CHAPTER 14. MULTIPLE INTEGRATION

Section 14.1 Double Integrals (page 759)

1. $f(x,y) = 5 - x - y$

$$R = 1 \times \Big[f(0,1) + f(0,2) + f(1,1) + f(1,2) + f(2,1) + f(2,2) \Big] = 4 + 3 + 3 + 2 + 2 + 1 = 15$$

2. $$R = 1 \times \Big[f(1,1) + f(1,2) + f(2,1) + f(2,2) + f(3,1) + f(3,2) \Big] = 3 + 2 + 2 + 1 + 1 + 0 = 9$$

3. $$R = 1 \times \Big[f(0,0) + f(0,1) + f(1,0) + f(1,1) + f(2,0) + f(2,1) \Big] = 5 + 4 + 4 + 3 + 3 + 2 = 21$$

4. $$R = 1 \times \Big[f(1,0) + f(1,1) + f(2,0) + f(2,1) + f(3,0) + f(3,1) \Big] = 4 + 3 + 3 + 2 + 2 + 1 = 15$$

5. $$R = 1 \times \Big[f(\tfrac12,\tfrac12) + f(\tfrac12,\tfrac32) + f(\tfrac32,\tfrac12) + f(\tfrac32,\tfrac32) + f(\tfrac52,\tfrac12) + f(\tfrac52,\tfrac32) \Big] = 4 + 3 + 3 + 2 + 2 + 1 = 15$$

6. $I = \iint_D (5 - x - y)\, dA$ is the volume of the solid in the figure.

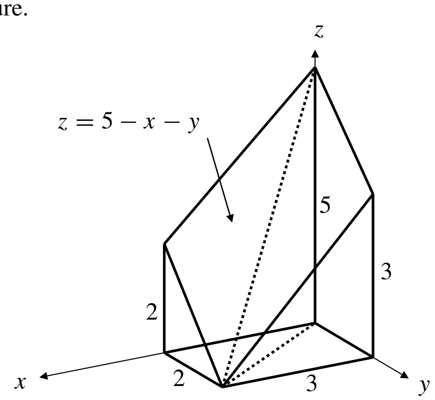


Fig. 14.1.6

The solid is split by the vertical plane through the z-axis and the point $(3,2,0)$ into two pyramids, each with a trapezoidal base; one pyramid's base is in the plane $y = 0$ and the other's is in the plane $z = 0$. I is the sum of the volumes of these pyramids:

$$I = \frac13 \left(\frac{5+2}{2}(3)(2) \right) + \frac13 \left(\frac{5+3}{2}(2)(3) \right) = 15.$$

7. $J = \iint_D 1\, dA$

$$R = 4 \times 1 \times \Big[5 + 5 + 5 + 5 + 4\Big] = 96$$

8. $R = 4 \times 1 \times \Big[4 + 4 + 4 + 3 + 0\Big] = 60$

9. $R = 4 \times 1 \times \Big[5 + 5 + 4 + 4 + 2\Big] = 80$

10. $J =$ area of disk $= \pi(5^2) \approx 78.54$

11. $R = 1 \times (e^{1/2} + e^{1/2} + e^{3/2} + e^{3/2} + e^{5/2} + e^{5/2}) \approx 32.63$

12. $f(x,y) = x^2 + y^2$

$$\begin{aligned} R = 4 \times 1 \times \Big[& f(\tfrac12,\tfrac12) + f(\tfrac32,\tfrac12) + f(\tfrac52,\tfrac12) + f(\tfrac72,\tfrac12) \\ & + f(\tfrac92,\tfrac12) + f(\tfrac12,\tfrac32) + f(\tfrac32,\tfrac32) + f(\tfrac52,\tfrac32) \\ & + f(\tfrac72,\tfrac32) + f(\tfrac92,\tfrac32) \\ & + f(\tfrac12,\tfrac52) + f(\tfrac32,\tfrac52) + f(\tfrac52,\tfrac52) + f(\tfrac72,\tfrac52) \\ & + f(\tfrac12,\tfrac72) + f(\tfrac32,\tfrac72) + f(\tfrac52,\tfrac72) + f(\tfrac12,\tfrac92) + f(\tfrac32,\tfrac92) \Big] \\ = {} & 918 \end{aligned}$$

13. $\iint_R dA =$ area of $R = 4 \times 5 = 20.$

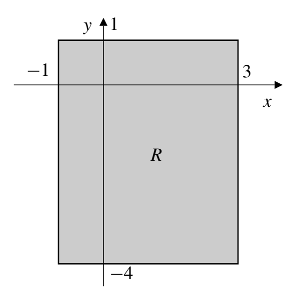


Fig. 14.1.13

14. $$\iint_D (x+3)\, dA = \iint_D x\, dA + 3 \iint_D dA = 0 + 3(\text{area of } D) = 3 \times \frac{\pi 2^2}{2} = 6\pi.$$

The integral of x over D is zero because D is symmetrical about $x = 0$.

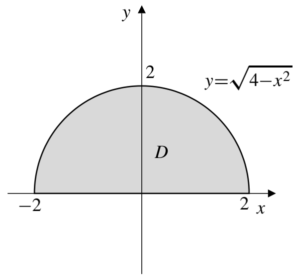


Fig. 14.1.14

15. T is symmetric about the line $x + y = 0$. Therefore, $\iint_T (x + y)\,dA = 0$.

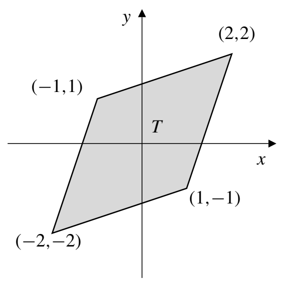


Fig. 14.1.15

16. $\iint_{|x|+|y|\le 1} \left(x^3\cos(y^2) + 3\sin y - \pi\right) dA$

$= 0 + 0 - \pi\Big($area bounded by $|x| + |y| = 1$

$= -\pi \times 4 \times \frac{1}{2}(1)(1) = -2\pi.$

(Each of the first two terms in the integrand is an odd function of one of the variables, and the square is symmetrical about each coordinate axis.)

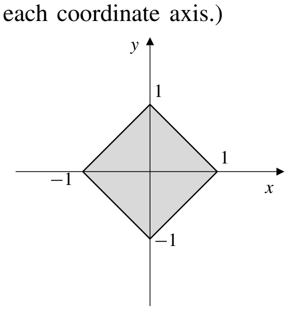


Fig. 14.1.16

17. $\iint_{x^2+y^2\le 1} (4x^2y^3 - x + 5)\,dA$

$= 0 - 0 + 5(\text{area of disk}) \qquad (\text{by symmetry})$

$= 5\pi.$

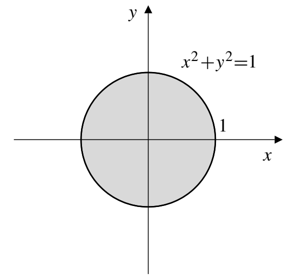


Fig. 14.1.17

18. $\iint_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}\,dA$

= volume of hemisphere shown in the figure

$= \frac{1}{2}\left(\frac{4}{3}\pi a^3\right) = \frac{2}{3}\pi a^3.$

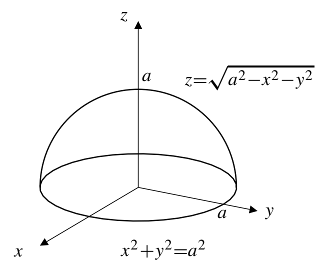


Fig. 14.1.18

19. $\iint_{x^2+y^2\le a^2} \left(a - \sqrt{x^2 + y^2}\right) dA$

= volume of cone shown in the figure

$= \frac{1}{3}\pi a^3.$

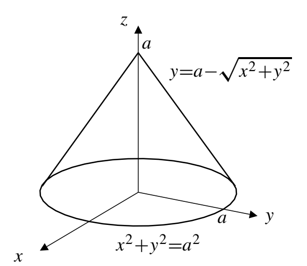


Fig. 14.1.19

20. By the symmetry of S with respect to x and y we have

$$\iint_S (x + y)\,dA = 2\iint_S x\,dA$$

$= 2 \times$ (volume of wedge shown in the figure)

$= 2 \times \frac{1}{2}(a^2)a = a^3.$

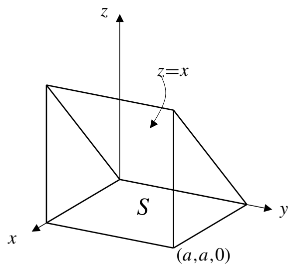


Fig. 14.1.20

21. $\iint_T (1 - x - y)\, dA$
$=$ volume of the tetrahedron shown in the figure
$= \frac{1}{3}\Big(\frac{1}{2}(1)(1)\Big)(1) = \frac{1}{6}.$

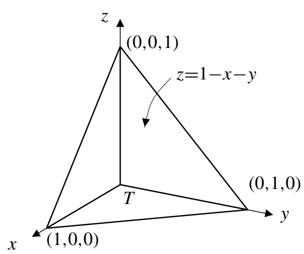


Fig. 14.1.21

22. $\iint_R \sqrt{b^2 - y^2}\, dA$
$=$ volume of the quarter cylinder shown in the figure
$= \frac{1}{4}(\pi b^2)a = \frac{1}{4}\pi a b^2.$

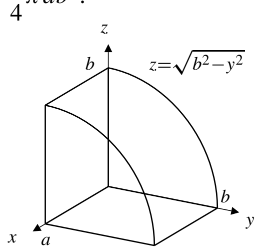


Fig. 14.1.22

Section 14.2 Iteration of Double Integrals in Cartesian Coordinates (page 766)

1.
$$\int_0^1 dx \int_0^x (xy + y^2)\, dy$$
$$= \int_0^1 dx \left(\frac{xy^2}{2} + \frac{y^3}{3} \right)\Bigg|_{y=0}^{y=x}$$
$$= \frac{5}{6} \int_0^1 x^3\, dx = \frac{5}{24}.$$

2.
$$\int_0^1 \int_0^y (xy + y^2)\, dx\, dy$$
$$= \int_0^1 \left(\frac{x^2 y}{2} + x y^2 \right)\Bigg|_{x=0}^{x=y} dy$$
$$= \frac{3}{2} \int_0^1 y^3\, dy = \frac{3}{8}.$$

3.
$$\int_0^\pi \int_{-x}^x \cos y\, dy\, dx$$
$$= \int_0^\pi \sin y \Bigg|_{y=-x}^{y=x} dx$$
$$= 2 \int_0^\pi \sin x\, dx = -2\cos x \Bigg|_0^\pi = 4.$$

4.
$$\int_0^2 dy \int_0^y y^2 e^{xy}\, dx$$
$$= \int_0^2 y^2\, dy \left(\frac{1}{y} e^{xy} \Bigg|_{x=0}^{x=y} \right)$$
$$= \int_0^2 y(e^{y^2} - 1)\, dy = \frac{e^{y^2} - y^2}{2} \Bigg|_0^2 = \frac{e^4 - 5}{2}.$$

5.
$$\iint_R (x^2 + y^2)\, dA = \int_0^a dx \int_0^b (x^2 + y^2)\, dy$$
$$= \int_0^a dx \left(x^2 y + \frac{y^3}{3} \right)\Bigg|_{y=0}^{y=b}$$
$$= \int_0^a \left(b x^2 + \frac{1}{3} b^3 \right) dx$$
$$= \frac{1}{3}\Big(b x^3 + b^3 x \Big)\Bigg|_0^a = \frac{1}{3}(a^3 b + a b^3).$$

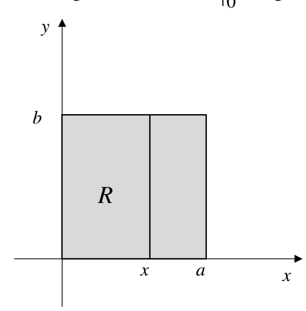


Fig. 14.2.5

6.
$$\iint_R x^2 y^2\, dA = \int_0^a x^2\, dx \int_0^b y^2\, dy$$
$$= \frac{a^3}{3} \frac{b^3}{3} = \frac{a^3 b^3}{9}.$$

7. $$\iint_S (\sin x + \cos y)\,dA$$
$$= \int_0^{\pi/2} dx \int_0^{\pi/2} (\sin x + \cos y)\,dy$$
$$= \int_0^{\pi/2} dx \left(y\sin x + \sin y\right)\Big|_{y=0}^{y=\pi/2}$$
$$= \int_0^{\pi/2} \left(\frac{\pi}{2}\sin x + 1\right)dx$$
$$= \left(-\frac{\pi}{2}\cos x + x\right)\Big|_0^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

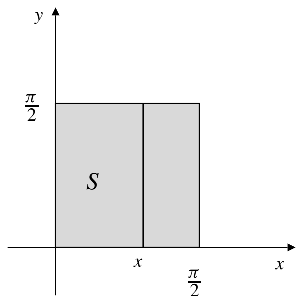


Fig. 14.2.7

8. $$\iint_T (x-3y)\,dA = \int_0^a dx \int_0^{b(1-(x/a))} (x-3y)\,dy$$
$$= \int_0^a dx \left(xy - \frac{3}{2}y^2\right)\Big|_{y=0}^{y=b(1-(x/a))}$$
$$= \int_0^a \left[b\left(x - \frac{x^2}{a}\right) - \frac{3}{2}b^2\left(1 - \frac{2x}{a} + \frac{x^2}{a^2}\right)\right]dx$$
$$= \left(b\frac{x^2}{2} - \frac{b}{a}\frac{x^3}{3} - \frac{3}{2}b^2x + \frac{3}{2}\frac{b^2x^2}{a} - \frac{1}{2}\frac{b^2x^3}{a^2}\right)\Big|_0^a$$
$$= \frac{a^2b}{6} - \frac{ab^2}{2}.$$

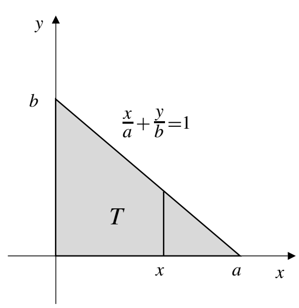


Fig. 14.2.8

9. $$\iint_R xy^2\,dA = \int_0^1 x\,dx \int_{x^2}^{\sqrt{x}} y^2\,dy$$
$$= \int_0^1 x\,dx \left(\frac{1}{3}y^3\right)\Big|_{y=x^2}^{y=\sqrt{x}}$$
$$= \frac{1}{3}\int_0^1 \left(x^{5/2} - x^7\right)dx$$
$$= \frac{1}{3}\left(\frac{2}{7}x^{7/2} - \frac{x^8}{8}\right)\Big|_0^1$$
$$= \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right) = \frac{3}{56}.$$

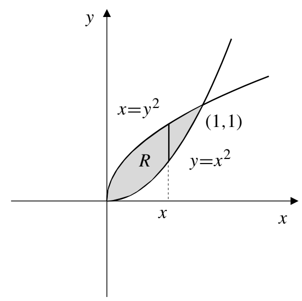


Fig. 14.2.9

10. $$\iint_D x\cos y\,dA$$
$$= \int_0^1 x\,dx \int_0^{1-x^2} \cos y\,dy$$
$$= \int_0^1 x\,dx\,(\sin y)\Big|_{y=0}^{y=1-x^2}$$
$$= \int_0^1 x\sin(1-x^2)\,dx \quad \text{Let } u = 1-x^2,\ du = -2x\,dx$$
$$= -\frac{1}{2}\int_1^0 \sin u\,du = \frac{1}{2}\cos u\Big|_1^0 = \frac{1-\cos(1)}{2}.$$

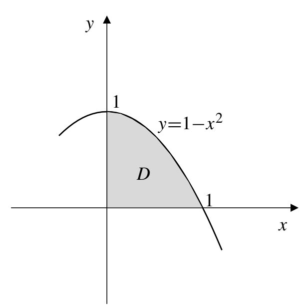


Fig. 14.2.10

11. For intersection: $xy = 1$, $2x + 2y = 5$.
Thus $2x^2 - 5x + 2 = 0$, or $(2x - 1)(x - 2) = 0$. The intersections are at $x = 1/2$ and $x = 2$. We have

$$\begin{aligned}
&\iint_D \ln x\, dA = \int_{1/2}^{2} \ln x\, dx \int_{1/x}^{(5/2)-x} dy \\
&= \int_{1/2}^{2} \ln x \left(\frac{5}{2} - x - \frac{1}{x}\right) dx \\
&= \int_{1/2}^{2} \ln x \left(\frac{5}{2} - x\right) dx - \frac{1}{2}\left(\ln x\right)^2\Big|_{1/2}^{2}
\end{aligned}$$

$$\begin{aligned}
U &= \ln x \quad & dV &= \left(\frac{5}{2} - x\right) dx \\
dU &= \frac{dx}{x} & V &= \frac{5}{2}x - \frac{x^2}{2}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}\left((\ln 2)^2 - (\ln \tfrac{1}{2})^2\right) + \left(\frac{5}{2}x - \frac{x^2}{2}\right)\ln x\Big|_{1/2}^{2} \\
&\qquad - \int_{1/2}^{2}\left(\frac{5}{2} - \frac{x}{2}\right) dx \\
&= (5-2)\ln 2 - \left(\frac{5}{4} - \frac{1}{8}\right)\ln\frac{1}{2} - \frac{15}{4} + \frac{15}{16} \\
&= \frac{33}{8}\ln 2 - \frac{45}{16}.
\end{aligned}$$

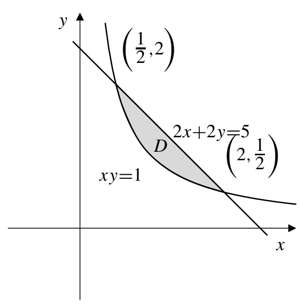


Fig. 14.2.11

12.
$$\begin{aligned}
&\iint_T \sqrt{a^2 - y^2}\, dA = \int_0^a \sqrt{a^2 - y^2}\, dy \int_y^a dx \\
&= \int_0^a (a - y)\sqrt{a^2 - y^2}\, dy \\
&= a\int_0^a \sqrt{a^2 - y^2}\, dy - \int_0^a y\sqrt{a^2 - y^2}\, dy
\end{aligned}$$

Let $u = a^2 - y^2$
$du = -2y\, dy$

$$\begin{aligned}
&= a\,\frac{\pi a^2}{4} + \frac{1}{2}\int_{a^2}^{0} u^{1/2}\, du \\
&= \frac{\pi a^3}{4} - \frac{1}{3}u^{3/2}\Big|_0^{a^2} = \left(\frac{\pi}{4} - \frac{1}{3}\right)a^3.
\end{aligned}$$

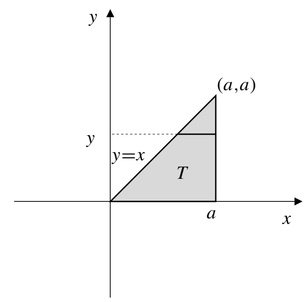


Fig. 14.2.12

13.
$$\begin{aligned}
&\iint_R \frac{x}{y}e^y\, dA = \int_0^1 \frac{e^y}{y}\, dy \int_y^{\sqrt{y}} x\, dx \\
&= \frac{1}{2}\int_0^1 (1 - y)e^y\, dy
\end{aligned}$$

$$\begin{aligned}
U &= 1 - y \quad & dV &= e^y\, dy \\
dU &= -dy & V &= e^y
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\left[(1 - y)e^y\Big|_0^1 + \int_0^1 e^y\, dy\right] \\
&= -\frac{1}{2} + \frac{1}{2}(e - 1) = \frac{e}{2} - 1.
\end{aligned}$$

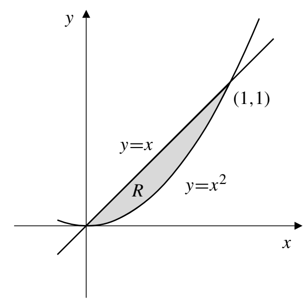


Fig. 14.2.13

14.
$$\begin{aligned}
\iint_T \frac{xy}{1 + x^4}\, dA &= \int_0^1 \frac{x}{1 + x^4}\, dx \int_0^x y\, dy \\
&= \frac{1}{2}\int_0^1 \frac{x^3}{1 + x^4}\, dx \\
&= \frac{1}{8}\ln(1 + x^4)\Big|_0^1 = \frac{\ln 2}{8}.
\end{aligned}$$

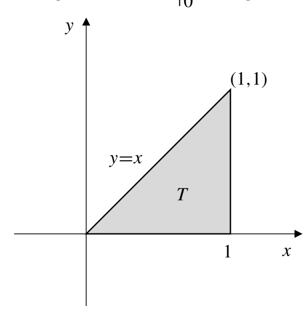


Fig. 14.2.14

15. $\int_0^1 dy \int_y^1 e^{-x^2}\,dx = \int_R e^{-x^2}\,dx \quad (R \text{ as shown})$

$$= \int_0^1 e^{-x^2}\,dx \int_0^x dy$$

$$= \int_0^1 xe^{-x^2}\,dx \qquad \text{Let } u = x^2, \quad du = 2x\,dx$$

$$= \frac{1}{2}\int_0^1 e^{-u}\,du = -\frac{1}{2}e^{-u}\Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right).$$

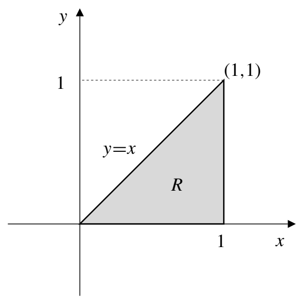


Fig. 14.2.15

16. $\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x}\,dx = \iint_R \frac{\sin x}{x}\,dA \quad (R \text{ as shown})$

$$= \int_0^{\pi/2} \frac{\sin x}{x}\,dx \int_0^x dy = \int_0^{\pi/2} \sin x\,dx = 1.$$

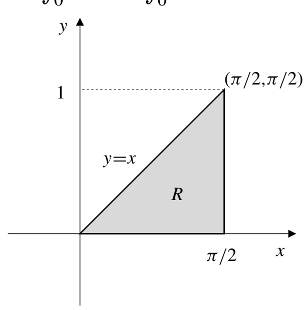


Fig. 14.2.16

17. $\int_0^1 dx \int_x^1 \frac{y^\lambda}{x^2+y^2}\,dy \qquad (\lambda > 0)$

$$= \iint_R \frac{y^\lambda}{x^2+y^2}\,dA \quad (R \text{ as shown})$$

$$= \int_0^1 y^\lambda\,dy \int_0^y \frac{dx}{x^2+y^2}$$

$$= \int_0^1 y^\lambda\,dy\,\frac{1}{y}\left(\tan^{-1}\frac{x}{y}\right)\Bigg|_{x=0}^{x=y}$$

$$= \frac{\pi}{4}\int_0^1 y^{\lambda-1}\,dy = \frac{\pi y^\lambda}{4\lambda}\Bigg|_0^1 = \frac{\pi}{4\lambda}.$$

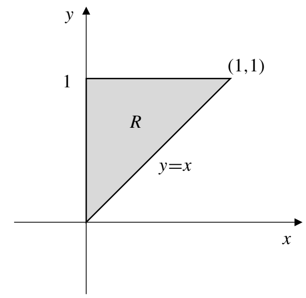


Fig. 14.2.17

18. $\int_0^1 dx \int_x^{x^{1/3}} \sqrt{1-y^4}\,dy$

$$= \iint_R \sqrt{1-y^4}\,dA \quad (R \text{ as shown})$$

$$= \int_0^1 y\sqrt{1-y^4}\,dy - \int_0^1 y^3\sqrt{1-y^4}\,dy$$

$$\text{Let } u = y^2, \quad du = 2y\,dy \qquad \text{Let } v = 1 - y^4, \quad dv = -4y^3\,dy$$

$$= \frac{1}{2}\int_0^1 \sqrt{1-u^2}\,du + \frac{1}{4}\int_1^0 v^{1/2}\,dv$$

$$= \frac{1}{2}\left(\frac{\pi}{4} \times 1^2\right) + \frac{1}{6}v^{3/2}\Bigg|_1^0 = \frac{\pi}{8} - \frac{1}{6}.$$

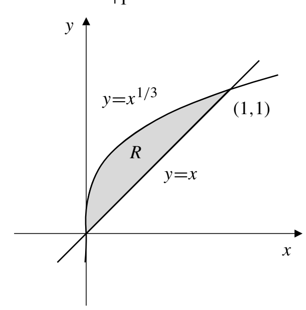


Fig. 14.2.18

19. $V = \int_0^1 dx \int_0^x (1-x^2)\,dy$

$$= \int_0^1 (1-x^2)x\,dx = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \text{ cu. units.}$$

20. $V = \int_0^1 dy \int_0^y (1-x^2)\,dx$

$$= \int_0^1 \left(y - \frac{y^3}{3}\right) dy = \frac{1}{2} - \frac{1}{12} = \frac{5}{12} \text{ cu. units.}$$

21. $$\begin{aligned} V &= \int_0^1 dx \int_0^{1-x} (1-x^2-y^2)\,dy \\ &= \int_0^1 \left((1-x^2)y - \frac{y^3}{3} \right)\bigg|_{y=0}^{y=1-x} dx \\ &= \int_0^1 \left((1-x^2)(1-x) - \frac{(1-x)^3}{3} \right) dx \\ &= \int_0^1 \left(\frac{2}{3} - 2x^2 + \frac{4x^3}{3} \right) dx = \frac{2}{3} - \frac{2}{3} + \frac{1}{3} = \frac{1}{3} \text{ cu. units.} \end{aligned}$$

22. $z = 1 - y^2$ and $z = x^2$ intersect on the cylinder $x^2 + y^2 = 1$. The volume lying below $z = 1 - y^2$ and above $z = x^2$ is

$$\begin{aligned} V &= \iint_{x^2+y^2\le 1} (1-y^2-x^2)\,dA \\ &= 4\int_0^1 dx \int_0^{\sqrt{1-x^2}} (1-x^2-y^2)\,dy \\ &= 4\int_0^1 dx \left((1-x^2)y - \frac{y^3}{3} \right)\bigg|_{y=0}^{y=\sqrt{1-x^2}} \\ &= \frac{8}{3}\int_0^1 (1-x^2)^{3/2}\,dx \quad \text{Let } x = \sin u \\ &\qquad\qquad\qquad\qquad\quad dx = \cos u\,du \\ &= \frac{8}{3}\int_0^{\pi/2} \cos^4 u\,du = \frac{2}{3}\int_0^{\pi/2} (1+\cos 2u)^2\,du \\ &= \frac{2}{3}\int_0^{\pi/2} \left(1 + 2\cos 2u + \frac{1+\cos 4u}{2} \right) du \\ &= \frac{2}{3}\,\frac{3}{2}\,\frac{\pi}{2} = \frac{\pi}{2} \text{ cu. units.} \end{aligned}$$

23. $$\begin{aligned} V &= \int_1^2 dx \int_0^x \frac{1}{x+y}\,dy \\ &= \int_1^2 dx \left(\ln(x+y)\bigg|_{y=0}^{y=x} \right) \\ &= \int_1^2 (\ln 2x - \ln x)\,dx = \ln 2 \int_1^2 dx = \ln 2 \text{ cu. units.} \end{aligned}$$

24. $$\begin{aligned} V &= \int_0^{\pi^{1/4}} dy \int_0^y x^2 \sin(y^4)\,dx \\ &= \frac{1}{3}\int_0^{\pi^{1/4}} y^3 \sin(y^4)\,dy \quad \text{Let } u = y^4 \\ &\qquad\qquad\qquad\qquad\quad du = 4y^3\,dy \\ &= \frac{1}{12}\int_0^{\pi} \sin u\,du = \frac{1}{6} \text{ cu. units.} \end{aligned}$$

25. $$\begin{aligned} \text{Vol} &= \iint_E (1-x^2-2y^2)\,dA \\ &= 4\int_0^1 dx \int_0^{\sqrt{(1-x^2)/2}} (1-x^2-2y^2)\,dy \\ &= 4\int_0^1 \left(\frac{1}{\sqrt{2}}(1-x^2)^{3/2} - \frac{2}{3}\,\frac{(1-x^2)^{3/2}}{2\sqrt{2}} \right) dx \\ &= \frac{4\sqrt{2}}{3}\int_0^1 (1-x^2)^{3/2}\,dx \quad \text{Let } x = \sin\theta \\ &\qquad\qquad\qquad\qquad\qquad dx = \cos\theta\,d\theta \\ &= \frac{4\sqrt{2}}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{4\sqrt{2}}{3}\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2} \right)^2 d\theta \\ &= \frac{\sqrt{2}}{3}\int_0^{\pi/2} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2} \right) d\theta \\ &= \frac{\sqrt{2}}{3}\left[\frac{3\theta}{2} + \sin 2\theta + \frac{1}{8}\sin 4\theta \right]\bigg|_0^{\pi/2} = \frac{\pi}{2\sqrt{2}} \text{ cu. units.} \end{aligned}$$

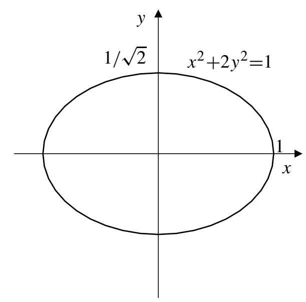


Fig. 14.2.25

26. $$\begin{aligned} \text{Vol} &= \iint_T \left(2 - \frac{x}{a} - \frac{y}{b} \right) dA \\ &= \int_0^a dx \int_0^{b(1-(x/a))} \left(2 - \frac{x}{a} - \frac{y}{b} \right) dy \\ &= \int_0^a \left[\left(2 - \frac{x}{a} \right) b \left(1 - \frac{x}{a} \right) - \frac{1}{2b} b^2 \left(1 - \frac{x}{a} \right)^2 \right] dx \\ &= \frac{b}{2}\int_0^a \left(3 - \frac{4x}{a} + \frac{x^2}{a^2} \right) dx \\ &= \frac{b}{2}\left(3x - \frac{2x^2}{a} + \frac{x^3}{3a^2} \right)\bigg|_0^a = \frac{2}{3}ab \text{ cu. units.} \end{aligned}$$

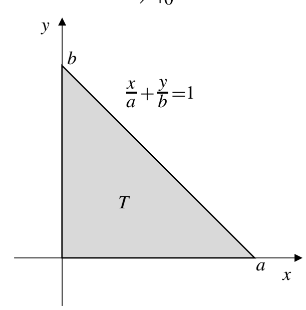


Fig. 14.2.26

27. Vol $= 8 \times$ part in the first octant

$$\begin{aligned} &= 8\int_0^a dx \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dy \\ &= 8\int_0^a (a^2-x^2)\,dx \\ &= 8\left(a^2x - \frac{x^3}{3}\right)\Bigg|_0^a = \frac{16}{3}a^3 \text{ cu. units.} \end{aligned}$$

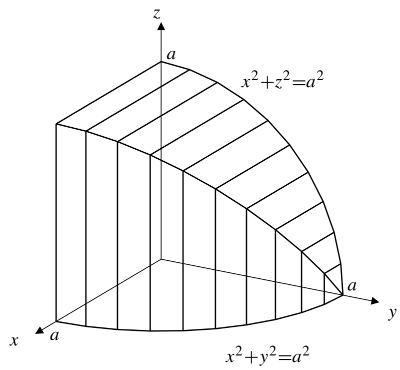


Fig. 14.2.27

28. The part of the plane $z = 8 - x$ lying inside the elliptic cylinder $x^2 = 2y^2 = 8$ lies above $z = 0$. The part of the plane $z = y - 4$ inside the cylinder lies below $z = 0$. Thus the required volume is

$$\begin{aligned} \text{Vol} &= \iint_{x^2+2y^2\le 8} \Big(8 - x - (y-4)\Big)\,dA \\ &= \iint_{x^2+2y^2\le 8} 12\,dA \qquad \text{(by symmetry)} \\ &= 12 \times \text{area of ellipse } \frac{x^2}{8} + \frac{y^2}{4} = 1 \\ &= 12 \times \pi(2\sqrt{2})(2) = 48\sqrt{2}\pi \text{ cu. units.} \end{aligned}$$

29. With $g(x)$ and $G(x)$ defined as in the statement of the problem, we have

$$\begin{aligned} \int_a^x G(u)\,du &= \int_a^x du \int_c^d f_1(u,t)\,dt \\ &= \int_c^d dt \int_a^x f_1(u,t)\,du \\ &= \int_c^d \Big(f(x,t) - f(a,t)\Big)\,dt = g(x) - C, \end{aligned}$$

where $C = \displaystyle\int_c^d f(a,t)\,dt$ is independent of x. Applying the Fundamental Theorem of Calculus we obtain

$$g'(x) = \frac{d}{dx}\int_a^x G(u)\,du = G(x).$$

30. Since $F'(x) = f(x)$ and $G'(x) = g(x)$ on $a \le x \le b$, we have

$$\begin{aligned} \iint_T f(x)g(x)\,dA &= \int_a^b f(x)\,dx \int_a^x G'(y)\,dy \\ &= \int_a^b f(x)\Big(G(x) - G(a)\Big)\,dx \\ &= \int_a^b f(x)G(x)\,dx - G(a)F(b) + G(a)F(a) \\ \iint_T f(x)g(x)\,dA &= \int_a^b g(y)\,dy \int_y^b F'(x)\,dx \\ &= \int_a^b g(y)\Big(F(b) - F(y)\Big)\,dy \\ &= F(b)G(b) - F(b)G(a) - \int_a^b F(y)g(y)\,dx. \end{aligned}$$

Thus

$$\int_a^b f(x)G(x)\,dx = F(b)G(b) - F(a)G(a) - \int_a^b g(y)F(y)\,dy.$$

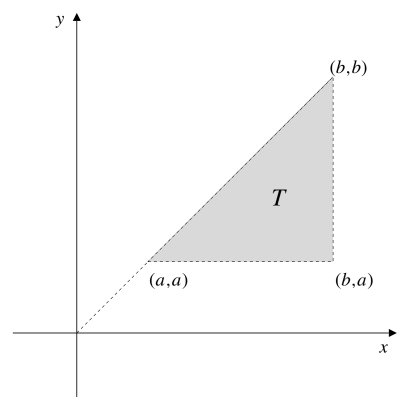


Fig. 14.2.30

Section 14.3 Improper Integrals and a Mean-Value Theorem (page 771)

1. $$\begin{aligned} \iint_Q e^{-x-y}\,dA &= \int_0^\infty e^{-x}\,dx \int_0^\infty e^{-y}\,dy \\ &= \left(\lim_{R\to\infty}(-e^{-x})\Big|_0^R\right)^2 = 1 \text{ (converges)} \end{aligned}$$

2. $$\begin{aligned} \iint_Q \frac{dA}{(1+x^2)(1+y^2)} &= \int_0^\infty \frac{dx}{1+x^2} \int_0^\infty \frac{dy}{1+y^2} \\ &= \left(\lim_{R\to\infty}(\tan^{-1}x)\Big|_0^R\right)^2 = \frac{\pi^2}{4} \end{aligned}$$

(converges)

3. $$\iint_S \frac{y}{1+x^2}\,dA = \int_0^1 y\,dy \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{1}{2}\left(\lim_{\substack{S\to-\infty\\ R\to\infty}} \tan^{-1}x)\Big|_S^R\right) = \frac{\pi}{2} \text{ (converges)}$$

4. $$\iint_T \frac{1}{x\sqrt{y}}\,dA = \int_0^1 \frac{dx}{x}\int_x^{2x}\frac{dy}{\sqrt{y}} = \int_0^1 \frac{2(\sqrt{2x}-\sqrt{x})}{x}\,dx = 2(\sqrt{2}-1)\int_0^1 \frac{dx}{\sqrt{x}} = 4(\sqrt{2}-1) \text{ (converges)}$$

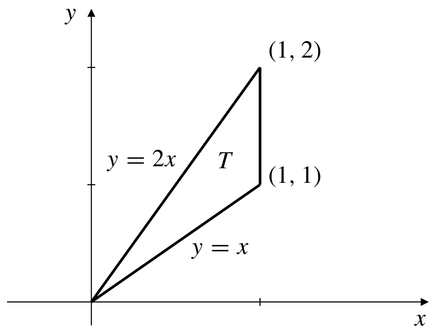


Fig. 14.3.4

5. $$\iint_Q \frac{x^2+y^2}{(1+x^2)(1+y^2)}\,dA = 2\iint_Q \frac{x^2\,dA}{(1+x^2)(1+y^2)} \quad \text{(by symmetry)}$$
$$= 2\int_0^\infty \frac{x^2\,dx}{1+x^2}\int_0^\infty \frac{dy}{1+y^2} = \pi\int_0^\infty \frac{x^2\,dx}{1+x^2},$$
which diverges to infinity, since $x^2/(1+x^2) \ge 1/2$ on $[1,\infty)$.

6. $$\iint_H \frac{dA}{1+x+y} = \int_0^\infty dx\int_0^1 \frac{1}{1+x+y}\,dy = \int_0^\infty \left(\ln(1+x+y)\Big|_{y=0}^{y=1}\right)dx = \int_0^\infty \ln\left(\frac{2+x}{1+x}\right)dx = \int_0^\infty \ln\left(1+\frac{1}{1+x}\right)dx.$$

Since $\displaystyle\lim_{u\to0+}\frac{\ln(1+u)}{u} = 1$, we have $\ln(1+u) \ge u/2$ on some interval $(0, u_0)$. Therefore

$$\ln\left(1+\frac{1}{1+x}\right) \ge \frac{1}{2(1+x)}$$

on some interval (x_0,∞), and

$$\int_0^\infty \ln\left(1+\frac{1}{1+x}\right)dx \ge \int_{x_0}^\infty \frac{1}{2(1+x)}\,dx,$$

which diverges to infinity. Thus the given double integral diverges to infinity by comparison.

7. $$\iint_{\mathbb{R}^2} e^{-(|x|+|y|)}\,dA = 4\iint_{\substack{x\ge0\\ y\ge0}} e^{-(x+y)}\,dA = 4\int_0^\infty e^{-x}\,dx\int_0^\infty e^{-y}\,dy = 4\left(\lim_{R\to\infty} -e^{-x}\Big|_0^R\right)^2 = 4$$

(The integral converges.)

8. On the strip S between the parallel lines $x+y=0$ and $x+y=1$ we have $e^{-|x+y|} = e^{-(x+y)} \ge 1/e$. Since S has infinite area,

$$\iint_S e^{-|x+y|}\,dA = \infty.$$

Since $e^{-|x+y|} > 0$ for all (x,y) in $\mathbb{R}^2$, we have

$$\iint_{\mathbb{R}^2} e^{-|x+y|}\,dA > \iint_S e^{-|x+y|}\,dA,$$

and the given integral diverges to infinity.

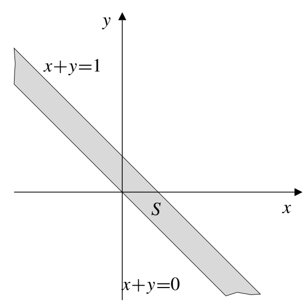


Fig. 14.3.8

9. $$\iint_T \frac{1}{x^3}e^{-y/x}\,dA = \int_1^\infty \frac{dx}{x^3}\int_0^x e^{-y/x}\,dy = \int_1^\infty \frac{dx}{x^3}\left(-xe^{-y/x}\Big|_{y=0}^{y=x}\right) = \left(1-\frac{1}{e}\right)\int_1^\infty \frac{dx}{x^2} = \left(1-\frac{1}{e}\right)\lim_{R\to\infty}\left(-\frac{1}{x}\Big|_1^R\right) = 1-\frac{1}{e}$$

(The integral converges.)

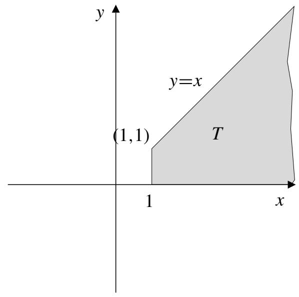


Fig. 14.3.9

10. $$\iint_T \frac{dA}{x^2+y^2} = \int_1^\infty dx \int_0^x \frac{dy}{x^2+y^2}$$
$$= \int_1^\infty dx \left(\frac{1}{x} \tan^{-1} \frac{y}{x} \bigg|_{y=0}^{y=x} \right)$$
$$= \frac{\pi}{4} \int_1^\infty \frac{dx}{x} = \infty$$
(The integral diverges to infinity.)

11. Since $e^{-xy} > 0$ on Q we have

$$\iint_Q e^{-xy}\, dA > \iint_R e^{-xy}\, dA,$$

where R satisfies $1 \le x < \infty$, $0 \le y \le 1/x$. Thus

$$\iint_Q e^{-xy}\, dA > \int_1^\infty dx \int_0^{1/x} e^{-xy}\, dy > \frac{1}{e} \int_1^\infty \frac{dx}{x} = \infty.$$

The given integral diverges to infinity.

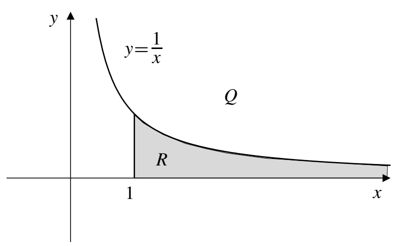
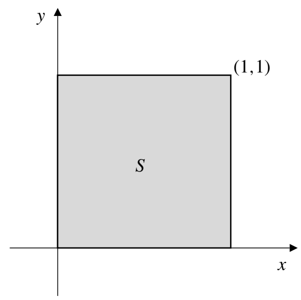


Fig. 14.3.11

12. $$\iint_R \frac{1}{x} \sin \frac{1}{x}\, dA = \int_{2/\pi}^\infty \frac{1}{x} \sin \frac{1}{x}\, dx \int_0^{1/x} dy$$
$$= \int_{2/\pi}^\infty \frac{1}{x^2} \sin \frac{1}{x}\, dx \quad \text{Let } u = 1/x, \quad du = -1/x^2\, dx$$
$$= -\int_{\pi/2}^0 \sin u\, du = \cos u \bigg|_{\pi/2}^0 = 1$$
(The integral converges.)

13. a) $$I = \iint_S \frac{dA}{x+y} = \int_0^1 dx \int_0^1 \frac{dy}{x+y}$$
$$= \int_0^1 dx \left(\ln(x+y) \bigg|_{y=0}^{y=1} \right)$$
$$= \lim_{c \to 0+} \Big[(x+1)\ln(x+1) - x \ln x \Big] \bigg|_c^1$$
$$= \lim_{c \to 0+} 2\ln 2 - 0 - (c+1)\ln(c+1) + c \ln c = 2 \ln 2.$$



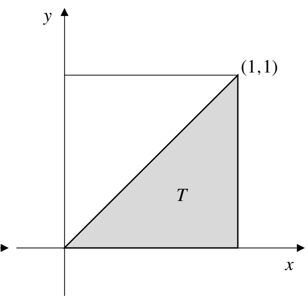


Fig. 14.3.13a Fig. 14.3.13b

b) $$I = 2\iint_T \frac{dA}{x+y} = 2 \lim c \to 0+ \int_c^1 dx \int_0^x \frac{dy}{x+y}$$
$$= 2 \lim c \to 0+ \int_c^1 dx \left(\ln(x+y) \bigg|_{y=0}^{y=x} \right)$$
$$= 2 \lim_{c \to 0+} \int_c^1 (\ln 2x - \ln x)\, dx = 2 \ln 2 \int_0^1 dx = 2 \ln 2.$$

14. $$\text{Vol} = \iint_S \frac{2xy}{x^2+y^2}\, dA$$
$$= 4 \iint_T \frac{2xy}{x^2+y^2}\, dA \quad (T \text{ as in \#9(b)})$$
$$= 4 \int_0^1 x\, dx \int_0^x \frac{y\, dy}{x^2+y^2} \quad \text{Let } u = x^2+y^2, \quad du = 2y\, dy$$
$$= 2 \int_0^1 x\, dx \int_{x^2}^{2x^2} \frac{du}{u}$$
$$= 2 \ln 2 \int_0^1 x\, dx = \ln 2 \text{ cu. units.}$$

15. $\displaystyle\iint_{D_k} \frac{dA}{x^a} = \int_0^1 \frac{dx}{x^a} \int_0^{x^k} dy = \int_0^1 x^{k-a}\, dx$, which converges if $k - a > -1$, that is, if $k > a - 1$.

16. $\displaystyle\iint_{D_k} y^b\, dA = \int_0^1 dx \int_0^{x^k} y^b\, dy = \int_0^1 \frac{x^{k(b+1)}}{b+1}\, dx$ if $b > -1$. This latter integral converges if $k(b+1) > -1$. Thus, the given integral converges if $b > -1$ and $k > -1/(b+1)$.

17. $\displaystyle\iint_{R_k} x^a\, dA = \int_1^\infty x^a\, dx \int_0^{x^k} dy = \int_1^\infty x^{k+a}\, dx$, which converges if $k + a < -1$, that is, if $k < -(a+1)$.

18. $\displaystyle\iint_{R_k} \frac{dA}{y^b} = \int_1^\infty dx \int_0^{x^k} \frac{dy}{y^b} = \int_1^\infty \frac{x^{k(1-b)}}{1-b}\,dx$ if $b < 1$.
This latter integral converges if $k(1-b) < -1$. Thus, the given integral converges if $b < 1$ and $k < -1/(1-b)$.

19. $\displaystyle\iint_{D_k} x^a y^b\,dA = \int_0^1 x^a\,dx \int_0^{x^k} y^b\,dy = \int_0^1 \frac{x^{a+(b+1)k}}{b+1}\,dx,$
if $b > -1$. This latter integral converges if $a + (b+1)k > -1$. Thus, the given integral converges if $b > -1$ and $k > -(a+1)/(b+1)$.

20. $\displaystyle\iint_{R_k} x^a y^b\,dA = \int_1^\infty x^a\,dx \int_0^{x^k} y^b\,dy = \int_1^\infty \frac{x^{a+(b+1)k}}{b+1}\,dx,$
if $b > -1$. This latter integral converges if $a + (b+1)k < -1$. Thus, the given integral converges if $b > -1$ and $k < -(a+1)/(b+1)$.

21. One iteration:

$$\begin{aligned}
\iint_S \frac{x-y}{(x+y)^3}\,dA &= \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy \quad \begin{array}{l}\text{Let } u = x+y\\ du = dy\end{array}\\
&= \int_0^1 dx \int_x^{x+1} \frac{2x-u}{u^3}\,du\\
&= \int_0^1 dx \left(\frac{1}{u} - \frac{x}{u^2}\right)\Bigg|_{u=x}^{u=x+1}\\
&= \int_0^1 \left(\frac{1}{x+1} - \frac{x}{(x+1)^2} - \frac{1}{x} + \frac{1}{x}\right) dx\\
&= \int_0^1 \frac{dx}{(x+1)^2} = -\frac{1}{x+1}\Bigg|_0^1 = \frac{1}{2}.
\end{aligned}$$

Other iteration:

$$\begin{aligned}
\iint_S \frac{x-y}{(x+y)^3}\,dA &= \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx \quad \begin{array}{l}\text{Let } u = x+y\\ du = dx\end{array}\\
&= \int_0^1 dy \int_y^{y+1} \frac{u-2y}{u^3}\,du\\
&= \int_0^1 dy \left(\frac{y}{u^2} - \frac{1}{u}\right)\Bigg|_{u=y}^{u=y+1}\\
&= \int_0^1 \left(\frac{y}{(y+1)^2} - \frac{1}{y+1} - \frac{1}{y} + \frac{1}{y}\right) dy\\
&= -\int_0^1 \frac{dx}{(y+1)^2} = \frac{1}{y+1}\Bigg|_0^1 = -\frac{1}{2}.
\end{aligned}$$

These seemingly contradictory results are explained by the fact that the given double integral is improper and does not, in fact, exist, that is, it does not converge. To see this, we calculate the integral over a certain subset of the square S, namely the triangle T defined by $0 < x < 1$, $0 < y < x$.

$$\begin{aligned}
\iint_T \frac{x-y}{(x+y)^3}\,dA &= \int_0^1 dx \int_0^x \frac{x-y}{(x+y)^3}\,dy\\
&\qquad \begin{array}{l}\text{Let } u = x+y\\ du = dy\end{array}\\
&= \int_0^1 dx \int_x^{2x} \frac{2x-u}{u^3}\,du\\
&= \int_0^1 dx \left(\frac{1}{u} - \frac{x}{u^2}\right)\Bigg|_{u=x}^{u=2x}\\
&= \frac{1}{4}\int_0^1 \frac{dx}{x}
\end{aligned}$$

which diverges to infinity.

22. The average value of x^2 over the rectangle R is

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} \iint_R x^2\,dA\\
&= \frac{1}{(b-a)(d-c)} \int_a^b x^2\,dx \int_c^d dy\\
&= \frac{1}{b-a}\,\frac{b^3-a^3}{3} = \frac{a^2+ab+b^2}{3}.
\end{aligned}$$

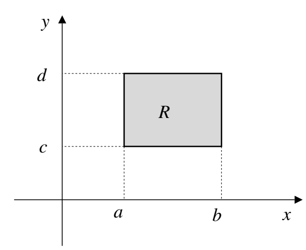


Fig. 14.3.22

23. The average value of $x^2 + y^2$ over the triangle T is

$$\begin{aligned}
&\frac{2}{a^2} \iint_T (x^2+y^2)\,dA\\
&= \frac{2}{a^2} \int_0^a dx \int_0^{a-x} (x^2+y^2)\,dy\\
&= \frac{2}{a^2} \int_0^a dx \left(x^2 y + \frac{y^3}{3}\right)\Bigg|_{y=0}^{y=a-x}\\
&= \frac{2}{3a^2} \int_0^a \left[3x^2(a-x) + (a-x)^3\right] dx\\
&= \frac{2}{3a^2} \int_0^a \left[a^3 - 3a^2 x + 6ax^2 - 4x^3\right] dx = \frac{a^2}{3}.
\end{aligned}$$

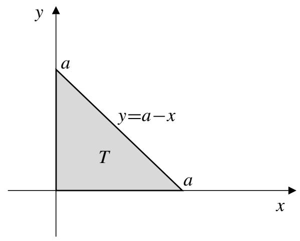


Fig. 14.3.23

24. The area of region R is

$$\int_0^1 (\sqrt{x} - x^2)\, dx = \frac{1}{3} \text{ sq. units.}$$

The average value of $1/x$ over R is

$$\begin{aligned} 3 \iint_R \frac{dA}{x} &= 3 \int_0^1 \frac{dx}{x} \int_{x^2}^{\sqrt{x}} dy \\ &= 3 \int_0^1 \left(x^{-1/2} - x\right) dx = \frac{9}{2}. \end{aligned}$$

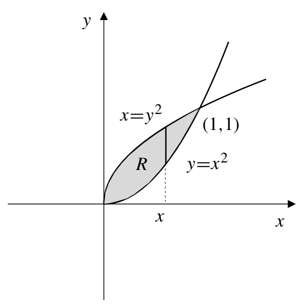


Fig. 14.3.24

25. The distance from (x, y) to the line $x + y = 0$ is $(x + y)/\sqrt{2}$. The average value of this distance over the quarter-disk Q is

$$\begin{aligned} &\frac{4}{\pi a^2} \iint_Q \frac{x+y}{\sqrt{2}}\, dA = \frac{4\sqrt{2}}{\pi a^2} \iint_Q x\, dA \\ &= \frac{4\sqrt{2}}{\pi a^2} \int_0^a x\, dx \int_0^{\sqrt{a^2-x^2}} dy \\ &= \frac{4\sqrt{2}}{\pi a^2} \int_0^a x\sqrt{a^2 - x^2}\, dx \quad \text{Let } u = a^2 - x^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad du = -2x\, dx \\ &= \frac{2\sqrt{2}}{\pi a^2} \int_0^{a^2} u^{1/2}\, du = \frac{4\sqrt{2}a}{3\pi}. \end{aligned}$$

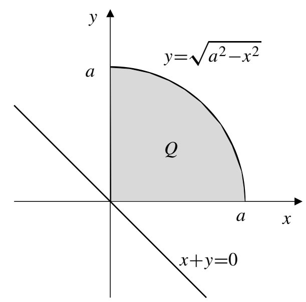


Fig. 14.3.25

26. Let R be the region $0 \le x < \infty$, $0 \le y \le 1/(1 + x^2)$. If $f(x, y) = x$, then

$$\int_R f(x, y)\, dA = \int_0^\infty x\, dx \int_0^{1/(1+x^2)} dy = \int_0^\infty \frac{x\, dx}{1 + x^2}$$

which diverges to infinity. Thus f has no average value on R.

27. If $f(x, y) = xy$ on the region R of the previous exercise, then

$$\begin{aligned} \iint_R f(x, y)\, dA &= \int_0^\infty x\, dx \int_0^{1/(1+x^2)} y\, dy \\ &= \frac{1}{2} \int_0^\infty \frac{x\, dx}{(1 + x^2)^2} \quad \text{Let } u = 1 + x^2 \\ &\qquad\qquad\qquad\qquad\quad du = 2x\, dx \\ &= \frac{1}{4} \int_1^\infty \frac{du}{u^2} = \frac{1}{4} \\ \text{Area } &= \int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2}. \end{aligned}$$

Thus $f(x, y)$ has average value $\dfrac{2}{\pi} \times \dfrac{1}{4} = \dfrac{1}{2\pi}$ on R.

28. The integral in Example 2 reduced to

$$\int_1^\infty \ln\left(1 + \frac{1}{x^2}\right) dx$$

$$\begin{aligned} U &= \ln\left(1 + \frac{1}{x^2}\right) \quad dV = dx \\ &\qquad\qquad\qquad\qquad V = x \\ dU &= -\frac{2\, dx}{x(x^2 + 1)} \end{aligned}$$

$$\begin{aligned} &= \lim_{R\to\infty} \left[x \ln\left(1 + \frac{1}{x^2}\right)\Bigg|_1^R + 2 \int_1^R \frac{dx}{1 + x^2} \right] \\ &= 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \ln 2 + \lim_{R\to\infty} \frac{\ln\Big(1 + (1/R^2)\Big)}{1/R} \\ &= \frac{\pi}{2} - \ln 2 + \lim_{R\to\infty} \frac{-(2/R^3)}{\Big(1 + (1/R^2)\Big)(-1/R^2)} \\ &= \frac{\pi}{2} - \ln 2. \end{aligned}$$

29. By the Mean-Value Theorem (Theorem 3),

$$\iint_{R_{hk}} f(x, y)\, dA = f(x_0, y_0)hk$$

for some point (x_0, y_0) in R_{hk}. Since $(x_0, y_0) \to (a, b)$ as $(h, k) \to (0, 0)$, and since f is continuous at (a, b), we have

$$\lim_{(h,k)\to(0,0)} \frac{1}{hk} \iint_{R_{hk}} f(x, y)\, dA = \lim_{(h,k)\to(0,0)} f(x_0, y_0) = f(a, b).$$

30. If $R = \{(x, y) : a \le x \le a+h,\ b \le y \le b+k\}$, then

$$\begin{aligned}
\iint_R f_{12}(x, y)\, dA &= \int_a^{a+h} dx \int_b^{b+k} f_{12}(x, y)\, dy \\
&= \int_a^{a+h} \Big[f_1(x, b+k) - f_1(x, b) \Big] dx \\
&= f(a+h, b+k) - f(a, b+k) - f(a+h, b) + f(a, b) \\
\iint_R f_{21}(x, y)\, dA &= \int_b^{b+k} dy \int_a^{a+h} f_{21}(x, y)\, dx \\
&= \int_b^{b+k} \Big[f_2(a+h, y) - f_2(a, y) \Big] dy \\
&= f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b).
\end{aligned}$$

Thus
$$\iint_R f_{12}(x, y)\, dA = \iint_R f_{21}(x, y)\, dA.$$

Divide both sides of this identity by hk and let $(h, k) \to (0, 0)$ to obtain, using the result of Exercise 31,
$$f_{12}(a, b) = f_{21}(a, b).$$

Section 14.4 Double Integrals in Polar Coordinates (page 780)

1. $$\iint_D (x^2 + y^2)\, dA = \int_0^{2\pi} d\theta \int_0^a r^2 r\, dr = 2\pi \frac{a^4}{4} = \frac{\pi a^4}{2}$$

2. $$\iint_D \sqrt{x^2 + y^2}\, dA = \int_0^{2\pi} d\theta \int_0^a r\, r\, dr = \frac{2\pi a^3}{3}$$

3. $$\iint_D \frac{dA}{\sqrt{x^2 + y^2}} = \int_0^{2\pi} d\theta \int_0^a \frac{r\, dr}{r} = 2\pi a$$

4. $$\iint_D |x|\, dA = 4 \int_0^{\pi/2} d\theta \int_0^a r\cos\theta\, r\, dr = 4 \sin\theta \Big|_0^{\pi/2} \frac{a^3}{3} = \frac{4a^3}{3}$$

5. $\displaystyle\iint_D x^2\, dA = \frac{\pi a^4}{4}$; by symmetry the value of this integral is half of that in Exercise 1.

6. $$\begin{aligned}
\iint_D x^2 y^2\, dA &= 4 \int_0^{\pi/2} d\theta \int_0^a r^4 \cos^2\theta \sin^2\theta\, r\, dr \\
&= \frac{a^6}{6} \int_0^{\pi/2} \sin^2(2\theta)\, d\theta \\
&= \frac{a^6}{12} \int_0^{\pi/2} \Big(1 - \cos(4\theta)\Big)\, d\theta = \frac{\pi a^6}{24}
\end{aligned}$$

7. $$\iint_Q y\, dA = \int_0^{\pi/2} d\theta \int_0^a r\sin\theta\, r\, dr = (-\cos\theta)\Big|_0^{\pi/2} \frac{a^3}{3} = \frac{a^3}{3}$$

8. $\displaystyle\iint_Q (x + y)\, dA = \frac{2a^3}{3}$; by symmetry, the value is twice that obtained in the previous exercise.

9. $$\iint_Q e^{x^2+y^2}\, dA = \int_0^{\pi/2} d\theta \int_0^a e^{r^2} r\, dr = \frac{\pi}{2}\left(\frac{1}{2} e^{r^2}\right)\Big|_0^a = \frac{\pi(e^{a^2} - 1)}{4}$$

10. $$\begin{aligned}
\iint_Q \frac{2xy}{x^2 + y^2}\, dA &= \int_0^{\pi/2} d\theta \int_0^a \frac{2r^2 \sin\theta\cos\theta}{r^2}\, r\, dr \\
&= \frac{a^2}{2} \int_0^{\pi/2} \sin(2\theta)\, d\theta = -\frac{a^2 \cos(2\theta)}{4}\Big|_0^{\pi/2} = \frac{a^2}{2}
\end{aligned}$$

11. $$\begin{aligned}
\iint_S (x + y)\, dA &= \int_0^{\pi/3} d\theta \int_0^a (r\cos\theta + r\sin\theta) r\, dr \\
&= \int_0^{\pi/3} (\cos\theta + \sin\theta)\, d\theta \int_0^a r^2\, dr \\
&= \frac{a^3}{3} (\sin\theta - \cos\theta)\Big|_0^{\pi/3} \\
&= \left[\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) - (-1)\right] \frac{a^3}{3} = \frac{(\sqrt{3} + 1)a^3}{6}
\end{aligned}$$

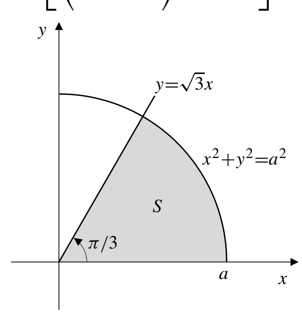


Fig. 14.4.11

12. $$\iint_S x\,dA = 2\int_0^{\pi/4} d\theta \int_{\sec\theta}^{\sqrt{2}} r\cos\theta\, r\,dr$$
$$= \frac{2}{3}\int_0^{\pi/4} \cos\theta\left(2\sqrt{2} - \sec^3\theta\right) d\theta$$
$$= \frac{4\sqrt{2}}{3}\sin\theta\bigg|_0^{\pi/4} - \frac{2}{3}\tan\theta\bigg|_0^{\pi/4}$$
$$= \frac{4}{3} - \frac{2}{3} = \frac{2}{3}$$

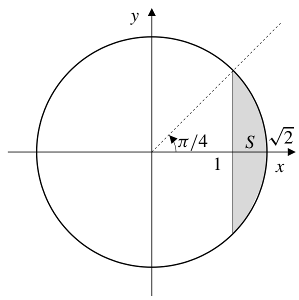


Fig. 14.4.12

13. $$\iint_T (x^2+y^2)\,dA = \int_0^{\pi/4} d\theta \int_0^{\sec\theta} r^3\,dr$$
$$= \frac{1}{4}\int_0^{\pi/4} \sec^4\theta\,d\theta$$
$$= \frac{1}{4}\int_0^{\pi/4} (1+\tan^2\theta)\sec^2\theta\,d\theta \qquad \begin{array}{l}\text{Let } u = \tan\theta \\ du = \sec^2\theta\,d\theta\end{array}$$
$$= \frac{1}{4}\int_0^1 (1+u^2)\,du$$
$$= \frac{1}{4}\left(u + \frac{u^3}{3}\right)\bigg|_0^1 = \frac{1}{3}$$

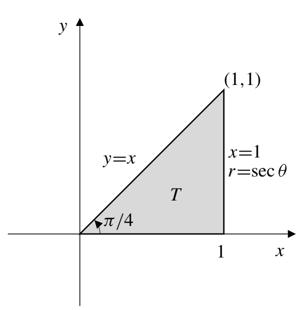


Fig. 14.4.13

14. $$\iint_{x^2+y^2\le 1} \ln(x^2+y^2)\,dA = \int_0^{2\pi} d\theta \int_0^1 (\ln r^2) r\,dr$$
$$= 4\pi \int_0^1 r\ln r\,dr$$
$$\begin{array}{ll} U = \ln r & dV = r\,dr \\ dU = \dfrac{dr}{r} & V = \dfrac{r^2}{2}\end{array}$$
$$= 4\pi\left[\frac{r^2}{2}\ln r\bigg|_0^1 - \frac{1}{2}\int_0^1 r\,dr\right]$$
$$= 4\pi\left[0 - 0 - \frac{1}{4}\right] = -\pi$$

(Note that the integral is improper, but converges since $\lim_{r\to 0+} r^2 \ln r = 0$.)

15. The average distance from the origin to points in the disk D: $x^2+y^2 \le a^2$ is

$$\frac{1}{\pi a^2}\iint_D \sqrt{x^2+y^2}\,dA = \frac{1}{\pi a^2}\int_0^{2\pi} d\theta \int_0^a r^2\,dr = \frac{2a}{3}.$$

16. The annular region R: $0 < a \le \sqrt{x^2+y^2} \le b$ has area $\pi(b^2-a^2)$. The average value of $e^{-(x^2+y^2)}$ over the region is

$$\frac{1}{\pi(b^2-a^2)}\iint_R e^{-(x^2+y^2)}\,dA$$
$$= \frac{1}{\pi(b^2-a^2)}\int_0^{2\pi} d\theta \int_a^b e^{-r^2} r\,dr \qquad \begin{array}{l}\text{Let } u = r^2 \\ du = 2r\,dr\end{array}$$
$$= \frac{1}{\pi(b^2-a^2)}(2\pi)\frac{1}{2}\int_{a^2}^{b^2} e^{-u}\,du$$
$$= \frac{1}{b^2-a^2}\left(e^{-a^2} - e^{-b^2}\right).$$

17. If D is the disk $x^2+y^2 \le 1$, then

$$\iint_D \frac{dA}{(x^2+y^2)^k} = \int_0^{2\pi} d\theta \int_0^1 r^{-2k} r\,dr = 2\pi\int_0^1 r^{1-2k}\,dr$$

which converges if $1-2k > -1$, that is, if $k < 1$. In this case the value of the integral is

$$2\pi\frac{r^{2-2k}}{2-2k}\bigg|_0^1 = \frac{\pi}{1-k}.$$

18. $$\iint_{\mathbb{R}^2} \frac{dA}{(1+x^2+y^2)^k}$$
$$= \int_0^{2\pi} d\theta \int_0^\infty \frac{r\,dr}{(1+r^2)^k} \qquad \begin{array}{l}\text{Let } u = 1+r^2 \\ du = 2r\,dr\end{array}$$
$$= \pi\int_1^\infty u^{-k}\,du = \frac{-\pi}{1-k} \text{ if } k > 1.$$

The integral converges to $\dfrac{\pi}{k-1}$ if $k > 1$.

19.
$$\begin{aligned}\iint_D xy\,dA &= \int_0^{\pi/4} d\theta \int_0^a r\cos\theta r\sin\theta r\,dr\\ &= \frac{1}{2}\int_0^{\pi/4}\sin 2\theta\,d\theta \int_0^a r^3\,dr\\ &= \frac{a^4}{8}\left(-\frac{\cos 2\theta}{2}\right)\Bigg|_0^{\pi/4} = \frac{a^4}{16}.\end{aligned}$$

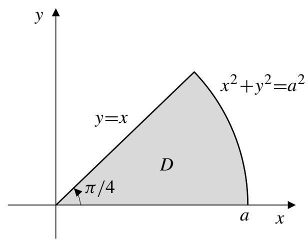


Fig. 14.4.19

20.
$$\begin{aligned}\iint_C y\,dA &= \int_0^{\pi} d\theta \int_0^{1+\cos\theta} r\sin\theta r\,dr\\ &= \frac{1}{3}\int_0^{\pi}\sin\theta(1+\cos\theta)^3\,d\theta \qquad \text{Let } u = 1+\cos\theta\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad du = -\sin\theta\,d\theta\\ &= \frac{1}{3}\int_0^2 u^3\,du = \frac{u^4}{12}\Bigg|_0^2 = \frac{4}{3}\end{aligned}$$

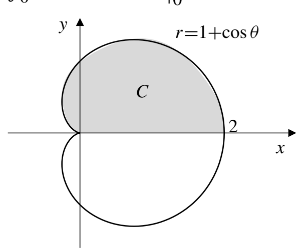


Fig. 14.4.20

21. The paraboloids $z = x^2 + y^2$ and $3z = 4 - x^2 - y^2$ intersect where $3(x^2+y^2) = 4 - (x^2+y^2)$, i.e., on the cylinder $x^2+y^2 = 1$. The volume they bound is given by

$$\begin{aligned}V &= \iint_{x^2+y^2\le 1}\left[\frac{4-x^2-y^2}{3} - (x^2+y^2)\right]dA\\ &= \int_0^{2\pi} d\theta \int_0^1 \left[\frac{4-r^2}{3} - r^2\right] r\,dr\\ &= \frac{8\pi}{3}\int_0^1 (r - r^3)\,dr\\ &= \frac{8\pi}{3}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 = \frac{2\pi}{3} \text{ cu. units.}\end{aligned}$$

22. One quarter of the required volume lies in the first octant. (See the figure.) In polar coordinates the cylinder $x^2 + y^2 = ax$ becomes $r = a\cos\theta$. Thus, the required volume is

$$\begin{aligned}V &= 4\iint_D \sqrt{a^2 - x^2 - y^2}\,dA\\ &= 4\int_0^{\pi/2} d\theta \int_0^{a\cos\theta}\sqrt{a^2 - r^2}\,r\,dr \qquad \text{Let } u = a^2 - r^2\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad du = -2r\,dr\\ &= 2\int_0^{\pi/2} d\theta \int_{a^2\sin^2\theta}^{a^2} u^{1/2}\,du\\ &= \frac{4}{3}\int_0^{\pi/2} d\theta \left(u^{3/2}\Big|_{a^2\sin^2\theta}^{a^2}\right)\\ &= \frac{4}{3}a^3\int_0^{\pi/2}(1 - \sin^3\theta)\,d\theta\\ &= \frac{4}{3}a^3\left(\frac{\pi}{2} - \int_0^{\pi/2}\sin\theta(1-\cos^2\theta)\,d\theta\right)\\ &\qquad\qquad \text{Let } v = \cos\theta\\ &\qquad\qquad dv = -\sin\theta\,d\theta\\ &= \frac{2\pi a^3}{3} - \frac{4a^3}{3}\int_0^1 (1 - v^2)\,dv\\ &= \frac{2\pi a^3}{3} - \frac{4a^3}{3}\left(v - \frac{v^3}{3}\right)\Bigg|_0^1\\ &= \frac{2\pi a^3}{3} - \frac{8a^3}{9} = \frac{2}{9}a^3(3\pi - 4) \text{ cu. units.}\end{aligned}$$

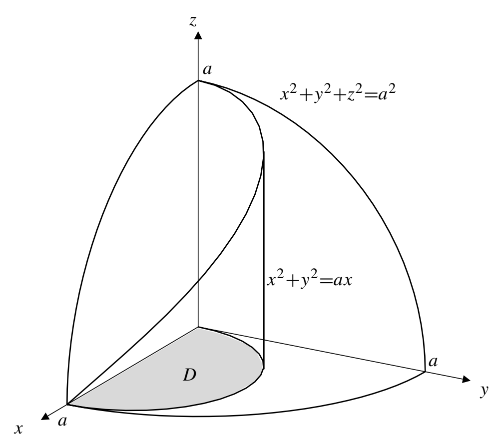


Fig. 14.4.22

23. The volume inside the sphere $x^2+y^2+z^2 = 2a^2$ and the cylinder $x^2+y^2 = a^2$ is

$$\begin{aligned}V &= 8\int_0^{\pi/2} d\theta \int_0^a \sqrt{2a^2 - r^2}\,r\,dr \qquad \text{Let } u = 2a^2 - r^2\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad du = -2r\,dr\\ &= 2\pi\int_{a^2}^{2a^2} u^{1/2}\,du = \frac{4\pi a^3}{3}\left(2\sqrt{2} - 1\right) \text{ cu. units.}\end{aligned}$$

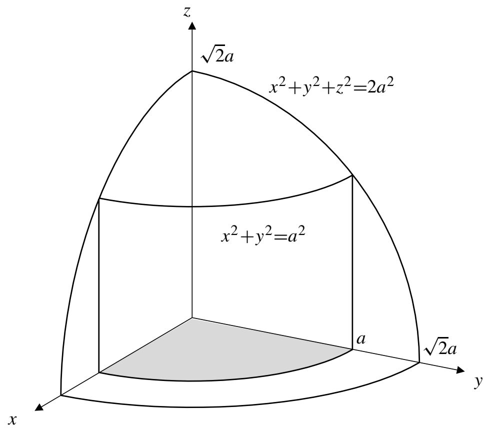


Fig. 14.4.23

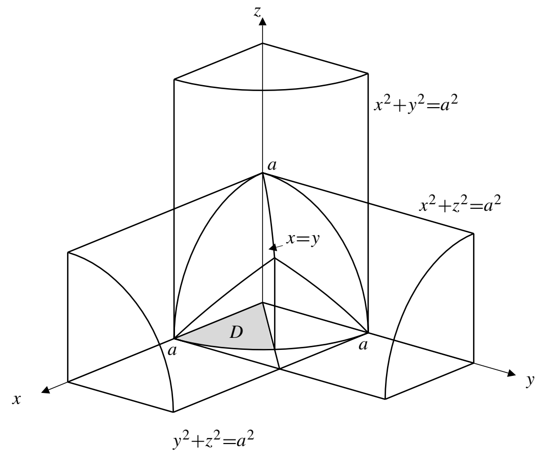


Fig. 14.4.25

24.
$$\begin{aligned}\text{Volume} &= \int_0^{2\pi} d\theta \int_0^2 (r\cos\theta + r\sin\theta + 4) r\,dr \\ &= \int_0^{2\pi} (\cos\theta + \sin\theta)\,d\theta \int_0^2 r^2\,dr + 8\pi \int_0^2 r\,dr \\ &= 0 + 4\pi(2^2) = 16\pi \text{ cu. units.}\end{aligned}$$

25. One eighth of the required volume lies in the first octant. This eighth is divided into two equal parts by the plane $x = y$. One of these parts lies above the circular sector D in the xy-plane specified in polar coordinate by $0 \le r \le a$, $0 \le \theta \le \pi/4$, and beneath the cylinder $z = \sqrt{a^2 - x^2}$. Thus, the total volume lying inside all three cylinders is

$$\begin{aligned}V &= 16 \iint_D \sqrt{a^2 - x^2}\,dA \\ &= 16 \int_0^{\pi/4} d\theta \int_0^a \sqrt{a^2 - r^2\cos^2\theta}\, r\,dr \\ &\qquad \text{Let } u = a^2 - r^2\cos^2\theta \\ &\qquad du = -2r\cos^2\theta\,dr \\ &= 8 \int_0^{\pi/4} \frac{d\theta}{\cos^2\theta} \int_{a^2\sin^2\theta}^{a^2} u^{1/2}\,du \\ &= \frac{16a^3}{3} \int_0^{\pi/4} \frac{1 - \sin^3\theta}{\cos^2\theta}\,d\theta \\ &= \frac{16a^3}{3} \int_0^{\pi/4} \left(\sec^2\theta - \frac{1 - \cos^2\theta}{\cos^2\theta}\sin\theta\right) d\theta \\ &= \frac{16a^3}{3} \left(\tan\theta - \frac{1}{\cos\theta} - \cos\theta\right)\Bigg|_0^{\pi/4} \\ &= \frac{16a^3}{3} \left(1 - 0 - \sqrt{2} + 1 - \frac{1}{\sqrt{2}} + 1\right) \\ &= 16\left(1 - \frac{1}{\sqrt{2}}\right) a^3 \text{ cu. units.}\end{aligned}$$

26. One quarter of the required volume V is shown in the figure. We have

$$\begin{aligned}V &= 4 \iint_D \sqrt{y}\,dA \\ &= 4 \int_0^{\pi/2} d\theta \int_0^{2\sin\theta} \sqrt{r\sin\theta}\, r\,dr \\ &= 4 \int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta \left(\frac{2}{5} r^{5/2}\Bigg|_0^{2\sin\theta}\right) \\ &= \frac{32\sqrt{2}}{5} \int_0^{\pi/2} \sin^3\,d\theta = \frac{64\sqrt{2}}{15} \text{ cu. units.}\end{aligned}$$

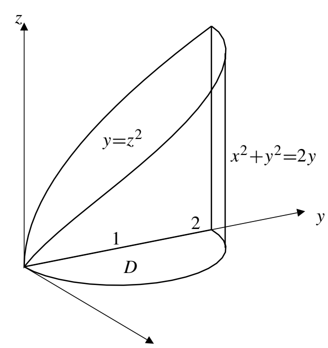


Fig. 14.4.26

27. By symmetry, we need only calculate the average distance from points in the sector S: $0 \le \theta \le \pi/4$,

$0 \le r \le 1$ to the line $x = 1$. This average value is

$$\begin{aligned}\frac{8}{\pi}\iint_S (1-x)\,dA &= \frac{8}{\pi}\int_0^{\pi/4} d\theta \int_0^1 (1 - r\cos\theta) r\,dr \\ &= \frac{8}{\pi}\left[\frac{\pi}{8} - \int_0^{\pi/4}\cos\theta\,d\theta \int_0^1 r^2\,dr\right] \\ &= 1 - \frac{8}{3\sqrt{2}\pi} = 1 - \frac{4\sqrt{2}}{3\pi} \text{ units.}\end{aligned}$$

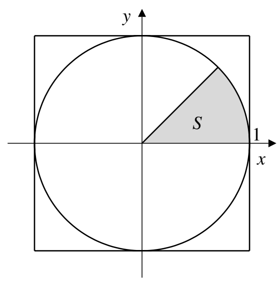


Fig. 14.4.27

28. The area of S is $(4\pi - 3\sqrt{3})/3$ sq. units. Thus

$$\begin{aligned}\bar{x} &= \frac{3}{4\pi - 3\sqrt{3}}\iint_S x\,dA \\ &= \frac{6}{4\pi - 3\sqrt{3}}\int_0^{\pi/3} d\theta \int_{\sec\theta}^2 r\cos\theta\, r\,dr \\ &= \frac{2}{4\pi - 3\sqrt{3}}\int_0^{\pi/3}\cos\theta(8 - \sec^3\theta)\,d\theta \\ &= \frac{2}{4\pi - 3\sqrt{3}}\left(4\sqrt{3} - \tan\theta\Big|_0^{\pi/3}\right) = \frac{6\sqrt{3}}{4\pi - 3\sqrt{3}}.\end{aligned}$$

The segment has centroid $\left(\dfrac{6\sqrt{3}}{4\pi - 3\sqrt{3}}, 0\right)$.

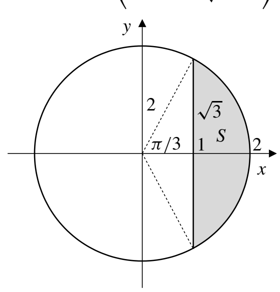


Fig. 14.4.28

29. Let E be the region in the first quadrant of the xy-plane bounded by the coordinate axes and the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The volume of the ellipsoid is

$$V = 8c\iint_E \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dx\,dy.$$

Let $x = au$, $y = bv$. Then

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = ab\,du\,dv.$$

The region E corresponds to the quarter disk Q: $u^2 + v^2 \le 1$, $u, v \ge 0$ in the uv-plane. Thus

$$\begin{aligned}V &= 8abc\iint_Q \sqrt{1 - u^2 - v^2}\,du\,dv \\ &= 8abc \times \left(\frac{1}{8} \times \text{volume of ball of radius 1}\right) \\ &= \frac{4}{3}\pi abc \text{ cu. units.}\end{aligned}$$

30. We use the same regions and change of variables as in the previous exercise. The required volume is

$$\begin{aligned}V &= \iint_E \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) dx\,dy \\ &= ab\iint_Q (1 - u^2 - v^2)\,du\,dv.\end{aligned}$$

Now transform to polar coordinates in the uv-plane: $u = r\cos\theta$, $v = r\sin\theta$.

$$\begin{aligned}V &= ab\int_0^{\pi/2} d\theta \int_0^1 (1 - r^2) r\,dr \\ &= \frac{\pi ab}{2}\left(\frac{r^2}{2} - \frac{r^4}{4}\right)\Bigg|_0^1 = \frac{\pi ab}{8} \text{ cu. units.}\end{aligned}$$

31. Let $x = \dfrac{u+v}{2}$, $y = \dfrac{u-v}{2}$, so that $x + y = u$ and $x - y = v$. We have

$$dx\,dy = \left|\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2}\end{vmatrix}\right| du\,dv = \frac{1}{2}\,du\,dv.$$

Under the above transformation the square $|x| + |y| \le a$ corresponds to the square S: $-a \le u \le a$, $-a \le v \le a$. Thus

$$\begin{aligned}\iint_{|x|+|y|\le a} e^{x+y}\,dA &= \frac{1}{2}\iint_S e^u\,du\,dv \\ &= \frac{1}{2}\int_{-a}^a e^u\,du \int_{-a}^a dv \\ &= a(e^a - e^{-a}) = 2a\sinh a.\end{aligned}$$

32. The parallelogram P bounded by $x + y = 1$, $x + y = 2$, $3x + 4y = 5$, and $3x + 4y = 6$ corresponds to the square S bounded by $u = 1$, $u = 2$,$v = 5$, and $v = 6$ under the transformation

$$u = x + y, \qquad v = 3x + 4y,$$

or, equivalently,

$$x = 4u - v, \qquad y = v - 3u.$$

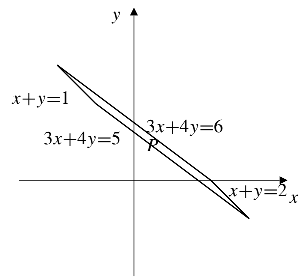


Fig. 14.4.32a

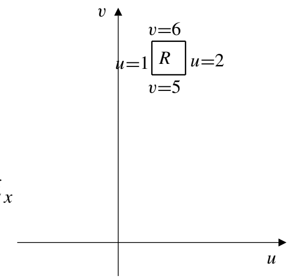


Fig. 14.4.32b

We have

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 4 & -1 \\ -3 & 1 \end{vmatrix} = 1,$$

so $dx\,dy = du\,dv$. Also

$$x^2 + y^2 = (4u-v)^2 + (v-3u)^2 = 25u^2 - 14uv + 2v^2.$$

Thus we have

$$\begin{aligned} \iint_P (x^2+y^2)\,dx\,dy &= \iint_S (25u^2 - 14uv + 2v^2)\,du\,dv \\ &= \int_1^2 du \int_5^6 (25u^2 - 14uv + 2v^2)\,dv = \frac{7}{2}. \end{aligned}$$

33. Let $u = xy$, $v = y/x$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = 2\frac{y}{x} = 2v,$$

so that $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{2v}$. The region D in the first quadrant of the xy-plane bounded by $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$ corresponds to the rectangle R in the uv-plane bounded by $u = 1$, $u = 4$, $v = 1$, and $v = 2$. Thus the area of D is given by

$$\begin{aligned} \iint_D dx\,dy &= \iint_R \frac{1}{2v}\,du\,dv \\ &= \frac{1}{2}\int_1^4 du \int_1^2 \frac{dv}{v} = \frac{3}{2}\ln 2 \text{ sq. units.} \end{aligned}$$

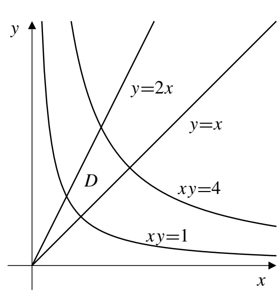


Fig. 14.4.33

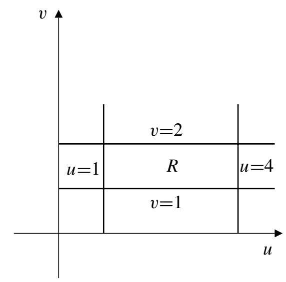


Fig. 14.4.33

34. Under the transformation $u = x^2 - y^2$, $v = xy$, the region R in the first quadrant of the xy-plane bounded by $y = 0$, $y = x$, $xy = 1$, and $x^2 - y^2 = 1$ corresponds to the square S in the uv-plane bounded by $u = 0$, $u = 1$, $v = 0$, and $v = 1$. Since

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2),$$

we therefore have

$$(x^2+y^2)\,dx\,dy = \frac{1}{2}\,du\,dv.$$

Hence,

$$\iint_R (x^2+y^2)\,dx\,dy = \iint_S \frac{1}{2}\,du\,dv = \frac{1}{2}.$$

35. $I = \displaystyle\iint_T e^{(y-x)/(y+x)}\,dA.$

a) $$\begin{aligned} I &= \int_0^{\pi/2} d\theta \int_0^{1/(\cos\theta+\sin\theta)} e^{\frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}}\,r\,dr \\ &= \frac{1}{2}\int_0^{\pi/2} e^{\frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta}} \frac{d\theta}{(\cos\theta+\sin\theta)^2} \\ &\qquad \text{Let } u = \frac{\cos\theta-\sin\theta}{\sin\theta+\cos\theta} \\ &\qquad du = -\frac{2\,d\theta}{(\sin\theta+\cos\theta)^2} \\ &= \frac{1}{4}\int_{-1}^{1} e^u\,du = \frac{e-e^{-1}}{4}. \end{aligned}$$

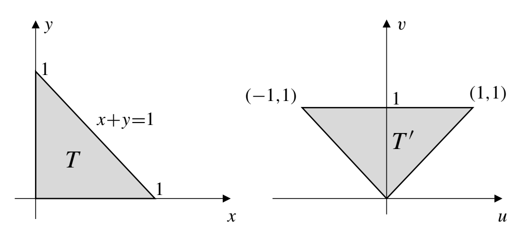


Fig. 14.4.35

b) If $u = y - x$, $v = y + x$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2,$$

so that $dA = dx\,dy = \frac{1}{2}\,du\,dv$. Also, T corresponds to the triangle T' bounded by $u = -v$, $u = v$, and $v = 1$. Thus

$$\begin{aligned} I &= \frac{1}{2}\iint_{T'} e^{u/v}\,du\,dv \\ &= \frac{1}{2}\int_0^1 dv \int_{-v}^{v} e^{u/v}\,du \\ &= \frac{1}{2}\int_0^1 dv \left(ve^{u/v}\right)\Big|_{-v}^{v} \\ &= \frac{1}{2}\left(e - e^{-1}\right)\int_0^1 v\,dv = \frac{e - e^{-1}}{4}. \end{aligned}$$

36. The region R whose area we must find is shown in part (a) of the figure. The change of variables $x = 3u$, $y = 2v$ maps the ellipse $4x^2 + 9y^2 = 36$ to the circle $u^2 + v^2 = 1$, and the line $2x + 3y = 1$ to the line $u + v = 1$. Thus it maps R to the region S in part (b) of the figure. Since

$$dx\,dy = \left| \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} \right| du\,dv = 6\,du\,dv,$$

the area of R is

$$A = \iint_R dx\,dy = 6\iint_S du\,dv.$$

But the area of S is $(\pi/4) - (1/2)$, so $A = (3\pi/2) - 3$ square units.

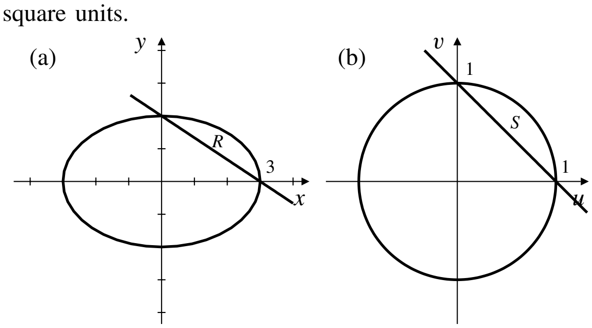


Fig. 14.4.36

37. $\mathrm{Erf}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-t^2}\,dt = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-s^2}\,ds$. Thus

$$\left(\mathrm{Erf}(x)\right)^2 = \frac{4}{\pi}\iint_S e^{-(s^2+t^2)}\,ds\,dt,$$

where S is the square $0 \le s \le x$, $0 \le t \le x$. By symmetry,

$$\left(\mathrm{Erf}(x)\right)^2 = \frac{8}{\pi}\iint_T e^{-(s^2+t^2)}\,ds\,dt,$$

where T is the triangle $0 \le s \le x$, $0 \le t \le s$.

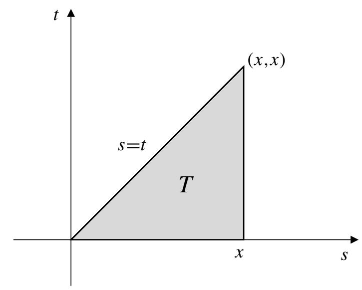


Fig. 14.4.37

Now transform to polar coordinates in the st-plane. We have

$$\begin{aligned} \left(\mathrm{Erf}(x)\right)^2 &= \frac{8}{\pi}\int_0^{\pi/4} d\theta \int_0^{x\sec\theta} e^{-r^2}\,r\,dr \\ &= \frac{4}{\pi}\int_0^{\pi/4} d\theta \left(-e^{-r^2}\right)\Big|_0^{x\sec\theta} \\ &= \frac{4}{\pi}\int_0^{\pi/4}\left(1 - e^{-x^2/\cos^2\theta}\right) d\theta. \end{aligned}$$

Since $\cos^2\theta \le 1$, we have $e^{-x^2/\cos^2\theta} \le e^{-x^2}$, so

$$\begin{aligned} \left(\mathrm{Erf}(x)\right)^2 &\ge 1 - e^{-x^2} \\ \mathrm{Erf}(x) &\ge \sqrt{1 - e^{-x^2}}. \end{aligned}$$

38. a) $\Gamma(x) = \displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$ Let $t = s^2$, $dt = 2s\,ds$

$$= 2\int_0^\infty s^{2x-1}e^{-s^2}\,ds.$$

b) $\Gamma\left(\frac{1}{2}\right) = 2\displaystyle\int_0^\infty e^{-s^2}\,ds = 2\,\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$

$$\Gamma\left(\tfrac{3}{2}\right) = \frac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$$

c) $B(x,y) = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad (x > 0,\ y > 0)$

let $t = \cos^2\theta$, $dt = -2\sin\theta\cos\theta\,d\theta$

$$= 2\int_0^{\pi/2} \cos^{2x-1}\theta\,\sin^{2y-1}\theta\,d\theta.$$

d) If Q is the first quadrant of the st-plane,

$$\Gamma(x)\Gamma(y) = \left(2\int_0^\infty s^{2x-1}e^{-s^2}\,ds\right)\left(2\int_0^\infty t^{2y-1}e^{-t^2}\,dt\right)$$
$$= 4\iint_Q s^{2x-1}t^{2y-1}e^{-(s^2+t^2)}\,ds\,dt$$
(change to polar coordinates)
$$= 4\int_0^{\pi/2} d\theta \int_0^\infty r^{2x-1}\cos^{2x-1}\theta\, r^{2y-1}\sin^{2y-1}\theta\, e^{-r^2} r\,dr$$
$$= \left(2\int_0^{\pi/2}\cos^{2x-1}\theta\sin^{2y-1}\theta\,d\theta\right) \times \left(2\int_0^\infty r^{2(x+y)-1}e^{-r^2}\,dr\right)$$
$$= B(x,y)\Gamma(x+y) \qquad \text{by (a) and (c).}$$

Thus $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Section 14.5 Triple Integrals (page 787)

1. R is symmetric about the coordinate planes and has volume $8abc$. Thus

$$\iiint_R (1+2x-3y)\,dV = \text{volume of } R + 0 - 0 = 8abc.$$

2.
$$\iiint_B xyz\,dV = \int_0^1 x\,dx \int_{-2}^0 y\,dy \int_1^4 z\,dz = \frac{1}{2}\left(-\frac{4}{2}\right)\left(\frac{16-1}{2}\right) = -\frac{15}{2}.$$

3. The hemispherical dome $x^2+y^2+z^2 \le 4$, $z \ge 0$, is symmetric about the planes $x=0$ and $y=0$. Therefore

$$\iiint_D (3+2xy)\,dV = 3\iiint_D dV + 2\iiint_D xy\,dV = 3\times\frac{2}{3}\pi(2^3) + 0 = 16\pi.$$

4.
$$\iiint_R x\,dV = \int_0^a x\,dx \int_0^{b\left(1-\frac{x}{a}\right)} dy \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz$$
$$= c\int_0^a x\,dx \int_0^{b\left(1-\frac{x}{a}\right)} \left(1-\frac{x}{a}-\frac{y}{b}\right) dy$$
$$= c\int_0^a x\left[b\left(1-\frac{x}{a}\right)^2 - \frac{b^2}{2b}\left(1-\frac{x}{a}\right)^2\right] dx$$
$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 x\,dx \qquad \text{Let } u = 1-(x/a),\ du = -(1/a)\,dx$$
$$= \frac{a^2bc}{2}\int_0^1 u^2(1-u)\,du = \frac{a^2bc}{24}.$$

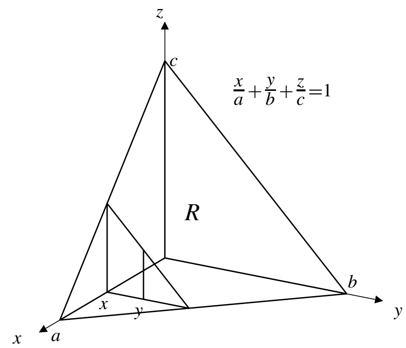


Fig. 14.5.4

5. R is the cube $0 \le x, y, z \le 1$. By symmetry,

$$\iiint_R (x^2+y^2)\,dV = 2\iiint_R x^2\,dV = 2\int_0^1 x^2\,dx\int_0^1 dy\int_0^1 dz = \frac{2}{3}.$$

6. As in Exercise 5,

$$\iiint_R (x^2+y^2+z^2)\,dV = 3\iiint_R x^2\,dV = \frac{3}{3} = 1.$$

7. The set R: $0 \le z \le 1-|x|-|y|$ is a pyramid, one quarter of which lies in the first octant and is bounded by the coordinate planes and the plane $x+y+z=1$. (See the figure.) By symmetry, the integral of xy over R is 0. Therefore,

$$\iiint_R (xy+z^2)\,dV = \iiint_R z^2\,dV$$
$$= 4\int_0^1 z^2\,dz\int_0^{1-z} dy\int_0^{1-z-y} dx$$
$$= 4\int_0^1 z^2\,dz\int_0^{1-z}(1-z-y)\,dy$$
$$= 4\int_0^1 z^2\left[(1-z)^2 - \frac{1}{2}(1-z)^2\right] dz$$
$$= 2\int_0^1 (z^2-2z^3+z^4)\,dz = \frac{1}{15}.$$

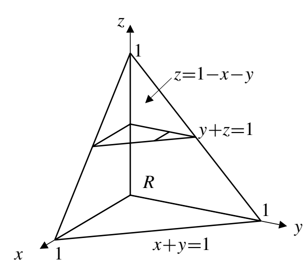


Fig. 14.5.7

8. R is the cube $0 \le x, y, z \le 1$. We have

$$\begin{aligned}
&\iiint_R yz^2 e^{-xyz}\, dV \\
&= \int_0^1 z\, dz \int_0^1 dy \left(-e^{-xyz}\right)\Big|_{x=0}^{x=1} \\
&= \int_0^1 z\, dz \int_0^1 (1 - e^{-yz})\, dy \\
&= \int_0^1 z \left(1 + \frac{1}{z} e^{-yz}\Big|_{y=0}^{y=1}\right) dz \\
&= \frac{1}{2} + \int_0^1 (e^{-z} - 1)\, dz \\
&= \frac{1}{2} - 1 - e^{-z}\Big|_0^1 = \frac{1}{2} - \frac{1}{e}.
\end{aligned}$$

9.
$$\begin{aligned}
\iiint_R \sin(\pi y^3)\, dV &= \int_0^1 \sin(\pi y^3)\, dy \int_0^y dz \int_0^y dx \\
&= \int_0^1 y^2 \sin(\pi y^3)\, dy = -\frac{\cos(\pi y^3)}{3\pi}\Big|_0^1 \\
&= \frac{2}{3\pi}.
\end{aligned}$$

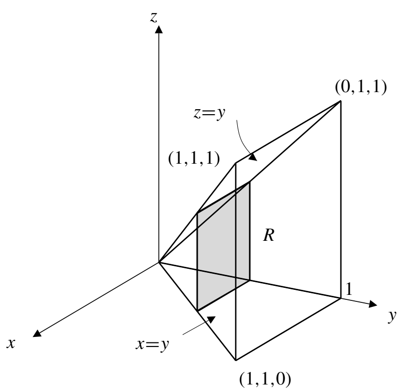


Fig. 14.5.9

10.
$$\begin{aligned}
\iiint_R y\, dV &= \int_0^1 y\, dy \int_{1-y}^1 dz \int_0^{2-y-z} dx \\
&= \int_0^1 y\, dy \int_{1-y}^1 (2 - y - z)\, dz \\
&= \int_0^1 y\, dy \left((2 - y)z - \frac{z^2}{2}\right)\Big|_{z=1-y}^{z=1} \\
&= \int_0^1 y \left((2 - y)y - \frac{1}{2}\left(1 - (1 - y)^2\right)\right) dy \\
&= \int_0^1 \frac{1}{2}\left(2y^2 - y^3\right) dy = \frac{5}{24}.
\end{aligned}$$

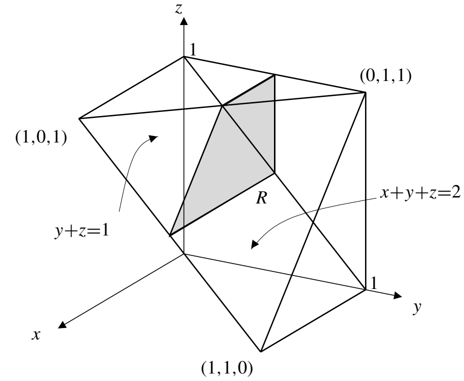


Fig. 14.5.10

11. R is bounded by $z = 1$, $z = 2$, $y = 0$, $y = z$, $x = 0$, and $x = y + z$. These bounds provide an iteration of the triple integral without our having to draw a diagram.

$$\begin{aligned}
&\iiint_R \frac{dV}{(x + y + z)^3} \\
&= \int_1^2 dz \int_0^z dy \int_0^{y+z} \frac{dx}{(x + y + z)^3} \\
&= \int_1^2 dz \int_0^z dy \left(\frac{-1}{2(x + y + z)^2}\right)\Big|_{x=0}^{x=y+z} \\
&= \frac{3}{8} \int_1^2 dz \int_0^z \frac{dy}{(y + z)^2} \\
&= \frac{3}{8} \int_1^2 \left(\frac{-1}{y + z}\right)\Big|_{y=0}^{y=z} dz \\
&= \frac{3}{16} \int_1^2 \frac{dz}{z} = \frac{3}{16} \ln 2.
\end{aligned}$$

12. We have

$$\begin{aligned}
&\iiint_R \cos x \cos y \cos z\, dV \\
&= \int_0^\pi \cos x\, dx \int_0^{\pi - x} \cos y\, dy \int_0^{\pi - x - y} \cos z\, dz \\
&= \int_0^\pi \cos x\, dx \int_0^{\pi - x} \cos y\, dy\, (\sin z)\Big|_{z=0}^{z=\pi - x - y} \\
&= \int_0^\pi \cos x\, dx \int_0^{\pi - x} \cos y \sin(x + y)\, dy \\
&\qquad \text{recall that } \sin a \cos b = \frac{1}{2}\Big(\sin(a + b) + \sin(a - b)\Big) \\
&= \int_0^\pi \cos x\, dx \int_0^{\pi - x} \frac{1}{2}\Big[\sin(x + 2y) + \sin x\Big] dy \\
&= \frac{1}{2} \int_0^\pi \cos x\, dx \left[-\frac{\cos(x + 2y)}{2} + y \sin x\right]\Bigg|_{y=0}^{y=\pi - x} \\
&= \frac{1}{2} \int_0^\pi \left(-\frac{\cos x \cos(2\pi - x)}{2} + \frac{\cos^2 x}{2}\right.
\end{aligned}$$

$$\left. + (\pi - x)\cos x \sin x \right) dx$$
$$= \frac{1}{2}\int_0^{\pi} \frac{\pi - x}{2} \sin 2x\, dx$$
$$U = \pi - x \qquad dV = \sin 2x\, dx$$
$$dU = -dx \qquad V = -\frac{\cos 2x}{2}$$
$$= \frac{1}{4}\left[-\frac{\pi - x}{2}\cos 2x \Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi} \cos 2x\, dx \right]$$
$$= \frac{1}{8}\left[\pi - \frac{\sin 2x}{2}\Big|_0^{\pi} \right] = \frac{\pi}{8}.$$

13. By Example 4 of Section 5.4, $\int_{-\infty}^{\infty} e^{-u^2}\, du = \sqrt{\pi}$. If $k > 0$, let $u = \sqrt{k}t$, so that $du = \sqrt{k}\, dt$. Thus

$$\int_{-\infty}^{\infty} e^{-kt^2}\, dt = \sqrt{\frac{\pi}{k}}.$$

Thus

$$\iiint_{\mathbb{R}^3} e^{-x^2-2y^2-3y^2}\, dV$$
$$= \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-2y^2}\, dy \int_{-\infty}^{\infty} e^{-3z^2}\, dz$$
$$= \sqrt{\pi}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\pi}{3}} = \frac{\pi^{3/2}}{\sqrt{6}}.$$

14. Let E be the elliptic disk bounded by $x^2 + 4y^2 = 4$. Then E has area $\pi(2)(1) = 2\pi$ square units. The volume of the region of 3-space lying above E and beneath the plane $z = 2 + x$ is

$$V = \iint_E (2 + x)\, dA = 2\iint_E dA = 4\pi \text{ cu. units,}$$

since $\iint_E x\, dA = 0$ by symmetry.

15.
$$\iiint_T x\, dV = \int_0^1 x\, dx \int_{1-x}^1 dy \int_{2-x-y}^1 dz$$
$$= \int_0^1 x\, dx \int_{1-x}^1 (x + y - 1)\, dy$$
$$= \int_0^1 x\left[\frac{(x-1)^2}{2} + x - \frac{1}{2} \right] dx$$
$$= \int_0^1 \frac{x^3}{2}\, dx = \frac{1}{8}.$$

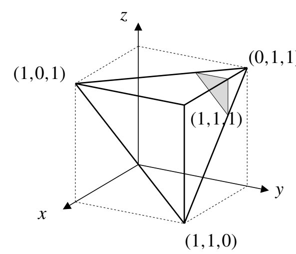


Fig. 14.5.15

16.

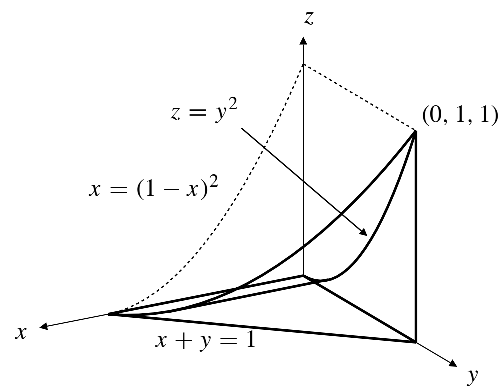


Fig. 14.5.16

$$\iiint_R f(x,y,z)\, dV = \int_0^1 dx \int_0^{1-x} dy \int_0^{y^2} f(x,y,z)\, dz$$
$$= \int_0^1 dy \int_0^{1-y} dx \int_0^{y^2} f(x,y,z)\, dz$$
$$= \int_0^1 dy \int_0^{y^2} dz \int_0^{1-y} f(x,y,z)\, dx$$
$$= \int_0^1 dz \int_{\sqrt{z}}^1 dy \int_0^{1-y} f(x,y,z)\, dx$$
$$= \int_0^1 dx \int_0^{(1-x)^2} dz \int_{\sqrt{z}}^{1-x} f(x,y,z)\, dy$$
$$= \int_0^1 dz \int_0^{1-\sqrt{z}} dx \int_{\sqrt{z}}^{1-x} f(x,y,z)\, dy.$$

17.
$$\int_0^1 dz \int_0^{1-z} dy \int_0^1 f(x,y,z)\, dx$$
$$= \iiint_R f(x,y,z)\, dV \qquad (R \text{ is the prism in the figure})$$
$$= \int_0^1 dx \int_0^1 dy \int_0^{1-y} f(x,y,z)\, dz.$$

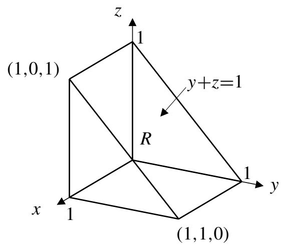


Fig. 14.5.17

18. $$\int_0^1 dz \int_z^1 dy \int_0^y f(x,y,z)\,dx$$
$$= \iiint_R f(x,y,z)\,dV \qquad (R \text{ is the pyramid in the figure})$$
$$= \int_0^1 dx \int_x^1 dy \int_0^y f(x,y,z)\,dz.$$

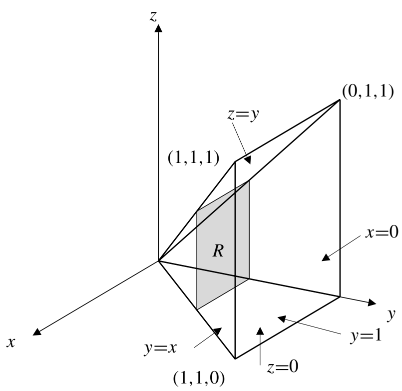


Fig. 14.5.18

19. $$\int_0^1 dz \int_z^1 dx \int_0^{x-z} f(x,y,z)\,dy$$
$$= \iiint_R f(x,y,z)\,dV \quad (R \text{ is the tetrahedron in the figure})$$
$$= \int_0^1 dx \int_0^x dy \int_0^{x-y} f(x,y,z)\,dz.$$

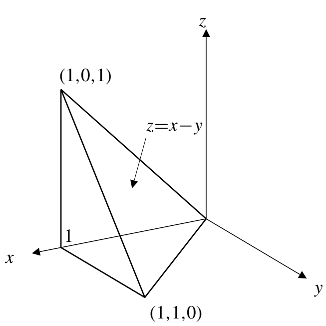


Fig. 14.5.19

20. $$\int_0^1 dy \int_0^{\sqrt{1-y^2}} dz \int_{y^2+z^2}^1 f(x,y,z)\,dx$$
$$= \iiint_R f(x,y,z)\,dV \quad (R \text{ is the paraboloid in the figure})$$
$$= \int_0^1 dx \int_0^{\sqrt{x}} dy \int_0^{\sqrt{x-y^2}} f(x,y,z)\,dz.$$

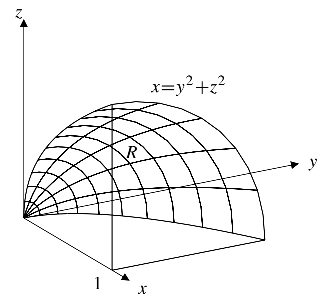


Fig. 14.5.20

21. $I = \int_0^1 dz \int_0^{1-z} dy \int_0^1 f(x,y,z)\,dx.$
The given iteration corresponds to
$$0 \le z \le 1, \quad 0 \le y \le 1-z, \quad 0 \le x \le 1.$$
Thus $0 \le x \le 1$, $0 \le y \le 1-0=1$, $0 \le z \le 1-y$, and
$$I = \int_0^1 dx \int_0^1 dy \int_0^{1-y} f(x,y,z)\,dz.$$

22. $I = \int_0^1 dz \int_z^1 dy \int_0^y f(x,y,z)\,dx.$
The given iteration corresponds to
$$0 \le z \le 1, \quad z \le y \le 1, \quad 0 \le x \le y.$$
Thus $0 \le x \le 1$, $x \le y \le 1$, $0 \le z \le y$, and
$$I = \int_0^1 dx \int_x^1 dy \int_0^y f(x,y,z)\,dz.$$

23. $I = \int_0^1 dz \int_z^1 dx \int_0^{x-z} f(x,y,z)\,dy.$
The given iteration corresponds to
$$0 \le z \le 1, \quad z \le x \le 1, \quad 0 \le y \le x-z.$$
Thus $0 \le x \le 1$, $0 \le y \le x$, $0 \le z \le x-y$, and
$$I = \int_0^1 dx \int_0^x dy \int_0^{x-y} f(x,y,z)\,dz.$$

24. $I = \int_0^1 dy \int_0^{\sqrt{1-y^2}} dz \int_{y^2+z^2}^1 f(x, y, z)\, dx.$
The given iteration corresponds to

$$0 \le y \le 1, \quad 0 \le z \le \sqrt{1-y^2}, \quad y^2 + z^2 \le x \le 1.$$

Thus $0 \le x \le 1$, $0 \le y \le \sqrt{x}$, $0 \le z \le \sqrt{x - y^2}$, and

$$I = \int_0^1 dx \int_0^{\sqrt{x}} dy \int_0^{\sqrt{x-y^2}} f(x, y, z)\, dz.$$

25. $I = \int_0^1 dy \int_y^1 dz \int_0^z f(x, y, z)\, dx.$
The given iteration corresponds to

$$0 \le y \le 1, \quad y \le z \le 1, \quad 0 \le x \le z.$$

Thus $0 \le x \le 1$, $x \le z \le 1$, $0 \le y \le z$, and

$$I = \int_0^1 dx \int_x^1 dz \int_0^z f(x, y, z)\, dy.$$

26.

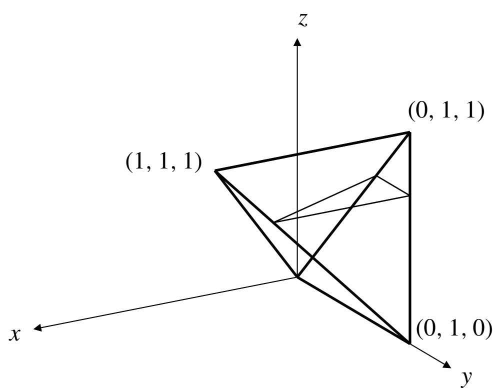


Fig. 14.5.26

$$I = \int_0^1 dx \int_x^1 dy \int_x^y f(x, y, z)\, dz = \iiint_P f(x, y, z)\, dV,$$

where P is the triangular pyramid (see the figure) with vertices at $(0, 0, 0)$, $(0, 1, 0)$, $(0, 1, 1)$, and $(1, 1, 1)$. If we we reiterate I to correspond to the horizontal slice shown then

$$\int_0^1 dz \int_z^1 dy \int_0^z f(x, y, z)\, dx.$$

27. $\int_0^1 dz \int_z^1 dx \int_0^x e^{x^3}\, dy$

$= \iiint_R e^{x^3}\, dV \qquad$ (R is the pyramid in the figure)

$= \int_0^1 e^{x^3} dx \int_0^x dy \int_0^x dz$

$= \int_0^1 x^2 e^{x^3}\, dx = \frac{e-1}{3}.$

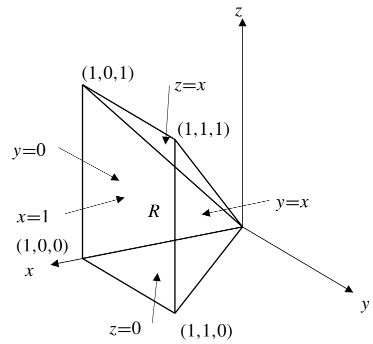


Fig. 14.5.27

28. $\int_0^1 dx \int_0^{1-x} dy \int_y^1 \frac{\sin(\pi z)}{z(2-z)}\, dz$

$= \iiint_R \frac{\sin(\pi z)}{z(2-z)}\, dV \qquad$ (R is the pyramid in the figure)

$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)}\, dz \int_0^z dy \int_0^{1-y} dx$

$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)}\, dz \int_0^z (1-y)\, dy$

$= \int_0^1 \frac{\sin(\pi z)}{z(2-z)} \left(z - \frac{z^2}{2} \right) dz$

$= \frac{1}{2} \int_0^1 \sin(\pi z)\, dz = \frac{1}{\pi}.$

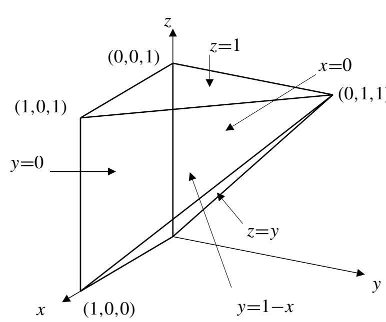


Fig. 14.5.28

29. The average value of $f(x, y, z)$ over R is

$$\bar{f} = \frac{1}{\text{volume of } R} \iiint_R f(x, y, z)\, dV.$$

If $f(x, y, z) = x^2 + y^2 + z^2$ and R is the cube $0 \le x, y, z \le 1$, then, by Exercise 6,

$$\bar{f} = \frac{1}{1} \iiint_R (x^2 + y^2 + z^2)\, dV = 1.$$

30. If the function $f(x, y, z)$ is continuous on a closed, bounded, connected set D in 3-space, then there exists a point (x_0, y_0, z_0) in D such that

$$\iiint_D f(x, y, z)\, dV = f(x_0, y_0, z_0) \times (\text{volume of } D).$$

Apply this with $D = B_\epsilon(a, b, c)$, which has volume $\frac{4}{3}\pi\epsilon^3$, to get

$$\iiint_{B_\epsilon(a,b,c)} f(x, y, z)\, dV = f(x_0, y_0, z_0)\frac{4}{3}\pi\epsilon^3$$

for some (x_0, y_0, z_0) in $B_\epsilon(a, b, c)$. Thus

$$\begin{aligned} &\lim_{\epsilon\to 0} \frac{3}{4\pi\epsilon^3} \iiint_{B_\epsilon(a,b,c)} f(x, y, z)\, dV \\ &= \lim_{\epsilon\to 0} f(x_0, y_0, z_0) = f(a, b, c) \end{aligned}$$

since f is continuous at (a, b, c).

Section 14.6 Change of Variables in Triple Integrals (page 795)

1. Spherical: $[4, \pi/3, 2\pi/3]$;
Cartesian: $(-\sqrt{3}, 3, -2)$; Cylindrical: $[2\sqrt{3}, 2\pi/3, 2]$.

2. Cartesian: $(2, -2, 1)$;
Cylindrical: $[2\sqrt{2}, -\pi/4, 1]$;
Spherical: $[3, \cos^{-1}(1/3), -\pi/4]$.

3. Cylindrical: $[2, \pi/6, -2]$;
Cartesian: $(\sqrt{3}, 1, -2]$; Spherical: $[2\sqrt{2}, 3\pi/4, \pi/6]$.

4. Spherical: $[1, \phi, \theta]$; Cylindrical: $[r, \pi/4, r]$.

$$\begin{aligned} x &= \sin\phi\cos\theta = r\cos\pi/4 = r/\sqrt{2} \\ y &= \sin\phi\sin\theta = r\sin\pi/4 = r/\sqrt{2} \\ z &= \cos\phi = r. \end{aligned}$$

Thus $x = y$, $\theta = \pi/4$, and $r = \sin\phi = \cos\phi$. Hence $\phi = \pi/4$, so $r = 1/\sqrt{2}$. Finally: $x = y = 1/2$, $z = 1/\sqrt{2}$.
Cartesian: $(1/2, 1/2, 1/\sqrt{2})$.

5. $\theta = \pi/2$ represents the half-plane $x = 0$, $y > 0$.

6. $\phi = 2\pi/3$ represents the lower half of the right-circular cone with vertex at the origin, axis along the z-axis, and semi-vertical angle $\pi/3$. Its Cartesian equation is $z = -\sqrt{(x^2 + y^2)/3}$.

7. $\phi = \pi/2$ represents the xy-plane.

8. $\rho = 4$ represents the sphere of radius 4 centred at the origin.

9. $r = 4$ represents the circular cylinder of radius 4 with axis along the z-axis.

10. $\rho = z$ represents the positive half of the z-axis.

11. $\rho = r$ represents the xy-plane.

12. $\rho = 2x$ represents the half-cone with vertex at the origin, axis along the positive x-axis, and semi-vertical angle $\pi/3$. Its Cartesian equation is $x = \sqrt{(y^2 + z^2)/3}$.

13. If $\rho = 2\cos\phi$, then $\rho^2 = 2\rho\cos\phi$, so

$$\begin{aligned} x^2 + y^2 + z^2 &= 2z \\ x^2 + y^2 + z^2 - 2z + 1 &= 1 \\ x^2 + y^2 + (z-1)^2 &= 1. \end{aligned}$$

Thus $\rho = 2\cos\phi$ represents the sphere of radius 1 centred at $(0, 0, 1)$.

14. $r = 2\cos\theta \Rightarrow x^2 + y^2 = r^2 = 2r\cos\theta = 2x$, or $(x-1)^2 + y^2 = 1$. Thus the given equation represents the circular cylinder of radius 1 with axis along the vertical line $x = 1$, $y = 0$.

15.
$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi\, d\phi \int_0^a R^2\, dR \\ &= \frac{2\pi a^3}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \text{ cu. units.} \end{aligned}$$

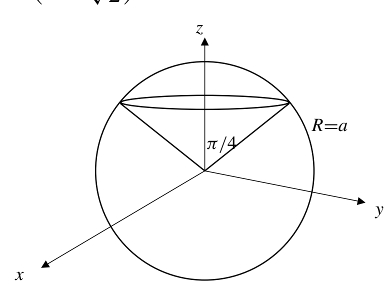


Fig. 14.6.15

16. The surface $z = \sqrt{r}$ intersects the sphere $r^2 + z^2 = 2$ where $r^2 + r - 2 = 0$. This equation has positive root $r = 1$. The required volume is

$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{\sqrt{r}}^{\sqrt{2-r^2}} dz \\ &= \int_0^{2\pi} d\theta \int_0^1 \left(\sqrt{2-r^2} - \sqrt{r}\right) r\,dr \\ &= 2\pi \left(\int_0^1 r\sqrt{2-r^2}\,dr - \frac{2}{5} \right) \quad \begin{array}{l} \text{Let } u = 2 - r^2 \\ du = -2r\,dr \end{array} \\ &= \pi \int_1^2 u^{1/2}\,du - \frac{4\pi}{5} \\ &= \frac{2\pi}{3}\left(2\sqrt{2} - 1\right) - \frac{4\pi}{5} = \frac{4\sqrt{2}\pi}{3} - \frac{22\pi}{15} \text{ cu. units.} \end{aligned}$$

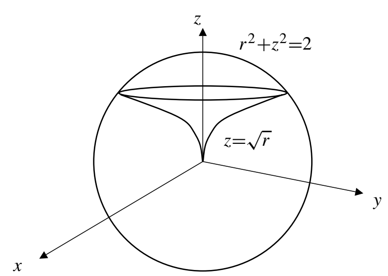


Fig. 14.6.16

17. The paraboloids $z = 10 - r^2$ and $z = 2(r^2 - 1)$ intersect where $r^2 = 4$, that is, where $r = 2$. The volume lying between these surfaces is

$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_0^2 [10 - r^2 - 2(r^2 - 1)] r\,dr \\ &= 2\pi \int_0^2 (12r - 3r^3)\,dr = 24\pi \text{ cu. units.} \end{aligned}$$

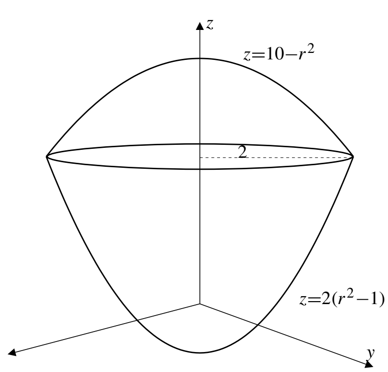


Fig. 14.6.17

18. The paraboloid $z = r^2$ intersects the sphere $r^2 + z^2 = 12$ where $r^4 + r^2 - 12 = 0$, that is, where $r = \sqrt{3}$. The required volume is

$$\begin{aligned} V &= \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} \left(\sqrt{12 - r^2} - r^2\right) r\,dr \\ &= 2\pi \int_0^{\sqrt{3}} r\sqrt{12 - r^2}\,dr - \frac{9\pi}{2} \quad \begin{array}{l} \text{Let } u = 12 - r^2 \\ du = -2r\,dr \end{array} \\ &= \pi \int_9^{12} u^{1/2}\,du - \frac{9\pi}{2} \\ &= \frac{2\pi}{3}\left(12^{3/2} - 27\right) - \frac{9\pi}{2} = 16\sqrt{3}\pi - \frac{45\pi}{2} \text{ cu. units.} \end{aligned}$$

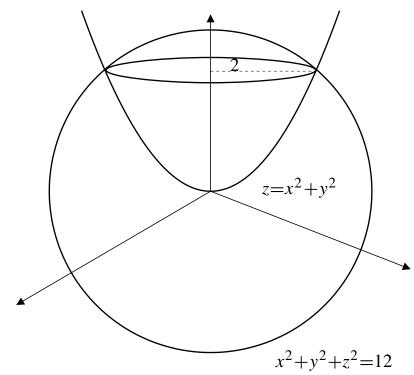


Fig. 14.6.18

19. One half of the required volume V lies in the first octant, inside the cylinder with polar equation $r = 2a\sin\theta$. Thus

$$\begin{aligned} V &= 2\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} (2a - r) r\,dr \\ &= 2a \int_0^{\pi/2} 4a^2 \sin^2\theta\,d\theta - \frac{2}{3}\int_0^{\pi/2} 8a^3 \sin^3\theta\,d\theta \\ &= 4a^3 \int_0^{\pi/2} (1 - \cos 2\theta)\,d\theta - \frac{16a^3}{3}\int_0^{\pi/2} \sin^3\theta\,d\theta \\ &= 2\pi a^3 - \frac{32a^3}{9} \text{ cu. units.} \end{aligned}$$

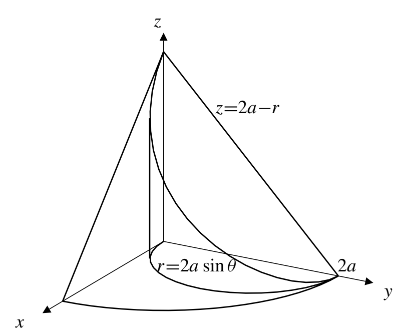


Fig. 14.6.19

20. The required volume V lies above $z = 0$, below $z = 1 - r^2$, and between $\theta = -\pi/4$ and $\theta = \pi/3$. Thus

$$V = \int_{-\pi/4}^{\pi/3} d\theta \int_0^1 (1 - r^2) r\, dr$$
$$= \frac{7\pi}{12}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{7\pi}{48} \text{ cu. units.}$$

21. Let R be the region in the first octant, inside the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and between the planes $y = 0$ and $y = x$. Under the transformation

$$x = au, \qquad y = bv, \qquad z = cw,$$

R corresponds to the region S in the first octant of uvw-space, inside the sphere

$$u^2 + v^2 + w^2 = 1,$$

and between the planes $v = 0$ and $bv = au$. Therefore, the volume of R is

$$V = \iiint_R dx\, dy\, dz = abc \iiint_S du\, dv\, dw.$$

Using spherical coordinates in uvw-space, S corresponds to

$$0 \le R \le 1, \qquad 0 \le \phi \le \frac{\pi}{2}, \qquad 0 \le \theta \le \tan^{-1}\frac{a}{b}.$$

Thus

$$V = abc \int_0^{\tan^{-1}(a/b)} d\theta \int_0^{\pi/2} \sin\phi\, d\phi \int_0^1 R^2\, dR$$
$$= \frac{1}{3} abc \tan^{-1}\frac{a}{b} \text{ cu. units.}$$

22. One eighth of the required volume V lies in the first octant. Call this region R. Under the transformation

$$x = au, \qquad y = bv, \qquad z = cw,$$

R corresponds to the region S in the first octant of uvw-space bounded by $w = 0$, $w = 1$, and $u^2 + v^2 - w^2 = 1$. Thus

$$V = 8abc \times (\text{volume of } S).$$

The volume of S can be determined by using horizontal slices:

$$V = 8abc \int_0^1 \frac{\pi}{4}(1 + w^2)\, dw = \frac{8}{3}\pi abc \text{ cu. units.}$$

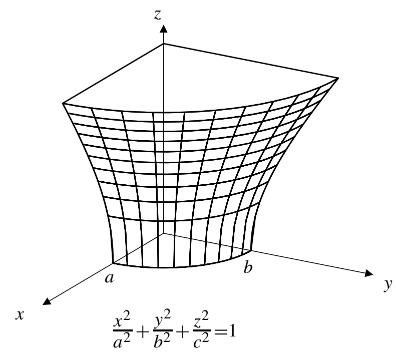


Fig. 14.6.22

23. Let $x = au$, $y = bv$, $z = w$. The indicated region R corresponds to the region S above the uv-plane and below the surface $w = 1 - u^2 - v^2$. We use polar coordinates in the uv-plane to calculate the volume V of R:

$$V = \iiint_R dV = ab \iiint_S du\, dv\, dw$$
$$= ab \int_0^{2\pi} d\theta \int_0^1 (1 - r^2) r\, dr = \frac{\pi ab}{2} \text{ cu. units.}$$

24.
$$\iiint_R (x^2 + y^2 + z^2)\, dV$$
$$= \int_0^{2\pi} d\theta \int_0^a r\, dr \int_0^h (r^2 + z^2)\, dz$$
$$= 2\pi \int_0^a \left(r^3 h + \frac{1}{3} r h^3\right) dr$$
$$= 2\pi \left(\frac{a^4 h}{4} + \frac{a^2 h^3}{6}\right) = \frac{\pi a^4 h}{2} + \frac{\pi a^2 h^3}{3}.$$

25.
$$\iiint_B (x^2 + y^2)\, dV$$
$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^a R^2 \sin^2\phi\, R^2\, dR$$
$$= 2\pi \int_0^{\pi} \sin^3\phi\, d\phi \int_0^a R^4\, dR$$
$$= 2\pi \left(\frac{4}{3}\right) \frac{a^5}{5} = \frac{8\pi a^5}{15}.$$

26.
$$\iiint_B (x^2 + y^2 + z^2)\, dV$$
$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^a R^4\, dR = \frac{4\pi a^5}{5}.$$

27.
$$\iiint_R (x^2 + y^2 + z^2)\, dV$$
$$= \int_0^{2\pi} d\theta \int_0^{\tan^{-1}(1/c)} \sin\phi\, d\phi \int_0^a R^4\, dR$$
$$= \frac{2\pi a^5}{5}\left[1 - \cos\left(\tan^{-1}\frac{1}{c}\right)\right] = \frac{2\pi a^5}{5}\left(1 - \frac{c}{\sqrt{c^2 + 1}}\right).$$

28. $$\iiint_R (x^2+y^2)\,dV$$
$$= \int_0^{2\pi} d\theta \int_0^{\tan^{-1}(1/c)} \sin^3\phi\,d\phi \int_0^a R^4\,dR$$
$$= \frac{2\pi a^5}{5}\int_0^{\tan^{-1}(1/c)} \sin\phi(1-\cos^2\phi)\,d\phi \quad \text{Let } u=\cos\phi, \ du=-\sin\phi\,d\phi$$
$$= \frac{2\pi a^5}{5}\int_{c/\sqrt{c^2+1}}^{1}(1-u^2)\,du$$
$$= \frac{2\pi a^5}{5}\left(u-\frac{u^3}{3}\right)\bigg|_{c/\sqrt{c^2+1}}^{1}$$
$$= \frac{2\pi a^5}{5}\left(\frac{2}{3}-\frac{c}{\sqrt{c^2+1}}+\frac{c^3}{3(c^2+1)^{3/2}}\right).$$

29. $z=r^2$ and $z=\sqrt{2-r^2}$ intersect where $r^4+r^2-2=0$, that is, on the cylinder $r=1$. Thus

$$\iiint_R z\,dV = \int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{r^2}^{\sqrt{2-r^2}} z\,dz$$
$$= \pi\int_0^1 (2-r^2-r^4)r\,dr = \frac{7\pi}{12}.$$

30. By symmetry, both integrals have the same value:

$$\iiint_R x\,dV = \iiint_R z\,dV$$
$$= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \cos\phi\sin\phi\,d\phi \int_0^a R^3\,dR$$
$$= \frac{\pi}{2}\left(\frac{1}{2}\right)\frac{a^4}{4} = \frac{\pi a^4}{16}.$$

31. $$\iiint_R x\,dV = \int_0^{\pi/2} d\theta \int_0^a r\,dr \int_0^{h(1-(r/a))} r\cos\theta\,dz$$
$$= h\int_0^{\pi/2}\cos\theta\,d\theta \int_0^a r^2\left(1-\frac{r}{a}\right)dr = \frac{ha^3}{12},$$
$$\iiint_R z\,dV = \int_0^{\pi/2} d\theta \int_0^a r\,dr \int_0^{h(1-(r/a))} z\,dz$$
$$= \frac{\pi h^2}{4}\int_0^a \left(1-\frac{r}{a}\right)^2 r\,dr$$
$$= \frac{\pi h^2}{4}\left(\frac{r^2}{2}-\frac{2r^3}{3a}+\frac{r^4}{4a^2}\right)\bigg|_0^a = \frac{\pi a^2h^2}{48}.$$

32. If

$$x=au, \qquad y=bv, \qquad z=cw,$$

then the volume of a region R in xyz-space is abc times the volume of the corresponding region S in uvw-space.

If R is the region inside the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

and above the plane $y+z=b$, then the corresponding region S lies inside the sphere

$$u^2+v^2+w^2=1$$

and above the plane $bv+cw=b$. The distance from the origin to this plane is

$$D=\frac{b}{\sqrt{b^2+c^2}} \qquad \text{(assuming } b>0)$$

by Example 7 of Section 1.4. By symmetry, the volume of S is equal to the volume lying inside the sphere $u^2+v^2+w^2=1$ and above the plane $w=D$. We calculate this latter volume by slicing; it is

$$\pi\int_D^1 (1-w^2)\,dw = \pi\left(w-\frac{w^3}{3}\right)\bigg|_D^1$$
$$= \pi\left(\frac{2}{3}-D+\frac{D^3}{3}\right).$$

Hence, the volume of R is

$$\pi abc\left(\frac{2}{3}-\frac{b}{\sqrt{b^2+c^2}}+\frac{b^3}{3(b^2+c^2)^{3/2}}\right) \text{ cu. units.}$$

33. By Example 10 of Section 3.5, we know that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

The required result follows if we add $\dfrac{\partial^2 u}{\partial z^2}$ to both sides.

34. Cylindrical and spherical coordinates are related by

$$z=\rho\cos\phi, \qquad r=\rho\sin\phi.$$

(The θ coordinates are identical in the two systems.) Observe that z, r, ρ, and ϕ play, respectively, the same roles that x, y, r, and θ play in the transformation from Cartesian to polar coordinates in the plane. We can exploit this correspondence to avoid repeating the calculations of partial derivatives of a function u, since the results correspond to calculations made (for a function z) in Example 10 of Section 3.5. Comparing with the calculations in that Example, we have

$$\frac{\partial u}{\partial \rho}=\cos\phi\frac{\partial u}{\partial z}+\sin\phi\frac{\partial u}{\partial r}$$
$$\frac{\partial u}{\partial \phi}=-\rho\sin\phi\frac{\partial u}{\partial z}+\rho\cos\phi\frac{\partial u}{\partial r}$$
$$\frac{\partial^2 u}{\partial \rho^2}=\cos^2\phi\frac{\partial^2 u}{\partial z^2}+2\cos\phi\sin\phi\frac{\partial^2 u}{\partial z\partial r}+\sin^2\phi\frac{\partial^2 u}{\partial r^2}$$
$$\frac{\partial^2 u}{\partial \phi^2}=-\rho\frac{\partial u}{\partial \rho}+\rho^2\bigg(\sin^2\phi\frac{\partial^2 u}{\partial z^2}$$
$$-2\cos\phi\sin\phi\frac{\partial^2 u}{\partial z\partial r}+\cos^2\phi\frac{\partial^2 u}{\partial r^2}\bigg).$$

Substituting these expressions into the expression for Δu given in the statement of this exercise in terms of spherical coordinates, we obtain the expression in terms of cylindrical coordinates established in the previous exercise:

$$\begin{aligned}&\frac{\partial^2 u}{\partial \rho^2}+\frac{2}{\rho}\frac{\partial u}{\partial \rho}+\frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi}+\frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}+\frac{1}{\rho^2\sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}\\ &=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}\\ &=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\Delta u\end{aligned}$$

by Exercise 33.

35. Consider the transformation

$$x=x(u,v,w),\quad y=y(u,v,w),\quad z=z(u,v,w),$$

and let P be the point in xyz-space corresponding to $u=a$, $v=b$, $w=c$. Fixing $v=b$, $w=c$, results in a parametric curve (with parameter u) through P. The vector

$$\overrightarrow{PQ}=\frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k}$$

and corresponding vectors

$$\begin{aligned}\overrightarrow{PR}&=\frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k}\\ \overrightarrow{PS}&=\frac{\partial x}{\partial w}\mathbf{i}+\frac{\partial y}{\partial w}\mathbf{j}+\frac{\partial z}{\partial w}\mathbf{k}\end{aligned}$$

span a parallelepiped in xyz-space corresponding to a rectangular box with volume $du\,dv\,dw$ in uvw-space. The parallelepiped has volume

$$|(\overrightarrow{PQ}\times\overrightarrow{PR})\bullet\overrightarrow{PS}|=\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw.$$

Thus

$$dV=dx\,dy\,dz=\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|du\,dv\,dw.$$

Section 14.7 Applications of Multiple Integrals (page 803)

1. $z=2x+2y,\quad \dfrac{\partial z}{\partial x}=2=\dfrac{\partial z}{\partial y}$

$$dS=\sqrt{1+2^2+2^2}\,dA=3\,dA$$
$$S=\iint_{x^2+y^2\le 1}3\,dA=3\pi(1^2)=3\pi \text{ sq. units.}$$

2. $z=(3x-4y)/5,\quad \dfrac{\partial z}{\partial x}=\dfrac{3}{5},\quad \dfrac{\partial z}{\partial y}=\dfrac{4}{5}$

$$dS=\sqrt{1+\frac{3^2+4^2}{5^2}}\,dA=\sqrt{2}\,dA$$
$$S=\iint_{(x/2)^2+y^2\le 1}\sqrt{2}\,dA=\sqrt{2}\pi(2)(1)=2\sqrt{2}\pi \text{ sq. units.}$$

3. $z=\sqrt{a^2-x^2-y^2}$

$$\frac{\partial z}{\partial x}=-\frac{x}{\sqrt{a^2-x^2-y^2}},\quad \frac{\partial z}{\partial y}=-\frac{y}{\sqrt{a^2-x^2-y^2}}$$
$$dS=\sqrt{1+\frac{x^2+y^2}{a^2-x^2-y^2}}\,dA=\frac{a}{\sqrt{a^2-x^2-y^2}}\,dA$$
$$\begin{aligned}S&=\iint_{x^2+y^2\le a^2}\frac{a\,dA}{\sqrt{a^2-x^2-y^2}}\quad\text{(use polars)}\\ &=a\int_0^{2\pi}d\theta\int_0^a\frac{r\,dr}{\sqrt{a^2-r^2}}\quad \begin{array}{l}\text{Let } u=a^2-r^2\\ du=-2r\,dr\end{array}\\ &=\pi a\int_0^{a^2}u^{-1/2}\,du=2\pi a^2 \text{ sq. units.}\end{aligned}$$

4. $z=2\sqrt{1-x^2-y^2}$

$$\frac{\partial z}{\partial x}=-\frac{2x}{\sqrt{1-x^2-y^2}},\quad \frac{\partial z}{\partial y}=-\frac{2y}{\sqrt{1-x^2-y^2}}$$
$$dS=\sqrt{1+\frac{4(x^2+y^2)}{1-x^2-y^2}}\,dA=\sqrt{\frac{1+3(x^2+y^2)}{1-x^2-y^2}}\,dA$$
$$\begin{aligned}S&=\iint_{x^2+y^2\le 1}dS\\ &=\int_0^{2\pi}d\theta\int_0^1\sqrt{\frac{1+3r^2}{1-r^2}}\,r\,dr\quad \begin{array}{l}\text{Let } u^2=1-r^2\\ u\,du=-r\,dr\end{array}\\ &=2\pi\int_0^1\sqrt{4-3u^2}\,du\quad \begin{array}{l}\text{Let } \sqrt{3}u=2\sin v\\ \sqrt{3}\,du=2\cos v\,dv\end{array}\\ &=2\pi\int_0^{\pi/3}(2\cos^2 v)\,\frac{2\,dv}{\sqrt{3}}\\ &=\frac{4\pi}{\sqrt{3}}\int_0^{\pi/3}(1+\cos 2v)\,dv\\ &=\frac{4\pi}{\sqrt{3}}\left(v+\frac{\sin 2v}{2}\right)\Bigg|_0^{\pi/3}=\frac{4\pi^2}{3\sqrt{3}}+\pi \text{ sq. units.}\end{aligned}$$

5. $3z^2=x^2+y^2,\quad 6z\dfrac{\partial z}{\partial x}=2x,\quad \dfrac{\partial z}{\partial x}=\dfrac{x}{3z},\quad \dfrac{\partial z}{\partial y}=\dfrac{y}{3z}$

$$dS=\sqrt{1+\frac{x^2+y^2}{9z^2}}\,dA=\sqrt{\frac{9z^2+3z^2}{9z^2}}\,dA=\frac{2}{\sqrt{3}}\,dA$$
$$S=\iint_{x^2+y^2\le 12}\frac{2}{\sqrt{3}}\,dA=\frac{2}{\sqrt{3}}\pi(12)=\frac{24\pi}{\sqrt{3}} \text{ sq. units.}$$

6. $z = 1 - x^2 - y^2, \quad \frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y$

$$dS = \sqrt{1 + 4x^2 + 4y^2}\, dA$$

$$\begin{aligned} S &= \iint_{x^2+y^2 \le 1,\ x \ge 0,\ y \ge 0} \sqrt{1 + 4(x^2 + y^2)}\, dA \\ &= \int_0^{\pi/2} d\theta \int_0^1 \sqrt{1 + 4r^2}\, r\, dr \quad \text{Let } u = 1 + 4r^2 \\ &\qquad\qquad\qquad\qquad\qquad du = 8r\, dr \\ &= \frac{\pi}{16} \int_1^5 u^{1/2}\, du \\ &= \frac{\pi}{16} \left(\frac{2}{3} u^{3/2} \right) \bigg|_1^5 = \frac{\pi(5\sqrt{5} - 1)}{24} \text{ sq. units.} \end{aligned}$$

7. The triangle is defined by $0 \le y \le 1$, $0 \le x \le y$.

$$z = y^2, \quad \frac{\partial z}{\partial y} = 2y, \quad dS = \sqrt{1 + 4y^2}\, dA$$

$$\begin{aligned} S &= \int_0^1 dy \int_0^y \sqrt{1 + 4y^2}\, dx \\ &= \int_0^1 y\sqrt{1 + 4y^2}\, dy \quad \text{Let } u = 1 + 4y^2 \\ &\qquad\qquad\qquad\qquad du = 8y\, dy \\ &= \frac{1}{8} \int_1^5 u^{1/2}\, du = \frac{1}{8} \left(\frac{2}{3} u^{3/2} \right) \bigg|_1^5 = \frac{5\sqrt{5} - 1}{12} \text{ sq. units.} \end{aligned}$$

8. $z = \sqrt{x}, \quad \frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x}}, \quad dS = \sqrt{1 + \frac{1}{4x}}\, dA$

$$\begin{aligned} S &= \int_0^1 dx \int_0^{\sqrt{x}} \sqrt{1 + \frac{1}{4x}}\, dy = \int_0^1 \sqrt{\frac{4x+1}{4x}} \sqrt{x}\, dx \\ &= \frac{1}{2} \int_0^1 \sqrt{4x + 1}\, dx \quad \text{Let } u = 4x + 1 \\ &\qquad\qquad\qquad\qquad du = 4\, dx \\ &= \frac{1}{8} \int_1^5 u^{1/2}\, du = \frac{1}{8} \left(\frac{2}{3} u^{3/2} \right) \bigg|_1^5 = \frac{5\sqrt{5} - 1}{12} \text{ sq. units.} \end{aligned}$$

9. $z^2 = 4 - x^2, \quad 2z \frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial x} = -\frac{x}{z}$

$$dS = \sqrt{1 + \frac{x^2}{z^2}}\, dA = \frac{2}{z}\, dA = \frac{2}{\sqrt{4 - x^2}}\, dA$$

(since $z \ge 0$ on the part of the surface whose area we want to find)

$$\begin{aligned} S &= \int_0^2 dx \int_0^x \frac{2}{\sqrt{4 - x^2}}\, dy \\ &= \int_0^2 \frac{2x}{\sqrt{4 - x^2}}\, dx \quad \text{Let } u = 4 - x^2 \\ &\qquad\qquad\qquad\qquad du = -2x\, dx \\ &= \int_0^4 u^{-1/2}\, du = 2\sqrt{u} \bigg|_0^4 = 4 \text{ sq. units.} \end{aligned}$$

10. The area elements on $z = 2xy$ and $z = x^2 + y^2$, respectively, are

$$\begin{aligned} dS_1 &= \sqrt{1 + (2y)^2 + (2x)^2}\, dA = \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy, \\ dS_2 &= \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy. \end{aligned}$$

Since these elements are equal, the area of the parts of both surfaces defined over any region of the xy-plane will be equal.

11. If $z = \frac{1}{2}(x^2 + y^2)$, then $dS = \sqrt{1 + x^2 + y^2}\, dA$. One-eighth of the part of the surface above $-1 \le x \le 1$, $-1 \le y \le 1$, lies above the triangle T: given by $0 \le x \le 1$, $0 \le y \le x$, or, in polar coordinates, by $0 \le \theta \le \pi/4$, $0 \le r \le 1/\cos\theta = \sec\theta$. Thus

$$\begin{aligned} S &= 8 \iint_T \sqrt{1 + x^2 + y^2}\, dA \\ &= 8 \int_0^{\pi/4} d\theta \int_0^{\sec\theta} \sqrt{1 + r^2}\, r\, dr \quad \text{Let } u = 1 + r^2 \\ &\qquad\qquad\qquad\qquad\qquad du = 2r\, dr \\ &= 4 \int_0^{\pi/4} d\theta \int_0^{1+\sec^2\theta} \sqrt{u}\, du \\ &= \frac{8}{3} \int_0^{\pi/4} \left[(1 + \sec^2\theta)^{3/2} - 1 \right] d\theta \\ &= \frac{8}{3} \int_0^{\pi/4} (1 + \sec^2\theta)^{3/2}\, d\theta - \frac{2\pi}{3}. \end{aligned}$$

Using a TI-85 numerical integration routine, we obtain the numerical value $S \approx 5.123$ sq. units.

12. As the figure suggests, the area of the canopy is the area of a hemisphere of radius $\sqrt{2}$ minus four times the area of half of a spherical cap cut off from the sphere $x^2 + y^2 + z^2 = 2$ by a plane at distance 1 from the origin, say the plane $z = 1$. Such a spherical cap, $z = \sqrt{2 - x^2 - y^2}$, lies above the disk $x^2 + y^2 \le 2 - 1 = 1$. Since $\frac{\partial z}{\partial x} = -x/z$ and $\frac{\partial z}{\partial y} = -y/z$ on it, the area of the spherical cap is

$$\begin{aligned} &\iint_{x^2+y^2 \le 1} \sqrt{1 + \frac{x^2 + y^2}{z^2}}\, dA \\ &= 2\sqrt{2}\pi \int_0^1 \frac{r\, dr}{\sqrt{2 - r^2}} \quad \text{Let } u = 2 - r^2 \\ &\qquad\qquad\qquad\qquad du = -2r\, dr \\ &= \sqrt{2}\pi \int_1^2 u^{-1/2}\, du = 2\sqrt{2}(\sqrt{2} - 1) = 4 - 2\sqrt{2}. \end{aligned}$$

Thus the area of the canopy is

$$S = 2\pi(\sqrt{2})^2 - 4 \times \frac{1}{2} \times (4 - 2\sqrt{2}) = 4(\pi + \sqrt{2}) - 8 \text{ sq. units.}$$

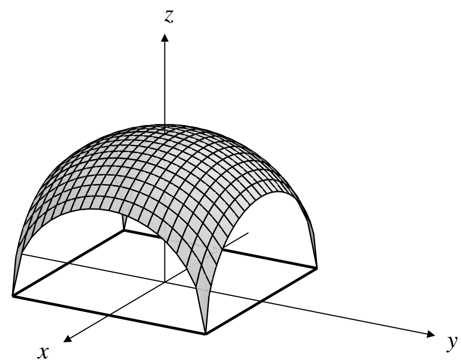


Fig. 14.7.12

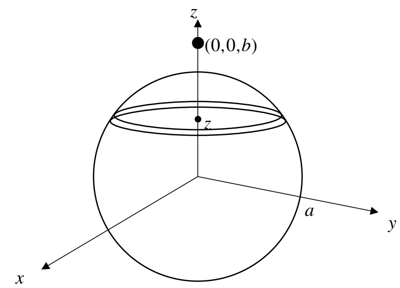


Fig. 14.7.14

13.
$$\begin{aligned}\text{Mass} &= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi \, d\phi \int_0^a \frac{A\rho^2 \, d\rho}{B+\rho^2} \\ &= 4\pi A \int_0^a \left(1 - \frac{B}{\rho^2 + B}\right) d\rho \\ &= 4\pi A \left(a - \sqrt{B} \tan^{-1} \frac{a}{\sqrt{B}}\right) \text{ units.}\end{aligned}$$

14. A slice of the ball at height z, having thickness dz, is a circular disk of radius $\sqrt{a^2 - z^2}$ and areal density $\delta\, dz$. As calculated in the text, this disk attracts mass m at $(0, 0, b)$ with vertical force

$$dF = 2\pi km\delta dz \left(1 - \frac{b-z}{\sqrt{a^2 - z^2 + (b-z)^2}}\right).$$

Thus the ball attracts m with vertical force

$$\begin{aligned}F &= 2\pi km\delta \int_{-a}^{a} \left(1 - \frac{b-z}{\sqrt{a^2+b^2-2bz}}\right) dz \\ &\qquad \text{let } v = a^2 + b^2 - 2bz, \quad dv = -2b\, dz \\ &\qquad \text{then } b - z = b - \frac{a^2+b^2-v}{2b} = \frac{b^2-a^2+v}{2b} \\ &= 2\pi km\delta \left[2a - \frac{1}{4b^2} \int_{(b-a)^2}^{(b+a)^2} \frac{b^2-a^2+v}{\sqrt{v}}\, dv\right] \\ &= 2\pi km\delta \Big[2a - \frac{b^2-a^2}{2b^2}\Big(b + a - (b-a)\Big) \\ &\qquad - \frac{1}{6b^2}\Big((b+a)^3 - (b-a)^3\Big)\Big] \\ &= \frac{4\pi km\delta a^3}{3b^2} = \frac{kmM}{b^2},\end{aligned}$$

where $M = (4/3)\pi a^3 \delta$ is the mass of the ball. Thus the ball attracts the external mass m as though the ball were a point mass M located at its centre.

15. The force is

$$\begin{aligned}F &= 2\pi km\delta \int_0^h \left(1 - \frac{b-z}{\sqrt{a^2 + (b-z)^2}}\right) dz \\ &\qquad \text{Let } u = a^2 + (b-z)^2 \\ &\qquad du = -2(b-z)\, dz \\ &= 2\pi km\delta \left(h - \frac{1}{2} \int_{a^2+(b-h)^2}^{a^2+b^2} \frac{du}{\sqrt{u}}\right) \\ &= 2\pi km\delta \Big(h - \sqrt{a^2+b^2} + \sqrt{a^2 + (b-h)^2}\Big).\end{aligned}$$

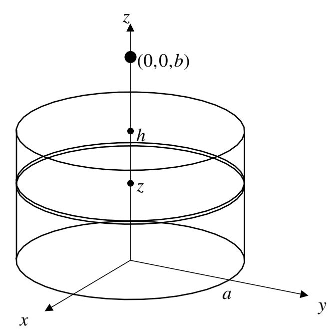


Fig. 14.7.15

16. The force is

$$\begin{aligned}F &= 2\pi km\delta \int_0^b \left(1 - \frac{b-z}{\sqrt{a^2(b-z)^2 + (b-z)^2}}\right) dz \\ &= 2\pi km\delta \int_0^b \left(1 - \frac{1}{\sqrt{a^2+1}}\right) dz \\ &= 2\pi km\delta b \left(1 - \frac{1}{\sqrt{a^2+1}}\right).\end{aligned}$$

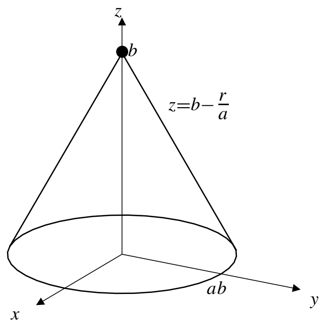


Fig. 14.7.16

17. The force is

$$F = 2\pi km\delta \int_0^a \left(1 - \frac{b-z}{\sqrt{a^2+b^2-2bz}}\right) dz$$

use the same substitution as in Exercise 2)

$$\begin{aligned}
&= 2\pi km\delta \left(a - \frac{1}{4b^2}\int_{(b-a)^2}^{a^2+b^2} \frac{b^2-a^2+v}{\sqrt{v}}\, dv\right) \\
&= 2\pi km\delta \bigg(a - \frac{b^2-a^2}{2b^2}\left(\sqrt{a^2+b^2}-(b-a)\right) \\
&\qquad - \frac{1}{6b^2}\left((a^2+b^2)^{3/2}-(b-a)^3\right)\bigg) \\
&= \frac{2\pi km\delta}{3b^2}\left(2b^3+a^3-(2b^2-a^2)\sqrt{a^2+b^2}\right).
\end{aligned}$$

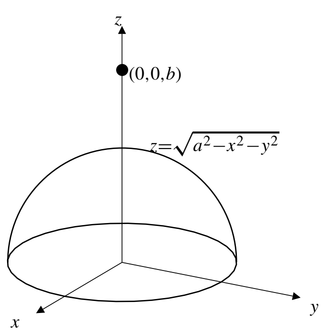


Fig. 14.7.17

18.

$$\begin{aligned}
m &= \int_0^a dx \int_0^a dy \int_0^a (x^2+y^2+z^2)\, dz \\
&= 3\int_0^a x^2\, dx \int_0^a dy \int_0^a dz = a^5 \\
M_{x=0} &= \int_0^a x\, dx \int_0^a dy \int_0^a (x^2+y^2+z^2)\, dz \\
&= \int_0^a x\, dx \int_0^a \left(a(x^2+y^2)+\frac{a^3}{3}\right) dy \\
&= \int_0^a \left(\frac{2a^4}{3}+a^2x^2\right) x\, dx = \frac{7a^6}{12}.
\end{aligned}$$

Thus $\bar{x} = M_{x=0}/m = \dfrac{7a}{12}$.

By symmetry, the centre of mass is $\left(\dfrac{7a}{12}, \dfrac{7a}{12}, \dfrac{7a}{12}\right)$.

19. Since the base triangle has centroid $\left(\dfrac{1}{3}, \dfrac{1}{3}, 0\right)$, the centroid of the prism is $\left(\dfrac{1}{3}, \dfrac{1}{3}, \dfrac{1}{2}\right)$.

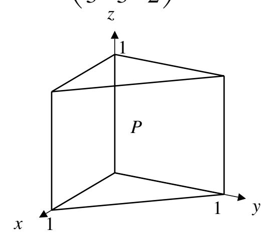


Fig. 14.7.19

20. Volume of region $= \displaystyle\int_0^{2\pi} d\theta \int_0^\infty e^{-r^2} r\, dr = \pi$. By symmetry, the moments about $x = 0$ and $y = 0$ are both zero. We have

$$\begin{aligned}
M_{z=0} &= \int_0^{2\pi} d\theta \int_0^\infty r\, dr \int_0^{e^{-r^2}} z\, dz \\
&= \pi \int_0^\infty re^{-2r^2}\, dr = \frac{\pi}{4}.
\end{aligned}$$

The centroid is $(0, 0, 1/4)$.

21. The volume is $\dfrac{1}{8}\left(\dfrac{4}{3}\pi a^3\right) = \dfrac{\pi a^3}{6}$. By symmetry, the moments about all three coordinate planes are equal. We have

$$\begin{aligned}
M_{z=0} &= \int_0^{\pi/2} d\theta \int_0^{\pi/2} \sin\phi\, d\phi \int_0^a \rho\cos\phi\, \rho^2\, d\rho \\
&= \frac{\pi a^4}{8}\int_0^{\pi/2} \sin\phi\cos\phi\, d\phi = \frac{\pi a^4}{16}.
\end{aligned}$$

Thus $\bar{z} = M_{z=0}/\text{volume} = 3a/8$.

The centroid is $\left(\dfrac{3a}{8}, \dfrac{3a}{8}, \dfrac{3a}{8}\right)$.

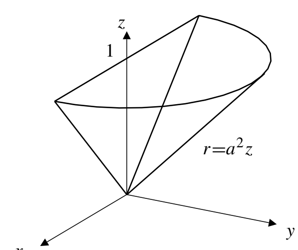


Fig. 14.7.21

22. The cube has centroid $(1/2, 1/2, 1/2)$. The tetrahedron lying above the plane $x + y + x = 2$ has centroid $(3/4, 3/4, 3/4)$ and volume $1/6$. Therefore the part of the cube lying below the plane has centroid (c, c, c) and volume $5/6$, where

$$\frac{5}{6}c + \frac{3}{4} \times \frac{1}{6} = \frac{1}{2} \times 1.$$

Thus $c = 9/20$; the centroid is $\left(\frac{9}{20}, \frac{9}{20}, \frac{9}{20}\right)$.

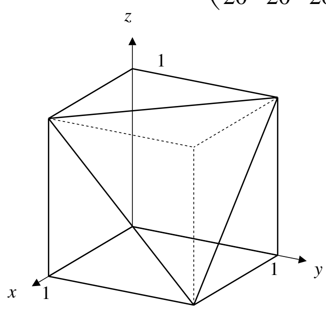


Fig. 14.7.22

23. The model still involves angular acceleration to spin the ball — it doesn't just fall. Part of the gravitational potential energy goes to producing this spin as the ball falls, even in the limiting case where the fall is vertical.

24.
$$I = \delta \int_0^{2\pi} d\theta \int_0^a r^3\,dr \int_0^h dz$$
$$= 2\pi\delta h\left(\frac{a^4}{4}\right) = \frac{\pi\delta h a^4}{2}.$$
$$m = \pi\delta a^2 h, \qquad \bar{D} = \sqrt{I/m} = \frac{a}{\sqrt{2}}.$$

25.
$$I = \delta \int_0^{2\pi} d\theta \int_0^a r\,dr \int_0^h (x^2+z^2)\,dz$$
$$= \delta \int_0^{2\pi} d\theta \int_0^a \left(hr^2\cos^2\theta + \frac{h^3}{3}\right) r\,dr$$
$$= \delta \int_0^{2\pi} \left(\frac{ha^4}{4}\cos^2\theta + \frac{h^3a^2}{6}\right) d\theta$$
$$= \delta\left(\frac{\pi h a^4}{4} + \frac{\pi h^3 a^2}{3}\right) = \pi\delta a^2 h\left(\frac{a^2}{4} + \frac{h^2}{3}\right)$$
$$m = \pi\delta a^2 h, \qquad \bar{D} = \sqrt{I/m} = \sqrt{\frac{a^2}{4} + \frac{h^2}{3}}.$$

26.
$$I = \delta \int_0^{2\pi} d\theta \int_0^a r^3\,dr \int_0^{h(1-(r/a))} dz$$
$$= 2\pi\delta h \int_0^a r^3\left(1 - \frac{r}{a}\right) dr = \frac{\pi\delta a^4 h}{10},$$
$$m = \frac{\pi\delta a^2 h}{3}, \qquad \bar{D} = \sqrt{I/m} = \sqrt{\frac{3}{10}}a.$$

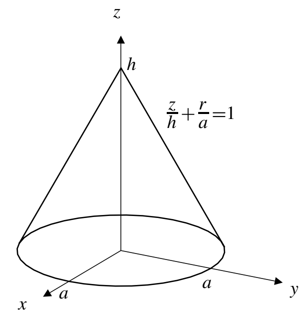


Fig. 14.7.26

27.
$$I = \delta \int_0^{2\pi} d\theta \int_0^a r\,dr \int_0^{h(1-(r/a))} (x^2+z^2)\,dz$$
$$= \delta \int_0^{2\pi} d\theta \int_0^a \left[h\left(1 - \frac{r}{a}\right) r^2\cos^2\theta + \frac{h^3}{3}\left(1 - \frac{r}{a}\right)^3\right] r\,dr$$
$$= \pi\delta h \int_0^a \left(r^3 - \frac{r^4}{a}\right) dr + \frac{2\pi\delta h^3}{3} \int_0^a r\left(1 - \frac{r}{h}\right)^3 dr$$
in the second integral put $u = 1 - (r/a)$
$$= \frac{\pi\delta a^4 h}{20} + \frac{2\pi\delta a^2 h^3}{3} \int_0^1 (1-u)u^3\,du$$
$$= \frac{\pi\delta a^4 h}{20} + \frac{2\pi\delta a^2 h^3}{60} = \frac{\pi\delta a^2 h}{60}(3a^2 + 2h^2),$$
$$m = \frac{\pi\delta a^2 h}{3}, \qquad \bar{D} = \sqrt{I/m} = \sqrt{\frac{3a^2+2h^2}{20}}.$$

28.
$$I = \delta \iiint_Q (x^2+y^2)\,dV$$
$$= 2\delta \int_0^a x^2\,dx \int_0^a dy \int_0^a dz = \frac{2\delta a^5}{3},$$
$$m = \delta a^3, \qquad \bar{D} = \sqrt{I/m} = \sqrt{\frac{2}{3}}a.$$

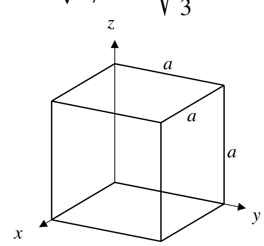


Fig. 14.7.28

29. The distance s from (x, y, z) to the line $x = y$, $z = 0$ satisfies $s^2 = u^2 + z^2$, where u is the distance from $(x, y, 0)$ to the line $x = y$ in the xy-plane. By Example 7 of Section 1.4 $u = |x - y|/\sqrt{2}$, so

$$s^2 = \frac{(x-y)^2}{2} + z^2.$$

The moment of inertia of the cube about this line is

$$\begin{aligned} I &= \delta\int_0^a dx\int_0^a dy\int_0^a \left(\frac{(x-y)^2}{2}+z^2\right)dz \\ &= \delta\int_0^a dx\int_0^a \left(\frac{a}{2}(x-y)^2+\frac{a^3}{3}\right)dy \quad \begin{array}{l}\text{Let } u=x-y\\ du=-dy\end{array} \\ &= \frac{\delta a^5}{3}+\frac{\delta a}{2}\int_0^a dx\int_{x-a}^x u^2\,du \\ &= \frac{\delta a^5}{3}+\frac{\delta a}{6}\int_0^a (3ax^2-3a^2x+a^3)\,dx \\ &= \frac{\delta a^5}{3}+\frac{\delta a}{6}\left(a^4-\frac{3a^4}{2}+a^4\right)=\frac{5\delta a^5}{12}, \end{aligned}$$

$$m=\delta a^3, \qquad \bar{D}=\sqrt{I/m}=\sqrt{\frac{5}{12}}\,a.$$

30. The line L through the origin parallel to the vector $\mathbf{v}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ is a diagonal of the cube Q. By Example 8 of Section 1.4, the distance from the point with position vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ to L is $s=|\mathbf{v}\times\mathbf{r}|/|\mathbf{v}|$. Thus, the square of the distance from (x,y,z) to L is

$$\begin{aligned} s^2 &= \frac{(x-y)^2+(y-z)^2+(z-x)^2}{3} \\ &= \frac{2}{3}\Big(x^2+y^2+z^2-xy-xz-yz\Big). \end{aligned}$$

We have

$$\iiint_Q x^2\,dV=\iiint_Q y^2\,dV=\iiint_Q z^2\,dV=\frac{a^5}{3}$$
$$\iiint_Q xy\,dV=\iiint_Q yz\,dV=\iiint_Q xz\,dV=\frac{a^5}{4}.$$

Therefore, the moment of inertia of Q about L is

$$I=\frac{2\delta}{3}\left(3\times\frac{a^5}{3}-3\times\frac{a^5}{4}\right)=\frac{\delta a^5}{6}.$$

The mass of Q is $m=\delta a^3$, so the radius of gyration is

$$\bar{D}=\sqrt{I/m}=\frac{a}{\sqrt{6}}.$$

31.
$$\begin{aligned} I &= \delta\int_{-a}^a dx\int_{-b}^b dy\int_{-c}^c (x^2+y^2)\,dz \\ &= 2\delta c\int_{-a}^a\left(2bx^2+\frac{2b^3}{3}\right)dx \\ &= \frac{8\delta abc}{3}(a^2+b^2), \end{aligned}$$

$$m=8\delta abc, \qquad \bar{D}=\sqrt{I/m}=\sqrt{\frac{a^2+b^2}{3}}.$$

32. $$I=\delta\int_0^{2\pi}d\theta\int_0^c dz\int_a^b r^3\,dr=\frac{\pi\delta c(b^4-a^4)}{2},$$

$$m=\pi\delta c(b^2-a^2), \qquad \bar{D}=\sqrt{\frac{b^2+a^2}{2}}.$$

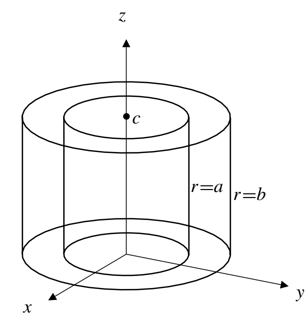


Fig. 14.7.32

33.
$$\begin{aligned} m &= 2\delta\int_0^{2\pi}d\theta\int_b^a r\,dr\int_0^{\sqrt{a^2-r^2}}dz \\ &= 4\pi\delta\int_b^a r\sqrt{a^2-r^2}\,dr \quad \begin{array}{l}\text{Let } u=a^2-r^2\\ du=-2r\,dr\end{array} \\ &= 2\pi\delta\int_0^{a^2-b^2}\sqrt{u}\,du=\frac{4\pi\delta}{3}(a^2-b^2)^{3/2}, \end{aligned}$$

$$\begin{aligned} I &= 2\delta\int_0^{2\pi}\int_b^a r^3\,dr\int_0^{\sqrt{a^2-r^2}}dz \\ &= 4\pi\delta\int_b^a r^3\sqrt{a^2-r^2}\,dr \quad \begin{array}{l}\text{Let } u=a^2-r^2\\ du=-2r\,dr\end{array} \\ &= 2\pi\delta\int_0^{a^2-b^2}(a^2-u)\sqrt{u}\,du \\ &= 2\pi\delta\left(\frac{2}{3}a^2(a^2-b^2)^{3/2}-\frac{2}{5}(a^2-b^2)^{5/2}\right) \\ &= 4\pi\delta(a^2-b^2)^{3/2}\frac{1}{15}(2a^2+3b^2)=\frac{1}{5}m(2a^2+3b^2). \end{aligned}$$

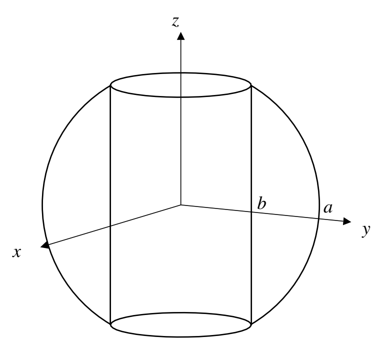


Fig. 14.7.33

34. By Exercise 26, the cylinder has moment of inertia

$$I = \frac{\pi\delta a^4 h}{2} = \frac{ma^2}{2},$$

where m is its mass. Following the method of Example 4(b), the kinetic energy of the cylinder rolling down the inclined plane with speed v is

$$\begin{aligned} KE &= \frac{1}{2}mv^2 + \frac{1}{2}I\Omega^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{4}ma^2\frac{v^2}{a^2} = \frac{3}{4}mv^2. \end{aligned}$$

The potential energy of the cylinder when it is at height h is mgh, so, by conservation of energy,

$$\frac{3}{4}mv^2 + mgh = \text{constant}.$$

Differentiating this equation with respect to time t, we obtain

$$\begin{aligned} 0 &= \frac{3}{2}mv\frac{dv}{dt} + mg\frac{dh}{dt} \\ &= \frac{3}{2}mv\frac{dv}{dt} + mgv\sin\alpha. \end{aligned}$$

Thus the cylinder rolls down the plane with acceleration

$$-\frac{dv}{dt} = \frac{2}{3}g\sin\alpha.$$

35. By Exercise 35, the ball with hole has moment of inertia

$$I = \frac{m}{5}(2a^2 + 3b^2)$$

about the axis of the hole. The kinetic energy of the rolling ball is

$$\begin{aligned} KE &= \frac{1}{2}mv^2 + \frac{m}{10}(2a^2+3b^2)\frac{v^2}{a^2} \\ &= mv^2\left(\frac{1}{2} + \frac{2a^2+3b^2}{10a^2}\right) = mv^2\,\frac{7a^2+3b^2}{10a^2}. \end{aligned}$$

By conservation of energy,

$$mv^2\,\frac{7a^2+3b^2}{10a^2} + mgh = \text{constant}.$$

Differentiating with respect to time, we obtain

$$\frac{7a^2+3b^2}{5a^2}\,mv\,\frac{dv}{dt} + mgv\sin\alpha = 0.$$

Thus the ball rolls down the plane (with its hole remaining horizontal) with acceleration

$$-\frac{dv}{dt} = \frac{5a^2}{7a^2+3b^2}\,g\sin\alpha.$$

36. The kinetic energy of the oscillating pendulum is

$$KE = \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2.$$

The potential energy is mgh, where h is the distance of C above A. In this case, $h = -a\cos\theta$. By conservation of energy,

$$\frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 - mga\cos\theta = \text{constant}.$$

Differentiating with respect to time t, we obtain

$$I\left(\frac{d\theta}{dt}\right)\frac{d^2\theta}{dt^2} + mga\sin\theta\left(\frac{d\theta}{dt}\right) = 0,$$

or

$$\frac{d^2\theta}{dt^2} + \frac{mga}{I}\sin\theta = 0.$$

For small oscillations we have $\sin\theta \approx \theta$, and the above equation is approximated by

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0,$$

where $\omega^2 = mga/I$. The period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mga}}.$$

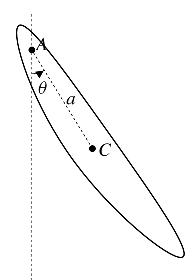


Fig. 14.7.36

37. If the centre of mass of B is at the origin, then

$$M_{x=0} = \iiint_B x\delta\,dV = 0.$$

If line L_0 is the z-axis, and L_k is the line $x = k$, $y = 0$, then the moment of inertia I_k of B about L_k is

$$\begin{aligned} I_k &= \iiint_B \left((x-k)^2 + y^2\right)\delta\, dV \\ &= \iiint_B (x^2 + y^2 + k^2 - 2kx)\,\delta\, dV \\ &= I_0 + k^2 m - 2kM_{x=0} = I_0 + k^2 m, \end{aligned}$$

where m is the mass of B and I_0 is the moment about L_0.

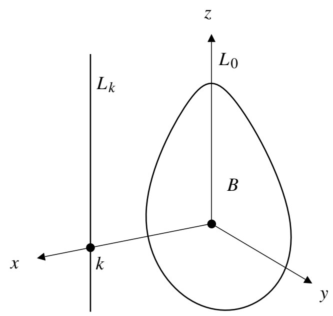


Fig. 14.7.37

38. The moment of inertia of the ball about the point where it contacts the plane is, by Example 4(b) and Exercise 39,

$$\begin{aligned} I &= \frac{8}{15}\pi\delta a^5 + \left(\frac{4}{3}\pi\delta a^3\right) a^2 \\ &= \left(\frac{2}{5} + 1\right) ma^2 = \frac{7}{5}ma^2. \end{aligned}$$

The kinetic energy of the ball, regarded as rotating about the point of contact with the plane, is therefore

$$KE = \frac{1}{2}I\Omega^2 = \frac{7}{10}ma^2\frac{v^2}{a^2} = \frac{7}{10}mv^2.$$

39. By Example 7 of Section 1.4, the distance from the point with position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to the straight line L through the origin parallel to the vector $\mathbf{a} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ is

$$s = \frac{|\mathbf{a}\times\mathbf{r}|}{|\mathbf{a}|}.$$

The moment of inertia of the body occupying region R about L is, therefore,

$$\begin{aligned} I &= \frac{1}{|\mathbf{a}|^2}\iiint_R |\mathbf{a}\times\mathbf{r}|^2\delta\, dV \\ &= \frac{1}{A^2+B^2+C^2}\iiint_R \Big[(Bz - Cy)^2 + (Cx - Az)^2 \\ &\quad + (Ay - Bx)^2\Big]\delta\, dV \\ &= \frac{1}{A^2+B^2+C^2}\Big[(B^2+C^2)P_{xx} + (A^2+C^2)P_{yy} \\ &\qquad + (A^2+B^2)P_{zz} - 2ABP_{xy} - 2ACP_{xz} - 2BCP_{yz}\Big]. \end{aligned}$$

Review Exercises 14 (page 804)

1. By symmetry,

$$\begin{aligned} \iint_R (x+y)\, dA &= 2\iint_R x\, dA = 2\int_0^1 x\, dx \int_{x^2}^{\sqrt{x}} dy \\ &= 2\int_0^1 (x^{3/2} - x^3)\, dx \\ &= 2\left(\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right)\Bigg|_0^1 = 2\left(\frac{2}{5} - \frac{1}{4}\right) = \frac{3}{10} \end{aligned}$$

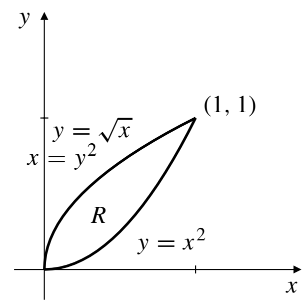


Fig. R-14.1

2.

$$\begin{aligned} \iint_P (x^2+y^2)\, dA &= \int_0^1 dy \int_y^{2+y} (x^2+y^2)\, dx \\ &= \int_0^1 \left(\frac{x^3}{3} + xy^2\right)\Bigg|_{x=y}^{x=2+y} dy \\ &= \int_0^1 \left(\frac{(2+y)^3}{3} + y^2(2+y) - \frac{y^3}{3} - y^3\right) dy \\ &= \int_0^1 \left(\frac{8}{3} + 4y + 4y^2\right) dy = \frac{8}{3} + 2 + \frac{4}{3} = 6 \end{aligned}$$

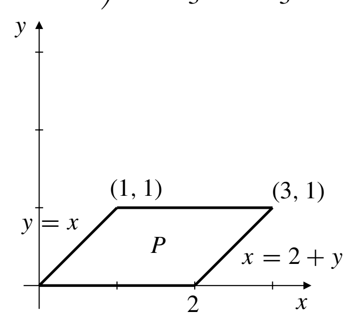


Fig. R-14.2

3.

$$\begin{aligned} \iint_D \frac{y}{x}\, dA &= \int_0^{\pi/4} d\theta \int_0^2 \tan\theta\, r\, dr \\ &= \ln\sec\theta\Bigg|_0^{\pi/4} \frac{r^2}{2}\Bigg|_0^2 = 2\ln\sqrt{2} = \ln 2 \end{aligned}$$

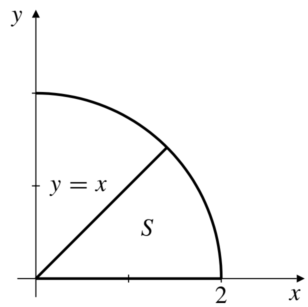


Fig. R-14.3

4. a) $$I = \int_0^{\sqrt{3}} dy \int_{y/\sqrt{3}}^{\sqrt{4-y^2}} e^{-x^2-y^2}\,dx = \iint_R e^{-x^2-y^2}\,dA$$
where R is as shown in the figure.

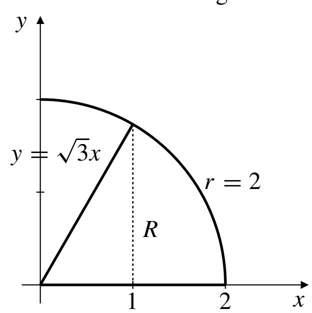


Fig. R-14.4

b) $$I = \int_0^1 dx \int_0^{\sqrt{3}x} e^{-x^2-y^2}\,dy + \int_1^2 dx \int_0^{\sqrt{4-x^2}} e^{-x^2-y^2}\,dy$$

c) $$I = \int_0^{\pi/3} d\theta \int_0^2 e^{-r^2} r\,dr$$

d) $$I = \frac{\pi}{3}\left(-\frac{e^{-r^2}}{2}\right)\Bigg|_0^2 = \frac{\pi(1-e^{-4})}{6}$$

5. The cone $z = k\sqrt{x^2+y^2}$ has semi-vertical angle $\phi_0 = \tan^{-1}(1/k)$. Thus the volume inside the cone and inside the sphere $x^2+y^2+z^2 = a^2$ is

$$V = \int_0^{2\pi} d\theta \int_0^{\phi_0} \sin\phi\,d\phi \int_0^a \rho^2\,d\rho = \frac{2\pi a^3}{3}(1-\cos\phi_0) = \frac{2\pi a^3}{3}\left(1 - \frac{k}{\sqrt{k^2+1}}\right).$$

To have
$$V = \frac{1}{4}\left(\frac{4}{3}\pi a^3\right) = \frac{\pi a^3}{3},$$
we need to ensure that
$$2\left(1 - \frac{k}{\sqrt{k^2+1}}\right) = 1.$$
Thus $k^2+1 = (2k)^2$, and so $3k^2 = 1$, and $k = 1/\sqrt{3}$.

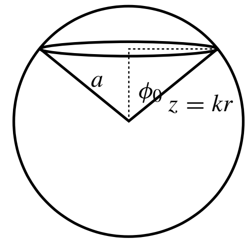


Fig. R-14.5

6. $$I = \int_0^2 dy \int_0^y f(x,y)\,dx + \int_2^6 dy \int_0^{\sqrt{6-y}} f(x,y)\,dx = \iint_R f(x,y)\,dA,$$
where R is as shown in the figure. Thus
$$I = \int_0^2 dx \int_x^{6-x^2} f(x,y)\,dy.$$

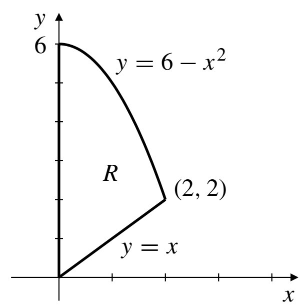


Fig. R-14.6

7. $$J = \int_0^1 dz \int_0^z dy \int_0^y f(x,y,z)\,dx$$
corresponds to the region
$$0 \le z \le 1, \quad 0 \le y \le z, \quad 0 \le x \le y,$$
which can also be expressed in the form
$$0 \le x \le 1, \quad x \le y \le 1, \quad y \le z \le 1.$$
Thus $J = \int_0^1 dx \int_x^1 dy \int_y^1 f(x,y,z)\,dz.$

8. A horizontal slice of the object at height z above the base, and having thickness dz, is a disk of radius $r = \frac{1}{2}(10-z)$ m. Its volume is
$$dV = \pi\frac{(10-z)^2}{4}\,dz \text{ m}^3.$$

The density of the slice is $\delta = kz^2$ kg/m^3. Since $\delta = 3,000$ when $z = 10$, we have $k = 30$.

a) The mass of the object is

$$\begin{aligned} m &= \int_0^{10} 30z^2 \frac{\pi}{4}(10-z)^2\,dz \\ &= \frac{15\pi}{2}\int_0^{10}(100z^2 - 20z^3 + z^4)\,dz \\ &= \frac{15\pi}{2}\left(\frac{100,000}{3} - 50,000 + 20,000\right) \approx 78,540 \text{ kg.} \end{aligned}$$

b) The moment of inertia (about its central axis) of the disk-shaped slice at height z is

$$dI = 30z^2\,dz \int_0^{2\pi} d\theta \int_0^{(10-z)/2} r^3\,dr.$$

Thus the moment of inertia about the whole solid cone is

$$I = \int_0^{10} 30z^2\,dz \int_0^{2\pi} d\theta \int_0^{(10-z)/2} r^3\,dr.$$

9. $$f(t) = \int_t^a e^{-x^2}\,dx$$

$$\begin{aligned} \bar{f} &= \frac{1}{a}\int_0^a f(t)\,dt = \frac{1}{a}\int_0^a dt \int_t^a e^{-x^2}\,dx \\ &= \frac{1}{a}\int_0^a e^{-x^2}\,dx \int_0^x dt = \frac{1}{a}\int_0^a xe^{-x^2}\,dx \\ &= \frac{1}{a}\left(-\frac{e^{-x^2}}{2}\right)\Bigg|_0^a = \frac{1-e^{-a^2}}{2a} \end{aligned}$$

10. If $f(x, y) = \lfloor x + y \rfloor$, then $f = 0$, 1, or 2, in parts of the quarter disk Q, as shown in the figure.

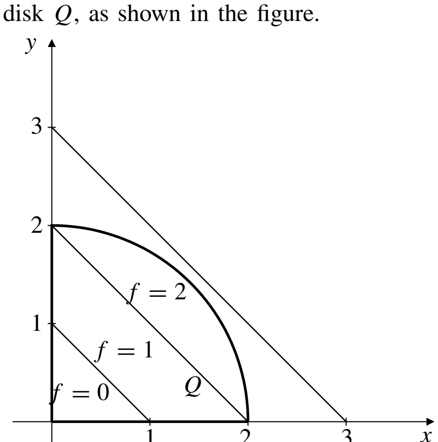


Fig. R-14.10

Thus

$$\iint_Q f(x, y)\,dA = 0\left(\frac{1}{2}\right) + 1\left(\frac{3}{2}\right) + 2(\pi - 2) = 2\pi - \frac{5}{2},$$

and $\bar{f} = \dfrac{1}{\pi}\left(2\pi - \dfrac{5}{2}\right) = 2 - \dfrac{5}{2\pi}$.

11. The sphere $x^2 + y^2 + z^2 = 6a^2$ and the paraboloid $z = (x^2 + y^2)/a$ intersect where $z^2 + az - 6a^2 = 0$, that is, where $(z + 3a)(z - 2a) = 0$. Only $z = 2a$ is possible; the plane $z = -3a$ does not intersect the sphere. If $z = 2a$, then $x^2 + y^2 = r^2 = 6a^2 - 4a^2 = 2a^2$, so the intersection is on the vertical cylinder of radius $\sqrt{2}a$ with axis on the z-axis. We have,

$$\begin{aligned} &\iiint_D (x^2 + y^2)\,dV \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}a} r^3\,dr \int_{r^2/a}^{\sqrt{6a^2-r^2}} dz \\ &= 2\pi \int_0^{\sqrt{2}a}\left[r^3\sqrt{6a^2 - r^2} - \frac{r^5}{a}\right] dr \\ &\qquad \text{Let } u = 6a^2 - r^2 \\ &\qquad du = -2r\,dr \\ &= \pi \int_{4a^2}^{6a^2} (6a^2 - u)\sqrt{u}\,du - \frac{\pi}{3a}(\sqrt{2}a)^6 \\ &= \pi\left(4a^2u^{3/2} - \frac{2}{5}u^{5/2}\right)\Bigg|_{4a^2}^{6a^2} - \frac{8}{3}\pi a^5 \\ &= \frac{8\pi}{15}(18\sqrt{6} - 41)a^5 \end{aligned}$$

12. The solid S lies above the region in the xy-plane bounded by the circle $x^2 + y^2 = 2ay$, which has polar equation $r = 2a\sin\theta$, $(0 \le \theta \le \pi)$. It lies below the cone
$z = \sqrt{x^2 + y^2} = r$. The moment of inertia of S about the z-axis is

$$\begin{aligned} I &= \iiint_S (x^2 + y^2)\,dV = \int_0^\pi d\theta \int_0^{2a\sin\theta} r^3\,dr \int_0^r dz \\ &= \int_0^\pi d\theta \int_0^{2a\sin\theta} r^4\,dr = \frac{32a^5}{5}\int_0^\pi \sin^5\theta\,d\theta \\ &= \frac{32a^5}{5}\int_0^\pi (1 - \cos^2\theta)^2 \sin\theta\,d\theta \quad \text{Let } u = \cos\theta \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad du = -\sin\theta\,d\theta \\ &= \frac{32a^5}{5}\int_{-1}^1 (1 - 2u^2 + u^4)\,du \\ &= \frac{64a^5}{5}\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{512a^5}{75}. \end{aligned}$$

13. A horizontal slice of D at height z is a right triangle with legs $(2 - z)/2$ and $2 - z$. Thus the volume of D is

$$V = \frac{1}{4}\int_0^1 (2 - z)^2\,dz = \frac{7}{12}.$$

Its moment about $z = 0$ is

$$M_{z=0} = \frac{1}{4}\int_0^1 z(2-z)^2\,dz$$
$$= \frac{1}{4}\int_0^1 (4z - 4z^2 + z^3)\,dz = \frac{11}{48}.$$

The z-coordinate of the centroid of D is

$$\bar{z} = \frac{11}{48}\Big/\frac{7}{12} = \frac{11}{28}.$$

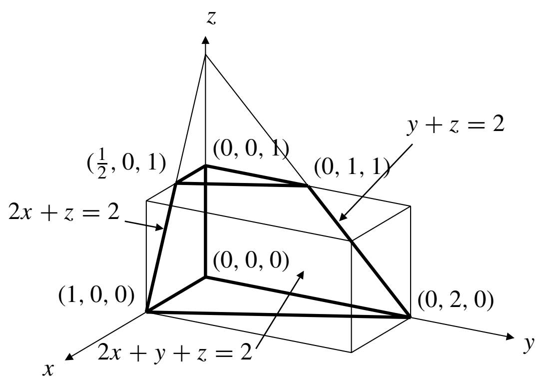


Fig. R-14.13

14. $$V = \iiint_S dV = \int_0^1 dy \int_0^{1-y} dz \int_0^{2-y-2z} dx$$
$$= \int_0^1 dy \int_0^{1-y} (2-y-2z)\,dz$$
$$= \int_0^1 [(2-y)(1-y) - (1-y)^2]\,dy$$
$$= \int_0^1 (1-y)\,dy = \frac{1}{2}$$
$$M_{x=0} = \iiint_S x\,dV = \int_0^1 dy \int_0^{1-y} dz \int_0^{2-y-2z} x\,dx$$
$$= \frac{1}{2}\int_0^1 dy \int_0^{1-y} [(2-y)^2 - 4(2-y)z + 4z^2]\,dz$$
$$= \frac{1}{2}\int_0^1 \left[(2-y)^2(1-y) - 2(2-y)(1-y)^2 + \frac{4}{3}(1-y)^3\right] dy \quad \text{Let } u = 1-y,\ du = -dy$$
$$= \frac{1}{2}\int_0^1 \left[(u+1)^2 u - 2(u+1)u^2 + \frac{4}{3}u^3\right] du$$
$$= \frac{1}{2}\int_0^1 \left[\frac{1}{3}u^3 + u\right] du = \frac{7}{24}$$
$$\bar{x} = \frac{7}{24}\Big/\frac{1}{2} = \frac{7}{12}$$

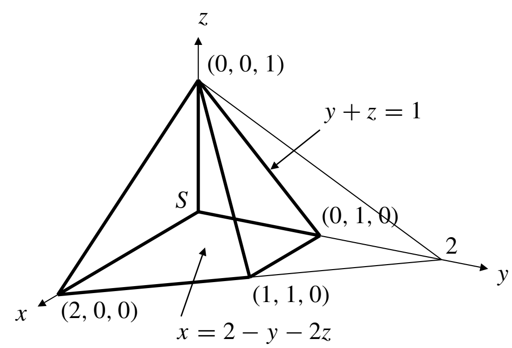


Fig. R-14.14

15. $$\iiint_S z\,dV = \int_0^1 z\,dz \int_0^{1+z} dy \int_0^{1+z-y} dx$$
$$= \int_0^1 z\,dz \int_0^{1+z} (1+z-y)\,dy$$
$$= \int_0^1 z\left[(1+z)^2 - \frac{(1+z)^2}{2}\right] dz$$
$$= \frac{1}{2}\int_0^1 (z + 2z^2 + z^3)\,dz = \frac{17}{24}$$

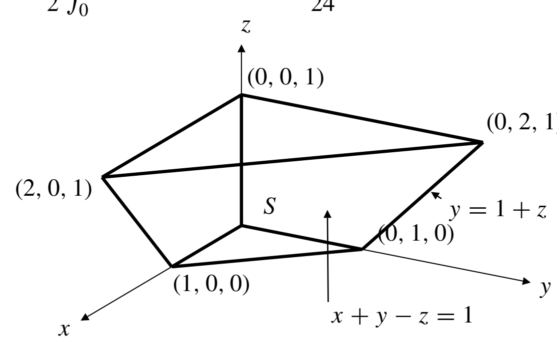


Fig. R-14.15

16. The plane $z = 2x$ intersects the paraboloid $z = x^2 + y^2$ on the circular cylinder $x^2 + y^2 = 2x$, (that is, $(x-1)^2 + y^2 = 1$), which has radius 1. Since $dS = \sqrt{1+2^2}\,dA = \sqrt{5}\,dA$ on the plane, the area of the part of the plane inside the paraboloid (and therefore inside the cylinder) is $\sqrt{5}$ times the area of a circle of radius 1, that is, $\sqrt{5}\pi$ square units.

17. As noted in the previous exercise, the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 2x$ is inside the vertical cylinder $x^2 + y^2 = 2x$, which has polar equation $r = 2\cos\theta$ $(-\pi/2 \le \theta \le \pi/2)$. On the paraboloid:

$$dS = \sqrt{1 + (2x)^2 + (2y)^2}\,dA = \sqrt{1+4r^2}\,r\,dr\,d\theta.$$

The area of that part of the paraboloid is

$$\begin{aligned} S &= \int_{-\pi/2}^{\pi/2} d\theta \int_0^{2\cos\theta} \sqrt{1+4r^2}\, r\, dr \quad \text{Let } u = 1+4r^2,\ du = 8r\,dr \\ &= \frac{1}{8}\int_{-\pi/2}^{\pi/2} d\theta \int_1^{1+16\cos^2\theta} u^{1/2}\, du \\ &= \frac{1}{4}\int_0^{\pi/2} \frac{2}{3}[(1+16\cos^2\theta)^{3/2} - 1]\, d\theta \\ &= \frac{1}{6}\int_0^{\pi/2} [(1+16\cos^2\theta)^{3/2} - 1]\, d\theta \\ &\approx 7.904 \text{ sq. units.} \end{aligned}$$

(using a TI-85 numerical integration function).

18. The region R inside the ellipsoid $\frac{x^2}{36}+\frac{y^2}{9}+\frac{z^2}{4}=1$ and above the plane $x+y+z=1$ is transformed by the change of variables

$$x = 6u, \quad y = 3v, \quad z = 2w$$

to the region S inside the sphere $u^2+v^2+w^2=1$ and above the plane $6u+3v+2w=1$. The distance from the origin to this plane is

$$D = \frac{1}{\sqrt{6^2+3^2+2^2}} = \frac{1}{7},$$

so, by symmetry, the volume of S is equal to the volume inside the sphere and above the plane $w = 1/7$, that is,

$$\int_{1/7}^{1} \pi(1-w^2)\, dw = \pi\left(w - \frac{w^3}{3}\right)\Big|_{1/7}^{1} = \frac{180\pi}{343} \text{ units}^3.$$

Since $|\partial(x,y,z)/\partial(u,v,w)| = 6\cdot3\cdot2 = 18$, the volume of R is $18 \times (180\pi/343) = 3240\pi/343 \approx 29.68$ cu. units.

Challenging Problems 14 (page 805)

1. This problem is similar to Review Exercise 18 above. The region R inside the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and above the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is transformed by the change of variables

$$x = au, \quad y = bv, \quad z = cw$$

to the region S inside the sphere $u^2+v^2+w^2=1$ and above the plane $u+v+w=1$. The distance from the origin to this plane is $\frac{1}{\sqrt{3}}$, so, by symmetry, the volume of S is equal to the volume inside the sphere and above the plane $w = 1/\sqrt{3}$, that is,

$$\begin{aligned} \int_{1/\sqrt{3}}^{1} \pi(1-w^2)\, dw &= \pi\left(w - \frac{w^3}{3}\right)\Big|_{1\sqrt{3}}^{1} \\ &= \frac{2\pi(9-4\sqrt{3})}{27} \text{ cu. units.} \end{aligned}$$

Since $|\partial(x,y,z)/\partial(u,v,w)| = abc$, the volume of R is $\frac{2\pi(9-4\sqrt{3})}{27}abc$ cu. units.

2. The plane $(x/a)+(y/b)+(z/c)=1$ intersects the ellipsoid
$(x/a)^2+(y/b)^2+(z/c)^2=1$ above the region R in the xy-plane bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\left(1-\frac{x}{a}-\frac{y}{b}\right)^2 = 1,$$

or, equivalently,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{xy}{ab}-\frac{x}{a}-\frac{y}{b} = 0.$$

Thus the area of the part of the plane lying inside the ellipsoid is

$$\begin{aligned} S &= \iint_R \sqrt{1+\frac{c^2}{a^2}+\frac{c^2}{b^2}}\, dx\, dy \\ &= \frac{\sqrt{a^2b^2+a^2c^2+b^2c^2}}{ab} (\text{area of } R). \end{aligned}$$

Under the transformation $x = a(u+v)$, $y = b(u-v)$, R corresponds to the ellipse in the uv-plane bounded by

$$\begin{aligned} &(u+v)^2+(u-v)^2+(u^2-v^2)-(u+v)-(u-v) = 0 \\ &3u^2+v^2-2u = 0 \\ &3\left(u^2-\frac{2}{3}u+\frac{1}{9}\right)+v^2 = \frac{1}{3} \\ &\frac{(u-1/3)^2}{1/9}+\frac{v^2}{1/3} = 1, \end{aligned}$$

an ellipse with area $\pi(1/3)(1/\sqrt{3}) = \pi/(3\sqrt{3})$ sq. units. Since

$$dx\, dy = \left|\begin{vmatrix} a & a \\ b & -b \end{vmatrix}\right| du\, dv = 2ab\, du\, dv,$$

we have

$$S = \frac{2\pi}{3\sqrt{3}}\sqrt{a^2b^2+a^2c^2+b^2c^2} \text{ sq. units.}$$

3. a) $$\frac{1}{1-xy} = 1 + xy + (xy)^2 + \cdots = \sum_{n=1}^{\infty}(xy)^{n-1}$$

$$\int_0^1 \int_0^1 \frac{dx\,dy}{1-xy} = \sum_{n=1}^{\infty} \int_0^1 x^{n-1}\,dx \int_0^1 y^{n-1}\,dy = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Remark: The series for $1/(1-xy)$ converges for $|xy| < 1$. Therefore the outer integral is improper (i.e., $\lim_{c\to 1-} \int_0^c dx$). We cannot do a detailed analysis of the convergence here, but the convergence of $\sum 1/n^2$ shows that the iterated double integral must converge.

b) Similarly,

$$\frac{1}{1+xy} = 1 - xy + (xy)^2 - \cdots = \sum_{n=1}^{\infty}(-xy)^{n-1}$$

$$\int_0^1 \int_0^1 \frac{dx\,dy}{1+xy} = \sum_{n=1}^{\infty}(-1)^{n-1} \int_0^1 x^{n-1}\,dx \int_0^1 y^{n-1}\,dy = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$$

$$\int_0^1 \int_0^1 \int_0^1 \frac{dx\,dy}{1-xyz} = \sum_{n=1}^{\infty} \int_0^1 x^{n-1}\,dx \int_0^1 y^{n-1}\,dy \int_0^1 z^{n-1}\,dz = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

$$\int_0^1 \int_0^1 \int_0^1 \frac{dx\,dy}{1+xyz} = \sum_{n=1}^{\infty}(-1)^{n-1} \int_0^1 x^{n-1}\,dx \int_0^1 y^{n-1}\,dy \int_0^1 z^{n-1}\,dz = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}.$$

4. Under the transformation $u = \mathbf{a} \bullet \mathbf{r}$, $v = \mathbf{b} \bullet \mathbf{r}$, $w = \mathbf{c} \bullet \mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the parallelepiped P corresponds to the rectangle R specified by $0 \le u \le d_1$, $0 \le v \le d_2$, $0 \le w \le d_3$. If $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ and similar expressions hold for $\mathbf{b}$ and $\mathbf{c}$, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{a} \bullet (\mathbf{b} \times \mathbf{c}).$$

Therefore

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw = \frac{du\,dv\,dw}{|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|},$$

and we have

$$\begin{aligned} &\iiint_P (\mathbf{a} \bullet \mathbf{r})(\mathbf{b} \bullet \mathbf{r})(\mathbf{c} \bullet \mathbf{r})\,dx\,dy\,dz \\ &= \iiint_R \frac{uvw}{|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|}\,du\,dv\,dw \\ &= \frac{1}{|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|} \int_0^{d_1} u\,du \int_0^{d_2} v\,dv \int_0^{d_3} w\,dw \\ &= \frac{d_1^2 d_2^2 d_3^2}{8|\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})|}. \end{aligned}$$

5. The volume V_0 removed from the ball is eight times the part in the first octant, which is itself split into two equal parts by the plane $x = y$:

$$\begin{aligned} V_0 &= 16 \int_0^1 dx \int_0^x \sqrt{4 - x^2 - y^2}\,dy \\ &= 16 \int_0^{\pi/4} d\theta \int_0^{\sec\theta} \sqrt{4 - r^2}\,r\,dr \qquad \text{Let } u = 4 - r^2,\ du = -2r\,dr \\ &= 8 \int_0^{\pi/4} d\theta \int_{4-\sec^2\theta}^{4} u^{1/2}\,du \\ &= \frac{16}{3} \int_0^{\pi/4} \left[8 - (4 - \sec^2\theta)^{3/2}\right] d\theta \\ &= \frac{32\pi}{3} - \frac{16}{3} \int_0^{\pi/4} \frac{(4\cos^2\theta - 1)^{3/2}}{\cos^3\theta}\,d\theta. \end{aligned}$$

Now the volume of the whole ball is $(4\pi/3)2^3 = 32\pi/3$, so the volume remaining after the hole is cut is

$$\begin{aligned} V &= \frac{32\pi}{3} - V_0 \\ &= \frac{16}{3} \int_0^{\pi/4} \frac{(3 - 4\sin^2\theta)^{3/2}}{(1 - \sin^2\theta)^2} \cos\theta\,d\theta \qquad \text{Let } v = \sin\theta,\ dv = \cos\theta\,d\theta \\ &= \frac{16}{3} \int_0^{1/\sqrt{2}} \frac{(3 - 4v^2)^{3/2}}{(1 - v^2)^2}\,dv. \end{aligned}$$

We submitted this last integral to Mathematica to obtain

$$V = \frac{4}{3}\Big(32\sin^{-1}\sqrt{\frac{2}{3}} - 2^{3/2} + 11\tan^{-1}(3 - 2^{3/2}) - 11\tan^{-1}(3 + 2^{3/2})\Big) \approx 18.9349.$$

6. Under the transformation $x = u^3$, $y = v^3$, $z = w^3$, the region R bounded by the surface $x^{2/3} + y^{2/3} + z^{2/3} = a^{2/3}$ gets mapped to the ball B bounded by $u^2 + v^2 + w^2 = a^{2/3}$. Assume that $a > 0$. Since

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 27u^2v^2w^2,$$

the volume of R is

$$V = 27 \iiint_B u^2 v^2 w^2 \, du \, dv \, dw.$$

Now switch to polar coordinates $[\rho, \phi, \theta]$ in uvw-space. Since

$$uvw = (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi),$$

we have

$$\begin{aligned}
V &= 27 \int_0^{2\pi} \cos^2\theta \sin^2\theta \, d\theta \int_0^{\pi} \sin^5\phi \cos^2\phi \, d\phi \int_0^{a^{1/3}} \rho^8 \, d\rho \\
&= 3a^3 \int_0^{2\pi} \frac{\sin^2(2\theta)}{4} \, d\theta \int_0^{\pi} (1-\cos^2\phi)^2 \cos^2\phi \sin\phi \, d\phi \\
&\qquad \text{Let } t = \cos\phi, \; dt = -\sin\phi \, d\phi \\
&= 3a^3 \int_0^{2\pi} \frac{1-\cos(4\theta)}{8} \, d\theta \int_{-1}^{1} (1-t^2)^2 t^2 \, dt \\
&= \frac{3a^3}{8}(2\pi)2 \int_0^1 (t^2 - 2t^4 + t^6) \, dt = \frac{4\pi a^3}{35} \text{ cu. units.}
\end{aligned}$$

7. One-eighth of the required volume lies in the first octant. Under the transformation $x = u^6$, $y = v^6$, $z = w^6$, the region first-octant R bounded by the surface $x^{1/3} + y^{1/3} + z^{1/3} = a^{1/3}$ and the coordinate planes gets mapped to the first octant part B of the ball bounded by $u^2 + v^2 + w^2 \le a^{1/3}$. Assume that $a > 0$. Since

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 6^3 u^5 v^5 w^5,$$

the required volume is

$$V = 8(6^3) \iiint_B u^5 v^5 w^5 \, du \, dv \, dw.$$

Now switch to polar coordinates $[\rho, \phi, \theta]$ in uvw-space. Since

$$uvw = (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho \cos\phi),$$

we have

$$\begin{aligned}
V &= 1,728 \int_0^{\pi/2} (\cos\theta \sin\theta)^5 \, d\theta \int_0^{\pi/2} (\sin^2\phi \cos\phi)^5 \sin\phi \, d\phi \\
&\qquad \times \int_0^{a^{1/6}} \rho^{17} \, d\rho \\
&= 96a^3 \int_0^{\pi/2} \frac{\sin^5(2\theta)}{32} \, d\theta \int_0^{\pi/2} \sin^{11}\phi (1-\sin^2\phi)^2 \cos\phi \, d\phi \\
&\qquad \text{Let } s = \sin\phi, \; ds = \cos\phi \, d\phi \\
&= 3a^3 \int_0^{\pi/2} (1-\cos^2(2\theta))^2 \sin(2\theta) \, d\theta \int_0^1 s^{11}(1-s^2)^2 \, ds \\
&\qquad \text{Let } t = \cos(2\theta), \; dt = -2\sin(2\theta) \, d\theta \\
&= \frac{3a^3}{2} \int_{-1}^{1} (1 - 2t^2 + t^4) \, dt \int_0^1 (s^{11} - 2s^{13} + s^{15}) \, ds \\
&= 3a^3 \left(1 - \frac{2}{3} + \frac{1}{5}\right)\left(\frac{1}{12} - \frac{1}{7} + \frac{1}{16}\right) = \frac{a^3}{210} \text{ cu. units.}
\end{aligned}$$