2 = 2e \ln 2$.
(Note: $\ln 2^{1/\ln 2} = \dfrac{1}{\ln 2} \ln 2 = 1 \Rightarrow 2^{1/\ln 2} = e$)
The tangent line has equation $y = 2e \ln 2(x - 1)$.

64. The tangent line to $y = a^x$ which passes through the origin is tangent at the point (b, a^b) where

$$\frac{a^b - 0}{b - 0} = \frac{d}{dx}a^x\bigg|_{x=b} = a^b \ln a.$$

Thus $\dfrac{1}{b} = \ln a$, so $a^b = a^{1/\ln a} = e$. The line $y = x$ will intersect $y = a^x$ provided the slope of this tangent line does not exceed 1, i.e., provided $\dfrac{e}{b} \le 1$, or $e \ln a \le 1$. Thus we need $a \le e^{1/e}$.

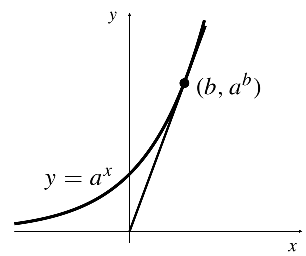

Fig. 3.3.64

65. $e^{xy} \ln \dfrac{x}{y} = x + \dfrac{1}{y}$

$e^{xy}(y + xy') \ln \dfrac{x}{y} + e^{xy} \dfrac{y}{x}\left(\dfrac{y - xy'}{y^2}\right) = 1 - \dfrac{1}{y^2}y'$

At $\left(e, \dfrac{1}{e}\right)$ we have

$e\left(\dfrac{1}{e} + ey'\right) 2 + e\dfrac{1}{e^2}(e - e^3 y') = 1 - e^2 y'$

$2 + 2e^2 y' + 1 - e^2 y' = 1 - e^2 y'$.

Thus the slope is $y' = -\dfrac{1}{e^2}$.

66. $xe^y + y - 2x = \ln 2 \Rightarrow e^y + xe^y y' + y' - 2 = 0$.
At $(1, \ln 2)$, $2 + 2y' + y' - 2 = 0 \Rightarrow y' = 0$.
Therefore, the tangent line is $y = \ln 2$.

67. $f(x) = Ax\cos\ln x + Bx\sin\ln x$

$$f'(x) = A\cos\ln x - A\sin\ln x + B\sin\ln x + B\cos\ln x$$
$$= (A+B)\cos\ln x + (B-A)\sin\ln x$$

If $A = B = \frac{1}{2}$ then $f'(x) = \cos\ln x$.

Therefore $\int \cos\ln x\,dx = \frac{1}{2}x\cos\ln x + \frac{1}{2}x\sin\ln x + C$.

If $B = \frac{1}{2}$, $A = -\frac{1}{2}$ then $f'(x) = \sin\ln x$.

Therefore $\int \sin\ln x\,dx = \frac{1}{2}x\sin\ln x - \frac{1}{2}x\cos\ln x + C$.

68. $F_{A,B}(x) = Ae^x\cos x + Be^x\sin x$

$$\frac{d}{dx}F_{A,B}(x)$$
$$= Ae^x\cos x - Ae^x\sin x + Be^x\sin x + Be^x\cos x$$
$$= (A+B)e^x\cos x + (B-A)e^x\sin x = F_{A+B,B-A}(x)$$

69. Since $\frac{d}{dx}F_{A,B}(x) = F_{A+B,B-A}(x)$ we have

a) $\frac{d^2}{dx^2}F_{A,B}(x) = \frac{d}{dx}F_{A+B,B-A}(x) = F_{2B,-2A}(x)$

b) $\frac{d^3}{dx^3}e^x\cos x = \frac{d^3}{dx^3}F_{1,0}(x) = \frac{d}{dx}F_{0,-2}(x)$
$= F_{-2,-2}(x) = -2e^x\cos x - 2e^x\sin x$

70. $\frac{d}{dx}(Ae^{ax}\cos bx + Be^{ax}\sin bx)$

$$= Aae^{ax}\cos bx - Abe^{ax}\sin bx + Bae^{ax}\sin bx + Bbe^{ax}\cos bx$$
$$= (Aa+Bb)e^{ax}\cos bx + (Ba-Ab)e^{ax}\sin bx.$$

(a) If $Aa+Bb = 1$ and $Ba-Ab = 0$, then $A = \frac{a}{a^2+b^2}$ and $B = \frac{b}{a^2+b^2}$. Thus

$$\int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}\left(ae^{ax}\cos bx + be^{ax}\sin bx\right) + C.$$

(b) If $Aa+Bb = 0$ and $Ba-Ab = 1$, then $A = \frac{-b}{a^2+b^2}$ and $B = \frac{a}{a^2+b^2}$. Thus

$$\int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}\left(ae^{ax}\sin bx - be^{ax}\cos bx\right) + C.$$

71. $\frac{d}{dx}\left[\ln\frac{1}{x} + \ln x\right] = \frac{1}{1/x}\left(\frac{-1}{x^2}\right) + \frac{1}{x} = -\frac{1}{x} + \frac{1}{x} = 0.$

Therefore $\ln\frac{1}{x} + \ln x = C$ (constant). Taking $x = 1$, we get $C = \ln 1 + \ln 1 = 0$. Thus $\ln\frac{1}{x} = -\ln x$.

72. $\ln\frac{x}{y} = \ln\left(x\,\frac{1}{y}\right) = \ln x + \ln\frac{1}{y} = \ln x - \ln y.$

73. $\frac{d}{dx}[\ln(x^r) - r\ln x] = \frac{rx^{r-1}}{x^r} - \frac{r}{x} = \frac{r}{x} - \frac{r}{x} = 0.$

Therefore $\ln(x^r) - r\ln x = C$ (constant). Taking $x = 1$, we get $C = \ln 1 - r\ln 1 = 0 - 0 = 0$. Thus $\ln(x^r) = r\ln x$.

74. Let $x > 0$, and $F(x)$ be the area bounded by $y = t^2$, the t-axis, $t = 0$ and $t = x$. For $h > 0$, $F(x+h) - F(x)$ is the shaded area in the following figure.

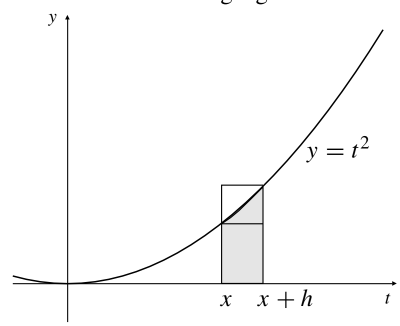

Fig. 3.3.74

Comparing this area with that of the two rectangles, we see that

$$hx^2 < F(x+h) - F(x) < h(x+h)^2.$$

Hence, the Newton quotient for $F(x)$ satisfies

$$x^2 < \frac{F(x+h) - F(x)}{h} < (x+h)^2.$$

Letting h approach 0 from the right (by the Squeeze Theorem applied to one-sided limits)

$$\lim_{h\to 0+}\frac{F(x+h) - F(x)}{h} = x^2.$$

If $h < 0$ and $0 < x + h < x$, then

$$(x+h)^2 < \frac{F(x+h) - F(x)}{h} < x^2,$$

so similarly,

$$\lim_{h\to 0-}\frac{F(x+h) - F(x)}{h} = x^2.$$

Combining these two limits, we obtain

$$\frac{d}{dx}F(x) = \lim_{h\to 0}\frac{F(x+h) - F(x)}{h} = x^2.$$

Therefore $F(x) = \int x^2\,dx = \frac{1}{3}x^3 + C$. Since $F(0) = C = 0$, therefore $F(x) = \frac{1}{3}x^3$. For $x = 2$, the area of the region is $F(2) = \frac{8}{3}$ square units.

75. a) The shaded area A in part (i) of the figure is less than the area of the rectangle (actually a square) with base from $t = 1$ to $t = 2$ and height $1/1 = 1$. Since $\ln 2 = A < 1$, we have $2 < e^1 = e$; i.e., $e > 2$.

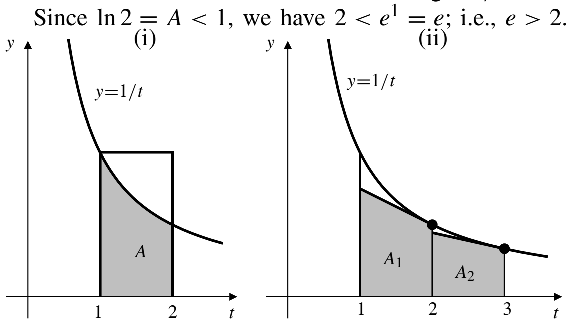

Fig. 3.3.75

b) If $f(t) = 1/t$, then $f'(t) = -1/t^2$ and $f''(t) = 2/t^3 > 0$ for $t > 0$. Thus $f'(t)$ is an increasing function of t for $t > 0$, and so the graph of $f(t)$ bends upward away from any of its tangent lines. (This kind of argument will be explored further in Chapter 5.)

c) The tangent to $y = 1/t$ at $t = 2$ has slope $-1/4$. Its equation is

$$y = \frac{1}{2} - \frac{1}{4}(x-2) \quad \text{or } y = 1 - \frac{x}{4}.$$

The tangent to $y = 1/t$ at $t = 3$ has slope $-1/9$. Its equation is

$$y = \frac{1}{3} - \frac{1}{9}(x-3) \quad \text{or } y = \frac{2}{3} - \frac{x}{9}.$$

d) The trapezoid bounded by $x = 1$, $x = 2$, $y = 0$, and $y = 1 - (x/4)$ has area

$$A_1 = \frac{1}{2}\left(\frac{3}{4} + \frac{1}{2}\right) = \frac{5}{8}.$$

The trapezoid bounded by $x = 2$, $x = 3$, $y = 0$, and $y = (2/3) - (x/9)$ has area

$$A_2 = \frac{1}{2}\left(\frac{4}{9} + \frac{1}{3}\right) = \frac{7}{18}.$$

e) $\ln 3 > A_1 + A_2 = \dfrac{5}{8} + \dfrac{7}{18} = \dfrac{73}{72} > 1$.
Thus $3 > e^1 = e$. Combining this with the result of (a) we conclude that $2 < e < 3$.

Section 3.4 Growth and Decay (page 187)

1. $\lim_{x\to\infty} x^3 e^{-x} = \lim_{x\to\infty} \dfrac{x^3}{e^x} = 0$ (exponential wins)

2. $\lim_{x\to\infty} x^{-3} e^{x} = \lim_{x\to\infty} \dfrac{e^x}{x^3} = \infty$

3. $\lim_{x\to\infty} \dfrac{2e^x - 3}{e^x + 5} = \lim_{x\to\infty} \dfrac{2 - 3e^{-x}}{1 + 5e^{-x}} = \dfrac{2-0}{1+0} = 2$

4. $\lim_{x\to\infty} \dfrac{x - 2e^{-x}}{x + 3e^{-x}} = \lim_{x\to\infty} \dfrac{1 - 2/(xe^x)}{1 + 3/(xe^x)} = \dfrac{1-0}{1+0} = 1$

5. $\lim_{x\to 0+} x \ln x = 0$ (power wins)

6. $\lim_{x\to 0+} \dfrac{\ln x}{x} = -\infty$

7. $\lim_{x\to 0} x(\ln|x|)^2 = 0$

8. $\lim_{x\to\infty} \dfrac{(\ln x)^3}{\sqrt{x}} = 0$ (power wins)

9. Let $N(t)$ be the number of bacteria present after t hours.
Then $N(0) = 100$, $N(1) = 200$.
Since $\dfrac{dN}{dt} = kN$ we have $N(t) = N(0)e^{kt} = 100e^{kt}$.
Thus $200 = 100e^k$ and $k = \ln 2$.
Finally, $N\left(\dfrac{5}{2}\right) = 100e^{(5/2)\ln 2} \approx 565.685$.
There will be approximately 566 bacteria present after another $1\frac{1}{2}$ hours.

10. Let $y(t)$ be the number of kg undissolved after t hours. Thus, $y(0) = 50$ and $y(5) = 20$. Since $y'(t) = ky(t)$, therefore $y(t) = y(0)e^{kt} = 50e^{kt}$. Then

$$20 = y(5) = 50e^{5k} \Rightarrow k = \tfrac{1}{5}\ln\tfrac{2}{5}.$$

If 90% of the sugar is dissolved at time T then $5 = y(T) = 50e^{kT}$, so

$$T = \frac{1}{k}\ln\frac{1}{10} = \frac{5\ln(0.1)}{\ln(0.4)} \approx 12.56.$$

Hence, 90% of the sugar will dissolved in about 12.56 hours.

11. Let $P(t)$ be the percentage undecayed after t years.
Thus $P(0) = 100$, $P(15) = 70$.
Since $\dfrac{dP}{dt} = kP$, we have $P(t) = P(0)e^{kt} = 100e^{kt}$.
Thus $70 = P(15) = 100e^{15k}$ so $k = \dfrac{1}{15}\ln(0.7)$.
The half-life T satisfies if $50 = P(T) = 100e^{kT}$, so
$T = \dfrac{1}{k}\ln(0.5) = \dfrac{15\ln(0.5)}{\ln(0.7)} \approx 29.15$.
The half-life is about 29.15 years.

12. Let $P(t)$ be the percentage remaining after t years. Thus $P'(t) = kP(t)$ and $P(t) = P(0)e^{kt} = 100e^{kt}$. Then,

$$50 = P(1690) = 100e^{1690k} \Rightarrow k = \frac{1}{1690} \ln \frac{1}{2} \approx 0.0004101.$$

a) $P(100) = 100e^{100k} \approx 95.98$, i.e., about 95.98% remains after 100 years.

b) $P(1000) = 100e^{1000k} \approx 66.36$, i.e., about 66.36% remains after 1000 years.

13. Let $P(t)$ be the percentage of the initial amount remaining after t years.
Then $P(t) = 100e^{kt}$ and $99.57 = P(1) = 100e^k$.
Thus $k = \ln(0.9957)$.
The half-life T satisfies $50 = P(T) = 100e^{kT}$,
so $T = \dfrac{1}{k}\ln(0.5) = \dfrac{\ln(0.5)}{\ln(0.995)} \approx 160.85$.
The half-life is about 160.85 years.

14. Let $N(t)$ be the number of bacteria in the culture t days after the culture was set up. Thus $N(3) = 3N(0)$ and $N(7) = 10 \times 10^6$. Since $N(t) = N(0)e^{kt}$, we have

$$3N(0) = N(3) = N(0)e^{3k} \Rightarrow k = \tfrac{1}{3}\ln 3.$$
$$10^7 = N(7) = N(0)e^{7k} \Rightarrow N(0) = 10^7 e^{-(7/3)\ln 3} \approx 770400.$$

There were approximately 770,000 bacteria in the culture initially. (Note that we are approximating a discrete quantity (number of bacteria) by a continuous quantity $N(t)$ in this exercise.)

15. Let $W(t)$ be the weight t days after birth.
Thus $W(0) = 4000$ and $W(t) = 4000e^{kt}$.
Also $4400 = W(14) = 4000e^{14k}$, is $k = \dfrac{1}{14}\ln(1.1)$.
Five days after birth, the baby weighs
$W(5) = 4000e^{(5/14)\ln(1.1)} \approx 4138.50 \approx 4139$ grams.

16. Since

$$I'(t) = kI(t) \Rightarrow I(t) = I(0)e^{kt} = 40e^{kt},$$
$$15 = I(0.01) = 40e^{0.01k} \Rightarrow k = \frac{1}{0.01}\ln\frac{15}{40} = 100\ln\frac{3}{8},$$

thus,

$$I(t) = 40\exp\left(100t\ln\frac{3}{8}\right) = 40\left(\frac{3}{8}\right)^{100t}.$$

17. $\$P$ invested at 4% compounded continuously grows to $\$P(e^{0.04})^7 = \$Pe^{0.28}$ in 7 years. This will be \$10,000 if $\$P = \$10,000e^{-0.28} = \$7,557.84$.

18. Let $y(t)$ be the value of the investment after t years. Thus $y(0) = 1000$ and $y(5) = 1500$. Since $y(t) = 1000e^{kt}$ and $1500 = y(5) = 1000e^{5k}$, therefore, $k = \frac{1}{5}\ln\frac{3}{2}$.

a) Let t be the time such that $y(t) = 2000$, i.e.,

$$1000e^{kt} = 2000$$
$$\Rightarrow \quad t = \frac{1}{k}\ln 2 = \frac{5\ln 2}{\ln(\frac{3}{2})} = 8.55.$$

Hence, the doubling time for the investment is about 8.55 years.

b) Let r% be the effective annual rate of interest; then

$$1000(1 + \frac{r}{100}) = y(1) = 1000e^k$$
$$\Rightarrow r = 100(e^k - 1) = 100[\exp(\tfrac{1}{5}\ln\tfrac{3}{2}) - 1]$$
$$= 8.447.$$

The effective annual rate of interest is about 8.45%.

19. Let the purchasing power of the dollar be $P(t)$ cents after t years.
Then $P(0) = 100$ and $P(t) = 100e^{kt}$.
Now $91 = P(1) = 100e^k$ so $k = \ln(0.91)$.
If $25 = P(t) = 100^{kt}$ then
$t = \dfrac{1}{k}\ln(0.25) = \dfrac{\ln(0.25)}{\ln(0.91)} \approx 14.7$.
The purchasing power will decrease to \$0.25 in about 14.7 years.

20. Let i% be the effective rate, then an original investment of $\$A$ will grow to $\$A\left(1 + \dfrac{i}{100}\right)$ in one year. Let r% be the nominal rate per annum compounded n times per year, then an original investment of $\$A$ will grow to

$$\$A\left(1 + \frac{r}{100n}\right)^n$$

in one year, if compounding is performed n times per year. For $i = 9.5$ and $n = 12$, we have

$$\$A\left(1 + \frac{9.5}{100}\right) = \$A\left(1 + \frac{r}{1200}\right)^{12}$$
$$\Rightarrow r = 1200\left(\sqrt[12]{1.095} - 1\right) = 9.1098.$$

The nominal rate of interest is about 9.1098%.

21. Let $x(t)$ be the number of rabbits on the island t years after they were introduced. Thus $x(0) = 1{,}000$, $x(3) = 3{,}500$, and $x(7) = 3{,}000$. For $t < 5$ we have $dx/dt = k_1 x$, so

$$x(t) = x(0)e^{k_1 t} = 1{,}000e^{k_1 t}$$
$$x(2) = 1{,}000e^{2k_1} = 3{,}500 \quad \Longrightarrow \quad e^{2k_1} = 3.5$$
$$x(5) = 1{,}000e^{5k_1} = 1{,}000\left(e^{2k_1}\right)^{5/2} = 1{,}000(3.5)^{5/2}$$
$$\approx 22{,}918.$$

For $t > 5$ we have $dx/dt = k_2 x$, so that

$$\begin{aligned}
x(t) &= x(5)e^{k_2(t-5)} \\
x(7) &= x(5)e^{2k_2} = 3{,}000 \quad \Longrightarrow \quad e^{2k_2} \approx \frac{3{,}000}{22{,}918} \\
x(10) &= x(5)3^{5k_2} = x(5)\left(e^{2k_2}\right)^{5/2} \approx 22{,}918\left(\frac{3{,}000}{22{,}918}\right)^{5/2} \\
&\approx 142.
\end{aligned}$$

so there are approximately 142 rabbits left after 10 years.

22. Let $N(t)$ be the number of rats on the island t months after the initial population was released and before the first cull. Thus $N(0) = R$ and $N(3) = 2R$. Since $N(t) = Re^{kt}$, we have $e^{3k} = 2$, so $e^k = 2^{1/3}$. Hence $N(5) = Re^{5k} = 2^{5/3}R$. After the first 1,000 rats are killed the number remaining is $2^{5/3}R - 1{,}000$. If this number is less than R, the number at the end of succeeding 5-year periods will decline. The minimum value of R for which this won't happen must satisfy $2^{5/3}R - 1{,}000 = R$, that is, $R = 1{,}000/(2^{5/3} - 1) \approx 459.8$. Thus $R = 460$ rats should be brought to the island initially.

23. $f'(x) = a + bf(x)$.

a) If $u(x) = a + bf(x)$, then $u'(x) = bf'(x) = b[a + bf(x)] = bu(x)$. This equation for u is the equation of exponential growth/decay. Thus

$$\begin{aligned}
u(x) &= C_1 e^{bx}, \\
f(x) &= \frac{1}{b}\left(C_1 e^{bx} - a\right) = Ce^{bx} - \frac{a}{b}.
\end{aligned}$$

b) If $\dfrac{dy}{dx} = a + by$ and $y(0) = y_0$, then, from part (a),

$$y = Ce^{bx} - \frac{a}{b}, \quad y_0 = Ce^0 - \frac{a}{b}.$$

Thus $C = y_0 + (a/b)$, and

$$y = \left(y_0 + \frac{a}{b}\right)e^{bx} - \frac{a}{b}.$$

24. a) The concentration $x(t)$ satisfies $\dfrac{dx}{dt} = a - bx(t)$. This says that $x(t)$ is increasing if it is less than a/b and decreasing if it is greater than a/b. Thus, the limiting concentration is a/b.

b) The differential equation for $x(t)$ resembles that of Exercise 21(b), except that $y(x)$ is replaced by $x(t)$, and b is replaced by $-b$. Using the result of Exercise 21(b), we obtain, since $x(0) = 0$,

$$\begin{aligned}
x(t) &= \left(x(0) - \frac{a}{b}\right)e^{-bt} + \frac{a}{b} \\
&= \frac{a}{b}\left(1 - e^{-bt}\right).
\end{aligned}$$

c) We will have $x(t) = \frac{1}{2}(a/b)$ if $1 - e^{-bt} = \frac{1}{2}$, that is, if $e^{-bt} = \frac{1}{2}$, or $-bt = \ln(1/2) = -\ln 2$. The time required to attain half the limiting concentration is $t = (\ln 2)/b$.

25. Let $T(t)$ be the reading t minutes after the Thermometer is moved outdoors. Thus $T(0) = 72$, $T(1) = 48$.
By Newton's law of cooling, $\dfrac{dT}{dt} = k(T - 20)$.
If $V(t) = T(t) - 20$, then $\dfrac{dV}{dt} = kV$, so
$V(t) = V(0)e^{kt} = 52e^{kt}$.
Also $28 = V(1) = 52e^k$, so $k = \ln(7/13)$.
Thus $V(5) = 52e^{5\ln(7/13)} \approx 2.354$. At $t = 5$ the thermometer reads about $T(5) = 20 + 2.354 = 22.35$°C.

26. Let $T(t)$ be the temperature of the object t minutes after its temperature was 45° C. Thus $T(0) = 45$ and $T(40) = 20$. Also $\dfrac{dT}{dt} = k(T + 5)$. Let $u(t) = T(t) + 5$, so $u(0) = 50$, $u(40) = 25$, and $\dfrac{du}{dt} = \dfrac{dT}{dt} = k(T + 5) = ku$. Thus,

$$\begin{aligned}
u(t) &= 50e^{kt}, \\
25 &= u(40) = 50e^{40k}, \\
\Rightarrow k &= \frac{1}{40}\ln\frac{25}{50} = \frac{1}{40}\ln\frac{1}{2}.
\end{aligned}$$

We wish to know t such that $T(t) = 0$, i.e., $u(t) = 5$, hence

$$\begin{aligned}
5 &= u(t) = 50e^{kt} \\
t &= \frac{40\ln\left(\frac{5}{50}\right)}{\ln\left(\frac{1}{2}\right)} = 132.88 \text{ min.}
\end{aligned}$$

Hence, it will take about $(132.88 - 40) = 92.88$ minutes more to cool to 0° C.

27. Let $T(t)$ be the temperature of the body t minutes after it was 5°.
Thus $T(0) = 5$, $T(4) = 10$. Room temperature = 20°.
By Newton's law of cooling (warming) $\dfrac{dT}{dt} = k(T - 20)$.
If $V(t) = T(t) - 20$ then $\dfrac{dV}{dt} = kV$,
so $V(t) = V(0)e^{kt} = -15e^{kt}$.
Also $-10 = V(4) = -15e^{4k}$, so $k = \dfrac{1}{4}\ln\left(\dfrac{2}{3}\right)$.
If $T(t) = 15°$, then $-5 = V(t) = -15e^{kt}$

$$\text{so } t = \frac{1}{k}\ln\left(\frac{1}{3}\right) = 4\frac{\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{2}{3}\right)} \approx 10.838.$$

It will take a further 6.84 minutes to warm to 15°C.

28. By the solution given for the logistic equation, we have

$$y_1 = \frac{Ly_0}{y_0 + (L - y_0)e^{-k}}, \qquad y_2 = \frac{Ly_0}{y_0 + (L - y_0)e^{-2k}}$$

Thus $y_1(L - y_0)e^{-k} = (L - y_1)y_0$, and
$y_2(L - y_0)e^{-2k} = (L - y_2)y_0$.
Square the first equation and thus eliminate e^{-k}:

$$\left(\frac{(L - y_1)y_0}{y_1(L - y_0)}\right)^2 = \frac{(L - y_2)y_0}{y_2(L - y_0)}$$

Now simplify: $y_0y_2(L - y_1)^2 = y_1^2(L - y_0)(L - y_2)$
$y_0y_2L^2 - 2y_1y_0y_2L + y_0y_1^2y_2 = y_1^2L^2 - y_1^2(y_0 + y_2)L + y_0y_1^2y_2$

Assuming $L \neq 0$, $\quad L = \dfrac{y_1^2(y_0 + y_2) - 2y_0y_1y_2}{y_1^2 - y_0y_2}$.
If $y_0 = 3$, $y_1 = 5$, $y_2 = 6$, then
$L = \dfrac{25(9) - 180}{25 - 18} = \dfrac{45}{7} \approx 6.429$.

29. The rate of growth of y in the logistic equation is

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right).$$

Since

$$\frac{dy}{dt} = -\frac{k}{L}\left(y - \frac{L}{2}\right)^2 + \frac{kL}{4},$$

thus $\dfrac{dy}{dt}$ is greatest when $y = \dfrac{L}{2}$.

30. The solution $y = \dfrac{Ly_0}{y_0 + (L - y_0)e^{-kt}}$ is valid on the largest interval containing $t = 0$ on which the denominator does not vanish.
If $y_0 > L$ then $y_0 + (L - y_0)e^{-kt} = 0$ if
$t = t^* = -\dfrac{1}{k}\ln\dfrac{y_0}{y_0 - L}$.
Then the solution is valid on (t^*, ∞).
$\lim_{t \to t^*+} y(t) = \infty$.

31. The solution

$$y = \frac{Ly_0}{y_0 + (L - y_0)e^{-kt}}$$

of the logistic equation is valid on any interval containing $t = 0$ and not containing any point where the denominator is zero. The denominator is zero if $y_0 = (y_0 - L)e^{-kt}$, that is, if

$$t = t^* = -\frac{1}{k}\ln\left(\frac{y_0}{y_0 - L}\right).$$

Assuming k and L are positive, but y_0 is negative, we have $t^* > 0$. The solution is therefore valid on $(-\infty, t^*)$. The solution approaches $-\infty$ as $t \to t^*-$.

32.

$$\begin{aligned} y(t) &= \frac{L}{1 + Me^{-kt}} \\ 200 = y(0) &= \frac{L}{1 + M} \\ 1,000 = y(1) &= \frac{L}{1 + Me^{-k}} \\ 10,000 &= \lim_{t \to \infty} y(t) = L \end{aligned}$$

Thus $200(1 + M) = L = 10,000$, so $M = 49$. Also $1,000(1 + 49e^{-k}) = L = 10,000$, so $e^{-k} = 9/49$ and $k = \ln(49/9) \approx 1.695$.

33.

$$y(3) = \frac{L}{1 + Me^{-3k}} = \frac{10,000}{1 + 49(9/49)^3} \approx 7671 \text{ cases}$$

$$y'(3) = \frac{LkMe^{-3k}}{(1 + Me^{-3k})^2} \approx 3,028 \text{ cases/week.}$$

Section 3.5 The Inverse Trigonometric Functions (page 195)

1. $\sin^{-1}\dfrac{\sqrt{3}}{2} = \dfrac{\pi}{3}$

2. $\cos^{-1}\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$

3. $\tan^{-1}(-1) = -\dfrac{\pi}{4}$

4. $\sec^{-1}\sqrt{2} = \dfrac{\pi}{4}$

5. $\sin(\sin^{-1} 0.7) = 0.7$

6.

$$\begin{aligned} \cos(\sin^{-1} 0.7) &= \sqrt{1 - \sin^2(\arcsin 0.7)} \\ &= \sqrt{1 - 0.49} = \sqrt{0.51} \end{aligned}$$

7. $\tan^{-1}\left(\tan\dfrac{2\pi}{3}\right) = \tan^{-1}(-\sqrt{3}) = -\dfrac{\pi}{3}$

8. $\sin^{-1}(\cos 40°) = 90° - \cos^{-1}(\cos 40°) = 50°$

9.

$$\begin{aligned} \cos^{-1}\Big(\sin(-0.2)\Big) &= \frac{\pi}{2} - \sin^{-1}\Big(\sin(-0.2)\Big) \\ &= \frac{\pi}{2} + 0.2 \end{aligned}$$

10. $\sin\left(\cos^{-1}(-\frac{1}{3})\right) = \sqrt{1-\cos^2(\arccos(-\frac{1}{3})}$
$$= \sqrt{1-\tfrac{1}{9}} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$$

11. $\cos\left(\tan^{-1}\frac{1}{2}\right) = \dfrac{1}{\sec\left(\tan^{-1}\frac{1}{2}\right)}$
$$= \frac{1}{\sqrt{1+\tan^2\left(\tan^{-1}\frac{1}{2}\right)}} = \frac{2}{\sqrt{5}}$$

12. $\tan(\tan^{-1} 200) = 200$

13. $\sin(\cos^{-1} x) = \sqrt{1-\cos^2(\cos^{-1} x)}$
$$= \sqrt{1-x^2}$$

14. $\cos(\sin^{-1} x) = \sqrt{1-\sin^2(\sin^{-1} x)} = \sqrt{1-x^2}$

15. $\cos(\tan^{-1} x) = \dfrac{1}{\sec(\tan^{-1} x)} = \dfrac{1}{\sqrt{1+x^2}}$

16. $\tan(\arctan x) = x \Rightarrow \sec(\arctan x) = \sqrt{1+x^2}$
$$\Rightarrow \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$
$$\Rightarrow \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$$

17. $\tan(\cos^{-1} x) = \dfrac{\sin(\cos^{-1} x)}{\cos(\cos^{-1} x}$
$$= \frac{\sqrt{1-x^2}}{x} \quad \text{(by \# 13)}$$

18. $\cos(\sec^{-1} x) = \dfrac{1}{x} \Rightarrow \sin(\sec^{-1} x) = \sqrt{1-\dfrac{1}{x^2}} = \dfrac{\sqrt{x^2-1}}{|x|}$
$$\Rightarrow \tan(\sec^{-1} x) = \sqrt{x^2-1}\,\operatorname{sgn} x$$
$$= \begin{cases} \sqrt{x^2-1} & \text{if } x \ge 1 \\ -\sqrt{x^2-1} & \text{if } x \le -1 \end{cases}$$

19. $y = \sin^{-1}\left(\dfrac{2x-1}{3}\right)$
$$y' = \frac{1}{\sqrt{1-\left(\frac{2x-1}{3}\right)^2}}\,\frac{2}{3}$$
$$= \frac{2}{\sqrt{9-(4x^2-4x+1)}}$$
$$= \frac{1}{\sqrt{2+x-x^2}}$$

20. $y = \tan^{-1}(ax+b), \qquad y' = \dfrac{a}{1+(ax+b)^2}.$

21. $y = \cos^{-1}\dfrac{x-b}{a}$
$$y' = -\frac{1}{\sqrt{1-\frac{(x-b)^2}{a^2}}}\,\frac{1}{a}$$
$$= \frac{-1}{\sqrt{a^2-(x-b)^2}} \quad \text{(assuming) } a > 0).$$

22. $f(x) = x\sin^{-1} x$
$$f'(x) = \sin^{-1} x + \frac{x}{\sqrt{1-x^2}}.$$

23. $f(t) = t\tan^{-1} t$
$$f'(t) = \tan^{-1} t + \frac{t}{1+t^2}$$

24. $u = z^2\sec^{-1}(1+z^2)$
$$\frac{du}{dz} = 2z\sec^{-1}(1+z^2) + \frac{z^2(2z)}{(1+z^2)\sqrt{(1+z^2)^2-1}}$$
$$= 2z\sec^{-1}(1+z^2) + \frac{2z^2\operatorname{sgn}(z)}{(1+z^2)\sqrt{z^2+2}}$$

25. $F(x) = (1+x^2)\tan^{-1} x$
$$F'(x) = 2x\tan^{-1} x + 1$$

26. $y = \sin^{-1}\left(\dfrac{a}{x}\right) \qquad (|x| > |a|)$
$$y' = \frac{1}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\left[-\frac{a}{x^2}\right] = -\frac{a}{|x|\sqrt{x^2-a^2}}$$

27. $G(x) = \dfrac{\sin^{-1} x}{\sin^{-1}(2x)}$
$$G'(x) = \frac{\sin^{-1}(2x)\dfrac{1}{\sqrt{1-x^2}} - \sin^{-1} x\dfrac{2}{\sqrt{1-4x^2}}}{\left(\sin^{-1}(2x)\right)^2}$$
$$= \frac{\sqrt{1-4x^2}\sin^{-1}(2x) - 2\sqrt{1-x^2}\sin^{-1} x}{\sqrt{1-x^2}\sqrt{1-4x^2}\left(\sin^{-1}(2x)\right)^2}$$

28. $H(t) = \dfrac{\sin^{-1} t}{\sin t}$
$$H'(t) = \frac{\sin t\left(\dfrac{1}{\sqrt{1-t^2}}\right) - \sin^{-1} t\,\cos t}{\sin^2 t}$$
$$= \frac{1}{(\sin t)\sqrt{1-t^2}} - \csc t\cot t\sin^{-1} t$$

29. $f(x) = (\sin^{-1} x^2)^{1/2}$
$$f'(x) = \frac{1}{2}(\sin^{-1} x^2)^{-1/2}\frac{2x}{\sqrt{1-x^4}}$$
$$= \frac{x}{\sqrt{1-x^4}\sqrt{\sin^{-1} x^2}}$$

30. $$y = \cos^{-1}\left(\frac{a}{\sqrt{a^2+x^2}}\right)$$
$$y' = -\left(1 - \frac{a^2}{a^2+x^2}\right)^{-1/2}\left[-\frac{a}{2}(a^2+x^2)^{-3/2}(2x)\right] = \frac{a\,\mathrm{sgn}\,(x)}{a^2+x^2}$$

31. $$y = \sqrt{a^2-x^2} + a\sin^{-1}\frac{x}{a}$$
$$y' = -\frac{x}{\sqrt{a^2-x^2}} + \frac{a}{\sqrt{1-\frac{x^2}{a^2}}}\frac{1}{a} = \frac{a-x}{\sqrt{a^2-x^2}} = \sqrt{\frac{a-x}{a+x}} \qquad (a>0)$$

32. $$y = a\cos^{-1}\left(1-\frac{x}{a}\right) - \sqrt{2ax-x^2} \qquad (a>0)$$
$$y' = -a\left[1-\left(1-\frac{x}{a}\right)^2\right]^{-1/2}\left(-\frac{1}{a}\right) - \frac{2a-2x}{2\sqrt{2ax-x^2}} = \frac{x}{\sqrt{2ax-x^2}}$$

33. $$\tan^{-1}\left(\frac{2x}{y}\right) = \frac{\pi x}{y^2}$$
$$\frac{1}{1+\frac{4x^2}{y^2}}\frac{2y-2xy'}{y^2} = \pi\frac{y^2-2xyy'}{y^4}$$
At $(1,2)$ $\frac{1}{2}\frac{4-2y'}{4} = \pi\frac{4-4y'}{16}$
$$8-4y' = 4\pi - 4\pi y' \Rightarrow y' = \frac{\pi-2}{\pi-1}$$
At $(1,2)$ the slope is $\frac{\pi-2}{\pi-1}$

34. If $y = \sin^{-1}x$, then $y' = \frac{1}{\sqrt{1-x^2}}$. If the slope is 2 then $\frac{1}{\sqrt{1-x^2}} = 2$ so that $x = \pm\frac{\sqrt{3}}{2}$. Thus the equations of the two tangent lines are
$$y = \frac{\pi}{3} + 2\left(x - \frac{\sqrt{3}}{2}\right) \text{ and } y = -\frac{\pi}{3} + 2\left(x + \frac{\sqrt{3}}{2}\right).$$

35. $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} > 0$ on $(-1,1)$.
Therefore, $\sin^{-1}$ is increasing.
$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2} > 0$ on $(-\infty,\infty)$.
Therefore $\tan^{-1}$ is increasing.
$\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}} < 0$ on $(-1,1)$.
Therefore $\cos^{-1}$ is decreasing.

36. Since the domain of $\sec^{-1}$ consists of two disjoint intervals $(-\infty,-1]$ and $[1,\infty)$, the fact that the derivative of $\sec^{-1}$ is positive wherever defined does not imply that $\sec^{-1}$ is increasing over its whole domain, only that it is increasing on each of those intervals taken independently. In fact, $\sec^{-1}(-1) = \pi > 0 = \sec^{-1}(1)$ even though $-1 < 1$.

37. $$\frac{d}{dx}\csc^{-1}x = \frac{d}{dx}\sin^{-1}\frac{1}{x} = \frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(-\frac{1}{x^2}\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

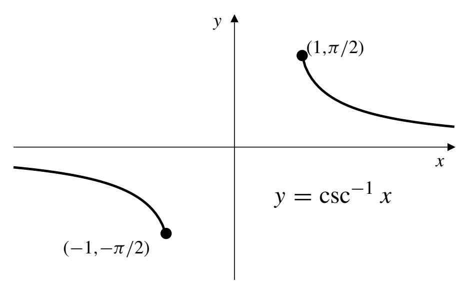

Fig. 3.5.37

38. $\cot^{-1}x = \arctan(1/x)$;
$$\frac{d}{dx}\cot^{-1}x = \frac{1}{1+\frac{1}{x^2}}\frac{-1}{x^2} = -\frac{1}{1+x^2}$$

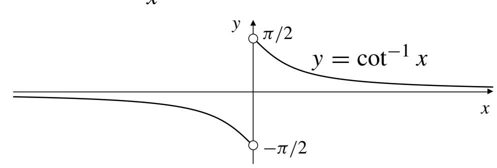

Fig. 3.5.38

Remark: the domain of $\cot^{-1}$ can be extended to include 0 by defining, say, $\cot^{-1}0 = \pi/2$. This will make $\cot^{-1}$ right-continuous (but not continuous) at $x = 0$. It is also possible to define $\cot^{-1}$ in such a way that it is continuous on the whole real line, but we would then lose the identity $\cot^{-1}x = \tan^{-1}(1/x)$, which we prefer to maintain for calculation purposes.

39. $\dfrac{d}{dx}(\tan^{-1} x + \cot^{-1} x) = \dfrac{d}{dx}\left(\tan^{-1} x + \tan^{-1} \dfrac{1}{x}\right)$

$$= \frac{1}{1+x^2} + \frac{1}{1+\frac{1}{x^2}}\left(-\frac{1}{x^2}\right) = 0 \text{ if } x \neq 0$$

Thus $\tan^{-1} x + \cot^{-1} x = C_1$ (const. for $x > 0$)

At $x = 1$ we have $\dfrac{\pi}{4} + \dfrac{\pi}{4} = C_1$

Thus $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$ for $x > 0$.

Also $\tan^{-1} x + \cot^{-1} x = C_2$ for $(x < 0)$.

At $x = -1$, we get $-\dfrac{\pi}{4} - \dfrac{\pi}{4} = C_2$.

Thus $\tan^{-1} x + \cot^{-1} x = -\dfrac{\pi}{2}$ for $x < 0$.

40. If $g(x) = \tan(\tan^{-1} x)$ then

$$g'(x) = \frac{\sec^2(\tan^{-1} x)}{1+x^2} = \frac{1 + [\tan(\tan^{-1} x)]^2}{1+x^2} = \frac{1+x^2}{1+x^2} = 1.$$

If $h(x) = \tan^{-1}(\tan x)$ then h is periodic with period π, and

$$h'(x) = \frac{\sec^2 x}{1+\tan^2 x} = 1$$

provided that $x \neq (k + \frac{1}{2})\pi$ where k is an integer. $h(x)$ is not defined at odd multiples of $\dfrac{\pi}{2}$.

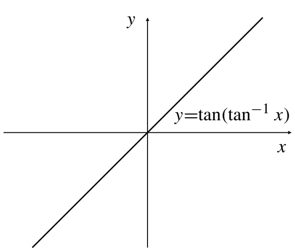

Fig. 3.5.40(a)

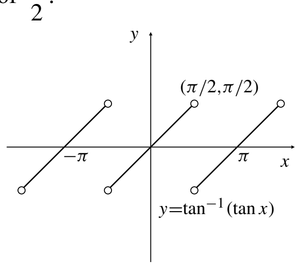

Fig. 3.5.40(b)

41. $\dfrac{d}{dx}\cos^{-1}(\cos x) = \dfrac{-1}{\sqrt{1-\cos^2 x}}(-\sin x)$

$$= \begin{cases} 1 & \text{if } \sin x > 0 \\ -1 & \text{if } \sin x < 0 \end{cases}$$

$\cos^{-1}(\cos x)$ is continuous everywhere and differentiable everywhere except at $x = n\pi$ for integers n.

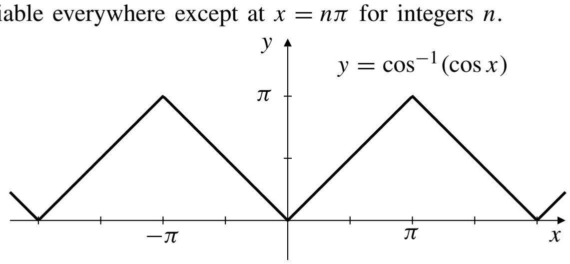

Fig. 3.5.41

42. $\dfrac{d}{dx}\sin^{-1}(\cos x) = \dfrac{1}{\sqrt{1-\cos^2 x}}(-\sin x)$

$$= \begin{cases} -1 & \text{if } \sin x > 0 \\ 1 & \text{if } \sin x < 0 \end{cases}$$

$\sin^{-1}(\cos x)$ is continuous everywhere and differentiable everywhere except at $x = n\pi$ for integers n.

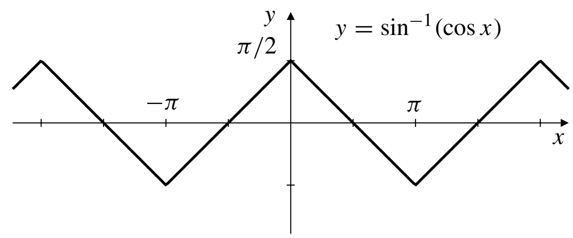

Fig. 3.5.42

43. $\dfrac{d}{dx}\tan^{-1}(\tan x) = \dfrac{1}{1+\tan^2 x}(\sec^2 x) = 1$ except at odd multiples of $\pi/2$.

$\tan^{-1}(\tan x)$ is continuous and differentiable everywhere except at $x = (2n+1)\pi/2$ for integers n. It is not defined at those points.

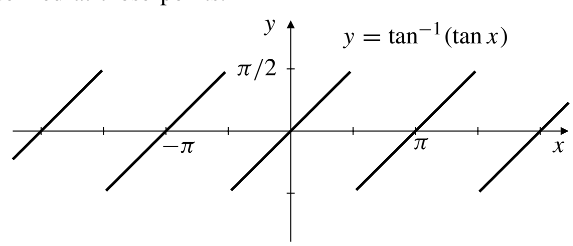

Fig. 3.5.43

44. $\dfrac{d}{dx}\tan^{-1}(\cot x) = \dfrac{1}{1+\cot^2 x}(-\csc^2 x) = -1$ except at integer multiples of π.

$\tan^{-1}(\cot x)$ is continuous and differentiable everywhere except at $x = n\pi$ for integers n. It is not defined at those points.

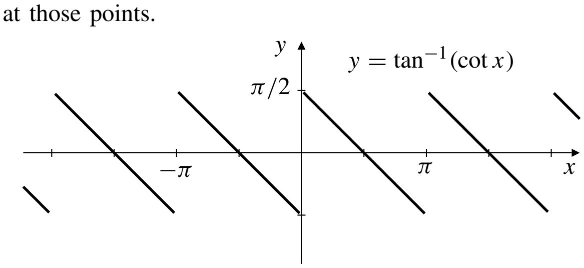

Fig. 3.5.44

45. If $|x| < 1$ and $y = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$, then $y > 0 \Leftrightarrow x > 0$ and

$$\begin{aligned}\tan y &= \frac{x}{\sqrt{1-x^2}}\\ \sec^2 y &= 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}\\ \sin^2 y &= 1 - \cos^2 y = 1 - (1-x^2) = x^2\\ \sin y &= x.\end{aligned}$$

Thus $y = \sin^{-1} x$ and $\sin^{-1} x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$.
An alternative method of proof involves showing that the derivative of the left side minus the right side is 0, and both sides are 0 at $x = 0$.

46. If $x \ge 1$ and $y = \tan^{-1}\sqrt{x^2-1}$, then $\tan y = \sqrt{x^2-1}$ and $\sec y = x$, so that $y = \sec^{-1} x$.
If $x \le -1$ and $y = \pi - \tan^{-1}\sqrt{x^2-1}$, then $\frac{\pi}{2} < y < \frac{3\pi}{2}$, so $\sec y < 0$. Therefore

$$\begin{aligned}\tan y &= \tan(\pi - \tan^{-1}\sqrt{x^2-1}) = -\sqrt{x^2-1}\\ \sec^2 y &= 1 + (x^2-1) = x^2\\ \sec y &= x,\end{aligned}$$

because both x and $\sec y$ are negative. Thus $y = \sec^{-1} x$ in this case also.

47. If $y = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$, then $y > 0 \Leftrightarrow x > 0$ and

$$\begin{aligned}\sin y &= \frac{x}{\sqrt{1+x^2}}\\ \cos^2 y &= 1 - \sin^2 y = 1 - \frac{x^2}{1+x^2} = \frac{1}{1+x^2}\\ \tan^2 y &= \sec^2 y - 1 = 1 + x^2 - 1 = x^2\\ \tan y &= x.\end{aligned}$$

Thus $y = \tan^{-1} x$ and $\tan^{-1} x = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$.

48. If $x \ge 1$ and $y = \sin^{-1}\dfrac{\sqrt{x^2-1}}{x}$, then $0 \le y < \frac{\pi}{2}$ and

$$\begin{aligned}\sin y &= \frac{\sqrt{x^2-1}}{x}\\ \cos^2 y &= 1 - \frac{x^2-1}{x^2} = \frac{1}{x^2}\\ \sec^2 y &= x^2.\end{aligned}$$

Thus $\sec y = x$ and $y = \sec^{-1} x$.
If $x \le -1$ and $y = \pi - \sin^{-1}\dfrac{\sqrt{x^2-1}}{x}$, then $\frac{\pi}{2} \le y < \frac{3\pi}{2}$ and $\sec y < 0$. Therefore

$$\begin{aligned}\sin y &= \sin\left(\pi - \sin^{-1}\frac{\sqrt{x^2-1}}{x}\right) = \frac{\sqrt{x^2-1}}{x}\\ \cos^2 y &= 1 - \frac{x^2-1}{x^2} = \frac{1}{x^2}\\ \sec^2 y &= x^2\\ \sec y &= x,\end{aligned}$$

because both x and $\sec y$ are negative. Thus $y = \sec^{-1} x$ in this case also.

49. $f'(x) \equiv 0$ on $(-\infty, -1)$
Thus $f(x) = \tan^{-1}\left(\dfrac{x-1}{x+1}\right) - \tan^{-1} x = C$ on $(-\infty, -1)$.

Evaluate the limit as $x \to -\infty$:

$$\lim_{x\to-\infty} f(x) = \tan^{-1} 1 - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$$

Thus $\tan^{-1}\left(\dfrac{x-1}{x+1}\right) - \tan^{-1} x = \dfrac{3\pi}{4}$ on $(-\infty, -1)$.

50. Since $f(x) = x - \tan^{-1}(\tan x)$ then

$$f'(x) = 1 - \frac{\sec^2 x}{1 + \tan^2 x} = 1 - 1 = 0$$

if $x \ne -(k + \frac{1}{2})\pi$ where k is an integer. Thus, f is constant on intervals not containing odd multiples of $\dfrac{\pi}{2}$. $f(0) = 0$ but $f(\pi) = \pi - 0 = \pi$. There is no contradiction here because $f'\left(\dfrac{\pi}{2}\right)$ is not defined, so f is not constant on the interval containing 0 and π.

51. $f(x) = x - \sin^{-1}(\sin x) \quad (-\pi \le x \le \pi)$

$$\begin{aligned}f'(x) &= 1 - \frac{1}{\sqrt{1-\sin^2 x}}\cos x\\ &= 1 - \frac{\cos x}{|\cos x|}\\ &= \begin{cases} 0 & \text{if } -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}\\ 2 & \text{if } -\pi < x < -\dfrac{\pi}{2} \text{ or } \dfrac{\pi}{2} < x < \pi\end{cases}\end{aligned}$$

Note: f is not differentiable at $\pm\dfrac{\pi}{2}$.

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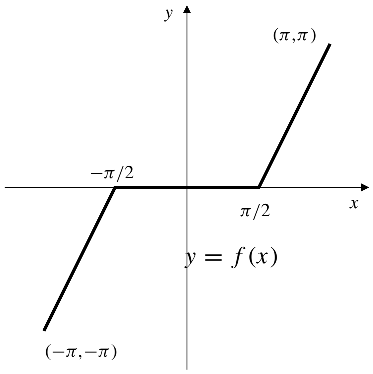

Fig. 3.5.51

52. $y' = \dfrac{1}{1+x^2} \Rightarrow y = \tan^{-1} x + C$

$y(0) = C = 1$

Thus, $y = \tan^{-1} x + 1$.

53. $\begin{cases} y' = \dfrac{1}{9+x^2} & \Rightarrow y = \dfrac{1}{3}\tan^{-1}\dfrac{x}{3} + C \\ y(3) = 2 & 2 = \dfrac{1}{3}\tan^{-1} 1 + C \qquad C = 2 - \dfrac{\pi}{12} \end{cases}$

Thus $y = \dfrac{1}{3}\tan^{-1}\dfrac{x}{3} + 2 - \dfrac{\pi}{12}$.

54. $y' = \dfrac{1}{\sqrt{1-x^2}} \Rightarrow y = \sin^{-1} x + C$

$y(\frac{1}{2}) = \sin^{-1}(\frac{1}{2}) + C = 1$

$\Rightarrow \dfrac{\pi}{6} + C = 1 \Rightarrow C = 1 - \dfrac{\pi}{6}$.

Thus, $y = \sin^{-1} x + 1 - \dfrac{\pi}{6}$.

55. $\begin{cases} y' = \dfrac{4}{\sqrt{25-x^2}} & \Rightarrow y = 4\sin^{-1}\dfrac{x}{5} + C \\ y(0) = 0 & 0 = 0 + C \Rightarrow C = 0 \end{cases}$

Thus $y = 4\sin^{-1}\dfrac{x}{5}$.

Section 3.6 Hyperbolic Functions (page 200)

1. $\dfrac{d}{dx}\operatorname{sech} x = \dfrac{d}{dx}\dfrac{1}{\cosh x}$

$= -\dfrac{1}{\cosh^2 x}\sinh x = -\operatorname{sech} x \tanh x$

$\dfrac{d}{dx}\operatorname{csch} x = \dfrac{d}{dx}\dfrac{1}{\sinh x}$

$= -\dfrac{1}{\sinh^2 x}\cosh x = -\operatorname{csch} x \coth x$

$\dfrac{d}{dx}\coth x = \dfrac{d}{dx}\dfrac{cosh}{\sinh x}$

$= \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\dfrac{1}{\sinh^2 x} = -\operatorname{csch}^2 x$

2. $\cosh x \cosh y + \sinh x \sinh y$

$= \frac{1}{4}[(e^x + e^{-x})(e^y + e^{-y}) + (e^x - e^{-x})(e^y - e^{-y})]$

$= \frac{1}{4}(2e^{x+y} + 2e^{-x-y}) = \frac{1}{2}(e^{x+y} + e^{-(x+y)})$

$= \cosh(x+y)$.

$\sinh x \cosh y + \cosh x \sinh y$

$= \frac{1}{4}[(e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y})]$

$= \frac{1}{2}(e^{x+y} - e^{-(x+y)}) = \sinh(x+y)$.

$\cosh(x-y) = \cosh[x + (-y)]$

$= \cosh x \cosh(-y) + \sinh x \sinh(-y)$

$= \cosh x \cosh y - \sinh x \sinh y$.

$\sinh(x-y) = \sinh[x + (-y)]$

$= \sinh x \cosh(-y) + \cosh x \sinh(-y)$

$= \sinh x \cosh y - \cosh x \sinh y$.

3. $\tanh(x \pm y) = \dfrac{\sinh(x \pm y)}{\cosh(x \pm y)}$

$= \dfrac{\sinh x \cosh y \pm \cosh x \sinh y}{\cosh x \cosh y \pm \sinh x \sinh y}$

$= \dfrac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$

4. $y = \coth x = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ $\qquad y = \operatorname{sech} x = \dfrac{2}{e^x + e^{-x}}$

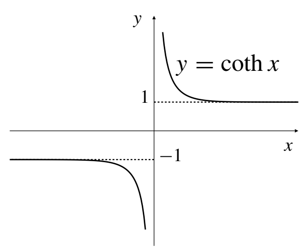

Fig. 3.6.4(a)

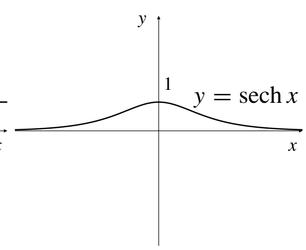

Fig. 3.6.4(b)

$y = \operatorname{csch} x = \dfrac{2}{e^x - e^{-x}}$

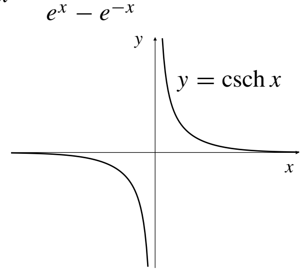

Fig. 3.6.4

5. $\dfrac{d}{dx}\sinh^{-1}x = \dfrac{d}{dx}\ln(x+\sqrt{x^2+1}) = \dfrac{1+\dfrac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$
$= \dfrac{1}{\sqrt{x^2+1}}$

$\dfrac{d}{dx}\cosh^{-1}x = \dfrac{d}{dx}\ln(x+\sqrt{x^2-1}) = \dfrac{1+\dfrac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}$
$= \dfrac{1}{\sqrt{x^2-1}}$

$\dfrac{d}{dx}\tanh^{-1}x = \dfrac{d}{dx}\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$
$= \dfrac{1}{2}\dfrac{1-x}{1+x}\dfrac{1-x-(1+x)(-1)}{(1-x)^2} = \dfrac{1}{1-x^2}$

$\int\dfrac{dx}{\sqrt{x^2+1}} = \sinh^{-1}x + C$

$\int\dfrac{dx}{\sqrt{x^2-1}} = \cosh^{-1}x + C \quad (x > 1)$

$\int\dfrac{dx}{1-x^2} = \tanh^{-1}x + C \quad (-1 < x < 1)$

6. Let $y = \sinh^{-1}\left(\dfrac{x}{a}\right) \Leftrightarrow x = a\sinh y \Rightarrow 1 = a(\cosh y)\dfrac{dy}{dx}$. Thus,

$$\frac{d}{dx}\sinh^{-1}\left(\frac{x}{a}\right) = \frac{1}{a\cosh y}$$
$$= \frac{1}{a\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{a^2+x^2}}$$
$$\int\frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a} + C. \qquad (a > 0)$$

Let $y = \cosh^{-1}\dfrac{x}{a} \Leftrightarrow x = a\,\mathrm{Cosh}\,y = a\cosh y$ for $y \ge 0$, $x \ge a$. We have $1 = a(\sinh y)\dfrac{dy}{dx}$. Thus,

$$\frac{d}{dx}\cosh^{-1}\frac{x}{a} = \frac{1}{a\sinh y}$$
$$= \frac{1}{a\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-a^2}}$$
$$\int\frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} + C. \qquad (a > 0,\ x \ge a)$$

Let $y = \tanh^{-1}\dfrac{x}{a} \Leftrightarrow x = a\tanh y \Rightarrow 1 = a(\mathrm{sech}^2\,y)\dfrac{dy}{dx}$. Thus,

$$\frac{d}{dx}\tanh^{-1}\frac{x}{a} = \frac{1}{a\,\mathrm{sech}^2\,y}$$
$$= \frac{a}{a^2 - a^2\tanh^2 x} = \frac{a}{a^2-x^2}$$
$$\int\frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\frac{x}{a} + C.$$

7. a) $\sinh\ln x = \dfrac{1}{2}(e^{\ln x} - e^{-\ln x}) = \dfrac{1}{2}\left(x - \dfrac{1}{x}\right) = \dfrac{x^2-1}{2x}$

b) $\cosh\ln x = \dfrac{1}{2}(e^{\ln x} + e^{-\ln x}) = \dfrac{1}{2}\left(x + \dfrac{1}{x}\right) = \dfrac{x^2+1}{2x}$

c) $\tanh\ln x = \dfrac{\sinh\ln x}{\cosh\ln x} = \dfrac{x^2-1}{x^2+1}$

d) $\dfrac{\cosh\ln x + \sinh\ln x}{\cosh\ln x - \sinh\ln x} = \dfrac{x^2+1+(x^2-1)}{(x^2+1)-(x^2-1)} = x^2$

8. $\mathrm{csch}^{-1}x = \sinh^{-1}(1/x) = \ln\left(\dfrac{1}{x} + \sqrt{\dfrac{1}{x^2}+1}\right)$ has domain and range consisting of all real numbers x except $x = 0$. We have

$$\frac{d}{dx}\mathrm{csch}^{-1}x = \frac{d}{dx}\sinh^{-1}\frac{1}{x}$$
$$= \frac{1}{\sqrt{1+\left(\frac{1}{x}\right)^2}}\left(\frac{-1}{x^2}\right) = \frac{-1}{|x|\sqrt{x^2+1}}.$$

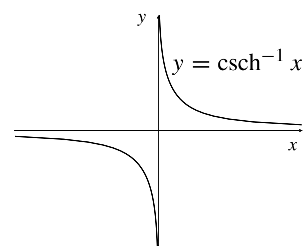

Fig. 3.6.8

9. $\coth^{-1}x = \tanh^{-1}\dfrac{1}{x} = \dfrac{1}{2}\ln\left(\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}\right) = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right)$, for $|x| > 1$. Also

$$\frac{d}{dx}\coth^{-1}x = \frac{d}{dx}\tanh^{-1}\frac{1}{x}$$
$$= \frac{1}{1-(1/x)^2}\,\frac{-1}{x^2} = \frac{-1}{x^2-1}.$$

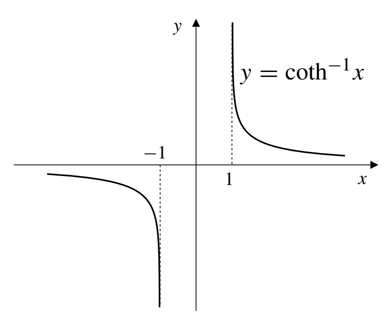

Fig. 3.6.9

10. Let $y = \mathrm{Sech}^{-1} x$ where $\mathrm{Sech}\, x = \mathrm{sech}\, x$ for $x \ge 0$. Hence, for $y \ge 0$,

$$x = \mathrm{sech}\, y \Leftrightarrow \frac{1}{x} = \cosh y$$
$$\Leftrightarrow \frac{1}{x} = \mathrm{Cosh}\, y \Leftrightarrow y = \mathrm{Cosh}^{-1} \frac{1}{x}.$$

Thus,

$$\mathrm{Sech}^{-1} x = \mathrm{Cosh}^{-1} \frac{1}{x}$$
$$\mathcal{D}(\mathrm{Sech}^{-1}) = \mathcal{R}(\mathrm{sech}) = (0, 1]$$
$$\mathcal{R}(\mathrm{Sech}^{-1}) = \mathcal{D}(\mathrm{sech}) = [0, \infty).$$

Also,

$$\frac{d}{dx}\mathrm{Sech}^{-1} x = \frac{d}{dx}\mathrm{Cosh}^{-1} \frac{1}{x}$$
$$= \frac{1}{\sqrt{\left(\frac{1}{x}\right)^2 - 1}}\left(\frac{-1}{x^2}\right) = \frac{-1}{x\sqrt{1-x^2}}.$$

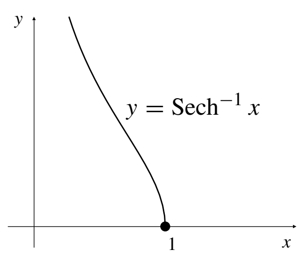

Fig. 3.6.10

11. $f_{A,B}(x) = Ae^{kx} + Be^{-kx}$
$f'_{A,B}(x) = kAe^{kx} - kBe^{-kx}$
$f''_{A,B}(x) = k^2 Ae^{kx} + k^2 Be^{-kx}$
Thus $f''_{A,B} - k^2 f_{A,B} = 0$

$$g_{C,D}(x) = C \cosh kx + D \sinh kx$$
$$g'_{C,D}(x) = kC \cosh kx + kD \sinh kx$$
$$g''_{C,D}(x) = k^2 C \cosh kx + k^2 D \sinh kx$$

Thus $g''_{C,D} - k^2 g_{C,D} = 0$
$\cosh kx + \sinh kx = e^{kx}$
$\cosh kx - \sinh kx = e^{-kx}$
Thus $f_{A,B}(x) = (A+B)\cosh kx + (A-B)\sinh kx$, that is,
$f_{A,B}(x) = g_{A+B,A-B}(x)$, and
$g_{C,D}(x) = \frac{C}{2}(e^{kx} + e^{-kx}) + \frac{D}{2}(e^{kx} - e^{-kx})$,
that is $g_{C,D}(x) = f_{(C+D)/2,(C-D)/2}(x)$.

12. Since

$$h_{L,M}(x) = L \cosh k(x-a) + M \sinh k(x-a)$$
$$h''_{L,M}(x) = Lk^2 \cosh k(x-a) + Mk^2 \sinh k(x-a)$$
$$= k^2 h_{L,M}(x)$$

hence, $h_{L,M}(x)$ is a solution of $y'' - k^2 y = 0$ and

$$h_{L,M}(x)$$
$$= \frac{L}{2}\left(e^{kx-ka} + e^{-kx+ka}\right) + \frac{M}{2}\left(e^{kx-ka} - e^{-kx+ka}\right)$$
$$= \left(\frac{L}{2}e^{-ka} + \frac{M}{2}e^{-ka}\right)e^{kx} + \left(\frac{L}{2}e^{ka} - \frac{M}{2}e^{ka}\right)e^{-kx}$$
$$= Ae^{kx} + Be^{-kx} = f_{A,B}(x)$$

where $A = \frac{1}{2}e^{-ka}(L+M)$ and $B = \frac{1}{2}e^{ka}(L-M)$.

13. $y'' - k^2 y = 0 \Rightarrow y = h_{L,M}(x)$
$= L \cosh k(x-a) + M \sinh k(x-a)$
$y(a) = y_0 \Rightarrow y_0 = L + 0 \Rightarrow L = y_0$,
$y'(a) = v_0 \Rightarrow v_0 = 0 + Mk \Rightarrow M = \frac{v_0}{k}$
Therefore $y = h_{y_0, v_0/k}(x)$
$= y_0 \cosh k(x-a) + (v_0/k)\sinh k(x-a)$.

Section 3.7 Second-Order Linear DEs with Constant Coefficients (page 206)

1. $y'' + 7y' + 10y = 0$
auxiliary eqn $r^2 + 7r + 10 = 0$
$(r+5)(r+2) = 0 \quad\Rightarrow\quad r = -5, -2$
$y = Ae^{-5t} + Be^{-2t}$

2. $y'' - 2y' - 3y = 0$
auxiliary eqn $r^2 - 2r - 3 = 0 \quad\Rightarrow\quad r = -1,\ r = 3$
$y = Ae^{-t} + Be^{3t}$

3. $y'' + 2y' = 0$
auxiliary eqn $r^2 + 2r = 0 \quad\Rightarrow\quad r = 0,\ -2$
$y = A + Be^{-2t}$

4. $4y'' - 4y' - 3y = 0$
$4r^2 - 4r - 3 = 0 \Rightarrow (2r+1)(2r-3) = 0$
Thus, $r_1 = -\frac{1}{2}$, $r_2 = \frac{3}{2}$, and $y = Ae^{-(1/2)t} + Be^{(3/2)t}$.

5. $y'' + 8y' + 16y = 0$
auxiliary eqn $r^2 + 8r + 16 = 0 \quad\Rightarrow\quad r = -4,\ -4$
$y = Ae^{-4t} + Bte^{-4t}$

6. $y'' - 2y' + y = 0$
$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$
Thus, $r = 1,\ 1$, and $y = Ae^t + Bte^t$.

7. $y'' - 6y' + 10y = 0$
auxiliary eqn $r^2 - 6r + 10 = 0 \quad\Rightarrow\quad r = 3 \pm i$
$y = Ae^{3t}\cos t + Be^{3t}\sin t$

8. $9y'' + 6y' + y = 0$

$9r^2 + 6r + 1 = 0 \Rightarrow (3r + 1)^2 = 0$

Thus, $r = -\frac{1}{3}, \ -\frac{1}{3}$, and $y = Ae^{-(1/3)t} + Bte^{-(1/3)t}$.

9.

$$\begin{aligned} & y'' + 2y' + 5y = 0 \\ \text{auxiliary eqn} \quad & r^2 + 2r + 5 = 0 \quad \Rightarrow \quad r = -1 \pm 2i \\ & y = Ae^{-t}\cos 2t + Be^{-t}\sin 2t \end{aligned}$$

10. For $y'' - 4y' + 5y = 0$ the auxiliary equation is $r^2 - 4r + 5 = 0$, which has roots $r = 2 \pm i$. Thus, the general solution of the DE is $y = Ae^{2t}\cos t + Be^{2t}\sin t$.

11. For $y'' + 2y' + 3y = 0$ the auxiliary equation is $r^2 + 2r + 3 = 0$, which has solutions $r = -1 \pm \sqrt{2}i$. Thus the general solution of the given equation is $y = Ae^{-t}\cos(\sqrt{2}t) + Be^{-t}\sin(\sqrt{2}t)$.

12. Given that $y'' + y' + y = 0$, hence $r^2 + r + 1 = 0$. Since $a = 1$, $b = 1$ and $c = 1$, the discriminant is $D = b^2 - 4ac = -3 < 0$ and $-(b/2a) = -\frac{1}{2}$ and $\omega = \sqrt{3}/2$. Thus, the general solution is $y = Ae^{-(1/2)t}\cos\left(\frac{\sqrt{3}}{2}t\right) + Be^{-(1/2)t}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

13. $\begin{cases} 2y'' + 5y' - 3y = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$

The DE has auxiliary equation $2r^2 + 5y - 3 = 0$, with roots $r = \frac{1}{2}$ and $r = -3$. Thus $y = Ae^{t/2} + Be^{-3t}$.

Now $1 = y(0) = A + B$, and $0 = y'(0) = \dfrac{A}{2} - 3B$.

Thus $B = 1/7$ and $A = 6/7$. The solution is $y = \dfrac{6}{7}e^{t/2} + \dfrac{1}{7}e^{-3t}$.

14. Given that $y'' + 10y' + 25y = 0$, hence $r^2 + 10r + 25 = 0 \Rightarrow (r + 5)^2 = 0 \Rightarrow r = -5$. Thus,

$$\begin{aligned} y &= Ae^{-5t} + Bte^{-5t} \\ y' &= -5e^{-5t}(A + Bt) + Be^{-5t}. \end{aligned}$$

Since

$$\begin{aligned} 0 &= y(1) = Ae^{-5} + Be^{-5} \\ 2 &= y'(1) = -5e^{-5}(A + B) + Be^{-5}, \end{aligned}$$

we have $A = -2e^5$ and $B = 2e^5$.

Thus, $y = -2e^5e^{-5t} + 2te^5e^{-5t} = 2(t - 1)e^{-5(t-1)}$.

15. $\begin{cases} y'' + 4y' + 5y = 0 \\ y(0) = 2 \\ y'(0) = 0 \end{cases}$

The auxiliary equation for the DE is $r^2 + 4r + 5 = 0$, which has roots $r = -2 \pm i$. Thus

$$\begin{aligned} y &= Ae^{-2t}\cos t + Be^{-2t}\sin t \\ y' &= (-2Ae^{-2t} + Be^{-2t})\cos t - (Ae^{-2t} + 2Be^{-2t})\sin t. \end{aligned}$$

Now $2 = y(0) = A \Rightarrow A = 2$, and

$2 = y'(0) = -2A + B \Rightarrow B = 6$.

Therefore $y = e^{-2t}(2\cos t + 6\sin t)$.

16. The auxiliary equation $r^2 - (2 + \epsilon)r + (1 + \epsilon)$ factors to $(r - 1 - \epsilon)(r - 1) = 0$ and so has roots $r = 1 + \epsilon$ and $r = 1$. Thus the DE $y'' - (2 + \epsilon)y' + (1 + \epsilon)y = 0$ has general solution $y = Ae^{(1+\epsilon)t} + Be^t$. The function $y_\epsilon(t) = \dfrac{e^{(1+\epsilon)t} - e^t}{\epsilon}$ is of this form with $A = -B = 1/\epsilon$. We have, substituting $\epsilon = h/t$,

$$\begin{aligned} \lim_{\epsilon \to 0} y_\epsilon(t) &= \lim_{\epsilon \to 0} \frac{e^{(1+\epsilon)t} - e^t}{\epsilon} \\ &= t \lim_{h \to 0} \frac{e^{t+h} - e^t}{h} \\ &= t\left(\frac{d}{dt}e^t\right) = t\,e^t \end{aligned}$$

which is, along with e^t, a solution of the CASE II DE $y'' - 2y' + y = 0$.

17. Given that $a > 0$, $b > 0$ and $c > 0$:

Case 1: If $D = b^2 - 4ac > 0$ then the two roots are

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Since

$$\begin{aligned} b^2 - 4ac &< b^2 \\ \pm\sqrt{b^2 - 4ac} &< b \\ -b \pm \sqrt{b^2 - 4ac} &< 0 \end{aligned}$$

therefore r_1 and r_2 are negative. The general solution is

$$y(t) = Ae^{r_1 t} + Be^{r_2 t}.$$

If $t \to \infty$, then $e^{r_1 t} \to 0$ and $e^{r_2 t} \to 0$.

Thus, $\lim_{t \to \infty} y(t) = 0$.

Case 2: If $D = b^2 - 4ac = 0$ then the two equal roots $r_1 = r_2 = -b/(2a)$ are negative. The general solution is

$$y(t) = Ae^{r_1 t} + Bte^{r_2 t}.$$

If $t \to \infty$, then $e^{r_1 t} \to 0$ and $e^{r_2 t} \to 0$ at a faster rate than $Bt \to \infty$. Thus, $\lim_{t \to \infty} y(t) = 0$.

Case 3: If $D = b^2 - 4ac < 0$ then the general solution is

$$y = Ae^{-(b/2a)t}\cos(\omega t) + Be^{-(b/2a)t}\sin(\omega t)$$

where $\omega = \dfrac{\sqrt{4ac - b^2}}{2a}$. If $t \to \infty$, then the amplitude of both terms $Ae^{-(b/2a)t} \to 0$ and $Be^{-(b/2a)t} \to 0$. Thus, $\lim_{t \to \infty} y(t) = 0$.

18. The auxiliary equation $ar^2+br+c=0$ has roots

$$r_1=\frac{-b-\sqrt{D}}{2a},\quad r_2=\frac{-b+\sqrt{D}}{2a},$$

where $D = b^2 - 4ac$. Note that $a(r_2-r_1)=\sqrt{D}=-(2ar_1+b)$. If $y=e^{r_1t}u$, then $y'=e^{r_1t}(u'+r_1u)$, and $y''=e^{r_1t}(u''+2r_1u'+r_1^2u)$. Substituting these expressions into the DE $ay''+by'+cy=0$, and simplifying, we obtain

$$e^{r_1t}(au''+2ar_1u'+bu')=0,$$

or, more simply, $u''-(r_2-r_1)u'=0$. Putting $v=u'$ reduces this equation to first order:

$$v'=(r_2-r_1)v,$$

which has general solution $v=Ce^{(r_2-r_1)t}$. Hence

$$u=\int Ce^{(r_2-r_1)t}\,dt=Be^{(r_2-r_1)t}+A,$$

and $y=e^{r_1t}u=Ae^{r_1t}+Be^{r_2t}$.

19. If $y=A\cos\omega t+B\sin\omega t$ then

$$\begin{aligned}y''+\omega^2y&=-A\omega^2\cos\omega t-B\omega^2\sin\omega t\\&\quad+\omega^2(A\cos\omega t+B\sin\omega t)=0\end{aligned}$$

for all t. So y is a solution of (†).

20. If $f(t)$ is any solution of (†) then $f''(t)=-\omega^2f(t)$ for all t. Thus,

$$\begin{aligned}&\frac{d}{dt}\left[\omega^2\big(f(t)\big)^2+\big(f'(t)\big)^2\right]\\&=2\omega^2f(t)f'(t)+2f'(t)f''(t)\\&=2\omega^2f(t)f'(t)-2\omega^2f(t)f'(t)=0\end{aligned}$$

for all t. Thus, $\omega^2\big(f(t)\big)^2+\big(f'(t)\big)^2$ is constant. (This can be interpreted as a conservation of energy statement.)

21. If $g(t)$ satisfies (†) and also $g(0)=g'(0)=0$, then by Exercise 20,

$$\begin{aligned}&\omega^2\big(g(t)\big)^2+\big(g'(t)\big)^2\\&=\omega^2\big(g(0)\big)^2+\big(g'(0)\big)^2=0.\end{aligned}$$

Since a sum of squares cannot vanish unless each term vanishes, $g(t)=0$ for all t.

22. If $f(t)$ is any solution of (†), let $g(t)=f(t)-A\cos\omega t-B\sin\omega t$ where $A=f(0)$ and $B\omega=f'(0)$. Then g is also solution of (†). Also $g(0)=f(0)-A=0$ and $g'(0)=f'(0)-B\omega=0$. Thus, $g(t)=0$ for all t by Exercise 24, and therefore $f(x)=A\cos\omega t+B\sin\omega t$. Thus, it is proved that every solution of (†) is of this form.

23. We are given that $k=-\dfrac{b}{2a}$ and $\omega^2=\dfrac{4ac-b^2}{4a^2}$ which is positive for Case III. If $y=e^{kt}u$, then

$$\begin{aligned}y'&=e^{kt}\big(u'+ku\big)\\y''&=e^{kt}\big(u''+2ku'+k^2u\big).\end{aligned}$$

Substituting into $ay''+by'+cy=0$ leads to

$$\begin{aligned}0&=e^{kt}\Big(au''+(2ka+b)u'+(ak^2+bk+c)u\Big)\\&=e^{kt}\Big(au''+0+((b^2/(4a)-(b^2/(2a)+c)u\Big)\\&=a\,e^{kt}\Big(u''+\omega^2u\Big).\end{aligned}$$

Thus u satisfies $u''+\omega^2u=0$, which has general solution

$$u=A\cos(\omega t)+B\sin(\omega t)$$

by the previous problem. Therefore $ay''+by'+cy=0$ has general solution

$$y=Ae^{kt}\cos(\omega t)+Be^{kt}\sin(\omega t).$$

24. Because $y''+4y=0$, therefore $y=A\cos 2t+B\sin 2t$. Now

$$\begin{aligned}y(0)&=2\Rightarrow A=2,\\y'(0)&=-5\Rightarrow B=-\tfrac{5}{2}.\end{aligned}$$

Thus, $y=2\cos 2t-\frac{5}{2}\sin 2t$.
circular frequency $=\omega=2$, frequency $=\dfrac{\omega}{2\pi}=\dfrac{1}{\pi}\approx 0.318$
period $=\dfrac{2\pi}{\omega}=\pi\approx 3.14$
amplitude $=\sqrt{(2)^2+(-\frac{5}{2})^2}\simeq 3.20$

25. $\begin{cases}y''+100y=0\\y(0)=0\\y'(0)=3\end{cases}$

$y=A\cos(10t)+B\sin(10t)$
$A=y(0)=0,\quad 10B=y'(0)=3$
$y=\dfrac{3}{10}\sin(10t)$

26. $y = \mathcal{A}\cos\Big(\omega(t-c)\Big) + \mathcal{B}\sin\Big(\omega(t-c)\Big)$
(easy to calculate $y'' + \omega^2 y = 0$)

$$\begin{aligned}
y &= \mathcal{A}\Big(\cos(\omega t)\cos(\omega c) + \sin(\omega t)\sin(\omega c)\Big)\\
&\quad + \mathcal{B}\Big(\sin(\omega t)\cos(\omega c) - \cos(\omega t)\sin(\omega c)\Big)\\
&= \Big(\mathcal{A}\cos(\omega c) - \mathcal{B}\sin(\omega c)\Big)\cos\omega t\\
&\quad + \Big(\mathcal{A}\sin(\omega c) + \mathcal{B}\cos(\omega c)\Big)\sin\omega t\\
&= A\cos\omega t + B\sin\omega t
\end{aligned}$$

where $A = \mathcal{A}\cos(\omega c) - \mathcal{B}\sin(\omega c)$ and
$B = \mathcal{A}\sin(\omega c) + \mathcal{B}\cos(\omega c)$

27. For $y'' + y = 0$, we have $y = A\sin t + B\cos t$. Since,

$$\begin{aligned}
y(2) &= 3 = A\sin 2 + B\cos 2\\
y'(2) &= -4 = A\cos 2 - B\sin 2,
\end{aligned}$$

therefore

$$\begin{aligned}
A &= 3\sin 2 - 4\cos 2\\
B &= 4\sin 2 + 3\cos 2.
\end{aligned}$$

Thus,

$$\begin{aligned}
y &= (3\sin 2 - 4\cos 2)\sin t + (4\sin 2 + 3\cos 2)\cos t\\
&= 3\cos(t-2) - 4\sin(t-2).
\end{aligned}$$

28. $\begin{cases} y'' + \omega^2 y = 0\\ y(a) = A\\ y'(a) = B\end{cases}$

$y = A\cos\Big(\omega(t-a)\Big) + \dfrac{B}{\omega}\sin\Big(\omega(t-a)\Big)$

29. From Example 9, the spring constant is $k = 9\times 10^4$ gm/sec^2. For a frequency of 10 Hz (i.e., a circular frequency $\omega = 20\pi$ rad/sec.), a mass m satisfying $\sqrt{k/m} = 20\pi$ should be used. So,

$$m = \frac{k}{400\pi^2} = \frac{9\times 10^4}{400\pi^2} = 22.8 \text{ gm}.$$

The motion is determined by

$$\begin{cases} y'' + 400\pi^2 y = 0\\ y(0) = -1\\ y'(0) = 2\end{cases}$$

therefore, $y = A\cos 20\pi t + B\sin 20\pi t$ and

$$\begin{aligned}
y(0) &= -1 \Rightarrow A = -1\\
y'(0) &= 2 \Rightarrow B = \frac{2}{20\pi} = \frac{1}{10\pi}.
\end{aligned}$$

Thus, $y = -\cos 20\pi t + \dfrac{1}{10\pi}\sin 20\pi t$, with y in cm and t in second, gives the displacement at time t. The amplitude is $\sqrt{(-1)^2 + (\frac{1}{10\pi})^2} \approx 1.0005$ cm.

30. Frequency $= \dfrac{\omega}{2\pi}$, $\omega^2 = \dfrac{k}{m}$ (k = spring const, m = mass)
Since the spring does not change, $\omega^2 m = k$ (constant)
For $m = 400$ gm, $\omega = 2\pi(24)$ (frequency = 24 Hz)
If $m = 900$ gm, then $\omega^2 = \dfrac{4\pi^2(24)^2(400)}{900}$
so $\omega = \dfrac{2\pi\times 24\times 2}{3} = 32\pi$.
Thus frequency $= \dfrac{32\pi}{2\pi} = 16$ Hz
For $m = 100$ gm, $\omega = \dfrac{4\pi^2(24)^2 400}{100}$
so $\omega = 96\pi$ and frequency $= \dfrac{\omega}{2\pi} = 48$ Hz.

31. Using the addition identities for cosine and sine,

$$\begin{aligned}
y &= e^{kt}[A\cos\omega(t-t_0)B\sin\omega(t-t_0)]\\
&= e^{kt}[A\cos\omega t\cos\omega t_0 + A\sin\omega t\sin\omega t_0\\
&\quad + B\sin\omega t\cos\omega t_0 - B\cos\omega t\sin\omega t_0]\\
&= e^{kt}[A_1\cos\omega t + B_1\sin\omega t],
\end{aligned}$$

where $A_1 = A\cos\omega t_0 - B\sin\omega t_0$ and $B_1 = A\sin\omega t_0 + B\cos\omega t_0$. Under the conditions of this problem we know that $e^{kt}\cos\omega t$ and $e^{kt}\sin\omega t$ are independent solutions of $ay'' + by' + cy = 0$, so our function y must also be a solution, and, since it involves two arbitrary constants, it is a general solution.

32. Expanding the hyperbolic functions in terms of exponentials,

$$\begin{aligned}
y &= e^{kt}[A\cosh\omega(t-t_0)B\sinh\omega(t-t_0)]\\
&= e^{kt}\left[\frac{A}{2}e^{\omega(t-t_0)} + \frac{A}{2}e^{-\omega(t-t_0)}\right.\\
&\quad \left. + \frac{B}{2}e^{\omega(t-t_0)} - \frac{B}{2}e^{-\omega(t-t_0)}\right]\\
&= A_1 e^{(k+\omega)t} + B_1 e^{(k-\omega)t}
\end{aligned}$$

where $A_1 = (A/2)e^{-\omega t_0} + (B/2)e^{-\omega t_0}$ and $B_1 = (A/2)e^{\omega t_0} - (B/2)e^{\omega t_0}$. Under the conditions of this problem we know that $Rr = k \pm \omega$ are the two real roots of the auxiliary equation $ar^2 + br + c = 0$, so $e^{(k\pm\omega)t}$ are independent solutions of $ay'' + by' + cy = 0$, and our function y must also be a solution. Since it involves two arbitrary constants, it is a general solution.

33. $\begin{cases} y'' + 2y' + 5y = 0\\ y(3) = 2\\ y'(3) = 0\end{cases}$

The DE has auxiliary equation $r^2 + 2r + 5 = 0$ with roots $r = -1 \pm 2i$. By the second previous problem, a general solution can be expressed in the form $y = e^{-t}[A\cos 2(t-3) + B\sin 2(t-3)]$ for which

$$\begin{aligned} y' &= -e^{-t}[A\cos 2(t-3) + B\sin 2(t-3)] \\ &\quad + e^{-t}[-2A\sin 2(t-3) + 2B\cos 2(t-3)]. \end{aligned}$$

The initial conditions give

$$\begin{aligned} 2 &= y(3) = e^{-3}A \\ 0 &= y'(3) = -e^{-3}(A + 2B) \end{aligned}$$

Thus $A = 2e^3$ and $B = -A/2 = -e^3$. The IVP has solution

$$y = e^{3-t}[2\cos 2(t-3) - \sin 2(t-3)].$$

34. $\begin{cases} y'' + 4y' + 3y = 0 \\ y(3) = 1 \\ y'(3) = 0 \end{cases}$

The DE has auxiliary equation $r^2 + 4r + 3 = 0$ with roots $r = -2 + 1 = -1$ and $r = -2 - 1 = -3$ (i.e. $k \pm \omega$, where $k = -2$ and $\omega = 1$). By the second previous problem, a general solution can be expressed in the form $y = e^{-2t}[A\cosh(t-3) + B\sinh(t-3)]$ for which

$$\begin{aligned} y' &= -2e^{-2t}[A\cosh(t-3) + B\sinh(t-3)] \\ &\quad + e^{-2t}[A\sinh(t-3) + B\cosh(t-3)]. \end{aligned}$$

The initial conditions give

$$\begin{aligned} 1 &= y(3) = e^{-6}A \\ 0 &= y'(3) = -e^{-6}(-2A + B) \end{aligned}$$

Thus $A = e^6$ and $B = 2A = 2e^6$. The IVP has solution

$$y = e^{6-2t}[\cosh(t-3) + 2\sinh(t-3)].$$

35. Let $u(x) = c - k^2 y(x)$. Then $u(0) = c - k^2 a$. Also $u'(x) = -k^2 y'(x)$, so $u'(0) = -k^2 b$. We have

$$u''(x) = -k^2 y''(x) = -k^2\Big(c - k^2 y(x)\Big) = -k^2 u(x)$$

This IVP for the equation of simple harmonic motion has solution

$$u(x) = (c - k^2 a)\cos(kx) - kb\sin(kx)$$

so that

$$\begin{aligned} y(x) &= \frac{1}{k^2}\Big(c - u(x)\Big) \\ &= \frac{c}{k^2}\Big(c - (c - k^2 a)\cos(kx) + kb\sin(kx)\Big) \\ &= \frac{c}{k^2}(1 - \cos(kx) + a\cos(kx) + \frac{b}{k}\sin(kx). \end{aligned}$$

36. Since $x'(0) = 0$ and $x(0) = 1 > 1/5$, the motion will be governed by $x'' = -x + (1/5)$ until such time $t > 0$ when $x'(t) = 0$ again.

Let $u = x - (1/5)$. Then $u'' = x'' = -(x - 1/5) = -u$, $u(0) = 4/5$, and $u'(0) = x'(0) = 0$. This simple harmonic motion initial-value problem has solution $u(t) = (4/5)\cos t$. Thus $x(t) = (4/5)\cos t + (1/4)$ and $x'(t) = u'(t) = -(4/5)\sin t$. These formulas remain valid until $t = \pi$ when $x'(t)$ becomes 0 again. Note that $x(\pi) = -(4/5) + (1/5) = -(3/5)$.

Since $x(\pi) < -(1/5)$, the motion for $t > \pi$ will be governed by $x'' = -x - (1/5)$ until such time $t > \pi$ when $x'(t) = 0$ again.

Let $v = x + (1/5)$. Then $v'' = x'' = -(x + 1/5) = -v$, $v(\pi) = -(3/5) + (1/5) = -(2/5)$, and $v'(\pi) = x'(\pi) = 0$. Thius initial-value problem has solution $v(t) = -(2/5)\cos(t - \pi) = (2/5)\cos t$, so that $x(t) = (2/5)\cos t - (1/5)$ and $x'(t) = -(2/5)\sin t$. These formulas remain valid for $t \ge \pi$ until $t = 2\pi$ when x' becomes 0 again. We have $x(2\pi) = (2/5) - (1/5) = 1/5$ and $x'(2\pi) = 0$.

The conditions for stopping the motion are met at $t = 2\pi$; the mass remains at rest thereafter. Thus

$$x(t) = \begin{cases} \frac{4}{5}\cos t + \frac{1}{5} & \text{if } 0 \le t \le \pi \\ \frac{2}{5}\cos t - \frac{1}{5} & \text{if } \pi < t \le 2\pi \\ \frac{1}{5} & \text{if } t > 2\pi \end{cases}$$

Review Exercises 3 (page 208)

1. $f(x) = 3x + x^3 \Rightarrow f'(x) = 3(1 + x^2) > 0$ for all x, so f is increasing and therefore one-to-one and invertible. Since $f(0) = 0$, therefore $f^{-1}(0) = 0$, and

$$\frac{d}{dx}(f^{-1})(x)\bigg|_{x=0} = \frac{1}{f'(f^{-1}(0))} = \frac{1}{f'(0)} = \frac{1}{3}.$$

2. $f(x) = \sec^2 x \tan x \Rightarrow f'(x) = 2\sec^2 x \tan^2 x + \sec^4 x > 0$ for x in $(-\pi/2, \pi/2)$, so f is increasing and therefore one-to-one and invertible there. The domain of f^{-1} is $(-\infty, \infty)$, the range of f. Since $f(\pi/4) = 2$, therefore $f^{-1}(2) = \pi/4$, and

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(\pi/4)} = \frac{1}{8}.$$

3. $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x}{e^{x^2}} = 0.$

4. Observe $f'(x) = e^{-x^2}(1-2x^2)$ is positive if $x^2 < 1/2$ and is negative if $x^2 > 1/2$. Thus f is increasing on $(-1/\sqrt{2}, 1/\sqrt{2})$ and is decreasing on $(-\infty, -1/\sqrt{2})$ and on $(1/\sqrt{2}, \infty)$.

5. The max and min values of f are $1/\sqrt{2e}$ (at $x = 1/\sqrt{2}$) and $-1/\sqrt{2e}$ (at $x = -1/\sqrt{2}$).

6. $y = e^{-x}\sin x$, $(0 \le x \le 2\pi)$ has a horizontal tangent where

$$0 = \frac{dy}{dx} = e^{-x}(\cos x - \sin x).$$

This occurs if $\tan x = 1$, so $x = \pi/4$ or $x = 5\pi/4$. The points are $(\pi/4, e^{-\pi/4}/\sqrt{2})$ and $(5\pi/4, -e^{-5\pi/4}/\sqrt{2})$.

7. If $f'(x) = x$ for all x, then

$$\frac{d}{dx}\frac{f(x)}{e^{x^2/2}} = \frac{f'(x) - xf(x)}{e^{x^2/2}} = 0.$$

Thus $f(x)/e^{x^2/2} = C$ (constant) for all x. Since $f(2) = 3$, we have $C = 3/e^2$ and $f(x) = (3/e^2)e^{x^2/2} = 3e^{(x^2/2)-2}$.

8. Let the length, radius, and volume of the clay cylinder at time t be ℓ, r, and V, respectively. Then $V = \pi r^2 \ell$, and

$$\frac{dV}{dt} = 2\pi r\ell\frac{dr}{dt} + \pi r^2\frac{d\ell}{dt}.$$

Since $dV/dt = 0$ and $d\ell/dt = k\ell$ for some constant $k > 0$, we have

$$2\pi r\ell\frac{dr}{dt} = -k\pi r^2\ell, \quad \Rightarrow \quad \frac{dr}{dt} = -\frac{kr}{2}.$$

That is, r is decreasing at a rate proportional to itself.

9. a) An investment of $\$P$ at $r\%$ compounded continuously grows to $\$Pe^{rT/100}$ in T years. This will be $\$2P$ provided $e^{rT/100} = 2$, that is, $rT = 100\ln 2$. If $T = 5$, then $r = 20\ln 2 \approx 13.86\%$.

b) Since the doubling time is $T = 100\ln 2/r$, we have

$$\Delta T \approx \frac{dT}{dr}\Delta r = -\frac{100\ln 2}{r^2}\Delta r.$$

If $r = 13.863\%$ and $\Delta r = -0.5\%$, then

$$\Delta T \approx -\frac{100\ln 2}{13.863^2}(-0.5) \approx 0.1803 \text{ years}.$$

The doubling time will increase by about 66 days.

10. a) $\lim_{h\to 0}\frac{a^h - 1}{h} = \lim_{h\to 0}\frac{a^{0+h} - a^0}{h} = \frac{d}{dx}a^x\Big|_{x=0} = \ln a.$

Putting $h = 1/n$, we get $\lim_{n\to\infty} n\left(a^{1/n} - 1\right) = \ln a$.

b) Using the technique described in the exercise, we calculate

$$2^{10}\left(2^{1/2^{10}} - 1\right) \approx 0.69338183$$
$$2^{11}\left(2^{1/2^{11}} - 1\right) \approx 0.69326449$$

Thus $\ln 2 \approx 0.693$.

11. $\frac{d}{dx}\left(f(x)\right)^2 = \left(f'(x)\right)^2$

$\Rightarrow 2f(x)f'(x) = \left(f'(x)\right)^2$

$\Rightarrow f'(x) = 0$ or $f'(x) = 2f(x)$.

Since $f(x)$ is given to be nonconstant, we have $f'(x) = 2f(x)$. Thus $f(x) = f(0)e^{2x} = e^{2x}$.

12. If $f(x) = (\ln x)/x$, then $f'(x) = (1 - \ln x)/x^2$. Thus $f'(x) > 0$ if $\ln x < 1$ (i.e., $x < e$) and $f'(x) < 0$ if $\ln x > 1$ (i.e., $x > e$). Since f is increasing to the left of e and decreasing to the right, it has a maximum value $f(e) = 1/e$ at $x = e$. Thus, if $x > 0$ and $x \ne e$, then

$$\frac{\ln x}{x} < \frac{1}{e}.$$

Putting $x = \pi$ we obtain $(\ln\pi)/\pi < 1/e$. Thus

$$\ln(\pi^e) = e\ln\pi < \pi = \pi\ln e = \ln e^\pi,$$

and $\pi^e < e^\pi$ follows because ln is increasing.

13. $y = x^x = e^{x\ln x} \;\Rightarrow\; y' = x^x(1 + \ln x)$. The tangent to $y = x^x$ at $x = a$ has equation

$$y = a^a + a^a(1 + \ln a)(x - a).$$

This line passes through the origin if $0 = a^a[1 - a(1 + \ln a)]$, that is, if $(1 + \ln a)a = 1$. Observe that $a = 1$ solves this equation. Therefore the slope of the line is $1^1(1 + \ln 1) = 1$, and the line is $y = x$.

14. a) $\frac{\ln x}{x} = \frac{\ln 2}{2}$ is satisfied if $x = 2$ or $x = 4$ (because $\ln 4 = 2\ln 2$).

b) The line $y = mx$ through the origin intersects the curve $y = \ln x$ at $(b, \ln b)$ if $m = (\ln b)/b$. The same line intersects $y = \ln x$ at a different point $(x, \ln x)$ if $(\ln x)/x = m = (\ln b)/b$. This equation will have only one solution $x = b$ if the line $y = mx$ intersects the curve $y = \ln x$ only once, at $x = b$, that is, if the line is tangent to the curve at $x = b$. In this case m is the slope of $y = \ln x$ at $x = b$, so

$$\frac{1}{b} = m = \frac{\ln b}{b}.$$

Thus $\ln b = 1$, and $b = e$.

15. Let the rate be $r\%$. The interest paid by account A is $1,000(r/100) = 10r$.
The interest paid by account B is $1,000(e^{r/100} - 1)$. This is \$10 more than account A pays, so

$$1,000(e^{r/100} - 1) = 10r + 10.$$

A TI-85 solve routine gives $r \approx 13.8165\%$.

16. If $y = \cos^{-1} x$, then $x = \cos y$ and $0 \le y \le \pi$. Thus

$$\tan y = \operatorname{sgn} x\sqrt{\sec^2 y - 1} = \operatorname{sgn} x\sqrt{\frac{1}{x^2} - 1} = \frac{\sqrt{1-x^2}}{x}.$$

Thus $\cos^{-1} x = \tan^{-1}((\sqrt{1-x^2})/x)$.

Since $\cot x = 1/\tan x$, $\cot^{-1} x = \tan^{-1}(1/x)$.

$$\begin{aligned}\csc^{-1} x &= \sin^{-1}\frac{1}{x} = \frac{\pi}{2} - \cos^{-1}\frac{1}{x}\\ &= \frac{\pi}{2} - \tan^{-1}\frac{\sqrt{1-(1/x)^2}}{1/x}\\ &= \frac{\pi}{2} - \operatorname{sgn} x \tan^{-1}\sqrt{x^2-1}.\end{aligned}$$

17. $\cos^{-1} x = \dfrac{\pi}{2} - \sin^{-1} x$.

If $y = \cot^{-1} x$, then $x = \cot y$ and $0 < y < \pi/2$. Thus

$$\begin{aligned}\csc y &= \operatorname{sgn} x\sqrt{1 + \cot^2 y} = \operatorname{sgn} x\sqrt{1+x^2}\\ \sin y &= \frac{\operatorname{sgn} x}{\sqrt{1+x^2}}.\end{aligned}$$

Thus $\cot^{-1} x = \sin^{-1}\dfrac{\operatorname{sgn} x}{\sqrt{1+x^2}} = \operatorname{sgn} x \sin^{-1}\dfrac{1}{\sqrt{1+x^2}}$.

$\csc^{-1} x = \sin^{-1}\dfrac{1}{x}$.

18. Let $T(t)$ be the temperature of the milk t minutes after it is removed from the refrigerator. Let $U(t) = T(t) - 20$. By Newton's law,

$$U'(t) = kU(t) \quad\Rightarrow\quad U(t) = U(0)e^{kt}.$$

Now $T(0) = 5 \Rightarrow U(0) = -15$ and $T(12) = 12 \Rightarrow U(12) = -8$. Thus

$$\begin{aligned}-8 &= U(12) = U(0)e^{12k} = -15e^{12k}\\ e^{12k} &= 8/15, \qquad k = \tfrac{1}{12}\ln(8/15).\end{aligned}$$

If $T(s) = 18$, then $U(s) = -2$, so $-2 = -15e^{sk}$. Thus $sk = \ln(2/15)$, and

$$s = \frac{\ln(2/15)}{k} = 12\frac{\ln(2/15)}{\ln(8/15)} \approx 38.46.$$

It will take another $38.46 - 12 = 26.46$ min for the milk to warm up to $18°$.

19. Let R be the temperature of the room, Let $T(t)$ be the temperature of the water t minutes after it is brought into the room. Let $U(t) = T(t) - R$. Then

$$U'(t) = kU(t) \quad\Rightarrow\quad U(t) = U(0)e^{kt}.$$

We have

$$\begin{aligned}&T(0) = 96 \Rightarrow U(0) = 96 - R\\ &T(10) = 60 \Rightarrow U(10) = 60 - R \Rightarrow 60 - R = (96 - R)e^{10k}\\ &T(20) = 40 \Rightarrow U(20) = 40 - R \Rightarrow 40 - R = (96 - R)e^{20k}.\end{aligned}$$

Thus

$$\begin{aligned}&\left(\frac{60-R}{96-R}\right)^2 = e^{20k} = \frac{40-R}{96-R}\\ &(60-R)^2 = (96-R)(40-R)\\ &3600 - 120R + R^2 = 3840 - 136R + R^2\\ &16R = 240 \qquad R = 15.\end{aligned}$$

Room temperature is $15°$.

20. Let $f(x) = e^x - 1 - x$. Then $f(0) = 0$ and by the MVT,

$$\frac{f(x)}{x} = \frac{f(x) - f(0)}{x - 0} = f'(c) = e^c - 1$$

for some c between 0 and x. If $x > 0$, then $c > 0$, and $f'(c) > 0$. If $x < 0$, then $c < 0$, and $f'(c) < 0$. In either case $f(x) = xf'(c) > 0$, which is what we were asked to show.

21. Suppose that for some positive integer k, the inequality

$$e^x > 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$$

holds for all $x > 0$. This is certainly true for $k = 1$, as shown in the previous exercise. Apply the MVT to

$$g(t) = e^t - 1 - t - \frac{t^2}{2!} - \cdots - \frac{t^{k+1}}{(k+1)!}$$

on the interval $(0, x)$ (where $x > 0$) to obtain

$$\frac{g(x)}{x} = \frac{g(x) - g(0)}{x - 0} = g'(c)$$

for some c in $(0, x)$. Since x and $g'(c)$ are both positive, so is $g(x)$. This completes the induction and shows the desired inequality holds for $x > 0$ for all positive integers k.

Challenging Problems 3 (page 209)

1. a) $(d/dx)x^x = x^x(1 + \ln x) > 0$ if $\ln x > -1$, that is, if $x > e^{-1}$. Thus x^x is increasing on $[e^{-1}, \infty)$.

b) Being increasing on $[e^{-1}, \infty)$, $f(x) = x^x$ is invertible on that interval. Let $g = f^{-1}$. If $y = x^x$, then $x = g(y)$. Note that $y \to \infty$ if and only if $x \to \infty$. We have

$$\begin{aligned}
\ln y &= x \ln x \\
\ln(\ln y) &= \ln x + \ln(\ln x) \\
\lim_{y\to\infty} \frac{g(y)\ln(\ln y)}{\ln y} &= \lim_{x\to\infty} \frac{x(\ln x + \ln(\ln x))}{x \ln x} \\
&= \lim_{x\to\infty} \left(1 + \frac{\ln(\ln x)}{\ln x}\right).
\end{aligned}$$

Now $\ln x < \sqrt{x}$ for sufficiently large x, so $\ln(\ln x) < \sqrt{\ln x}$ for sufficiently large x. Therefore, $0 < \dfrac{\ln(\ln x)}{\ln x} < \dfrac{1}{\sqrt{\ln x}} \to 0$ as $x \to \infty$, and so

$$\lim_{y\to\infty} \frac{g(y)\ln(\ln y)}{\ln y} = 1 + 0 = 1.$$

2. $\dfrac{dv}{dt} = -g - kv$.

a) Let $u(t) = -g - kv(t)$. Then $\dfrac{du}{dt} = -k\dfrac{dv}{dt} = -ku$, and

$$\begin{aligned}
u(t) &= u(0)e^{-kt} = -(g + kv_0)e^{-kt} \\
v(t) &= -\frac{1}{k}\Big(g + u(t)\Big) = -\frac{1}{k}\Big(g - (g + kv_0)e^{-kt}\Big).
\end{aligned}$$

b) $\lim_{t\to\infty} v(t) = -g/k$

c) $$\begin{aligned}
\frac{dy}{dt} &= v(t) = -\frac{g}{k} + \frac{g + kv_0}{k} e^{-kt}, \qquad y(0) = y_0 \\
y(t) &= -\frac{gt}{k} - \frac{g + kv_0}{k^2} e^{-kt} + C \\
y_0 &= -0 - \frac{g + kv_0}{k^2} + C \ \Rightarrow\ C = y_0 + \frac{g + kv_0}{k^2} \\
y(t) &= y_0 - \frac{gt}{k} + \frac{g + kv_0}{k^2}\left(1 - e^{-kt}\right)
\end{aligned}$$

3. $\dfrac{dv}{dt} = -g + kv^2$ $(k > 0)$

a) Let $u = 2t\sqrt{gk}$. If $v(t) = \sqrt{\dfrac{g}{k}}\,\dfrac{1 - e^u}{1 + e^u}$, then

$$\begin{aligned}
\frac{dv}{dt} &= \sqrt{\frac{g}{k}}\,\frac{(1 + e^u)(-e^u) - (1 - e^u)e^u}{(1 + e^u)^2}\, 2\sqrt{gk} \\
&= \frac{-4ge^u}{(1 + e^u)^2} \\
kv^2 - g &= g\left(\frac{(1 - e^u)^2}{(1 + e^u)^2} - 1\right) \\
&= \frac{-4ge^u}{(1 + e^u)^2} = \frac{dv}{dt}.
\end{aligned}$$

Thus $v(t) = \sqrt{\dfrac{g}{k}}\,\dfrac{1 - e^{2t\sqrt{gk}}}{1 + e^{2t\sqrt{gk}}}$.

b) $$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \sqrt{\frac{g}{k}}\,\frac{e^{-2t\sqrt{gk}} - 1}{e^{-2t\sqrt{gk}} + 1} = -\sqrt{\frac{g}{k}}$$

c) If $y(t) = y_0 + \sqrt{\dfrac{g}{k}}\,t - \dfrac{1}{k}\ln\dfrac{1 + e^{2t\sqrt{gk}}}{2}$, then $y(0) = y_0$ and

$$\begin{aligned}
\frac{dy}{dt} &= \sqrt{\frac{g}{k}} - \frac{1}{k}\,\frac{2\sqrt{gk}e^{2t\sqrt{gk}}}{1 + e^{2t\sqrt{gk}}} \\
&= \sqrt{\frac{g}{k}}\,\frac{1 - e^{2t\sqrt{gk}}}{1 + e^{2t\sqrt{gk}}} = v(t).
\end{aligned}$$

Thus $y(t)$ gives the height of the object at time t during its fall.

4. If $p = e^{-bt}y$, then $\dfrac{dp}{dt} = e^{-bt}\left(\dfrac{dy}{dt} - by\right)$.

The DE $\dfrac{dp}{dt} = kp\left(1 - \dfrac{p}{e^{-bt}M}\right)$ therefore transforms to

$$\begin{aligned}
\frac{dy}{dt} &= by + kpe^{bt}\left(1 - \frac{p}{e^{-bt}M}\right) \\
&= (b + k)y - \frac{ky^2}{M} = Ky\left(1 - \frac{y}{L}\right),
\end{aligned}$$

where $K = b + k$ and $L = \dfrac{b + k}{k}M$. This is a standard Logistic equation with solution (as obtained in Section 3.4) given by

$$y = \frac{Ly_0}{y_0 + (L - y_0)e^{-Kt}},$$

where $y_0 = y(0) = p(0) = p_0$. Converting this solution back in terms of the function $p(t)$, we obtain

$$\begin{aligned}
p(t) &= \frac{Lp_0e^{-bt}}{p_0 + (L - p_0)e^{-(b+k)t}} \\
&= \frac{(b + k)Mp_0}{p_0ke^{bt} + \Big((b + k)M - kp_0\Big)e^{-kt}}.
\end{aligned}$$

Since p represents a percentage, we must have $(b + k)M/k < 100$.

If $k = 10$, $b = 1$, $M = 90$, and $p_0 = 1$, then $\dfrac{b + k}{k}M = 99 < 100$. The numerator of the final expression for $p(t)$ given above is a constant. Therefore $p(t)$ will be largest when the derivative of the denominator,

$$f(t) = p_0ke^{bt} + \Big((b + k)M - kp_0\Big)e^{-kt} = 10e^t + 980e^{-10t}$$

is zero. Since $f'(t) = 10e^t - 9{,}800e^{-10t}$, this will happen at $t = \ln(980)/11$. The value of p at this t is approximately 48.1. Thus the maximum percentage of potential clients who will adopt the technology is about 48.1%.