CHAPTER 2. DIFFERENTIATION

Section 2.1 Tangent Lines and Their Slopes (page 98)

1. Slope of $y = 3x - 1$ at $(1, 2)$ is

$$m = \lim_{h\to 0} \frac{3(1+h) - 1 - (3\times 1 - 1)}{h} = \lim_{h\to 0} \frac{3h}{h} = 3.$$

The tangent line is $y - 2 = 3(x - 1)$, or $y = 3x - 1$. (The tangent to a straight line at any point on it is the same straight line.)

2. Since $y = x/2$ is a straight line, its tangent at any point $(a, a/2)$ on it is the same line $y = x/2$.

3. Slope of $y = 2x^2 - 5$ at $(2, 3)$ is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{2(2+h)^2 - 5 - (2(2^2) - 5)}{h} \\ &= \lim_{h\to 0} \frac{8 + 8h + 2h^2 - 8}{h} \\ &= \lim_{h\to 0} (8 + 2h) = 8 \end{aligned}$$

Tangent line is $y - 3 = 8(x - 2)$ or $y = 8x - 13$.

4. The slope of $y = 6 - x - x^2$ at $x = -2$ is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{6 - (-2+h) - (-2+h)^2 - 4}{h} \\ &= \lim_{h\to 0} \frac{3h - h^2}{h} = \lim_{h\to 0} (3 - h) = 3. \end{aligned}$$

The tangent line at $(-2, 4)$ is $y = 3x + 10$.

5. Slope of $y = x^3 + 8$ at $x = -2$ is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{(-2+h)^3 + 8 - (-8 + 8)}{h} \\ &= \lim_{h\to 0} \frac{-8 + 12h - 6h^2 + h^3 + 8 - 0}{h} \\ &= \lim_{h\to 0} \left(12 - 6h + h^2\right) = 12 \end{aligned}$$

Tangent line is $y - 0 = 12(x + 2)$ or $y = 12x + 24$.

6. The slope of $y = \dfrac{1}{x^2 + 1}$ at $(0, 1)$ is

$$m = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{h^2+1} - 1\right) = \lim_{h\to 0} \frac{-h}{h^2+1} = 0.$$

The tangent line at $(0, 1)$ is $y = 1$.

7. Slope of $y = \sqrt{x+1}$ at $x = 3$ is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \cdot \frac{\sqrt{4+h} + 2}{\sqrt{4+h} + 2} \\ &= \lim_{h\to 0} \frac{4 + h - 4}{h\left(\sqrt{h+h} + 2\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} = \frac{1}{4}. \end{aligned}$$

Tangent line is $y - 2 = \dfrac{1}{4}(x - 3)$, or $x - 4y = -5$.

8. The slope of $y = \dfrac{1}{\sqrt{x}}$ at $x = 9$ is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{\sqrt{9+h}} - \frac{1}{3}\right) \\ &= \lim_{h\to 0} \frac{3 - \sqrt{9+h}}{3h\sqrt{9+h}} \cdot \frac{3 + \sqrt{9+h}}{3 + \sqrt{9+h}} \\ &= \lim_{h\to 0} \frac{9 - 9 - h}{3h\sqrt{9+h}(3 + \sqrt{9+h})} \\ &= -\frac{1}{3(3)(6)} = -\frac{1}{54}. \end{aligned}$$

The tangent line at $(9, \frac{1}{3})$ is $y = \frac{1}{3} - \frac{1}{54}(x - 9)$, or $y = \frac{1}{2} - \frac{1}{54}x$.

9. Slope of $y = \dfrac{2x}{x+2}$ at $x = 2$ is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{\dfrac{2(2+h)}{2+h+2} - 1}{h} \\ &= \lim_{h\to 0} \frac{4 + 2h - 2 - h - 2}{h(2 + h + 2)} \\ &= \lim_{h\to 0} \frac{h}{h(4+h)} = \frac{1}{4}. \end{aligned}$$

Tangent line is $y - 1 = \dfrac{1}{4}(x - 2)$,
or $x - 4y = -2$.

10. The slope of $y = \sqrt{5 - x^2}$ at $x = 1$ is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{\sqrt{5 - (1+h)^2} - 2}{h} \\ &= \lim_{h\to 0} \frac{5 - (1+h)^2 - 4}{h\left(\sqrt{5 - (1+h)^2} + 2\right)} \\ &= \lim_{h\to 0} \frac{-2 - h}{\sqrt{5 - (1+h)^2} + 2} = -\frac{1}{2} \end{aligned}$$

The tangent line at $(1, 2)$ is $y = 2 - \frac{1}{2}(x - 1)$, or $y = \frac{5}{2} - \frac{1}{2}x$.

11. Slope of $y = x^2$ at $x = x_0$ is

$$m = \lim_{h\to 0} \frac{(x_0+h)^2 - x_0^2}{h} = \lim_{h\to 0} \frac{2x_0h + h^2}{h} = 2x_0.$$

Tangent line is $y - x_0^2 = 2x_0(x - x_0)$,
or $y = 2x_0x - x_0^2$.

12. The slope of $y = \dfrac{1}{x}$ at $(a, \frac{1}{a})$ is

$$m = \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{a+h} + \frac{1}{a}\right) = \lim_{h\to 0} \frac{a-a-h}{h(a+h)(a)} = -\frac{1}{a^2}.$$

The tangent line at $(a, \dfrac{1}{a})$ is $y = \dfrac{1}{a} - \dfrac{1}{a^2}(x-a)$, or
$y = \dfrac{2}{a} - \dfrac{x}{a^2}$.

13. Since $\lim_{h\to 0} \dfrac{\sqrt{|0+h|} - 0}{h} = \lim_{h\to 0} \dfrac{1}{|h|\operatorname{sgn}(h)}$ does not exist (and is not ∞ or $-\infty$), the graph of $f(x) = \sqrt{|x|}$ has no tangent at $x = 0$.

14. The slope of $f(x) = (x-1)^{4/3}$ at $x = 1$ is

$$m = \lim_{h\to 0} \frac{(1+h-1)^{4/3} - 0}{h} = \lim_{h\to 0} h^{1/3} = 0.$$

The graph of f has a tangent line with slope 0 at $x = 1$. Since $f(1) = 0$, the tangent has equation $y = 0$

15. The slope of $f(x) = (x+2)^{3/5}$ at $x = -2$ is

$$m = \lim_{h\to 0} \frac{(-2+h+2)^{3/5} - 0}{h} = \lim_{h\to 0} h^{-2/5} = \infty.$$

The graph of f has vertical tangent $x = -2$ at $x = -2$.

16. The slope of $f(x) = |x^2 - 1|$ at $x = 1$ is
$m = \lim_{h\to 0} \dfrac{|(1+h)^2 - 1| - |1-1|}{h} = \lim_{h\to 0} \dfrac{|2h + h^2|}{h}$,
which does not exist, and is not $-\infty$ or ∞. The graph of f has no tangent at $x = 1$.

17. If $f(x) = \begin{cases} \sqrt{x} & \text{if } x \ge 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$, then

$$\lim_{h\to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0+} \frac{\sqrt{h}}{h} = \infty$$
$$\lim_{h\to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0-} \frac{-\sqrt{-h}}{h} = \infty$$

Thus the graph of f has a vertical tangent $x = 0$.

18. The slope of $y = x^2 - 1$ at $x = x_0$ is

$$m = \lim_{h\to 0} \frac{[(x_0+h)^2 - 1] - (x_0^2 - 1)}{h}$$
$$= \lim_{h\to 0} \frac{2x_0h + h^2}{h} = 2x_0.$$

If $m = -3$, then $x_0 = -\frac{3}{2}$. The tangent line with slope $m = -3$ at $(-\frac{3}{2}, \frac{5}{4})$ is $y = \frac{5}{4} - 3(x + \frac{3}{2})$, that is, $y = -3x - \frac{13}{4}$.

19. a) Slope of $y = x^3$ at $x = a$ is

$$m = \lim_{h\to 0} \frac{(a+h)^3 - a^3}{h}$$
$$= \lim_{h\to 0} \frac{a^3 + 3a^2h + 3ah^2 + h^3 - a^3}{h}$$
$$= \lim_{h\to 0} (3a^2 + 3ah + h^2) = 3a^2$$

b) We have $m = 3$ if $3a^2 = 3$, i.e., if $a = \pm 1$.
Lines of slope 3 tangent to $y = x^3$ are
$y = 1 + 3(x-1)$ and $y = -1 + 3(x+1)$, or
$y = 3x - 2$ and $y = 3x + 2$.

20. The slope of $y = x^3 - 3x$ at $x = a$ is

$$m = \lim_{h\to 0} \frac{1}{h}\Big[(a+h)^3 - 3(a+h) - (a^3 - 3a)\Big]$$
$$= \lim_{h\to 0} \frac{1}{h}\Big[a^3 + 3a^2h + 3ah^2 + h^3 - 3a - 3h - a^3 + 3a\Big]$$
$$= \lim_{h\to 0} [3a^2 + 3ah + h^2 - 3] = 3a^2 - 3.$$

At points where the tangent line is parallel to the x-axis, the slope is zero, so such points must satisfy $3a^2 - 3 = 0$. Thus, $a = \pm 1$. Hence, the tangent line is parallel to the x-axis at the points $(1, -2)$ and $(-1, 2)$.

21. The slope of the curve $y = x^3 - x + 1$ at $x = a$ is

$$m = \lim_{h\to 0} \frac{(a+h)^3 - (a+h) + 1 - (a^3 - a + 1)}{h}$$
$$= \lim_{h\to 0} \frac{3a^2h + 3ah^2 + a^3 - h}{h}$$
$$= \lim_{h\to 0} (3a^2 + 3ah + h^2 - 1) = 3a^2 - 1.$$

The tangent at $x = a$ is parallel to the line $y = 2x + 5$ if $3a^2 - 1 = 2$, that is, if $a = \pm 1$. The corresponding points on the curve are $(-1, 1)$ and $(1, 1)$.

22. The slope of the curve $y = 1/x$ at $x = a$ is

$$m = \lim_{h\to 0} \frac{\dfrac{1}{a+h} - \dfrac{1}{a}}{h} = \lim_{h\to 0} \frac{a - (a+h)}{ah(a+h)} = -\frac{1}{a^2}.$$

The tangent at $x = a$ is perpendicular to the line
$y = 4x - 3$ if $-1/a^2 = -1/4$, that is, if $a = \pm 2$. The corresponding points on the curve are $(-2, -1/2)$ and $(2, 1/2)$.

23. The slope of the curve $y = x^2$ at $x = a$ is

$$m = \lim_{h\to 0} \frac{(a+h)^2 - a^2}{h} = \lim_{h\to 0}(2a+h) = 2a.$$

The normal at $x = a$ has slope $-1/(2a)$, and has equation

$$y - a^2 = -\frac{1}{2a}(x-a), \quad \text{or} \quad \frac{x}{2a} + y = \frac{1}{2} + a^2.$$

This is the line $x + y = k$ if $2a = 1$, and so $k = (1/2) + (1/2)^2 = 3/4$.

24. The curves $y = kx^2$ and $y = k(x-2)^2$ intersect at $(1, k)$. The slope of $y = kx^2$ at $x = 1$ is

$$m_1 = \lim_{h\to 0} \frac{k(1+h)^2 - k}{h} = \lim_{h\to 0}(2+h)k = 2k.$$

The slope of $y = k(x-2)^2$ at $x = 1$ is

$$m_2 = \lim_{h\to 0} \frac{k(2-(1+h))^2 - k}{h} = \lim_{h\to 0}(-2+h)k = -2k.$$

The two curves intersect at right angles if $2k = -1/(-2k)$, that is, if $4k^2 = 1$, which is satisfied if $k = \pm 1/2$.

25. Horizontal tangents at $(0,0)$, $(3,108)$, and $(5,0)$.

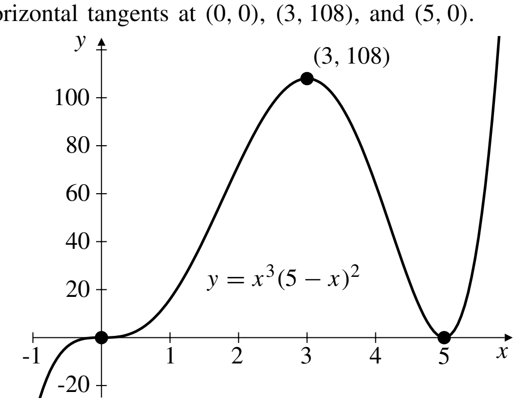


Fig. 2.1.25

26. Horizontal tangent at $(-1, 8)$ and $(2, -19)$.

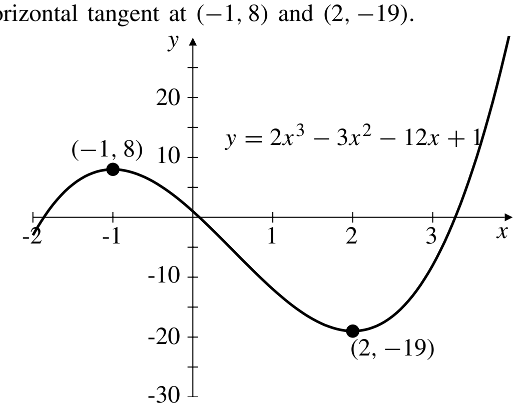


Fig. 2.1.26

27. Horizontal tangent at $(-1/2, 5/4)$. No tangents at $(-1, 1)$ and $(1, -1)$.

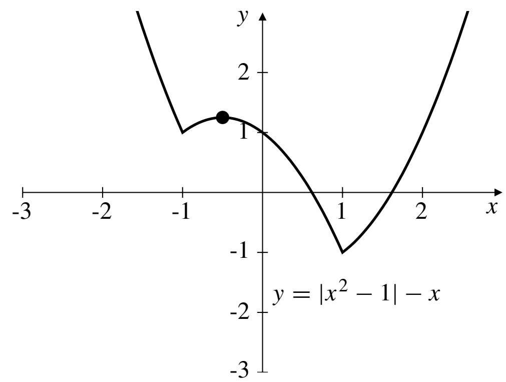


Fig. 2.1.27

28. Horizontal tangent at $(a, 2)$ and $(-a, -2)$ for all $a > 1$. No tangents at $(1, 2)$ and $(-1, -2)$.

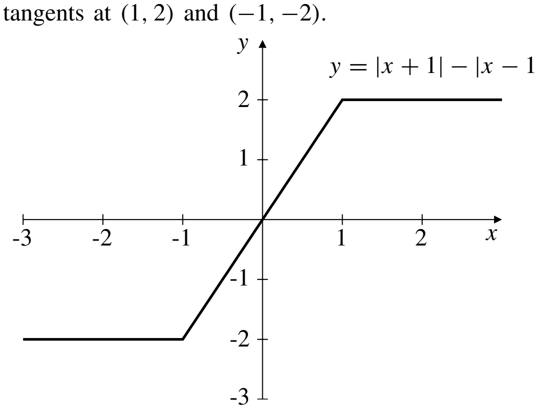


Fig. 2.1.28

29. Horizontal tangent at $(0, -1)$. The tangents at $(\pm 1, 0)$ are vertical.

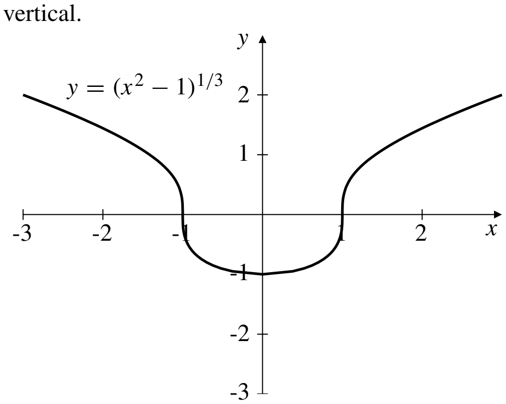


Fig. 2.1.29

30. Horizontal tangent at $(0, 1)$. No tangents at $(-1, 0)$ and $(1, 0)$.

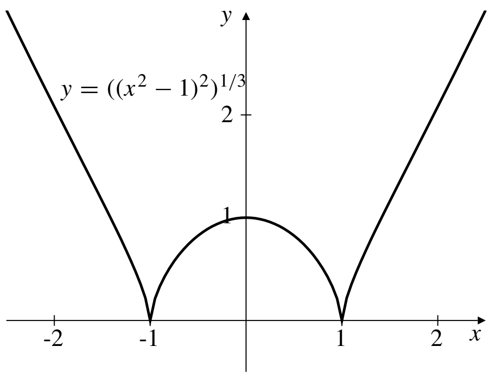


Fig. 2.1.30

31. The graph of the function $f(x) = x^{2/3}$ (see Figure 2.1.7 in the text) has a cusp at the origin O, so does not have a tangent line there. However, the angle between OP and the positive y-axis does $\to 0$ as P approaches 0 along the graph. Thus the answer is NO.

32. The slope of $P(x)$ at $x = a$ is

$$m = \lim_{h\to 0} \frac{P(a+h) - P(a)}{h}.$$

Since $P(a+h) = a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n$ and $P(a) = a_0$, the slope is

$$\begin{aligned} m &= \lim_{h\to 0} \frac{a_0 + a_1 h + a_2 h^2 + \cdots + a_n h^n - a_0}{h} \\ &= \lim_{h\to 0} a_1 + a_2 h + \cdots + a_n h^{n-1} = a_1. \end{aligned}$$

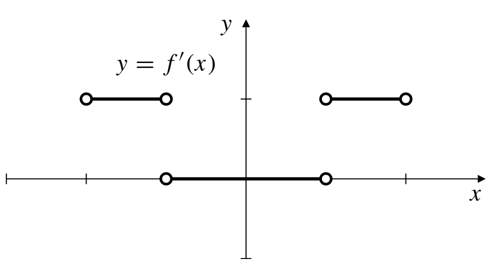
Thus the line $y = \ell(x) = m(x-a) + b$ is tangent to $y = P(x)$ at $x = a$ if and only if $m = a_1$ and $b = a_0$, that is, if and only if

$$\begin{aligned} P(x) - \ell(x) &= a_2(x-a)^2 + a_3(x-a)^3 + \cdots + a_n(x-a)^n \\ &= (x-a)^2\Big[a_2 + a_3(x-a) + \cdots + a_n(x-a)^{n-2}\Big] \\ &= (x-a)^2 Q(x) \end{aligned}$$

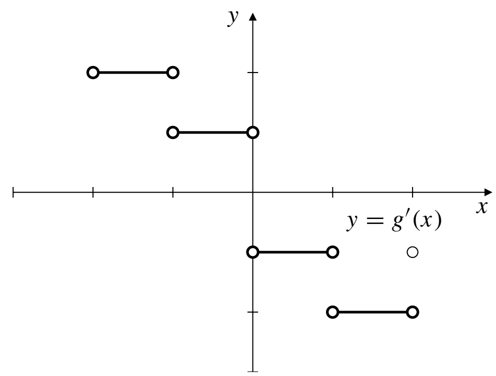
where Q is a polynomial.

Section 2.2 The Derivative (page 105)

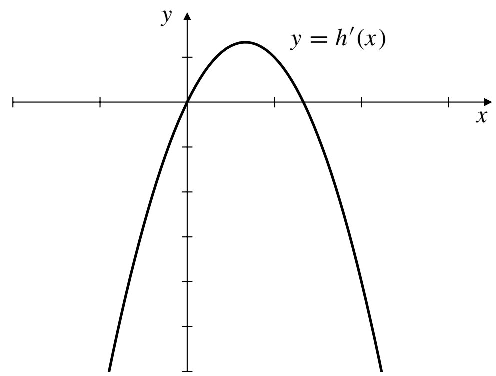
1.



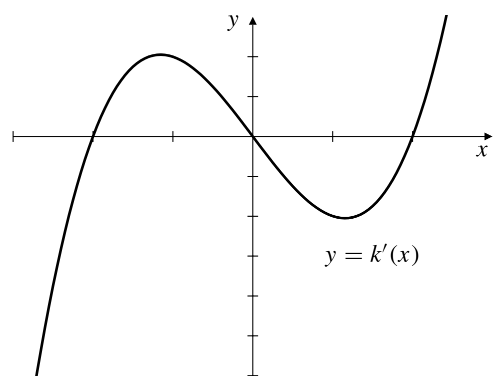
2.



3.



4.



5. Assuming the tick marks are spaced 1 unit apart, the function f is differentiable on the intervals $(-2,-1)$, $(-1,1)$, and $(1,2)$.

6. Assuming the tick marks are spaced 1 unit apart, the function g is differentiable on the intervals $(-2,-1)$, $(-1,0)$, $(0,1)$, and $(1,2)$.

7. $y = f(x)$ has its minimum at $x = 3/2$ where $f'(x) = 0$

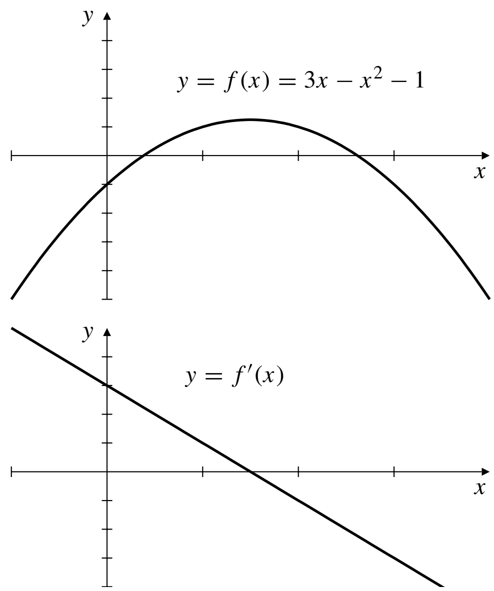


Fig. 2.2.7

8. $y = f(x)$ has horizontal tangents at the points near $1/2$ and $3/2$ where $f'(x) = 0$

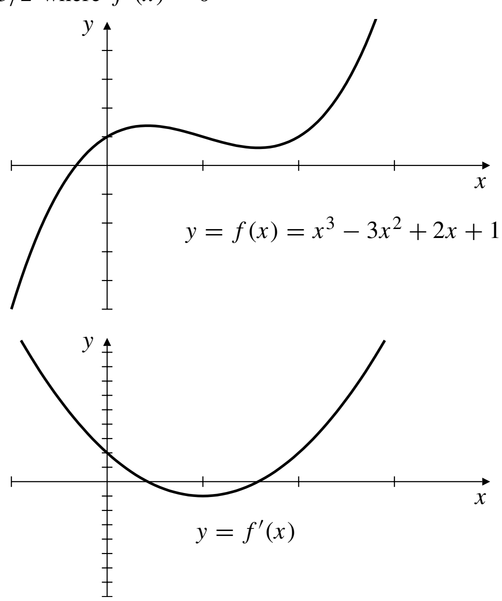


Fig. 2.2.8

9. $y = f(x)$ fails to be differentiable at $x = -1$, $x = 0$, and $x = 1$. It has horizontal tangents at two points, one between -1 and 0 and the other between 0 and 1.

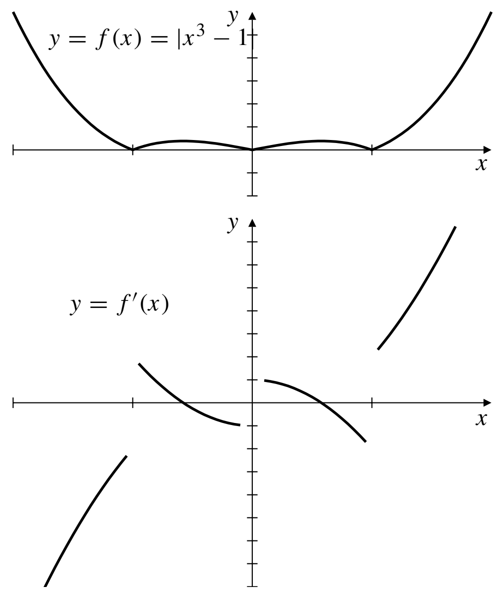


Fig. 2.2.9

10. $y = f(x)$ is constant on the intervals $(-\infty, -2)$, $(-1, 1)$, and $(2, \infty)$. It is not differentiable at $x = \pm 2$ and $x = \pm 1$.

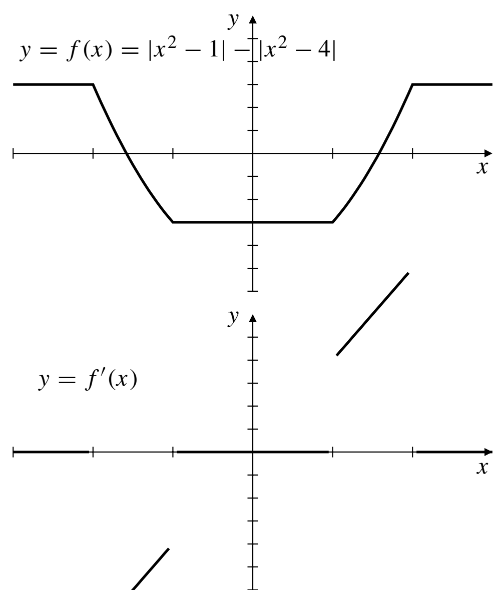


Fig. 2.2.10

11. $y = x^2 - 3x$

$$y' = \lim_{h\to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2 - 3h}{h} = 2x - 3$$

12. $f(x) = 1 + 4x - 5x^2$

$$f'(x) = \lim_{h\to 0} \frac{1 + 4(x+h) - 5(x+h)^2 - (1 + 4x - 5x^2)}{h}$$
$$= \lim_{h\to 0} \frac{4h - 10xh - 5h^2}{h} = 4 - 10x$$

13. $f(x) = x^3$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2$$

14. $s = \dfrac{1}{3+4t}$

$$\frac{ds}{dt} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{3+4(t+h)} - \frac{1}{3+4t}\right]$$
$$= \lim_{h\to 0} \frac{3+4t-3-4t-4h}{h(3+4t)[3+(4t+h)]} = -\frac{4}{(3+4t)^2}$$

15. $F(t) = \sqrt{2t+1}$

$$F'(t) = \lim_{h\to 0} \frac{\sqrt{2(t+h)+1} - \sqrt{2t+1}}{h}$$
$$= \lim_{h\to 0} \frac{2t+2h+1-2t-1}{h\left(\sqrt{2(t+h)+1} + \sqrt{2t+1}\right)}$$
$$= \lim_{h\to 0} \frac{2}{\sqrt{2(t+h)+1} + \sqrt{2t+1}}$$
$$= \frac{1}{\sqrt{2t+1}}$$

16. $f(x) = \frac{3}{4}\sqrt{2-x}$

$$f'(x) = \lim_{h\to 0} \frac{\frac{3}{4}\sqrt{2-(x+h)} - \frac{3}{4}\sqrt{2-x}}{h}$$
$$= \lim_{h\to 0} \frac{3}{4}\left[\frac{2-x-h-2+x}{h(\sqrt{2-(x+h)} + \sqrt{2-x})}\right]$$
$$= -\frac{3}{8\sqrt{2-x}}$$

17. $y = x + \dfrac{1}{x}$

$$y' = \lim_{h\to 0} \frac{x+h+\dfrac{1}{x+h} - x - \dfrac{1}{x}}{h}$$
$$= \lim_{h\to 0} \left(1 + \frac{x-x-h}{h(x+h)x}\right)$$
$$= 1 + \lim_{h\to 0} \frac{-1}{(x+h)x} = 1 - \frac{1}{x^2}$$

18. $z = \dfrac{s}{1+s}$

$$\frac{dz}{ds} = \lim_{h\to 0} \frac{1}{h}\left[\frac{s+h}{1+s+h} - \frac{s}{1+s}\right]$$
$$= \lim_{h\to 0} \frac{(s+h)(1+s) - s(1+s+h)}{h(1+s)(1+s+h)} = \frac{1}{(1+s)^2}$$

19. $F(x) = \dfrac{1}{\sqrt{1+x^2}}$

$$F'(x) = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{1+(x+h)^2}} - \dfrac{1}{\sqrt{1+x^2}}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{1+x^2} - \sqrt{1+(x+h)^2}}{h\sqrt{1+(x+h)^2}\sqrt{1+x^2}}$$
$$= \lim_{h\to 0} \frac{1+x^2-1-x^2-2hx-h^2}{h\sqrt{1+(x+h)^2}\sqrt{1+x^2}\left(\sqrt{1+x^2} + \sqrt{1+(x+h)^2}\right)}$$
$$= \frac{-2x}{2(1+x^2)^{3/2}} = -\frac{x}{(1+x^2)^{3/2}}$$

20. $y = \dfrac{1}{x^2}$

$$y' = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{(x+h)^2} - \frac{1}{x^2}\right]$$
$$= \lim_{h\to 0} \frac{x^2 - (x+h)^2}{hx^2(x+h)^2} = -\frac{2}{x^3}$$

21. $y = \dfrac{1}{\sqrt{1+x}}$

$$y'(x) = \lim_{h\to 0} \frac{\dfrac{1}{\sqrt{1+x+h}} - \dfrac{1}{\sqrt{1+x}}}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{1+x} - \sqrt{1+x+h}}{h\sqrt{1+x+h}\sqrt{1+x}}$$
$$= \lim_{h\to 0} \frac{1+x-1-x-h}{h\sqrt{1+x+h}\sqrt{1+x}\left(\sqrt{1+x} + \sqrt{1+x+h}\right)}$$
$$= \lim_{h\to 0} -\frac{1}{\sqrt{1+x+h}\sqrt{1+x}\left(\sqrt{1+x} + \sqrt{1+x+h}\right)}$$
$$= -\frac{1}{2(1+x)^{3/2}}$$

22. $f(t) = \dfrac{t^2-3}{t^2+3}$

$$f'(t) = \lim_{h\to 0} \frac{1}{h}\left(\frac{(t+h)^2-3}{(t+h)^2+3} - \frac{t^2-3}{t^2+3}\right)$$
$$= \lim_{h\to 0} \frac{[(t+h)^2-3](t^2+3) - (t^2-3)[(t+h)^2+3]}{h(t^2+3)[(t+h)^2+3]}$$
$$= \lim_{h\to 0} \frac{12th + 6h^2}{h(t^2+3)[(t+h)^2+3]} = \frac{12t}{(t^2+3)^2}$$

23. Since $f(x) = x \operatorname{sgn} x = |x|$, for $x \neq 0$, f will become continuous at $x = 0$ if we define $f(0) = 0$. However, f will still not be differentiable at $x = 0$ since $|x|$ is not differentiable at $x = 0$.

24. Since $g(x) = x^2 \operatorname{sgn} x = x|x| = \begin{cases} x^2 & \text{if } x > 0 \\ -x^2 & \text{if } x < 0 \end{cases}$, g will become continuous and differentiable at $x = 0$ if we define $g(0) = 0$.

25. $h(x) = |x^2 + 3x + 2|$ fails to be differentiable where $x^2 + 3x + 2 = 0$, that is, at $x = -2$ and $x = -1$. Note: both of these are single zeros of $x^2 + 3x + 2$. If they were higher order zeros (i.e. if $(x+2)^n$ or $(x+1)^n$ were a factor of $x^2 + 3x + 2$ for some integer $n \ge 2$) then h would be differentiable at the corresponding point.

26. $y = x^3 - 2x$

x	$\dfrac{f(x) - f(1)}{x - 1}$
0.9	0.71000
0.99	0.97010
0.999	0.99700
0.9999	0.99970

x	$\dfrac{f(x) - f(1)}{x - 1}$
1.1	1.31000
1.01	1.03010
1.001	1.00300
1.0001	1.00030

$$\begin{aligned}\frac{d}{dx}(x^3 - 2x)\bigg|_{x=1} &= \lim_{h\to 0} \frac{(1+h)^3 - 2(1+h) - (-1)}{h}\\ &= \lim_{h\to 0} \frac{h + 3h^2 + h^3}{h}\\ &= \lim_{h\to 0} 1 + 3h + h^2 = 1\end{aligned}$$

27. $f(x) = 1/x$

x	$\dfrac{f(x) - f(2)}{x - 2}$
1.9	−0.26316
1.99	−0.25126
1.999	−0.25013
1.9999	−0.25001

x	$\dfrac{f(x) - f(2)}{x - 2}$
2.1	−0.23810
2.01	−0.24876
2.001	−0.24988
2.0001	−0.24999

$$\begin{aligned}f'(2) &= \lim_{h\to 0} \frac{\dfrac{1}{2+h} - 2}{h} = \lim_{h\to 0} \frac{2 - (2+h)}{h(2+h)2}\\ &= \lim_{h\to 0} -\frac{1}{(2+h)2} = -\frac{1}{4}\end{aligned}$$

28. The slope of $y = 5 + 4x - x^2$ at $x = 2$ is

$$\begin{aligned}\frac{dy}{dx}\bigg|_{x=2} &= \lim_{h\to 0} \frac{5 + 4(2+h) - (2+h)^2 - 9}{h}\\ &= \lim_{h\to 0} \frac{-h^2}{h} = 0.\end{aligned}$$

Thus, the tangent line at $x = 2$ has the equation $y = 9$.

29. $y = \sqrt{x+6}$. Slope at $(3, 3)$ is

$$m = \lim_{h\to 0} \frac{\sqrt{9+h} - 3}{h} = \lim_{h\to 0} \frac{9 + h - 9}{h\left(\sqrt{9+h} + 3\right)} = \frac{1}{6}.$$

Tangent line is $y - 3 = \frac{1}{6}(x - 3)$, or $x - 6y = -15$.

30. The slope of $y = \dfrac{t}{t^2 - 2}$ at $t = -2$ and $y = -1$ is

$$\begin{aligned}\frac{dy}{dt}\bigg|_{t=-2} &= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2+h}{(-2+h)^2 - 2} - (-1)\right]\\ &= \lim_{h\to 0} \frac{-2 + h + [(-2+h)^2 - 2]}{h[(-2+h)^2 - 2]} = -\frac{3}{2}.\end{aligned}$$

Thus, the tangent line has the equation $y = -1 - \frac{3}{2}(t+2)$, that is, $y = -\frac{3}{2}t - 4$.

31. $y = \dfrac{2}{t^2 + t}$ Slope at $t = a$ is

$$\begin{aligned}m &= \lim_{h\to 0} \frac{\dfrac{2}{(a+h)^2 + (a+h)} - \dfrac{2}{a^2 + a}}{h}\\ &= \lim_{h\to 0} \frac{2(a^2 + a - a^2 - 2ah - h^2 - a - h)}{h[(a+h)^2 + a + h](a^2 + a)}\\ &= \lim_{h\to 0} \frac{-4a - 2h - 2}{[(a+h)^2 + a + h](a^2 + a)}\\ &= -\frac{4a + 2}{(a^2 + a)^2}\end{aligned}$$

Tangent line is $y = \dfrac{2}{a^2 + a} - \dfrac{2(2a+1)}{(a^2 + a)^2}(t - a)$

32. $f'(x) = -17x^{-18}$ for $x \ne 0$

33. $g'(t) = 22t^{21}$ for all t

34. $\dfrac{dy}{dx} = \dfrac{1}{3}x^{-2/3}$ for $x \ne 0$

35. $\dfrac{dy}{dx} = -\dfrac{1}{3}x^{-4/3}$ for $x \ne 0$

36. $\dfrac{d}{dt}t^{-2.25} = -2.25t^{-3.25}$ for $t > 0$

37. $\dfrac{d}{ds}s^{119/4} = \dfrac{119}{4}s^{115/4}$ for $s > 0$

38. $\dfrac{d}{ds}\sqrt{s}\bigg|_{s=9} = \dfrac{1}{2\sqrt{s}}\bigg|_{s=9} = \dfrac{1}{6}.$

39. $F(x) = \dfrac{1}{x}$, $F'(x) = -\dfrac{1}{x^2}$, $F'\left(\dfrac{1}{4}\right) = -16$

40. $f'(8) = -\dfrac{2}{3}x^{-5/3}\bigg|_{x=8} = -\dfrac{1}{48}$

41. $\dfrac{dy}{dt}\bigg|_{t=4} = \dfrac{1}{4}t^{-3/4}\bigg|_{t=4} = \dfrac{1}{8\sqrt{2}}$

42. The slope of $y = \sqrt{x}$ at $x = x_0$ is

$$\frac{dy}{dx}\bigg|_{x=x_0} = \frac{1}{2\sqrt{x_0}}.$$

Thus, the equation of the tangent line is $y = \sqrt{x_0} + \dfrac{1}{2\sqrt{x_0}}(x - x_0)$, that is, $y = \dfrac{x + x_0}{2\sqrt{x_0}}$.

43. Slope of $y = \dfrac{1}{x}$ at $x = a$ is $-\dfrac{1}{x^2}\bigg|_{x=a} = \dfrac{1}{a^2}$.
Normal has slope a^2, and equation $y - \dfrac{1}{a} = a^2(x - a)$,
or $y = a^2 x - a^3 + \dfrac{1}{a}$

44. The intersection points of $y = x^2$ and $x + 4y = 18$ satisfy

$$\begin{aligned} 4x^2 + x - 18 &= 0 \\ (4x + 9)(x - 2) &= 0. \end{aligned}$$

Therefore $x = -\frac{9}{4}$ or $x = 2$.
The slope of $y = x^2$ is $m_1 = \dfrac{dy}{dx} = 2x$.
At $x = -\dfrac{9}{4}$, $m_1 = -\dfrac{9}{2}$. At $x = 2$, $m_1 = 4$.
The slope of $x + 4y = 18$, i.e. $y = -\frac{1}{4}x + \frac{18}{4}$, is $m_2 = -\frac{1}{4}$.
Thus, at $x = 2$, the product of these slopes is $(4)(-\frac{1}{4}) = -1$. So, the curve and line intersect at right angles at that point.

45. Let the point of tangency be (a, a^2). Slope of tangent is
$\dfrac{d}{dx}x^2\bigg|_{x=a} = 2a$
This is the slope from (a, a^2) to $(1, -3)$, so
$\dfrac{a^2 + 3}{a - 1} = 2a$, and

$$\begin{aligned} a^2 + 3 &= 2a^2 - 2a \\ a^2 - 2a - 3 &= 0 \\ a &= 3 \text{ or } -1 \end{aligned}$$

The two tangent lines are
(for $a = 3$): $y - 9 = 6(x - 3)$ or $6x - 9$
(for $a = -1$): $y - 1 = -2(x + 1)$ or $y = -2x - 1$

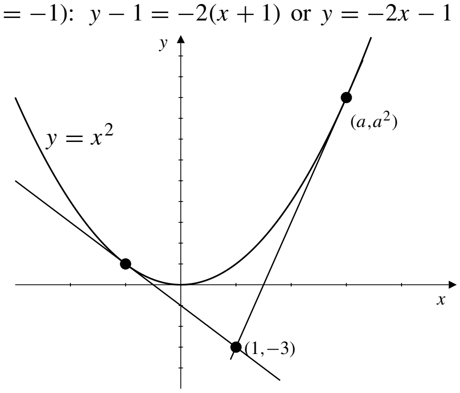


Fig. 2.2.45

46. The slope of $y = \dfrac{1}{x}$ at $x = a$ is

$$\frac{dy}{dx}\bigg|_{x=a} = -\frac{1}{a^2}.$$

If the slope is -2, then $-\dfrac{1}{a^2} = -2$, or $a = \pm\dfrac{1}{\sqrt{2}}$.
Therefore, the equations of the two straight lines are
$y = \sqrt{2} - 2\left(x - \dfrac{1}{\sqrt{2}}\right)$ and $y = -\sqrt{2} - 2\left(x + \dfrac{1}{\sqrt{2}}\right)$,
or $y = -2x \pm 2\sqrt{2}$.

47. Let the point of tangency be $(a, \sqrt{a})$
Slope of tangent is $\dfrac{d}{dx}\sqrt{x}\bigg|_{x=a} = \dfrac{1}{2\sqrt{a}}$
Thus $\dfrac{1}{2\sqrt{a}} = \dfrac{\sqrt{a} - 0}{a + 2}$, so $a + 2 = 2a$, and $a = 2$.
The required slope is $\dfrac{1}{2\sqrt{2}}$.

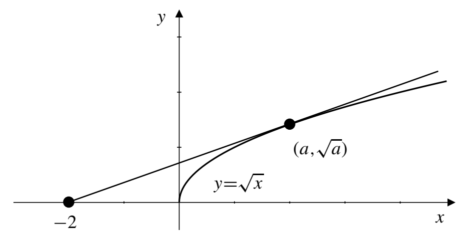


Fig. 2.2.47

48. If a line is tangent to $y = x^2$ at (t, t^2), then its slope is $\dfrac{dy}{dx}\bigg|_{x=t} = 2t$. If this line also passes through (a, b), then its slope satisfies

$$\frac{t^2 - b}{t - a} = 2t, \quad \text{that is } t^2 - 2at + b = 0.$$

Hence $t = \dfrac{2a \pm \sqrt{4a^2 - 4b}}{2} = a \pm \sqrt{a^2 - b}$.
If $b < a^2$, i.e. $a^2 - b > 0$, then $t = a \pm \sqrt{a^2 - b}$ has two real solutions. Therefore, there will be two distinct tangent lines passing through (a, b) with equations $y = b + 2\left(a \pm \sqrt{a^2 - b}\right)(x - a)$. If $b = a^2$, then $t = a$. There will be only one tangent line with slope $2a$ and equation $y = b + 2a(x - a)$.
If $b > a^2$, then $a^2 - b < 0$. There will be no real solution for t. Thus, there will be no tangent line.

49. Suppose f is odd: $f(-x) = -f(x)$. Then

$$f'(-x) = \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} = \lim_{h\to 0} -\frac{f(x-h) - f(x)}{h}$$

(let $h = -k$)

$$= \lim_{k\to 0} \frac{f(x+k) - f(x)}{k} = f'(x)$$

Thus f' is even.

Now suppose f is even: $f(-x) = f(x)$. Then

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} \\ &= \lim_{k\to 0} \frac{f(x+k) - f(x)}{-k} \\ &= -f'(x) \end{aligned}$$

so f' is odd.

50. Let $f(x) = x^{-n}$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^{-n} - x^{-n}}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\frac{1}{(x+h)^n} - \frac{1}{x^n}\right) \\ &= \lim_{h\to 0} \frac{x^n - (x+h)^n}{hx^n(x+h)^n} \\ &= \lim_{h\to 0} \frac{x-(x+h)}{hx^n((x+h)^n} \times \\ &\qquad \left(x^{n-1} + x^{n-2}(x+h) + \cdots + (x+h)^{n-1}\right) \\ &= -\frac{1}{x^{2n}} \times nx^{n-1} = -nx^{-(n+1)}. \end{aligned}$$

51. $f(x) = x^{1/3}$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^{1/3} - x^{1/3}}{h} \\ &\quad \times \frac{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}} \\ &= \lim_{h\to 0} \frac{x+h-x}{h[(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}]} \\ &= \lim_{h\to 0} \frac{1}{(x+h)^{2/3} + (x+h)^{1/3}x^{1/3} + x^{2/3}} \\ &= \frac{1}{3x^{2/3}} = \frac{1}{3}x^{-2/3} \end{aligned}$$

52. Let $f(x) = x^{1/n}$. Then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(x+h)^{1/n} - x^{1/n}}{h} \qquad (\text{let } x+h = a^n,\ x = b^n) \\ &= \lim_{a\to b} \frac{a-b}{a^n - b^n} \\ &= \lim_{a\to b} \frac{1}{a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}} \\ &= \frac{1}{nb^{n-1}} = \frac{1}{n}x^{(1/n)-1}. \end{aligned}$$

53.

$$\begin{aligned} \frac{d}{dx}x^n &= \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\Big[x^n + \frac{n}{1}x^{n-1}h + \frac{n(n-1)}{1\times 2}x^{n-2}h^2 \\ &\quad + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-3}h^3 + \cdots + h^n - x^n\Big] \\ &= \lim_{h\to 0}\Big(nx^{n-1} + h\Big[\frac{n(n-1)}{1\times 2}x^{n-2}h \\ &\quad + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-3}h^2 + \cdots + h^{n-1}\Big]\Big) \\ &= nx^{n-1} \end{aligned}$$

54. Let

$$\begin{aligned} f'(a+) &= \lim_{h\to 0+} \frac{f(a+h) - f(a)}{h} \\ f'(a-) &= \lim_{h\to 0-} \frac{f(a+h) - f(a)}{h} \end{aligned}$$

If $f'(a+)$ is finite, call the half-line with equation $y = f(a) + f'(a+)(x-a)$, $(x \ge a)$, the *right tangent line* to the graph of f at $x = a$. Similarly, if $f'(a-)$ is finite, call the half-line $y = f(a) + f'(a-)(x-a)$, $(x \le a)$, the *left tangent line*. If $f'(a+) = \infty$ (or $-\infty$), the right tangent line is the half-line $x = a$, $y \ge f(a)$ (or $x = a$, $y \le f(a)$). If $f'(a-) = \infty$ (or $-\infty$), the right tangent line is the half-line $x = a$, $y \le f(a)$ (or $x = a$, $y \ge f(a)$).

The graph has a tangent line at $x = a$ if and only if $f'(a+) = f'(a-)$. (This includes the possibility that both quantities may be $+\infty$ or both may be $-\infty$.) In this case the right and left tangents are two opposite halves of the same straight line. For $f(x) = x^{2/3}$, $f'(x) = \frac{2}{3}x^{-1/3}$. At $(0,0)$, we have $f'(0+) = +\infty$ and $f'(0-) = -\infty$. In this case both left and right tangents are the *positive* y-axis, and the curve does not have a tangent line at the origin.

For $f(x) = |x|$, we have

$$f'(x) = \operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0. \end{cases}$$

At $(0,0)$, $f'(0+) = 1$, and $f'(0-) = -1$. In this case the right tangent is $y = x$, $(x \ge 0)$, and the left tangent is $y = -x$, $(x \le 0)$. There is no tangent line.

Section 2.3 Differentiation Rules (page 113)

1. $y = 3x^2 - 5x - 7, \quad y' = 6x - 5.$

2. $y = 4x^{1/2} - \dfrac{5}{x}, \quad y' = 2x^{-1/2} + 5x^{-2}$

3. $f(x) = Ax^2 + Bx + C, \quad f'(x) = 2Ax + B.$

4. $f(x) = \dfrac{6}{x^3} + \dfrac{2}{x^2} - 2, \quad f'(x) = -\dfrac{18}{x^4} - \dfrac{4}{x^3}$

5. $z = \dfrac{s^5 - s^3}{15}, \quad \dfrac{dz}{dx} = \dfrac{1}{3}s^4 - \dfrac{1}{5}s^2.$

6. $y = x^{45} - x^{-45} \quad y' = 45x^{44} + 45x^{-46}$

7. $g(t) = t^{1/3} + 2t^{1/4} + 3t^{1/5}$

$$g'(t) = \frac{1}{3}t^{-2/3} + \frac{1}{2}t^{-3/4} + \frac{3}{5}t^{-4/5}$$

8. $y = 3\sqrt[3]{t^2} - \dfrac{2}{\sqrt{t^3}} = 3t^{2/3} - 2t^{-3/2}$

$$\frac{dy}{dt} = 2t^{-1/3} + 3t^{-5/2}$$

9. $u = \dfrac{3}{5}x^{5/3} - \dfrac{5}{3}x^{-3/5}$

$$\frac{du}{dx} = x^{2/3} + x^{-8/5}$$

10. $F(x) = (3x - 2)(1 - 5x)$

$$F'(x) = 3(1 - 5x) + (3x - 2)(-5) = 13 - 30x$$

11. $y = \sqrt{x}\left(5 - x - \dfrac{x^2}{3}\right) = 5\sqrt{x} - x^{3/2} - \dfrac{1}{3}x^{5/2}$

$$y' = \frac{5}{2\sqrt{x}} - \frac{3}{2}\sqrt{x} - \frac{5}{6}x^{3/2}$$

12. $g(t) = \dfrac{1}{2t - 3}, \quad g'(t) = -\dfrac{2}{(2t - 3)^2}$

13. $y = \dfrac{1}{x^2 + 5x}$

$$y' = -\frac{1}{(x^2 + 5x)^2}(2x + 5) = -\frac{2x + 5}{(x^2 + 5x)^2}$$

14. $y = \dfrac{4}{3 - x}, \quad y' = \dfrac{4}{(3 - x)^2}$

15. $f(t) = \dfrac{\pi}{2 - \pi t}$

$$f'(t) = -\frac{\pi}{(2 - \pi t)^2}(-\pi) = \frac{\pi^2}{(2 - \pi t)^2}$$

16. $g(y) = \dfrac{2}{1 - y^2}, \quad g'(y) = \dfrac{4y}{(1 - y^2)^2}$

17. $f(x) = \dfrac{1 - 4x^2}{x^3} = x^{-3} - \dfrac{4}{x}$

$$f'(x) = -3x^{-4} + 4x^{-2} = \frac{4x^2 - 3}{x^4}$$

18. $g(u) = \dfrac{u\sqrt{u} - 3}{u^2} = u^{-1/2} - 3u^{-2}$

$$g'(u) = -\frac{1}{2}u^{-3/2} + 6u^{-3} = \frac{12 - u\sqrt{u}}{2u^3}$$

19. $y = \dfrac{2 + t + t^2}{\sqrt{t}} = 2t^{-1/2} + \sqrt{t} + t^{3/2}$

$$\frac{dy}{dt} = -t^{-3/2} + \frac{1}{2\sqrt{t}} + \frac{3}{2}\sqrt{t} = \frac{3t^2 + t - 2}{2t\sqrt{t}}$$

20. $z = \dfrac{x - 1}{x^{2/3}} = x^{1/3} - x^{-2/3}$

$$\frac{dz}{dx} = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{x + 2}{3x^{5/3}}$$

21. $f(x) = \dfrac{3 - 4x}{3 + 4x}$

$$f'(x) = \frac{(3 + 4x)(-4) - (3 - 4x)(4)}{(3 + 4x)^2} = -\frac{24}{(3 + 4x)^2}$$

22. $z = \dfrac{t^2 + 2t}{t^2 - 1}$

$$z' = \frac{(t^2 - 1)(2t + 2) - (t^2 + 2t)(2t)}{(t^2 - 1)^2} = -\frac{2(t^2 + t + 1)}{(t^2 - 1)^2}$$

23. $s = \dfrac{1 + \sqrt{t}}{1 - \sqrt{t}}$

$$\frac{ds}{dt} = \frac{(1 - \sqrt{t})\frac{1}{2\sqrt{t}} - (1 + \sqrt{t})(-\frac{1}{2\sqrt{t}})}{(1 - \sqrt{t})^2} = \frac{1}{\sqrt{t}(1 - \sqrt{t})^2}$$

24. $f(x) = \dfrac{x^3 - 4}{x + 1}$

$$f'(x) = \frac{(x + 1)(3x^2) - (x^3 - 4)(1)}{(x + 1)^2} = \frac{2x^3 + 3x^2 + 4}{(x + 1)^2}$$

25. $f(x) = \dfrac{ax + b}{cx + d}$

$$f'(x) = \frac{(cx + d)a - (ax + b)c}{(cx + d)^2} = \frac{ad - bc}{(cx + d)^2}$$

26. $$F(t) = \frac{t^2+7t-8}{t^2-t+1}$$
$$F'(t) = \frac{(t^2-t+1)(2t+7)-(t^2+7t-8)(2t-1)}{(t^2-t+1)^2} = \frac{-8t^2+18t-1}{(t^2-t+1)^2}$$

27. $$f(x) = (1+x)(1+2x)(1+3x)(1+4x)$$
$$f'(x) = (1+2x)(1+3x)(1+4x)+2(1+x)(1+3x)(1+4x) + 3(1+x)(1+2x)(1+4x)+4(1+x)(1+2x)(1+3x)$$
OR
$$\begin{aligned} f(x) &= [(1+x)(1+4x)]\,[(1+2x)(1+3x)] \\ &= (1+5x+4x^2)(1+5x+6x^2) \\ &= 1+10x+25x^2+10x^2(1+5x)+24x^4 \\ &= 1+10x+35x^2+50x^3+24x^4 \end{aligned}$$
$$f'(x) = 10+70x+150x^2+96x^3$$

28. $$f(r) = (r^{-2}+r^{-3}-4)(r^2+r^3+1)$$
$$f'(r) = (-2r^{-3}-3r^{-4})(r^2+r^3+1) + (r^{-2}+r^{-3}-4)(2r+3r^2)$$
or
$$f(r) = -2+r^{-1}+r^{-2}+r^{-3}+r-4r^2-4r^3$$
$$f'(r) = -r^{-2}-2r^{-3}-3r^{-4}+1-8r-12r^2$$

29. $$y = (x^2+4)(\sqrt{x}+1)(5x^{2/3}-2)$$
$$y' = 2x(\sqrt{x}+1)(5x^{2/3}-2) + \frac{1}{2\sqrt{x}}(x^2+4)(5x^{2/3}-2) + \frac{10}{3}x^{-1/3}(x^2+4)(\sqrt{x}+1)$$

30. $$y = \frac{(x^2+1)(x^3+2)}{(x^2+2)(x^3+1)} = \frac{x^5+x^3+2x^2+2}{x^5+2x^3+x^2+2}$$
$$\begin{aligned} y' &= \frac{(x^5+2x^3+x^2+2)(5x^4+3x^2+4x)}{(x^5+2x^3+x^2+2)^2} - \frac{(x^5+x^3+2x^2+2)(5x^4+6x^2+2x)}{(x^5+2x^3+x^2+2)^2} \\ &= \frac{2x^7-3x^6-3x^4-6x^2+4x}{(x^5+2x^3+x^2+2)^2} \\ &= \frac{2x^7-3x^6-3x^4-6x^2+4x}{(x^2+2)^2(x^3+1)^2} \end{aligned}$$

31. $$y = \frac{x}{2x+\dfrac{1}{3x+1}} = \frac{3x^2+x}{6x^2+2x+1}$$
$$y' = \frac{(6x^2+2x+1)(6x+1)-(3x^2+x)(12x+2)}{(6x^2+2x+1)^2} = \frac{6x+1}{(6x^2+2x+1)^2}$$

32. $$f(x) = \frac{(\sqrt{x}-1)(2-x)(1-x^2)}{\sqrt{x}(3+2x)} = \left(1-\frac{1}{\sqrt{x}}\right)\cdot\frac{2-x-2x^2+x^3}{3+2x}$$
$$\begin{aligned} f'(x) &= \left(\frac{1}{2}x^{-3/2}\right)\frac{2-x-2x^2+x^3}{3+2x} + \left(1-\frac{1}{\sqrt{x}}\right) \\ &\quad\times \frac{(3+2x)(-1-4x+3x^2)-(2-x-2x^2+x^3)(2)}{(3+2x)^2} \\ &= \frac{(2-x)(1-x^2)}{2x^{3/2}(3+2x)} + \left(1-\frac{1}{\sqrt{x}}\right)\frac{4x^3+5x^2-12x-7}{(3+2x)^2} \end{aligned}$$

33. $$\frac{d}{dx}\left(\frac{x^2}{f(x)}\right)\bigg|_{x=2} = \frac{f(x)(2x)-x^2f'(x)}{[f(x)]^2}\bigg|_{x=2} = \frac{4f(2)-4f'(2)}{[f(2)]^2} = -\frac{4}{4} = -1$$

34. $$\frac{d}{dx}\left(\frac{f(x)}{x^2}\right)\bigg|_{x=2} = \frac{x^2f'(x)-2xf(x)}{x^4}\bigg|_{x=2} = \frac{4f'(2)-4f(2)}{16} = \frac{4}{16} = \frac{1}{4}$$

35. $$\frac{d}{dx}\left(x^2f(x)\right)\bigg|_{x=2} = \left(2xf(x)+x^2f'(x)\right)\bigg|_{x=2} = 4f(2)+4f'(2) = 20$$

36. $$\begin{aligned} \frac{d}{dx}\left(\frac{f(x)}{x^2+f(x)}\right)\bigg|_{x=2} &= \frac{(x^2+f(x))f'(x)-f(x)(2x+f'(x))}{(x^2+f(x))^2}\bigg|_{x=2} \\ &= \frac{(4+f(2))f'(2)-f(2)(4+f'(2))}{(4+f(2))^2} = \frac{18-14}{6^2} = \frac{1}{9} \end{aligned}$$

37. $$\begin{aligned} \frac{d}{dx}\left(\frac{x^2-4}{x^2+4}\right)\bigg|_{x=-2} &= \frac{d}{dx}\left(1-\frac{8}{x^2+4}\right)\bigg|_{x=-2} \\ &= \frac{8}{(x^2+4)^2}(2x)\bigg|_{x=-2} \\ &= -\frac{32}{64} = -\frac{1}{2} \end{aligned}$$

38. $$\begin{aligned} &\frac{d}{dt}\left[\frac{t(1+\sqrt{t})}{5-t}\right]\bigg|_{t=4} \\ &= \frac{d}{dt}\left[\frac{t+t^{3/2}}{5-t}\right]\bigg|_{t=4} \\ &= \frac{(5-t)(1+\frac{3}{2}t^{1/2})-(t+t^{3/2})(-1)}{(5-t)^2}\bigg|_{t=4} \\ &= \frac{(1)(4)-(12)(-1)}{(1)^2} = 16 \end{aligned}$$

39. $f(x) = \dfrac{\sqrt{x}}{x+1}$

$f'(x) = \dfrac{(x+1)\dfrac{1}{2\sqrt{x}} - \sqrt{x}(1)}{(x+1)^2}$

$f'(2) = \dfrac{\dfrac{3}{2\sqrt{2}} - \sqrt{2}}{9} = -\dfrac{1}{18\sqrt{2}}$

40. $\dfrac{d}{dt}[(1+t)(1+2t)(1+3t)(1+4t)]\Big|_{t=0}$

$= (1)(1+2t)(1+3t)(1+4t) + (1+t)(2)(1+3t)(1+4t)+$

$(1+t)(1+2t)(3)(1+4t) + (1+t)(1+2t)(1+3t)(4)\Big|_{t=0}$

$= 1+2+3+4 = 10$

41. $y = \dfrac{2}{3-4\sqrt{x}}$, $y' = -\dfrac{2}{\left(3-4\sqrt{x}\right)^2}\left(-\dfrac{4}{2\sqrt{x}}\right)$

Slope of tangent at $(1,-2)$ is $m = \dfrac{8}{(-1)^2 2} = 4$

Tangent line has the equation $y = -2 + 4(x-1)$ or $y = 4x - 6$

42. For $y = \dfrac{x+1}{x-1}$ we calculate

$$y' = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2} = -\frac{2}{(x-1)^2}.$$

At $x = 2$ we have $y = 3$ and $y' = -2$. Thus, the equation of the tangent line is $y = 3 - 2(x-2)$, or $y = -2x + 7$. The normal line is $y = 3 + \frac{1}{2}(x-2)$, or $y = \frac{1}{2}x + 2$.

43. $y = x + \dfrac{1}{x}$, $y' = 1 - \dfrac{1}{x^2}$

For horizontal tangent: $0 = y' = 1 - \dfrac{1}{x^2}$ so $x^2 = 1$ and $x = \pm 1$

The tangent is horizontal at $(1,2)$ and at $(-1,-2)$

44. If $y = x^2(4-x^2)$, then

$$y' = 2x(4-x^2) + x^2(-2x) = 8x - 4x^3 = 4x(2-x^2).$$

The slope of a horizontal line must be zero, so $4x(2-x^2) = 0$, which implies that $x = 0$ or $x = \pm\sqrt{2}$.
At $x = 0$, $y = 0$ and at $x = \pm\sqrt{2}$, $y = 4$.
Hence, there are two horizontal lines that are tangent to the curve. Their equations are $y = 0$ and $y = 4$.

45. $y = \dfrac{1}{x^2+x+1}$, $y' = -\dfrac{2x+1}{(x^2+x+1)^2}$

For horizontal tangent we want $0 = y' = -\dfrac{2x+1}{(x^2+x+1)^2}$. Thus $2x+1 = 0$ and $x = -\dfrac{1}{2}$

The tangent is horizontal only at $\left(-\dfrac{1}{2}, \dfrac{4}{3}\right)$.

46. If $y = \dfrac{x+1}{x+2}$, then

$$y' = \frac{(x+2)(1) - (x+1)(1)}{(x+2)^2} = \frac{1}{(x+2)^2}.$$

In order to be parallel to $y = 4x$, the tangent line must have slope equal to 4, i.e.,

$$\frac{1}{(x+2)^2} = 4, \qquad \text{or } (x+2)^2 = \tfrac{1}{4}.$$

Hence $x + 2 = \pm\frac{1}{2}$, and $x = -\frac{3}{2}$ or $-\frac{5}{2}$. At $x = -\frac{3}{2}$, $y = -1$, and at $x = -\frac{5}{2}$, $y = 3$.
Hence, the tangent is parallel to $y = 4x$ at the points $\left(-\frac{3}{2}, -1\right)$ and $\left(-\frac{5}{2}, 3\right)$.

47. Let the point of tangency be $(a, \frac{1}{a})$. The slope of the tangent is $-\dfrac{1}{a^2} = \dfrac{b - \frac{1}{a}}{0 - a}$. Thus $b - \frac{1}{a} = \frac{1}{a}$ and $a = \dfrac{2}{b}$.

Tangent has slope $-\dfrac{b^2}{4}$ so has equation $y = b - \dfrac{b^2}{4}x$.

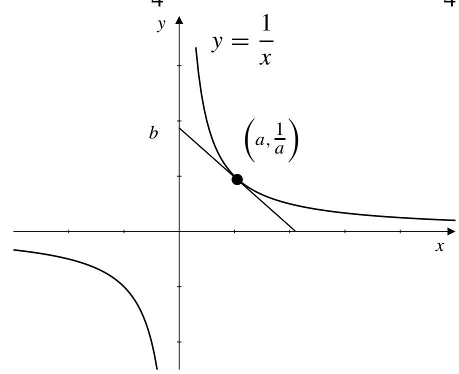


Fig. 2.3.47

48. Since $\dfrac{1}{\sqrt{x}} = y = x^2 \Rightarrow x^{5/2} = 1$, therefore $x = 1$ at the intersection point. The slope of $y = x^2$ at $x = 1$ is $2x\Big|_{x=1} = 2$. The slope of $y = \dfrac{1}{\sqrt{x}}$ at $x = 1$ is

$$\frac{dy}{dx}\Big|_{x=1} = -\frac{1}{2}x^{-3/2}\Big|_{x=1} = -\frac{1}{2}.$$

The product of the slopes is $(2)\left(-\frac{1}{2}\right) = -1$. Hence, the two curves intersect at right angles.

49. The tangent to $y = x^3$ at (a, a^3) has equation $y = a^3 + 3a^2(x - a)$, or $y = 3a^2x - 2a^3$. This line passes through $(2, 8)$ if $8 = 6a^2 - 2a^3$ or, equivalently, if $a^3 - 3a^2 + 4 = 0$. Since $(2, 8)$ lies on $y = x^3$, $a = 2$ must be a solution of this equation. In fact it must be a double root; $(a-2)^2$ must be a factor of $a^3 - 3a^2 + 4$. Dividing by this factor, we find that the other factor is $a + 1$, that is,

$$a^3 - 3a^2 + 4 = (a-2)^2(a+1).$$

The two tangent lines to $y = x^3$ passing through $(2, 8)$ correspond to $a = 2$ and $a = -1$, so their equations are $y = 12x - 16$ and $y = 3x + 2$.

50. The tangent to $y = x^2/(x-1)$ at $(a, a^2/(a-1))$ has slope

$$m = \left.\frac{(x-1)2x - x^2(1)}{(x-1)^2}\right|_{x=a} = \frac{a^2 - 2a}{(a-1)^2}.$$

The equation of the tangent is

$$y - \frac{a^2}{a-1} = \frac{a^2 - 2a}{(a-1)^2}(x - a).$$

This line passes through $(2, 0)$ provided

$$0 - \frac{a^2}{a-1} = \frac{a^2 - 2a}{(a-1)^2}(2 - a),$$

or, upon simplification, $3a^2 - 4a = 0$. Thus we can have either $a = 0$ or $a = 4/3$. There are two tangents through $(2, 0)$. Their equations are $y = 0$ and $y = -8x + 16$.

51.
$$\begin{aligned}\frac{d}{dx}\sqrt{f(x)} &= \lim_{h\to 0}\frac{\sqrt{f(x+h)} - \sqrt{f(x)}}{h}\\ &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\frac{1}{\sqrt{f(x+h)} + \sqrt{f(x)}}\\ &= \frac{f'(x)}{2\sqrt{f(x)}}\end{aligned}$$
$$\frac{d}{dx}\sqrt{x^2+1} = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$$

52. $f(x) = |x^3| = \begin{cases} x^3 & \text{if } x \ge 0 \\ -x^3 & \text{if } x < 0\end{cases}$. Therefore f is differentiable everywhere except *possibly* at $x = 0$, However,

$$\begin{aligned}\lim_{h\to 0+}\frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0+} h^2 = 0\\ \lim_{h\to 0-}\frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0-}(-h^2) = 0.\end{aligned}$$

Thus $f'(0)$ exists and equals 0. We have

$$f'(x) = \begin{cases} 3x^2 & \text{if } x \ge 0 \\ -3x^2 & \text{if } x < 0.\end{cases}$$

53. To be proved: $\frac{d}{dx}x^{n/2} = \frac{n}{2}x^{(n/2)-1}$ for $n = 1, 2, 3, \ldots$.
Proof: It is already known that the case $n = 1$ is true: the derivative of $x^{1/2}$ is $(1/2)x^{-1/2}$.
Assume that the formula is valid for $n = k$ for some positive integer k:

$$\frac{d}{dx}x^{k/2} = \frac{k}{2}x^{(k/2)-1}.$$

Then, by the Product Rule and this hypothesis,

$$\begin{aligned}\frac{d}{dx}x^{(k+1)/2} &= \frac{d}{dx}x^{1/2}x^{k/2}\\ &= \frac{1}{2}x^{-1/2}x^{k/2} + \frac{k}{2}x^{1/2}x^{(k/2)-1} = \frac{k+1}{2}x^{(k+1)/2-1}.\end{aligned}$$

Thus the formula is also true for $n = k + 1$. Therefore it is true for all positive integers n by induction.
For negative $n = -m$ (where $m > 0$) we have

$$\begin{aligned}\frac{d}{dx}x^{n/2} &= \frac{d}{dx}\frac{1}{x^{m/2}}\\ &= \frac{-1}{x^m}\frac{m}{2}x^{(m/2)-1}\\ &= -\frac{m}{2}x^{-(m/2)-1} = \frac{n}{2}x^{(n/2)-1}.\end{aligned}$$

54. To be proved:

$$\begin{aligned}&(f_1f_2\cdots f_n)'\\ &= f_1'f_2\cdots f_n + f_1f_2'\cdots f_n + \cdots + f_1f_2\cdots f_n'\end{aligned}$$

Proof: The case $n = 2$ is just the Product Rule. Assume the formula holds for $n = k$ for some integer $k > 2$. Using the Product Rule and this hypothesis we calculate

$$\begin{aligned}&(f_1f_2\cdots f_kf_{k+1})'\\ &= [(f_1f_2\cdots f_k)f_{k+1}]'\\ &= (f_1f_2\cdots f_k)'f_{k+1} + (f_1f_2\cdots f_k)f_{k+1}'\\ &= (f_1'f_2\cdots f_k + f_1f_2'\cdots f_k + \cdots + f_1f_2\cdots f_k')f_{k+1}\\ &\qquad + (f_1f_2\cdots f_k)f_{k+1}'\\ &= f_1'f_2\cdots f_kf_{k+1} + f_1f_2'\cdots f_kf_{k+1} + \cdots\\ &\qquad + f_1f_2\cdots f_k'f_{k+1} + f_1f_2\cdots f_kf_{k+1}'\end{aligned}$$

so the formula is also true for $n = k + 1$. The formula is therefore for all integers $n \ge 2$ by induction.

Section 2.4 The Chain Rule (page 118)

1. $y = (2x+3)^6, \quad y' = 6(2x+3)^5 2 = 12(2x+3)^5$

2.
$$y = \left(1 - \frac{x}{3}\right)^{99}$$
$$y' = 99\left(1 - \frac{x}{3}\right)^{98}\left(-\frac{1}{3}\right) = -33\left(1 - \frac{x}{3}\right)^{98}$$

3. $f(x) = (4 - x^2)^{10}$
$f'(x) = 10(4 - x^2)^9(-2x) = -20x(4 - x^2)^9$

4. $\dfrac{dy}{dx} = \dfrac{d}{dx}\sqrt{1 - 3x^2} = \dfrac{-6x}{2\sqrt{1 - 3x^2}} = -\dfrac{3x}{\sqrt{1 - 3x^2}}$

5. $F(t) = \left(2 + \dfrac{3}{t}\right)^{-10}$

$F'(t) = -10\left(2 + \dfrac{3}{t}\right)^{-11}\dfrac{-3}{t^2} = \dfrac{30}{t^2}\left(2 + \dfrac{3}{t}\right)^{-11}$

6. $z = (1 + x^{2/3})^{3/2}$
$z' = \frac{3}{2}(1 + x^{2/3})^{1/2}(\frac{2}{3}x^{-1/3}) = x^{-1/3}(1 + x^{2/3})^{1/2}$

7. $y = \dfrac{3}{5 - 4x}$
$y' = -\dfrac{3}{(5 - 4x)^2}(-4) = \dfrac{12}{(5 - 4x)^2}$

8. $y = (1 - 2t^2)^{-3/2}$
$y' = -\frac{3}{2}(1 - 2t^2)^{-5/2}(-4t) = 6t(1 - 2t^2)^{-5/2}$

9. $y = |1 - x^2|, \quad y' = -2x\,\mathrm{sgn}\,(1 - x^2) = \dfrac{2x^3 - 2x}{|1 - x^2|}$

10. $f(t) = |2 + t^3|$
$f'(t) = [\mathrm{sgn}\,(2 + t^3)](3t^2) = \dfrac{3t^2(2 + t^3)}{|2 + t^3|}$

11. $y = 4x + |4x - 1|$
$y' = 4 + 4(\mathrm{sgn}\,(4x - 1))$
$= \begin{cases} 8 & \text{if } x > \frac{1}{4} \\ 0 & \text{if } x < \frac{1}{4} \end{cases}$

12. $y = (2 + |x|^3)^{1/3}$
$y' = \frac{1}{3}(2 + |x|^3)^{-2/3}(3|x|^2)\mathrm{sgn}\,(x)$
$= |x|^2(2 + |x|^3)^{-2/3}\left(\dfrac{x}{|x|}\right) = x|x|(2 + |x|^3)^{-2/3}$

13. $y = \dfrac{1}{2 + \sqrt{3x + 4}}$
$y' = -\dfrac{1}{\left(2 + \sqrt{3x + 4}\right)^2}\left(\dfrac{3}{2\sqrt{3x + 4}}\right)$
$= -\dfrac{3}{2\sqrt{3x + 4}\left(2 + \sqrt{3x + 4}\right)^2}$

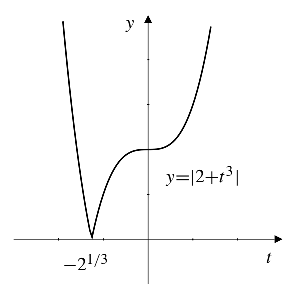
14. $f(x) = \left(1 + \sqrt{\dfrac{x - 2}{3}}\right)^4$

$f'(x) = 4\left(1 + \sqrt{\dfrac{x - 2}{3}}\right)^3\left(\dfrac{1}{2}\sqrt{\dfrac{3}{x - 2}}\right)\left(\dfrac{1}{3}\right)$
$= \dfrac{2}{3}\sqrt{\dfrac{3}{x - 2}}\left(1 + \sqrt{\dfrac{x - 2}{3}}\right)^3$

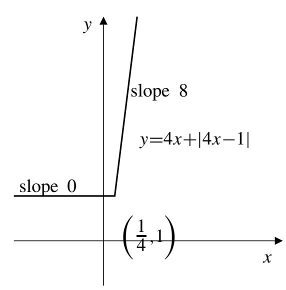
15. $z = \left(u + \dfrac{1}{u - 1}\right)^{-5/3}$
$\dfrac{dz}{du} = -\dfrac{5}{3}\left(u + \dfrac{1}{u - 1}\right)^{-8/3}\left(1 - \dfrac{1}{(u - 1)^2}\right)$
$= -\dfrac{5}{3}\left(1 - \dfrac{1}{(u - 1)^2}\right)\left(u + \dfrac{1}{u - 1}\right)^{-8/3}$

16. $y = \dfrac{x^5\sqrt{3 + x^6}}{(4 + x^2)^3}$
$y' = \dfrac{1}{(4 + x^2)^6}\left((4 + x^2)^3\left[5x^4\sqrt{3 + x^6} + x^5\left(\dfrac{3x^5}{\sqrt{3 + x^6}}\right)\right]\right.$
$\left. - x^5\sqrt{3 + x^6}\left[3(4 + x^2)^2(2x)\right]\right)$
$= \dfrac{(4 + x^2)\left[5x^4(3 + x^6) + 3x^{10}\right] - x^5(3 + x^6)(6x)}{(4 + x^2)^4\sqrt{3 + x^6}}$
$= \dfrac{60x^4 - 3x^6 + 32x^{10} + 2x^{12}}{(4 + x^2)^4\sqrt{3 + x^6}}$

17.



18.



19. $\dfrac{d}{dx}x^{1/4} = \dfrac{d}{dx}\sqrt{\sqrt{x}} = \dfrac{1}{2\sqrt{\sqrt{x}}} \times \dfrac{1}{2\sqrt{x}} = \dfrac{1}{4}x^{-3/4}$

20. $\dfrac{d}{dx}x^{3/4} = \dfrac{d}{dx}\sqrt{x\sqrt{x}} = \dfrac{1}{2\sqrt{x\sqrt{x}}}\left(\sqrt{x} + \dfrac{x}{2\sqrt{x}}\right) = \dfrac{3}{4}x^{-1/4}$

21. $\dfrac{d}{dx}x^{3/2} = \dfrac{d}{dx}\sqrt{x^3} = \dfrac{1}{2\sqrt{x^3}}(3x^2) = \dfrac{3}{2}x^{1/2}$

22. $\dfrac{d}{dt}f(2t + 3) = 2f'(2t + 3)$

23. $\dfrac{d}{dx}f(5x - x^2) = (5 - 2x)f'(5x - x^2)$

24. $$\frac{d}{dx}\left[f\left(\frac{2}{x}\right)\right]^3 = 3\left[f\left(\frac{2}{x}\right)\right]^2 f'\left(\frac{2}{x}\right)\left(\frac{-2}{x^2}\right) = -\frac{2}{x^2}f'\left(\frac{2}{x}\right)\left[f\left(\frac{2}{x}\right)\right]^2$$

25. $$\frac{d}{dx}\sqrt{3+2f(x)} = \frac{2f'(x)}{2\sqrt{3+2f(x)}} = \frac{f'(x)}{\sqrt{3+2f(x)}}$$

26. $$\frac{d}{dt}f(\sqrt{3+2t}) = f'(\sqrt{3+2t})\frac{2}{2\sqrt{3+2t}} = \frac{1}{\sqrt{3+2t}}f'(\sqrt{3+2t})$$

27. $$\frac{d}{dx}f(3+2\sqrt{x}) = \frac{1}{\sqrt{x}}f'(3+2\sqrt{x})$$

28. $$\frac{d}{dt}f\Big(2f\big(3f(x)\big)\Big) = f'\Big(2f\big(3f(x)\big)\Big)\cdot 2f'\big(3f(x)\big)\cdot 3f'(x) = 6f'(x)f'\big(3f(x)\big)f'\Big(2f\big(3f(x)\big)\Big)$$

29. $$\frac{d}{dx}f\big(2-3f(4-5t)\big) = f'\big(2-3f(4-5t)\big)\big(-3f'(4-5t)\big)(-5) = 15f'(4-5t)f'\big(2-3f(4-5t)\big)$$

30. $$\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\bigg|_{x=-2} = \frac{(x^2+1)\dfrac{x}{\sqrt{x^2-1}} - \sqrt{x^2-1}(2x)}{(x^2+1)^2}\Bigg|_{x=-2} = \frac{(5)\left(-\dfrac{2}{\sqrt{3}}\right) - \sqrt{3}(-4)}{25} = \frac{2}{25\sqrt{3}}$$

31. $$\frac{d}{dt}\sqrt{3t-7}\bigg|_{t=3} = \frac{3}{2\sqrt{3t-7}}\bigg|_{t=3} = \frac{3}{2\sqrt{2}}$$

32. $$f(x) = \frac{1}{\sqrt{2x+1}}$$
$$f'(4) = -\frac{1}{(2x+1)^{3/2}}\bigg|_{x=4} = -\frac{1}{27}$$

33. $$y = (x^3+9)^{17/2}$$
$$y'\Big|_{x=-2} = \frac{17}{2}(x^3+9)^{15/2}3x^2\Big|_{x=-2} = \frac{17}{2}(12) = 102$$

34. $$F(x) = (1+x)(2+x)^2(3+x)^3(4+x)^4$$
$$\begin{aligned}F'(x) = {}&(2+x)^2(3+x)^3(4+x)^4 + \\ &2(1+x)(2+x)(3+x)^3(4+x)^4 + \\ &3(1+x)(2+x)^2(3+x)^2(4+x)^4 + \\ &4(1+x)(2+x)^2(3+x)^3(4+x)^3\end{aligned}$$
$$\begin{aligned}F'(0) &= (2^2)(3^3)(4^4) + 2(1)(2)(3^3)(4^4) + \\ &\quad 3(1)(2^2)(3^2)(4^4) + 4(1)(2^2)(3^3)(4^3) \\ &= 4(2^2\cdot 3^3\cdot 4^4) = 110,592\end{aligned}$$

35. $$y = \Big(x + \big((3x)^5 - 2\big)^{-1/2}\Big)^{-6}$$
$$\begin{aligned}y' &= -6\Big(x + \big((3x)^5-2\big)^{-1/2}\Big)^{-7} \\ &\quad\times\Big(1 - \frac{1}{2}\big((3x)^5-2\big)^{-3/2}\big(5(3x)^4 3\big)\Big) \\ &= -6\Big(1 - \frac{15}{2}(3x)^4\big((3x)^5-2\big)^{-3/2}\Big) \\ &\quad\times\Big(x + \big((3x)^5-2\big)^{-1/2}\Big)^{-7}\end{aligned}$$

36. The slope of $y = \sqrt{1+2x^2}$ at $x = 2$ is
$$\frac{dy}{dx}\bigg|_{x=2} = \frac{4x}{2\sqrt{1+2x^2}}\bigg|_{x=2} = \frac{4}{3}.$$
Thus, the equation of the tangent line at $(2,3)$ is $y = 3 + \frac{4}{3}(x-2)$, or $y = \frac{4}{3}x + \frac{1}{3}$.

37. Slope of $y = (1+x^{2/3})^{3/2}$ at $x = -1$ is
$$\frac{3}{2}(1+x^{2/3})^{1/2}\left(\frac{2}{3}x^{-1/3}\right)\bigg|_{x=-1} = -\sqrt{2}$$
The tangent line at $(-1, 2^{3/2})$ has equation $y = 2^{3/2} - \sqrt{2}(x+1)$.

38. The slope of $y = (ax+b)^8$ at $x = \dfrac{b}{a}$ is
$$\frac{dy}{dx}\bigg|_{x=b/a} = 8a(ax+b)^7\Big|_{x=b/a} = 1024ab^7.$$
The equation of the tangent line at $x = \dfrac{b}{a}$ and $y = (2b)^8 = 256b^8$ is
$$y = 256b^8 + 1024ab^7\left(x - \frac{b}{a}\right), \text{ or } y = 2^{10}ab^7x - 3\times 2^8b^8.$$

39. Slope of $y = 1/(x^2-x+3)^{3/2}$ at $x = -2$ is
$$-\frac{3}{2}(x^2-x+3)^{-5/2}(2x-1)\bigg|_{x=-2} = -\frac{3}{2}(9^{-5/2})(-5) = \frac{5}{162}$$
The tangent line at $(-2, \frac{1}{27})$ has equation
$$y = \frac{1}{27} + \frac{5}{162}(x+2).$$

40. Given that $f(x) = (x-a)^m(x-b)^n$ then

$$f'(x) = m(x-a)^{m-1}(x-b)^n + n(x-a)^m(x-b)^{n-1}$$
$$= (x-a)^{m-1}(x-b)^{n-1}(mx - mb + nx - na).$$

If $x \neq a$ and $x \neq b$, then $f'(x) = 0$ if and only if

$$mx - mb + nx - na = 0,$$

which is equivalent to

$$x = \frac{n}{m+n}a + \frac{m}{m+n}b.$$

This point lies lies between a and b.

41. $x(x^4 + 2x^2 - 2)/(x^2+1)^{5/2}$

42. $4(7x^4 - 49x^2 + 54)/x^7$

43. $857, 592$

44. $5/8$

45. The Chain Rule does *not* enable you to calculate the derivatives of $|x|^2$ and $|x^2|$ at $x = 0$ directly as a composition of two functions, one of which is $|x|$, because $|x|$ is not differentiable at $x = 0$. However, $|x|^2 = x^2$ and $|x^2| = x^2$, so both functions are differentiable at $x = 0$ and have derivative 0 there.

46. It may happen that $k = g(x+h) - g(x) = 0$ for values of h arbitrarily close to 0 so that the division by k in the "proof" is not justified.

Section 2.5 Derivatives of Trigonometric Functions (page 123)

1. $\dfrac{d}{dx}\csc x = \dfrac{d}{dx}\dfrac{1}{\sin x} = -\dfrac{\cos x}{\sin^2 x} = -\csc x \cot x$

2. $\dfrac{d}{dx}\cot x = \dfrac{d}{dx}\dfrac{\cos x}{\sin x} = \dfrac{-\cos^2 x - \sin^2 x}{\sin^2 x} = -csc^2 x$

3. $y = \cos 3x, \quad y' = -3\sin 3x$

4. $y = \sin\dfrac{x}{5}, \quad y' = \dfrac{1}{5}\cos\dfrac{x}{5}.$

5. $y = \tan \pi x, \quad y' = \pi \sec^2 \pi x$

6. $y = \sec ax, \quad y' = a \sec ax \tan ax.$

7. $y = \cot(4-3x), \quad y' = 3\csc^2(4-3x)$

8. $\dfrac{d}{dx}\sin\dfrac{\pi - x}{3} = -\dfrac{1}{3}\cos\dfrac{\pi - x}{3}$

9. $f(x) = \cos(s - rx), \quad f'(x) = r\sin(s - rx)$

10. $y = \sin(Ax + B), \quad y' = A\cos(Ax+B)$

11. $\dfrac{d}{dx}\sin(\pi x^2) = 2\pi x\cos(\pi x^2)$

12. $\dfrac{d}{dx}\cos(\sqrt{x}) = -\dfrac{1}{2\sqrt{x}}\sin(\sqrt{x})$

13. $y = \sqrt{1+\cos x}, \quad y' = \dfrac{-\sin x}{2\sqrt{1+\cos x}}$

14.
$$\frac{d}{dx}\sin(2\cos x) = \cos(2\cos x)(-2\sin x)$$
$$= -2\sin x\cos(2\cos x)$$

15. $f(x) = \cos(x + \sin x)$
$f'(x) = -(1+\cos x)\sin(x + \sin x)$

16. $g(\theta) = \tan(\theta\sin\theta)$
$g'(\theta) = (\sin\theta + \theta\cos\theta)\sec^2(\theta\sin\theta)$

17. $u = \sin^3(\pi x/2), \quad u' = \dfrac{3\pi}{2}\cos(\pi x/2)\sin^2(\pi x/2)$

18. $y = \sec(1/x), \quad y' = -(1/x^2)\sec(1/x)\tan(1/x)$

19. $F(t) = \sin at\cos at \quad (= \dfrac{1}{2}\sin 2at)$
$F'(t) = a\cos at\cos at - a\sin at\sin at$
$(= a\cos 2at)$

20. $G(\theta) = \dfrac{\sin a\theta}{\cos b\theta}$
$$G'(\theta) = \frac{a\cos b\theta\cos a\theta + b\sin a\theta\sin b\theta}{\cos^2 b\theta}.$$

21. $\dfrac{d}{dx}\Big(\sin(2x) - \cos(2x)\Big) = 2\cos(2x) + 2\sin(2x)$

22.
$$\frac{d}{dx}(\cos^2 x - \sin^2 x) = \frac{d}{dx}\cos(2x)$$
$$= -2\sin(2x) = -4\sin x\cos x$$

23. $\dfrac{d}{dx}(\tan x + \cot x) = \sec^2 x - \csc^2 x$

24. $\dfrac{d}{dx}(\sec x - \csc x) = \sec x\tan x + \csc x\cot x$

25. $\dfrac{d}{dx}(\tan x - x) = \sec^2 x - 1 = \tan^2 x$

26. $\dfrac{d}{dx}\tan(3x)\cot(3x) = \dfrac{d}{dx}(1) = 0$

27. $\dfrac{d}{dt}(t\cos t - \sin t) = \cos t - t\sin t - \cos t = -t\sin t$

28. $\dfrac{d}{dt}(t\sin t + \cos t) = \sin t + t\cos t - \sin t = t\cos t$

29.
$$\frac{d}{dx}\frac{\sin x}{1+\cos x} = \frac{(1+\cos x)(\cos x) - \sin(x)(-\sin x)}{(1+\cos x)^2}$$
$$= \frac{\cos x + 1}{(1+\cos x)^2} = \frac{1}{1+\cos x}$$

30.
$$\frac{d}{dx}\frac{\cos x}{1+\sin x} = \frac{(1+\sin x)(-\sin x) - \cos(x)(\cos x)}{(1+\sin x)^2}$$
$$= \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

31. $\dfrac{d}{dx}x^2\cos(3x) = 2x\cos(3x) - 3x^2\sin(3x)$

32. $g(t) = \sqrt{(\sin t)/t}$
$$g'(t) = \frac{1}{2\sqrt{(\sin t)/t}} \times \frac{t\cos t - \sin t}{t^2} = \frac{t\cos t - \sin t}{2t^{3/2}\sqrt{\sin t}}$$

33. $v = \sec(x^2)\tan(x^2)$
$v' = 2x\sec(x^2)\tan^2(x^2) + 2x\sec^3(x^2)$

34. $z = \dfrac{\sin\sqrt{x}}{1+\cos\sqrt{x}}$
$$z' = \frac{(1+\cos\sqrt{x})(\cos\sqrt{x}/2\sqrt{x}) - (\sin\sqrt{x})(-\sin\sqrt{x}/2\sqrt{x})}{(1+\cos\sqrt{x})^2} = \frac{1+\cos\sqrt{x}}{2\sqrt{x}(1+\cos\sqrt{x})^2} = \frac{1}{2\sqrt{x}(1+\cos\sqrt{x})}$$

35. $\dfrac{d}{dt}\sin(\cos(\tan t)) = -(\sec^2 t)(\sin(\tan t))\cos(\cos(\tan t))$

36. $f(s) = \cos(s + \cos(s + \cos s))$
$$f'(s) = -[\sin(s + \cos(s + \cos s))] \times [1 - (\sin(s+\cos s))(1-\sin s)]$$

37. Differentiate both sides of $\sin(2x) = 2\sin x\cos x$ and divide by 2 to get $\cos(2x) = \cos^2 x - \sin^2 x$.

38. Differentiate both sides of $\cos(2x) = \cos^2 x - \sin^2 x$ and divide by -2 to get $\sin(2x) = 2\sin x\cos x$.

39. Slope of $y = \sin x$ at $(\pi, 0)$ is $\cos\pi = -1$. Therefore the tangent and normal lines to $y = \sin x$ at $(\pi, 0)$ have equations $y = -(x-\pi)$ and $y = x - \pi$, respectively.

40. The slope of $y = \tan(2x)$ at $(0, 0)$ is $2\sec^2(0) = 2$. Therefore the tangent and normal lines to $y = \tan(2x)$ at $(0, 0)$ have equations $y = 2x$ and $y = -x/2$, respectively.

41. The slope of $y = \sqrt{2}\cos(x/4)$ at $(\pi, 1)$ is $-(\sqrt{2}/4)\sin(\pi/4) = -1/4$. Therefore the tangent and normal lines to $y = \sqrt{2}\cos(x/4)$ at $(\pi, 1)$ have equations $y = 1 - (x-\pi)/4$ and $y = 1 + 4(x-\pi)$, respectively.

42. The slope of $y = \cos^2 x$ at $(\pi/3, 1/4)$ is $-\sin(2\pi/3) = -\sqrt{3}/2$. Therefore the tangent and normal lines to $y = \tan(2x)$ at $(0, 0)$ have equations
$y = (1/4) - (\sqrt{3}/2)(x - (\pi/3))$ and
$y = (1/4) + (2/\sqrt{3})(x - (\pi/3))$, respectively.

43. Slope of $y = \sin(x^\circ) = \sin\left(\dfrac{\pi x}{180}\right)$ is $y' = \dfrac{\pi}{180}\cos\left(\dfrac{\pi x}{180}\right)$. At $x = 45$ the tangent line has equation
$$y = \frac{1}{\sqrt{2}} + \frac{\pi}{180\sqrt{2}}(x-45).$$

44. For $y = \sec(x^\circ) = \sec\left(\dfrac{x\pi}{180}\right)$ we have
$$\frac{dy}{dx} = \frac{\pi}{180}\sec\left(\frac{x\pi}{180}\right)\tan\left(\frac{x\pi}{180}\right).$$
At $x = 60$ the slope is $\dfrac{\pi}{180}(2\sqrt{3}) = \dfrac{\pi\sqrt{3}}{90}$.
Thus, the normal line has slope $-\dfrac{90}{\pi\sqrt{3}}$ and has equation $y = 2 - \dfrac{90}{\pi\sqrt{3}}(x-60)$.

45. The slope of $y = \tan x$ at $x = a$ is $\sec^2 a$. The tangent there is parallel to $y = 2x$ if $\sec^2 a = 2$, or $\cos a = \pm 1/\sqrt{2}$. The only solutions in $(-\pi/2, \pi/2)$ are $a = \pm\pi/4$. The corresponding points on the graph are $(\pi/4, 1)$ and $(-\pi/4, 1)$.

46. The slope of $y = \tan(2x)$ at $x = a$ is $2\sec^2(2a)$. The tangent there is normal to $y = -x/8$ if $2\sec^2(2a) = 8$, or $\cos(2a) = \pm 1/2$. The only solutions in $(-\pi/4, \pi/4)$ are $a = \pm\pi/6$. The corresponding points on the graph are $(\pi/6, \sqrt{3})$ and $(-\pi/6, -\sqrt{3})$.

47. $\dfrac{d}{dx}\sin x = \cos x = 0$ at odd multiples of $\pi/2$.
$\dfrac{d}{dx}\cos x = -\sin x = 0$ at multiples of π.
$\dfrac{d}{dx}\sec x = \sec x\tan x = 0$ at multiples of π.
$\dfrac{d}{dx}\csc x = -\csc x\cot x = 0$ at odd multiples of $\pi/2$.
Thus each of these functions has horizontal tangents at infinitely many points on its graph.

48. $\dfrac{d}{dx}\tan x = \sec^2 x = 0$ nowhere.
$\dfrac{d}{dx}\cot x = -\csc^2 x = 0$ nowhere.
Thus neither of these functions has a horizontal tangent.

49. $y = x + \sin x$ has a horizontal tangent at $x = \pi$ because $dy/dx = 1 + \cos x = 0$ there.

50. $y = 2x + \sin x$ has no horizontal tangents because $dy/dx = 2 + \cos x \ge 1$ everywhere.

51. $y = x + 2\sin x$ has horizontal tangents at $x = 2\pi/3$ and $x = 4\pi/3$ because $dy/dx = 1 + 2\cos x = 0$ at those points.

52. $y = x + 2\cos x$ has horizontal tangents at $x = \pi/6$ and $x = 5\pi/6$ because $dy/dx = 1 - 2\sin x = 0$ at those points.

53. $\displaystyle\lim_{x\to 0}\frac{tan(2x)}{x} = \lim_{x\to 0}\frac{\sin(2x)}{2x}\,\frac{2}{\cos(2x)} = 1 \times 2 = 2$

54. $\displaystyle\lim_{x\to\pi}\sec(1+\cos x) = \sec(1-1) = \sec 0 = 1$

55. $\displaystyle\lim_{x\to 0}x^2\csc x\cot x = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)^2\cos x = 1^2 \times 1 = 1$

56. $\lim_{x\to 0} \cos\left(\dfrac{\pi - \pi\cos^2 x}{x^2}\right) = \lim_{x\to 0} \cos\pi\left(\dfrac{\sin x}{x}\right)^2 = \cos\pi = -1$

57. $\lim_{h\to 0} \dfrac{1-\cos h}{h^2} = \lim_{h\to 0} \dfrac{2\sin^2(h/2)}{h^2} = \lim_{h\to 0} \dfrac{1}{2}\left(\dfrac{\sin(h/2)}{h/2}\right)^2 = \dfrac{1}{2}$

58. f will be differentiable at $x = 0$ if

$$2\sin 0 + 3\cos 0 = b, \qquad \text{and}$$
$$\frac{d}{dx}(2\sin x + 3\cos x)\bigg|_{x=0} = a.$$

Thus we need $b = 3$ and $a = 2$.

59. There are infinitely many lines through the origin that are tangent to $y = \cos x$. The two with largest slope are shown in the figure.

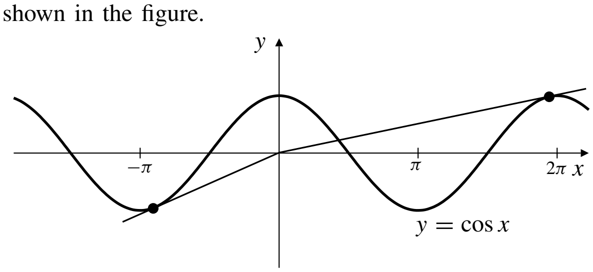


Fig. 2.5.59

The tangent to $y = \cos x$ at $x = a$ has equation $y = \cos a - (\sin a)(x - a)$. This line passes through the origin if $\cos a = -a\sin a$. We use a calculator with a "solve" function to find solutions of this equation near $a = -\pi$ and $a = 2\pi$ as suggested in the figure. The solutions are $a \approx -2.798386$ and $a \approx 6.121250$. The slopes of the corresponding tangents are given by $-\sin a$, so they are 0.336508 and 0.161228 to six decimal places.

60. 1

61. $-\sqrt{2\pi+3}(2\pi^{3/2} - 4\pi + 3)/\pi$

62. a) As suggested by the figure in the problem, the square of the length of chord AP is $(1-\cos\theta)^2 + (0-\sin\theta)^2$, and the square of the length of arc AP is θ^2. Hence

$$(1+\cos\theta)^2 + \sin^2\theta < \theta^2,$$

and, since squares cannot be negative, each term in the sum on the left is less than θ^2. Therefore

$$0 \le |1-\cos\theta| < |\theta|, \qquad 0 \le |\sin\theta| < |\theta|.$$

Since $\lim_{\theta\to 0}|\theta| = 0$, the squeeze theorem implies that

$$\lim_{\theta\to 0} 1-\cos\theta = 0, \qquad \lim_{\theta\to 0}\sin\theta = 0.$$

From the first of these, $\lim_{\theta\to 0}\cos\theta = 1$.

b) Using the result of (a) and the addition formulas for cosine and sine we obtain

$$\lim_{h\to 0}\cos(\theta_0 + h) = \lim_{h\to 0}(\cos\theta_0\cos h - \sin\theta_0\sin h) = \cos\theta_0$$
$$\lim_{h\to 0}\sin(\theta_0 + h) = \lim_{h\to 0}(\sin\theta_0\cos h + \cos\theta_0\sin h) = \sin\theta_0.$$

This says that cosine and sine are continuous at any point θ_0.

Section 2.6 The Mean-Value Theorem (page 131)

1. $f(x) = x^2, \quad f'(x) = 2x$

$$b + a = \frac{b^2 - a^2}{b-a} = \frac{f(b)-f(a)}{b-a}$$
$$= f'(c) = 2c \quad \Rightarrow c = \frac{b+a}{2}$$

2. If $f(x) = \dfrac{1}{x}$, and $f'(x) = -\dfrac{1}{x^2}$ then

$$\frac{f(2)-f(1)}{2-1} = \frac{1}{2} - 1 = -\frac{1}{2} = -\frac{1}{c^2} = f'(c)$$

where $c = \sqrt{2}$ lies between 1 and 2.

3. $f(x) = x^3 - 3x + 1$, $f'(x) = 3x^2 - 3$, $a = -2$, $b = 2$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-2)}{4}$$
$$= \frac{8-6+1-(-8+6+1)}{4}$$
$$= \frac{4}{4} = 1$$

$f'(c) = 3c^2 - 3$

$3c^2 - 3 = 1 \Rightarrow 3c^2 = 4 \Rightarrow c = \pm\dfrac{2}{\sqrt{3}}$

(Both points will be in $(-2, 2)$.)

4. If $f(x) = \cos x + (x^2/2)$, then $f'(x) = x - \sin x > 0$ for $x > 0$. By the MVT, if $x > 0$, then $f(x) - f(0) = f'(c)(x-0)$ for some $c > 0$, so $f(x) > f(0) = 1$. Thus $\cos x + (x^2/2) > 1$ and $\cos x > 1 - (x^2/2)$ for $x > 0$. Since both sides of the inequality are even functions, it must hold for $x < 0$ as well.

5. Let $f(x) = \tan x$. If $0 < x < \pi/2$, then by the MVT $f(x) - f(0) = f'(c)(x-0)$ for some c in $(0, \pi/2)$. Thus $\tan x = x\sec^2 c > x$, since $secc > 1$.

6. Let $f(x) = (1+x)^r - 1 - rx$ where $r > 1$.
Then $f'(x) = r(1+x)^{r-1} - r$.
If $-1 \le x < 0$ then $f'(x) < 0$; if $x > 0$, then $f'(x) > 0$.
Thus $f(x) > f(0) = 0$ if $-1 \le x < 0$ or $x > 0$.
Thus $(1+x)^r > 1 + rx$ if $-1 \le x < 0$ or $x > 0$.

7. Let $f(x) = (1+x)^r$ where $0 < r < 1$. Thus, $f'(x) = r(1+x)^{r-1}$. By the Mean-Value Theorem, for $x \geq -1$, and $x \neq 0$,

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$
$$\Rightarrow \frac{(1-x)^r - 1}{x} = r(1+c)^{r-1}$$

for some c between 0 and x. Thus,
$(1+x)^r = 1 + rx(1+c)^{r-1}$.
If $-1 \leq x < 0$, then $c < 0$ and $0 < 1+c < 1$. Hence

$$\begin{aligned} (1+c)^{r-1} &> 1 \qquad \text{(since } r-1 < 0\text{)},\\ rx(1+c)^{r-1} &< rx \qquad \text{(since } x < 0\text{)}. \end{aligned}$$

Hence, $(1+x)^r < 1 + rx$.
If $x > 0$, then

$$\begin{aligned} c &> 0\\ 1 + c &> 1\\ (1+c)^{r-1} &< 1\\ rx(1+c)^{r-1} &< rx. \end{aligned}$$

Hence, $(1+x)^r < 1 + rx$ in this case also.
Hence, $(1+x)^r < 1+rx$ for either $-1 \leq x < 0$ or $x > 0$.

8. If $f(x) = x^2 + 2x + 2$ then $f'(x) = 2x + 2 = 2(x+1)$. Evidently, $f'(x) > 0$ if $x > -1$ and $f'(x) < 0$ if $x < -1$. Therefore, f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

9. $f(x) = x^3 - 4x + 1$
$f'(x) = 3x^2 - 4$
$f'(x) > 0$ if $|x| > \dfrac{2}{\sqrt{3}}$
$f'(x) < 0$ if $|x| < \dfrac{2}{\sqrt{3}}$
f is increasing on $(-\infty, -\dfrac{2}{\sqrt{3}})$ and $(\dfrac{2}{\sqrt{3}}, \infty)$.
f is decreasing on $(-\dfrac{2}{\sqrt{3}}, \dfrac{2}{\sqrt{3}})$.

10. If $f(x) = x^3 + 4x + 1$, then $f'(x) = 3x^2 + 4$. Since $f'(x) > 0$ for all real x, hence $f(x)$ is increasing on the whole real line, i.e., on $(-\infty, \infty)$.

11. $f(x) = (x^2 - 4)^2$
$f'(x) = 2x2(x^2 - 4) = 4x(x-2)(x+2)$
$f'(x) > 0$ if $x > 2$ or $-2 < x < 0$
$f'(x) < 0$ if $x < -2$ or $0 < x < 2$
f is increasing on $(-2, 0)$ and $(2, \infty)$.
f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

12. If $f(x) = \dfrac{1}{x^2+1}$ then $f'(x) = \dfrac{-2x}{(x^2+1)^2}$. Evidently, $f'(x) > 0$ if $x < 0$ and $f'(x) < 0$ if $x > 0$. Therefore, f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

13. $f(x) = x^3(5-x)^2$

$$\begin{aligned} f'(x) &= 3x^2(5-x)^2 + 2x^3(5-x)(-1)\\ &= x^2(5-x)(15-5x)\\ &= 5x^2(5-x)(3-x) \end{aligned}$$

$f'(x) > 0$ if $x < 0$, $0 < x < 3$, or $x > 5$
$f'(x) < 0$ if $3 < x < 5$
f is increasing on $(-\infty, 3)$ and $(5, \infty)$.
f is decreasing on $(3, 5)$.

14. If $f(x) = x - 2\sin x$, then $f'(x) = 1 - 2\cos x = 0$ at $x = \pm\pi/3 + 2n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$.
f is decreasing on $(-\pi/3 + 2n\pi, \pi + 2n\pi)$.
f is increasing on $(\pi/3 + 2n\pi, -\pi/3 + 2(n+1)\pi)$ for integers n.

15. If $f(x) = x + \sin x$, then $f'(x) = 1 + \cos x \geq 0$
$f'(x) = 0$ only at isolated points $x = \pm\pi, \ \pm 3\pi, \ldots$.
Hence f is increasing everywhere.

16. If $x_1 < x_2 < \ldots < x_n$ belong to I, and $f(x_i) = 0$, $(1 \leq i \leq n)$, then there exists y_i in (x_i, x_{i+1}) such that $f'(y_i) = 0$, $(1 \leq i \leq n-1)$ by MVT.

17. There is no guarantee that the MVT applications for f and g yield the same c.

18. For $x \neq 0$, we have $f'(x) = 2x\sin(1/x) - \cos(1/x)$ which has no limit as $x \to 0$. However, $f'(0) = \lim_{h\to 0} f(h)/h = \lim_{h\to 0} h\sin(1/h) = 0$ does exist even though f' cannot be continuous at 0.

19. If f' exists on $[a, b]$ and $f'(a) \neq f'(b)$, let us assume, without loss of generality, that $f'(a) > k > f'(b)$. If $g(x) = f(x) - kx$ on $[a, b]$, then g is continuous on $[a, b]$ because f, having a derivative, must be continuous there. By the Max-Min Theorem, g must have a maximum value (and a minimum value) on that interval. Suppose the maximum value occurs at c. Since $g'(a) > 0$ we must have $c > a$; since $g'(b) < 0$ we must have $c < b$. By Theorem 14, we must have $g'(c) = 0$ and so $f'(c) = k$. Thus f' takes on the (arbitrary) intermediate value k.

20. $f(x) = \begin{cases} x + 2x^2\sin(1/x) & \text{if } x \neq 0\\ 0 & \text{if } x = 0. \end{cases}$

a)
$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}\\ &= \lim_{h\to 0} \frac{h + 2h^2\sin(1/h)}{h}\\ &= \lim_{h\to 0}(1 + 2h\sin(1/h)) = 1, \end{aligned}$$
because $|2h\sin(1/h)| \leq 2|h| \to 0$ as $h \to 0$.

b) For $x \neq 0$, we have

$$f'(x) = 1 + 4x\sin(1/x) - 2\cos(1/x).$$

There are numbers x arbitrarily close to 0 where $f'(x) = -1$; namely, the numbers $x = \pm 1/(2n\pi)$, where $n = 1, 2, 3, \ldots$. Since $f'(x)$ is continuous at every $x \neq 0$, it is negative in a small interval about every such number. Thus f cannot be increasing on any interval containing $x = 0$.

Section 2.7 Using Derivatives (page 136)

1. If $y = x^2$, then $\Delta y \approx 2x\,\Delta x$. If $\Delta x = (2/100)x$, then $\Delta y \approx (4/100)x^2 = (4/100)y$, so y increases by about 4%.

2. If $y = 1/x$, then $\Delta y \approx (-1/x^2)\,\Delta x$. If $\Delta x = (2/100)x$, then $\Delta y \approx (-2/100)/x = (-2/100)y$, so y decreases by about 2%.

3. If $y = 1/x^2$, then $\Delta y \approx (-2/x^3)\,\Delta x$. If $\Delta x = (2/100)x$, then $\Delta y \approx (-4/100)/x^2 = (-4/100)y$, so y decreases by about 4%.

4. If $y = x^3$, then $\Delta y \approx 3x^2\,\Delta x$. If $\Delta x = (2/100)x$, then $\Delta y \approx (6/100)x^3 = (6/100)y$, so y increases by about 6%.

5. If $y = \sqrt{x}$, then $\Delta y \approx (1/2\sqrt{x})\,\Delta x$. If $\Delta x = (2/100)x$, then $\Delta y \approx (1/100)\sqrt{x} = (1/100)y$, so y increases by about 1%.

6. If $y = x^{-2/3}$, then $\Delta y \approx (-2/3)x^{-5/3}\,\Delta x$. If $\Delta x = (2/100)x$, then $\Delta y \approx (-4/300)x^{2/3} = (-4/300)y$, so y decreases by about 1.33%.

7. If $V = \frac{4}{3}\pi r^3$, then $\Delta V = 4\pi r^2\,\Delta r$. If r increases by 2%, then $\Delta r = 2r/100$ and $\Delta V \approx 8\pi r^3/100$. Therefore $\Delta V/V \approx 6/100$. The volume increases by about 6%.

8. If V is the volume and x is the edge length of the cube then $V = x^3$. Thus $\Delta V \approx 3x^2\,\Delta x$. $\Delta V = -(6/100)V$, then $-6x^3/100 = 3x^2\,\Delta x$, so $\Delta x \approx -(2/100)x$. The edge of the cube decreases by about 2%.

9. Rate change of Area A with respect to side s, where $A = s^2$, is $\dfrac{dA}{ds} = 2s$. When $s = 4$ ft, the area is changing at rate 8 ft^2/ft.

10. If $A = s^2$, then $s = \sqrt{A}$ and $ds/dA = 1/(2\sqrt{A})$. If $A = 16$ m^2, then the side is changing at rate $ds/dA = 1/8$ m/m^2.

11. The diameter D and area A of a circle are related by $D = 2\sqrt{A/\pi}$. The rate of change of diameter with respect to area is $dD/dA = \sqrt{1/(\pi A)}$ units per square unit.

12. Since $A = \pi D^2/4$, the rate of change of area with respect to diameter is $dA/dD = \pi D/2$ square units per unit.

13. Rate of change of $V = \dfrac{4}{3}\pi r^3$ with respect to radius r is $\dfrac{dV}{dr} = 4\pi r^2$. When $r = 2$ m, this rate of change is 16π m^3/m.

14. Let A be the area of a square, s be its side length and L be its diagonal. Then, $L^2 = s^2 + s^2 = 2s^2$ and $A = s^2 = \frac{1}{2}L^2$, so $\dfrac{dA}{dL} = L$. Thus, the rate of change of the area of a square with respect to its diagonal L is L.

15. If the radius of the circle is r then $C = 2\pi r$ and $A = \pi r^2$.
Thus $C = 2\pi\sqrt{\dfrac{A}{\pi}} = 2\sqrt{\pi}\sqrt{A}$.
Rate of change of C with respect to A is
$\dfrac{dC}{dA} = \dfrac{\sqrt{\pi}}{\sqrt{A}} = \dfrac{1}{r}$.

16. Let s be the side length and V be the volume of a cube. Then $V = s^3 \Rightarrow s = V^{1/3}$ and $\dfrac{ds}{dV} = \frac{1}{3}V^{-2/3}$. Hence, the rate of change of the side length of a cube with respect to its volume V is $\frac{1}{3}V^{-2/3}$.

17. If $f(x) = x^2 - 4$, then $f'(x) = 2x$. The critical point of f is $x = 0$. f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

18. If $f(x) = x^3 - 12x + 1$, then $f'(x) = 3(x^2 - 4)$. The critical points of f are $x = \pm 2$. f is increasing on $(-\infty, -2)$ and $(2, \infty)$ where $f'(x) > 0$, and is decreasing on $(-2, 2)$ where $f'(x) < 0$.

19. If $y = x^3 + 6x^2$, then $y' = 3x^2 + 12x = 3x(x + 4)$. The critical points of y are $x = 0$ and $x = -4$. y is increasing on $(-\infty, -4)$ and $(0, \infty)$ where $y' > 0$, and is decreasing on $(-4, 0)$ where $y' < 0$.

20. If $y = 1 - x - x^5$, then $y' = -1 - 5x^4 < 0$ for all x. Thus y has no critical points and is decreasing on the whole real line.

21. $f(x) = x^3$ is increasing on $(-\infty, 0)$ and $(0, \infty)$ because $f'(x) = 3x^2 > 0$ there. But $f(x_1) < f(0) = 0 < f(x_2)$ whenever $x_1 < 0 < x_2$, so f is also increasing on intervals containing the origin.

22. If $f(x) = x + 2\sin x$, then $f'(x) = 1 + 2\cos x > 0$ if $\cos x > -1/2$. Thus f is increasing on the intervals $(-(4\pi/3) + 2n\pi, (4\pi/3) + 2n\pi)$ where n is any integer.

23. CPs $x = 0.535898$ and $x = 7.464102$

24. CPs $x = -1.366025$ and $x = 0.366025$

25. CPs $x = -0.518784$ and $x = 0$

26. CP $x = 0.521350$

27. Volume in tank is $V(t) = 350(20 - t)^2$ L at t min.

a) At $t = 5$, water volume is changing at rate

$$\left.\frac{dV}{dt}\right|_{t=5} = -700(20-t)\Big|_{t=5} = -10,500.$$

Water is draining out at 10,500 L/min at that time.
At $t = 15$, water volume is changing at rate

$$\left.\frac{dV}{dt}\right|_{t=15} = -700(20-t)\Big|_{t=15} = -3,500.$$

Water is draining out at 3,500 L/min at that time.

b) Average rate of change between $t = 5$ and $t = 15$ is

$$\frac{V(15)-V(5)}{15-5} = \frac{350\times(25-225)}{10} = -7,000.$$

The average rate of draining is 7,000 L/min over that interval.

28. Flow rate $F = kr^4$, so $\Delta F \approx 4kr^3\,\Delta r$. If $\Delta F = F/10$, then

$$\Delta r \approx \frac{F}{40kr^3} = \frac{kr^4}{40kr^3} = 0.025r.$$

The flow rate will increase by 10% if the radius is increased by about 2.5%.

29. $F = k/r^2$ implies that $dF/dr = -2k/r^3$. Since $dF/dr = 1$ pound/mi when $r = 4,000$ mi, we have $2k = 4,000^3$. If $r = 8,000$, we have $dF/dr = -(4,000/8,000)^3 = -1/8$. At $r = 8,000$ mi F decreases with respect to r at a rate of 1/8 pounds/mi.

30. If price = \$$p$, then revenue is \$$R = 4,000p - 10p^2$.

a) Sensitivity of R to p is $dR/dp = 4,000 - 20p$. If $p = 100$, 200, and 300, this sensitivity is 2,000 \$/\$, 0 \$/\$, and $-2,000$ \$/\$ respectively.

b) The distributor should charge \$200. This maximizes the revenue.

31. Cost is \$$C(x) = 8,000 + 400x - 0.5x^2$ if x units are manufactured.

a) Marginal cost if $x = 100$ is
$C'(100) = 400 - 100 = \$300$.

b) $C(101) - C(100) = 43,299.50 - 43,000 = \$299.50$ which is approximately $C'(100)$.

32. Daily profit if production is x sheets per day is \$$P(x)$ where

$$P(x) = 8x - 0.005x^2 - 1,000.$$

a) Marginal profit $P'(x) = 8 - 0.01x$. This is positive if $x < 800$ and negative if $x > 800$.

b) To maximize daily profit, production should be 800 sheets/day.

33. $C = \dfrac{80,000}{n} + 4n + \dfrac{n^2}{100}$
$\dfrac{dC}{dn} = -\dfrac{80,000}{n^2} + 4 + \dfrac{n}{50}$.

(a) $n = 100$, $\dfrac{dC}{dn} = -2$. Thus, the marginal cost of production is $-\$2$.

(b) $n = 300$, $\dfrac{dC}{dn} = \dfrac{82}{9} \approx 9.11$. Thus, the marginal cost of production is approximately \$9.11.

34. Daily profit $P = 13x - Cx = 13x - 10x - 20 - \dfrac{x^2}{1000}$

$$= 3x - 20 - \frac{x^2}{1000}$$

Graph of P is a parabola opening downward. P will be maximum where the slope is zero:

$$0 = \frac{dP}{dx} = 3 - \frac{2x}{1000} \quad \text{so } x = 1500$$

Should extract 1500 tonnes of ore per day to maximize profit.

35. One of the components comprising $C(x)$ is usually a fixed cost, \$$S$, for setting up the manufacturing operation. On a per item basis, this fixed cost \$$S/x$, decreases as the number x of items produced increases, especially when x is small. However, for large x other components of the total cost may increase on a per unit basis, for instance labour costs when overtime is required or maintenance costs for machinery when it is over used.

Let the average cost be $A(x) = \dfrac{C(x)}{x}$. The minimal average cost occurs at point where the graph of $A(x)$ has a horizontal tangent:

$$0 = \frac{dA}{dx} = \frac{xC'(x) - C(x)}{x^2}.$$

Hence, $xC'(x) - C(x) = 0 \Rightarrow C'(x) = \dfrac{C(x)}{x} = A(x)$.
Thus the marginal cost $C'(x)$ equals the average cost at the minimizing value of x.

36. If $y = Cp^{-r}$, then the elasticity of y is

$$-\frac{p}{y}\frac{dy}{dp} = -\frac{p}{Cp^{-r}}(-r)Cp^{-r-1} = r.$$

Section 2.8 Higher-Order Derivatives (page 140)

1. $y = (3-2x)^7$
$y' = -14(3-2x)^6$
$y'' = 168(3-2x)^5$
$y''' = -1680(3-2x)^4$

2. $y = x^2 - \dfrac{1}{x}$ $\quad y'' = 2 - \dfrac{2}{x^3}$
$y' = 2x + \dfrac{1}{x^2}$ $\quad y''' = \dfrac{6}{x^4}$

3. $y = \dfrac{6}{(x-1)^2} = 6(x-1)^{-2}$
$y' = -12(x-1)^{-3}$
$y'' = 36(x-1)^{-4}$
$y''' = -144(x-1)^{-5}$

4. $y = \sqrt{ax+b}$ $\quad y'' = -\dfrac{a^2}{4(ax+b)^{3/2}}$
$y' = \dfrac{a}{2\sqrt{ax+b}}$ $\quad y''' = \dfrac{3a^3}{8(ax+b)^{5/2}}$

5. $y = x^{1/3} - x^{-1/3}$
$y' = \dfrac{1}{3}x^{-2/3} + \dfrac{1}{3}x^{-4/3}$
$y'' = -\dfrac{2}{9}x^{-5/3} - \dfrac{4}{9}x^{-7/3}$
$y''' = \dfrac{10}{27}x^{-8/3} + \dfrac{28}{27}x^{-10/3}$

6. $y = x^{10} + 2x^8$ $\quad y'' = 90x^8 + 112x^6$
$y' = 10x^9 + 16x^7$ $\quad y''' = 720x^7 + 672x^5$

7. $y = (x^2+3)\sqrt{x} = x^{5/2} + 3x^{1/2}$
$y' = \dfrac{5}{2}x^{3/2} + \dfrac{3}{2}x^{-1/2}$
$y'' = \dfrac{15}{4}x^{1/2} - \dfrac{3}{4}x^{-3/2}$
$y''' = \dfrac{15}{8}x^{-1/2} + \dfrac{9}{8}x^{-5/2}$

8. $y = \dfrac{x-1}{x+1}$ $\quad y'' = -\dfrac{4}{(x+1)^3}$
$y' = \dfrac{2}{(x+1)^2}$ $\quad y''' = \dfrac{12}{(x+1)^4}$

9. $y = \tan x$ $\quad y'' = 2\sec^2 x \tan x$
$y' = \sec^2 x$ $\quad y''' = 2\sec^4 x + 4\sec^2 x \tan^2 x$

10. $y = \sec x$ $\quad y'' = \sec x \tan^2 x + \sec^3 x$
$y' = \sec x \tan x$ $\quad y''' = \sec x \tan^3 x + 5\sec^3 x \tan x$

11. $y = \cos(x^2)$ $\quad y'' = -2\sin(x^2) - 4x^2\cos(x^2)$
$y' = -2x\sin(x^2)$ $\quad y''' = -12x\cos(x^2) + 8x^3\sin(x^2)$

12. $y = \dfrac{\sin x}{x}$
$y' = \dfrac{\cos x}{x} - \dfrac{\sin x}{x^2}$
$y'' = \dfrac{(2-x^2)\sin x}{x^3} - \dfrac{2\cos x}{x^2}$
$y''' = \dfrac{(6-x^2)\cos x}{x^3} + \dfrac{3(x^2-2)\sin x}{x^4}$

13. $f(x) = \dfrac{1}{x} = x^{-1}$
$f'(x) = -x^{-2}$
$f''(x) = 2x^{-3}$
$f'''(x) = -3!x^{-4}$
$f^{(4)}(x) = 4!x^{-5}$
Guess: $f^{(n)}(x) = (-1)^n n!x^{-(n+1)}$ $(*)$
Proof: (*) is valid for $n = 1$ (and 2, 3, 4).
Assume $f^{(k)}(x) = (-1)^k k!x^{-(k+1)}$ for some $k \ge 1$
Then $f^{(k+1)}(x) = (-1)^k k!\Big(-(k+1)\Big)x^{-(k+1)-1}$
$= (-1)^{k+1}(k+1)!x^{-((k+1)+1)}$ which is (*) for $n = k+1$.
Therefore, (*) holds for $n = 1, 2, 3, \ldots$ by induction.

14. $f(x) = \dfrac{1}{x^2} = x^{-2}$
$f'(x) = -2x^{-3}$
$f''(x) = -2(-3)x^{-4} = 3!x^{-4}$
$f^{(3)}(x) = -2(-3)(-4)x^{-5} = -4!x^{-5}$
Conjecture:

$$f^{(n)}(x) = (-1)^n(n+1)!x^{-(n+2)} \qquad \text{for } n = 1, 2, 3, \ldots$$

Proof: Evidently, the above formula holds for $n = 1$, 2 and 3. Assume it holds for $n = k$, i.e., $f^{(k)}(x) = (-1)^k(k+1)!x^{-(k+2)}$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}f^{(k)}(x) \\ &= (-1)^k(k+1)![(-1)(k+2)]x^{-(k+2)-1} \\ &= (-1)^{k+1}(k+2)!x^{-[(k+1)+2]}. \end{aligned}$$

Thus, the formula is also true for $n = k+1$. Hence it is true for $n = 1, 2, 3, \ldots$ by induction.

15. $f(x) = \dfrac{1}{2-x} = (2-x)^{-1}$
$f'(x) = +(2-x)^{-2}$
$f''(x) = 2(2-x)^{-3}$
$f'''(x) = +3!(2-x)^{-4}$
Guess: $f^{(n)}(x) = n!(2-x)^{-(n+1)}$ $(*)$
Proof: (*) holds for $n = 1, 2, 3$.
Assume $f^{(k)}(x) = k!(2-x)^{-(k+1)}$ (i.e., (*) holds for $n = k$)
Then $f^{(k+1)}(x) = k!\Big(-(k+1)(2-x)^{-(k+1)-1}(-1)\Big)$
$= (k+1)!(2-x)^{-((k+1)+1)}$.
Thus (*) holds for $n = k+1$ if it holds for k.
Therefore, (*) holds for $n = 1, 2, 3, \ldots$ by induction.

16. $f(x)=\sqrt{x}=x^{1/2}$
$f'(x)=\frac{1}{2}x^{-1/2}$
$f''(x)=\frac{1}{2}(-\frac{1}{2})x^{-3/2}$
$f'''(x)=\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})x^{-5/2}$
$f^{(4)}(x)=\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})x^{-7/2}$
Conjecture:

$$f^{(n)}(x)=(-1)^{n-1}\frac{1\cdot3\cdot5\cdots(2n-3)}{2^n}x^{-(2n-1)/2}\quad(n\geq2).$$

Proof: Evidently, the above formula holds for $n=2,3$ and 4. Assume that it holds for $n=k$, i.e.

$$f^{(k)}(x)=(-1)^{k-1}\frac{1\cdot3\cdot5\cdots(2k-3)}{2^k}x^{-(2k-1)/2}.$$

Then

$$\begin{aligned}f^{(k+1)}(x)&=\frac{d}{dx}f^{(k)}(x)\\&=(-1)^{k-1}\frac{1\cdot3\cdot5\cdots(2k-3)}{2^k}\cdot\left[\frac{-(2k-1)}{2}\right]x^{-[(2k-1)/2]-1}\\&=(-1)^{(k+1)-1}\frac{1\cdot3\cdot5\cdots(2k-3)[2(k+1)-3]}{2^{k+1}}x^{-[2(k+1)-1]/2}.\end{aligned}$$

Thus, the formula is also true for $n=k+1$. Hence, it is true for $n\geq2$ by induction.

17. $f(x)=\dfrac{1}{a+bx}=(a+bx)^{-1}$
$f'(x)=-b(a+bx)^{-2}$
$f''(x)=2b^2(a+bx)^{-3}$
$f'''(x)=-3!b^3(a+bx)^{-4}$
Guess: $f^{(n)}(x)=(-1)^n n!b^n(a+bx)^{-(n+1)}\quad(*)$
Proof: (*) holds for $n=1,2,3$
Assume (*) holds for $n=k$:
$f^{(k)}(x)=(-1)^k k!b^k(a+bx)^{-(k+1)}$
Then
$$\begin{aligned}f^{(k+1)}(x)&=(-1)^k k!b^k\Big(-(k+1)\Big)(a+bx)^{-(k+1)-1}(b)\\&=(-1)^{k+1}(k+1)!b^{k+1}(a+bx)^{((k+1)+1)}\end{aligned}$$
So (*) holds for $n=k+1$ if it holds for $n=k$.
Therefore, (*) holds for $n=1,2,3,4,\ldots$ by induction.

18. $f(x)=x^{2/3}$
$f'(x)=\frac{2}{3}x^{-1/3}$
$f''(x)=\frac{2}{3}(-\frac{1}{3})x^{-4/3}$
$f'''(x)=\frac{2}{3}(-\frac{1}{3})(-\frac{4}{3})x^{-7/3}$
Conjecture:
$f^{(n)}(x)=2(-1)^{n-1}\dfrac{1\cdot4\cdot7\cdot\cdots\cdot(3n-5)}{3^n}x^{-(3n-2)/3}$ for $n\geq2$.
Proof: Evidently, the above formula holds for $n=2$ and 3. Assume that it holds for $n=k$, i.e.

$$f^{(k)}(x)=2(-1)^{k-1}\frac{1\cdot4\cdot7\cdot\cdots\cdot(3k-5)}{3^k}x^{-(3k-2)/3}.$$

Then,

$$\begin{aligned}f^{(k+1)}(x)&=\frac{d}{dx}f^{(k)}(x)\\&=2(-1)^{k-1}\frac{1\cdot4\cdot7\cdot\cdots\cdot(3k-5)}{3^k}\cdot\left[\frac{-(3k-2)}{3}\right]x^{-[(3k-2)/3]-1}\\&=2(-1)^{(k+1)-1}\frac{1\cdot4\cdot7\cdot\cdots\cdot(3k-5)[3(k+1)-5]}{3^(k+1)}x^{-[3(k+1)-2]/3}.\end{aligned}$$

Thus, the formula is also true for $n=k+1$. Hence, it is true for $n\geq2$ by induction.

19.
$$\begin{aligned}f(x)&=\cos(ax)\\f'(x)&=-a\sin(ax)\\f''(x)&=-a^2\cos(ax)\\f'''(x)&=a^3\sin(ax)\\f^{(4)}(x)&=a^4\cos(ax)=a^4f(x)\end{aligned}$$
It follows that $f^{(n)}(x)=a^4f^{(n-4)}(x)$ for $n\geq4$, and

$$f^{(n)}(x)=\begin{cases}a^n\cos(ax)&\text{if } n=4k\\-a^n\sin(ax)&\text{if } n=4k+1\\-a^n\cos(ax)&\text{if } n=4k+2\\a^n\sin(ax)&\text{if } n=4k+3\end{cases}\quad(k=0,\ 1,\ 2,\ \ldots)$$

Differentiating any of these four formulas produces the one for the next higher value of n, so induction confirms the overall formula.

20.
$$\begin{aligned}f(x)&=x\cos x\\f'(x)&=\cos x-x\sin x\\f''(x)&=-2\sin x-x\cos x\\f'''(x)&=-3\cos x+x\sin x\\f^{(4)}(x)&=4\sin x+x\cos x\end{aligned}$$
This suggests the formula (for $k=0,\ 1,\ 2,\ \ldots$)

$$f^{(n)}(x)=\begin{cases}n\sin x+x\cos x&\text{if } n=4k\\n\cos x-x\sin x&\text{if } n=4k+1\\-n\sin x-x\cos x&\text{if } n=4k+2\\-n\cos x+x\sin x&\text{if } n=4k+3\end{cases}$$

Differentiating any of these four formulas produces the one for the next higher value of n, so induction confirms the overall formula.

21.
$$\begin{aligned}f(x)&=x\sin(ax)\\f'(x)&=\sin(ax)+ax\cos(ax)\\f''(x)&=2a\cos(ax)-a^2x\sin(ax)\\f'''(x)&=-3a^2\sin(ax)-a^3x\cos(ax)\\f^{4)}(x)&=-4a^3\cos(ax)+a^4x\sin(ax)\end{aligned}$$
This suggests the formula

$$f^{(n)}(x)=\begin{cases}-na^{n-1}\cos(ax)+a^nx\sin(ax)&\text{if } n=4k\\na^{n-1}\sin(ax)+a^nx\cos(ax)&\text{if } n=4k+1\\na^{n-1}\cos(ax)-a^nx\sin(ax)&\text{if } n=4k+2\\-na^{n-1}\sin(ax)-a^nx\cos(ax)&\text{if } n=4k+3\end{cases}$$

for $k = 0, 1, 2, \ldots$. Differentiating any of these four formulas produces the one for the next higher value of n, so induction confirms the overall formula.

22. $f(x) = \dfrac{1}{|x|} = |x|^{-1}$. Recall that $\dfrac{d}{dx}|x| = \operatorname{sgn} x$, so

$$f'(x) = -|x|^{-2}\operatorname{sgn} x.$$

If $x \neq 0$ we have

$$\frac{d}{dx}\operatorname{sgn} x = 0 \quad \text{and} \quad (\operatorname{sgn} x)^2 = 1.$$

Thus we can calculate successive derivatives of f using the product rule where necessary, but will get only one nonzero term in each case:

$$\begin{aligned} f''(x) &= 2|x|^{-3}(\operatorname{sgn} x)^2 = 2|x|^{-3} \\ f^{(3)}(x) &= -3!|x|^{-4}\operatorname{sgn} x \\ f^{(4)}(x) &= 4!|x|^{-5}. \end{aligned}$$

The pattern suggests that

$$f^{(n)}(x) = \begin{cases} -n!|x|^{-(n+1)}\operatorname{sgn} x & \text{if } n \text{ is odd} \\ n!|x|^{-(n+1)} & \text{if } n \text{ is even} \end{cases}$$

Differentiating this formula leads to the same formula with n replaced by $n+1$ so the formula is valid for all $n \geq 1$ by induction.

23.
$$\begin{aligned} f(x) &= \sqrt{1-3x} = (1-3x)^{1/2} \\ f'(x) &= \frac{1}{2}(-3)(1-3x)^{-1/2} \\ f''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)(-3)^2(1-3x)^{-3/2} \\ f'''(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-3)^3(1-3x)^{-5/2} \\ f^{(4)}(x) &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-3)^4(1-3x)^{7/2} \end{aligned}$$

Guess: $f^{(n)}(x) = -\dfrac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2^n}3^n$

$(1-3x)^{-(2n-1)/2} \quad (*)$

Proof: (*) is valid for $n = 2, 3, 4$, (but not $n = 1$)

Assume (*) holds for $n = k$ for some integer $k \geq 2$

i.e., $f^{(k)}(x) = -\dfrac{1 \times 3 \times 5 \times \ldots \times (2k-3)}{2^k}3^k$

$(1-3x)^{-(2k-1)/2}$

Then $f^{(k+1)}(x) = -\dfrac{1 \times 3 \times 5 \times \cdots \times (2k-3)}{2^k}3^k$

$$\left(-\frac{2(k-1)}{2}\right)(1-3x)^{-(2k-1)/2-1}(-3)$$

$$= -\frac{1 \times 3 \times 5 \times \cdots \Big(2(k+1)-1\Big)}{2^{k+1}}3^{k+1}$$

$(1-3x)^{-(2(k+1)-1)/2}$

Thus (*) holds for $n = k+1$ if it holds for $n = k$. Therefore, (*) holds for $n = 2, 3, 4, \ldots$ by induction.

24. If $y = \tan(kx)$, then $y' = k\sec^2(kx)$ and

$$\begin{aligned} y'' &= 2k^2 sec^2(kx) tan(kx) \\ &= 2k^2(1+\tan^2(kx))\tan(kx) = 2k^2 y(1+y^2). \end{aligned}$$

25. If $y = \sec(kx)$, then $y' = k\sec(kx)\tan(kx)$ and

$$\begin{aligned} y'' &= k^2(\sec^2(kx)\tan^2(kx) + \sec^3(kx)) \\ &= k^2 y(2\sec^2(kx) - 1) = k^2 y(2y^2 - 1). \end{aligned}$$

26. To be proved: if $f(x) = \sin(ax+b)$, then

$$f^{(n)}(x) = \begin{cases} (-1)^k a^n \sin(ax+b) & \text{if } n = 2k \\ (-1)^k a^n \cos(ax+b) & \text{if } n = 2k+1 \end{cases}$$

for $k = 0, 1, 2, \ldots$ Proof: The formula works for $k = 0$ ($n = 2 \times 0 = 0$ and $n = 2 \times 0 + 1 = 1$):

$$\begin{cases} f^{(0)}(x) = f(x) = (-1)^0 a^0 \sin(ax+b) = \sin(ax+b) \\ f^{(1)}(x) = f'(x) = (-1)^0 a^1 \cos(ax+b) = a\cos(ax+b) \end{cases}$$

Now assume the formula holds for some $k \geq 0$.
If $n = 2(k+1)$, then

$$\begin{aligned} f^{(n)}(x) = \frac{d}{dx}f^{(n-1)}(x) &= \frac{d}{dx}f^{(2k+1)}(x) \\ &= \frac{d}{dx}\Big((-1)^k a^{2k+1}\cos(ax+b)\Big) \\ &= (-1)^{k+1}a^{2k+2}\sin(ax+b) \end{aligned}$$

and if $n = 2(k+1)+1 = 2k+3$, then

$$\begin{aligned} f^{(n)}(x) &= \frac{d}{dx}\Big((-1)^{k+1}a^{2k+2}\sin(ax+b) \\ &= (-1)^{k+1}a^{2k+3}\cos(ax+b). \end{aligned}$$

Thus the formula also holds for $k+1$. Therefore it holds for all positive integers k by induction.

27. If $y = \tan x$, then

$$y' = \sec^2 x = 1 + \tan^2 x = 1 + y^2 = P_2(y),$$

where P_2 is a polynomial of degree 2. Assume that $y^{(n)} = P_{n+1}(y)$ where P_{n+1} is a polynomial of degree $n+1$. The derivative of any polynomial is a polynomial of one lower degree, so

$$y^{(n+1)} = \frac{d}{dx}P_{n+1}(y) = P_n(y)\frac{dy}{dx} = P_n(y)(1+y^2) = P_{n+2}(y),$$

a polynomial of degree $n+2$. By induction, $(d/dx)^n \tan x = P_{n+1}(\tan x)$, a polynomial of degree $n+1$ in $\tan x$.

28. $$(fg)'' = (f'g + fg') = f''g + f'g' + f'g' + fg''$$
$$= f''g + 2f'g' + fg''$$

29. $$\begin{aligned}(fg)^{(3)} &= \frac{d}{dx}(fg)'' \\ &= \frac{d}{dx}[f''g + 2f'g' + fg''] \\ &= f^{(3)}g + f''g' + 2f''g' + 2f'g'' + f'g'' + fg^{(3)} \\ &= f^{(3)}g + 3f''g' + 3f'g'' + fg^{(3)}.\end{aligned}$$
$$\begin{aligned}(fg)^{(4)} &= \frac{d}{dx}(fg)^{(3)} \\ &= \frac{d}{dx}[f^{(3)}g + 3f''g' + 3f'g'' + fg^{(3)}] \\ &= f^{(4)}g + f^{(3)}g' + 3f^{(3)}g' + 3f''g'' + 3f''g'' \\ &\quad + 3f'g^{(3)} + f'g^{(3)} + fg^{(4)} \\ &= f^{(4)}g + 4f^{(3)}g' + 6f''g'' + 4f'g^{(3)} + fg^{(4)}.\end{aligned}$$
$$\begin{aligned}(fg)^{(n)} &= f^{(n)}g + nf(n-1)g' + \frac{n!}{2!(n-2)!}f^{(n-2)}g'' \\ &\quad + \frac{n!}{3!(n-3)!}f^{(n-3)}g^{(3)} + \cdots + nf'g^{(n-1)} + fg^{(n)} \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}f^{(n-k)}g^{(k)}.\end{aligned}$$

30. Let a, b, and c be three points in I where f vanishes; that is, $f(a) = f(b) = f(c) = 0$. Suppose $a < b < c$. By the Mean-Value Theorem, there exist points r in (a, b) and s in (b, c) such that $f'(r) = f'(s) = 0$. By the Mean-Value Theorem applied to f' on $[r, s]$, there is some point t in (r, s) (and therefore in I) such that $f''(t) = 0$.

31. If $f^{(n)}$ exists on interval I and f vanishes at $n + 1$ distinct points of I, then $f^{(n)}$ vanishes at at least one point of I.
Proof: True for $n = 2$ by Exercise 8.
Assume true for $n = k$. (Induction hypothesis)
Suppose $n = k + 1$, i.e., f vanishes at $k + 2$ points of I and $f^{(k+1)}$ exists.
By Exercise 7, f' vanishes at $k + 1$ points of I.
By the induction hypothesis, $f^{(k+1)} = (f')^{(k)}$ vanishes at a point of I so the statement is true for $n = k + 1$.
Therefore the statement is true for all $n \ge 2$ by induction.
(case $n = 1$ is just MVT.)

32. Given that $f(0) = f(1) = 0$ and $f(2) = 1$:

a) By MVT,
$$f'(a) = \frac{f(2) - f(0)}{2 - 0} = \frac{1 - 0}{2 - 0} = \frac{1}{2}$$
for some a in $(0, 2)$.

b) By MVT, for some r in $(0, 1)$,
$$f'(r) = \frac{f(1) - f(0)}{1 - 0} = \frac{0 - 0}{1 - 0} = 0.$$
Also, for some s in $(1, 2)$,
$$f'(s) = \frac{f(2) - f(1)}{2 - 1} = \frac{1 - 0}{2 - 1} = 1.$$
Then, by MVT applied to f' on the interval $[r, s]$, for some b in (r, s),
$$\begin{aligned}f''(b) &= \frac{f'(s) - f'(r)}{s - r} = \frac{1 - 0}{s - r} \\ &= \frac{1}{s - r} > \frac{1}{2}\end{aligned}$$
since $s - r < 2$.

c) Since $f''(x)$ exists on $[0, 2]$, therefore $f'(x)$ is continuous there. Since $f'(r) = 0$ and $f'(s) = 1$, and since $0 < \frac{1}{7} < 1$, the Intermediate-Value Theorem assures us that $f'(c) = \frac{1}{7}$ for some c between r and s.

Section 2.9 Implicit Differentiation (page 145)

1. $xy - x + 2y = 1$
Differentiate with respect to x:
$y + xy' - 1 + 2y' = 0$
Thus $y' = \dfrac{1 - y}{2 + x}$

2. $x^3 + y^3 = 1$
$3x^2 + 3y^2y' = 0$, so $y' = -\dfrac{x^2}{y^2}$.

3. $x^2 + xy = y^3$
Differentiate with respect to x:
$2x + y + xy' = 3y^2y'$
$y' = \dfrac{2x + y}{3y^2 - x}$

4. $x^3y + xy^5 = 2$
$3x^2y + x^3y' + y^5 + 5xy^4y' = 0$
$y' = \dfrac{-3x^2y - y^5}{x^3 + 5xy^4}$

5. $x^2y^3 = 2x - y$
$2xy^3 + 3x^2y^2y' = 2 - y'$
$y' = \dfrac{2 - 2xy^3}{3x^2y^2 + 1}$

6. $x^2 + 4(y - 1)^2 = 4$
$2x + 8(y - 1)y' = 0$, so $y' = \dfrac{x}{4(1 - y)}$

7. $\dfrac{x-y}{x+y} = \dfrac{x^2}{y} + 1 = \dfrac{x^2+y}{y}$
Thus $xy - y^2 = x^3 + x^2y + xy + y^2$, or $x^3 + x^2y + 2y^2 = 0$
Differentiate with respect to x:
$3x^2 + 2xy + x^2y' + 4yy' = 0$
$y' = -\dfrac{3x^2+2xy}{x^2+4y}$

8. $x\sqrt{x+y} = 8 - xy$
$\sqrt{x+y} + x\dfrac{1}{2\sqrt{x+y}}(1+y') = -y - xy'$
$2(x+y) + x(1+y') = -2\sqrt{x+y}(y+xy')$
$y' = -\dfrac{3x+2y+2y\sqrt{x+y}}{x+2x\sqrt{x+y}}$

9. $2x^2 + 3y^2 = 5$
$4x + 6yy' = 0$
At $(1, 1)$: $4 + 6y' = 0$, $y' = -\dfrac{2}{3}$
Tangent line: $y - 1 = -\dfrac{2}{3}(x-1)$ or $2x + 3y = 5$

10. $x^2y^3 - x^3y^2 = 12$
$2xy^3 + 3x^2y^2y' - 3x^2y^2 - 2x^3yy' = 0$
At $(-1, 2)$: $-16 + 12y' - 12 + 4y' = 0$, so the slope is
$y' = \dfrac{12+16}{12+4} = \dfrac{28}{16} = \dfrac{7}{4}$.
Thus, the equation of the tangent line is
$y = 2 + \frac{7}{4}(x+1)$, or $7x - 4y + 15 = 0$.

11. $\dfrac{x}{y} + \left(\dfrac{y}{x}\right)^3 = 2$
$x^4 + y^4 = 2x^3y$
$4x^3 + 4y^3y' = 6x^2y + 2x^3y'$
at $(-1, -1)$: $-4 - 4y' = -6 - 2y'$
$2y' = 2$, $y' = 1$
Tangent line: $y + 1 = 1(x+1)$ or $y = x$.

12. $x + 2y + 1 = \dfrac{y^2}{x-1}$
$1 + 2y' = \dfrac{(x-1)2yy' - y^2(1)}{(x-1)^2}$
At $(2, -1)$ we have $1 + 2y' = -2y' - 1$ so $y' = -\frac{1}{2}$.
Thus, the equation of the tangent is
$y = -1 - \frac{1}{2}(x-2)$, or $x + 2y = 0$.

13. $2x + y - \sqrt{2}\sin(xy) = \pi/2$
$2 + y' - \sqrt{2}\cos(xy)(y + xy') = 0$
At $(\pi/4, 1)$: $2 + y' - (1 + (\pi/4)y') = 0$, so
$y' = -4/(4-\pi)$. The tangent has equation

$$y = 1 - \frac{4}{4-\pi}\left(x - \frac{\pi}{4}\right).$$

14. $\tan(xy^2) = (2/\pi)xy$
$(\sec^2(xy^2))(y^2 + 2xyy') = (2/\pi)(y + xy')$.
At $(-\pi, 1/2)$: $2((1/4) - \pi y') = (1/\pi) - 2y'$, so
$y' = (\pi - 2)/(4\pi(\pi-1))$. The tangent has equation

$$y = \frac{1}{2} + \frac{\pi - 2}{4\pi(\pi-1)}(x + \pi).$$

15. $x\sin(xy - y^2) = x^2 - 1$
$\sin(xy - y^2) + x(\cos(xy - y^2))(y + xy' - 2yy') = 2x$.
At $(1, 1)$: $0 + (1)(1)(1 - y') = 2$, so $y' = -1$. The tangent has equation $y = 1 - (x-1)$, or $y = 2 - x$.

16. $\cos\left(\dfrac{\pi y}{x}\right) = \dfrac{x^2}{y} - \dfrac{17}{2}$
$\left[-\sin\left(\dfrac{\pi y}{x}\right)\right]\dfrac{\pi(xy' - y)}{x^2} = \dfrac{2xy - x^2y'}{y^2}$.
At $(3, 1)$: $-\dfrac{\sqrt{3}}{2}\,\dfrac{\pi(3y'-1)}{9} = 6 - 9y'$,
so $y' = (108 - \sqrt{3}\pi)/(162 - 3\sqrt{3}\pi)$. The tangent has equation

$$y = 1 + \frac{108 - \sqrt{3}\pi}{162 - 3\sqrt{3}\pi}(x - 3).$$

17. $xy = x + y$
$y + xy' = 1 + y' \Rightarrow y' = \dfrac{y-1}{1-x}$
$y' + y' + xy'' = y''$
Therefore, $y'' = \dfrac{2y'}{1-x} = \dfrac{2(y-1)}{(1-x)^2}$

18. $x^2 + 4y^2 = 4$, $\quad 2x + 8yy' = 0$, $\quad 2 + 8(y')^2 + 8yy'' = 0$.
Thus, $y' = \dfrac{-x}{4y}$ and

$$y'' = \frac{-2 - 8(y')^2}{8y} = -\frac{1}{4y} - \frac{x^2}{16y^3} = \frac{-4y^2 - x^2}{16y^3} = -\frac{1}{4y^3}.$$

19. $x^3 - y^2 + y^3 = x$
$3x^2 - 2yy' + 3y^2y' = 1 \Rightarrow y' = \dfrac{1-3x^2}{3y^2-2y}$
$6x - 2(y')^2 - 2yy'' + 6y(y')^2 + 3y^2y'' = 0$

$$y'' = \frac{(2-6y)(y')^2 - 6x}{3y^2 - 2y} = \frac{(2-6y)\dfrac{(1-3x^2)^2}{(3y^2-2y)^2} - 6x}{3y^2-2y}$$
$$= \frac{(2-6y)(1-3x^2)^2}{(3y^2-2y)^3} - \frac{6x}{3y^2-2y}$$

20. $x^3 - 3xy + y^3 = 1$
$3x^2 - 3y - 3xy' + 3y^2y' = 0$
$6x - 3y' - 3y' - 3xy'' + 6y(y')^2 + 3y^2y'' = 0$
Thus

$$\begin{aligned} y' &= \frac{y - x^2}{y^2 - x} \\ y'' &= \frac{-2x + 2y' - 2y(y')^2}{y^2 - x} \\ &= \frac{2}{y^2 - x}\left[-x + \left(\frac{y - x^2}{y^2 - x}\right) - y\left(\frac{y - x^2}{y^2 - x}\right)^2\right] \\ &= \frac{2}{y^2 - x}\left[\frac{-2xy}{(y^2 - x)^2}\right] = \frac{4xy}{(x - y^2)^3}. \end{aligned}$$

21. $x^2 + y^2 = a^2$
$2x + 2yy' = 0$ so $x + yy' = 0$ and $y' = -\dfrac{x}{y}$
$1 + y'y' + yy'' = 0$ so

$$\begin{aligned} y'' &= -\frac{1 + (y')^2}{y} = -\frac{1 + \dfrac{x^2}{y^2}}{y} \\ &= -\frac{y^2 + x^2}{y^3} = -\frac{a^2}{y^3} \end{aligned}$$

22. $Ax^2 + By^2 = C$
$2Ax + 2Byy' = 0 \Rightarrow y' = -\dfrac{Ax}{By}$
$2A + 2B(y')^2 + 2Byy'' = 0.$
Thus,

$$\begin{aligned} y'' = \frac{-A - B(y')^2}{By} &= \frac{-A - B\left(\dfrac{Ax}{By}\right)^2}{By} \\ &= \frac{-A(By^2 + Ax^2)}{B^2y^3} = -\frac{AC}{B^2y^3}. \end{aligned}$$

23. Maple gives 0 for the value.

24. Maple gives the slope as $\dfrac{206}{55}$.

25. Maple gives the value -26.

26. Maple gives the value $-\dfrac{855,000}{371,293}$.

27. Ellipse: $x^2 + 2y^2 = 2$
$2x + 4yy' = 0$
Slope of ellipse: $y'_E = -\dfrac{x}{2y}$
Hyperbola: $2x^2 - 2y^2 = 1$
$4x - 4yy' = 0$
Slope of hyperbola: $y'_H = \dfrac{x}{y}$
At intersection points $\begin{cases} x^2 + 2y^2 = 2 \\ 2x^2 - 2y^2 = 1 \end{cases}$
$3x^2 = 3$ so $x^2 = 1$, $y^2 = \dfrac{1}{2}$
Thus $y'_E y'_H = -\dfrac{x}{2y}\dfrac{x}{y} = -\dfrac{x^2}{2y^2} = -1$
Therefore the curves intersect at right angles.

28. The slope of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is found from

$$\frac{2x}{a^2} + \frac{2y}{b^2}y' = 0, \quad \text{i.e. } y' = -\frac{b^2x}{a^2y}.$$

Similarly, the slope of the hyperbola $\dfrac{x^2}{A^2} - \dfrac{y^2}{B^2} = 1$ at (x, y) satisfies

$$\frac{2x}{A^2} - \frac{2y}{B^2}y' = 0, \quad \text{or } y' = \frac{B^2x}{A^2y}.$$

If the point (x, y) is an intersection of the two curves, then

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{x^2}{A^2} - \frac{y^2}{B^2} \\ x^2\left(\frac{1}{A^2} - \frac{1}{a^2}\right) &= y^2\left(\frac{1}{B^2} + \frac{1}{b^2}\right). \end{aligned}$$

Thus, $\dfrac{x^2}{y^2} = \dfrac{b^2 + B^2}{B^2b^2} \cdot \dfrac{A^2a^2}{a^2 - A^2}$.
Since $a^2 - b^2 = A^2 + B^2$, therefore $B^2 + b^2 = a^2 - A^2$, and $\dfrac{x^2}{y^2} = \dfrac{A^2a^2}{B^2b^2}$. Thus, the product of the slope of the two curves at (x, y) is

$$-\frac{b^2x}{a^2y} \cdot \frac{B^2x}{A^2y} = -\frac{b^2B^2}{a^2A^2} \cdot \frac{A^2a^2}{B^2b^2} = -1.$$

Therefore, the curves intersect at right angles.

29. If $z = \tan(x/2)$, then

$$1 = \sec^2(x/2)\,\frac{1}{2}\frac{dx}{dz} = \frac{1 + \tan^2(x/2)}{2}\frac{dx}{dz} = \frac{1 + z^2}{2}\frac{dx}{dz}.$$

Thus $dx/dz = 2/(1 + z^2)$. Also

$$\begin{aligned} \cos x &= 2\cos^2(x/2) - 1 = \frac{2}{\sec^2(x/2)} - 1 \\ &= \frac{2}{1 + z^2} - 1 = \frac{1 - z^2}{1 + z^2} \\ \sin x &= 2\sin(x/2)\cos(x/2) = \frac{2\tan(x/2)}{1 + \tan^2(x/2)} = \frac{2z}{1 + z^2}. \end{aligned}$$

30. $\frac{x-y}{x+y} = \frac{x}{y} + 1 \Leftrightarrow xy - y^2 = x^2 + xy + xy + y^2$
$\Leftrightarrow x^2 + 2y^2 + xy = 0$
Differentiate with respect to x:

$$2x + 4yy' + y + xy' = 0 \quad \Rightarrow \quad y' = -\frac{2x+y}{4y+x}.$$

However, since $x^2 + 2y^2 + xy = 0$ can be written

$$x + xy + \frac{1}{4}y^2 + \frac{7}{4}y^2 = 0, \text{ or } (x + \frac{y}{2})^2 + \frac{7}{4}y^2 = 0,$$

the only solution is $x = 0$, $y = 0$, and these values do not satisfy the original equation. There are no points on the given curve.

Section 2.10 Antiderivatives and Initial-Value Problems (page 151)

1. $\int 5\,dx = 5x + C$

2. $\int x^2\,dx = \frac{1}{3}x^3 + C$

3. $\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

4. $\int x^{12}\,dx = \frac{1}{13}x^{13} + C$

5. $\int x^3\,dx = \frac{1}{4}x^4 + C$

6. $\int (x + \cos x)\,dx = \frac{x^2}{2} + \sin x + C$

7. $\int \tan x \cos x\,dx = \int \sin x\,dx = -\cos x + C$

8. $\int \frac{1 + \cos^3 x}{\cos^2 x}\,dx = \int (\sec^2 x + \cos x)\,dx = \tan x + \sin x + C$

9. $\int (a^2 - x^2)\,dx = a^2 x - \frac{1}{3}x^3 + C$

10. $\int (A + Bx + Cx^2)\,dx = Ax + \frac{B}{2}x^2 + \frac{C}{3}x^3 + K$

11. $\int (2x^{1/2} + 3x^{1/3}\,dx = \frac{4}{3}x^{3/2} + \frac{9}{4}x^{4/3} + C$

12. $\int \frac{6(x-1)}{x^{4/3}}\,dx = \int (6x^{-1/3} - 6x^{-4/3})\,dx$
$= 9x^{2/3} + 18x^{-1/3} + C$

13. $\int \left(\frac{x^3}{3} - \frac{x^2}{2} + x - 1\right)\,dx = \frac{1}{12}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2 - x + C$

14. $105 \int (1 + t^2 + t^4 + t^6)\,dt$
$= 105(t + \frac{1}{3}t^3 + \frac{1}{5}t^5 + \frac{1}{7}t^7) + C$
$= 105t + 35t^3 + 21t^5 + 15t^7 + C$

15. $\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C$

16. $\int \sin\left(\frac{x}{2}\right)\,dx = -2\cos\left(\frac{x}{2}\right) + C$

17. $\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x} + C$

18. $\int \sec(1-x)\tan(1-x)\,dx = -\sec(1-x) + C$

19. $\int \sqrt{2x+3}\,dx = \frac{1}{3}(2x+3)^{3/2} + C$

20. Since $\frac{d}{dx}\sqrt{x+1} = \frac{1}{2\sqrt{x+1}}$, therefore

$$\int \frac{4}{\sqrt{x+1}}\,dx = 8\sqrt{x+1} + C.$$

21. $\int 2x \sin(x^2)\,dx = -\cos(x^2) + C$

22. Since $\frac{d}{dx}\sqrt{x^2+1} = \frac{x}{\sqrt{x^2+1}}$, therefore

$$\int \frac{2x}{\sqrt{x^2+1}}\,dx = 2\sqrt{x^2+1} + C.$$

23. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

24. $\int \sin x \cos x\,dx = \int \frac{1}{2}\sin(2x)\,dx = -\frac{1}{4}\cos(2x) + C$

25. $\int \cos^2 x\,dx = \int \frac{1 + \cos(2x)}{2}\,dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$

26. $\int \sin^2 x\,dx = \int \frac{1 - \cos(2x)}{2}\,dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C$

27. $\begin{cases} y' = x - 2 & \Rightarrow y = \frac{1}{2}x^2 - 2x + C \\ y(0) = 3 & \Rightarrow 3 = 0 + C \text{ therefore } C = 3 \end{cases}$
Thus $y = \frac{1}{2}x^2 - 2x + 3$ for all x.

28. Given that

$$\begin{cases} y' = x^{-2} - x^{-3} \\ y(-1) = 0, \end{cases}$$

then $y = \int (x^{-2} - x^{-3})\,dx = -x^{-1} + \frac{1}{2}x^{-2} + C$
and $0 = y(-1) = -(-1)^{-1} + \frac{1}{2}(-1)^{-2} + C$ so $C = -\frac{3}{2}$.
Hence, $y(x) = -\dfrac{1}{x} + \dfrac{1}{2x^2} - \dfrac{3}{2}$ which is valid on the interval $(-\infty, 0)$.

29. $\begin{cases} y' = 3\sqrt{x} & \Rightarrow \quad y = 2x^{3/2} + C \\ y(4) = 1 & \Rightarrow \quad 1 = 16 + C \text{ so } C = -15 \end{cases}$
Thus $y = 2x^{3/2} - 15$ for $x > 0$.

30. Given that
$$\begin{cases} y' = x^{1/3} \\ y(0) = 5, \end{cases}$$
then $y = \int x^{1/3}\,dx = \frac{3}{4}x^{4/3} + C$ and $5 = y(0) = C$.
Hence, $y(x) = \frac{3}{4}x^{4/3} + 5$ which is valid on the whole real line.

31. Since $y' = Ax^2 + Bx + C$ we have
$y = \dfrac{A}{3}x^3 + \dfrac{B}{2}x^2 + Cx + D$. Since $y(1) = 1$, therefore
$1 = y(1) = \dfrac{A}{3} + \dfrac{B}{2} + C + D$. Thus $D = 1 - \dfrac{A}{3} - \dfrac{B}{2} - C$, and
$y = \dfrac{A}{3}(x^3 - 1) + \dfrac{B}{2}(x^2 - 1) + C(x - 1) + 1$ for all x

32. Given that
$$\begin{cases} y' = x^{-9/7} \\ y(1) = -4, \end{cases}$$
then $y = \int x^{-9/7}\,dx = -\frac{7}{2}x^{-2/7} + C$.
Also, $-4 = y(1) = -\frac{7}{2} + C$, so $C = -\frac{1}{2}$. Hence,
$y = -\frac{7}{2}x^{-2/7} - \frac{1}{2}$, which is valid in the interval $(0, \infty)$.

33. For $\begin{cases} y' = \cos x \\ y(\pi/6) = 2 \end{cases}$, we have
$$\begin{aligned} y &= \int \cos x\,dx = \sin x + C \\ 2 &= \sin\frac{\pi}{6} + C = \frac{1}{2} + C \quad \Longrightarrow \quad C = \frac{3}{2} \\ y &= \sin x + \frac{3}{2} \qquad \text{(for all } x\text{)}. \end{aligned}$$

34. For $\begin{cases} y' = \sin(2x) \\ y(\pi/2) = 1 \end{cases}$, we have
$$\begin{aligned} y &= \int \sin(2x)\,dx = -\frac{1}{2}\cos(2x) + C \\ 1 &= -\frac{1}{2}\cos\pi + C = \frac{1}{2} + C \quad \Longrightarrow \quad C = \frac{1}{2} \\ y &= \frac{1}{2}\Big(1 - \cos(2x)\Big) \qquad \text{(for all } x\text{)}. \end{aligned}$$

35. For $\begin{cases} y' = \sec^2 x \\ y(0) = 1 \end{cases}$, we have
$$\begin{aligned} y &= \int \sec^2 x\,dx = \tan x + C \\ 1 &= \tan 0 + C = C \quad \Longrightarrow \quad C = 1 \\ y &= \tan x + 1 \qquad \text{(for } -\pi/2 < x < \pi/2\text{)}. \end{aligned}$$

36. For $\begin{cases} y' = \sec^2 x \\ y(\pi) = 1 \end{cases}$, we have
$$\begin{aligned} y &= \int \sec^2 x\,dx = \tan x + C \\ 1 &= \tan\pi + C = C \quad \Longrightarrow \quad C = 1 \\ y &= \tan x + 1 \qquad \text{(for } \pi/2 < x < 3\pi/2\text{)}. \end{aligned}$$

37. Since $y'' = 2$, therefore $y' = 2x + C_1$.
Since $y'(0) = 5$, therefore $5 = 0 + C_1$, and $y' = 2x + 5$.
Thus $y = x^2 + 5x + C_2$.
Since $y(0) = -3$, therefore $-3 = 0 + 0 + C_2$, and $C_2 = -3$.
Finally, $y = x^2 + 5x - 3$, for all x.

38. Given that
$$\begin{cases} y'' = x^{-4} \\ y'(1) = 2 \\ y(1) = 1, \end{cases}$$
then $y' = \int x^{-4}\,dx = -\frac{1}{3}x^{-3} + C$.
Since $2 = y'(1) = -\frac{1}{3} + C$, therefore $C = \frac{7}{3}$,
and $y' = -\frac{1}{3}x^{-3} + \frac{7}{3}$. Thus
$$y = \int \left(-\tfrac{1}{3}x^{-3} + \tfrac{7}{3}\right) dx = \tfrac{1}{6}x^{-2} + \tfrac{7}{3}x + D,$$
and $1 = y(1) = \frac{1}{6} + \frac{7}{3} + D$, so that $D = -\frac{3}{2}$. Hence, $y(x) = \frac{1}{6}x^{-2} + \frac{7}{3}x - \frac{3}{2}$, which is valid in the interval $(0, \infty)$.

39. Since $y'' = x^3 - 1$, therefore $y' = \dfrac{1}{4}x^4 - x + C_1$.
Since $y'(0) = 0$, therefore $0 = 0 - 0 + C_1$, and
$y' = \dfrac{1}{4}x^4 - x$.
Thus $y = \dfrac{1}{20}x^5 - \dfrac{1}{2}x^2 + C_2$.
Since $y(0) = 8$, we have $8 = 0 - 0 + C_2$.
Hence $y = \dfrac{1}{20}x^5 - \dfrac{1}{2}x^2 + 8$ for all x.

40. Given that
$$\begin{cases} y'' = 5x^2 - 3x^{-1/2} \\ y'(1) = 2 \\ y(1) = 0, \end{cases}$$

we have $y' = \int 5x^2 - 3x^{-1/2}\,dx = \frac{5}{3}x^3 - 6x^{1/2} + C$.
Also, $2 = y'(1) = \frac{5}{3} - 6 + C$ so that $C = \dfrac{19}{3}$. Thus, $y' = \frac{5}{3}x^3 - 6x^{1/2} + \frac{19}{3}$, and

$$y = \int \left(\tfrac{5}{3}x^3 - 6x^{1/2} + \tfrac{19}{3}\right) dx = \tfrac{5}{12}x^4 - 4x^{3/2} + \tfrac{19}{3}x + D.$$

Finally, $0 = y(1) = \frac{5}{12} - 4 + \frac{19}{3} + D$ so that $D = -\frac{11}{4}$. Hence, $y(x) = \frac{5}{12}x^4 - 4x^{3/2} + \frac{19}{3}x - \frac{11}{4}$.

41. For $\begin{cases} y'' = \cos x \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$ we have

$$\begin{aligned} y' &= \int \cos x\,dx = \sin x + C_1 \\ 1 &= \sin 0 + \mathbb{C}_1 \quad \Longrightarrow \quad C_1 = 1 \\ y &= \int (\sin x + 1)\,dx = -\cos x + x + C_2 \\ 0 &= -\cos 0 + 0 + C_2 \quad \Longrightarrow \quad C_2 = 1 \\ y &= 1 + x - \cos x. \end{aligned}$$

42. For $\begin{cases} y'' = x + \sin x \\ y(0) = 2 \\ y'(0) = 0 \end{cases}$ we have

$$\begin{aligned} y' &= \int (x + \sin x)\,dx = \frac{x^2}{2} - \cos x + C_1 \\ 0 &= 0 - \cos 0 + \mathbb{C}_1 \quad \Longrightarrow \quad C_1 = 1 \\ y &= \int \left(\frac{x^2}{2} - \cos x + 1\right) dx = \frac{x^3}{6} - \sin x + x + C_2 \\ 2 &= 0 - \sin 0 + 0 + C_2 \quad \Longrightarrow \quad C_2 = 2 \\ y &= \frac{x^3}{6} - \sin x + x + 2. \end{aligned}$$

43. Let $y = Ax + \dfrac{B}{x}$. Then $y' = A - \dfrac{B}{x^2}$, and $y'' = \dfrac{2B}{x^3}$. Thus, for all $x \neq 0$,

$$x^2y'' + xy' - y = \frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} = 0.$$

We will also have $y(1) = 2$ and $y'(1) = 4$ provided

$$A + B = 2, \quad \text{and} \quad A - B = 4.$$

These equations have solution $A = 3$, $B = -1$, so the initial value problem has solution $y = 3x - (1/x)$.

44. Let r_1 and r_2 be distinct rational roots of the equation
$ar(r-1) + br + c = 0$
Let $y = Ax^{r_1} + Bx^{r_2} \qquad (x > 0)$
Then $y' = Ar_1x^{r_1-1} + Br_2x^{r_2-1}$,
and $y'' = Ar_1(r_1-1)x^{r_1-2} + Br_2(r_2-1)x^{r_2-2}$. Thus
$ax^2y'' + bxy' + cy$

$$\begin{aligned} &= ax^2(Ar_1(r_1-1)x^{r_1-2} + Br_2(r_2-1)x^{r_2-2} \\ &\quad + bx(Ar_1x^{r_1-1} + Br_2x^{r_2-1}) + c(Ax^{r_1} + Bx^{r_2}) \\ &= A\Big(ar_1(r_1-1) + br_1 + c\Big)x^{r_1} \\ &\quad + B(ar_2(r_2-1) + br_2 + c\Big)x^{r_2} \\ &= 0x^{r_1} + 0x^{r_2} \equiv 0 \quad (x > 0) \end{aligned}$$

45. $\begin{cases} 4x^2y'' + 4xy' - y = 0 \quad (*) \quad \Rightarrow a = 4, b = 4, c = -1 \\ y(4) = 2 \\ y'(4) = -2 \end{cases}$

Auxilary Equation: $4r(r-1) + 4r - 1 = 0$

$$4r^2 - 1 = 0$$

$$r = \pm\frac{1}{2}$$

By #31, $y = Ax^{1/2} + Bx^{-1/2}$ solves $(*)$ for $x > 0$.
Now $y' = \dfrac{A}{2}x^{-1/2} - \dfrac{B}{2}x^{-3/2}$
Substitute the initial conditions:

$$\begin{aligned} 2 &= 2A + \frac{B}{2} \quad &\Rightarrow 1 = A + \frac{B}{4} \\ -2 &= \frac{A}{4} - \frac{B}{16} \quad &\Rightarrow -8 = A - \frac{B}{4}. \end{aligned}$$

Hence $9 = \dfrac{B}{2}$, so $B = 18$, $A = -\dfrac{7}{2}$.
Thus $y = -\dfrac{7}{2}x^{1/2} + 18x^{-1/2}$ (for $x > 0$).

46. Consider

$$\begin{cases} x^2y'' - 6y = 0 \\ y(1) = 1 \\ y'(1) = 1. \end{cases}$$

Let $y = x^r$, $y' = rx^{r-1}$, $y'' = r(r-1)x^{r-2}$. Substituting these expressions into the differential equation we obtain

$$\begin{aligned} x^2[r(r-1)x^{r-2}] - 6x^r &= 0 \\ [r(r-1) - 6]x^r &= 0. \end{aligned}$$

Since this equation must hold for all $x > 0$, we must have

$$\begin{aligned} r(r-1) - 6 &= 0 \\ r^2 - r - 6 &= 0 \\ (r-3)(r+2) &= 0. \end{aligned}$$

There are two roots: $r_1 = -2$, and $r_2 = 3$. Thus the differential equation has solutions of the form
$y = Ax^{-2} + Bx^3$. Then $y' = -2Ax^{-3} + 3Bx^2$. Since $1 = y(1) = A + B$ and $1 = y'(1) = -2A + 3B$, therefore $A = \frac{2}{5}$ and $B = \frac{3}{5}$. Hence, $y = \frac{2}{5}x^{-2} + \frac{3}{5}x^3$.

Section 2.11 Velocity and Acceleration (page 157)

1. $x = t^2 - 4t + 3$, $v = \dfrac{dx}{dt} = 2t - 4$, $a = \dfrac{dv}{dt} = 2$

a) particle is moving: to the right for $t > 2$

b) to the left for $t < 2$

c) particle is always accelerating to the right

d) never accelerating to the left

e) particle is speeding up for $t > 2$

f) slowing down for $t < 2$

g) the acceleration is 2 at all times

h) average velocity over $0 \le t \le 4$ is

$$\frac{x(4) - x(0)}{4 - 0} = \frac{16 - 16 + 3 - 3}{4} = 0$$

2. $x = 4 + 5t - t^2$, $v = 5 - 2t$, $a = -2$.

a) The point is moving to the right if $v > 0$, i.e., when $t < \frac{5}{2}$.

b) The point is moving to the left if $v < 0$, i.e., when $t > \frac{5}{2}$.

c) The point is accelerating to the right if $a > 0$, but $a = -2$ at all t; hence, the point never accelerates to the right.

d) The point is accelerating to the left if $a < 0$, i.e., for all t.

e) The particle is speeding up if v and a have the same sign, i.e., for $t > \frac{5}{2}$.

f) The particle is slowing down if v and a have opposite sign, i.e., for $t < \frac{5}{2}$.

g) Since $a = -2$ at all t, $a = -2$ at $t = \frac{5}{2}$ when $v = 0$.

h) The average velocity over $[0, 4]$ is $\dfrac{x(4) - x(0)}{4} = \dfrac{8 - 4}{4} = 1$.

3. $x = t^3 - 4t + 1$, $v = \dfrac{dx}{dt} = 3t^2 - 4$, $a = \dfrac{dv}{dt} = 6t$

a) particle moving: to the right for $t < -2/\sqrt{3}$ or $t > 2/\sqrt{3}$,

b) to the left for $-2/\sqrt{3} < t < 2/\sqrt{3}$

c) particle is accelerating: to the right for $t > 0$

d) to the left for $t < 0$

e) particle is speeding up for $t > 2/\sqrt{3}$ or for $-2/\sqrt{3} < t < 0$

f) particle is slowing down for $t < -2/\sqrt{3}$ or for $0 < t < 2/\sqrt{3}$

g) velocity is zero at $t = \pm 2/\sqrt{3}$. Acceleration at these times is $\pm 12/\sqrt{3}$.

h) average velocity on $[0, 4]$ is $\dfrac{4^3 - 4 \times 4 + 1 - 1}{4 - 0} = 12$

4. $x = \dfrac{t}{t^2 + 1}$, $\quad v = \dfrac{(t^2 + 1)(1) - (t)(2t)}{(t^2 + 1)^2} = \dfrac{1 - t^2}{(t^2 + 1)^2}$,

$$a = \frac{(t^2 + 1)^2(-2t) - (1 - t^2)(2)(t^2 + 1)(2t)}{(t^2 + 1)^4} = \frac{2t(t^2 - 3)}{(t^2 + 1)^3}.$$

a) The point is moving to the right if $v > 0$, i.e., when $1 - t^2 > 0$, or $-1 < t < 1$.

b) The point is moving to the left if $v < 0$, i.e., when $t < -1$ or $t > 1$.

c) The point is accelerating to the right if $a > 0$, i.e., when $2t(t^2 - 3) > 0$, that is, when $t > \sqrt{3}$ or $-\sqrt{3} < t < 0$.

d) The point is accelerating to the left if $a < 0$, i.e., for $t < -\sqrt{3}$ or $0 < t < \sqrt{3}$.

e) The particle is speeding up if v and a have the same sign, i.e., for $t < -\sqrt{3}$, or $-1 < t < 0$ or $1 < t < \sqrt{3}$.

f) The particle is slowing down if v and a have opposite sign, i.e., for $-\sqrt{3} < t < -1$, or $0 < t < 1$ or $t > \sqrt{3}$.

g) $v = 0$ at $t = \pm 1$. At $t = -1$, $a = \dfrac{-2(-2)}{(2)^3} = \dfrac{1}{2}$.

At $t = 1$, $a = \dfrac{2(-2)}{(2)^3} = -\dfrac{1}{2}$.

h) The average velocity over $[0, 4]$ is $\dfrac{x(4) - x(0)}{4} = \dfrac{\frac{4}{17} - 0}{4} = \dfrac{1}{17}$.

5. $y = 9.8t - 4.9t^2$ metres (t in seconds)
velocity $v = \dfrac{dy}{dt} = 9.8 - 9.8t$
acceleration $a = \dfrac{dv}{dt} = -9.8$
The acceleration is 9.8 m/s^2 downward at all times.
Ball is at maximum height when $v = 0$, i.e., at $t = 1$.
Thus maximum height is $y\Big|_{t=1} = 9.8 - 4.9 = 4.9$ metres.
Ball strikes the ground when $y = 0$, $(t > 0)$, i.e., $0 = t(9.8 - 4.9t)$ so $t = 2$.
Velocity at $t = 2$ is $9.8 - 9.8(2) = -9.8$ m/s.
Ball strikes the ground travelling at 9.8 m/s (downward).

6. Given that $y = 100 - 2t - 4.9t^2$, the time t at which the ball reaches the ground is the positive root of the equation $y = 0$, i.e., $100 - 2t - 4.9t^2 = 0$, namely,

$$t = \frac{-2 + \sqrt{4 + 4(4.9)(100)}}{9.8} \approx 4.318 \text{ s.}$$

The average velocity of the ball is $\dfrac{-100}{4.318} = -23.16$ m/s.
Since $-23.159 = v = -2 - 9.8t$, then $t \simeq 2.159$ s.

7. $D = t^2$, D in metres, t in seconds
velocity $v = \dfrac{dD}{dt} = 2t$
Aircraft becomes airborne if
$v = 200 \text{ km/h} = \dfrac{200,000}{3600} = \dfrac{500}{9}$ m/s.
Time for aircraft to become airborne is $t = \dfrac{250}{9}$ s, that is, about 27.8 s.
Distance travelled during takeoff run is $t^2 \approx 771.6$ metres.

8. Let $y(t)$ be the height of the projectile t seconds after it is fired upward from ground level with initial speed v_0. Then

$$y''(t) = -9.8, \quad y'(0) = v_0, \quad y(0) = 0.$$

Two antidifferentiations give

$$y = -4.9t^2 + v_0 t = t(v_0 - 4.9t).$$

Since the projectile returns to the ground at $t = 10$ s, we have $y(10) = 0$, so $v_0 = 49$ m/s. On Mars, the acceleration of gravity is 3.72 m/s^2 rather than 9.8 m/s^2, so the height of the projectile would be

$$y = -1.86t^2 + v_0 t = t(49 - 1.86t).$$

The time taken to fall back to ground level on Mars would be $t = 49/1.86 \approx 26.3$ s.

9. The height of the ball after t seconds is $y(t) = -(g/2)t^2 + v_0 t$ m if its initial speed was v_0 m/s. Maximum height h occurs when $dy/dt = 0$, that is, at $t = v_0/g$. Hence

$$h = -\frac{g}{2} \cdot \frac{v_0^2}{g^2} + v_0 \cdot \frac{v_0}{g} = \frac{v_0^2}{2g}.$$

An initial speed of $2v_0$ means the maximum height will be $4v_0^2/2g = 4h$. To get a maximum height of $2h$ an initial speed of $\sqrt{2}v_0$ is required.

10. To get to $3h$ metres above Mars, the ball would have to be thrown upward with speed

$$v_M = \sqrt{6g_M h} = \sqrt{6g_M v_0^2/(2g)} = v_0\sqrt{3g_M/g}.$$

Since $g_M = 3.72$ and $g = 9.80$, we have $v_M \approx 1.067v_0$ m/s.

11. If the cliff is h ft high, then the height of the rock t seconds after it falls is $y = h - 16t^2$ ft. The rock hits the ground $(y = 0)$ at time $t = \sqrt{h/16}$ s. Its speed at that time is $v = -32t = -8\sqrt{h} = -160$ ft/s. Thus $\sqrt{h} = 20$, and the cliff is $h = 400$ ft high.

12. If the cliff is h ft high, then the height of the rock t seconds after it is thrown down is $y = h - 32t - 16t^2$ ft. The rock hits the ground $(y = 0)$ at time

$$t = \frac{-32 + \sqrt{32^2 + 64h}}{32} = -1 + \frac{1}{4}\sqrt{16 + h} \text{ s.}$$

Its speed at that time is

$$v = -32 - 32t = -8\sqrt{16 + h} = -160 \text{ ft/s.}$$

Solving this equation for h gives the height of the cliff as 384 ft.

13. Let $x(t)$ be the distance travelled by the train in the t seconds after the brakes are applied. Since $d^2x/dt^2 = -1/6$ m/s^2 and since the initial speed is $v_0 = 60$ km/h $= 100/6$ m/s, we have

$$x(t) = -\frac{1}{12}t^2 + \frac{100}{6}t.$$

The speed of the train at time t is $v(t) = -(t/6) + (100/6)$ m/s, so it takes the train 100 s to come to a stop. In that time it travels $x(100) = -100^2/12 + 100^2/6 = 100^2/12 \approx 833$ metres.

14. $x = At^2 + Bt + C$, $v = 2At + B$.
The average velocity over $[t_1, t_2]$ is
$$\frac{x(t_2) - x(t_1)}{t_2 - t_1}$$
$$= \frac{At_2^2 + Bt_1 + C - At_1^2 - Bt_1 - C}{t_2 - t_1}$$
$$= \frac{A(t_2^2 - t_1^2) + B(t_2 - t_1)}{(t_2 - t_1)}$$
$$= \frac{A(t_2 + t_1)(t_2 - t_1) + B(t_2 - t_1)}{(t_2 - t_1)}$$
$= A(t_2 + t_1) + B$.
The instantaneous velocity at the midpoint of $[t_1, t_2]$ is
$$v\left(\frac{t_2 + t_1}{2}\right) = 2A\left(\frac{t_2 + t_1}{2}\right) + B = A(t_2 + t_1) + B.$$
Hence, the average velocity over the interval is equal to the instantaneous velocity at the midpoint.

15. $$s = \begin{cases} t^2 & 0 \le t \le 2 \\ 4t - 4 & 2 < t < 8 \\ -68 + 20t - t^2 & 8 \le t \le 10 \end{cases}$$
Note: s is continuous at 2 and 8 since $2^2 = 4(2) - 4$ and $4(8) - 4 = -68 + 160 - 64$
$$\text{velocity } v = \frac{ds}{dt} = \begin{cases} 2t & \text{if } 0 < t < 2 \\ 4 & \text{if } 2 < t < 8 \\ 20 - 2t & \text{if } 8 < t < 10 \end{cases}$$
Since $2t \to 4$ as $t \to 2-$, therefore, v is continuous at 2 $((v(2) = 4)$.
Since $20 - 2t \to 4$ as $t \to 8+$, therefore v is continuous at 8 $(v(8) = 4)$. Hence the velocity is continuous for $0 < t < 10$
$$\text{acceleration } a = \frac{dv}{dt} = \begin{cases} 2 & \text{if } 0 < t < 2 \\ 0 & \text{if } 2 < t < 8 \\ -2 & \text{if } 8 < t < 10 \end{cases}$$
is discontinuous at $t = 2$ and $t = 8$
Maximum velocity is 4 and is attained on the interval $2 \le t \le 8$.

16. This exercise and the next three refer to the following figure depicting the velocity of a rocket fired from a tower as a function of time since firing.

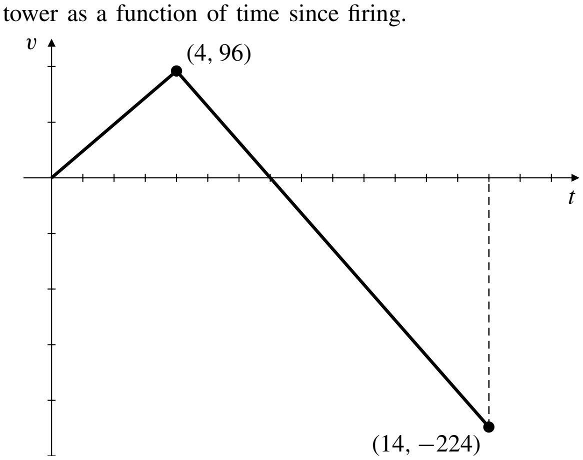


Fig. 2.11.16

The rocket's acceleration while its fuel lasted is the slope of the first part of the graph, namely $96/4 = 24$ ft/s.

17. The rocket was rising for the first 7 seconds.

18. As suggested in Example 1 on page 154 of the text, the distance travelled by the rocket while it was falling from its maximum height to the ground is the area between the velocity graph and the part of the t-axis where $v < 0$. The area of this triangle is $(1/2)(14 - 7)(224) = 784$ ft. This is the maximum height the rocket achieved.

19. The distance travelled upward by the rocket while it was rising is the area between the velocity graph and the part of the t-axis where $v > 0$, namely $(1/2)(7)(96) = 336$ ft. Thus the height of the tower from which the rocket was fired is $784 - 336 = 448$ ft.

20. Let $s(t)$ be the distance the car travels in the t seconds after the brakes are applied. Then $s''(t) = -t$ and the velocity at time t is given by
$$s'(t) = \int (-t)\, dt = -\frac{t^2}{2} + C_1,$$
where $C_1 = 20$ m/s (that is, 72km/h) as determined in Example 6. Thus
$$s(t) = \int \left(20 - \frac{t^2}{2}\right) dt = 20t - \frac{t^3}{6} + C_2,$$
where $C_2 = 0$ because $s(0) = 0$. The time taken to come to a stop is given by $s'(t) = 0$, so it is $t = \sqrt{40}$ s. The distance travelled is
$$s = 20\sqrt{40} - \frac{1}{6}40^{3/2} \approx 84.3 \text{ m.}$$

Review Exercises 2 (page 158)

1. $y = (3x + 1)^2$
$$\frac{dy}{dx} = \lim_{h\to 0} \frac{(3x + 3h + 1)^2 - (3x + 1)^2}{h}$$
$$= \lim_{h\to 0} \frac{9x^2 + 18xh + 9h^2 + 6x + 6h + 1 - (9x^2 + 6x + 1)}{h}$$
$$= \lim_{h\to 0} (18x + 9h + 6) = 18x + 6$$

2. $$\frac{d}{dx}\sqrt{1 - x^2} = \lim_{h\to 0} \frac{\sqrt{1 - (x + h)^2} - \sqrt{1 - x^2}}{h}$$
$$= \lim_{h\to 0} \frac{1 - (x + h)^2 - (1 - x^2)}{h(\sqrt{1 - (x + h)^2} + \sqrt{1 - x^2})}$$
$$= \lim_{h\to 0} \frac{-2x - h}{\sqrt{1 - (x + h)^2} + \sqrt{1 - x^2}} = -\frac{x}{\sqrt{1 - x^2}}$$

3. $f(x) = 4/x^2$

$$f'(2) = \lim_{h\to 0} \frac{\dfrac{4}{(2+h)^2} - 1}{h}$$
$$= \lim_{h\to 0} \frac{4 - (4 + 4h + h^2)}{h(2+h)^2} = \lim_{h\to 0} \frac{-4-h}{(2+h)^2} = -1$$

4. $g(t) = \dfrac{t-5}{1+\sqrt{t}}$

$$g'(9) = \lim_{h\to 0} \frac{\dfrac{4+h}{1+\sqrt{9+h}} - 1}{h}$$
$$= \lim_{h\to 0} \frac{(3+h-\sqrt{9+h})(3+h+\sqrt{9+h})}{h(1+\sqrt{9+h})(3+h+\sqrt{9+h})}$$
$$= \lim_{h\to 0} \frac{9+6h+h^2-(9+h)}{h(1+\sqrt{9+h})(3+h+\sqrt{9+h})}$$
$$= \lim_{h\to 0} \frac{5+h}{(1+\sqrt{9+h})(3+h+\sqrt{9+h})}$$
$$= \frac{5}{24}$$

5. The tangent to $y = \cos(\pi x)$ at $x = 1/6$ has slope

$$\left.\frac{dy}{dx}\right|_{x=1/6} = -\pi \sin\frac{\pi}{6} = -\frac{\pi}{2}.$$

Its equation is

$$y = \frac{\sqrt{3}}{2} - \frac{\pi}{2}\left(x - \frac{1}{6}\right).$$

6. At $x = \pi$ the curve $y = \tan(x/4)$ has slope $(\sec^2(\pi/4))/4 = 1/2$. The normal to the curve there has equation $y = 1 - 2(x - \pi)$.

7. $$\frac{d}{dx}\frac{1}{x - \sin x} = -\frac{1-\cos x}{(x-\sin x)^2}$$

8. $$\frac{d}{dx}\frac{1+x+x^2+x^3}{x^4} = \frac{d}{dx}(x^{-4}+x^{-3}+x^{-2}+x^{-1})$$
$$= -4x^{-5} - 3x^{-4} - 2x^{-3} - x^{-2}$$
$$= -\frac{4+3x+2x^2+x^3}{x^5}$$

9. $$\frac{d}{dx}(4-x^{2/5})^{-5/2} = -\frac{5}{2}(4-x^{2/5})^{-7/2}\left(-\frac{2}{5}x^{-3/5}\right)$$
$$= x^{-3/5}(4-x^{2/5})^{-7/2}$$

10. $$\frac{d}{dx}\sqrt{2+\cos^2 x} = \frac{-2\cos x \sin x}{2\sqrt{2+\cos^2 x}} = \frac{-\sin x \cos x}{\sqrt{2+\cos^2 x}}$$

11. $$\frac{d}{d\theta}(\tan\theta - \theta\sec^2\theta) = \sec^2\theta - \sec^2\theta - 2\theta\sec^2\theta\tan\theta$$
$$= -2\theta\sec^2\theta\tan\theta$$

12. $$\frac{d}{dt}\frac{\sqrt{1+t^2}-1}{\sqrt{1+t^2}+1}$$
$$= \frac{(\sqrt{1+t^2}+1)\dfrac{t}{\sqrt{1+t^2}} - (\sqrt{1+t^2}-1)\dfrac{t}{\sqrt{1+t^2}}}{(\sqrt{1+t^2}+1)^2}$$
$$= \frac{2t}{\sqrt{1+t^2}(\sqrt{1+t^2}+1)^2}$$

13. $$\lim_{h\to 0}\frac{(x+h)^{20}-x^{20}}{h} = \frac{d}{dx}x^{20} = 20x^{19}$$

14. $$\lim_{x\to 2}\frac{\sqrt{4x+1}-3}{x-2} = \lim_{h\to 0} 4\frac{\sqrt{9+4h}-3}{4h}$$
$$= \left.\frac{d}{dx}4\sqrt{x}\right|_{x=9} = \frac{4}{2\sqrt{9}} = \frac{2}{3}$$

15. $$\lim_{x\to\pi/6}\frac{\cos(2x)-(1/2)}{x-\pi/6} = \lim_{h\to 0} 2\frac{\cos((\pi/3)+2h)-\cos(\pi/3)}{2h}$$
$$= 2\left.\frac{d}{dx}\cos x\right|_{x=\pi/3}$$
$$= -2\sin(\pi/3) = -\sqrt{3}$$

16. $$\lim_{x\to -a}\frac{(1/x^2)-(1/a^2)}{x+a} = \lim_{h\to 0}\frac{\dfrac{1}{(-a+h)^2} - \dfrac{1}{(-a)^2}}{h}$$
$$= \left.\frac{d}{dx}\frac{1}{x^2}\right|_{x=-a} = \frac{2}{a^3}$$

17. $$\frac{d}{dx}f(3-x^2) = -2xf'(3-x^2)$$

18. $$\frac{d}{dx}[f(\sqrt{x})]^2 = 2f(\sqrt{x})f'(\sqrt{x})\frac{1}{2\sqrt{x}} = \frac{f(\sqrt{x})f'(\sqrt{x})}{\sqrt{x}}$$

19. $$\frac{d}{dx}f(2x)\sqrt{g(x/2)} = 2f'(2x)\sqrt{g(x/2)} + \frac{f(2x)g'(x/2)}{4\sqrt{g(x/2)}}$$

20. $$\frac{d}{dx}\frac{f(x)-g(x)}{f(x)+g(x)}$$
$$= \frac{1}{(f(x)+g(x))^2}\Big[f(x)+g(x))(f'(x)-g'(x))$$
$$- (f(x)-g(x))(f'(x)+g'(x)\Big]$$
$$= \frac{2(f'(x)g(x)-f(x)g'(x))}{(f(x)+g(x))^2}$$

21. $$\frac{d}{dx}f(x+(g(x))^2) = (1+2g(x)g'(x))f'(x+(g(x))^2)$$

22. $$\frac{d}{dx}f\left(\frac{g(x^2)}{x}\right) = \frac{2x^2g'(x^2)-g(x^2)}{x^2}f'\left(\frac{g(x^2)}{x}\right)$$

23. $$\frac{d}{dx}f(\sin x)g(\cos x)$$
$$= (\cos x)f'(\sin x)g(\cos x) - (\sin x)f(\sin x)g'(\cos x)$$

24. $$\frac{d}{dx}\sqrt{\frac{\cos f(x)}{\sin g(x)}}$$
$$= \frac{1}{2}\sqrt{\frac{\sin g(x)}{\cos f(x)}}$$
$$\times \frac{-f'(x)\sin f(x)\sin g(x) - g'(x)\cos f(x)\cos g(x)}{(\sin g(x))^2}$$

25. If $x^3y + 2xy^3 = 12$, then $3x^2y + x^3y' + 2y^3 + 6xy^2y' = 0$. At $(2, 1)$: $12 + 8y' + 2 + 12y' = 0$, so the slope there is $y' = -7/10$. The tangent line has equation $y = 1 - \frac{7}{10}(x-2)$ or $7x + 10y = 24$.

26. $3\sqrt{2}x\sin(\pi y) + 8y\cos(\pi x) = 2$
$3\sqrt{2}\sin(\pi y) + 3\pi\sqrt{2}x\cos(\pi y)y' + 8y'\cos(\pi x)$
$-8\pi y\sin(\pi x) = 0$
At $(1/3, 1/4)$: $3 + \pi y' + 4y' - \pi\sqrt{3} = 0$, so the slope there is $y' = \dfrac{\pi\sqrt{3}-3}{\pi+4}$.

27. $$\int \frac{1+x^4}{x^2}\,dx = \int\left(\frac{1}{x^2} + x^2\right)dx = -\frac{1}{x} + \frac{x^3}{3} + C$$

28. $$\int \frac{1+x}{\sqrt{x}}\,dx = \int (x^{-1/2} + x^{1/2})\,dx = 2\sqrt{x} + \frac{2}{3}x^{3/2} + C$$

29. $$\int \frac{2+3\sin x}{\cos^2 x}\,dx = \int (2\sec^2 x + 3\sec x\tan x)\,dx$$
$$= 2\tan x + 3\sec x + C$$

30. $$\int (2x+1)^4\,dx = \int (16x^4 + 32x^3 + 24x^2 + 8x + 1)\,dx$$
$$= \frac{16x^5}{5} + 8x^4 + 8x^3 + 4x^2 + x + C$$
or, equivalently,
$$\int (2x+1)^4\,dx = \frac{(2x+1)^5}{10} + C$$

31. If $f'(x) = 12x^2 + 12x^3$, then $f(x) = 4x^3 + 3x^4 + C$. If $f(1) = 0$, then $4 + 3 + C = 0$, so $C = -7$ and $f(x) = 4x^3 + 3x^4 - 7$.

32. If $g'(x) = \sin(x/3) + \cos(x/6)$, then
$$g(x) = -3\cos(x/3) + 6\sin(x/6) + C.$$
If $(\pi, 2)$ lies on $y = g(x)$, then $-(3/2) + 3 + C = 2$, so $C = 1/2$ and $g(x) = -3\cos(x/3) + 6\sin(x/6) + (1/2)$.

33. $$\frac{d}{dx}(x\sin x + \cos x) = \sin x + x\cos x - \sin x = x\cos x$$
$$\frac{d}{dx}(x\cos x - \sin x) = \cos x - x\sin x - \cos x = -x\sin x$$
$$\int x\cos x\,dx = x\sin x + \cos x + C$$
$$\int x\sin x\,dx = -x\cos x + \sin x + C$$

34. If $f'(x) = f(x)$ and $g(x) = xf(x)$, then
$$g'(x) = f(x) + xf'(x) = (1+x)f(x)$$
$$g''(x) = f(x) + (1+x)f'(x) = (2+x)f(x)$$
$$g'''(x) = f(x) + (2+x)f'(x) = (3+x)f(x)$$
Conjecture: $g^{(n)}(x) = (n+x)f(x)$ for $n = 1, 2, 3, \ldots$
Proof: The formula is true for $n = 1$, 2, and 3 as shown above. Suppose it is true for $n = k$; that is, suppose $g^{(k)}(x) = (k+x)f(x)$. Then
$$g^{(k+1)}(x) = \frac{d}{dx}\Big((k+x)f(x)\Big)$$
$$= f(x) + (k+x)f'(x) = ((k+1)+x)f(x).$$
Thus the formula is also true for $n = k+1$. It is therefore true for all positive integers n by induction.

35. The tangent to $y = x^3 + 2$ at $x = a$ has equation $y = a^3 + 2 + 3a^2(x-a)$, or $y = 3a^2x - 2a^3 + 2$. This line passes through the origin if $0 = -2a^3 + 2$, that is, if $a = 1$. The line then has equation $y = 3x$.

36. The tangent to $y = \sqrt{2+x^2}$ at $x = a$ has slope $a/\sqrt{2+a^2}$ and equation
$$y = \sqrt{2+a^2} + \frac{a}{\sqrt{2+a^2}}(x-a).$$
This line passes through $(0, 1)$ provided
$$1 = \sqrt{2+a^2} - \frac{a^2}{\sqrt{2+a^2}}$$
$$\sqrt{2+a^2} = 2 + a^2 - a^2 = 2$$
$$2 + a^2 = 4$$
The possibilities are $a = \pm\sqrt{2}$, and the equations of the corrresponding tangent lines are $y = 1 \pm (x/\sqrt{2})$.

37. $$\frac{d}{dx}\Big(\sin^n x\sin(nx)\Big)$$
$$= n\sin^{n-1}x\cos x\sin(nx) + n\sin^n x\cos(nx)$$
$$= n\sin^{n-1}x[\cos x\sin(nx) + \sin x\cos(nx)]$$
$$= n\sin^{n-1}x\sin((n+1)x)$$
$y = \sin^n x\sin(nx)$ has a horizontal tangent at $x = m\pi/(n+1)$, for any integer m.

38. $\frac{d}{dx}\Big(\sin^n x\cos(nx)\Big)$
$= n\sin^{n-1}x\cos x\cos(nx) - n\sin^n x\sin(nx)$
$= n\sin^{n-1}x[\cos x\cos(nx) - \sin x\sin(nx)]$
$= n\sin^{n-1}x\cos((n+1)x)$
$\frac{d}{dx}\Big(\cos^n x\sin(nx)\Big)$
$= -n\cos^{n-1}x\sin x\sin(nx) + n\cos^n x\cos(nx)$
$= n\cos^{n-1}x[\cos x\cos(nx) - \sin x\sin(nx)]$
$= n\cos^{n-1}x\cos((n+1)x)$
$\frac{d}{dx}\Big(\cos^n x\cos(nx)\Big)$
$= -n\cos^{n-1}x\sin x\cos(nx) - n\cos^n x\sin(nx)$
$= -n\cos^{n-1}x[\sin x\cos(nx) + \cos x\sin(nx)]$
$= -n\cos^{n-1}x\sin((n+1)x)$

39. $Q = (0,1)$. If $P = (a, a^2)$ on the curve $y = x^2$, then the slope of $y = x^2$ at P is $2a$, and the slope of PQ is $(a^2-1)/a$. PQ is normal to $y = x^2$ if $a = 0$ or $[(a^2-1)/a](2a) = -1$, that is, if $a = 0$ or $a^2 = 1/2$. The points P are $(0,0)$ and $(\pm 1/\sqrt{2}, 1/2)$. The distances from these points to Q are 1 and $\sqrt{3}/2$, respectively. The distance from Q to the curve $y = x^2$ is the shortest of these distances, namely $\sqrt{3}/2$ units.

40. The average profit per tonne if x tonnes are exported is $P(x)/x$, that is the slope of the line joining $(x, P(x))$ to the origin. This slope is maximum if the line is tangent to the graph of $P(x)$. In this case the slope of the line is $P'(x)$, the marginal profit.

41. $F(r) = \begin{cases} \dfrac{mgR^2}{r^2} & \text{if } r \geq R \\ mkr & \text{if } 0 \leq r < R \end{cases}$

a) For continuity of $F(r)$ at $r = R$ we require $mg = mkR$, so $k = g/R$.

b) As r increases from R, F changes at rate

$$\frac{d}{dr}\frac{mgR^2}{r^2}\bigg|_{r=R} = -\frac{2mgR^2}{R^3} = -\frac{2mg}{R}.$$

As r decreases from R, F changes at rate

$$-\frac{d}{dr}(mkr)\bigg|_{r=R} = -mk = -\frac{mg}{R}.$$

Observe that this rate is half the rate at which F decreases when r increases from R.

42. $PV = kT$. Differentiate with respect to P holding T constant to get

$$V + P\frac{dV}{dP} = 0$$

Thus the isothermal compressibility of the gas is

$$\frac{1}{V}\frac{dV}{dP} = \frac{1}{V}\left(-\frac{V}{P}\right) = -\frac{1}{P}.$$

43. Let the building be h m high. The height of the first ball at time t during its motion is

$$y_1 = h + 10t - 4.9t^2.$$

It reaches maximum height when $dy_1/dt = 10 - 9.8t = 0$, that is, at $t = 10/9.8$ s. The maximum height of the first ball is

$$y_1 = h + \frac{100}{9.8} - \frac{4.9 \times 100}{(9.8)^2} = h + \frac{100}{19.6}.$$

The height of the second ball at time t during its motion is

$$y_2 = 20t - 4.9t^2.$$

It reaches maximum height when $dy_2/dt = 20 - 9.8t = 0$, that is, at $t = 20/9.8$ s. The maximum height of the second ball is

$$y_2 = \frac{400}{9.8} - \frac{4.9 \times 400}{(9.8)^2} = \frac{400}{19.6}.$$

These two maximum heights are equal, so

$$h + \frac{100}{19.6} = \frac{400}{19.6},$$

which gives $h = 300/19.6 \approx 15.3$ m as the height of the building.

44. The first ball has initial height 60 m and initial velocity 0, so its height at time t is

$$y_1 = 60 - 4.9t^2 \text{ m}.$$

The second ball has initial height 0 and initial velocity v_0, so its height at time t is

$$y_2 = v_0 t - 4.9t^2 \text{ m}.$$

The two balls collide at a height of 30 m (at time T, say). Thus

$$30 = 60 - 4.9T^2$$
$$30 = v_0 T - 4.9T^2.$$

Thus $v_0 T = 60$ and $T^2 = 30/4.9$. The initial upward speed of the second ball is

$$v_0 = \frac{60}{T} = 60\sqrt{\frac{4.9}{30}} \approx 24.25 \text{ m/s}.$$

At time T, the velocity of the first ball is

$$\left.\frac{dy_1}{dt}\right|_{t=T} = -9.8T \approx -24.25 \text{ m/s}.$$

At time T, the velocity of the second ball is

$$\left.\frac{dy_2}{dt}\right|_{t=T} = v_0 - 9.8T = 0 \text{ m/s}.$$

45. Let the car's initial speed be v_0. The car decelerates at 20 ft/s^2 starting at $t = 0$, and travels distance s in time t, where $d^2s/dt^2 = -20$. Thus

$$\frac{ds}{dt} = v_0 - 20t$$
$$x = v_0t - 10t^2.$$

The car stops at time $t = v_0/20$. The stopping distance is $s = 160$ ft, so

$$160 = \frac{v_0^2}{20} - \frac{v_0^2}{40} = \frac{v_0^2}{40}.$$

The car's initial speed cannot exceed $v_0 = \sqrt{160 \times 40} = 80$ ft/s.

46. $P = 2\pi\sqrt{L/g} = 2\pi L^{1/2}g^{-1/2}$.

a) If L remains constant, then

$$\Delta P \approx \frac{dP}{dg}\,\Delta g = -\pi L^{1/2}g^{-3/2}\,\Delta g$$
$$\frac{\Delta P}{P} \approx \frac{-\pi L^{1/2}g^{-3/2}}{2\pi L^{1/2}g^{-1/2}}\,\Delta g = -\frac{1}{2}\,\frac{\Delta g}{g}.$$

If g increases by 1%, then $\Delta g/g = 1/100$, and $\Delta P/P = -1/200$. Thus P decreases by 0.5%.

b) If g remains constant, then

$$\Delta P \approx \frac{dP}{dL}\,\Delta L = \pi L^{-1/2}g^{-1/2}\,\Delta L$$
$$\frac{\Delta P}{P} \approx \frac{\pi L^{-1/2}g^{-1/2}}{2\pi L^{1/2}g^{-1/2}}\,\Delta L = \frac{1}{2}\,\frac{\Delta L}{L}.$$

If L increases by 2%, then $\Delta L/L = 2/100$, and $\Delta P/P = 1/100$. Thus P increases by 1%.

Challenging Problems 2 (page 159)

1. The line through (a, a^2) with slope m has equation $y = a^2 + m(x - a)$. It intersects $y = x^2$ at points x that satisfy

$$x^2 = a^2 + mx - ma, \quad \text{or}$$
$$x^2 - mx + ma - a^2 = 0$$

In order that this quadratic have only one solution $x = a$, the left side must be $(x - a)^2$, so that $m = 2a$. The tangent has slope $2a$.
This won't work for more general curves whose tangents can intersect them at more than one point.

2. $f'(x) = 1/x$, $f(2) = 9$.

a)
$$\begin{aligned}\lim_{x\to2}\frac{f(x^2+5)-f(9)}{x-2} &= \lim_{h\to0}\frac{f(9+4h+h^2)-f(9)}{h}\\ &= \lim_{h\to0}\frac{f(9+4h+h^2)-f(9)}{4h+h^2}\times\frac{4h+h^2}{h}\\ &= \lim_{k\to0}\frac{f(9+k)-f(9)}{k}\times\lim_{h\to0}(4+h)\\ &= f'(9)\times4=\frac{4}{9}\end{aligned}$$

b)
$$\begin{aligned}\lim_{x\to2}\frac{\sqrt{f(x)}-3}{x-2} &= \lim_{h\to0}\frac{\sqrt{f(2+h)}-3}{h}\\ &= \lim_{h\to0}\frac{f(2+h)-9}{h}\times\frac{1}{\sqrt{f(2+h)}+3}\\ &= f'(2)\times\frac{1}{6}=\frac{1}{12}.\end{aligned}$$

3. $f'(4) = 3$, $g'(4) = 7$, $g(4) = 4$, $g(x) \neq 4$ if $x \neq 4$.

a)
$$\begin{aligned}\lim_{x\to4}\Big(f(x)-f(4)\Big) &= \lim_{x\to4}\frac{f(x)-f(4)}{x-4}(x-4)\\ &= f'(4)(4-4)=0\end{aligned}$$

b)
$$\begin{aligned}\lim_{x\to4}\frac{f(x)-f(4)}{x^2-16} &= \lim_{x\to4}\frac{f(x)-f(4)}{x-4}\times\frac{1}{x+4}\\ &= f'(4)\times\frac{1}{8}=\frac{3}{8}\end{aligned}$$

c)
$$\begin{aligned}\lim_{x\to4}\frac{f(x)-f(4)}{\sqrt{x}-2} &= \lim_{x\to4}\frac{f(x)-f(4)}{x-4}\times(\sqrt{x}+2)\\ &= f'(4)\times4=12\end{aligned}$$

d)
$$\begin{aligned}\lim_{x\to4}\frac{f(x)-f(4)}{\dfrac{1}{x}-\dfrac{1}{4}} &= \lim_{x\to4}\frac{f(x)-f(4)}{x-4}\times\frac{x-4}{(4-x)/4x}\\ &= f'(4)\times(-16)=-48\end{aligned}$$

e)
$$\begin{aligned}\lim_{x\to4}\frac{f(x)-f(4)}{g(x)-4} &= \lim_{x\to4}\frac{\dfrac{f(x)-f(4)}{x-4}}{\dfrac{g(x)-g(4)}{x-4}}\\ &= \frac{f'(4)}{g'(4)}=\frac{3}{7}\end{aligned}$$

f)
$$\begin{aligned}&\lim_{x\to4}\frac{f(g(x))-f(4)}{x-4}\\ &= \lim_{x\to4}\frac{f(g(x))-f(4)}{g(x)-4}\times\frac{g(x)-g(4)}{x-4}\\ &= f'(g(4))\times g'(4)=f'(4)\times g'(4)=3\times7=21\end{aligned}$$

4. $f(x) = \begin{cases} x & \text{if } x = 1,\ 1/2,\ 1/3,\ \ldots \\ x^2 & \text{otherwise}\end{cases}$.

a) f is continuous except at $1/2$, $1/3$, $1/4$, ... It is continuous at $x = 1$ and $x = 0$ (and everywhere else). Note that

$$\lim_{x\to 1} x^2 = 1 = f(1),$$
$$\lim_{x\to 0} x^2 = \lim_{x\to 0} x = 0 = f(0)$$

b) If $a = 1/2$ and $b = 1/3$, then

$$\frac{f(a)+f(b)}{2} = \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) = \frac{5}{12}.$$

If $1/3 < x < 1/2$, then $f(x) = x^2 < 1/4 < 5/12$. Thus the statement is FALSE.

c) By (a) f cannot be differentiable at $x = 1/2$, $1/2$, ... It is not differentiable at $x = 0$ either, since

$$\lim_{h\to 0} h - 0h = 1 \ne 0 = \lim_{h\to 0} \frac{h^2-0}{h}.$$

f is differentiable elsewhere, including at $x = 1$ where its derivative is 2.

5. If $h \ne 0$, then

$$\left|\frac{f(h)-f(0)}{h}\right| = \frac{|f(h)|}{|h|} > \frac{\sqrt{|h|}}{|h|} \to \infty$$

as $h \to 0$. Therefore $f'(0)$ does not exist.

6. Given that $f'(0) = k$, $f(0) \ne 0$, and $f(x+y) = f(x)f(y)$, we have

$$f(0) = f(0{+}0) = f(0)f(0) \quad\Longrightarrow\quad f(0) = 0 \quad\text{or}\quad f(0) = 1.$$

Thus $f(0) = 1$.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x)f(h)-f(x)}{h} = f(x)f'(0) = kf(x). \end{aligned}$$

7. Given that $g'(0) = k$ and $g(x+y) = g(x)+g(y)$, then

a) $g(0) = g(0+0) = g(0)+g(0)$. Thus $g(0) = 0$.

b) $$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{g(x)+g(h)-g(x)}{h} = \lim_{h\to 0}\frac{g(h)-g(0)}{h} \\ &= g'(0) = k. \end{aligned}$$

c) If $h(x) = g(x) - kx$, then $h'(x) = g'(x) - k = 0$ for all x. Thus $h(x)$ is constant for all x. Since $h(0) = g(0) - 0 = 0$, we have $h(x) = 0$ for all x, and $g(x) = kx$.

8. a) $$\begin{aligned} f'(x) &= \lim_{k\to 0}\frac{f(x+k)-f(x)}{k} \quad (\text{let } k = -h) \\ &= \lim_{h\to 0}\frac{f(x-h)-f(x)}{-h} = \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}. \\ f'(x) &= \frac{1}{2}\big(f'(x)+f'(x)\big) \\ &= \frac{1}{2}\left(\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\right. \\ &\qquad \left. + \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}\right) \\ &= \lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h}. \end{aligned}$$

b) The change of variables used in the first part of (a) shows that

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \textit{and} \quad \lim_{h\to 0}\frac{f(x)-f(x-h)}{h}$$

are always equal if either exists.

c) If $f(x) = |x|$, then $f'(0)$ does not exist, but

$$\lim_{h\to 0}\frac{f(0+h)-f(0-h)}{2h} = \lim_{h\to 0}\frac{|h|-|h|}{h} = \lim_{h\to 0}\frac{0}{h} = 0.$$

9. The tangent to $y = x^3$ at $x = 3a/2$ has equation

$$y = \frac{27a^3}{8} + \frac{27}{4a^2}\left(x - \frac{3a}{2}\right).$$

This line passes through $(a, 0)$ because

$$\frac{27a^3}{8} + \frac{27}{4a^2}\left(a - \frac{3a}{2}\right) = 0.$$

If $a \ne 0$, the x-axis is another tangent to $y = x^3$ that passes through $(a, 0)$.

The number of tangents to $y = x^3$ that pass through (x_0, y_0) is

three, if $x_0 \ne 0$ and y_0 is between 0 and x_0^3;

two, if $x_0 \ne 0$ and either $y_0 = 0$ or $y_0 = x_0^3$;

one, otherwise.

This is the number of distinct real solutions b of the cubic equation $2b^3 - 3b^2x_0 + y_0 = 0$, which states that the tangent to $y = x^3$ at (b, b^3) passes through (x_0, y_0).

10. By symmetry, any line tangent to both curves must pass through the origin.

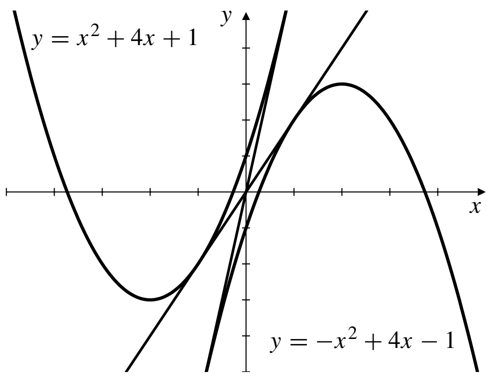


Fig. C-2.10

The tangent to $y = x^2 + 4x + 1$ at $x = a$ has equation

$$\begin{aligned} y &= a^2 + 4a + 1 + (2a + 4)(x - a) \\ &= (2a + 4)x - (a^2 - 1), \end{aligned}$$

which passes through the origin if $a = \pm 1$. The two common tangents are $y = 6x$ and $y = 2x$.

11. The slope of $y = x^2$ at $x = a$ is $2a$.
The slope of the line from $(0, b)$ to (a, a^2) is $(a^2 - b)/a$.
This line is normal to $y = x^2$ if either $a = 0$ or $2a((a^2 - b)/a) = -1$, that is, if $a = 0$ or $2a^2 = 2b - 1$.
There are three real solutions for a if $b > 1/2$ and only one ($a = 0$) if $b \le 1/2$.

12. The point $Q = (a, a^2)$ on $y = x^2$ that is closest to $P = (3, 0)$ is such that PQ is normal to $y = x^2$ at Q. Since PQ has slope $a^2/(a - 3)$ and $y = x^2$ has slope $2a$ at Q, we require

$$\frac{a^2}{a - 3} = -\frac{1}{2a},$$

which simplifies to $2a^3 + a - 3 = 0$. Observe that $a = 1$ is a solution of this cubic equation. Since the slope of $y = 2x^3 + x - 3$ is $6x^2 + 1$, which is always positive, the cubic equation can have only one real solution. Thus $Q = (1, 1)$ is the point on $y = x^2$ that is closest to P. The distance from P to the curve is $|PQ| = \sqrt{5}$ units.

13. The curve $y = x^2$ has slope $m = 2a$ at (a, a^2). The tangent there has equation

$$y = a^2 + m(x - a) = mx - \frac{m^2}{4}.$$

The curve $y = Ax^2 + Bx + C$ has slope $m = 2Aa + B$ at $(a, Aa^2 + Ba + C)$. Thus $a = (m - B)/(2A)$, and the tangent has equation

$$\begin{aligned} y &= Aa^2 + Ba + C + m(x - a) \\ &= mx + \frac{(m - B)^2}{4A} + \frac{B(m - B)}{2A} + C - \frac{m(m - B)}{2A} \\ &= mx + C + \frac{(m - B)^2}{4A} - \frac{(m - B)^2}{2A} \\ &= mx + f(m), \end{aligned}$$

where $f(m) = C - (m - B)^2/(4A)$.

14. Parabola $y = x^2$ has tangent $y = 2ax - a^2$ at (a, a^2). Parabola $y = Ax^2 + Bx + C$ has tangent

$$y = (2Ab + B)x - Ab^2 + C$$

at $(b, Ab^2 + Bb + C)$. These two tangents coincide if

$$\begin{aligned} 2Ab + B &= 2a \qquad (*) \\ Ab^2 - C &= a^2. \end{aligned}$$

The two curves have one (or more) common tangents if $(*)$ has real solutions for a and b. Eliminating a between the two equations leads to

$$(2Ab + B)^2 = 4Ab^2 - 4C,$$

or, on simplification,

$$4A(A - 1)b^2 + 4ABb + (B^2 + 4C) = 0.$$

This quadratic equation in b has discriminant

$$D = 16A^2B^2 - 16A(A-1)(B^2+4C) = 16A(B^2 - 4(A-1)C).$$

There are five cases to consider:

CASE I. If $A = 1$, $B \ne 0$, then $(*)$ gives

$$b = -\frac{B^2 + 4C}{4B}, \quad a = \frac{B^2 - 4C}{4B}.$$

There is a single common tangent in this case.

CASE II. If $A = 1$, $B = 0$, then $(*)$ forces $C = 0$, which is not allowed. There is no common tangent in this case.

CASE III. If $A \ne 1$ but $B^2 = 4(A - 1)C$, then

$$b = \frac{-B}{2(A - 1)} = a.$$

There is a single common tangent, and since the points of tangency on the two curves coincide, the two curves are tangent to each other.

CASE IV. If $A \ne 1$ and $B^2 - 4(A - 1)C < 0$, there are no real solutions for b, so there can be no common tangents.

CASE V. If $A \ne 1$ and $B^2 - 4(A - 1)C > 0$, there are two distinct real solutions for b, and hence two common tangent lines.

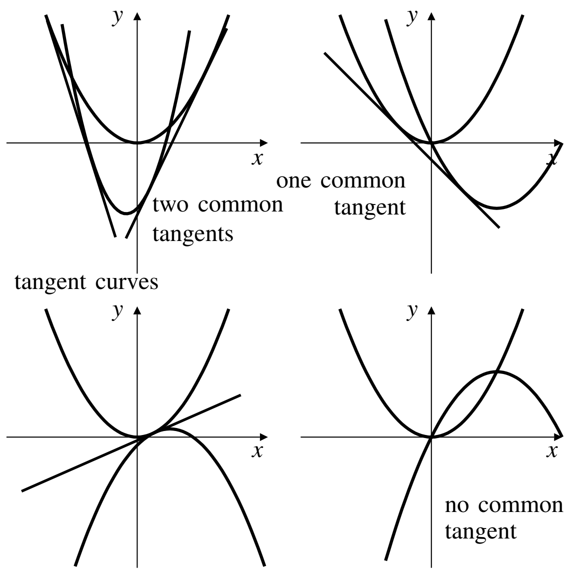


Fig. C-2.14

15. a) The tangent to $y = x^3$ at (a, a^3) has equation

$$y = 3a^2x - 2a^3.$$

For intersections of this line with $y = x^3$ we solve

$$\begin{aligned} x^3 - 3a^2x + 2a^3 &= 0 \\ (x-a)^2(x+2a) &= 0. \end{aligned}$$

The tangent also intersects $y = x^3$ at (b, b^3), where $b = -2a$.

b) The slope of $y = x^3$ at $x = -2a$ is $3(-2a)^2 = 12a^2$, which is four times the slope at $x = a$.

c) If the tangent to $y = x^3$ at $x = a$ were also tangent at $x = b$, then the slope at b would be four times that at a and the slope at a would be four times that at b. This is clearly impossible.

d) No line can be tangent to the graph of a cubic polynomial $P(x)$ at two distinct points a and b, because if there was such a double tangent $y = L(x)$, then $(x-a)^2(x-b)^2$ would be a factor of the cubic polynomial $P(x) - L(x)$, and cubic polynomials do not have factors that are 4th degree polynomials.

16. a) $y = x^4 - 2x^2$ has horizontal tangents at points x satisfying $4x^3 - 4x = 0$, that is, at $x = 0$ and $x = \pm 1$. The horizontal tangents are $y = 0$ and $y = -1$. Note that $y = -1$ is a double tangent; it is tangent at the two points $(\pm 1, -1)$.

b) The tangent to $y = x^4 - 2x^2$ at $x = a$ has equation

$$\begin{aligned} y &= a^4 - 2a^2 + (4a^3 - 4a)(x - a) \\ &= 4a(a^2 - 1)x - 3a^4 + 2a^2. \end{aligned}$$

Similarly, the tangent at $x = b$ has equation

$$y = 4b(b^2 - 1)x - 3b^4 + 2b^2.$$

These tangents are the same line (and hence a double tangent) if

$$\begin{aligned} 4a(a^2 - 1) &= 4b(b^2 - 1) \\ -3a^4 + 2a^2 &= -3b^4 + 2b^2. \end{aligned}$$

The second equation says that either $a^2 = b^2$ or $3(a^2 + b^2) = 2$; the first equation says that $a^3 - b^3 = a - b$, or, equivalently, $a^2 + ab + b^2 = 1$.
If $a^2 = b^2$, then $a = -b$ ($a = b$ is not allowed). Thus $a^2 = b^2 = 1$ and the two points are $(\pm 1, -1)$ as discovered in part (a).
If $a^2 + b^2 = 2/3$, then $ab = 1/3$. This is not possible since it implies that

$$0 = a^2 + b^2 - 2ab = (a - b)^2 > 0.$$

Thus $y = -1$ is the only double tangent to $y = x^4 - 2x^2$.

c) If $y = Ax + B$ is a double tangent to $y = x^4 - 2x^2 + x$, then $y = (A - 1)x + B$ is a double tangent to $y = x^4 - 2x^2$. By (b) we must have $A - 1 = 0$ and $B = -1$. Thus the only double tangent to $y = x^4 - 2x^2 + x$ is $y = x - 1$.

17. a) The tangent to

$$y = f(x) = ax^4 + bx^3 + cx^2 + dx + e$$

at $x = p$ has equation

$$y = (4ap^3 + 3bp^2 + 2cp + d)x - 3ap^4 - 2bp^3 - cp^2 + e.$$

This line meets $y = f(x)$ at $x = p$ (a double root), and

$$x = \frac{-2ap - b \pm \sqrt{b^2 - 4ac - 4abp - 8a^2p^2}}{2a}.$$

These two latter roots are equal (and hence correspond to a double tangent) if the expression under the square root is 0, that is, if

$$8a^2p^2 + 4abp + 4ac - b^2 = 0.$$

This quadratic has two real solutions for p provided its discriminant is positive, that is, provided

$$16a^2b^2 - 4(8a^2)(4ac - b^2) > 0.$$

This condition simplifies to

$$3b^2 > 8ac.$$

For example, for $y = x^4 - 2x^2 + x - 1$, we have $a = 1$, $b = 0$, and $c = -2$, so $3b^2 = 0 > -16 = 8ac$, and the curve has a double tangent.

b) From the discussion above, the second point of tangency is

$$q = \frac{-2ap - b}{2a} = -p - \frac{b}{2a}.$$

The slope of PQ is

$$\frac{f(q) - f(p)}{q - p} = \frac{b^3 - 4abc + 8a^2 d}{8a^2}.$$

Calculating $f'((p+q)/2)$ leads to the same expression, so the double tangent PQ is parallel to the tangent at the point horizontally midway between P and Q.

c) The inflection points are the real zeros of

$$f''(x) = 2(6ax^2 + 3bx + c).$$

This equation has distinct real roots provided $9b^2 > 24ac$, that is, $3b^2 > 8ac$. The roots are

$$r = \frac{-3b - \sqrt{9b^2 - 24ac}}{12a}$$
$$s = \frac{-3b + \sqrt{9b^2 - 24ac}}{12a}.$$

The slope of the line joining these inflection points is

$$\frac{f(s) - f(r)}{s - r} = \frac{b^3 - 4abc + 8a^2 d}{8a^2},$$

so this line is also parallel to the double tangent.

18. a) Claim: $\dfrac{d^n}{dx^n}\cos(ax) = a^n \cos\left(ax + \dfrac{n\pi}{2}\right)$.

Proof: For $n = 1$ we have

$$\frac{d}{dx}\cos(ax) = -a\sin(ax) = a\cos\left(ax + \frac{\pi}{2}\right),$$

so the formula above is true for $n = 1$. Assume it is true for $n = k$, where k is a positive integer. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\cos(ax) &= \frac{d}{dx}\left[a^k \cos\left(ax + \frac{k\pi}{2}\right)\right]\\ &= a^k\left[-a\sin\left(ax + \frac{k\pi}{2}\right)\right]\\ &= a^{k+1}\cos\left(ax + \frac{(k+1)\pi}{2}\right).\end{aligned}$$

Thus the formula holds for $n = 1, 2, 3, \ldots$ by induction.

b) Claim: $\dfrac{d^n}{dx^n}\sin(ax) = a^n \sin\left(ax + \dfrac{n\pi}{2}\right)$.

Proof: For $n = 1$ we have

$$\frac{d}{dx}\sin(ax) = a\cos(ax) = a\sin\left(ax + \frac{\pi}{2}\right),$$

so the formula above is true for $n = 1$. Assume it is true for $n = k$, where k is a positive integer. Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\sin(ax) &= \frac{d}{dx}\left[a^k \sin\left(ax + \frac{k\pi}{2}\right)\right]\\ &= a^k\left[a\cos\left(ax + \frac{k\pi}{2}\right)\right]\\ &= a^{k+1}\sin\left(ax + \frac{(k+1)\pi}{2}\right).\end{aligned}$$

Thus the formula holds for $n = 1, 2, 3, \ldots$ by induction.

c) Note that

$$\begin{aligned}\frac{d}{dx}(\cos^4 x + \sin^4 x) &= -4\cos^3 x \sin x + 4\sin^3 x \cos x\\ &= -4\sin x \cos x(\cos^2 - \sin^2 x)\\ &= -2\sin(2x)\cos(2x)\\ &= -\sin(4x) = \cos\left(4x + \frac{\pi}{2}\right).\end{aligned}$$

It now follows from part (a) that

$$\frac{d^n}{dx^n}(\cos^4 x + \sin^4 x) = 4^{n-1}\cos\left(4x + \frac{n\pi}{2}\right).$$

19.

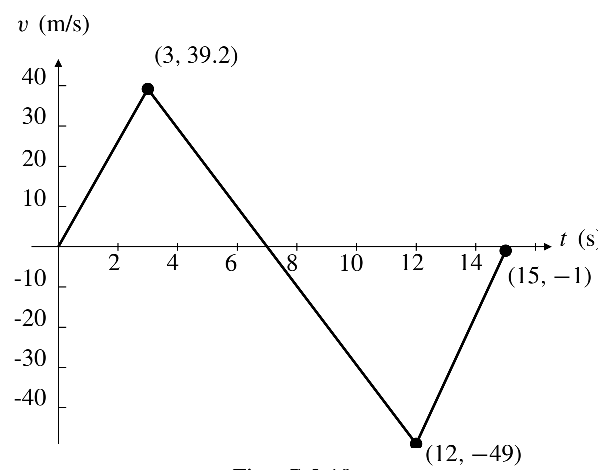


Fig. C-2.19

a) The fuel lasted for 3 seconds.

b) Maximum height was reached at $t = 7$ s.

c) The parachute was deployed at $t = 12$ s.

d) The upward acceleration in $[0, 3]$ was $39.2/3 \approx 13.07$ m/s^2.

e) The maximum height achieved by the rocket is the distance it fell from $t = 7$ to $t = 15$. This is the area under the t-axis and above the graph of v on that interval, that is,

$$\frac{12-7}{2}(49) + \frac{49+1}{2}(15-12) = 197.5 \text{ m}.$$

f) During the time interval $[0, 7]$, the rocket rose a distance equal to the area under the velocity graph and above the t-axis, that is,

$$\frac{1}{2}(7-0)(39.2) = 137.2 \text{ m}.$$

Therefore the height of the tower was $197.5 - 137.2 = 60.3$ m.