CHAPTER 17. ORDINARY DIFFERENTIAL EQUATIONS

Section 17.1 Classifying Differential Equations (page 902)

1. $\dfrac{dy}{dx} = 5y$: 1st order, linear, homogeneous.

2. $\dfrac{d^2y}{dx^2} + x = y$: 2nd order, linear, nonhomogeneous.

3. $y\dfrac{dy}{dx} = x$: 1st order, nonlinear.

4. $y''' + xy' = x\sin x$: 3rd order, linear, nonhomogeneous.

5. $y'' + x\sin x\, y' = y$: 2nd order, linear, homogeneous.

6. $y'' + 4y' - 3y = 2y^2$: 2nd order, nonlinear.

7. $\dfrac{d^3y}{dt^3} + t\dfrac{dy}{dt} + t^2y = t^3$:
3rd order, linear, nonhomogeneous.

8. $\cos x\dfrac{dx}{dt} + x\sin t = 0$: 1st order, nonlinear, homogeneous.

9. $y^{(4)} + e^xy'' = x^3y'$: 4th order, linear, homogeneous.

10. $x^2y'' + e^xy' = \dfrac{1}{y}$: 2nd order, nonlinear.

11. If $y = \cos x$, then $y'' + y = -\cos x + \cos x = 0$.
If $y = \sin x$, then $y'' + y = -\sin x + \sin x = 0$. Thus $y = \cos x$ and $y = \sin x$ are both solutions of $y'' + y = 0$. This DE is linear and homogeneous, so any function of the form
$$y = A\cos x + B\sin x,$$
where A and B are constants, is a solution also. Therefore $\sin x - \cos x$ is a solution ($A = -1$, $B = 1$), and
$$\sin(x+3) = \sin 3\cos x + \cos 3\sin x$$
is a solution, but $\sin 2x$ is not since it cannot be represented in the form $A\cos x + B\sin x$.

12. If $y = e^x$, then $y'' - y = e^x - e^x = 0$; if $y = e^{-x}$, then $y'' - y = e^{-x} - e^{-x} = 0$. Thus e^x and e^{-x} are both solutions of $y'' - y = 0$. Since $y'' - y = 0$ is linear and homogeneous, any function of the form
$$y = Ae^x + Be^{-x}$$
is also a solution. Thus $\cosh x = \frac{1}{2}(e^x + e^{-x})$ is a solution, but neither $\cos x$ nor x^e is a solution.

13. Given that $y_1 = \cos(kx)$ is a solution of $y'' + k^2y = 0$, we suspect that $y_2 = \sin(kx)$ is also a solution. This is easily verified since
$$y_2'' + k^2y_2 = -k^2\sin(kx) + k^2\sin(kx) = 0.$$
Since the DE is linear and homogeneous,
$$y = Ay_1 + By_2 = A\cos(kx) + B\sin(kx)$$
is a solution for any constants A and B. It will satisfy
$$\begin{aligned} 3 &= y(\pi/k) = A\cos(\pi) + B\sin(\pi) = -A \\ 3 &= y'(\pi/k) = -Ak\sin(\pi) + Bk\cos(\pi) = -Bk, \end{aligned}$$
provided $A = -3$ and $B = -3/k$. The required solution is
$$y = -3\cos(kx) - \frac{3}{k}\sin(kx).$$

14. Given that $y_1 = e^{kx}$ is a solution of $y'' - k^2y = 0$, we suspect that $y_2 = e^{-kx}$ is also a solution. This is easily verified since
$$y_2'' - k^2y_2 = k^2e^{-kx} - k^2e^{-kx} = 0.$$
Since the DE is linear and homogeneous,
$$y = Ay_1 + By_2 = Ae^{kx} + Be^{-kx}$$
is a solution for any constants A and B. It will satisfy
$$\begin{aligned} 0 &= y(1) = Ae^k + Be^{-k} \\ 2 &= y'(1) = Ake^k - Bke^{-k}, \end{aligned}$$
provided $A = e^{-k}/k$ and $B = -e^k/k$. The required solution is
$$y = \frac{1}{k}e^{k(x-1)} - \frac{1}{k}e^{-k(x-1)}.$$

15. By Exercise 11, $y = A\cos x + B\sin x$ is a solution of $y'' + y = 0$ for any choice of the constants A and B. This solution will satisfy
$$\begin{aligned} 0 &= y(\pi/2) - 2y(0) = B - 2A, \\ 3 &= y(\pi/4) = \frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}}, \end{aligned}$$
provided $A = \sqrt{2}$ and $B = 2\sqrt{2}$. The required solution is
$$y = \sqrt{2}\cos x + 2\sqrt{2}\sin x.$$

16. $y = e^{rx}$ is a solution of the equation $y'' - y' - 2y = 0$ if $r^2 e^{rx} - re^{rx} - 2e^{rx} = 0$, that is, if $r^2 - r - 2 = 0$. This quadratic has two roots, $r = 2$, and $r = -1$. Since the DE is linear and homogeneous, the function $y = Ae^{2x} + Be^{-x}$ is a solution for any constants A and B. This solution satisfies

$$1 = y(0) = A + B, \quad 2 = y'(0) = 2A - B,$$

provided $A = 1$ and $B = 0$. Thus, the required solution is $y = e^{2x}$.

17. If $y = y_1(x) = x$, then $y_1' = 1$ and $y_1'' = 0$. Thus $y_1'' + y_1 = 0 + x = x$. By Exercise 11 we know that $y_2 = A\cos x + B\sin x$ satisfies the homogeneous DE $y'' + y = 0$. Therefore, by Theorem 2,

$$y = y_1(x) + y_2(x) = x + A\cos x + B\sin x$$

is a solution of $y'' + y = x$. This solution satisfies

$$1 = y(\pi) = \pi - A, \quad 0 = y'(\pi) = 1 - B,$$

provided $A = \pi - 1$ and $B = 1$. Thus the required solution is $y = x + (\pi - 1)\cos x + \sin x$.

18. If $y = y_1(x) = -e$, then $y_1' = 0$ and $y_1'' = 0$. Thus $y_1'' - y_1 = 0 + e = e$. By Exercise 12 we know that $y_2 = Ae^x + Be^{-x}$ satisfies the homogeneous DE $y'' - y = 0$. Therefore, by Theorem 2,

$$y = y_1(x) + y_2(x) = -e + Ae^x + Be^{-x}$$

is a solution of $y'' - y = e$. This solution satisfies

$$0 = y(1) = Ae + \frac{B}{e} - e, \quad 1 = y'(1) = Ae - \frac{B}{e},$$

provided $A = (e+1)/(2e)$ and $B = e(e-1)/2$. Thus the required solution is $y = -e + \frac{1}{2}(e+1)e^{x-1} + \frac{1}{2}(e-1)e^{1-x}$.

Section 17.2 Solving First-Order Equations (page 907)

1. $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ Let $y = vx$

$$v + x\frac{dv}{dx} = \frac{x(1+v)}{x(1-v)}$$
$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v^2}{1-v}$$
$$\int \frac{1-v}{1+v^2}\,dv = \int \frac{dx}{x}$$
$$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln|x| + C_1$$
$$\tan^{-1}(y/x) - \frac{1}{2}\ln\frac{x^2+y^2}{x^2} = \ln|x| + C_1$$
$$2\tan^{-1}(y/x) - \ln(x^2+y^2) = C.$$

2. $\dfrac{dy}{dx} = \dfrac{xy}{x^2+2y^2}$ Let $y = vx$

$$v + x\frac{dv}{dx} = \frac{vx^2}{(1+2v^2)x^2}$$
$$x\frac{dv}{dx} = \frac{v}{1+2v^2} - v = -\frac{2v^3}{1+2v^2}$$
$$\int \frac{1+2v^2}{v^3}\,dv = -2\int \frac{dx}{x}$$
$$-\frac{1}{2v^2} + 2\ln|v| = -2\ln|x| + C_1$$
$$-\frac{x^2}{2y^2} + 2\ln|y| = C_1$$
$$x^2 - 4y^2\ln|y| = Cy^2.$$

3. $\dfrac{dy}{dx} = \dfrac{x^2+xy+y^2}{x^2}$ Let $y = vx$

$$v + x\frac{dv}{dx} = \frac{x^2(1+v+v^2)}{x^2}$$
$$\int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$
$$\tan^{-1} v = \ln|x| + C$$
$$\frac{y}{x} = \tan\big(\ln|x| + C\big)$$
$$y = x\tan\big(\ln|x| + C\big).$$

4. $\dfrac{dy}{dx} = \dfrac{x^3+3xy^2}{3x^2y+y^3}$ Let $y = vx$

$$v + x\frac{dv}{dx} = \frac{x^3(1+3v^2)}{x^3(3v+v^3)}$$
$$x\frac{dv}{dx} = \frac{1+3v^2}{3v+v^3} - v = \frac{1-v^4}{v(3+v^2)}$$
$$\int \frac{(3+v^2)v\,dv}{1-v^4} = \int \frac{dx}{x} \qquad \text{Let } u = v^2,\ du = 2v\,dv$$
$$\frac{1}{2}\int \frac{3+u}{1-u^2}\,du = \ln|x| + C_1$$
$$\frac{3}{4}\ln\left|\frac{u+1}{u-1}\right| - \frac{1}{4}\ln|1-u^2| = \ln|x| + C_1$$
$$3\ln\left|\frac{y^2+x^2}{y^2-x^2}\right| - \ln\left|\frac{x^4-y^4}{x^4}\right| = 4\ln|x| + C_2$$
$$\ln\left|\left(\frac{x^2+y^2}{x^2-y^2}\right)^3 \frac{1}{x^4-y^4}\right| = C_2$$
$$\ln\left|\frac{(x^2+y^2)^2}{(x^2-y^2)^4}\right| = C_2$$
$$x^2 + y^2 = C(x^2-y^2)^2.$$

5. $x\dfrac{dy}{dx} = y + x\cos^2\left(\dfrac{y}{x}\right)$ (let $y = vx$)

$$xv + x^2\frac{dv}{dx} = vx + x\cos^2 v$$
$$x\frac{dv}{dx} = \cos^2 v$$
$$\sec^2 v\,dv = \frac{dx}{x}$$
$$\tan v = \ln|x| + \ln|C|$$
$$\tan\left(\frac{y}{x}\right) = \ln|Cx|$$
$$y = x\tan^{-1}(\ln|Cx|).$$

6. $\dfrac{dy}{dx} = \dfrac{y}{x} - e^{-y/x}$ (let y=vx)

$$v + x\frac{dv}{dx} = v - e^{-v}$$
$$e^v\,dv = -\frac{dx}{x}$$
$$e^v = -\ln|x| + \ln|C|$$
$$e^{y/x} = \ln\left|\frac{C}{x}\right|$$
$$y = x\ln\ln\left|\frac{C}{x}\right|.$$

7. We require $\dfrac{dy}{dx} = \dfrac{2x}{1+y^2}$. Thus

$$\int(1+y^2)\,dy = \int 2x\,dx$$
$$y + \frac{1}{3}y^3 = x^2 + C.$$

Since $(2, 3)$ lies on the curve, $12 = 4 + C$. Thus $C = 8$ and $y + \frac{1}{3}y^3 - x^2 = 8$, or $3y + y^3 - 3x^2 = 24$.

8. $\dfrac{dy}{dx} = 1 + \dfrac{2y}{x}$ Let $y = vx$

$$v + x\frac{dv}{dx} = 1 + 2v$$
$$x\frac{dv}{dx} = 1 + v$$
$$\int\frac{dv}{1+v} = \int\frac{dx}{x}$$
$$\ln|1+v| = \ln|x| + C_1$$
$$1 + \frac{y}{x} = Cx \quad\Rightarrow\quad x + y = Cx^2.$$

Since $(1, 3)$ lies on the curve, $4 = C$. Thus the curve has equation $x + y = 4x^2$.

9. If $\xi = x - x_0$, $\eta = y - y_0$, and

$$\frac{dy}{dx} = \frac{ax+by+c}{ex+fy+g},$$

then

$$\begin{aligned}\frac{d\eta}{d\xi} = \frac{dy}{dx} &= \frac{a(\xi+x_0)+b(\eta+y_0)+c}{e(\xi+x_0)+f(\eta+y_0)+g}\\ &= \frac{a\xi+b\eta+(ax_0+by_0+c)}{e\xi+f\eta+(ex_0+fy_0+g)}\\ &= \frac{a\xi+b\eta}{e\xi+f\eta}\end{aligned}$$

provided x_0 and y_0 are chosen such that

$$ax_0 + by_0 + c = 0, \qquad\text{and}\qquad ex_0 + fy_0 + g = 0.$$

10. The system $x_0 + 2y_0 - 4 = 0$, $2x_0 - y_0 - 3 = 0$ has solution $x_0 = 2$, $y_0 = 1$. Thus, if $\xi = x - 2$ and $\eta = y - 1$, where

$$\frac{dy}{dx} = \frac{x+2y-4}{2x-y-3},$$

then

$$\frac{d\eta}{d\xi} = \frac{\xi+2\eta}{2\xi-\eta} \qquad \text{Let } \eta = v\xi$$
$$v + \xi\frac{dv}{d\xi} = \frac{1+2v}{2-v}$$
$$\xi\frac{dv}{d\xi} = \frac{1+2v}{2-v} - v = \frac{1+v^2}{2-v}$$
$$\int\left(\frac{2-v}{1+v^2}\right)dv = \int\frac{d\xi}{\xi}$$
$$2\tan^{-1}v - \frac{1}{2}\ln(1+v^2) = \ln|\xi| + C_1$$
$$4\tan^{-1}\frac{\eta}{\xi} - \ln(\xi^2+\eta^2) = C.$$

Hence the solution of the original equation is

$$4\tan^{-1}\frac{y-1}{x-2} - \ln\Big((x-2)^2 + (y-1)^2\Big) = C.$$

11. $(xy^2 + y)\,dx + (x^2y + x)\,dy = 0$

$$d\left(\frac{1}{2}x^2y^2 + xy\right) = 0$$
$$x^2y^2 + 2xy = C.$$

12. $(e^x\sin y + 2x)\,dx + (e^x\cos y + 2y)\,dy = 0$

$$d(e^x\sin y + x^2 + y^2) = 0$$
$$e^x\sin y + x^2 + y^2 = C.$$

13. $e^{xy}(1+xy)\,dx + x^2e^{xy}\,dy = 0$

$$d\Big(xe^{xy}\Big) = 0 \quad\Rightarrow\quad xe^{xy} = C.$$

14. $\left(2x + 1 - \dfrac{y^2}{x^2}\right)dx + \dfrac{2y}{x}\,dy = 0$

$$d\left(x^2 + x + \frac{y^2}{x}\right) = 0$$
$$x^2 + x + \frac{y^2}{x} = C.$$

15. $(x^2+2y)\,dx - x\,dy = 0$

$M = x^2+2y, \qquad N = -x$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{3}{x}$ (indep. of y)

$\frac{d\mu}{\mu} = -\frac{3}{x}\,dx \quad \Rightarrow \quad \mu = \frac{1}{x^3}$

$\left(\frac{1}{x}+\frac{2y}{x^3}\right)dx - \frac{1}{x^2}\,dy = 0$

$d\left(\ln|x| - \frac{y}{x^2}\right) = 0$

$\ln|x| - \frac{y}{x^2} = C_1$

$y = x^2\ln|x| + Cx^2.$

16. $(xe^x + x\ln y + y)\,dx + \left(\frac{x^2}{y} + x\ln x + x\sin y\right)dy = 0$

$M = xe^x + x\ln y + y, \qquad N = \frac{x^2}{y} + x\ln x + x\sin y$

$\frac{\partial M}{\partial y} = \frac{x}{y} + 1, \qquad \frac{\partial N}{\partial x} = \frac{2x}{y} + \ln x + 1 + \sin y$

$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{N}\left(-\frac{x}{y} - \ln x - \sin y\right) = -\frac{1}{x}$

$\frac{d\mu}{\mu} = -\frac{1}{x}\,dx \quad \Rightarrow \quad \mu = \frac{1}{x}$

$\left(e^x + \ln y + \frac{y}{x}\right)dx + \left(\frac{x}{y} + \ln x + \sin y\right)dy$

$d\left(e^x + x\ln y + y\ln x - \cos y\right) = 0$

$e^x + x\ln y + y\ln x - \cos y = C.$

17. If $\mu(y)M(x,y)\,dx + \mu(y)N(x,y)\,dy$ is exact, then

$$\frac{\partial}{\partial y}\Big(\mu(y)M(x,y)\Big) = \frac{\partial}{\partial x}\Big(\mu(y)N(x,y)\Big)$$
$$\mu'(y)M + \mu\frac{\partial M}{\partial y} = \mu\frac{\partial N}{\partial x}$$
$$\frac{\mu'}{\mu} = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Thus M and N must be such that

$$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

depends only on y.

18. $2y^2(x+y^2)\,dx + xy(x+6y^2)\,dy = 0$

$(2xy^2+2y^4)\mu(y)\,dx + (x^2y+6xy^3)\mu(y)\,dy = 0$

$\frac{\partial M}{\partial y} = (4xy+8y^3)\mu(y) + (2xy^2+2y^4)\mu'(y)$

$\frac{\partial N}{\partial x} = (2xy+6y^3)\mu(y).$

For exactness we require

$(2xy^2+2y^4)\mu'(y) = [(2xy+6y^3)-(4xy+8y^3)]\mu(y)$

$y(2xy+2y^3)\mu'(y) = -(2xy+2y^3)\mu(y)$

$y\mu'(y) = -\mu(y) \quad \Rightarrow \quad \mu(y) = \frac{1}{y}$

$(2xy+2y^3)\,dx + (x^2+6xy^2)\,dy = 0$

$d(x^2y+2xy^3) = 0 \quad \Rightarrow \quad x^2y+2xy^3 = C.$

19. Consider $y\,dx - (2x+y^3e^y)\,dy = 0$.

Here $M = y$, $N = -2x - y^3e^y$, $\frac{\partial M}{\partial y} = 1$, and $\frac{\partial N}{\partial x} = -2$.

Thus

$$\frac{\mu'}{\mu} = -\frac{3}{y} \quad \Rightarrow \quad \mu = \frac{1}{y^3}$$
$$\frac{1}{y^2}\,dx - \left(\frac{2x}{y^3} + e^y\right)dy = 0$$
$$d\left(\frac{x}{y^2} - e^y\right) = 0$$
$$\frac{x}{y^2} - e^y = C, \qquad \text{or} \qquad x - y^2e^y = Cy^2.$$

20. If $\mu(xy)$ is an integrating factor for $M\,dx + N\,dy = 0$, then

$$\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N), \qquad \text{or}$$
$$x\mu'(xy)M + \mu(xy)\frac{\partial M}{\partial y} = y\mu'(xy)N + \mu(xy)\frac{\partial N}{\partial x}.$$

Thus M and N will have to be such that the right-hand side of the equation

$$\frac{\mu'(xy)}{\mu(xy)} = \frac{1}{xM-yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

depends only on the product xy.

21. For $\left(x\cos x + \frac{y^2}{x}\right)dx - \left(\frac{x\sin x}{y} + y\right)dy$ we have

$$M = x\cos x + \frac{y^2}{x}, \qquad N = -\frac{x\sin x}{y} - y$$
$$\frac{\partial M}{\partial y} = \frac{2y}{x}, \qquad \frac{\partial N}{\partial x} = -\frac{\sin x}{y} - \frac{x\cos x}{y}$$
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -\left(\frac{\sin x}{y} + \frac{x\cos x}{y} + \frac{2y}{x}\right)$$
$$xM - yN = x^2\cos x + y^2 + x\sin x + y^2$$
$$\frac{1}{xM-yN}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{1}{xy}.$$

Thus, an integrating factor is given by

$$\frac{\mu'(t)}{\mu(t)} = -\frac{1}{t} \qquad \Rightarrow \qquad \mu(t) = \frac{1}{t}.$$

We multiply the original equation by $1/(xy)$ to make it exact:

$$\left(\frac{\cos x}{y} + \frac{y}{x^2}\right) dx - \left(\frac{\sin x}{y^2} + \frac{1}{x}\right) dy = 0$$
$$d\left(\frac{\sin x}{y} - \frac{y}{x}\right) = 0$$
$$\frac{\sin x}{y} - \frac{y}{x} = C.$$

The solution is $x \sin x - y^2 = Cxy$.

Section 17.3 Existence, Uniqueness, and Numerical Methods (page 915)

A computer spreadsheet was used in Exercises 1–12. The intermediate results appearing in the spreadsheet are not shown in these solutions.

1. We start with $x_0 = 1$, $y_0 = 0$, and calculate

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h(x_n + y_n).$$

a) For $h = 0.2$ we get $x_5 = 2$, $y_5 = 1.97664$.

b) For $h = 0.1$ we get $x_{10} = 2$, $y_{10} = 2.187485$.

c) For $h = 0.05$ we get $x_{20} = 2$, $y_{20} = 2.306595$.

2. We start with $x_0 = 1$, $y_0 = 0$, and calculate

$$x_{n+1} = x_n + h, \qquad u_{n+1} = y_n + h(x_n + y_n)$$
$$y_{n+1} = y_n + \frac{h}{2}(x_n + y_n + x_{n+1} + u_{n+1}).$$

a) For $h = 0.2$ we get $x_5 = 2$, $y_5 = 2.405416$.

b) For $h = 0.1$ we get $x_{10} = 2$, $y_{10} = 2.428162$.

c) For $h = 0.05$ we get $x_{20} = 2$, $y_{20} = 2.434382$.

3. We start with $x_0 = 1$, $y_0 = 0$, and calculate

$$\begin{aligned} x_{n+1} &= x_n + h \\ p_n &= x_n + y_n \\ q_n &= x_n + \frac{h}{2} + y_n + \frac{h}{2}p_n \\ r_n &= x_n + \frac{h}{2} + y_n + \frac{h}{2}q_n \\ q_n &= x_n + h + y_n + hr_n \\ y_{n+1} &= y_n + \frac{h}{6}(p_n + 2q_n + 2r_n + s_n). \end{aligned}$$

a) For $h = 0.2$ we get $x_5 = 2$, $y_5 = 2.436502$.

b) For $h = 0.1$ we get $x_{10} = 2$, $y_{10} = 2.436559$.

c) For $h = 0.05$ we get $x_{20} = 2$, $y_{20} = 2.436563$.

4. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$x_{n+1} = x_n + h, \qquad y_{n+1} = hx_n\, e^{-y_n}.$$

a) For $h = 0.2$ we get $x_{10} = 2$, $y_{10} = 1.074160$.

b) For $h = 0.1$ we get $x_{20} = 2$, $y_{20} = 1.086635$.

5. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$x_{n+1} = x_n + h, \qquad u_{n+1} = y_n + hx_n e^{-y_n}$$
$$y_{n+1} = y_n + \frac{h}{2}(x_n e^{-y_n} + x_{n+1}e^{-u_{n+1}}.$$

a) For $h = 0.2$ we get $x_{10} = 2$, $y_{10} = 1.097897$.

b) For $h = 0.1$ we get $x_{20} = 2$, $y_{20} = 1.098401$.

6. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$\begin{aligned} x_{n+1} &= x_n + h \\ p_n &= x_n e^{-y_n} \\ q_n &= \left(x_n + \frac{h}{2}\right) e^{-(y_n + (h/2)p_n} \\ r_n &= \left(x_n + \frac{h}{2}\right) e^{-(y_n + (h/2)q_n} \\ s_n &= (x_n + h)e^{-(y_n + hr_n)} \\ y_{n+1} &= y_n + \frac{h}{6}(p_n + 2q_n + 2r_n + s_n). \end{aligned}$$

a) For $h = 0.2$ we get $x_{10} = 2$, $y_{10} = 1.098614$.

b) For $h = 0.1$ we get $x_{20} = 2$, $y_{20} = 1.098612$.

7. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h \cos y_n.$$

a) For $h = 0.2$ we get $x_5 = 1$, $y_5 = 0.89441$.

b) For $h = 0.1$ we get $x_{10} = 1$, $y_{10} = 0.87996$.

c) For $h = 0.05$ we get $x_{20} = 1$, $y_{20} = 0.872831$.

8. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$x_{n+1} = x_n + h, \qquad u_{n+1} = y_n + h \cos y_n$$
$$y_{n+1} = y_n + \frac{h}{2}(\cos y_n + \cos u_{n+1}).$$

a) For $h = 0.2$ we get $x_5 = 1$, $y_5 = 0.862812$.

b) For $h = 0.1$ we get $x_{10} = 1$, $y_{10} = 0.865065$.

c) For $h = 0.05$ we get $x_{20} = 1$, $y_{20} = 0.865598$.

9. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$\begin{aligned} x_{n+1} &= x_n + h \\ p_n &= \cos y_n \\ q_n &= \cos(y_n + (h/2)p_n) \\ r_n &= \cos(y_n + (h/2)q_n) \\ q_n &= \cos(y_n + hr_n) \\ y_{n+1} &= y_n + \frac{h}{6}(p_n + 2q_n + 2r_n + s_n). \end{aligned}$$

a) For $h = 0.2$ we get $x_5 = 1$, $y_5 = 0.865766$.

b) For $h = 0.1$ we get $x_{10} = 1$, $y_{10} = 0.865769$.

c) For $h = 0.05$ we get $x_{20} = 1$, $y_{20} = 0.865769$.

10. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$x_{n+1} = x_n + h, \qquad y_{n+1} = y_n + h\cos(x_n^2).$$

a) For $h = 0.2$ we get $x_5 = 1$, $y_5 = 0.944884$.

b) For $h = 0.1$ we get $x_{10} = 1$, $y_{10} = 0.926107$.

c) For $h = 0.05$ we get $x_{20} = 1$, $y_{20} = 0.915666$.

11. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$\begin{aligned} x_{n+1} &= x_n + h, \qquad u_{n+1} = y_n + h\cos(x_n^2) \\ y_{n+1} &= y_n + \frac{h}{2}(\cos(x_n^2) + \cos(x_{n+1}^2)). \end{aligned}$$

a) For $h = 0.2$ we get $x_5 = 1$, $y_5 = 0.898914$.

b) For $h = 0.1$ we get $x_{10} = 1$, $y_{10} = 0.903122$.

c) For $h = 0.05$ we get $x_{20} = 1$, $y_{20} = 0.904174$.

12. We start with $x_0 = 0$, $y_0 = 0$, and calculate

$$\begin{aligned} x_{n+1} &= x_n + h \\ p_n &= \cos(x_n^2) \\ q_n &= \cos((x_n + (h/2))^2) \\ r_n &= \cos((x_n + (h/2))^2) \\ q_n &= \cos((x_n + h)^2) \\ y_{n+1} &= y_n + \frac{h}{6}(p_n + 2q_n + 2r_n + s_n). \end{aligned}$$

a) For $h = 0.2$ we get $x_5 = 1$, $y_5 = 0.904524$.

b) For $h = 0.1$ we get $x_{10} = 1$, $y_{10} = 0.904524$.

c) For $h = 0.05$ we get $x_{20} = 1$, $y_{20} = 0.904524$.

13.
$$\begin{aligned} & y(x) = 2 + \int_1^x \Big(y(t)\Big)^2\,dt \\ & \frac{dy}{dx} = \Big(y(x)\Big)^2, \qquad y(1) = 2 + 0 = 2 \\ & \frac{dy}{y^2} = dx \ \Rightarrow\ -\frac{1}{y(x)} = x + C \\ & -\frac{1}{2} = 1 + C \ \Rightarrow\ C = -\frac{3}{2} \\ & y = -\frac{1}{x - (3/2)} = \frac{2}{3 - 2x}. \end{aligned}$$

14.
$$\begin{aligned} & u(x) = 1 + 3\int_2^x t^2 u(t)\,dt \\ & \frac{du}{dx} = 3x^2 u(x), \qquad u(2) = 1 + 0 = 1 \\ & \frac{du}{u} = 3x^2\,dx \ \Rightarrow\ \ln u = x^3 + C \\ & 0 = \ln 1 = \ln u(2) = 2^3 + C \ \Rightarrow\ C = -8 \\ & u = e^{x^3 - 8}. \end{aligned}$$

15. For the problem $y' = f(x)$, $y(a) = 0$, the 1-step Runge-Kutta method with $h = b - a$ gives:

$$\begin{aligned} & x_0 = a, \qquad y_0 = 0, \qquad x_1 = x_0 + h = b \\ & p_0 = f(a), \quad q_0 = f\left(a + \frac{h}{2}\right) = f\left(\frac{a+b}{2}\right) = r_0 \\ & s_0 = f(a + h) = f(b) \\ & y_1 = y_0 + \frac{h}{6}(p_0 + 2q_0 + 2r_0 + s_0) \\ & \quad = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right), \end{aligned}$$

which is the Simpson's Rule approximation to $\int_a^b f(x)\,dx$ based on 2 subintervals of length $h/2$.

16. If $\phi(0) = A \ge 0$ and $\phi'(x) \ge k\phi(x)$ on an interval $[0, X]$, where $k > 0$ and $X > 0$, then

$$\frac{d}{dx}\left(\frac{\phi(x)}{e^{kx}}\right) = \frac{e^{kx}\phi'(x) - ke^{kx}\phi(x)}{e^{2kx}} \ge 0.$$

Thus $\phi(x)/e^{kx}$ is increasing on $[0, X]$. Since its value at $x = 0$ is $\phi(0) = A \ge 0$, therefore $\phi(x)/e^{kx} \ge A$ on $[0, X]$, and $\phi(x) \ge Ae^{kx}$ there.

17. a) Suppose $u' = u^2$, $y' = x + y^2$, and $v' = 1 + v^2$ on $[0, X]$, where $u(0) = y(0) = v(0) = 1$, and $X > 0$ is such that $v(x)$ is defined on $[0, X]$. (In part (b) below, we will show that $X < 1$, and we assume this fact now.) Since all three functions are increasing on $[0, X]$, we have $u(x) \ge 1$, $y(x) \ge 1$, and $v(x) \ge 1$ on $[0, X]$.

If $\phi(x) = y(x) - u(x)$, then $\phi(0) = 0$ and

$$\begin{aligned}\phi'(x) &= x + y^2 - u^2 \geq y^2 - u^2 \\ &\geq (y+u)(y-u) \geq 2\phi\end{aligned}$$

on $[0, X]$. By Exercise 16, $\phi(x) \geq 0$ on $[0, X]$, and so $u(x) \leq y(x)$ there.

Similarly, since $X < 1$, if $\phi(x) = v(x) - y(x)$, then $\phi(0) = 0$ and

$$\begin{aligned}\phi'(x) &= 1 + v^2 - x - y^2 \geq v^2 - y^2 \\ &\geq (v+y)(v-y) \geq 2\phi\end{aligned}$$

on $[0, X]$, so $y(x) \leq v(x)$ there.

b) The IVP $u' = u^2$, $u(0) = 1$ has solution $u(x) = \dfrac{1}{1-x}$, obtained by separation of variables. This solution is valid for $x < 1$.

The IVP $v' = 1 + v^2$, $v(0) = 1$ has solution $v(x) = \tan\left(x + \frac{\pi}{4}\right)$, also obtained by separation of variables. It is valid only for $-3\pi/4 < x < \pi/4$. Observe that $\pi/4 < 1$, proving the assertion made about v in part (a). By the result of part (a), the solution of the IVP $y' = x + y^2$, $y(0) = 1$, increases on an interval $[0, X]$ and $\to \infty$ as $x \to X$ from the left, where X is some number in the interval $[\pi/4, 1]$.

c) Here are some approximations to $y(x)$ for values of x near 0.9 obtained by the Runge-Kutta method with $x_0 = 0$ and $y_0 = 1$:

For $h = 0.05$

$n = 17$	$x_n = 0.85$	$y_n = 12.37139$
$n = 18$	$x_n = 0.90$	$y_n = 31.777317$
$n = 19$	$x_n = 0.95$	$y_n = 4071.117315.$

For $h = 0.02$

$n = 43$	$x_n = 0.86$	$y_n = 14.149657$
$n = 44$	$x_n = 0.88$	$y_n = 19.756061$
$n = 45$	$x_n = 0.90$	$y_n = 32.651029$
$n = 46$	$x_n = 0.92$	$y_n = 90.770048$
$n = 47$	$x_n = 0.94$	$y_n = 34266.466629.$

For $h = 0.01$

$n = 86$	$x_n = 0.86$	$y_n = 14.150706$
$n = 87$	$x_n = 0.87$	$y_n = 16.493286$
$n = 88$	$x_n = 0.88$	$y_n = 19.761277$
$n = 89$	$x_n = 0.89$	$y_n = 24.638758$
$n = 90$	$x_n = 0.90$	$y_n = 32.703853$
$n = 91$	$x_n = 0.91$	$y_n = 48.591332$
$n = 92$	$x_n = 0.92$	$y_n = 94.087476$
$n = 93$	$x_n = 0.93$	$y_n = 636.786465$
$n = 94$	$x_n = 0.94$	$y_n = 2.8399 \times 10^{11}.$

The values are still in reasonable agreement at $x = 0.9$, but they start to diverge quickly thereafter. This suggests that X is slightly greater than 0.9.

Section 17.4 Differential Equations of Second Order (page 919)

1. If $y_1 = e^x$, then $y_1'' - 3y_1' + 2y_1 = e^x(1 - 3 + 2) = 0$, so y_1 is a solution of the DE $y'' - 3y' + 2y = 0$. Let $y = e^x v$. Then

$$\begin{aligned}y' &= e^x(v' + v), \qquad y'' = e^x(v'' + 2v' + v) \\ y'' - 3y' + 2y &= e^x(v'' + 2v' + v - 3v' - 3v + 2v) \\ &= e^x(v'' - v').\end{aligned}$$

y satisfies $y'' - 3y' + 2y = 0$ provided $w = v'$ satisfies $w' - w = 0$. This equation has solution $v' = w = C_1 e^x$, so $v = C_1 e^x + C_2$. Thus the given DE has solution $y = e^x v = C_1 e^{2x} + C_2 e^x$.

2. If $y_1 = e^{-2x}$, then $y_1'' - y_1' - 6y_1 = e^{-2x}(4 + 2 - 6) = 0$, so y_1 is a solution of the DE $y'' - y' - 6y = 0$. Let $y = e^{-2x} v$. Then

$$\begin{aligned}y' &= e^{-2x}(v' - 2v), \qquad y'' = e^{-2x}(v'' - 4v' + 4v) \\ y'' - y' - 6y &= e^{-2x}(v'' - 4v' + 4v - v' + 2v - 6v) \\ &= e^x(v'' - 5v').\end{aligned}$$

y satisfies $y'' - y' - 6y = 0$ provided $w = v'$ satisfies $w' - 5w = 0$. This equation has solution $v' = w = (C_1/5)e^{5x}$, so $v = C_1 e^{5x} + C_2$. Thus the given DE has solution $y = e^{-2x} v = C_1 e^{3x} + C_2 e^{-2x}$.

3. If $y_1 = x$ on $(0, \infty)$, then

$$x^2 y_1'' + 2x y_1' - 2y_1 = 0 + 2x - 2x = 0,$$

so y_1 is a solution of the DE $x^2 y'' + 2xy' - 2y = 0$. Let $y = xv(x)$. Then

$$\begin{aligned}y' &= xv' + v, \qquad y'' = xv'' + 2v' \\ x^2 y'' + 2xy' - 2y &= x^3 v'' + 2x^2 v' + 2x^2 v' + 2xv - 2xv \\ &= x^2(xv'' + 4v').\end{aligned}$$

y satisfies $x^2 y'' + 2xy' - 2y = 0$ provided $w = v'$ satisfies $xw' + 4w = 0$.
This equation has solution $v' = w = -3C_1 x^{-4}$ (obtained by separation of variables), so $v = C_1 x^{-3} + C_2$. Thus the given DE has solution $y = xv = C_1 x^{-2} + C_2 x$.

4. If $y_1 = x^2$ on $(0, \infty)$, then

$$x^2 y_1'' - 3xy_1' + 4y_1 = 2x^2 - 6x^2 + 4x^2 = 0,$$

so y_1 is a solution of the DE $x^2y'' - 3xy' + 4y = 0$. Let $y = x^2 v(x)$. Then

$$\begin{aligned} y' &= x^2v' + 2xv, \qquad y'' = x^2v'' + 4xv' + 2v \\ x^2y'' - 3xy' + 4y &= x^4v'' + 4x^3v' + 2x^2v \\ &\quad - 3x^3v' - 6x^2v + 4x^2v \\ &= x^3(xv'' + v'). \end{aligned}$$

y satisfies $x^2y'' - 3xy' + 4y = 0$ provided $w = v'$ satisfies $xw' + w = 0$. This equation has solution $v' = w = C_1/x$ (obtained by separation of variables), so $v = C_1 \ln x + C_2$. Thus the given DE has solution
$y = x^2 v = C_1 x^2 \ln x + C_2 x^2$.

5. If $y = x$, then $y' = 1$ and $y'' = 0$. Thus

$$x^2y'' - x(x+2)y' + (x+2)y = 0.$$

Now let $y = xv(x)$. Then

$$y' = v + xv', \qquad y'' = 2v' + xv''.$$

Substituting these expressions into the differential equation we get

$$\begin{aligned} 2x^2v' + x^3v'' - x^2v - 2xv - x^3v' \\ - 2x^2v' + x^2v + 2xv = 0 \\ x^3v'' - x^3v' = 0, \qquad \text{or } v'' - v' = 0, \end{aligned}$$

which has solution $v = C_1 + C_2e^x$. Hence the general solution of the given differential equation is

$$y = C_1x + C_2xe^x.$$

6. If $y = x^{-1/2}\cos x$, then

$$\begin{aligned} y' &= -\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x \\ y'' &= \frac{3}{4}x^{-5/2}\cos x + x^{-3/2}\sin x - x^{-1/2}\cos x. \end{aligned}$$

Thus

$$\begin{aligned} & x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y \\ &= \frac{3}{4}x^{-1/2}\cos x + x^{1/2}\sin x - x^{3/2}\cos x \\ &\quad - \frac{1}{2}x^{-1/2}\cos x - x^{1/2}\sin x + x^{3/2}\cos x - \frac{1}{4}x^{-1/2}\cos x \\ &= 0. \end{aligned}$$

Therefore $y = x^{-1/2}\cos x$ is a solution of the Bessel equation

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0. \qquad (*)$$

Now let $y = x^{-1/2}(\cos x)v(x)$. Then

$$\begin{aligned} y' &= -\frac{1}{2}x^{-3/2}(\cos x)v - x^{-1/2}(\sin x)v + x^{-1/2}(\cos x)v' \\ y'' &= \frac{3}{4}x^{-5/2}(\cos x)v + x^{-3/2}(\sin x)v - x^{-3/2}(\cos x)v' \\ &\quad - x^{-1/2}(\cos x)v - 2x^{-1/2}(\sin x)v' + x^{-1/2}(\cos x)v''. \end{aligned}$$

If we substitute these expressions into the equation $(*)$, many terms cancel out and we are left with the equation

$$(\cos x)v'' - 2(\sin x)v' = 0.$$

Substituting $u = v'$, we rewrite this equation in the form

$$\begin{aligned} &(\cos x)\frac{du}{dx} = 2(\sin x)u \\ &\int \frac{du}{u} = 2\int \tan x\,dx \quad \Rightarrow \quad \ln|u| = 2\ln|\sec x| + C_0. \end{aligned}$$

Thus $v' = u = C_1\sec^2 x$, from which we obtain

$$v = C_1\tan x + C_2.$$

Thus the general solution of the Bessel equation $(*)$ is

$$y = x^{-1/2}(\cos x)v = C_1x^{-1/2}\sin x + C_2x^{-1/2}\cos x.$$

7. If $y_1 = y$ and $y_2 = y'$ where y satisfies

$$y'' + a_1(x)y' + a_0(x)y = f(x),$$

then $y_1' = y_2$ and $y_2' = -a_0y_1 - a_1y_2 + f$. Thus

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

8. If y satisfies

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x),$$

then let

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \quad \ldots \quad y_n = y^{(n-1)}.$$

Therefore

$$\begin{aligned} &y_1' = y_2, \quad y_2' = y_3, \quad \ldots \quad y_{n-2}' = y_{n-1}, \quad \text{and} \\ &y_n' = -a_0y_1 - a_1y_2 - a_2y_3 - \cdots - a_{n-1}y_n + f, \end{aligned}$$

and we have

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_n \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix}.$$

9. If $\mathbf{y} = C_1 e^{\lambda x}\mathbf{v}$, then

$$\mathbf{y}' = C_1 \lambda e^{\lambda x}\mathbf{v} = C_1 e^{\lambda x}\mathcal{A}\mathbf{v} = \mathcal{A}\mathbf{y}$$

provided λ and $\mathbf{v}$ satisfy $\mathcal{A}\mathbf{v} = \lambda\mathbf{v}$.

10.
$$\begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 6 - 5\lambda + \lambda^2 - 2$$
$$= \lambda^2 - 5\lambda + 4$$
$$= (\lambda - 1)(\lambda - 4) = 0$$

if $\lambda = 1$ or $\lambda = 4$.

Let $\mathcal{A} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$.

If $\lambda = 1$ and $\mathcal{A}\mathbf{v} = \mathbf{v}$, then

$$\mathcal{A} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow v_1 + v_2 = 0.$$

Thus we may take $\mathbf{v} = \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

If $\lambda = 4$ and $\mathcal{A}\mathbf{v} = 4\mathbf{v}$, then

$$\mathcal{A} = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 4\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \Leftrightarrow 2v_1 - v_2 = 0.$$

Thus we may take $\mathbf{v} = \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

By the result of Exercise 9, $\mathbf{y} = e^x\mathbf{v}_1$ and $\mathbf{y} = e^{4x}\mathbf{v}_2$ are solutions of the homogeneous linear system $\mathbf{y}' = \mathcal{A}\mathbf{y}$. Therefore the general solution of the system is

$$\mathbf{y} = C_1 e^x \mathbf{v}_1 + C_2 e^{4x}\mathbf{v}_2,$$

that is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 e^x \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{4x}\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \text{or}$$
$$y_1 = C_1 e^x + C_2 e^{4x}$$
$$y_2 = -C_1 e^x + 2C_2 e^{4x}.$$

Section 17.5 Linear Differential Equations with Constant Coefficients (page 923)

1. $y''' - 4y'' + 3y' = 0$
Auxiliary: $r^3 - 4r^2 + 3r = 0$
$r(r-1)(r-3) = 0 \Rightarrow r = 0, 1, 3$
General solution: $y = C_1 + C_2 e^t + C_3 e^{3t}$.

2. $y^{(4)} - 2y'' + y = 0$
Auxiliary: $r^4 - 2r^2 + 1 = 0$
$(r^2 - 1)^2 = 0 \Rightarrow r = -1, -1, 1, 1$
General solution: $y = C_1 e^{-t} + C_2 t e^{-t} + C_3 e^t + C_4 t e^t$.

3. $y^{(4)} + 2y'' + y = 0$
Auxiliary: $r^4 + 2r^2 + 1 = 0$
$(r^2 + 1)^2 = 0 \Rightarrow r = -i, -i, i, i$
General solution:
$y = C_1 \cos t + C_2 \sin t + C_3 t \cos t + C_4 t \sin t$.

4. $y^{(4)} + 4y^{(3)} + 6y'' + 4y' + y = 0$
Auxiliary: $r^4 + 4r^3 + 6r^2 + 4r + 1 = 0$
$(r+1)^4 = 0 \Rightarrow r = -1, -1, -1, -1$
General solution: $y = e^{-t}(C_1 + C_2 t + C_3 t^2 + C_4 t^3)$.

5. If $y = e^{2t}$, then $y''' - 2y' - 4y = e^{2t}(8 - 4 - 4) = 0$. The auxiliary equation for the DE is $r^3 - 2r - 4 = 0$, for which we already know that $r = 2$ is a root. Dividing the left side by $r - 2$, we obtain the quotient $r^2 + 2r + 2$. Hence the other two auxiliary roots are $-1 \pm i$.
General solution: $y = C_1 e^{2t} + C_2 e^{-t}\cos t + C_3 e^{-t}\sin t$.

6. Aux. eqn: $(r^2 - r - 2)^2(r^2 - 4)^2 = 0$
$(r+1)^2(r-2)^2(r-2)^2(r+2)^2 = 0$
$r = 2, 2, 2, 2, -1, -1, -2, -2.$
The general solution is

$$y = e^{2t}(C_1 + C_2 t + C_3 t^2 + C_4 t^3) + e^{-t}(C_5 + C_6 t) + e^{-2t}(C_7 + C_8 t).$$

7. $x^2 y'' - xy' + y = 0$
aux: $r(r-1) - r + 1 = 0$
$r^2 - 2r + 1 = 0$
$(r-1)^2 = 0, \quad r = 1, 1.$
Thus $y = Ax + Bx\ln x$.

8. $x^2 y'' - xy' - 3y = 0$
$r(r-1) - r - 3 = 0 \Rightarrow r^2 - 2r - 3 = 0$
$\Rightarrow (r-3)(r+1) = 0 \Rightarrow r_1 = -1$ and $r_2 = 3$
Thus, $y = Ax^{-1} + Bx^3$.

9. $$x^2y'' + xy' - y = 0$$
aux: $r(r-1) + r - 1 = 0 \quad \Rightarrow \quad r = \pm 1$
$$y = Ax + \frac{B}{x}.$$

10. Consider $x^2y'' - xy' + 5y = 0$. Since $a = 1$, $b = -1$, and $c = 5$, therefore $(b-a)^2 < 4ac$. Then $k = (a-b)/2a = 1$ and $\omega^2 = 4$. Thus, the general solution is $y = Ax\cos(2\ln x) + Bx\sin(2\ln x)$.

11. $$x^2y'' + xy' = 0$$
aux: $r(r-1) + r = 0 \quad \Rightarrow \quad r = 0,\ 0.$
Thus $y = A + B\ln x$.

12. Given that $x^2y'' + xy' + y = 0$. Since $a = 1$, $b = 1$, $c = 1$ therefore $(b-a)^2 < 4ac$. Then $k = (a-b)/2a = 0$ and $\omega^2 = 1$. Thus, the general solution is $y = A\cos(\ln x) + B\sin(\ln x)$.

13. $x^3y''' + xy' - y = 0$.
Trying $y = x^r$ leads to the auxiliary equation
$$r(r-1)(r-2) + r - 1 = 0$$
$$r^3 - 3r^2 + 3r - 1 = 0$$
$$(r-1)^3 = 0 \ \Rightarrow \ r = 1, 1, 1.$$

Thus $y = x$ is a solution. To find the general solution, try $y = xv(x)$. Then
$$y' = xv' + v, \quad y'' = xv'' + 2v', \quad y''' = xv''' + 3v''.$$

Now $x^3y''' + xy' - y = x^4v''' + 3x^3v'' + x^2v' + xv - xv$
$= x^2(x^2v''' + 3xv'' + v')$,
and y is a solution of the given equation if $v' = w$ is a solution of $x^2w'' + 3xw' + w = 0$. This equation has auxiliary equation $r(r-1) + 3r + 1 = 0$, that is $(r+1)^2 = 0$, so its solutions are
$$v' = w = \frac{C_2}{x} + \frac{2C_3\ln x}{x}$$
$$v = C_1 + C_2\ln x + C_3(\ln x)^2.$$

The general solution of the given equation is, therefore,
$$y = C_1x + C_2x\ln x + C_3x(\ln x)^2.$$

Section 17.6 Nonhomogeneous Linear Equations (page 929)

1. $y'' + y' - 2y = 1$.
The auxiliary equation for $y'' + y' - 2y = 0$ is $r^2 + r - 2 = 0$, which has roots $r = -2$ and $r = 1$. Thus the complementary function is
$$y_h = C_1e^{-2x} + C_2e^x.$$

For a particular solution y_p of the given equation try $y = A$. This satisfies the given equation if $A = -1/2$. Thus the general solution of the given equation is
$$y = -\frac{1}{2} + C_1e^{-2x} + C_2e^x.$$

2. $y'' + y' - 2y = x$.
The complementary function is $y_h = C_1e^{-2x} + C_2e^x$, as shown in Exercise 1. For a particular solution try $y = Ax + B$. Then $y' = A$ and $y'' = 0$, so y satisfies the given equation if
$$x = A - 2(Ax + B) = A - 2B - 2Ax.$$

We require $A - 2B = 0$ and $-2A = 1$, so $A = -1/2$ and $B = -1/4$. The general solution of the given equation is
$$y = -\frac{2x+1}{4} + C_1e^{-2x} + C_2e^x.$$

3. $y'' + y' - 2y = e^{-x}$.
The complementary function is $y_h = C_1e^{-2x} + C_2e^x$, as shown in Exercise 1. For a particular solution try $y = Ae^{-x}$. Then $y' = -Ae^{-x}$ and $y'' = Ae^{-x}$, so y satisfies the given equation if
$$e^{-x} = e^{-x}(A - A - 2A) = -2Ae^{-x}.$$

We require $A = -1/2$. The general solution of the given equation is
$$y = -\frac{1}{2}e^{-x} + C_1e^{-2x} + C_2e^x.$$

4. $y'' + y' - 2y = e^x$.
The complementary function is $y_h = C_1e^{-2x} + C_2e^x$, as shown in Exercise 1. For a particular solution try $y = Axe^x$. Then
$$y' = Ae^x(1+x), \qquad y'' = Ae^x(2+x),$$

so y satisfies the given equation if
$$e^x = Ae^x(2 + x + 1 + x - 2x) = 3Ae^x.$$

We require $A = 1/3$. The general solution of the given equation is
$$y = \frac{1}{3}xe^x + C_1e^{-2x} + C_3e^x.$$

5. $y'' + 2y' + 5y = x^2$.
The homogeneous equation has auxiliary equation $r^2 + 2r + 5 = 0$ with roots $r = -1 \pm 2i$. Thus the complementary function is

$$y_h = C_1 e^{-x}\cos(2x) + C_2 e^{-x}\sin(2x).$$

For a particular solution, try $y = Ax^2 + Bx + C$. Then $y' = 2Ax + B$ and $y'' = 2A$. We have

$$\begin{aligned} x^2 &= y'' + 2y' + 5y \\ &= 2A + 4Ax + 2B + 5Ax^2 + 5Bx + 5C. \end{aligned}$$

Thus we require $5A = 1$, $4A + 5B = 0$, and $2A + 2B + 5C = 0$. This gives $A = 1/5$, $B = -4/25$, and $C = -2/125$. The given equation has general solution

$$y = \frac{x^2}{5} - \frac{4x}{25} - \frac{2}{125} + e^{-x}(C_1\cos(2x) + C_2\sin(2x)).$$

6. $y'' + 4y = x^2$. The complementary function is $y = C_1\cos(2x) + C_2\sin(2x)$. For the given equation, try $y = Ax^2 + Bx + C$. Then

$$x^2 = y'' + 4y = 2A + 4Ax^2 + 4Bx + 4C$$

Thus $2A + 4C = 0$, $4A = 1$, $4B = 0$, and we have $A = \frac{1}{4}$, $B = 0$, and $C = -\frac{1}{8}$. The given equation has general solution

$$y = \frac{1}{4}x^2 - \frac{1}{8} + C_1\cos(2x) + C_2\sin(2x).$$

7. $y'' - y' - 6y = e^{-2x}$.
The homogeneous equation has auxiliary equation $r^2 - r - 6 = 0$ with roots $r = -2$ and $r = 3$. Thus the complementary function is

$$y_h = C_1 e^{-2x} + C_2 e^{3x}.$$

For a particular solution, try $y = Axe^{-2x}$. Then $y' = e^{-2x}(A - 2Ax)$ and $y'' = e^{-2x}(-4A + 4Ax)$. We have

$$\begin{aligned} e^{-2x} &= y'' - y' - 6y \\ &= e^{-2x}(-4A + 4Ax - A + 2Ax - 6Ax) = -5Ae^{-2x}. \end{aligned}$$

Thus we require $A = -1/5$. The given equation has general solution

$$y = -\frac{1}{5}xe^{-2x} + C_1 e^{-2x} + C_2 e^{3x}.$$

8. $y'' + 4y' + 4y = e^{-2x}$.
The homogeneous equation has auxiliary equation $r^2 + 4r + 4 = 0$ with roots $r = -2, -2$. Thus the complementary function is

$$y_h = C_1 e^{-2x} + C_2 x e^{-2x}.$$

For a particular solution, try $y = Ax^2 e^{-2x}$. Then $y' = e^{-2x}(2Ax - 2Ax^2)$ and $y'' = e^{-2x}(2A - 8Ax + 4Ax^2)$. We have

$$\begin{aligned} e^{-2x} &= y'' + 4y' + 4y \\ &= e^{-2x}(2A - 8Ax + 4Ax^2 + 8Ax - 8Ax^2 + 4Ax^2) \\ &= 2Ae^{-2x}. \end{aligned}$$

Thus we require $A = 1/2$. The given equation has general solution

$$y = e^{-2x}\left(\frac{x^2}{2} + C_1 + C_2 x\right).$$

9. $y'' + 2y' + 2y = e^x \sin x$.
The homogeneous equation has auxiliary equation $r^2 + 2r + 2 = 0$ with roots $r = -1 \pm i$. Thus the complementary function is

$$y_h = C_1 e^{-x}\cos x + C_2 e^{-x}\sin x.$$

For a particular solution, try $y = Ae^x\cos x + Be^x\sin x$. Then

$$\begin{aligned} y' &= (A + B)e^x\cos x + (B - A)e^x\sin x \\ y'' &= 2Be^x\cos x - 2Ae^x\sin x. \end{aligned}$$

This satisfies the nonhomogeneous DE if

$$\begin{aligned} e^x\sin x &= y'' + 2y' + 2y \\ &= e^x\cos x(2B + 2(A + B) + 2A) \\ &\qquad + e^x\sin x(-2A + 2(B - A) + 2B) \\ &= e^x\cos x(4A + 4B) + e^x\sin x(4B - 4A). \end{aligned}$$

Thus we require $A + B = 0$ and $4(B - A) = 1$, that is, $B = -A = 1/8$. The given equation has general solution

$$y = \frac{e^x}{8}(\sin x - \cos x) + e^{-x}(C_1\cos x + C_2\sin x).$$

10. $y'' + 2y' + 2y = e^{-x}\sin x$.
The complementary function is the same as in Exercise 9, but for a particular solution we try

$$\begin{aligned} y &= Axe^{-x}\cos x + Bxe^{-x}\sin x \\ y' &= e^{-x}\cos x(A - Ax + Bx) + e^{-x}\sin x(B - Bx - Ax) \\ y'' &= e^{-x}\cos x(2B - 2Bx - 2A) \\ &\qquad + e^{-x}\sin x(2Ax - 2A - 2B). \end{aligned}$$

This satisfies the nonhomogeneous DE if

$$\begin{aligned} e^{-x}\sin x &= y'' + 2y' + 2y \\ &= 2Be^{-x}\cos x - 2Ae^{-x}\sin x. \end{aligned}$$

Thus we require $B = 0$ and $A = -1/2$. The given equation has general solution

$$y = -\frac{1}{2}xe^{-x}\cos x + e^{-x}(C_1\cos x + C_2\sin x).$$

11. $y'' + y' = 4 + 2x + e^{-x}$.
The homogeneous equation has auxiliary equation $r^2 + r = 0$ with roots $r = 0$ and $r = -1$. Thus the complementary function is $y_h = C_1 + C_2e^{-x}$. For a particular solution, try $y = Ax + Bx^2 + Cxe^{-x}$. Then

$$\begin{aligned} y' &= A + 2Bx + e^{-x}(C - Cx) \\ y'' &= 2B + e^{-x}(-2C + Cx). \end{aligned}$$

This satisfies the nonhomogeneous DE if

$$\begin{aligned} 4 + 2x + e^{-x} &= y'' + y' \\ &= A + 2B + 2Bx - Ce^{-x}. \end{aligned}$$

Thus we require $A + 2B = 4$, $2B = 2$, and $-C = 1$, that is, $A = 2$, $B = 1$, $C = -1$. The given equation has general solution

$$y = 2x + x^2 - xe^{-x} + C_1 + C_2e^{-x}.$$

12. $y'' + 2y' + y = xe^{-x}$.
The homogeneous equation has auxiliary equation $r^2 + 2r + 1 = 0$ with roots $r = -1$ and $r = -1$. Thus the complementary function is $y_h = C_1e^{-x} + C_2xe^{-x}$. For a particular solution, try $y = e^{-x}(Ax^2 + Bx^3)$. Then

$$\begin{aligned} y' &= e^{-x}(2Ax + (3B - A)x^2 - Bx^3) \\ y'' &= e^{-x}(2A + (6B - 4A)x - (6B - A)x^2 + Bx^3). \end{aligned}$$

This satisfies the nonhomogeneous DE if

$$\begin{aligned} xe^{-x} &= y'' + 2y' + y \\ &= e^{-x}(2A + 6Bx). \end{aligned}$$

Thus we require $A = 0$ and $B = 1/6$. The given equation has general solution

$$y = \frac{1}{6}x^3e^{-x} + C_1e^{-x} + C_2xe^{-x}.$$

13. $y'' + y' - 2y = e^{-x}$.
The complementary function is $y_h = C_1e^{-2x} + C_2e^x$. For a particular solution use

$$y_p = e^{-2x}u_1(x) + e^xu_2(x),$$

where the coefficients u_1 and u_2 satisfy

$$\begin{aligned} -2e^{-2x}u_1' + e^xu_2' &= e^{-x} \\ e^{-2x}u_1' + e^xu_2' &= 0. \end{aligned}$$

Thus

$$\begin{aligned} u_1' &= -\frac{1}{3}e^x & u_2' &= \frac{1}{3}e^{-2x} \\ u_1 &= -\frac{1}{3}e^x & u_2 &= -\frac{1}{6}e^{-2x}. \end{aligned}$$

Thus $y_p = -\frac{1}{3}e^{-x} - \frac{1}{6}e^{-x} = -\frac{1}{2}e^{-x}$. The general solution of the given equation is

$$y = -\frac{1}{2}e^{-x} + C_1e^{-2x} + C_2e^x.$$

14. $y'' + y' - 2y = e^x$.
The complementary function is $y_h = C_1e^{-2x} + C_2e^x$. For a particular solution use

$$y_p = e^{-2x}u_1(x) + e^xu_2(x),$$

where the coefficients u_1 and u_2 satisfy

$$\begin{aligned} -2e^{-2x}u_1' + e^xu_2' &= e^x \\ e^{-2x}u_1' + e^xu_2' &= 0. \end{aligned}$$

Thus

$$\begin{aligned} u_1' &= -\frac{1}{3}e^{3x} & u_2' &= \frac{1}{3} \\ u_1 &= -\frac{1}{9}e^{3x} & u_2 &= \frac{1}{3}x. \end{aligned}$$

Thus $y_p = -\frac{1}{9}e^x + \frac{1}{3}xe^x$. The general solution of the given equation is

$$\begin{aligned} y &= -\frac{1}{9}e^x + \frac{1}{3}xe^x + C_1e^{-2x} + C_2e^x \\ &= \frac{1}{3}xe^x + C_1e^{-2x} + C_3e^x. \end{aligned}$$

15. $x^2y'' + xy' - y = x^2$.
If $y = Ax^2$, then $y' = 2Ax$ and $y'' = 2A$. Thus

$$\begin{aligned} x^2 &= x^2y'' + xy' - y \\ &= 2Ax^2 + 2Ax^2 - Ax^2 = 3Ax^2, \end{aligned}$$

so $A = 1/3$. A particular solution of the given equation is $y = x^2/3$. The auxiliary equation for the homogeneous equation $x^2y'' + xy' - y = 0$ is $4r(r-1) + r - 1 = 0$, or $r^2 - 1 = 0$, which has solutions $r = \pm 1$. Thus the general solution of the given equation is

$$y = \frac{1}{3}x^2 + C_1x + \frac{C_2}{x}.$$

16. $x^2y'' + xy' - y = x^r$ has a solution of the form $y = Ax^r$ provided $r \neq \pm 1$. If this is the case, then

$$x^r = Ax^r\Big(r(r-1) + r - 1\Big) = Ax^r(r^2 - 1).$$

Thus $A = 1/(r^2 - 1)$ and a particular solution of the DE is

$$y = \frac{1}{r^2 - 1}x^r.$$

17. $x^2y'' + xy' - y = x$.
Try $y = Ax\ln x$. Then $y' = A(\ln x + 1)$ and $y'' = A/x$. We have

$$x = x^2\frac{A}{x} + xA(\ln x + 1) - Ax\ln x = 2Ax.$$

Thus $A = 1/2$. The complementary function was obtained in Exercise 15. The given equation has general solution

$$y = \frac{1}{2}x\ln x + C_1x + \frac{C_2}{x}.$$

18. $x^2y'' + xy' - y = x$.
Try $y = xu_1(x) + \frac{1}{x}u_2(x)$, where u_1 and u_2 satisfy

$$xu_1' + \frac{u_2'}{x} = 0, \quad u_1' - \frac{u_2'}{x^2} = \frac{1}{x}.$$

Solving these equations for u_1' and u_2', we get

$$u_2' = -\frac{x}{2}, \qquad u_1' = \frac{1}{2x}.$$

Thus $u_1 = \frac{1}{2}\ln x$ and $u_2 = -\frac{x^2}{4}$. A particular solution is

$$y = \frac{1}{2}x\ln x - \frac{x}{4}.$$

The term $-x/4$ can be absorbed into the term C_1x in the complementary function, so the general solution is

$$y = \frac{1}{2}x\ln x + C_1x + \frac{C_2}{x}.$$

19. $x^2y'' - (2x + x^2)y' + (2 + x)y = x^3$.
Since x and xe^x are independent solutions of the corresponding homogeneous equation, we can write a solution of the given equation in the form

$$y = xu_1(x) + xe^xu_2(x),$$

where u_1 and u_2 are chosen to satisfy

$$xu_1' + xe^xu_2' = 0, \quad u_1' + (1 + x)e^xu_2' = x.$$

Solving these equations for u_1' and u_2', we get $u_1' = -1$ and $u_2' = e^{-x}$. Thus $u_1 = -x$ and $u_2 = -e^{-x}$. The particular solution is $y = -x^2 - x$. Since $-x$ is a solution of the homogeneous equation, we can absorb that term into the complementary function and write the general solution of the given DE as

$$y = -x^2 + C_1x + C_2xe^x.$$

20. $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = x^{3/2}$.
A particular solution can be obtained in the form

$$y = x^{-1/2}(\cos x)u_1(x) + x^{-1/2}(\sin x)u_2(x),$$

where u_1 and u_2 satisfy

$$\begin{aligned}
&x^{-1/2}(\cos x)u_1' + x^{-1/2}(\sin x)u_2' = 0\\
&\left(-\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x\right)u_1'\\
&\qquad - \left(\frac{1}{2}x^{-3/2}\sin x - x^{-1/2}\cos x\right)u_2' = x^{-1/2}.
\end{aligned}$$

We can simplify these equations by dividing the first by $x^{-1/2}$, and adding the first to $2x$ times the second, then dividing the result by $2x^{1/2}$. The resulting equations are

$$\begin{aligned}
(\cos x)u_1' + (\sin x)u_2' &= 0\\
-(\sin x)u_1' + (\cos x)u_2' &= 1,
\end{aligned}$$

which have solutions $u_1' = -\sin x$, $u_2' = \cos x$, so that $u_1 = \cos x$ and $u_2 = \sin x$. Thus a particular solution of the given equation is

$$y = x^{-1/2}\cos^2 x + x^{-1/2}\sin^2 x = x^{-1/2}.$$

The general solution is

$$y = x^{-1/2}\Big(1 + C_2\cos x + C_2\sin x\Big).$$

Section 17.7 Series Solutions of Differential Equations (page 933)

1. $y'' = (x-1)^2 y$. Try

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n (x-1)^n. \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2} \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n \\ 0 &= y'' - (x-1)^2 y \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=0}^{\infty} a_n(x-1)^{n+2} \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-1)^n - \sum_{n=2}^{\infty} a_{n-2}(x-1)^n \\ &= 2a_2 + 6a_3(x-1) \\ &\quad + \sum_{n=2}^{\infty}\Big[(n+2)(n+1)a_{n+2} - a_{n-2}\Big](x-1)^n. \end{aligned}$$

Thus $a_2 = a_3 = 0$, and $a_{n+2} = \dfrac{a_{n-2}}{(n+1)(n+2)}$ for $n \ge 2$. Given a_0 and a_1 we have

$$\begin{aligned} a_4 &= \frac{a_0}{3\times 4} \\ a_8 &= \frac{a_4}{7\times 8} = \frac{a_0}{3\times 4\times 7\times 8} \\ &\ \vdots \\ a_{4n} &= \frac{a_0}{3\times 4\times 7\times 8\times\cdots\times(4n-1)(4n)} \\ &= \frac{a_0}{4^n n!\times 3\times 7\times\cdots\times(4n-1)} \\ a_5 &= \frac{a_1}{4\times 5} \\ a_9 &= \frac{a_5}{8\times 9} = \frac{a_1}{4\times 5\times 8\times 9} \\ &\ \vdots \\ a_{4n+1} &= \frac{a_1}{4\times 5\times 8\times 9\times\cdots\times(4n)(4n+1)} \\ &= \frac{a_1}{4^n n!\times 5\times 9\times\cdots\times(4n+1)} \\ a_{4n+3} &= a_{4n+2} = \cdots = a_3 = a_2 = 0. \end{aligned}$$

The solution is

$$\begin{aligned} y &= a_0\left(1 + \sum_{n=1}^{\infty}\frac{(x-1)^{4n}}{4^n n!\times 3\times 7\times\cdots\times(4n-1)}\right) \\ &\quad + a_1\left(x - 1 + \sum_{n=1}^{\infty}\frac{(x-1)^{4n+1}}{4^n n!\times 5\times 9\times\cdots\times(4n+1)}\right). \end{aligned}$$

2. $y'' = xy$. Try $\sum_{n=0}^{\infty} a_n x^n$. Then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n. \end{aligned}$$

Thus we have

$$\begin{aligned} 0 &= y'' - xy \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty} a_n x^{n+1} \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} a_{n-1}x^n \\ &= 2a_2 + \sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{n+2} - a_{n-1}\Big]x^n. \end{aligned}$$

Thus $a_2 = 0$ and $a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)}$ for $n \ge 1$. Given a_0 and a_1, we have

$$\begin{aligned} a_3 &= \frac{a_0}{2\times 3} \\ a_6 &= \frac{a_3}{5\times 6} = \frac{a_0}{2\times 3\times 5\times 6} = \frac{1\times 4\times a_0}{6!} \\ a_9 &= \frac{a_6}{8\times 9} = \frac{1\times 4\times 7\times a_0}{9!} \\ &\ \vdots \\ a_{3n} &= \frac{1\times 4\times\cdots\times(3n-2)a_0}{(3n)!} \\ a_4 &= \frac{a_1}{3\times 4} = \frac{2\times a_1}{4!} \\ a_7 &= \frac{a_4}{6\times 7} = \frac{2\times 5\times a_1}{7!} \\ &\ \vdots \\ a_{3n+1} &= \frac{2\times 5\times\cdots\times(3n-1)a_1}{(3n+1)!} \\ 0 &= a_2 = a_5 = a_8 = \cdots = a_{3n+2}. \end{aligned}$$

Thus the general solution of the given equation is

$$\begin{aligned} y &= a_0\left(1 + \sum_{n=1}^{\infty}\frac{1\times 4\times\cdots\times(3n-2)}{(3n)!}x^{3n}\right) \\ &\quad + a_1\sum_{n=1}^{\infty}\frac{2\times 5\times\cdots\times(3n-1)}{(3n+1)!}x^{3n+1}. \end{aligned}$$

3. $\begin{cases} y'' + xy' + 2y = 0 \\ y(0) = 1 \\ y'(0) = 2 \end{cases}$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.$$

Substituting these expressions into the differential equation, we get

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=1}^{\infty} n a_n x^n$$
$$+ 2\sum_{n=0}^{\infty} a_n x^n = 0, \qquad \text{so}$$
$$2a_2 + 2 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n+2)a_n]x^n = 0.$$

It follows that

$$a_2 = -1, \qquad a_{n+2} = -\frac{a_n}{n+1}, \qquad n = 1, 2, 3, \ldots.$$

Since $a_0 = y(0) = 1$, and $a_1 = y'(0) = 2$, we have

$$\begin{array}{ll} a_0 = 1 & a_1 = 2 \\ a_2 = -1 & a_3 = -\frac{2}{2} \\ a_4 = \frac{1}{3} & a_5 = \frac{2}{2\times 4} \\ a_6 = -\frac{1}{3\times 5} & a_7 = -\frac{2}{2\times 4\times 6} \\ a_8 = \frac{1}{3\times 5\times 7} & a_9 = \frac{2}{2\times 4\times 6\times 8}. \end{array}$$

The patterns here are obvious:

$$a_{2n} = \frac{(-1)^n}{3\times 5\times\cdots\times(2n-1)} = \frac{(-1)^n 2^n n!}{(2n)!} \qquad a_{2n+1} = \frac{(-1)^n 2}{2^n n!}$$

Thus $y = \sum_{n=0}^{\infty}(-1)^n\left[\frac{2^n n! x^{2n}}{(2n)!} + \frac{x^{2n+1}}{2^{n-1}n!}\right]$.

4. If $y = \sum_{n=0}^{\infty} a_n x^n$, then $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

Thus,

$$\begin{aligned} 0 &= y'' + xy' + y \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + x\sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n \\ &= 2a_2 + a_0 + \sum_{n=1}^{\infty}\Big[(n+2)(n+1)a_{n+2} + (n+1)a_n\Big]x^n. \end{aligned}$$

Since coefficients of all powers of x must vanish, therefore $2a_2 + a_0 = 0$ and, for $n \ge 1$,

$$(n+2)(n+1)a_{n+2} + (n+1)a_n = 0,$$
$$\text{that is,} \quad a_{n+2} = \frac{-a_n}{n+2}.$$

If $y(0) = 1$, then $a_0 = 1$, $a_2 = \frac{-1}{2}$, $a_4 = \frac{1}{2^2\cdot 2!}$, $a_6 = \frac{-1}{2^3\cdot 3!}$, $a_8 = \frac{1}{2^4\cdot 4!}, \ldots$. If $y'(0) = 0$, then $a_1 = a_3 = a_5 = \ldots = 0$. Hence,

$$y = 1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 - \frac{1}{48}x^6 + \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\cdot n!}x^{2n}.$$

5. $y'' + (\sin x)y = 0$, $y(0) = 1$, $y'(0) = 0$. Try

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots.$$

Then $a_0 = 1$ and $a_1 = 0$. We have

$$\begin{aligned} y'' &= 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + \cdots \\ (\sin x)y &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\ &\quad \times (1 + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots) \\ &= x + \left(a_2 - \frac{1}{6}\right)x^3 + a_3x^4 \\ &\quad + \left(a_4 - \frac{1}{6}a_2 + \frac{1}{120}\right)x^5 + \cdots. \end{aligned}$$

Hence we must have $2a_2 = 0$, $6a_3 + 1 = 0$, $12a_4 = 0$, $20a_5 + a_2 - \frac{1}{6} = 0, \ldots$. That is, $a_2 = 0$, $a_4 = 0$, $a_3 = -\frac{1}{6}$, $a_5 = \frac{1}{120}$. The solution is

$$y = 1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots.$$

6. $(1 - x^2)y'' - xy' + 9y = 0$, $y(0) = 0$, $y'(0) = 1$. Try

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Then $a_0 = 0$ and $a_1 = 1$. We have

$$\begin{aligned}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \\
0 &= (1-x^2)y'' - xy' + 9y \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{\infty} n(n-1)a_n x^n \\
&\quad - \sum_{n=1}^{\infty} n a_n x^n + 9\sum_{n=0}^{\infty} a_n x^n \\
&= 2a_2 + 9a_0 + (6a_3 + 8a_1)x \\
&\quad + \sum_{n=2}^{\infty}\Big[(n+2)(n+1)a_{n+2} - (n^2-9)a_n\Big]x^n.
\end{aligned}$$

Thus $2a_2 + 9a_0 = 0$, $6a_3 + 8a_1 = 0$, and

$$a_{n+2} = \frac{(n^2-9)a_n}{(n+1)(n+2)}.$$

Therefore we have

$$\begin{aligned}
&a_2 = a_4 = a_6 = \cdots = 0 \\
&a_3 = -\frac{4}{3}, \quad a_5 = 0 = a_7 = a_9 = \cdots.
\end{aligned}$$

The initial-value problem has solution

$$y = x - \frac{4}{3}x^3.$$

7. $3xy'' + 2y' + y = 0$.
Since $x = 0$ is a regular singular point of this equation, try

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+\mu} \qquad (a_0 = 1) \\
y' &= \sum_{n=0}^{\infty} (n+\mu) a_n x^{n+\mu-1} \\
y'' &= \sum_{n=0}^{\infty} (n+\mu)(n+\mu-1) a_n x^{n+\mu-2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
0 &= 3xy'' + 2y' + y \\
&= \sum_{n=0}^{\infty}\Big[3(n+\mu)^2 - (n+\mu)\Big]a_n x^{n+\mu-1} + \sum_{n=1}^{\infty} a_{n-1}x^{n+\mu-1} \\
&= (3\mu^2 - \mu)x^{\mu-1} \\
&\quad + \sum_{n=1}^{\infty}\Big[\Big(3(n+\mu)^2 - (n+\mu)\Big)a_n + a_{n-1}\Big]x^{n+\mu-1}.
\end{aligned}$$

Thus $3\mu^2 - \mu = 0$ and $a_n = -\dfrac{a_{n-1}}{3(n+\mu)^2 - (n+\mu)}$ for $n \ge 1$. There are two cases: $\mu = 0$ and $\mu = 1/3$.

CASE I. $\mu = 0$. Then $a_n = -\dfrac{a_{n-1}}{n(3n-1)}$. Since $a_0 = 1$ we have

$$\begin{aligned}
a_1 &= -\frac{1}{1\times 2}, \qquad a_2 = \frac{1}{1\times 2\times 2\times 5} \\
a_3 &= -\frac{1}{1\times 2\times 2\times 5\times 3\times 8} \\
&\vdots \\
a_n &= \frac{(-1)^n}{n!\times 2\times 5\times\cdots\times(3n-1)}.
\end{aligned}$$

One series solution is

$$y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!\times 2\times 5\times\cdots\times(3n-1)}.$$

CASE II. $\mu = \dfrac{1}{3}$. Then

$$a_n = \frac{-a_{n-1}}{3\left(n+\frac{1}{3}\right)^2 - \left(n+\frac{1}{3}\right)} = \frac{-a_{n-1}}{n(3n+1)}.$$

Since $a_0 = 1$ we have

$$\begin{aligned}
a_1 &= -\frac{1}{1\times 4}, \qquad a_2 = \frac{1}{1\times 4\times 2\times 7} \\
a_3 &= -\frac{1}{1\times 4\times 2\times 7\times 3\times 10} \\
&\vdots \\
a_n &= \frac{(-1)^n}{n!\times 1\times 4\times 7\times\cdots\times(3n+1)}.
\end{aligned}$$

A second series solution is

$$y = x^{1/3}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!\times 1\times 4\times 7\times\cdots\times(3n+1)}\right).$$

8. $xy'' + y' + xy = 0$.
Since $x = 0$ is a regular singular point of this equation, try

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+\mu} \qquad (a_0 = 1) \\
y' &= \sum_{n=0}^{\infty} (n+\mu) a_n x^{n+\mu-1} \\
y'' &= \sum_{n=0}^{\infty} (n+\mu)(n+\mu-1) a_n x^{n+\mu-2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
0 &= xy'' + y' + xy \\
&= \sum_{n=0}^{\infty}\Big[(n+\mu)(n+\mu-1)+(n+\mu)\Big]a_n x^{n+\mu-1} \\
&\quad + \sum_{n=0}^{\infty} a_n x^{n+\mu+1} \\
&= \sum_{n=0}^{\infty}(n+\mu)^2 a_n x^{n+\mu-1} + \sum_{n=2}^{\infty} a_{n-2}x^{n+\mu-1} \\
&= \mu^2 x^{\mu-1} + (1+\mu)^2 a_1 x^{\mu} \\
&\quad + \sum_{n=2}^{\infty}\Big[(n+\mu)^2 a_n + a_{n-2}\Big]x^{n+\mu-1}.
\end{aligned}$$

Thus $\mu = 0$, $a_1 = 0$, and $a_n = -\dfrac{a_{n-2}}{n^2}$ for $n \ge 2$.
It follows that $0 = a_1 = a_3 = a_5 = \cdots$, and, since $a_0 = 1$,

$$a_2 = -\frac{1}{2^2}, \qquad a_4 = \frac{1}{2^2 4^2}, \ \ldots$$
$$a_{2n} = \frac{(-1)^n}{2^2 4^2 \cdots (2n)^2} = \frac{(-1)^n}{2^{2n}(n!)^2}.$$

One series solution is

$$y = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

Review Exercises 17 (page 934)

1. $\dfrac{dy}{dx} = 2xy$

$\dfrac{dy}{y} = 2x\,dx \ \Rightarrow\ \ln|y| = x^2 + C_1$

$y = Ce^{x^2}$

2. $\dfrac{dy}{dx} = e^{-y}\sin x$

$e^y\,dy = \sin x\,dx \ \Rightarrow\ e^y = -\cos x + C$

$y = \ln(C - \cos x)$

3. $\dfrac{dy}{dx} = x + 2y \ \Rightarrow\ \dfrac{dy}{dx} - 2y = x$

$\dfrac{d}{dx}(e^{-2x}y) = e^{-2x}\left(\dfrac{dy}{dx} - 2y\right) = xe^{-2x}$

$e^{-2x}y = \displaystyle\int xe^{-2x}\,dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + C$

$y = -\dfrac{x}{2} - \dfrac{1}{4} + Ce^{2x}$

4. $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2xy}$ (let $y = xv(x)$)

$v + x\dfrac{dv}{dx} = \dfrac{1+v^2}{2v}$

$x\dfrac{dv}{dx} = \dfrac{1+v^2}{2v} - v = \dfrac{1-v^2}{2v}$

$\dfrac{2v\,dv}{v^2-1} = -\dfrac{dx}{x}$

$\ln(v^2-1) = \ln\dfrac{1}{x} + \ln C = \ln\dfrac{C}{x}$

$\dfrac{y^2}{x^2} - 1 = \dfrac{C}{x} \ \Rightarrow\ y^2 - x^2 = Cx$

5. $\dfrac{dy}{dx} = \dfrac{x+y}{y-x}$

$(x+y)\,dx + (x-y)\,dy = 0$ (exact)

$d\left(\dfrac{x^2}{2} + xy - \dfrac{y^2}{2}\right) = 0$

$x^2 + 2xy - y^2 = C$

6. $\dfrac{dy}{dx} = -\dfrac{y+e^x}{x+e^y}$

$(y+e^x)\,dx + (x+e^y)\,dy = 0$ (exact)

$d\left(xy + e^x + e^y\right) = 0$

$xy + e^x + e^y = C$

7. $\dfrac{d^2y}{dt^2} = \left(\dfrac{dy}{dt}\right)^2$ (let $p = dy/dt$)

$\dfrac{dp}{dt} = p^2 \ \Rightarrow\ \dfrac{dp}{p^2} = dt$

$\dfrac{1}{p} = C_1 - t$

$\dfrac{dy}{dt} = p = \dfrac{1}{C_1 - t}$

$y = \displaystyle\int \frac{dt}{C_1 - t} = -\ln|t - C_1| + C_2$

8. $2\dfrac{d^2y}{dt^2} + 5\dfrac{dy}{dt} + 2y = 0$

Aux: $2r^2 + 5r + 2 = 0 \ \Rightarrow\ r = -1/2,\ -2$

$y = C_1e^{-t/2} + C_2e^{-2t}$

9. $4y'' - 4y' + 5y = 0$

Aux: $4r^2 - 4r + 5 = 0$

$(2r-1)^2 + 4 = 0 \ \Rightarrow\ r = \dfrac{1}{2} \pm i$

$y = C_1e^{x/2}\cos x + C_2e^{x/2}\sin x$

10. $2x^2y'' + y = 0$

Aux: $2r(r-1) + 1 = 0$

$2r^2 - 2r + 1 = 0 \ \Rightarrow\ r = \dfrac{1}{2}(1 \pm i)$

$y = C_1|x|^{1/2}\cos\left(\frac{1}{2}\ln|x|\right) + C_2|x|^{1/2}\sin\left(\frac{1}{2}\ln|x|\right)$

11. $t^2\dfrac{d^2y}{dt^2} - t\dfrac{dy}{dt} + 5y = 0$

Aux: $r(r-1) - r + 5 = 0$

$(r-1)^2 + 4 = 0 \ \Rightarrow\ r = 1 \pm 2i$

$y = C_1 t\cos(2\ln|t|) + C_2 t\sin(2\ln|t|)$

12. $\dfrac{d^3y}{dt^3} + 8\dfrac{d^2y}{dt^2} + 16\dfrac{dy}{dt} = 0$

Aux: $r^3 + 8r^2 + 16r = 0$

$r(r+4)^2 = 0 \ \Rightarrow\ r = 0, -4, -4$

$y = C_1 + C_2e^{-4t} + C_3te^{-4t}$

13. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = e^x + e^{3x}$

Aux: $r^2 - 5r + 6 = 0 \ \Rightarrow\ r = 2, 3.$

Complementary function: $y = C_1e^{2x} + C_2e^{3x}$.

Particular solution: $y = Ae^x + Bxe^{3x}$

$y' = Ae^x + B(1+3x)e^{3x}$

$y'' = Ae^x + B(6+9x)e^{3x}$

$$\begin{aligned} e^x + e^{3x} &= Ae^x(1-5+6) \\ &\qquad + Be^{3x}(6+9x-5-15x+6x) \\ &= 2Ae^x + Be^{3x}. \end{aligned}$$

Thus $A = 1/2$ and $B = 1$. The general solution is

$$y = \frac{1}{2}e^x + xe^{3x} + C_1e^{2x} + C_2e^{3x}.$$

14. $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = xe^{2x}$

Same complementary function as in Exercise 13: $C_1e^{2x} + C_2e^{3x}$. For a particular solution we try $y = (Ax^2 + Bx)e^{2x}$. Substituting this into the given DE leads to

$$xe^{2x} = (2A - B)e^{2x} - 2Axe^{2x},$$

so that we need $A = -1/2$ and $B = 2A = -1$. The general solution is

$$y = -\left(\frac{1}{2}x^2 + x\right)e^{2x} + C_1e^{2x} + C_2e^{3x}.$$

15. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x^2$

Aux: $r^2 + 2r + 1 = 0$ has solutions $r = -1, -1$.

Complementary function: $y = C_1e^{-x} + C_2xe^{-x}$.

Particular solution: try $y = Ax^2 + Bx + C$. Then

$$x^2 = 2A + 2(2Ax + B) + Ax^2 + Bx + C.$$

Thus $A = 1$, $B = -4$, $C = 6$. The general solution is

$$y = x^2 - 4x + 6 + C_1e^{-x} + C_2xe^{-x}.$$

16. $x^2\dfrac{d^2y}{dx^2} - 2y = x^3.$

The corresponding homogeneous equation has auxiliary equation $r(r-1) - 2 = 0$, with roots $r = 2$ and $r = -1$, so the complementary function is $y = C_1x^2 + C_2/x$. A particular solution of the non-homogeneous equation can have the form $y = Ax^3$. Substituting this into the DE gives

$$6Ax^3 - 2Ax^3 = x^3,$$

so that $A = 1/4$. The general solution is

$$y = \frac{1}{4}x^3 + C_1x^2 + \frac{C_2}{x}.$$

17. $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}, \quad y(2) = 1$

$y^2\,dy = x^2\,dx$

$y^3 = x^3 + C$

$1 = 8 + C \ \Rightarrow\ C = -7$

$y^3 = x^3 - 7 \ \Rightarrow\ y = (x^3 - 7)^{1/3}$

18. $\dfrac{dy}{dx} = \dfrac{y^2}{x^2}, \quad y(2) = 1$

$\dfrac{dy}{y^2} = \dfrac{dx}{x^2} \ \Rightarrow\ -\dfrac{1}{y} = -\dfrac{1}{x} - C$

$1 = \dfrac{1}{2} + C \ \Rightarrow\ C = \dfrac{1}{2}$

$y = \left(\dfrac{1}{x} + \dfrac{1}{2}\right)^{-1} = \dfrac{2x}{x+2}$

19. $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}$, $y(0) = 1$. Let $y = xv(x)$. Then

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{v}{1+v^2} \\ x\frac{dv}{dx} &= \frac{v}{1+v^2} - v = -\frac{v^3}{1+v^2} \\ -\frac{1+v^2}{v^3}\,dv &= \frac{dx}{x} \\ \frac{1}{2v^2} - \ln|v| &= \ln|x| + \ln C \\ \frac{x^2}{y^2} = \frac{1}{v^2} &= \ln(Cvx)^2 = \ln(C^2y^2) \\ C^2y^2 = e^{x^2/y^2}, &\quad y(0) = 1 \Rightarrow C^2 = 1 \\ y^2 = e^{x^2/y^2}, &\quad \text{or } y = e^{x^2/(2y^2)}. \end{aligned}$$

20. $\frac{dy}{dx} + (\cos x)y = 2\cos x, \qquad y(\pi) = 1$

$\frac{d}{dx}\left(e^{\sin x}y\right) = e^{\sin x}\left(\frac{dy}{dx} + (\cos x)y\right) = 2\cos x e^{\sin x}$

$e^{\sin x}y = 2e^{\sin x} + C$

$y = 2 + Ce^{-\sin x}$

$1 = 2 + Ce^0 \Rightarrow C = -1$

$y = 2 - e^{-\sin x}$

21. $y'' + 3y' + 2y = 0,\ y(0) = 1,\ y'(0) = 2$
Aux: $r^2 + 3r + 2 = 0 \Rightarrow r = -1, -2.$

$$\begin{aligned} y &= Ae^{-x} + Be^{-2x} \quad \Rightarrow \quad 1 = A + B \\ y' &= -Ae^{-x} - 2Be^{-2x} \quad \Rightarrow \quad 2 = -A - 2B. \end{aligned}$$

Thus $B = -3$, $A = 4$. The solution is $y = 4e^{-x} - 3e^{-2x}$.

22. $y'' + 2y' + (1+\pi^2)y = 0,\ y(1) = 0,\ y'(1) = \pi$
Aux: $r^2 + 2r + 1 + \pi^2 = 0 \Rightarrow r = -1 \pm \pi i.$

$$\begin{aligned} y &= Ae^{-x}\cos(\pi x) + Be^{-x}\sin(\pi x) \\ y' &= e^{-x}\cos(\pi x)(-A + B\pi) + e^{-x}\sin(\pi x)(-B - A\pi). \end{aligned}$$

Thus $-Ae^{-1} = 0$ and $(A - B\pi)e^{-1} = \pi$, so that $A = 0$ and $B = -e$. The solution is $y = -e^{1-x}\sin(\pi x)$.

23. $y'' + 10y' + 25y = 0,\ y(1) = e^{-5},\ y'(1) = 0$
Aux: $r^2 + 10r + 25 = 0 \Rightarrow r = -5,\ -5.$

$$\begin{aligned} y &= Ae^{-5x} + Bxe^{-5x} \\ y' &= -5Ae^{-5x} + B(1-5x)e^{-5x}. \end{aligned}$$

We require $e^{-5} = (A+B)e^{-5}$ and $0 = e^{-5}(-5A - 4B)$. Thus $A + B = 1$ and $-5A = 4B$, so that $B = 5$ and $A = -4$. The solution is $y = -4e^{-5x} + 5xe^{-5x}$.

24. $x^2y'' - 3xy' + 4y = 0,\ y(e) = e^2,\ y'(e) = 0$
Aux: $r(r-1) - 3r + 4 = 0$, or $(r-2)^2 = 0$, so that $r = 2,\ 2.$

$$\begin{aligned} y &= Ax^2 + Bx^2\ln x \\ y' &= 2Ax + 2Bx\ln x + Bx. \end{aligned}$$

We require $e^2 = Ae^2 + Be^2$ and $0 = 2Ae + 3Be$. Thus $A + B = 1$ and $2A = -3B$, so that $A = 3$ and $B = -2$. The solution is $y = 3x^2 - 2x^2\ln x$, valid for $x > 0$.

25. $\frac{d^2y}{dt^2} + 4y = 8e^{2t},\ y(0) = 1,\ y'(0) = -2$
Complementary function: $y = C_1\cos(2t) + C_2\sin(2t)$.
Particular solution: $y = Ae^{2t}$, provided $4A + 4A = 8$, that is, $A = 1$. Thus

$$\begin{aligned} y &= e^{2t} + C_1\cos(2t) + C_2\sin(2t) \\ y' &= 2e^{2t} - 2C_1\sin(2t) + 2C_2\cos(2t). \end{aligned}$$

We require $1 = y(0) = 1 + C_1$ and $-2 = y'(0) = 2 + 2C_2$. Thus $C_1 = 0$ and $C_2 = -2$. The solution is $y = e^{2t} - 2\sin(2t)$.

26. $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 3y = 6 + 7e^{x/2},\ y(0) = 0,\ y'(0) = 1$
Aux: $2r^2 + 5r - 3 = 0 \Rightarrow r = 1/2,\ -3.$
Complementary function: $y = C_1e^{x/2} + C_2e^{-3x}$.
Particular solution: $y = A + Bxe^{x/2}$

$$\begin{aligned} y' &= Be^{x/2}\left(1 + \frac{x}{2}\right) \\ y'' &= Be^{x/2}\left(1 + \frac{x}{4}\right). \end{aligned}$$

We need

$$Be^{x/2}\left(2 + \frac{x}{2} + 5 + \frac{5x}{2} - 3x\right) - 3A = 6 + 7e^{x/2}.$$

This is satisfied if $A = -2$ and $B = 1$. The general solution of the DE is

$$y = -2 + xe^{x/2} + C_1e^{x/2} + C_2e^{-3x}.$$

Now the initial conditions imply that

$$\begin{aligned} 0 &= y(0) = -2 + C_1 + C_2 \\ 1 &= y'(0) = 1 + \frac{C_1}{2} - 3C_2, \end{aligned}$$

which give $C_1 = 12/7$, $C_2 = 2/7$. Thus the IVP has solution

$$y = -2 + xe^{x/2} + \frac{1}{7}(12e^{x/2} + 2e^{-3x}).$$

27. $[(x+A)e^x\sin y + \cos y]\,dx + x[e^x\cos y + B\sin y]\,dy = 0$ is $M\,dx + N\,dy$. We have

$$\begin{aligned} \frac{\partial M}{\partial y} &= (x+A)e^x\cos y - \sin y \\ \frac{\partial N}{\partial x} &= e^x\cos y + B\sin y + xe^x\cos y. \end{aligned}$$

These expressions are equal (and the DE is exact) if $A = 1$ and $B = -1$. If so, the left side of the DE is $d\phi(x,y)$, where

$$\phi(x,y) = xe^x\sin y + x\cos y.$$

The general solution is $xe^x\sin y + x\cos y = C$.

28. $(x^2 + 3y^2)\,dx + xy\,dy = 0$. Multiply by x^n:

$$x^n(x^2 + 3y^2)\,dx + x^{n+1}y\,dy = 0$$

is exact provided $6x^ny = (n+1)x^ny$, that is, provided $n = 5$. In this case the left side is $d\phi$, where

$$\phi(x,y) = \frac{1}{2}x^6y^2 + \frac{1}{8}x^8.$$

The general solution of the given DE is

$$4x^6y^2 + x^8 = C.$$

29. $x^2y'' - x(2 + x\cot x)y' + (2 + x\cot x)y = 0$
If $y = x$, then $y' = 1$ and $y'' = 0$, so the DE is clearly satisfied by y. To find a second, independent solution, try $y = xv(x)$. Then $y' = v + xv'$, and $y'' = 2v' + xv''$. Substituting these expressions into the given DE, we obtain

$$\begin{aligned}&2x^2v' + x^3v'' - (xv + x^2v')(2 + x\cot x)\\&\qquad + xv(2 + x\cot x) = 0\\&x^3v'' - x^3v'\cot x = 0,\end{aligned}$$

or, putting $w = v'$, $w' = (\cot x)w$, that is,

$$\begin{aligned}\frac{dw}{w} &= \frac{\cos x\,dx}{\sin x}\\ \ln w &= \ln\sin x + \ln C_2\\ v' = w &= C_2\sin x \quad\Rightarrow\quad v = C_1 - C_2\cos x.\end{aligned}$$

A second solution of the DE is $x\cos x$, and the general solution is

$$y = C_1x + C_2x\cos x.$$

30. $x^2y'' - x(2 + x\cot x)y' + (2 + x\cot x)y = x^3\sin x$
Look for a particular solution of the form $y = xu_1(x) + x\cos x u_2(x)$, where

$$\begin{aligned}xu_1' + x\cos x u_2' &= 0\\ u_1' + (\cos x - x\sin x)u_2' &= x\sin x.\end{aligned}$$

Divide the first equation by x and subtract from the second equation to get

$$-x\sin x u_2' = x\sin x.$$

Thus $u_2' = -1$ and $u_2 = -x$. The first equation now gives $u_1' = \cos x$, so that $u_1 = \sin x$. The general solution of the DE is

$$y = x\sin x - x^2\cos x + C_1x + C_2x\cos x.$$

31. Suppose $y' = f(x, y)$ and $y(x_0) = y_0$, where $f(x, y)$ is continuous on the whole xy-plane and satisfies $|f(x, y)| \le K$ there. By the Fundamental Theorem of Calculus, we have

$$\begin{aligned}y(x) - y_0 &= y(x) - y(x_0)\\ &= \int_{x_0}^{x} y'(t)\,dt = \int_{x_0}^{x} f\Big(t, y(t)\Big)\,dt.\end{aligned}$$

Therefore,

$$|y(x) - y_0| \le K|x - x_0|.$$

Thus $y(x)$ is bounded above and below by the lines $y = y_0 \pm K(x - x_0)$, and cannot have a vertical asymptote anywhere.

Remark: we don't seem to have needed the continuity of $\partial f/\partial y$, only the continuity of f (to enable the use of the Fundamental Theorem).