# CHAPTER 13. APPLICATIONS OF PARTIAL DERIVATIVES

## Section 13.1 Extreme Values (page 714)

**1.** $f(x,y) = x^2 + 2y^2 - 4x + 4y$
$f_1(x,y) = 2x - 4 = 0$ if $x = 2$
$f_2(x,y) = 4y + 4 = 0$ if $y = -1$.
Critical point is $(2,-1)$. Since $f(x,y) \to \infty$ as $x^2 + y^2 \to \infty$, $f$ has a local (and absolute) minimum value at that critical point.

**2.** $f(x,y) = xy - x + y$, $f_1 = y - 1$, $f_2 = x + 1$
$A = f_{11} = 0$, $B = f_{12} = 1$, $C = f_{22} = 0$.
Critical point $(-1,1)$ is a saddle point since $B^2 - AC > 0$.

**3.** $f(x,y) = x^3 + y^3 - 3xy$
$f_1(x,y) = 3(x^2 - y)$, $f_2(x,y) = 3(y^2 - x)$.
For critical points: $x^2 = y$ and $y^2 = x$. Thus $x^4 - x = 0$, that is, $x(x-1)(x^2+x+1) = 0$. Thus $x = 0$ or $x = 1$. The critical points are $(0,0)$ and $(1,1)$. We have

$$A = f_{11}(x,y) = 6x, \qquad B = f_{12}(x,y) = -3,$$
$$C = f_{22}(x,y) = 6y.$$

At $(0,0)$: $A = C = 0$, $B = -3$. Thus $AC < B^2$, and $(0,0)$ is a saddle point of $f$.
At $(1,1)$: $A = C = 6$, $B = -3$, so $AC > B^2$. Thus $f$ has a local minimum value at $(1,1)$.

**4.** $f(x,y) = x^4 + y^4 - 4xy$, $f_1 = 4(x^3 - y)$, $f_2 = 4(y^3 - x)$

$A = f_{11} = 12x^2$, $B = f_{12} = -4$, $C = f_{22} = 12y^2$.
For critical points: $x^3 = y$ and $y^3 = x$. Thus $x^9 = x$, or $x(x^8 - 1) = 0$, and $x = 0$, 1, or $-1$. The critical points are $(0,0)$, $(1,1)$ and $(-1,-1)$.
At $(0,0)$, $B^2 - AC = 16 - 0 > 0$, so $(0,0)$ is a saddle point.
At $(1,1)$ and $(-1,-1)$, $B^2 - AC = 16 - 144 < 0$, $A > 0$, so $f$ has local minima at these points.

**5.** $f(x,y) = \dfrac{x}{y} + \dfrac{8}{x} - y$
$f_1(x,y) = \dfrac{1}{y} - \dfrac{8}{x^2} = 0$ if $8y = x^2$
$f_2(x,y) = -\dfrac{x}{y^2} - 1 = 0$ if $x = -y^2$.
For critical points: $8y = x^2 = y^4$, so $y = 0$ or $y = 2$. $f(x,y)$ is not defined when $y = 0$, so the only critical point is $(-4,2)$. At $(-4,2)$ we have

$$A = f_{11} = \frac{16}{x^3} = -\frac{1}{4}, \qquad B = f_{12} = -\frac{1}{y^2} = -\frac{1}{4},$$
$$C = f_{22} = \frac{2x}{y^3} = -1.$$

Thus $B^2 - AC = \dfrac{1}{16} - \dfrac{1}{4} < 0$, and $(-4,2)$ is a local maximum.

**6.** $f(x,y) = \cos(x+y)$, $f_1 = -\sin(x+y) = f_2$.
All points on the lines $x + y = n\pi$ ($n$ is an integer) are critical points. If $n$ is even, $f = 1$ at such points; if $n$ is odd, $f = -1$ there. Since $-1 \le f(x,y) \le 1$ at all points in $\mathbb{R}^2$, $f$ must have local and absolute maximum values at points $x + y = n\pi$ with $n$ even, and local and absolute minimum values at such points with $n$ odd.

**7.** $f(x,y) = x\sin y$. For critical points we have

$$f_1 = \sin y = 0, \qquad f_2 = x\cos y = 0.$$

Since $\sin y$ and $\cos y$ cannot vanish at the same point, the only critical points correspond to $x = 0$ and $\sin y = 0$. They are $(0, n\pi)$, for all integers $n$. All are saddle points.

**8.** $f(x,y) = \cos x + \cos y$, $f_1 = -\sin x$, $f_2 = -\sin y$
$A = f_{11} = -\cos x$, $B = f_{12} = 0$, $C = f_{22} = -\cos y$.
The critical points are points $(m\pi, n\pi)$, where $m$ and $n$ are integers.
Here $B^2 - AC = -\cos(m\pi)\cos(n\pi) = (-1)^{m+n+1}$ which is negative if $m+n$ is even, and positive if $m+n$ is odd.
If $m+n$ is odd then $f$ has a saddle point at $(m\pi, n\pi)$.
If $m+n$ is even and $m$ is odd then $f$ has a local (and absolute) minimum value, $-2$, at $(m\pi, n\pi)$. If $m+n$ is even and $m$ is even then $f$ has a local (and absolute) maximum value, 2, at $(m\pi, n\pi)$.

**9.**
$$f(x,y) = x^2 y e^{-(x^2+y^2)}$$
$$f_1(x,y) = 2xy(1-x^2)e^{-(x^2+y^2)}$$
$$f_2(x,y) = x^2(1-2y^2)e^{-(x^2+y^2)}$$
$$A = f_{11}(x,y) = 2y(1 - 5x^2 + 2x^4)e^{-(x^2+y^2)}$$
$$B = f_{12}(x,y) = 2x(1-x^2)(1-2y^2)e^{-(x^2+y^2)}$$
$$C = f_{22}(x,y) = 2x^2y(2y^2 - 3)e^{-(x^2+y^2)}.$$

For critical points:

$$xy(1-x^2) = 0$$
$$x^2(1-2y^2) = 0.$$

The critical points are $(0,y)$ for all $y$, $(\pm 1, 1/\sqrt{2})$, and $(\pm 1, -1/\sqrt{2})$.
Evidently, $f(0,y) = 0$. Also $f(x,y) > 0$ if $y > 0$ and $x \ne 0$, and $f(x,y) < 0$ if $y < 0$ and $x \ne 0$. Thus $f$ has a local minimum at $(0,y)$ if $y > 0$, and a local maximum if $y < 0$. The origin is a saddle point.
At $(\pm 1, 1/\sqrt{2})$: $A = C = -2\sqrt{2}e^{-3/2}$, $B = 0$, and so $AC > B^2$. Thus $f$ has local maximum values at these two points.

At $(\pm 1, -1/\sqrt{2})$: $A = C = 2\sqrt{2}e^{-3/2}$, $B = 0$, and so $AC > B^2$. Thus $f$ has local minimum values at these two points.

Since $f(x, y) \to 0$ as $x^2 + y^2 \to \infty$, the value $f(\pm 1, 1/\sqrt{2}) = e^{-3/2}/\sqrt{2}$ is the absolute maximum value for $f$, and the value $f(\pm 1, -1/\sqrt{2}) = -e^{-3/2}/\sqrt{2}$ is the absolute minimum value.

**10.** $$f(x, y) = \frac{xy}{2 + x^4 + y^4}$$
$$f_1 = \frac{(2 + x^4 + y^4)y - xy4x^3}{(2 + x^4 + y^4)^2} = \frac{y(2 + y^4 - 3x^4)}{(2 + x^4 + y^4)^2}$$
$$f_2 = \frac{x(2 + x^4 - 3y^4)}{(2 + x^4 + y^4)^2}.$$
For critical points, $y(2 + y^4 - 3x^4) = 0$ and $x(2 + x^4 - 3y^4) = 0$.
One critical point is $(0, 0)$. Since $f(0, 0) = 0$ but $f(x, y) > 0$ in the first quadrant and $f(x, y) < 0$ in the second quadrant, $(0, 0)$ must be a saddle point of $f$.
Any other critical points must satisfy $2 + y^4 - 3x^4 = 0$ and $2 + x^4 - 3y^4 = 0$, that is, $y^4 = x^4$, or $y = \pm x$. Thus $2 - 2x^4 = 0$ and $x = \pm 1$. Therefore there are four other critical points: $(1, 1)$, $(-1, -1)$, $(1, -1)$ and $(-1, 1)$. $f$ is positive at the first two of these, and negative at the other two. Since $f(x, y) \to 0$ as $x^2 + y^2 \to \infty$, $f$ must have maximum values at $(1, 1)$ and $(-1, -1)$, and minimum values at $(1, -1)$ and $(-1, 1)$.

**11.** $$f(x, y) = xe^{-x^3+y^3}$$
$$f_1(x, y) = (1 - 3x^3)e^{-x^3+y^3}$$
$$f_2(x, y) = 3xy^2e^{-x^3+y^3}$$
$$A = f_{11}(x, y) = 3x^2(3x^3 - 4)e^{-x^3+y^3}$$
$$B = f_{12}(x, y) = -3y^2(3x^3 - 1)e^{-x^3+y^3}$$
$$C = f_{22}(x, y) = 3xy(3y^3 + 2)e^{-x^3+y^3}$$
For critical points: $3x^3 = 1$ and $3xy^2 = 0$. The only critical point is $(3^{-1/3}, 0)$. At that point we have $B = C = 0$ so the second derivative test is inconclusive.
However, note that $f(x, y) = f(x, 0)e^{y^3}$, and $e^{y^3}$ has an inflection point at $y = 0$. Therefore $f(x, y)$ has neither a maximum nor a minimum value at $(3^{-1/3}, 0)$, so has a saddle point there.

**12.** $$f(x, y) = \frac{x^2}{x^2 + y^2}$$
$$f_1(x, y) = \frac{(x^2 + y^2)2x - 2x^3}{(x^2 + y^2)^2} = \frac{2xy^2}{(x^2 + y^2)^2}$$
$$f_2(x, y) = -\frac{2x^2y}{(x^2 + y^2)^2}.$$
Both partial derivatives are zero at all points of the coordinate axes. Also $f(x, 0) = 1$ for $x \neq 0$, and $f(0, y) = 0$ for $y \neq 0$.
Evidently $0 \le f(x, y) \le 1$ for all $(x, y) \neq (0, 0)$.
Thus, $f$ has absolute maximum value 1 at all points $(x, 0)$ for $x \neq 0$, and absolute minimum value 0 at all points $(0, y)$ for all $y \neq 0$.

**13.** $$f(x, y) = \frac{xy}{x^2 + y^2}$$
$$f_1(x, y) = \frac{(x^2 + y^2)y - 2x^2y}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2}$$
$$f_2(x, y) = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \quad \text{(by symmetry).}$$
Both partial derivatives are zero at all points of the lines $y = \pm x$ for $x \neq 0$. Also $f(x, x) = \frac{1}{2}$, and $f(x, -x) = -\frac{1}{2}$ for $x \neq 0$.
Since $x^2 \pm 2xy + y^2 = (x \pm y)^2 \ge 0$, we have $|xy| \le \frac{1}{2}(x^2 + y^2)$ for all $(x, y) \neq (0, 0)$, so $|f(x, y)| \le \frac{1}{2}$ on its domain.
Thus, $f$ has absolute maximum value $\frac{1}{2}$ at all points $(x, x)$ for $x \neq 0$, and absolute minimum value $-\frac{1}{2}$ at all points $(x, -x)$ for all $x \neq 0$.

**14.** $$f(x, y) = \frac{1}{1 - x + y + x^2 + y^2} = \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 + \frac{1}{2}}.$$
Evidently $f$ has absolute maximum value 2 at $\left(\frac{1}{2}, -\frac{1}{2}\right)$.
Since
$$f_1(x, y) = \frac{1 - 2x}{(1 - x + y + x^2 + y^2)^2}$$
$$f_2(x, y) = -\frac{1 + 2y}{(1 - x + y + x^2 + y^2)^2},$$
$\left(\frac{1}{2}, -\frac{1}{2}\right)$ is the only critical point of $f$.

**15.** $$f(x, y) = \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{(x + 1)(y + 1)(x + y)}{x^2y^2}$$
$$f_1(x, y) = -\frac{(y + 1)(xy + x + 2y)}{x^3y^2}$$
$$f_2(x, y) = -\frac{(x + 1)(xy + y + 2x)}{x^2y^3}$$
$$A = f_{11}(x, y) = \frac{2(y + 1)(xy + x + 3y)}{x^4y^2}$$
$$B = f_{12}(x, y) = \frac{2(xy + x + y)}{x^3y^3}$$
$$C = f_{22}(x, y) = \frac{2(x + 1)(xy + y + 3x)}{x^2y^4}.$$

For critical points:

$$\begin{aligned} &y = -1 \quad \text{or} \quad xy + x + 2y = 0, \\ \text{and} \quad &x = -1 \quad \text{or} \quad xy + y + 2x = 0. \end{aligned}$$

If $y = -1$, then $x = -1$ or $x - 1 = 0$.
If $x = -1$, then $y = -1$ or $y - 1 = 0$.
If $x \neq -1$ and $y \neq -1$, then $x - y = 0$, so $x^2 + 3x = 0$. Thus $x = 0$ or $x = -3$. However, the definition of $f$ excludes $x = 0$. Thus, the only critical points are

$$(1, -1), \quad (-1, 1), \quad (-1, -1), \quad \text{and} \quad (-3, -3).$$

At $(1, -1)$, $(-1, 1)$, and $(-1, -1)$ we have $AC = 0$ and $B \neq 0$. Therefore these three points are saddle points of $f$.
At $(-3, -3)$, $A = C = 4/243$ and $B = 2/243$, so $AC > B^2$. Therefore $f$ has a local minimum value at $(-3, -3)$.

**16.** $f(x, y, z) = xyz - x^2 - y^2 - z^2$. For critical points we have

$$0 = f_1 = yz - 2x, \quad 0 = f_2 = xz - 2y, \quad 0 = f_3 = xy - 2z.$$

Thus $xyz = 2x^2 = 2y^2 = 2z^2$, so $x^2 = y^2 = z^2$. Hence $x^3 = \pm 2x^2$, and $x = \pm 2$ or 0. Similarly for $y$ and $z$. The only critical points are $(0, 0, 0)$, $(2, 2, 2)$, $(-2, -2, 2)$, $(-2, 2, -2)$, and $(2, -2, -2)$.

Let $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, where $u^2 + v^2 + w^2 = 1$. Then

$$\begin{aligned} D_{\mathbf{u}} f(x, y, z) &= (yz - 2x)u + (xz - 2y)v + (xy - 2z)w \\ D_{\mathbf{u}}\Big(D_{\mathbf{u}} f(x, y, z)\Big) &= (-2u + zv + yw)u \\ &\quad + (zu - 2v + xw)v + (yu + xv - 2w)w. \end{aligned}$$

At $(0, 0, 0)$, $D_{\mathbf{u}}\Big(D_{\mathbf{u}} f(0, 0, 0)\Big) = -2u^2 - 2v^2 - 2w^2 < 0$ for $\mathbf{u} \neq \mathbf{0}$, so $f$ has a local maximum value at $(0, 0, 0)$.

At $(2, 2, 2)$, we have

$$\begin{aligned} D_{\mathbf{u}}\Big(D_{\mathbf{u}} f(2, 2, 2)\Big) &= (-2u + 2v + 2w)u + (2u - 2v + 2w)v \\ &\quad + (2u + 2v - 2w)w \\ &= -2(u^2 + v^2 + w^2) + 4(uv + vw + wu) \\ &= -2[(u - v - w)^2 - 4vw] \\ &\begin{cases} < 0 & \text{if } v = w = 0,\ u \neq 0 \\ > 0 & \text{if } v = w \neq 0,\ u - v - w = 0. \end{cases} \end{aligned}$$

Thus $(2, 2, 2)$ is a saddle point.

At $(2, -2, -2)$, we have

$$\begin{aligned} D_{\mathbf{u}}\Big(D_{\mathbf{u}} f\Big) &= -2(u^2 + v^2 + w^2 + 2uv + 2uw - 2vw) \\ &= -2[(u + v + w)^2 - 4vw] \\ &\begin{cases} < 0 & \text{if } v = w = 0,\ u \neq 0 \\ > 0 & \text{if } v = w \neq 0,\ u + v + w = 0. \end{cases} \end{aligned}$$

Thus $(2, -2, -2)$ is a saddle point. By symmetry, so are the remaining two critical points.

**17.**
$$\begin{aligned} f(x, y, z) &= xy + x^2 z - x^2 - y - z^2 \\ f_1(x, y, z) &= y + 2x(z - 1) \\ f_2(x, y, z) &= x - 1 \\ f_3(x, y, z) &= x^2 - 2z. \end{aligned}$$

The only critical point is $\left(1, 1, \frac{1}{2}\right)$. We have

$$\begin{aligned} D &= f\left(1 + h, 1 + k, \tfrac{1}{2} + m\right) - f\left(1, 1, \tfrac{1}{2}\right) \\ &= 1 + h + k + hk + \frac{1 + 2h + h^2}{2} + (1 + 2h + h^2)m \\ &\quad - 1 - 2h - h^2 - 1 - k - \frac{1}{4} - m - m^2 - \left(-\frac{3}{4}\right) \\ &= \frac{h^2(2m - 1) + 2h(k + 2m) - 2m^2}{2}. \end{aligned}$$

If $m = h$ and $k = 0$, then $D = \dfrac{h^2(1 + 2h)}{2} > 0$ for small $|h|$.
If $h = k = 0$, then $D = -m^2 < 0$ for $m \neq 0$.
Thus $f$ has a saddle point at $\left(1, 1, \frac{1}{2}\right)$.

**18.**
$$\begin{aligned} f(x, y, z) &= 4xyz - x^4 - y^4 - z^4 \\ D &= f(1 + h, 1 + k, 1 + m) - f(1, 1, 1) \\ &= 4(1 + h)(1 + k)(1 + m) - (1 + h)^4 - (1 + k)^4 \\ &\quad - (1 + m)^4 - 1 \\ &= 4(1 + h + k + m + hk + hm + km + hkm) \\ &\quad - (1 + 4h + 6h^2 + 4h^3 + h^4) \\ &\quad - (1 + 4k + 6k^2 + 4k^3 + k^4) \\ &\quad - (1 + 4m + 6m^2 + 4m^3 + m^4) - 1 \\ &= 4(hk + hm + km) - 6(h^2 + k^2 + m^2) + \cdots, \end{aligned}$$

where $\cdots$ stands for terms of degree 3 and 4 in the variables $h$, $k$, and $m$. Completing some squares among the quadratic terms we obtain

$$D = -2\Big[(h-k)^2 + (k-m)^2 + (h-m)^2 + h^2 + k^2 + m^2\Big] + \cdots$$

which is negative if $|h|$, $|k|$ and $|m|$ are small and not all 0. (This is because the terms of degree 3 and 4 are smaller in size than the quadratic terms for small values of the variables.)
Hence $f$ has a local maximum value at $(1, 1, 1)$.

**19.**
$$\begin{aligned} f(x, y) &= xye^{-(x^2 + y^4)} \\ f_1(x, y) &= y(1 - 2x^2)e^{-(x^2 + y^4)} \\ f_2(x, y) &= x(1 - 4y^4)e^{-(x^2 + y^4)} \end{aligned}$$

For critical points $y(1-2x^2)=0$ and $x(1-4y^4)=0$. The critical points are

$$(0,0), \qquad \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad \left(\pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

We have

$$\begin{aligned}
&f(0,0)=0\\
&f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2}e^{-3/4} > 0\\
&f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{2}e^{-3/4} < 0
\end{aligned}$$

Since $f(x,y)\to 0$ as $x^2+y^2\to\infty$, the maximum and minimum values of $f$ are $\frac{1}{2}e^{-3/4}$ and $-\frac{1}{2}e^{-3/4}$ respectively.

**20.** 
$$\begin{aligned}
f(x,y) &= \frac{x}{1+x^2+y^2}\\
f_1(x,y) &= \frac{1+y^2-x^2}{(1+x^2+y^2)^2}\\
f_2(x,y) &= \frac{-2xy}{(1+x^2+y^2)^2}.
\end{aligned}$$

For critical points, $x^2-y^2=1$, and $xy=0$. The critical points are $(\pm1,0)$. $f(\pm1,0)=\pm\frac{1}{2}$.
Since $f(x,y)\to 0$ as $x^2+y^2\to\infty$, the maximum and minimum values of $f$ are $1/2$ and $-1/2$ respectively.

**21.** 
$$\begin{aligned}
f(x,y,z) &= xyze^{-(x^2+y^2+z^2)}\\
f_1(x,y,z) &= yz(1-2x^2)e^{-(x^2+y^2+z^2)}\\
f_2(x,y,z) &= xz(1-2y^2)e^{-(x^2+y^2+z^2)}\\
f_3(x,y,z) &= xy(1-2z^2)e^{-(x^2+y^2+z^2)}.
\end{aligned}$$

Any critical point must satisfy

$$\begin{aligned}
yz(1-2x^2)=0 &\quad \text{i.e., } y=0 \text{ or } z=0 \text{ or } x=\pm\frac{1}{\sqrt{2}}\\
xz(1-2y^2)=0 &\quad \text{i.e., } x=0 \text{ or } z=0 \text{ or } y=\pm\frac{1}{\sqrt{2}}\\
xy(1-2z^2)=0 &\quad \text{i.e., } x=0 \text{ or } y=0 \text{ or } z=\pm\frac{1}{\sqrt{2}}.
\end{aligned}$$

Since $f(x,y,z)$ is positive at some points, negative at others, and approaches 0 as $(x,y,z)$ recedes to infinity, $f$ must have maximum and minimum values at critical points. Since $f(x,y,z)=0$ if $x=0$ or $y=0$ or $z=0$, the maximum and minimum values must occur among the eight critical points where $x=\pm1/\sqrt{2}$, $y=\pm1/\sqrt{2}$, and $z=\pm1/\sqrt{2}$. At four of these points, $f$ has the value $\frac{1}{2\sqrt{2}}e^{-3/2}$, the maximum value. At the other four $f$ has the value $-\frac{1}{2\sqrt{2}}e^{-3/2}$, the minimum value.

**22.** 
$$\begin{aligned}
&f(x,y) = x+8y+\frac{1}{xy}, \qquad (x>0, \quad y>0)\\
&f_1(x,y) = 1-\frac{1}{x^2y} = 0 \quad\Rightarrow\quad x^2y=1\\
&f_2(x,y) = 8-\frac{1}{xy^2} = 0 \quad\Rightarrow\quad 8xy^2=1.
\end{aligned}$$

The critical points must satisfy

$$\frac{x}{y} = \frac{x^2y}{xy^2} = 8,$$

that is, $x=8y$. Also, $x^2y=1$, so $64y^3=1$.
Thus $y=1/4$, and $x=2$; the critical point is $\left(2,\frac{1}{4}\right)$.
Since $f(x,y)\to\infty$ if $x\to 0+$, $y\to 0+$, or $x^2+y^2\to\infty$, the critical point must give a minimum value for $f$. The minimum value is $f\left(2,\frac{1}{4}\right)=2+2+2=6$.

**23.** Let the length, width, and height of the box be $x$, $y$, and $z$, respectively. Then $V=xyz$. The total surface area of the bottom and sides is

$$\begin{aligned}
S &= xy+2xz+2yz = xy+2(x+y)\frac{V}{xy}\\
&= xy+\frac{2V}{x}+\frac{2V}{y},
\end{aligned}$$

where $x>0$ and $y>0$. Since $S\to\infty$ as $x\to 0+$ or $y\to 0+$ or $x^2+y^2\to\infty$, $S$ must have a minimum value at a critical point in the first quadrant. For CP:

$$\begin{aligned}
0 &= \frac{\partial S}{\partial x} = y-\frac{2V}{x^2}\\
0 &= \frac{\partial S}{\partial y} = x-\frac{2V}{y^2}.
\end{aligned}$$

Thus $x^2y=2V=xy^2$, so that $x=y=(2V)^{1/3}$ and $z=V/(2V)^{2/3}=2^{-2/3}V^{1/3}$.

**24.** Let the length, width, and height of the box be $x$, $y$, and $z$, respectively. Then $V=xyz$. If the top and side walls cost \$$k$ per unit area, then the total cost of materials for the box is

$$\begin{aligned}
C &= 2kxy+kxy+2kxz+2kyz\\
&= k\left[3xy+2(x+y)\frac{V}{xy}\right] = k\left[3xy+\frac{2V}{x}+\frac{2V}{y}\right],
\end{aligned}$$

where $x>0$ and $y>0$. Since $C\to\infty$ as $x\to 0+$ or $y\to 0+$ or $x^2+y^2\to\infty$, $C$ must have a minimum value at a critical point in the first quadrant. For CP:

$$\begin{aligned}
0 &= \frac{\partial C}{\partial x} = k\left(3y-\frac{2V}{x^2}\right)\\
0 &= \frac{\partial C}{\partial y} = k\left(3x-\frac{2V}{y^2}\right).
\end{aligned}$$

Thus $3x^2y = 2V = 3xy^2$, so that $x = y = (2V/3)^{1/3}$ and $z = V/(2V/3)^{2/3} = (9V/4)^{1/3}$.

**25.** Let $(x, y, z)$ be the coordinates of the corner of the box that is in the first octant of space. Thus $x, y, z \geq 0$, and

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The volume of the box is

$$V = (2x)(2y)(2z) = 8cxy\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

for $x \geq 0$, $y \geq 0$, and $(x^2/a^2)+(y^2/b^2) \leq 1$. For analysis it is easier to deal with $V^2$ than with $V$:

$$V^2 = 64c^2\left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right).$$

Since $V = 0$ if $x = 0$ or $y = 0$ or $(x^2/a^2) + (y^2/b^2) = 1$, the maximum value of $V^2$, and hence of $V$, will occur at a critical point of $V^2$ where $x > 0$ and $y > 0$. For CP:

$$\begin{aligned} 0 = \frac{\partial V^2}{\partial x} &= 64c^2\left(2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right) \\ &= 128c^2xy^2\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) \\ 0 = \frac{\partial V^2}{\partial y} &= 128c^2x^2y\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right). \end{aligned}$$

Hence we must have

$$\frac{2x^2}{a^2} + \frac{y^2}{b^2} = 1 = \frac{x^2}{a^2} + \frac{2y^2}{b^2},$$

so that $x^2/a^2 = y^2/b^2 = 1/3$, and $x = a/\sqrt{3}$, $y = b/\sqrt{3}$. The largest box has volume

$$V = \frac{8abc}{3}\sqrt{1 - \frac{1}{3} - \frac{1}{3}} = \frac{8abc}{3\sqrt{3}} \text{ cubic units.}$$

**26.** Given that $a > 0$, $b > 0$, $c > 0$, and $a + b + c = 30$, we want to maximize

$$P = ab^2c^3 = (30 - b - c)b^2c^3 = 30b^2c^3 - b^3c^3 - b^2c^4.$$

Since $P = 0$ if $b = 0$ or $c = 0$ or $b + c = 30$ (i.e., $a = 0$), the maximum value of $P$ will occur at a critical point $(b, c)$ satisfying $b > 0$, $c > 0$, and $b + c < 30$. For CP:

$$\begin{aligned} 0 &= \frac{\partial P}{\partial b} = 60bc^3 - 3b^2c^3 - 2bc^4 = bc^3(60 - 3b - 2c) \\ 0 &= \frac{\partial P}{\partial c} = 90b^2c^2 - 3b^3c^2 - 4b^2c^3 = b^2c^2(90 - 3b - 4c). \end{aligned}$$

Hence $9b + 6c = 180 = 6b + 8c$, from which we obtain $3b = 2c = 30$. The three numbers are $b = 10$, $c = 15$, and $a = 30 - 10 - 15 = 5$.

**27.** Differentiate the given equation

$$e^{2zx-x^2} - 3e^{2zy+y^2} = 2$$

with respect to $x$ and $y$, regarding $z$ as a function of $x$ and $y$:

$$\begin{aligned} &e^{2zx-x^2}\left(2x\frac{\partial z}{\partial x} + 2z - 2x\right) - 3e^{2zy+y^2}\left(2y\frac{\partial z}{\partial x}\right) = 0 \qquad (*) \\ &e^{2zx-x^2}\left(2x\frac{\partial z}{\partial y}\right) - 3e^{2zy+y^2}\left(2y\frac{\partial z}{\partial y} + 2z + 2y\right) = 0 \qquad (**) \end{aligned}$$

For a critical point we have $\dfrac{\partial z}{\partial x} = 0$ and $\dfrac{\partial z}{\partial y} = 0$, and it follows from the equations above that $z = x$ and $z = -y$. Substituting these into the given equation, we get

$$\begin{aligned} &e^{z^2} - 3e^{-z^2} = 2 \\ &(e^{z^2})^2 - 2e^{z^2} - 3 = 0 \\ &(e^{z^2} - 3)(e^{z^2} + 1) = 0. \end{aligned}$$

Thus $e^{z^2} = 3$ or $e^{z^2} = -1$. Since $e^{z^2} = -1$ is not possible, we have $e^{z^2} = 3$, so $z = \pm\sqrt{\ln 3}$.
The critical points are $(\sqrt{\ln 3}, -\sqrt{\ln 3})$, and $(-\sqrt{\ln 3}, \sqrt{\ln 3})$.

**28.** We will use the second derivative test to classify the two critical points calculated in Exercise 25. To calculate the second partials

$$A = \frac{\partial^2 z}{\partial x^2}, \qquad B = \frac{\partial^2 z}{\partial x \partial y}, \qquad C = \frac{\partial^2 z}{\partial y^2},$$

we differentiate the expressions $(*)$, and $(**)$ obtained in Exercise 25.
Differentiating $(*)$ with respect to $x$, we obtain

$$\begin{aligned} &e^{2zx-x^2}\left[\left(2x\frac{\partial z}{\partial x} + 2z - 2x\right)^2 \right. \\ &\qquad \left. + 4\frac{\partial z}{\partial x} + 2x\frac{\partial^2 z}{\partial x^2} - 2\right] \\ &\quad - 3e^{2zy+y^2}\left[\left(2y\frac{\partial z}{\partial x}\right)^2 + 2y\frac{\partial^2 z}{\partial x^2}\right] = 0. \end{aligned}$$

At a critical point, $\dfrac{\partial z}{\partial x} = 0$, $z = x$, $z = -y$, and $z^2 = \ln 3$, so

$$\begin{aligned} 3\left(2x\frac{\partial^2 z}{\partial x^2} - 2\right) - \frac{3}{3}\left(2y\frac{\partial^2 z}{\partial x^2}\right) &= 0, \\ A = \frac{\partial^2 z}{\partial x^2} = \frac{6}{6x - 2y}. \end{aligned}$$

Differentiating $(**)$ with respect to $y$ gives

$$\begin{aligned} &e^{2zx-x^2}\left[\left(2x\frac{\partial z}{\partial y}\right)^2 + 2x\frac{\partial^2 z}{\partial y^2}\right] \\ &- 3e^{2zy+y^2}\left[\left(2y\frac{\partial z}{\partial y} + 2z + 2y\right)^2 + 4\frac{\partial z}{\partial y} + 2y\frac{\partial^2 z}{\partial y^2} + 2\right] = 0, \end{aligned}$$

and evaluation at a critical point gives

$$\begin{aligned} 3\left(2x\frac{\partial^2 z}{\partial y^2}\right) - \frac{3}{3}\left(2y\frac{\partial^2 z}{\partial y^2} + 2\right) &= 0, \\ C = \frac{\partial^2 z}{\partial y^2} = \frac{2}{6x - 2y}. \end{aligned}$$

Finally, differentiating $(*)$ with respect to $y$ gives

$$\begin{aligned} &e^{2zx-x^2}\left[\left(2x\frac{\partial z}{\partial x} + 2z - 2x\right)\left(2x\frac{\partial z}{\partial y}\right) + 2x\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial z}{\partial y}\right] \\ &- 3e^{2zy+y^2}\left[\left(2y\frac{\partial z}{\partial y} + 2z + 2y\right)\left(2y\frac{\partial z}{\partial x}\right) + 2\frac{\partial z}{\partial x} + 2y\frac{\partial^2 z}{\partial x \partial y}\right] = 0, \end{aligned}$$

and, evaluating at a critical point,

$$(6x - 2y)\frac{\partial^2 z}{\partial x \partial y} = 0,$$

so that

$$B = \frac{\partial^2 z}{\partial x \partial y} = 0.$$

At the critical point $(\sqrt{\ln 3}, -\sqrt{\ln 3})$ we have

$$A = \frac{6}{8\ln 3}, \qquad B = 0, \qquad C = \frac{2}{8\ln 3},$$

so $B^2 - AC < 0$, and $f$ has a local minimum at that critical point.
At the critical point $(-\sqrt{\ln 3}, \sqrt{\ln 3})$ we have

$$A = -\frac{6}{8\ln 3}, \qquad B = 0, \qquad C = -\frac{2}{8\ln 3},$$

so $B^2 - AC < 0$, and $f$ has a local maximum at that critical point.

**29.** $f(x, y) = (y - x^2)(y - 3x^2) = y^2 - 4x^2y + 3x^4$
$f_1(x, y) = -8xy + 12x^3 = 4x(3x^2 - 2y)$
$f_2(x, y) = 2y - 4x^2.$

Since $f_1(0, 0) = f_2(0, 0) = 0$, therefore $(0, 0)$ is a critical point of $f$.
Let $g(x) = f(x, kx) = k^2x^2 - 4kx^3 + 3x^4$. Then

$$\begin{aligned} g'(x) &= 2k^2x - 12kx^2 + 12x^3 \\ g''(x) &= 2k^2 - 24kx + 36x^2. \end{aligned}$$

Since $g'(0) = 0$ and $g''(0) = 2k^2 > 0$ for $k \neq 0$, $g$ has a local minimum value at $x = 0$. Thus $f(x, kx)$ has a local minimum at $x = 0$ if $k \neq 0$. Since $f(x, 0) = 3x^4$ and $f(0, y) = y^2$ both have local minimum values at $(0, 0)$, $f$ has a local minimum at $(0, 0)$ when restricted to any straight line through the origin.
However, on the curve $y = 2x^2$ we have

$$f(x, 2x^2) = x^2(-x^2) = -x^4,$$

which has a local maximum value at the origin. Therefore $f$ does *not* have an (unrestricted) local minimum value at $(0, 0)$.

Note that $A = f_{11}(0, 0) = (-8y + 36x^2)\Big|_{(0,0)} = 0$

$$B = f_{12}(0, 0) = -8x\Big|_{(0,0)} = 0.$$

Thus $AC = B^2$, and the second derivative test is indeterminate at the origin.

**30.** We have

$$\begin{aligned} Q(u, v) &= Au^2 + 2Buv + Cv^2 \\ &= A\left(u^2 + \frac{2B}{A}uv + \frac{B^2}{A^2}v^2\right) + \left(C - \frac{B^2}{A}\right)v^2 \\ &= A\left(u + \frac{Bv}{A}\right)^2 + \frac{AC - B^2}{A}v^2. \end{aligned}$$

If $\begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 > 0$, both terms above have the same sign, positive if $A > 0$ and negative if $A < 0$, ensuring that $Q$ is positive definite or negative definite respectively, since the two terms cannot both vanish if $(u, v) \neq (0, 0)$. If $AC - B^2 < 0$, $Q(u, v)$ is a difference of squares, and must be indefinite.

**31.** Let

$$Q(u, v, w) = Au^2 + Bv^2 + Cw^2 + 2Duv + 2Euw + 2Fvw$$

and let

$$K_1 = A, \qquad K_2 = \begin{vmatrix} A & D \\ D & B \end{vmatrix} = AB - D^2$$
$$K_3 = \begin{vmatrix} A & D & E \\ D & B & F \\ E & F & C \end{vmatrix} = ABC + 2DEF - BE^2 - CD^2 - AF^2.$$

Suppose that $K_1 \neq 0$, $K_2 \neq 0$, and $K_3 \neq 0$. We have

$$\begin{aligned} &Q(u, v, w) \\ &= A\left[u^2 + 2u\frac{Dv + Ew}{A} + \left(\frac{Dv + Ew}{A}\right)^2\right] \\ &\quad + \frac{AB - D^2}{A}v^2 + \frac{AC - E^2}{A}w^2 + \frac{2(AF - DE)}{A}vw \\ &= A\left(u + \frac{Dv + Ew}{A}\right)^2 \\ &\quad + \frac{AB - D^2}{A}\left(v^2 + \frac{2(AF - DE)}{AB - D^2}vw + \left(\frac{AF - DE}{AB - D^2}\right)^2 w^2\right) \\ &\quad + \left[\frac{AC - E^2}{A} - \frac{(AF - DE)^2}{A(AB - D^2)}\right]w^2 \\ &= A\left(u + \frac{Dv + Ew}{A}\right)^2 + \frac{AB - D^2}{A}\left(v + \frac{AF - DE}{AB - D^2}w\right)^2 \\ &\quad + \frac{A(ABC - BE^2 - AF^2 - CD^2 + 2DEF)}{A(AB - D^2)}w^2 \\ &= K_1\left(u + \frac{Dv + Ew}{A}\right)^2 + \frac{K_2}{K_1}\left(v + \frac{AF - DE}{AB - D^2}w\right)^2 \\ &\quad + \frac{K_3}{K_2}w^2. \end{aligned}$$

If $K_1 > 0$, $K_2 > 0$, and $K_3 > 0$, then all three squares the last expression above have positive coefficients, and so $Q$ is positive definite. If $K_1 < 0$, $K_2 > 0$, and $K_3 < 0$, then all three squares the last expression above have negative coefficients, and so $Q$ is negative definite. In all other cases where none of the $K_i = 0$, the coefficients of the squares are not all of the same sign so choices of $(u, v, w)$ can be made which make the expression either positive or negative, and $Q$ is indefinite.

If $f$ has continuous partial derivatives of order two and $(a, b, c)$ is a critical point of $f(x, y, z)$, let

$$\begin{aligned} A &= f_{11}(a, b, c), & D &= f_{12}(a, b, c), \\ B &= f_{22}(a, b, c), & E &= f_{23}(a, b, c), \\ C &= f_{33}(a, b, c), & F &= f_{23}(a, b, c). \end{aligned}$$

Then $f$ has a local minimum value at $(a, b, c)$ if $K_1 > 0$, $K_2 > 0$, and $K_3 > 0$, a local maximum value at $(a, b, c)$ if $K_1 < 0$, $K_2 > 0$, and $K_3 < 0$, and a saddle point at $(a, b, c)$ if $K_1, K_2, K_3$ are all nonzero but satisfy neither of the above conditions.

## Section 13.2 Extreme Values of Functions Defined on Restricted Domains (page 720)

**1.** $f(x, y) = x - x^2 + y^2$ on
$R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le 1\}$.
For critical points:

$$0 = f_1(x, y) = 1 - 2x, \qquad 0 = f_2(x, y) = 2y.$$

The only CP is $(1/2, 0)$, which lies on the boundary of $R$.
The boundary consists of four segments; we investigate each.
On $x = 0$ we have $f(x, y) = f(0, y) = y^2$ for $0 \le y \le 1$, which has minimum value 0 and maximum value 1.
On $y = 0$ we have $f(x, y) = f(x, 0) = x - x^2 = g(x)$ for $0 \le x \le 2$. Since $g'(x) = 1 - 2x = 0$ at $x = 1/2$, $g(1/2) = 1/4$, $g(0) = 0$, and $g(2) = -2$, the maximum and minimum values of $f$ on the boundary segment $y = 0$ are 1/4 and $-2$ respectively.
On $x = 2$ we have $f(x, y) = f(2, y) = -2 + y^2$ for $0 \le y \le 1$, which has minimum value $-2$ and maximum value $-1$.
On $y = 1$, $f(x, y) = f(x, 1) = x - x^2 + 1 = g(x) + 1$ for $0 \le x \le 2$. Thus the maximum and minimum values of $f$ on the boundary segment $y = 1$ are 5/4 and $-1$ respectively.
Overall, $f$ has maximum value $5/4$ and minimum value $-2$ on the rectangle $R$.

**2.** $f(x, y) = xy - 2x$ on
$R = \{(x, y) : -1 \le x \le 1,\ 0 \le y \le 1\}$.
For critical points:

$$0 = f_1(x, y) = y - 2, \qquad 0 = f_2(x, y) = x.$$

The only CP is $(0, 2)$, which lies outside $R$. Therefore the maximum and minimum values of $f$ on $R$ lie on one of the four boundary segments of $R$.
On $x = -1$ we have $f(-1, y) = 2 - y$ for $0 \le y \le 1$, which has maximum value 2 and minimum value 1.
On $x = 1$ we have $f(1, y) = y - 2$ for $0 \le y \le 1$, which has maximum value $-1$ and minimum value $-2$.
On $y = 0$ we have $f(x, 0) = -2x$ for $-1 \le x \le 1$, which has maximum value 2 and minimum value $-2$.
On $y = 1$ we have $f(x, 1) = -x$ for $-1 \le x \le 1$, which has maximum value 1 and minimum value $-1$.
Thus the maximum and minimum values of $f$ on the rectangle $R$ are 2 and $-2$ respectively.

**3.** $f(x, y) = xy - y^2$ on $D = \{(x, y) : x^2 + y^2 \le 1\}$.
For critical points:

$$0 = f_1(x, y) = y, \qquad 0 = f_2(x, y) = x - 2y.$$

The only CP is $(0,0)$, which lies inside $D$. We have $f(0,0)=0$.
The boundary of $D$ is the circle $x=\cos t$, $y=\sin t$, $-\pi\le t\le\pi$. On this circle we have

$$\begin{aligned} g(t) &= f(\cos t,\sin t)=\cos t\sin t-\sin^2 t\\ &= \frac{1}{2}\Big[\sin 2t+\cos 2t-1\Big], \qquad (-\pi\le t\le\pi).\\ g(0) &= g(2\pi)=0\\ g'(t) &= \cos 2t-\sin 2t. \end{aligned}$$

The critical points of $g$ satisfy $\cos 2t=\sin 2t$, that is, $\tan 2t=1$, so $2t=\pm\dfrac{\pi}{4}$ or $\pm\dfrac{5\pi}{4}$, and $t=\pm\dfrac{\pi}{8}$ or $\pm\dfrac{5\pi}{8}$. We have

$$\begin{aligned} g\left(\frac{\pi}{8}\right) &= \frac{1}{2\sqrt{2}}-\frac{1}{2}+\frac{1}{2\sqrt{2}}=\frac{1}{\sqrt{2}}-\frac{1}{2}>0\\ g\left(-\frac{\pi}{8}\right) &= -\frac{1}{2\sqrt{2}}-\frac{1}{2}+\frac{1}{2\sqrt{2}}=-\frac{1}{2}\\ g\left(\frac{5\pi}{8}\right) &= -\frac{1}{2\sqrt{2}}-\frac{1}{2}-\frac{1}{2\sqrt{2}}=-\frac{1}{\sqrt{2}}-\frac{1}{2}\\ g\left(-\frac{5\pi}{8}\right) &= \frac{1}{2\sqrt{2}}-\frac{1}{2}-\frac{1}{2\sqrt{2}}=-\frac{1}{2}. \end{aligned}$$

Thus the maximum and minimum values of $f$ on the disk $D$ are $\dfrac{1}{\sqrt{2}}-\dfrac{1}{2}$ and $-\dfrac{1}{\sqrt{2}}-\dfrac{1}{2}$ respectively.

**4.** $f(x,y)=x+2y$ on the closed disk $x^2+y^2\le 1$. Since $f_1=1$ and $f_2=2$, $f$ has no critical points, and the maximum and minimum values of $f$, which must exist because $f$ is continuous on a closed, bounded set in the plane, must occur at boundary points of the domain, that is, points of the circle $x^2+y^2=1$. This circle can be parametrized $x=\cos t$, $y=\sin t$, so that

$$f(x,y)=f(\cos t,\sin t)=\cos t+2\sin t=g(t), \text{ say.}$$

For critical points of $g$: $0=g'(t)=-\sin t+2\cos t$. Thus $\tan t=2$, and $x=\pm 1/\sqrt{5}$, $y=\pm 2/\sqrt{5}$. The critical points are $(-1/\sqrt{5},-2/\sqrt{5})$, where $f$ has value $-\sqrt{5}$, and $(1/\sqrt{5},2/\sqrt{5})$, where $f$ has value $\sqrt{5}$. Thus the maximum and minimum values of $f(x,y)$ on the disk are $\sqrt{5}$ and $-\sqrt{5}$ respectively.

**5.** $f(x,y)=xy-x^3y^2$ on the square $S$: $0\le x\le 1$, $0\le y\le 1$.
$f_1=y-3x^2y^2=y(1-3x^2y)$,
$f_2=x-2x^3y=x(1-2x^2y)$.
$(0,0)$ is a critical point. Any other critical points must satisfy $3x^2y=1$ and $2x^2y=1$, that is, $x^2y=0$. Therefore $(0,0)$ is the only critical point, and it is on the boundary of $S$. We need therefore only consider the values of $f$ on the boundary of $S$.
On the sides $x=0$ and $y=0$ of $S$, $f(x,y)=0$.
On the side $x=1$ we have $f(1,y)=y-y^2=g(y)$, $(0\le y\le 1)$. $g$ has maximum value $1/4$ at its critical point $y=1/2$.
On the side $y=1$ we have $f(x,1)=x-x^3=h(x)$, $(0\le x\le 1)$. $h$ has critical point given by $1-3x^2=0$; only $x=1/\sqrt{3}$ is on the side of $S$.
$h\left(\dfrac{1}{\sqrt{3}}\right)=\dfrac{2}{3\sqrt{3}}>\dfrac{1}{4}$.
On the square $S$, $f(x,y)$ has minimum value 0 (on the sides $x=0$ and $y=0$ and at the corner $(1,1)$ of the square), and maximum value $2/(3\sqrt{3})$ at the point $(1/\sqrt{3},1)$. There is a smaller local maximum value at $(1,1/2)$.

**6.** $f(x,y)=xy(1-x-y)$ on the triangle $T$ shown in the figure. Evidently $f(x,y)=0$ on all three boundary segements of $T$, and $f(x,y)>0$ inside $T$. Thus the minimum value of $f$ on $T$ is 0, and the maximum value must occur at an interior critical point. For critical points:

$$0=f_1(x,y)=y(1-2x-y), \qquad 0=f_2(x,y)=x(1-x-2y).$$

The only critical points are $(0,0)$, $(1,0)$ and $(0,1)$, which are on the boundary of $T$, and $(1/3,1/3)$, which is inside $T$. The maximum value of $f$ over $T$ is $f(1/3,1/3)=1/27$.

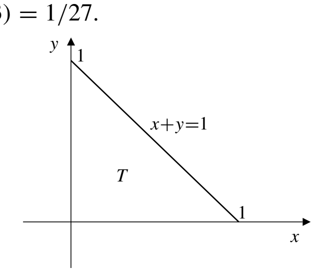


Fig. 13.2.6

**7.** Since $-1\le f(x,y)=\sin x\cos y\le 1$ everywhere, and since $f(\pi/2,0)=1$, $f(3\pi/2,0)=-1$, and both $(\pi/2,0)$ and $(3\pi/2,0)$ belong to the triangle bounded by $x=0$, $y=0$ and $x+y=2\pi$, therefore the maximum and minimum values of $f$ over that triangle must be 1 and $-1$ respectively.

**8.** $f(x,y)=\sin x\sin y\sin(x+y)$ on the triangle $T$ shown in the figure. Evidently $f(x,y)=0$ on the boundary of $T$, and $f(x,y)>0$ at all points inside $T$. Thus the minimum value of $f$ on $T$ is zero, and the maximum value must occur at an interior critical point. For critical points inside $T$ we must have

$$\begin{aligned} 0 &= f_1(x,y)=\cos x\sin y\sin(x+y)+\sin x\sin y\cos(x+y)\\ 0 &= f_2(x,y)=\sin x\cos y\sin(x+y)+\sin x\sin y\cos(x+y). \end{aligned}$$

Therefore $\cos x \sin y = \cos y \sin x$, which implies $x = y$ for points inside $T$, and

$$\begin{aligned}
&\cos x \sin x \sin 2x + \sin^2 x \cos 2x = 0\\
&2\sin^2 x \cos^2 x + 2\sin^2 x \cos^2 x - \sin^2 x = 0\\
&4\cos^2 x = 1.
\end{aligned}$$

Thus $\cos x = \pm 1/2$, and $x = \pm\pi/3$. The interior critical point is $(\pi/3, \pi/3)$, where $f$ has the value $3\sqrt{3}/8$. This is the maximum value of $f$ on $T$.

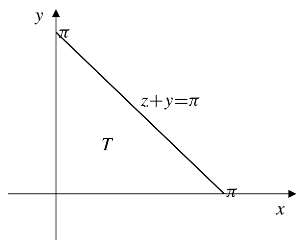


Fig. 13.2.8

**9.** $T = (x+y)e^{-x^2-y^2}$ on $D = \{(x, y) : x^2 + y^2 \le 1\}$.
For critical points:

$$\begin{aligned}
0 &= \frac{\partial T}{\partial x} = \Big(1 - 2x(x+y)\Big)e^{-x^2-y^2}\\
0 &= \frac{\partial T}{\partial y} = \Big(1 - 2y(x+y)\Big)e^{-x^2-y^2}.
\end{aligned}$$

The critical points are given by
$2x(x+y) = 1 = 2y(x+y)$, which forces $x = y$ and $4x^2 = 1$, so $x = y = \pm\frac{1}{2}$.
The two critical points are $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, both of which lie inside $D$. $T$ takes the values $\pm e^{-1/2}$ at these points.

On the boundary of $D$, $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$, so that

$$T = (\cos t + \sin t)e^{-1} = g(t), \qquad (0 \le t \le 2\pi).$$

We have $g(0) = g(2\pi) = e^{-1}$. For critical points of $g$:

$$0 = g'(t) = (\cos t - \sin t)e^{-1},$$

so $\tan t = 1$ and $t = \pi/4$ or $t = 5\pi/4$. Observe that $g(\pi/4) = \sqrt{2}e^{-1}$, and $g(5\pi/4) = -\sqrt{2}e^{-1}$.
Since $e^{-1/2} > \sqrt{2}e^{-1}$ (because $e > 2$), the maximum and minimum values of $T$ on the disk are $\pm e^{-1/2}$, the values at the interior critical points.

**10.** $f(x, y) = \dfrac{x-y}{1+x^2+y^2}$ on the half-plane $y \ge 0$.
For critical points:

$$\begin{aligned}
0 = f_1(x, y) &= \frac{1 - x^2 + y^2 + 2xy}{(1+x^2+y^2)^2}\\
0 = f_2(x, y) &= \frac{-1 - x^2 + y^2 - 2xy}{(1+x^2+y^2)^2}.
\end{aligned}$$

Any critical points must satisfy $1 - x^2 + y^2 + 2xy = 0$ and $-1 - x^2 + y^2 - 2xy = 0$, and hence $x^2 = y^2$ and $2xy = -1$. Therefore $y = -x = \pm 1/\sqrt{2}$. The only critical point in the region $y \ge 0$ is $(-1/\sqrt{2}, 1/\sqrt{2})$, where $f$ has the value $-1/\sqrt{2}$.

On the boundary $y = 0$ we have

$$f(x, 0) = \frac{x}{1+x^2} = g(x), \qquad (-\infty < x < \infty).$$

Evidently, $g(x) \to 0$ as $x \to \pm\infty$.
Since $g'(x) = \dfrac{1-x^2}{(1+x^2)^2}$, the critical points of $g$ are $x = \pm 1$. We have $g(\pm 1) = \pm\frac{1}{2}$.
The maximum and minimum values of $f$ on the upper half-plane $y \ge 0$ are $1/2$ and $-1/\sqrt{2}$ respectively.

**11.** Let $f(x, y, z) = xy^2 + yz^2$ on the ball $B$: $x^2+y^2+z^2 \le 1$.
First look for interior critical points:

$$0 = f_1 = y^2, \quad 0 = f_2 = 2xy + z^2, \quad 0 = f_3 = 2yz.$$

All points on the $x$-axis are CPs, and $f = 0$ at all such points.

Now consider the boundary sphere $z^2 = 1 - x^2 - y^2$. On it

$$f(x, y, z) = xy^2 + y(1-x^2-y^2) = xy^2 + y - x^2y - y^3 = g(x, y),$$

where $g$ is defined for $x^2 + y^2 \le 1$. Look for interior CPs of $g$:

$$\begin{aligned}
0 &= g_1 = y^2 - 2xy = y(y - 2x)\\
0 &= g_2 = 2xy + 1 - x^2 - 3y^2.
\end{aligned}$$

Case I: $y = 0$. Then $g = 0$ and $f = 0$.
Case II: $y = 2x$. Then $4x^2 + 1 - x^2 - 12x^2 = 0$, so $9x^2 = 1$ and $x = \pm 1/3$. This case produces critical points

$$\left(\frac{1}{3}, \frac{2}{3}, \pm\frac{2}{3}\right), \quad \text{where } f = \frac{4}{9}, \quad \text{and}$$
$$\left(-\frac{1}{3}, -\frac{2}{3}, \pm\frac{2}{3}\right), \quad \text{where } f = -\frac{4}{9}.$$

Now we must consider the boundary $x^2 + y^2 = 1$ of the domain of $g$. Here

$$g(x, y) = xy^2 = x(1-x^2) = x - x^3 = h(x)$$

for $-1 \le x \le 1$. At the endpoints $x = \pm 1$, $h = 0$, so $g = 0$ and $f = 0$. For CPs of $h$:

$$0 = h'(x) = 1 - 3x^2,$$

so $x = \pm 1/\sqrt{3}$ and $y = \pm\sqrt{2/3}$. The value of $h$ at such points is $\pm 2/(3\sqrt{3})$. However $2/(3\sqrt{3}) < 4/9$, so the maximum value of $f$ is 4/9, and the minimum value is $-4/9$.

**12.** Let $f(x, y, z) = xz + yz$ on the ball $x^2 + y^2 + z^2 \le 1$. First look for interior critical points:

$$0 = f_1 = z, \quad 0 = f_2 = z, \quad 0 = f_3 = x + y.$$

All points on the line $z = 0$, $x + y = 0$ are CPs, and $f = 0$ at all such points.

Now consider the boundary sphere $x^2 + y^2 + z^2 = 1$. On it

$$f(x, y, z) = (x + y)z = \pm(x + y)\sqrt{1 - x^2 - y^2} = g(x, y),$$

where $g$ has domain $x^2 + y^2 \le 1$. On the boundary of its domain, $g$ is identically 0, although $g$ takes both positive and negative values at some points inside its domain. Therefore, we need consider only critical points of $g$ in $x^2 + y^2 < 1$. For such CPs:

$$\begin{aligned} 0 = g_1 &= \sqrt{1 - x^2 - y^2} + \frac{(x + y)(-2x)}{2\sqrt{1 - x^2 - y^2}} \\ &= \frac{1 - x^2 - y^2 - x^2 - xy}{\sqrt{1 - x^2 - y^2}} \\ 0 = g_2 &= \frac{1 - x^2 - y^2 - xy - y^2}{\sqrt{1 - x^2 - y^2}}. \end{aligned}$$

Therefore $2x^2 + y^2 + xy = 1 = x^2 + 2y^2 + xy$, from which $x^2 = y^2$.
Case I: $x = -y$. Then $g = 0$, so $f = 0$.
Case II: $x = y$. Then $2x^2 + x^2 + x^2 = 1$, so $x^2 = 1/4$ and $x = \pm 1/2$. $g$ (which is really two functions depending on our choice of the "+" or "−" sign) has four CPs, two corresponding to $x = y = 1/2$ and two to $x = y = -1/2$. The values of $g$ at these four points are $\pm 1/\sqrt{2}$.

Since we have considered all points where $f$ can have extreme values, we conclude that the maximum value of $f$ on the ball is $1/\sqrt{2}$ (which occurs at the boundary points $\pm(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}})$) and minimum value $-1/\sqrt{2}$ (which occurs at the boundary points $\pm(\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}})$).

**13.** $f(x, y) = xye^{-xy}$ on $Q = \{(x, y) : x \ge 0,\ y \ge 0\}$.
Since $f(x, kx) = kx^2 e^{-kx^2} \to 0$ as $x \to \infty$ if $k > 0$, and $f(x, 0) = f(0, y) = 0$, we have $f(x, y) \to 0$ as $(x, y)$ recedes to infinity along any straight line from the origin lying in the first quadrant $Q$.
However, $f\left(x, \dfrac{1}{x}\right) = 1$ and $f(x, 0) = 0$ for all $x > 0$, even though the points $\left(x, \dfrac{1}{x}\right)$ and $(x, 0)$ become arbitrarily close together as $x$ increases. Thus $f$ does not have a limit as $x^2 + y^2 \to \infty$.
Observe that $f(x, y) = re^{-r} = g(r)$ on the hyperbola $xy = r > 0$. Since $g(r) \to 0$ as $r$ approaches 0 or $\infty$, and

$$g'(r) = (1 - r)e^{-r} = 0 \quad \Rightarrow \quad r = 1,$$

$f(x, y)$ is everywhere on $Q$ less than $g(1) = 1/e$. Thus $f$ does have a maximum value on $Q$.

**14.** $f(x, y) = xy^2 e^{-xy}$ on $Q = \{(x, y) : x \ge 0,\ y \ge 0\}$.
As in Exercise 13, $f(x, 0) = f(0, y) = 0$ and $\lim_{x\to\infty} f(x, kx) = k^2 x^3 e^{-x^2} = 0$.
Also, $f(0, y) = 0$ while $f\left(\dfrac{1}{y}, y\right) = \dfrac{y}{e} \to \infty$ as $y \to \infty$, so that $f$ has no limit as $x^2 + y^2 \to \infty$ in $Q$, and $f$ has no maximum value on $Q$.

**15.** If brewery A produces $x$ litres per month and brewery B produces $y$ litres per month, then the monthly profits of the two breweries are given by

$$P = 2x - \frac{2x^2 + y^2}{10^6}, \qquad Q = 2y - \frac{4y^2 + x^2}{2 \times 10^6}.$$

STRATEGY I. Each brewery selects its production level to maximize its own profit, and assumes its competitor does the same.
Then A chooses $x$ to satisfy

$$0 = \frac{\partial P}{\partial x} = 2 - \frac{4x}{10^6} \quad \Rightarrow \quad x = 5 \times 10^5.$$

B chooses $y$ to satisfy

$$0 = \frac{\partial Q}{\partial y} = 2 - \frac{8y}{2 \times 10^6} \quad \Rightarrow \quad y = 5 \times 10^5.$$

The total profit of the two breweries under this strategy is

$$\begin{aligned} P + Q &= 10^6 - \frac{3 \times 25 \times 10^{10}}{10^6} + 10^6 - \frac{5 \times 25 \times 10^{10}}{2 \times 10^6} \\ &= \$625,000. \end{aligned}$$

STRATEGY II. The two breweries cooperate to maximize the total profit

$$T = P + Q = 2x + 2y - \frac{5x^2 + 6y^2}{2 \times 10^6}$$

by choosing $x$ and $y$ to satisfy

$$\begin{aligned} 0 &= \frac{\partial T}{\partial x} = 2 - \frac{10x}{2 \times 10^6}, \\ 0 &= \frac{\partial T}{\partial y} = 2 - \frac{12y}{2 \times 10^6}. \end{aligned}$$

Thus $x = 4 \times 10^5$ and $y = \frac{1}{3} \times 10^6$.
In this case the total monthly profit is

$$\begin{aligned} P + Q &= 8 \times 10^5 + \frac{2}{3} \times 10^6 - \frac{80 \times 10^{10} + \frac{2}{3} \times 10^{12}}{2 \times 10^6} \\ &\approx \$733,333. \end{aligned}$$

Observe that the total profit is larger if the two breweries cooperate and fix prices to maximize it.

**16.** Let the dimensions be as shown in the figure. Then $2x + y = 100$, the length of the fence. For maximum area $A$ of the enclosure we will have $x > 0$ and $0 < \theta < \pi/2$. Since $h = x\cos\theta$, the area $A$ is

$$\begin{aligned} A &= xy\cos\theta + 2 \times \frac{1}{2}(x\sin\theta)(x\cos\theta) \\ &= x(100 - 2x)\cos\theta + x^2 \sin\theta\cos\theta \\ &= (100x - 2x^2)\cos\theta + \frac{1}{2}x^2 \sin 2\theta. \end{aligned}$$

We look for a critical point of $A$ satisfying $x > 0$ and $0 < \theta < \pi/2$.

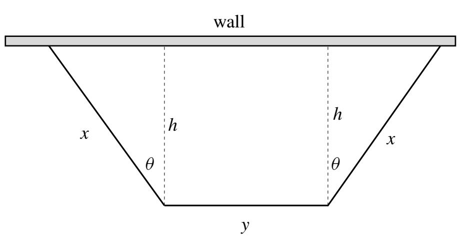


Fig. 13.2.16

$$\begin{aligned} 0 = \frac{\partial A}{\partial x} &= (100 - 4x)\cos\theta + x\sin 2\theta \\ &\Rightarrow \cos\theta(100 - 4x + 2x\sin\theta) = 0 \\ &\Rightarrow 4x - 2x\sin\theta = 100 \Rightarrow x = \frac{50}{2 - \sin\theta} \\ 0 = \frac{\partial A}{\partial \theta} &= -(100x - 2x^2)\sin\theta + x^2 \cos 2\theta \\ &\Rightarrow x(1 - 2\sin^2\theta) + 2x\sin\theta - 100\sin\theta = 0. \end{aligned}$$

Substituting the first equation into the second we obtain

$$\begin{aligned} &\frac{50}{2 - \sin\theta}\Big(1 - 2\sin^2\theta + 2\sin\theta\Big) - 100\sin\theta = 0 \\ &50(1 - 2\sin^2\theta + 2\sin\theta) = 100(2\sin\theta - \sin^2\theta) \\ &50 = 100\sin\theta. \end{aligned}$$

Thus $\sin\theta = 1/2$, and $\theta = \pi/6$.
Therefore $x = \frac{50}{2 - (1/2)} = \frac{100}{3}$, and $y = 100 - 2x = \frac{100}{3}$.

The maximum area for the enclosure is

$$A = \left(\frac{100}{3}\right)^2 \frac{\sqrt{3}}{2} + \left(\frac{100}{3}\right)^2 \frac{1}{2}\frac{\sqrt{3}}{2} = \frac{2500}{\sqrt{3}}$$

square units. All three segments of the fence will be the same length, and the bend angles will be 120°.

**17.** To maximize $Q(x, y) = 2x + 3y$ subject to

$$x \ge 0, \quad y \ge 0, \quad y \le 5, \quad x + 2y \le 12, \quad 4x + y \le 12.$$

The constraint region is shown in the figure.

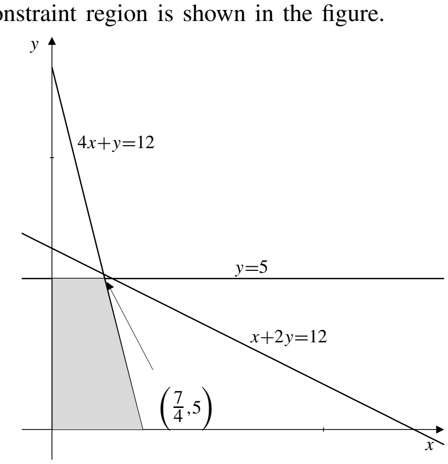


Fig. 13.2.17

Observe that any point satisfying $y \le 5$ and $4x + y \le 12$ automatically satisfies $x + 2y \le 12$. Since $y = 5$ and $4x + y = 12$ intersect at $\left(\frac{7}{4}, 5\right)$, the maximum value of $Q(x, y)$ subject to the given constraints is

$$Q\left(\frac{7}{4}, 5\right) = \frac{7}{2} + 15 = \frac{37}{2}.$$

**18.** Minimize $F(x, y, z) = 2x + 3y + 4z$ subject to

$$\begin{aligned} x &\ge 0, & y &\ge 0, & z &\ge 0, \\ x + y &\ge 2, & y + z &\ge 2, & x + z &\ge 2. \end{aligned}$$

Here the constraint region has vertices $(1, 1, 1)$, $(2, 2, 0)$, $(2, 0, 2)$, and $(0, 2, 2)$. Since $F(1, 1, 1) = 9$, $F(2, 2, 0) = 10$, $F(2, 0, 2) = 12$, and $F(0, 2, 2) = 14$, the minimum value of $F$ subject to the constraints is 9.

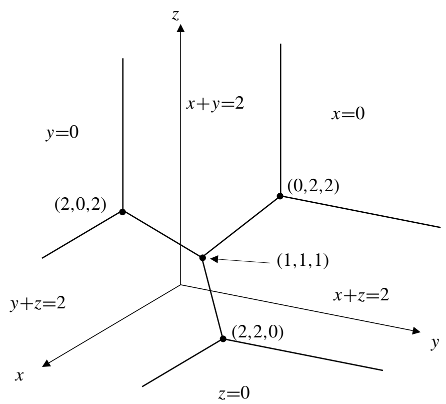


Fig. 13.2.18

**19.** Suppose that $x$ kg of deluxe fabric and $y$ kg of standard fabric are produced. Then the total revenue is

$$R = 3x + 2y.$$

The constraints imposed by raw material availability are

$$\begin{aligned}
\frac{20}{100}x + \frac{10}{100}y \le 2,000, &\iff 2x + y \le 20,000\\
\frac{50}{100}x + \frac{40}{100}y \le 6,000, &\iff 5x + 4y \le 60,000\\
\frac{30}{100}x + \frac{50}{100}y \le 6,000, &\iff 3x + 5y \le 60,000.
\end{aligned}$$

The lines $2x + y = 20,000$ and $5x + 4y = 60,000$ intersect at the point $\left(\frac{20,000}{3}, \frac{20,000}{3}\right)$, which satisfies $3x + 5y \le 60,000$, so lies in the constraint region. We have

$$f\left(\frac{20,000}{3}, \frac{20,000}{3}\right) \approx 33,333.$$

The lines $2x + y = 20,000$ and $3x + 5y = 60,000$ intersect at the point $\left(\frac{40,000}{7}, \frac{60,000}{7}\right)$, which does not satisfy $5x + 4y \le 60,000$ and so does not lie in the constraint region.

The lines $5x + 4y = 60,000$ and $3x + 5y = 60,000$ intersect at the point $\left(\frac{60,000}{13}, \frac{120,000}{13}\right)$, which satisfies $2x + y \le 20,000$ and so lies in the constraint region. We have

$$f\left(\frac{60,000}{13}, \frac{120,000}{13}\right) \approx 32,307.$$

To produce the maximum revenue, the manufacturer should produce $20,000/3 \approx 6,667$ kg of each grade of fabric.

**20.** If the developer builds $x$ houses, $y$ duplex units, and $z$ apartments, his profit will be

$$P = 40,000x + 20,000y + 16,000z.$$

The legal constraints imposed require that

$$\frac{x}{6} + \frac{y}{8} + \frac{z}{12} \le 10, \quad \text{that is } 4x + 3y + 2z \le 240,$$

and also

$$z \ge x + y.$$

Evidently we must also have $x \ge 0$, $y \ge 0$, and $z \ge 0$.

The planes $4x + 3y + 2z = 240$ and $z = x + y$ intersect where $6x + 5y = 240$. Thus the constraint region has vertices $(0, 0, 0)$, $(40, 0, 40)$, $(0, 48, 48)$, and $(0, 0, 120)$, which yield revenues of \$0, \$2,240,000, \$1,728,000, and \$1,920,000 respectively.

For maximum profit, the developer should build 40 houses, no duplex units, and 40 apartments.

## Section 13.3 Lagrange Multipliers (page 728)

**1.** First we observe that $f(x, y) = x^3y^5$ must have a maximum value on the line $x + y = 8$ because if $x \to -\infty$ then $y \to \infty$ and if $x \to \infty$ then $y \to -\infty$. In either case $f(x, y) \to -\infty$.

Let $L = x^3y^5 + \lambda(x + y - 8)$. For CPs of $L$:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 3x^2y^5 + \lambda\\
0 &= \frac{\partial L}{\partial y} = 5x^3y^4 + \lambda\\
0 &= \frac{\partial L}{\partial \lambda} = x + y - 8.
\end{aligned}$$

The first two equations give $3x^2y^5 = 5x^3y^4$, so that either $x = 0$ or $y = 0$ or $3y = 5x$. If $x = 0$ or $y = 0$ then $f(x, y) = 0$. If $3y = 5x$, then $x + \frac{5}{3}x = 8$, so $8x = 24$ and $x = 3$. Then $y = 5$, and $f(x, y) = 3^3 5^5 = 84,375$. This is the maximum value of $f$ on the line.

**2.** a) Let $D$ be the distance from $(3, 0)$ to the point $(x, y)$ on the curve $y = x^2$. Then

$$D^2 = (x - 3)^2 + y^2 = (x - 3)^2 + x^4.$$

For a minimum, $0 = \frac{dD^2}{dx} = 2(x - 3) + 4x^3$. Thus $2x^3 + x - 3 = 0$. Clearly $x = 1$ is a root of this cubic equation. Since

$$\frac{2x^3 + x - 3}{x - 1} = 2x^2 + 2x + 3,$$

and $2x^2+2x+3$ has negative discriminant, $x=1$ is the only critical point. Thus the minimum distance from $(3,0)$ to $y=x^2$ is $D=\sqrt{(-2)^2+1^4}=\sqrt{5}$ units.

b) We want to minimize $D^2=(x-3)^2+y^2$ subject to the constraint $y=x^2$. Let $L=(x-3)^2+y^2+\lambda(x^2-y)$. For critical points of $L$ we want

$$\begin{aligned} 0&=\frac{\partial L}{\partial x}=2(x-3)+2\lambda x \\ &\qquad\Rightarrow\ (1+\lambda)x-3=0 \qquad (A)\\ 0&=\frac{\partial L}{\partial y}=2y-\lambda \qquad (B)\\ 0&=\frac{\partial L}{\partial \lambda}=x^2-y. \qquad (C)\end{aligned}$$

Eliminating $\lambda$ from (A) and (B), we get $x+2xy-3=0$.
Substituting (C) then leads to $2x^3+x-3=0$, or $(x-1)(2x^2+2x+3)=0$. The only real solution is $x=1$, so the point on $y=x^2$ closest to $(3,0)$ is $(1,1)$.
Thus the minimum distance from $(3,0)$ to $y=x^2$ is $D=\sqrt{(1-3)^2+1^2}=\sqrt{5}$ units.

**3.** Let $(X,Y,Z)$ be the point on the plane $x+2y+2z=3$ closest to $(0,0,0)$.

a) The vector $\nabla(x+2y+2z)=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is perpendicular to the plane, so must be parallel to the vector $X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ from the origin to $(X,Y,Z)$. Thus

$$X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}=t(\mathbf{i}+2\mathbf{j}+2\mathbf{k}),$$

for some scalar $t$. Thus $X=t$, $Y=2t$, $Z=2t$, and, since $(X,Y,Z)$ lies on the plane,

$$3=X+2Y+2Z=t+4t+4t=9t.$$

Thus $t=\frac{1}{3}$, and we have $X=\frac{1}{3}$ and $Y=Z=\frac{2}{3}$. The minimum distance from the origin to the plane is therefore $\frac{1}{3}\sqrt{1+4+4}=1$ unit.

b) $(X,Y,Z)$ must minimize the square of the distance from the origin to $(x,y,z)$ on the plane. Thus it is a critical point of $S=x^2+y^2+z^2$. Since $x+2y+2z=3$, we have $x=3-2(y+z)$, and

$$S=S(y,z)=\Big(3-2(y+z)\Big)^2+y^2+z^2.$$

The critical points of this function are given by

$$\begin{aligned}0&=\frac{\partial S}{\partial y}=-4\Big(3-2(y+z)\Big)+2y=-12+10y+8z\\ 0&=\frac{\partial S}{\partial z}=-4\Big(3-2(y+z)\Big)+2z=-12+8y+10z.\end{aligned}$$

Therefore $Y=Z=\frac{2}{3}$ and $X=\frac{1}{3}$, and the distance is 1 unit as in part (a).

c) The point $(X,Y,Z)$ must be a critical point of the Lagrangian function

$$L=x^2+y^2+z^2+\lambda(x+2y+2z-3).$$

To find these critical points we have

$$\begin{aligned}0&=\frac{\partial L}{\partial x}=2x+\lambda\\ 0&=\frac{\partial L}{\partial y}=2y+2\lambda\\ 0&=\frac{\partial L}{\partial z}=2z+2\lambda\\ 0&=\frac{\partial L}{\partial \lambda}=x+2y+2z-3.\end{aligned}$$

The first three equations yield $y=z=-\lambda$, $x=-\lambda/2$. Substituting these into the fourth equation we get $\lambda=-\frac{2}{3}$, so that the critical point is once again $\left(\frac{1}{3},\frac{2}{3},\frac{2}{3}\right)$, whose distance from the origin is 1 unit.

**4.** Let $f(x,y,z)=x+y-z$, and define the Lagrangian

$$L=x+y-z+\lambda(x^2+y^2+z^2-1).$$

Solutions to the constrained problem will be found among the critical points of $L$. To find these we have

$$\begin{aligned}0&=\frac{\partial L}{\partial x}=1+2\lambda x,\\ 0&=\frac{\partial L}{\partial y}=1+2\lambda y,\\ 0&=\frac{\partial L}{\partial z}=-1+2\lambda z,\\ 0&=\frac{\partial L}{\partial \lambda}=x^2+y^2+z^2-1.\end{aligned}$$

Therefore $2\lambda x=2\lambda y=-2\lambda z$. Either $\lambda=0$ or $x=y=-z$. $\lambda=0$ is not possible. (It implies $0=1$ from the first equation.) From $x=y=-z$ we obtain $1=x^2+y^2+z^2=3x^2$, so $x=\pm\dfrac{1}{\sqrt{3}}$. $L$ has critical points at $\left(\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}},-\dfrac{1}{\sqrt{3}}\right)$ and $\left(\dfrac{1}{-\sqrt{3}},-\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}\right)$. At the first $f=\sqrt{3}$, which is the maximum value of $f$ on the sphere; at the second $f=-\sqrt{3}$, which is the minimum value.

**5.** The distance $D$ from $(2,1,-2)$ to $(x,y,z)$ is given by

$$D^2=(x-2)^2+(y-1)^2+(z+2)^2.$$

We can extremize $D$ by extremizing $D^2$. If $(x, y, z)$ lies on the sphere $x^2+y^2+z^2=1$, we should look for critical points of the Lagrangian

$$L = (x-2)^2 + (y-1)^2 + (z+2)^2 + \lambda(x^2+y^2+z^2-1).$$

Thus

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2(x-2) + 2\lambda x \quad \Leftrightarrow \quad x = \frac{2}{1+\lambda} \\
0 &= \frac{\partial L}{\partial y} = 2(y-1) + 2\lambda y \quad \Leftrightarrow \quad y = \frac{1}{1+\lambda} \\
0 &= \frac{\partial L}{\partial z} = 2(z+2) + 2\lambda z \quad \Leftrightarrow \quad z = \frac{-2}{1+\lambda} \\
0 &= \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1.
\end{aligned}$$

Substituting the solutions of the first three equations into the fourth, we obtain

$$\begin{aligned}
&\frac{1}{(1+\lambda)^2}(4+1+4) = 1 \\
&(1+\lambda)^2 = 9 \\
&1+\lambda = \pm 3.
\end{aligned}$$

Thus we must consider the two points $P = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, and $Q = \left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ for giving extreme values for $D$. At $P$, $D = 2$. At $Q$, $D = 4$. Thus the greatest and least distances from $(2, 1, -2)$ to the sphere $x^2+y^2+z^2=1$ are 4 units and 2 units respectively.

**6.** Let $L = x^2+y^2+z^2+\lambda(xyz^2-2)$. For critical points:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2x + \lambda yz^2 \quad \Leftrightarrow \quad -\lambda xyz^2 = 2x^2 \\
0 &= \frac{\partial L}{\partial y} = 2y + \lambda xz^2 \quad \Leftrightarrow \quad -\lambda xyz^2 = 2y^2 \\
0 &= \frac{\partial L}{\partial z} = 2z + 2\lambda xyz \quad \Leftrightarrow \quad -\lambda xyz^2 = z^2 \\
0 &= \frac{\partial L}{\partial \lambda} = xyz^2 - 2.
\end{aligned}$$

From the first three equations, $x^2 = y^2$ and $z^2 = 2x^2$. The fourth equation then gives $x^2y^24z^4 = 4$, or $x^8 = 1$. Thus $x^2 = y^2 = 1$ and $z^2 = 2$.
The shortest distance from the origin to the surface $xyz^2 = 2$ is

$$\sqrt{1+1+2} = 2 \text{ units.}$$

**7.** We want to minimize $V = \dfrac{4\pi abc}{3}$ subject to the constraint $\dfrac{1}{a^2}+\dfrac{4}{b^2}+\dfrac{1}{c^2} = 1$. Note that $abc$ cannot be zero. Let

$$L = \frac{4\pi abc}{3} + \lambda\left(\frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}-1\right).$$

For critical points of $L$:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial a} = \frac{4\pi bc}{3} - \frac{2\lambda}{a^3} \quad \Leftrightarrow \quad \frac{2\pi abc}{3} = \frac{\lambda}{a^2} \\
0 &= \frac{\partial L}{\partial b} = \frac{4\pi ac}{3} - \frac{8\lambda}{b^3} \quad \Leftrightarrow \quad \frac{2\pi abc}{3} = \frac{4\lambda}{b^2} \\
0 &= \frac{\partial L}{\partial c} = \frac{4\pi ab}{3} - \frac{2\lambda}{c^3} \quad \Leftrightarrow \quad \frac{2\pi abc}{3} = \frac{\lambda}{c^2} \\
0 &= \frac{\partial L}{\partial \lambda} = \frac{1}{a^2}+\frac{4}{b^2}+\frac{1}{c^2}-1.
\end{aligned}$$

$abc \neq 0$ implies $\lambda \neq 0$, and so we must have

$$\frac{1}{a^2} = \frac{4}{b^2} = \frac{1}{c^2} = \frac{1}{3},$$

so $a = \pm\sqrt{3}$, $b = \pm 2\sqrt{3}$, and $c = \pm\sqrt{3}$.

**8.** Let $L = x^2+y^2+\lambda(3x^2+2xy+3y^2-16)$. We have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2x + 6\lambda x + 2\lambda y \qquad (A) \\
0 &= \frac{\partial L}{\partial y} = 2y + 6\lambda y + 2\lambda x. \qquad (B)
\end{aligned}$$

Multiplying $(A)$ by $y$ and $(B)$ by $x$ and subtracting we get

$$2\lambda(y^2 - x^2) = 0.$$

Thus, either $\lambda = 0$, or $y = x$, or $y = -x$.
$\lambda = 0$ is not possible, since it implies $x = 0$ and $y = 0$, and the point $(0, 0)$ does not lie on the given ellipse.
If $y = x$, then $8x^2 = 16$, so $x = y = \pm\sqrt{2}$.
If $y = -x$, then $4x^2 = 16$, so $x = -y = \pm 2$.
The points on the ellipse nearest the origin are $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$. The points farthest from the origin are $(2, -2)$ and $(-2, 2)$. The major axis of the ellipse lies along $y = -x$ and has length $4\sqrt{2}$. The minor axis lies along $y = x$ and has length 4.

**9.** Let $L = xyz + \lambda(x^2+y^2+z^2-12)$. For CPs of $L$:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = yz + 2\lambda x \qquad (A) \\
0 &= \frac{\partial L}{\partial y} = xz + 2\lambda y \qquad (B) \\
0 &= \frac{\partial L}{\partial z} = xy + 2\lambda z \qquad (C) \\
0 &= \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-12. \qquad (D)
\end{aligned}$$

Multiplying equations (A), (B), and (C) by $x$, $y$, and $z$, respectively, and subtracting in pairs, we conclude that $\lambda x^2 = \lambda y^2 = \lambda z^2$, so that either $\lambda = 0$ or $x^2 = y^2 = z^2$. If $\lambda = 0$, then (A) implies that $yz = 0$, so $xyz = 0$. If $x^2 = y^2 = z^2$, then (D) gives $3x^2 = 12$, so $x^2 = 4$. We obtain eight points $(x, y, z)$ where each coordinate is either 2 or $-2$. At four of these points $xyz$ =8, which is the maximum value of $xyz$ on the sphere. At the other four $xyz = -8$, which is the minimum value.

**10.** Let $L = x + 2y - 3z + \lambda(x^2 + 4y^2 + 9z^2 - 108)$. For CPs of $L$:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 1 + 2\lambda x && (A)\\
0 &= \frac{\partial L}{\partial y} = 2 + 8\lambda y && (B)\\
0 &= \frac{\partial L}{\partial z} = -3 + 18\lambda z && (C)\\
0 &= \frac{\partial L}{\partial \lambda} = x^2 + 4y^2 + 9z^2 - 108. && (D)
\end{aligned}$$

From (A), (B), and (C),

$$\lambda = -\frac{1}{2x} = -\frac{2}{8y} = \frac{3}{18z},$$

so $x = 2y = -3z$. From (D):

$$x^2 + 4\left(\frac{x^2}{4}\right) + 9\left(\frac{x^2}{9}\right) = 108,$$

so $x^2 = 36$, and $x = \pm 6$. There are two CPs: $(6, 3, -2)$ and $(-6, -3, 2)$. At the first, $x + 2y - 3z = 18$, the maximum value, and at the second, $x + 2y - 3z = -18$, the minimum value.

**11.** Let $L = x + \lambda(x + y - z) + \mu(x^2 + 2y^2 + 2z^2 - 8)$. For critical points of $L$:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 1 + \lambda + 2\mu x && (A)\\
0 &= \frac{\partial L}{\partial y} = \lambda + 4\mu y && (B)\\
0 &= \frac{\partial L}{\partial z} = -\lambda + 4\mu z && (C)\\
0 &= \frac{\partial L}{\partial \lambda} = x + y - z && (D)\\
0 &= \frac{\partial L}{\partial \mu} = x^2 + 2y^2 + 2z^2 - 8. && (E)
\end{aligned}$$

From (B) and (C) we have $\mu(y + z) = 0$. Thus $\mu = 0$ or $y + z = 0$.

CASE I. $\mu = 0$. Then $\lambda = 0$ by (B), and $1 = 0$ by (A), so this case is not possible.

CASE II. $y + z = 0$. Then $z = -y$ and, by (D), $x = -2y$. Therefore, by (E), $4y^2 + 2y^2 + 2y^2 = 8$, and so $y = \pm 1$. From this case we obtain two points: $(2, -1, 1)$ and $(-2, 1, -1)$.
The function $f(x, y, z) = x$ has maximum value 2 and minimum value $-2$ when restricted to the curve $x + y = z$, $x^2 + 2y^2 + 2z^2 = 8$.

**12.** Let $L = x^2 + y^2 + z^2 + \lambda(x^2 + y^2 - z^2) + \mu(x - 2z - 3)$. For critical points of $L$:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2x(1 + \lambda) + \mu && (A)\\
0 &= \frac{\partial L}{\partial y} = 2y(1 + \lambda) && (B)\\
0 &= \frac{\partial L}{\partial z} = 2z(1 - \lambda) - 2\mu && (C)\\
0 &= \frac{\partial L}{\partial \lambda} = x^2 + y^2 - z^2 && (D)\\
0 &= \frac{\partial L}{\partial \mu} = x - 2z - 3. && (E)
\end{aligned}$$

From (B), either $y = 0$ or $\lambda = -1$.

CASE I. $y = 0$. Then (D) implies $x = \pm z$.
If $x = z$ then (E) implies $z = -3$, so we get the point $(-3, 0, -3)$.
If $x = -z$ then (E) implies $z = -1$, so we get the point $(1, 0, -1)$.

CASE II. $\lambda = -1$. Then (A) implies $\mu = 0$ and (C) implies $z = 0$. By (D), $x = y = 0$, and this contradicts (E), so this case is not possible.

If $f(x, y, z) = x^2 + y^2 + z^2$, then $f(-3, 0, -3) = 18$ is the maximum value of $f$ on the ellipse $x^2 + y^2 = z^2$, $x - 2z = 3$, and $f(1, 0, -1) = 2$ is the minimum value.

**13.** Let $L = 4 - z + \lambda(x^2 + y^2 - 8) + \mu(x + y + z - 1)$. For critical points of $L$:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2\lambda x + \mu && (A)\\
0 &= \frac{\partial L}{\partial y} = 2\lambda y + \mu && (B)\\
0 &= \frac{\partial L}{\partial z} = -1 + \mu && (C)\\
0 &= \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 8 && (D)\\
0 &= \frac{\partial L}{\partial \mu} = x + y + z - 1. && (E)
\end{aligned}$$

From (C), $\mu = 1$. From (A) and (B), $\lambda(x - y) = 0$, so either $\lambda = 0$ or $x = y$.

CASE I. $\lambda = 0$. Then $\mu = 0$ by (A), and this contradicts (C), so this case is not possible.

CASE II. $x = y$. Then $x = y = \pm 2$ by (D).
If $x = y = 2$, then $z = -3$ by (E).
If $x = y = -2$, then $z = 5$ by (E).
Thus we have two points, $(2, 2, -3)$ and $(-2, -2, 5)$, where $f(x, y, z) = 4 - z$ takes the values 7 (maximum), and $-1$ (minimum) respectively.

**14.** The max and min values of $f(x, y, z) = x + y^2 z$ subject to the constraints $y^2 + z^2 = 2$ and $z = x$ will be found among the critical points of

$$L = x + y^2 z + \lambda(y^2 + z^2 - 2) + \mu(z - x).$$

Thus

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 1 - \mu = 0, \\
0 &= \frac{\partial L}{\partial y} = 2yz + 2\lambda y = 0, \\
0 &= \frac{\partial L}{\partial z} = y^2 + 2\lambda z + \mu = 0, \\
0 &= \frac{\partial L}{\partial \lambda} = y^2 + z^2 - 2, \\
0 &= \frac{\partial L}{\partial \mu} = z - x.
\end{aligned}$$

From the first equation $\mu = 1$. From the second, either $y = 0$ or $z = -\lambda$.
If $y = 0$ then $z^2 = 2$, $z = x$, so critical points are $(\sqrt{2}, 0, \sqrt{2})$ and $(-\sqrt{2}, 0, -\sqrt{2})$. $f$ has the values $\pm\sqrt{2}$ at these points. If $z = -\lambda$ then $y^2 - 2z^2 + 1 = 0$. Thus $2z^2 - 1 = 2 - z^2$, or $z^2 = 1$, $z = \pm 1$. This leads to critical points $(1, \pm 1, 1)$ and $(-1, \pm 1, -1)$ where $f$ has values $\pm 2$. The maximum value of $f$ subject to the constraints is 2; the minimum value is $-2$.

**15.** Let

$$\begin{aligned}
L = (x-a)^2 + (y-b)^2 + (z-c)^2 + \lambda(x-y) + \mu(y-z) \\
+ \sigma(a+b) + \tau(c-2).
\end{aligned}$$

For critical points of $L$, we have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 2(x-a) + \lambda && (A) \\
0 &= \frac{\partial L}{\partial y} = 2(y-b) - \lambda + \mu && (B) \\
0 &= \frac{\partial L}{\partial z} = 2(z-c) - \mu && (C) \\
0 &= \frac{\partial L}{\partial a} = -2(x-a) + \sigma && (D) \\
0 &= \frac{\partial L}{\partial b} = -2(y-b) + \sigma && (E) \\
0 &= \frac{\partial L}{\partial c} = -2(z-c) + \tau && (F) \\
0 &= \frac{\partial L}{\partial \lambda} = x - y && (G) \\
0 &= \frac{\partial L}{\partial \mu} = y - z && (H) \\
0 &= \frac{\partial L}{\partial \sigma} = a + b && (I) \\
0 &= \frac{\partial L}{\partial \tau} = c - 2. && (J)
\end{aligned}$$

Subtracting (D) and (E) we get $x - y = a - b$. From (G), $x = y$, and therefore $a = b$. From (I), $a = b = 0$, and from (J), $c = 2$.
Adding (A), (B) and (C), we get $x+y+z = a+b+c = 2$.
From (G) and (H), $x = y = z = 2/3$.
The minimum distance between the two lines is

$$\sqrt{\left(\frac{2}{3}-0\right)^2 + \left(\frac{2}{3}-0\right)^2 + \left(\frac{2}{3}-2\right)^2} = \sqrt{\frac{24}{9}} = \frac{2\sqrt{6}}{3} \text{ units.}$$

**16.** Let $L = x_1 + x_2 + \cdots + x_n + \lambda(x_1^2 + x_2^2 + \cdots + x_n^2 - 1)$. For critical points of $L$ we have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x_1} = 1 + 2\lambda x_1, \quad \ldots \quad 0 = \frac{\partial L}{\partial x_n} = 1 + 2\lambda x_n \\
0 &= \frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 + \cdots + x_n^2 - 1.
\end{aligned}$$

The first $n$ equations give

$$x_1 = x_2 = \cdots = x_n = -\frac{1}{2\lambda},$$

and the final equation gives

$$\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} + \cdots + \frac{1}{4\lambda^2} = 1,$$

so that $4\lambda^2 = n$, and $\lambda = \pm\sqrt{n}/2$.
The maximum and minimum values of $x_1 + x_2 + \cdots + x_n$ subject to $x_1^2 + \cdots + x_n^2 = 1$ are $\pm\frac{n}{2\lambda}$, that is, $\sqrt{n}$ and $-\sqrt{n}$ respectively.

**17.** Let $L = x_1 + 2x_2 + \cdots + nx_n + \lambda(x_1^2 + x_2^2 + \cdots + x_n^2 - 1)$. For critical points of $L$ we have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x_1} = 1 + 2\lambda x_1 \quad \Leftrightarrow \quad x_1 = -\frac{1}{2\lambda} \\
0 &= \frac{\partial L}{\partial x_2} = 2 + 2\lambda x_2 \quad \Leftrightarrow \quad x_2 = -\frac{2}{2\lambda} \\
0 &= \frac{\partial L}{\partial x_3} = 3 + 2\lambda x_3 \quad \Leftrightarrow \quad x_3 = -\frac{3}{2\lambda} \\
&\vdots \\
0 &= \frac{\partial L}{\partial x_n} = n + 2\lambda x_n \quad \Leftrightarrow \quad x_n = -\frac{n}{2\lambda} \\
0 &= \frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 + \cdots + x_n^2 - 1.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{4\lambda^2} + \frac{4}{4\lambda^2} + \frac{9}{4\lambda^2} + \cdots + \frac{n^2}{4\lambda^2} = 1 \\
&4\lambda^2 = 1 + 4 + 9 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \\
&\lambda = \pm\frac{1}{2}\sqrt{\frac{n(n+1)(2n+1)}{6}}.
\end{aligned}$$

Thus the maximum and minimum values of $x_1 + 2x_2 + \cdots + nx_n$ over the hypersphere $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$ are

$$\pm\sqrt{\frac{6}{n(n+1)(2n+1)}}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$
$$= \pm\sqrt{\frac{n(n+1)(2n+1)}{6}}.$$

**18.** Let the width, depth, and height of the box be $x$, $y$ and $z$ respectively. We want to minimize the surface area

$$S = xy + 2xz + 2yz$$

subject to the constraint that $xyz = V$, where $V$ is a given positive volume. Let

$$L = xy + 2xz + 2yz + \lambda(xyz - V).$$

For critical points of $L$,

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = y + 2z + \lambda yz \quad \Leftrightarrow \quad -\lambda xyz = xy + 2xz \\
0 &= \frac{\partial L}{\partial y} = x + 2z + \lambda xz \quad \Leftrightarrow \quad -\lambda xyz = xy + 2yz \\
0 &= \frac{\partial L}{\partial z} = 2x + 2y + \lambda xy \quad \Leftrightarrow \quad -\lambda xyz = 2xz + 2yz \\
0 &= \frac{\partial L}{\partial \lambda} = xyz - V.
\end{aligned}$$

From the first three equations, $xy = 2xz = 2yz$. Since $x$, $y$, and $z$ are all necessarily positive, we must therefore have $x = y = 2z$. Thus the most economical box with no top has width and depth equal to twice the height.

**19.** We want to maximize $V = xyz$ subject to $4x+2y+z = 2$. Let

$$L = xyz + \lambda(4x + 2y + z - 2).$$

For critical points of $L$,

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = yz + 4\lambda \quad \Leftrightarrow \quad xyz + 4\lambda x = 0 \\
0 &= \frac{\partial L}{\partial y} = xz + 2\lambda \quad \Leftrightarrow \quad xyz + 2\lambda y = 0 \\
0 &= \frac{\partial L}{\partial z} = xy + \lambda \quad \Leftrightarrow \quad xyz + \lambda z = 0 \\
0 &= \frac{\partial L}{\partial \lambda} = 4x + 2y + z - 2 = 0.
\end{aligned}$$

The first three equations imply that $z = 2y = 4x$ (since we cannot have $\lambda = 0$ if $V$ is positive). The fourth equation then implies that $12x = 2$. Hence $x = 1/6$, $y = 1/3$, and $z = 2/3$.
The largest box has volume

$$V = \frac{1}{6} \times \frac{1}{3} \times \frac{2}{3} = \frac{1}{27} \text{ cubic units.}$$

**20.** We want to maximize $xyz$ subject to $xy+2yz+3xz = 18$. Let

$$L = xyz + \lambda(xy + 2yz + 3xz - 18).$$

For critical points of $L$,

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = yz + \lambda(y + 3z) \quad \Leftrightarrow \quad -xyz = \lambda(xy + 3xz) \\
0 &= \frac{\partial L}{\partial y} = xz + \lambda(x + 2z) \quad \Leftrightarrow \quad -xyz = \lambda(xy + 2yz) \\
0 &= \frac{\partial L}{\partial z} = xy + \lambda(2y + 3x) \quad \Leftrightarrow \quad -xyz = \lambda(2yz + 3xz) \\
0 &= \frac{\partial L}{\partial \lambda} = xy + 2yz + 3xz - 18.
\end{aligned}$$

From the first three equations $xy = 2yz = 3xz$. From the fourth equation, the sum of these expressions is 18. Thus

$$xy = 2yz = 3xz = 6.$$

Thus the maximum volume of the box is

$$V = xyz = \sqrt{(xy)(yz)(xz)} = \sqrt{6 \times 3 \times 2} = 6 \text{ cubic units.}$$

**21.** Let the width, depth, and height of the box be $x$, $y$, and $z$ as shown in the figure. Let the cost per unit area of the back and sides be \$$k$. Then the cost per unit area of the front and bottom is \$$5k$. We want to minimize

$$C = 5k(xz + xy) + k(2yz + xz)$$

subject to the constraint $xyz = V$ (constant). Let

$$L = k(5xy + 6xz + 2yz) + \lambda(xyz - V).$$

For critical points of $L$,

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = 5ky + 6kz + \lambda yz \quad \Leftrightarrow \quad -\lambda xyz = 5kxy + 6kxz \\
0 &= \frac{\partial L}{\partial y} = 5kx + 2kz + \lambda xz \quad \Leftrightarrow \quad -\lambda xyz = 5kxy + 2kyz \\
0 &= \frac{\partial L}{\partial z} = 6kx + 2ky + \lambda xy \quad \Leftrightarrow \quad -\lambda xyz = 6kxz + 2kyz \\
0 &= \frac{\partial L}{\partial \lambda} = xyz - V.
\end{aligned}$$

From the first three of these equations we obtain $5xy = 6xz = 2yz$. Thus $y = 3x$ and $z = \dfrac{5x}{2}$. From the fourth equation, $V = xyz = \dfrac{15}{2}x^3$.
The largest box has width $\left(\dfrac{2V}{15}\right)^{1/3}$, depth $3\left(\dfrac{2V}{15}\right)^{1/3}$, and height $\dfrac{5}{2}\left(\dfrac{2V}{15}\right)^{1/3}$.

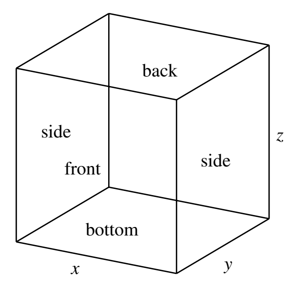


Fig. 13.3.21

**22.** $f(x, y, z) = xy + z^2$ on $B = \{(x, y, z) : x^2 + y^2 + z^2 \le 1\}$.
For critical points of $f$,

$$0 = f_1(x, y, z) = y, \qquad 0 = f_2(x, y, z) = x,$$
$$0 = f_3(x, y, z) = 2z.$$

Thus the only critical point is the interior point $(0, 0, 0)$, where $f$ has the value 0, evidently neither a maximum nor a minimum. The maximum and minimum must therefore occur on the boundary of $B$, that is, on the sphere
$x^2 + y^2 + z^2 = 1$. Let

$$L = xy + z^2 + \lambda(x^2 + y^2 + z^2 - 1).$$

For critical points of $L$,

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = y + 2\lambda x && (A)\\
0 &= \frac{\partial L}{\partial y} = x + 2\lambda y && (B)\\
0 &= \frac{\partial L}{\partial z} = 2z(1 + \lambda) && (C)\\
0 &= \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1. && (D)
\end{aligned}$$

From (C) either $z = 0$ or $\lambda = -1$.

CASE I. $z = 0$. (A) and (B) imply that $y^2 = x^2$ and (D) then implies that $x^2 = y^2 = 1/2$. At the four points

$$\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right)$$

$f$ takes the values $\frac{1}{2}$ and $-\frac{1}{2}$.

CASE II. $\lambda = -1$. (A) and (B) imply that $x = y = 0$, and so by (D), $z = \pm 1$. $f$ has the value 1 at the points $(0, 0, \pm 1)$.
Thus the maximum and minimum values of $f$ on $B$ are 1 and $-1/2$ respectively.

**23.** In this problem we do the boundary analysis for Exercise 22 using the suggested parametrization of the sphere
$x^2 + y^2 + z^2 = 1$. We have

$$\begin{aligned}
f(x, y, z) &= xy + z^2\\
&= \sin^2\phi \sin\theta \cos\theta + \cos^2\phi\\
&= \frac{1}{2}\sin^2\phi \sin 2\theta + \cos^2\phi\\
&= g(\phi, \theta)
\end{aligned}$$

for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$. For critical points of $g$,

$$\begin{aligned}
0 = g_1(\phi, \theta) &= \sin\phi\cos\phi\sin 2\theta - 2\sin\phi\cos\phi\\
&= \sin\phi\cos\phi(\sin 2\theta - 2)\\
0 = g_2(\phi, \theta) &= \sin^2\phi\cos 2\theta.
\end{aligned}$$

The first of these equations implies that either $\sin\phi = 0$ or $\cos\phi = 0$.
If $\sin\phi = 0$, then both equations are satisfied. Since $\cos\phi = \pm 1$ in this case, we have $g(\phi, \theta) = 1$.
If $\cos\phi = 0$, then $\sin\phi = \pm 1$, and the second equation requires $\cos 2\theta = 0$. Thus $\theta = \pm\frac{\pi}{4}$ or $\pm\frac{3\pi}{4}$. In this case $g(\phi, \theta) = \pm\frac{1}{2}$.
Again we find that $f(x, y, z) = xy + z^2$ has maximum value 1 and minimum value $-\frac{1}{2}$ when restricted to the surface of the ball $B$. These are the maximum and minimum values for the whole ball as noted in Exercise 22.

**24.** Let $L = \sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2} + \lambda(x + y + z - \pi)$. Then

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x} = \frac{1}{2}\cos\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2} + \lambda && (A)\\
0 &= \frac{\partial L}{\partial y} = \frac{1}{2}\sin\frac{x}{2}\cos\frac{y}{2}\sin\frac{z}{2} + \lambda && (B)\\
0 &= \frac{\partial L}{\partial z} = \frac{1}{2}\sin\frac{x}{2}\sin\frac{y}{2}\cos\frac{z}{2} + \lambda. && (C)
\end{aligned}$$

For any triangle we must have $0 \le x \le \pi$, $0 \le y \le \pi$ and $0 \le z \le \pi$. Also

$$P = \sin\frac{x}{2}\sin\frac{y}{2}\sin\frac{z}{2}$$

is 0 if any of $x$, $y$ or $z$ is 0 or $\pi$. Subtracting equations $(A)$ and $(B)$ gives

$$\frac{1}{2}\sin\frac{z}{2}\sin\frac{x - y}{2} = 0.$$

It follows that we must have $x = y$; all other possibilities lead to a zero value for $P$. Similarly, $y = z$. Thus the triangle for which $P$ is maximum must be equilateral:
$x = y = z = \pi/3$. Since $\sin(\pi/3) = 1/2$, the maximum value of $P$ is 1/8.

**25.** We are given that $g_2(a,b) \neq 0$, and therefore that the equation $g(x,y) = C$ has a solution of the form $y = h(x)$ valid near $(a,b)$. Since $g\big(x, h(x)\big) = C$ holds identically for $x$ near $a$, we must have

$$0 = \left(\frac{d}{dx} g\big(x, h(x)\big)\right)\bigg|_{x=a} = g_1(a,b) + g_2(a,b)h'(a).$$

If $f(x,y)$, subject to the constraint $g(x,y) = C$, has an extreme value at $(a,b)$, then $F(x) = f\big(x, h(x)\big)$ has an extreme value at $x = a$, so

$$0 = F'(a) = f_1(a,b) + f_2(a,b)h'(a).$$

Together these equations imply that
$g_1(a,b)f_2(a,b) = g_2(a,b)f_1(a,b)$, and therefore that

$$\frac{f_1(a,b)}{g_1(a,b)} = \frac{f_2(a,b)}{g_2(a,b)} = -\lambda \quad \text{(say)}.$$

(Since $g_2(a,b) \neq 0$, therefore, if $g_1(a,b) = 0$, then $f_1(a,b) = 0$ also.) It follows that

$$0 = f_1(a,b) + \lambda g_1(a,b), \qquad 0 = f_2(a,b) + \lambda g_2(a,b),$$

so $(a,b)$ is a critical point of $L = f(x,y) + \lambda g(x,y)$.

**26.** As can be seen in the figure, the minimum distance from $(0,-1)$ to points of the semicircle $y = \sqrt{1-x^2}$ is $\sqrt{2}$, the closest points to $(0,-1)$ on the semicircle being $(\pm 1, 0)$. These points will not be found by the method of Lagrange multipliers because the level curve $f(x,y) = 2$ of the function $f$ giving the square of the distance from $(x,y)$ to $(0,-1)$ is not tangent to the semicircle at $(\pm 1, 0)$. This could only have happened because $(\pm 1, 0)$ are *endpoints* of the semicircle.

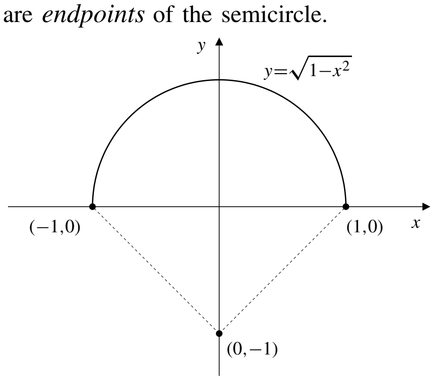


Fig. 13.3.26

**27.** If $f(x,y)$ has an extreme value on $g(x,y) = 0$ at a point $(x_0, y_0)$ where $\nabla g \neq \mathbf{0}$, and if $\nabla f$ exists at that point, then $\nabla f(x_0, y_0)$ must be parallel to $\nabla g(x_0, y_0)$;

$$\nabla f(x_0, y_0) + \lambda \nabla g(x_0, y_0) = 0$$

as shown in the text. The argument given there holds whether or not $\nabla f(x_0, y_0)$ is $\mathbf{0}$. However, if

$$\nabla f(x_0, y_0) = \mathbf{0}$$

then we will have $\lambda = 0$.

## Section 13.4 The Method of Least Squares (page 734)

**1.** If the power plant is located at $(x,y)$, then $x$ and $y$ should minimize (and hence be a critical point of)

$$S = \sum_{i=1}^{n}\Big[(x - x_i)^2 + (y - y_2)^2\Big].$$

Thus we must have

$$0 = \frac{\partial S}{\partial x} = 2\sum_{i=1}^{n}(x - x_i) = 2\left(nx - \sum_{i=1}^{n} x_i\right)$$
$$0 = \frac{\partial S}{\partial y} = 2\sum_{i=1}^{n}(y - y_i) = 2\left(ny - \sum_{i=1}^{n} y_i\right).$$

Thus $x = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i = \bar{x}$, and $y = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} y_i = \bar{y}$.
Place the power plant at the position whose coordinates are the averages of the coordinates of the machines.

**2.** We want to minimize $S = \sum_{i=1}^{n}(ax_i^2 - y_i)^2$. Thus

$$0 = \frac{dS}{da} = \sum_{i=1}^{n} 2(ax_i^2 - y_i)x_i^2$$
$$= 2\sum_{i=1}^{n}(ax_i^4 - x_i^2 y_i),$$

and $a = \left(\sum_{i=1}^{n} x_i^2 y_i\right) \Big/ \left(\sum_{i=1}^{n} x_i^4\right)$.

**3.** We minimize $S = \sum_{i=1}^{n}(ae^{x_i} - y_i)^2$. Thus

$$0 = \frac{dS}{da} = 2\sum_{i=1}^{n}(ae^{x_i} - y_i)e^{x^i},$$

and $a = \left(\sum_{i=1}^{n} y_i e^{x_i}\right) \Big/ \left(\sum_{i=1}^{n} e^{2x_i}\right)$.

**4.** We choose $a$, $b$, and $c$ to minimize

$$S = \sum_{i=1}^{n}\Big(ax_i + by_i + c - z_i\Big)^2.$$

Thus

$$0 = \frac{\partial S}{\partial a} = 2\sum_{i=1}^{n}(ax_i + by_i + c - z_i)x_i$$
$$0 = \frac{\partial S}{\partial b} = 2\sum_{i=1}^{n}(ax_i + by_i + c - z_i)y_i$$
$$0 = \frac{\partial S}{\partial c} = 2\sum_{i=1}^{n}(ax_i + by_i + c - z_i).$$

Let $A = \sum x_i^2$, $B = \sum x_i y_i$, $C = \sum x_i$, $D = \sum y_i^2$, $E = \sum y_i$, $F = \sum x_i z_i$, $G = \sum y_i z_i$, and $H = \sum z_i$. In terms of these quantities the above equations become

$$\begin{array}{ccccccc} Aa & + & Bb & + & Cc & = & F \\ Ba & + & Db & + & Ec & = & G \\ Ca & + & Eb & + & nc & = & H. \end{array}$$

By Cramer's Rule (Theorem 5 of Section 1.6) the solution is

$$a = \frac{1}{\Delta}\begin{vmatrix} F & B & C \\ G & D & E \\ H & E & n \end{vmatrix}, \qquad b = \frac{1}{\Delta}\begin{vmatrix} A & F & C \\ B & G & E \\ C & H & n \end{vmatrix},$$
$$c = \frac{1}{\Delta}\begin{vmatrix} A & B & F \\ B & D & G \\ C & E & H \end{vmatrix}, \quad \text{where } \Delta = \begin{vmatrix} A & B & C \\ B & D & E \\ C & E & n \end{vmatrix}.$$

**5.** If $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{w} = (1, \ldots, 1)$, and $\mathbf{p} = a\mathbf{x} + b\mathbf{y} + c\mathbf{w}$, we want to choose $a$, $b$, and $c$ so that $\mathbf{p}$ is the vector projection of $\mathbf{z}$ onto the subspace of $\mathbb{R}^3$ spanned by $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{w}$. Thus $\mathbf{p} - \mathbf{z}$ must be perpendicular to each of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{w}$:

$$(\mathbf{p} - \mathbf{z}) \bullet \mathbf{x} = 0, \quad (\mathbf{p} - \mathbf{z}) \bullet \mathbf{y} = 0, \quad (\mathbf{p} - \mathbf{z}) \bullet \mathbf{w} = 0.$$

When written in terms of the components of the vectors involved, these three equations are the same as the equations for $a$, $b$, and $c$ encountered in Exercise 4, and so they have the same solution as given for that exercise.

**6.** The relationship $y = p + qx^2$ is linear in $p$ and $q$, so we choose $p$ and $q$ to minimize

$$S = \sum_{i=1}^{n}(p + qx_i^2 - y_i)^2.$$

Thus

$$0 = \frac{\partial S}{\partial p} = 2\sum_{i=1}^{n}(p + qx_i^2 - y_i)$$
$$0 = \frac{\partial S}{\partial q} = 2\sum_{i=1}^{n}(p + qx_i^2 - y_i)x_i^2,$$

that is,

$$\begin{array}{ccccc} np & + & \left(\sum x_i^2\right) q & = & \sum y_i \\ \left(\sum x_i^2\right) p & + & \left(\sum x_i^4\right) q & = & \sum x_i^2 y_i, \end{array}$$

so

$$p = \frac{\left(\sum y_i\right)\left(\sum x_i^4\right) - \left(\sum x_i^2 y_i\right)\left(\sum x_i^2\right)}{n\left(\sum x_i^4\right) - \left(\sum x_i^2\right)^2}$$
$$q = \frac{n\left(\sum x_i^2 y_i\right) - \left(\sum y_i\right)\left(\sum x_i^2\right)}{n\left(\sum x_i^4\right) - \left(\sum x_i^2\right)^2}.$$

This is the result obtained by direct linear regression. (No transformation of variables was necessary.)

**7.** We transform $y = pe^{qx}$ into the form $\ln y = \ln p + qx$, which is linear in $\ln p$ and $q$. We let $\eta_i = \ln y_i$ and use the regression line $\eta = a + bx$ obtained from the data $(x_i, \eta_i)$, with $b = q$ and $a = \ln p$.
Using the formulas for $a$ and $b$ obtained in the text, we have

$$\ln p = a = \frac{n\left(\sum x_i \ln y_i\right) - \left(\sum x_i\right)\left(\sum \ln y_i\right)}{n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}$$
$$q = b = \frac{\left(\sum x_i^2\right)\left(\sum \ln y_i\right) - \left(\sum x_i\right)\left(\sum x_i \ln y_i\right)}{n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}$$
$$p = e^a.$$

These values of $p$ and $q$ are not the same values that minimize the expression

$$S = \sum_{i=1}^{n}(y_i - pe^{qx_i})^2.$$

**8.** We transform $y = \ln(p + qx)$ into the form $e^y = p + qx$, which is linear in $p$ and $q$. We let $\eta_i = e^{y_i}$ and use the regression line $\eta = ax + b$ obtained from the data $(x_i, \eta_i)$, with $a = q$ and $b = p$.
Using the formulas for $a$ and $b$ obtained in the text, we have

$$q = a = \frac{n\left(\sum x_i e^{y_i}\right) - \left(\sum x_i\right)\left(\sum e^{y_i}\right)}{n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}$$
$$p = b = \frac{\left(\sum x_i^2\right)\left(\sum e^{y_i}\right) - \left(\sum x_i\right)\left(\sum x_i e^{y_i}\right)}{n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}.$$

These values of $p$ and $q$ are not the same values that minimize the expression

$$S = \sum_{i=1}^{n}\Big(\ln(p + qx_i) - y_i\Big)^2.$$

**9.** The relationship $y = px + qx^2$ is linear in $p$ and $q$, so we choose $p$ and $q$ to minimize

$$S = \sum_{i=1}^{n}(px_i + qx_i^2 - y_i)^2.$$

Thus

$$0 = \frac{\partial S}{\partial p} = 2\sum_{i=1}^{n}(px_i + qx_i^2 - y_i)x_i$$
$$0 = \frac{\partial S}{\partial q} = 2\sum_{i=1}^{n}(px_i + qx_i^2 - y_i)x_i^2,$$

that is,

$$\begin{array}{ccccc} \left(\sum x_i^2\right)p & + & \left(\sum x_i^3\right)q & = & \sum x_i y_i \\ \left(\sum x_i^3\right)p & + & \left(\sum x_i^4\right)q & = & \sum x_i^2 y_i, \end{array}$$

so

$$p = \frac{\left(\sum x_i y_i\right)\left(\sum x_i^4\right) - \left(\sum x_i^2 y_i\right)\left(\sum x_i^3\right)}{\left(\sum x_i^2\right)\left(\sum x_i^4\right) - \left(\sum x_i^3\right)^2}$$
$$q = \frac{\left(\sum x_i^2\right)\left(\sum x_i^2 y_i\right) - \left(\sum x_i y_i\right)\left(\sum x_i^3\right)}{\left(\sum x_i^2\right)\left(\sum x_i^4\right) - \left(\sum x_i^3\right)^2}.$$

This is the result obtained by direct linear regression. (No transformation of variables was necessary.)

**10.** We transform $y = \sqrt{(px+q)}$ into the form $y^2 = px + q$, which is linear in $p$ and $q$. We let $\eta_i = y_i^2$ and use the regression line $\eta = ax + b$ obtained from the data $(x_i, \eta_i)$, with $a = p$ and $b = q$.
Using the formulas for $a$ and $b$ obtained in the text, we have

$$p = a = \frac{n\left(\sum x_i y_i^2\right) - \left(\sum x_i\right)\left(\sum y_i^2\right)}{n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}$$
$$q = b = \frac{\left(\sum x_i^2\right)\left(\sum y_i^2\right) - \left(\sum x_i\right)\left(\sum x_i y_i^2\right)}{n\left(\sum x_i^2\right) - \left(\sum x_i\right)^2}.$$

These values of $p$ and $q$ are not the same values that minimize the expression

$$S = \sum_{i=1}^{n}\left(\sqrt{px_i + q} - y_i\right)^2.$$

**11.** The relationship $y = pe^x + qe^{-x}$ is linear in $p$ and $q$, so we choose $p$ and $q$ to minimize

$$S = \sum_{i=1}^{n}\left(pe^{x_i} + qe^{-x_i} - y_i\right)^2.$$

Thus

$$0 = \frac{\partial S}{\partial p} = 2\sum_{i=1}^{n}\left(pe^{x_i} + qe^{-x_i} - y_i\right)e^{x_i}$$
$$0 = \frac{\partial S}{\partial q} = 2\sum_{i=1}^{n}\left(pe^{x_i} + qe^{-x_i} - y_i\right)e^{-x_i}.$$

that is,

$$\begin{array}{ccccc} \left(\sum e^{2x_i}\right)p & + & nq & = & \sum e^{x_i} y_i \\ np & + & \left(\sum e^{-2x_i}\right)q & = & \sum e^{-x_i} y_i, \end{array}$$

so

$$p = \frac{\left(\sum e^{-2x_i}\right)\left(\sum e^{x_i} y_i\right) - n\left(\sum e^{-x_i} y_i\right)}{\left(\sum e^{2x_i}\right)\left(\sum e^{-2x_i}\right) - n^2}$$
$$q = \frac{\left(\sum e^{2x_i}\right)\left(\sum e^{-x_i} y_i\right) - n\left(\sum e^{x_i} y_i\right)}{\left(\sum e^{2x_i}\right)\left(\sum e^{-2x_i}\right) - n^2}.$$

This is the result obtained by direct linear regression. (No transformation of variables was necessary.)

**12.** We use the result of Exercise 6. We have $n = 6$ and

$$\sum x_i^2 = 115, \qquad \sum x_i^4 = 4051,$$
$$\sum y_i = 55.18, \qquad \sum x_i^2 y_i = 1984.50.$$

Therefore

$$\begin{aligned} p &= \frac{\left(\sum y_i\right)\left(\sum x_i^4\right) - \left(\sum x_i^2 y_i\right)\left(\sum x_i^2\right)}{n\left(\sum x_i^4\right) - \left(\sum x_i^2\right)^2} \\ &= \frac{55.18 \times 4051 - 1984.50 \times 115}{6 \times 4051 - 115^2} \approx -0.42 \\ q &= \frac{n\left(\sum x_i^2 y_i\right) - \left(\sum y_i\right)\left(\sum x_i^2\right)}{n\left(\sum x_i^4\right) - \left(\sum x_i^2\right)^2} \\ &= \frac{6 \times 1984.50 - 55.18 \times 115}{6 \times 4051 - 115^2} \approx 0.50. \end{aligned}$$

We have (approximately) $y = -0.42 + 0.50x^2$. The predicted value of $y$ at $x = 5$ is $-0.42 + 0.50 \times 25 \approx 12.1$.

**13.** Choose $a$, $b$, and $c$ to minimize

$$S = \sum_{i=1}^{n}\left(ax_i^2 + bx_i + c - y_i\right)^2.$$

Thus

$$0 = \frac{\partial S}{\partial a} = 2\sum_{i=1}^{n}(ax_i^2 + bx_i + c - y_i)x_i^2$$
$$0 = \frac{\partial S}{\partial b} = 2\sum_{i=1}^{n}(ax_i^2 + bx_i + c - y_i)x_i$$
$$0 = \frac{\partial S}{\partial c} = 2\sum_{i=1}^{n}(ax_i^2 + bx_i + c - y_i).$$

Let $A = \sum x_i^4$, $B = \sum x_i^3$, $C = \sum x_i^2$, $D = \sum x_i$, $H = \sum x_i^2 y_i$, $I = \sum x_i y_i$, and $J = \sum y_i$. In terms of these quantities the above equations become

$$\begin{array}{ccccccc} Aa & + & Bb & + & Cc & = & H \\ Ba & + & Cb & + & Dc & = & I \\ Ca & + & Db & + & nc & = & J. \end{array}$$

By Cramer's Rule (Theorem 5 of Section 1.6) the solution is

$$a = \frac{1}{\Delta}\begin{vmatrix} H & B & C \\ I & C & D \\ J & D & n \end{vmatrix}, \qquad b = \frac{1}{\Delta}\begin{vmatrix} A & H & C \\ B & I & D \\ C & J & n \end{vmatrix},$$
$$c = \frac{1}{\Delta}\begin{vmatrix} A & B & H \\ B & C & I \\ C & D & J \end{vmatrix}, \quad \text{where } \Delta = \begin{vmatrix} A & B & C \\ B & C & D \\ C & D & n \end{vmatrix}.$$

**14.** Since $y = pe^x + q + re^{-x}$ is equivalent to

$$e^x y = p(e^x)^2 + qe^x + r,$$

we let $\xi_i = e^{x_i}$ and $\eta_i = e^{x_i} y_i$ for $i = 1, 2, \ldots, n$. We then have $p = a$, $q = b$, and $r = c$, where $a$, $b$, and $c$ are the values calculated by the formulas in Exercise 13, but for the data $(\xi_i, \eta_i)$ instead of $(x_i, y_i)$.

**15.** To minimize $I = \int_0^1 (ax^2 - x^3)^2\, dx$, we choose $a$ so that

$$\begin{aligned} 0 = \frac{dI}{da} &= \int_0^1 2(ax^2 - x^3)x^2\, dx \\ &= \left(2a\frac{x^5}{5} - \frac{2x^6}{6}\right)\Bigg|_0^1 = \frac{2a}{5} - \frac{1}{3}. \end{aligned}$$

Thus $a = 5/6$, and the minimum value of $I$ is

$$\begin{aligned} &\int_0^1 \left(\frac{25x^4}{36} - \frac{5x^5}{3} + x^6\right) dx \\ &= \frac{5}{36} - \frac{5}{18} + \frac{1}{7} = \frac{1}{252}. \end{aligned}$$

**16.** To maximize $I = \int_0^\pi \Big(ax(\pi - x) - \sin x\Big)^2 dx$, we choose $a$ so that

$$\begin{aligned} 0 = \frac{dI}{da} &= \int_0^\pi 2\Big(ax(\pi - x) - \sin x\Big)x(\pi - x)\, dx \\ &= 2a\int_0^\pi x^2(\pi - x)^2\, dx - 2\int_0^\pi x(\pi - x)\sin x\, dx \\ &= \frac{\pi^5 a}{15} - 8. \end{aligned}$$

(We have omited the details of evaluation of these integrals.) Hence $a = 120/\pi^5$. The minimum value of $I$ is

$$\int_0^\pi \left(\frac{120}{\pi^5}x(\pi - x) - \sin x\right)^2 dx = \frac{\pi}{2} - \frac{480}{\pi^5} \approx 0.00227.$$

**17.** To minimize $I = \int_0^1 (ax^2 + b - x^3)^2\, dx$, we choose $a$ and $b$ so that

$$\begin{aligned} 0 = \frac{\partial I}{\partial a} &= \int_0^1 2(ax^2 + b - x^3)x^2\, dx = \frac{2a}{5} + \frac{2b}{3} - \frac{1}{3} \\ 0 = \frac{\partial I}{\partial b} &= \int_0^1 2(ax^2 + b - x^3)\, dx = \frac{2a}{3} + 2b - \frac{1}{2}. \end{aligned}$$

Solving these two equations, we get $a = 15/16$ and $b = -1/16$. The minimum value of $I$ is

$$\int_0^1 \left(\frac{15x^2}{16} - \frac{1}{16} - x^3\right)^2 dx = \frac{1}{448}.$$

**18.** To minimize $\int_0^1 (x^3 - ax^2 - bx - c)^2\, dx$, choose $a$, $b$ and $c$ so that

$$\begin{aligned} 0 &= 2\int_0^1 (x^3 - ax^2 - bx - c)(-x^2)\, dx \\ 0 &= 2\int_0^1 (x^3 - ax^2 - bx - c)(-x)\, dx \\ 0 &= 2\int_0^1 (x^3 - ax^2 - bx - c)(-1)\, dx, \end{aligned}$$

that is,

$$\begin{aligned} \frac{a}{5} + \frac{b}{4} + \frac{c}{3} &= \frac{1}{6} \\ \frac{a}{4} + \frac{b}{3} + \frac{c}{2} &= \frac{1}{5} \\ \frac{a}{3} + \frac{b}{2} + c &= \frac{1}{4} \end{aligned}$$

for which the solution is $a = \frac{3}{2}$, $b = -\frac{3}{5}$, and $c = \frac{1}{20}$.

**19.** To minimize $\int_0^\pi (\sin x - ax^2 - bx)^2\, dx$ we choose $a$ and $b$ so that

$$\begin{aligned} 0 &= 2\int_0^\pi (\sin x - ax^2 - bx)(-x^2)\, dx \\ 0 &= 2\int_0^\pi (\sin x - ax^2 - bx)(-x)\, dx. \end{aligned}$$

We omit the details of the evaluation of the integrals. The result of the evaluation is that $a$ and $b$ satisfy

$$\begin{aligned} \frac{\pi^5}{5}a + \frac{\pi^4}{4}b &= \pi^2 - 4 \\ \frac{\pi^4}{4}a + \frac{\pi^3}{3}b &= \pi, \end{aligned}$$

for which the solution is

$$\begin{aligned} a &= \frac{20}{\pi^5}(\pi^2 - 16) \\ b &= \frac{12}{\pi^4}(20 - \pi^2). \end{aligned}$$

**20.** $J = \int_{-1}^{1} (x - a \sin \pi x - b \sin 2\pi x - c \sin 3\pi x)^2 \, dx.$

To minimize $J$, choose $a$, $b$, and $c$ to satisfy

$$\begin{aligned}
0 &= \frac{\partial J}{\partial a} \\
&= -2 \int_{-1}^{1} (x - a \sin \pi x - b \sin 2\pi x - c \sin 3\pi x) \sin \pi x \, dx \\
&= \frac{2}{\pi}(\pi a - 2) \\
0 &= \frac{\partial J}{\partial b} \\
&= -2 \int_{-1}^{1} (x - a \sin \pi x - b \sin 2\pi x - c \sin 3\pi x) \sin 2\pi x \, dx \\
&= \frac{2}{\pi}(\pi b + 1) \\
0 &= \frac{\partial J}{\partial c} \\
&= -2 \int_{-1}^{1} (x - a \sin \pi x - b \sin 2\pi x - c \sin 3\pi x) \sin 3\pi x \, dx \\
&= \frac{2}{3\pi}(3\pi c - 2).
\end{aligned}$$

We have omitted the details of evaluation of these integrals, but note that

$$\int_{-1}^{1} \sin m\pi x \sin n\pi x \, dx = 0$$

if $m$ and $n$ are different integers.

The equations above imply that $a = 2/\pi$, $b = -1/\pi$, and $c = 2/(3\pi)$. These are the values that minimize $J$.

**21.** To minimize

$$I = \int_0^{\pi} \left( f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} a_k \cos kx \right)^2 \, dx$$

we require

$$0 = \frac{\partial I}{\partial a_0} = 2 \int_0^{\pi} \left( f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} a_k \cos kx \right) \left( -\frac{1}{2} \right) \, dx,$$

and

$$0 = \frac{\partial I}{\partial a_n} = 2 \int_0^{\pi} \left( f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} a_k \cos kx \right) (-\cos nx) \, dx$$

for $n = 1, 2, \ldots$. Thus

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) \, dx,$$

and, since

$$\int_0^{\pi} \cos kx \, \cos nx \, dx = \begin{cases} 0 & \text{if } k \neq n \\ \dfrac{\pi}{2} & \text{if } k = n = 1, 2, \ldots \end{cases}$$

we also have

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \, \cos nx \, dx \qquad (n = 1, 2, \ldots).$$

**22.** The Fourier sine series coefficients for $f(x) = x$ on $(0, \pi)$ are

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) \, dx = (-1)^{n-1} \frac{2}{n}$$

for $n = 1, 2, \ldots$. Thus the series is

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{2}{n} \sin nx.$$

Since $x$ and the functions $\sin nx$ are all odd functions, we would also expect the series to converge to $x$ on $(-\pi, 0)$.

**23.** The Fourier cosine series coefficients for $f(x) = x$ on $(0, \pi)$ are

$$\begin{aligned}
a_0 &= \frac{2}{\pi} \int_0^{\pi} x \, dx = \pi \\
a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos(nx) \, dx = -\frac{2\left(1 - (-1)^n\right)}{n^2 \pi} \\
&= \begin{cases} 0 & \text{if } n \geq 2 \text{ is even} \\ -\dfrac{4}{n^2 \pi} & \text{if } n \geq 1 \text{ is odd.} \end{cases}
\end{aligned}$$

Thus the Fourier cosine series is

$$\pi - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos((2n+1)x)}{(2n+1)^2}.$$

Since the terms of this series are all even functions, and the series converges to $x$ if $0 < x < \pi$, it will converge to $-x = |x|$ if $-\pi < x < 0$.

Remark: since $|x|$ is continuous at $x = 0$, the series also converges at $x = 0$ to 0. It follows that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{4}.$$

**24.** We are given that $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n$.
To motivate the method, look at a special case, $n = 5$ say.

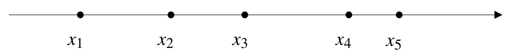


Fig. 13.4.24

If $x = x_3$, then

$$\begin{aligned}&\sum_{i=1}^{5}|x - x_i|\\&= (x_3 - x_1) + (x_3 - x_2) + 0 + (x_4 - x_3) + (x_5 - x_3)\\&= (x_5 - x_1) + (x_4 - x_2).\end{aligned}$$

If $x$ moves away from $x_3$ in either direction, then

$$\sum_{i=1}^{5}|x - x_i| = (x_5 - x_1) + (x_4 - x_2) + |x - x_3|.$$

Thus the minimum sum occurs if $x = x_3$.
In general, if $n$ is odd, then $\sum_{i=1}^{n}|x - x_i|$ is minimum if $x = x_{(n+1)/2}$, the middle point of the set of points $\{x_1, x_2, \ldots, x_n\}$. The value of $x$ is unique in this case. If $n$ is even and $x$ satisfies $x_{n/2} \le x \le x_{(n/2)+1}$, then

$$\sum_{i=1}^{n}|x - x_i| = \sum_{i=1}^{n/2}|x_{n+1-i} - x_i|,$$

and the sum will increase if $x$ is outside that interval. In this case the value of $x$ which minimizes the sum is not unique unless it happens that $x_{n/2} = x_{(n/2)+1}$.

## Section 13.5 Parametric Problems (page 743)

**1.**

$$\begin{aligned}F(x) &= \int_0^1 t^x\,dt = \frac{1}{x+1} \qquad (x > -1)\\ F'(x) &= \int_0^1 t^x \ln t\,dt = -\frac{1}{(x+1)^2}\\ F''(x) &= \int_0^1 t^x (\ln t)^2\,dt = \frac{2}{(x+1)^3}\\ &\ \vdots\\ F^{(n)}(x) &= \int_0^1 t^x (\ln t)^n\,dt = \frac{(-1)^n n!}{(x+1)^{n+1}}.\end{aligned}$$

**2.**

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi} \qquad \begin{aligned}&\text{Let } u = xt\\ &du = x\,dt\end{aligned}$$

$$\int_{-\infty}^{\infty} e^{-x^2t^2}\,dt = \frac{\sqrt{\pi}}{x}.$$

Differentiate with respect to $x$:

$$\begin{aligned}\int_{-\infty}^{\infty} -2xt^2 e^{-t^2x^2}\,dt &= -\frac{\sqrt{\pi}}{x^2}\\ \int_{-\infty}^{\infty} t^2 e^{-x^2t^2}\,dt &= \frac{\sqrt{\pi}}{2x^3}. \qquad (*)\end{aligned}$$

If $x = 1$ we get $\displaystyle\int_{-\infty}^{\infty} t^2 e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}$.
Differentiate $(*)$ with respect to $x$ again:

$$\int_{-\infty}^{\infty} -2xt^4 e^{-x^2t^2}\,dt = -\frac{3\sqrt{\pi}}{2x^4}.$$

Divide by $-2$ and let $x = 1$:

$$\int_{-\infty}^{\infty} t^4 e^{-t^2}\,dt = \frac{3\sqrt{\pi}}{4}.$$

**3.** Let $\displaystyle I(x, y) = \int_{-\infty}^{\infty} \frac{e^{-xt^2} - e^{-yt^2}}{t^2}\,dt$, where $x > 0$ and $y > 0$. Then

$$\begin{aligned}\frac{\partial I}{\partial x} &= -\int_{-\infty}^{\infty} e^{-xt^2}\,dt \qquad \begin{aligned}&\text{Let } \sqrt{x}t = s\\ &\sqrt{x}\,dt = ds\end{aligned}\\ &= -\frac{1}{\sqrt{x}}\int_{-\infty}^{\infty} e^{-s^2}\,ds = -\frac{\sqrt{\pi}}{\sqrt{x}}.\end{aligned}$$

Similarly, $\dfrac{\partial I}{\partial y} = \dfrac{\sqrt{\pi}}{\sqrt{y}}$. Now

$$\begin{aligned}I(x, y) &= -\sqrt{\pi}\int \frac{dx}{\sqrt{x}} = -2\sqrt{\pi x} + C_1(y)\\ \frac{\sqrt{\pi}}{\sqrt{y}} &= \frac{\partial I}{\partial y} = \frac{\partial C_1}{\partial y} \Rightarrow C_1(y) = 2\sqrt{\pi y} + C_2\\ I(x, y) &= 2\sqrt{\pi}\left(\sqrt{y} - \sqrt{x}\right) + C_2.\end{aligned}$$

But $I(x, x) = 0$. Therefore $C_2 = 0$, and

$$I(x, y) = \int_{-\infty}^{\infty} \frac{e^{-xt^2} - e^{-yt^2}}{t^2}\,dt = 2\sqrt{\pi}\left(\sqrt{y} - \sqrt{x}\right).$$

**4.** Let $\displaystyle I(x, y) = \int_0^1 \frac{t^x - t^y}{\ln t}\,dt$, where $x > -1$ and $y > -1$. Then

$$\begin{aligned}\frac{\partial I}{\partial x} &= \int_0^1 t^x\,dt = \frac{1}{x+1}\\ \frac{\partial I}{\partial y} &= -\frac{1}{y+1}.\end{aligned}$$

Thus

$$\begin{aligned}I(x, y) &= \int \frac{dx}{x+1} = \ln(x+1) + C_1(y)\\ \frac{-1}{y+1} &= \frac{\partial I}{\partial y} = \frac{\partial C_1}{\partial y} \Rightarrow C_1(y) = -\ln(y+1) + C_2\\ I(x, y) &= \ln\left(\frac{x+1}{y+1}\right) + C_2.\end{aligned}$$

But $I(x,x) = 0$, so $C_2 = 0$. Thus

$$I(x,y) = \int_0^1 \frac{t^x - t^y}{\ln t}\,dt = \ln\left(\frac{x+1}{y+1}\right)$$

for $x > -1$ and $y > -1$.

**5.** $\displaystyle\int_0^\infty e^{-xt}\sin t\,dt = \frac{1}{1+x^2}$ if $x > 0$.
Multiply by $-1$ and differentiate with respect to $x$ twice:

$$\int_0^\infty t e^{-xt}\sin t\,dt = \frac{2x}{(1+x^2)^2}$$
$$\int_0^\infty t^2 e^{-xt}\sin t\,dt = \frac{2(3x^2-1)}{(1+x^2)^3}.$$

**6.**
$$F(x) = \int_0^\infty e^{-xt}\frac{\sin t}{t}\,dt$$
$$F'(x) = \int_0^\infty -e^{-xt}\sin t\,dt = -\frac{1}{1+x^2} \quad (x > 0).$$

Therefore $F(x) = -\displaystyle\int \frac{dx}{1+x^2} = -\tan^{-1}x + C$.
Now, make the change of variable $xt = s$ in the integral defining $F(x)$, and obtain

$$F(x) = \int_0^\infty e^{-s}\frac{\sin(s/x)}{s/x}\frac{ds}{x} = \int_0^\infty \frac{e^{-s}}{s}\sin\frac{s}{x}\,ds.$$

Since $|\sin(s/x)| \le s/x$ if $s > 0$, $x > 0$, we have

$$|F(x)| \le \frac{1}{|x|}\int_0^\infty e^{-s}\,ds = \frac{1}{|x|} \to 0 \quad \text{as } x \to \infty.$$

Hence $-\dfrac{\pi}{2} + C = 0$, and $C = \dfrac{\pi}{2}$. Therefore

$$F(x) = \int_0^\infty e^{-xt}\frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}x.$$

In particular, $\int_0^\infty \dfrac{\sin t}{t}\,dt = \lim_{x\to 0} F(x) = \dfrac{\pi}{2}$.

**7.** $\displaystyle\int_0^\infty \frac{dt}{x^2+t^2} = \frac{1}{x}\tan^{-1}\frac{t}{x}\Big|_0^\infty = \frac{\pi}{2x}$ for $x > 0$.
Differentiate with respect to $x$:

$$\int_0^\infty \frac{-2x\,dt}{(x^2+t^2)^2} = -\frac{\pi}{2x^2}$$
$$\int_0^\infty \frac{dt}{(x^2+t^2)^2} = \frac{\pi}{4x^3}.$$

Differentiate with respect to $x$ again:

$$\int_0^\infty \frac{-4x\,dt}{(x^2+t^2)^3} = -\frac{3\pi}{4x^4}$$
$$\int_0^\infty \frac{dt}{(x^2+t^2)^3} = \frac{3\pi}{16x^5}.$$

**8.** $\displaystyle\int_0^x \frac{dt}{x^2+t^2} = \frac{1}{x}\tan^{-1}\frac{t}{x}\Big|_0^x = \frac{\pi}{4x}$ for $x > 0$.
Differentiate with respect to $x$:

$$\frac{1}{2x^2} + \int_0^x \frac{-2x\,dt}{(x^2+t^2)^2} = -\frac{\pi}{4x^2}$$
$$\int_0^x \frac{dt}{(x^2+t^2)^2} = -\frac{1}{2x}\left[-\frac{\pi}{4x^2} - \frac{1}{2x^2}\right] = \frac{\pi}{8x^3} + \frac{1}{4x^3}.$$

Differentiate with respect to $x$ again:

$$\frac{1}{4x^4} + \int_0^x \frac{-4x\,dt}{(x^2+t^2)^3} = -\frac{3}{x^4}\left[\frac{\pi}{8} + \frac{1}{4}\right]$$
$$\int_0^x \frac{dt}{(x^2+t^2)^3} = -\frac{1}{4x}\left[-\frac{3\pi}{8x^4} - \frac{3}{4x^4} - \frac{1}{4x^4}\right] = \frac{3\pi}{32x^5} + \frac{1}{4x^5}.$$

**9.**
$$f(x) = 1 + \int_a^x (x-t)^n f(t)\,dt \quad\Rightarrow\quad f(a) = 1$$
$$f'(x) = n\int_a^x (x-t)^{n-1} f(t)\,dt$$
$$f''(x) = n(n-1)\int_a^x (x-t)^{n-2} f(t)\,dt$$
$$\vdots$$
$$f^{(n)}(x) = n!\int_a^x f(t)\,dt$$
$$f^{(n+1)}(x) = n!\,f(x) \quad\Rightarrow\quad f^{(n+1)}(a) = n!\,f(a) = n!.$$

**10.**
$$f(x) = Cx + D + \int_0^x (x-t)f(t)\,dt \quad\Rightarrow\quad f(0) = D$$
$$f'(x) = C + \int_0^x f(t)\,dt \quad\Rightarrow\quad f'(0) = C$$
$$f''(x) = f(x) \quad\Rightarrow\quad f(x) = A\cosh x + B\sinh x$$
$$D = f(0) = A, \qquad C = f'(0) = B$$
$$\Rightarrow f(x) = D\cosh x + C\sinh x.$$

**11.**
$$f(x) = x + \int_0^x (x-2t)f(t)\,dt \quad\Rightarrow\quad f(0) = 0$$
$$f'(x) = 1 - xf(x) + \int_0^x f(t)\,dt \quad\Rightarrow\quad f'(0) = 1$$
$$f''(x) = -f(x) - xf'(x) + f(x) = -xf'(x).$$

If $u = f'(x)$, then $\dfrac{du}{u} = -x\,dx$, so $\ln u = -\dfrac{x^2}{2} + \ln C_1$.
Therefore

$$f'(x) = u = C_1 e^{-x^2/2}.$$

We have $1 = f'(0) = C_1$, so $f'(x) = e^{-x^2/2}$ and

$$f(x) = \int_0^x e^{-t^2/2}\,dt + C_2.$$

But $0 = f(0) = C_2$, and so

$$f(x) = \int_0^x e^{-t^2/2}\,dt.$$

**12.** $f(x) = 1 + \int_0^1 (x+t)f(t)\,dt$

$f'(x) = \int_0^1 f(t)\,dt = C, \quad$ say,

since the integral giving $f'(x)$ does not depend on $x$. Thus $f(x) = A + Cx$, where $A = f(0)$. Substituting this expression into the given equation, we obtain

$$\begin{aligned} A + Cx &= 1 + \int_0^1 (x+t)(A+Ct)\,dt \\ &= 1 + Ax + \frac{A}{2} + \frac{Cx}{2} + \frac{C}{3}. \end{aligned}$$

Therefore

$$\frac{A}{2} - 1 - \frac{C}{3} + x\left(\frac{C}{2} - A\right) = 0.$$

This can hold for all $x$ only if

$$\frac{A}{2} - 1 - \frac{C}{3} = 0 \quad\text{and}\quad \frac{C}{2} - A = 0.$$

Thus $C = 2A$ and $\frac{A}{2} - \frac{2A}{3} = 1$, so that $A = -6$ and $C = -12$. Therefore $f(x) = -6 - 12x$.

**13.** We eliminate $c$ from the pair of equations

$$\begin{aligned} f(x,y,c) &= 2cx - c^2 - y = 0 \\ \frac{\partial}{\partial c} f(x,y,c) &= 2x - 2c = 0. \end{aligned}$$

Thus $c = x$ and $2x^2 - x^2 - y = 0$. The envelope is $y = x^2$.

**14.** We eliminate $c$ from the pair of equations

$$\begin{aligned} f(x,y,c) &= y - (x-c)\cos c - \sin c = 0 \\ \frac{\partial}{\partial c} f(x,y,c) &= \cos c + (x-c)\sin c - \cos c = 0. \end{aligned}$$

Thus $c = x$ and $y - 0 - \sin x = 0$.
The envelope is $y = \sin x$.

**15.** We eliminate $c$ from the pair of equations

$$\begin{aligned} f(x,y,c) &= x\cos c + y\sin c - 1 = 0 \\ \frac{\partial}{\partial c} f(x,y,c) &= -x\sin c + y\cos c = 0. \end{aligned}$$

Squaring and adding these equations yields $x^2 + y^2 = 1$, which is the equation of the envelope.

**16.** We eliminate $c$ from the pair of equations

$$\begin{aligned} f(x,y,c) &= \frac{x}{\cos c} + \frac{y}{\sin c} - 1 = 0 \\ \frac{\partial}{\partial c} f(x,y,c) &= \frac{x\sin c}{\cos^2 c} - \frac{y\cos c}{\sin^2 c} = 0. \end{aligned}$$

From the second equation, $y = x\tan^3 c$. Thus

$$\frac{x}{\cos c}(1 + \tan^2 c) = 1$$

which implies that $x = \cos^3 c$, and hence $y = \sin^3 c$. The envelope is the astroid $x^{2/3} + y^{2/3} = 1$.

**17.** We eliminate $c$ from the pair of equations

$$\begin{aligned} f(x,y,c) &= c + (x-c)^2 - y = 0 \\ \frac{\partial}{\partial c} f(x,y,c) &= 1 + 2(c-x) = 0. \end{aligned}$$

Thus $c = x - \frac{1}{2}$. The envelope is the line $y = x - \frac{1}{4}$.

**18.** We eliminate $c$ from the pair of equations

$$\begin{aligned} f(x,y,c) &= (x-c)^2 + (y-c)^2 - 1 = 0 \\ \frac{\partial}{\partial c} f(x,y,c) &= 2(c-x) + 2(c-y) = 0. \end{aligned}$$

Thus $c = (x+y)/2$, and

$$\left(\frac{x-y}{2}\right)^2 + \left(\frac{y-x}{2}\right)^2 = 1$$

or $x - y = \pm\sqrt{2}$. These two parallel lines constitute the envelope of the given family which consists of circles of radius 1 with centres along the line $y = x$.

**19.** Not every one-parameter family of curves in the plane has an envelope. The family of parabolas $y = x^2 + c$ evidently does not. (See the figure.) If we try to calculate the envelope by eliminating $c$ from the equations

$$\begin{aligned} f(x,y,c) &= y - x^2 - c = 0 \\ \frac{\partial}{\partial c} f(x,y,c) &= -1 = 0, \end{aligned}$$

we fail because the second equation is contradictory.

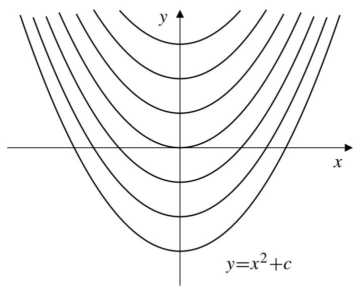


Fig. 13.5.19

**20.** The curve $x^2+(y-c)^2=kc^2$ is a circle with centre $(0,c)$ and radius $\sqrt{k}c$, provided $k>0$. Consider the system:

$$f(x,y,c)=x^2+(y-c)^2-kc^2=0$$
$$\frac{\partial}{\partial c}f(x,y,c)=-2(y-c)-2kc=0.$$

The second equation implies that $y-c=-kc$, and the first equation then says that $x^2=k(1-k)c^2$. This is only possible if $0\le k\le 1$.
The cases $k=0$ and $k=1$ are degenerate. If $k=0$ the "curves" are just points on the $y$-axis. If $k=1$ the curves are circles, all of which are tangent to the $x$-axis at the origin. There is no reasonable envelope in either case. If $0<k<1$, the envelope is the pair of lines given by $x^2=\dfrac{k}{1-k}y^2$, that is, the lines $\sqrt{1-k}x=\pm\sqrt{k}y$. These lines make angle $\sin^{-1}\sqrt{k}$ with the $y$-axis.

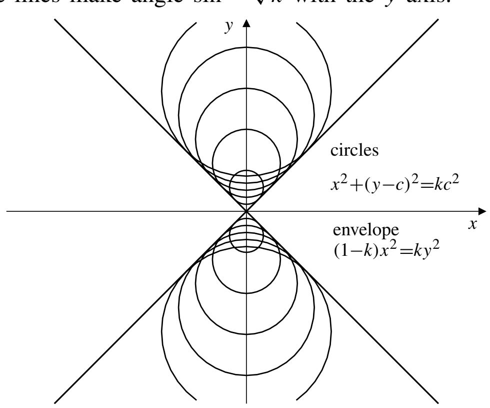


Fig. 13.5.20

**21.** We eliminate $c$ from the equations

$$f(x,y,c)=y^3-(x+c)^2=0$$
$$\frac{\partial}{\partial c}f(x,y,c)=-2(x+c)=0.$$

Thus $x=-c$, and we obtain the equation $y=0$ for the envelope. However, this is not really an envelope at all. The curves $y^3=(x+c)^2$ all have cusps along the $x$-axis; none of them is tangent to the axis.

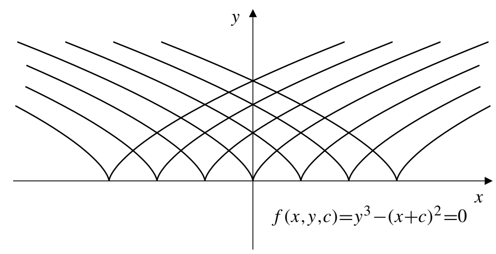


Fig. 13.5.21

**22.** If the family of surfaces $f(x,y,z,\lambda,\mu)=0$ has an envelope, that envelope will have parametric equations

$$x=x(\lambda,\mu),\qquad y=y(\lambda,\mu),\qquad z=z(\lambda,\mu),$$

giving the point on the envelope where the envelope is tangent to the particular surface in the family having parameter values $\lambda$ and $\mu$. Thus

$$f\Big(x(\lambda,\mu),y(\lambda,\mu),z(\lambda,\mu),\lambda,\mu\Big)=0.$$

Differentiating with respect to $\lambda$, we obtain

$$f_1\frac{\partial x}{\partial\lambda}+f_2\frac{\partial y}{\partial\lambda}+f_3\frac{\partial z}{\partial\lambda}+f_4=0.$$

However, since for fixed $\mu$, the parametric curve

$$x=x(t,\mu),\qquad y=y(t,\mu),\qquad z=z(t,\mu)$$

is tangent to the surface $f(x,y,z,\lambda,\mu)=0$ at $t=\lambda$, its tangent vector there,

$$\mathbf{T}=\frac{\partial x}{\partial\lambda}\mathbf{i}+\frac{\partial y}{\partial\lambda}\mathbf{j}+\frac{\partial z}{\partial\lambda}\mathbf{k},$$

is perpendicular to the normal

$$\mathbf{N}=\nabla f=f_1\mathbf{i}+f_2\mathbf{j}+f_3\mathbf{k},$$

so

$$f_1\frac{\partial x}{\partial\lambda}+f_2\frac{\partial y}{\partial\lambda}+f_3\frac{\partial z}{\partial\lambda}=0.$$

Hence we must also have $\dfrac{\partial f}{\partial\lambda}=f_4(x,y,z,\lambda,\mu)=0$.
Similarly, $\dfrac{\partial f}{\partial\mu}=0$.
The parametric equations of the envelope must therefore satisfy the three equations

$$f(x,y,z,\lambda,\mu)=0$$
$$\frac{\partial}{\partial\lambda}f(x,y,z,\lambda,\mu)=0$$
$$\frac{\partial}{\partial\mu}f(x,y,z,\lambda,\mu)=0.$$

The envelope can be found by eliminating $\lambda$ and $\mu$ from these three equations.

**23.** To find the envelope we eliminate $\lambda$ and $\mu$ from the equations

$$\begin{aligned} x\sin\lambda\cos\mu + y\sin\lambda\sin\mu + z\cos\lambda &= 1 \qquad (1)\\ x\cos\lambda\cos\mu + y\cos\lambda\sin\mu - z\sin\lambda &= 0 \qquad (2)\\ -x\sin\lambda\sin\mu + y\sin\lambda\cos\mu &= 0. \qquad (3)\end{aligned}$$

Multiplying (1) by $\cos\lambda$ and (2) by $\sin\lambda$ and subtracting the two gives

$$z = \cos\lambda.$$

Therefore (2) and (3) can be rewritten

$$\begin{aligned} x\cos\mu + y\sin\mu &= \sin\lambda\\ x\sin\mu - y\cos\mu &= 0.\end{aligned}$$

Squaring and adding these equations gives

$$x^2 + y^2 = \sin^2\lambda.$$

Therefore

$$x^2 + y^2 + z^2 = \sin^2\lambda + \cos^2\lambda = 1;$$

the envelope is the sphere of radius 1 centred at the origin.

**24.** $(x-\lambda)^2 + (y-\mu)^2 + z^2 = \dfrac{\lambda^2+\mu^2}{2}.$
Differentiate with respect to $\lambda$ and $\mu$:

$$-2(x-\lambda) = \lambda, \qquad -2(y-\mu) = \mu.$$

Thus $\lambda = 2x$, $\mu = 2y$, and

$$x^2 + y^2 + z^2 = 2x^2 + 2y^2.$$

The envelope is the cone $z^2 = x^2 + y^2$.

**25.** $y + \epsilon\sin(\pi y) = x \quad \Rightarrow \quad y = y(\epsilon, x)$

$$\frac{\partial y}{\partial \epsilon} + \sin(\pi y) + \pi\epsilon\cos(\pi y)\frac{\partial y}{\partial \epsilon} = 0$$

$$\frac{\partial^2 y}{\partial \epsilon^2} + 2\pi\cos(\pi y)\frac{\partial y}{\partial \epsilon} - \pi^2\epsilon\sin(\pi y)\left(\frac{\partial y}{\partial \epsilon}\right)^2 + \pi\epsilon\cos(\pi y)\frac{\partial^2 y}{\partial \epsilon^2} = 0.$$

If $\epsilon = 0$ then $y = x$, so $y(x,0) = x$. Also, at $\epsilon = 0$,

$$y_\epsilon(x,0)(1+0) = -\sin(\pi y(x,0)) = -\sin(\pi x),$$

that is, $y_\epsilon(x,0) = -\sin(\pi x)$. Also,

$$\begin{aligned} y_{\epsilon\epsilon}(x,0)(1+0) &= -2\pi\cos(\pi x)y_\epsilon(x,0) + 0\\ &= 2\pi\cos(\pi x)\sin(\pi x) = \pi\sin(2\pi x).\end{aligned}$$

Thus

$$\begin{aligned} y = y(x,\epsilon) &= y(x,0) + \epsilon y_\epsilon(x,0) + \frac{\epsilon^2}{2!}y_{\epsilon\epsilon}(x,0) + \cdots\\ &= x - \epsilon\sin(\pi x) + \frac{\epsilon^2}{2}\pi\sin(2\pi x) + \cdots\end{aligned}$$

**26.** $y^2 + \epsilon e^{-y^2} = 1 + x^2$

$$2yy_\epsilon + e^{-y^2} - 2y\epsilon e^{-y^2}y_\epsilon = 0$$

$$2y\left(1 - \epsilon e^{-y^2}\right)y_\epsilon + e^{-y^2} = 0$$

$$\begin{aligned} &2y_\epsilon\left(1-\epsilon e^{-y^2}\right)y_\epsilon - 2ye^{-y^2}y_\epsilon + 2y\left(2y\epsilon e^{-y^2}y_\epsilon\right)y_\epsilon\\ &\quad + 2y\left(1-\epsilon e^{-y^2}\right)y_{\epsilon\epsilon} - 2ye^{-y^2}y_\epsilon = 0.\end{aligned}$$

At $\epsilon = 0$ we have $y(x,0) = \sqrt{1+x^2}$, and

$$\begin{aligned} &2\sqrt{1+x^2}\,y_\epsilon(x,0) + e^{-(1+x^2)} = 0\\ &y_\epsilon(x,0) = -\frac{1}{2\sqrt{1+x^2}}e^{-(1+x^2)}\\ &2y_\epsilon^2 - 4ye^{-y^2}y_\epsilon + 2yy_{\epsilon\epsilon} = 0\\ &yy_{\epsilon\epsilon} = 2yy_\epsilon e^{-y^2} - y_\epsilon^2\\ &y_{\epsilon\epsilon}(x,0) = -\left(\frac{1}{\sqrt{1+x^2}} + \frac{1}{4(1+x^2)^{3/2}}\right)e^{-2(1+x^2)}.\end{aligned}$$

Thus

$$\begin{aligned} y &= y(x,\epsilon) = y(x,0) + \epsilon y_\epsilon(x,0) + \frac{\epsilon^2}{2!}y_{\epsilon\epsilon}(x,0) + \cdots\\ &= \sqrt{1+x^2} - \frac{\epsilon}{2\sqrt{1+x^2}}e^{-(1+x^2)}\\ &\qquad - \frac{\epsilon^2}{2}\left(\frac{1}{\sqrt{1+x^2}} + \frac{1}{4(1+x^2)^{3/2}}\right)e^{-2(1+x^2)} + \cdots.\end{aligned}$$

**27.** $2y + \dfrac{\epsilon x}{1+y^2} = 1$

$$2y_\epsilon + \frac{x}{1+y^2} - \frac{2\epsilon xyy_\epsilon}{(1+y^2)^2} = 0$$

$$2y_{\epsilon\epsilon} - \frac{4xyy_\epsilon}{(1+y^2)^2} - \epsilon\frac{\partial}{\partial\epsilon}\left(\frac{2xyy_\epsilon}{(1+y^2)^2}\right) = 0.$$

At $\epsilon = 0$ we have $y(x,0) = \dfrac{1}{2}$, and

$$\begin{aligned} y_\epsilon(x,0) &= -\frac{1}{2}\frac{x}{1+\frac{1}{4}} = -\frac{2x}{5}\\ y_{\epsilon\epsilon} &= \frac{1}{2}\frac{4x\left(\frac{1}{2}\right)\left(-\frac{2x}{5}\right)}{\left(1+\frac{1}{4}\right)^2} = -\frac{32x^2}{125}.\end{aligned}$$

Thus

$$\begin{aligned} y = y(x,\epsilon) &= y(x,0) + \epsilon y_\epsilon(x,0) + \frac{\epsilon^2}{2!} y_{\epsilon\epsilon}(x,0) + \cdots \\ &= \frac{1}{2} - \frac{2\epsilon x}{5} - \frac{16\epsilon^2 x^2}{125} + \cdots. \end{aligned}$$

**28.** Let $y(x,\epsilon)$ be the solution of $y + \epsilon y^5 = \frac{1}{2}$. Then we have

$$\begin{aligned} & y_\epsilon\left(1 + 5\epsilon y^4\right) + y^5 = 0 \\ & y_{\epsilon\epsilon}\left(1 + 5\epsilon y^4\right) + 20\epsilon y^3 y_\epsilon^2 + 10 y^4 y_\epsilon = 0 \\ & y_{\epsilon\epsilon\epsilon}\left(1 + 5\epsilon y^4\right) + y_{\epsilon\epsilon}\left(60\epsilon y^3 y_\epsilon + 15 y^4\right) \\ & \qquad + 60\epsilon y_\epsilon^3 y^2 + 60 y^3 y_\epsilon^2 = 0. \end{aligned}$$

At $\epsilon = 0$ we have

$$\begin{aligned} y(x,0) &= \frac{1}{2} \\ y_\epsilon(x,0) &= -\frac{1}{32} \\ y_{\epsilon\epsilon}(x,0) &= -\frac{10}{16}\left(-\frac{1}{32}\right) = \frac{5}{16^2} \\ y_{\epsilon\epsilon\epsilon}(x,0) &= -\frac{5}{16^2}\left(\frac{15}{16}\right) - \frac{60}{8}\left(-\frac{1}{32}\right)^2 = -\frac{105}{4096}. \end{aligned}$$

For $\epsilon = \frac{1}{100}$ we have

$$\begin{aligned} y &= \frac{1}{2} - \frac{1}{32} \times \frac{1}{100} + \frac{5}{256} \times \frac{1}{2 \times 100^2} \\ &\qquad - \frac{105}{4096} \times \frac{1}{6 \times 100^3} + \cdots \\ &\approx 0.49968847 \end{aligned}$$

with error less than $10^{-8}$ in magnitude.

**29.** Let $x(\epsilon)$ and $y(\epsilon)$ be the solution of

$$\begin{aligned} x + 2y + \epsilon e^{-x} &= 3 \\ x - y + \epsilon e^{-y} &= 0. \end{aligned}$$

Thus

$$\begin{aligned} x' + 2y' + e^{-x} - \epsilon e^{-x} x' &= 0 \\ x' - y' + e^{-y} - \epsilon e^{-y} y' &= 0 \\ x'' + 2y'' - 2e^{-x} x' + \epsilon e^{-x}(x')^2 - \epsilon e^{-x} x'' &= 0 \\ x'' - y'' - 2e^{-y} y' + \epsilon e^{-y}(y')^2 - \epsilon e^{-y} y'' &= 0. \end{aligned}$$

At $\epsilon = 0$ we have

$$\left.\begin{aligned} x + 2y &= 3 \\ x - y &= 0 \end{aligned}\right\} \Rightarrow x = y = 1$$

$$\left.\begin{aligned} x' + 2y' &= -\frac{1}{e} \\ x' - y' &= -\frac{1}{e} \end{aligned}\right\} \Rightarrow \begin{aligned} x' &= -\frac{1}{e} \\ y' &= 0 \end{aligned}$$

$$\left.\begin{aligned} x'' + 2y'' &= -\frac{2}{e^2} \\ x'' - y'' &= 0 \end{aligned}\right\} \Rightarrow x'' = y'' = \frac{-2}{3e^2}.$$

Thus

$$x = 1 - \frac{\epsilon}{e} - \frac{\epsilon^2}{3e^2} + \cdots, \qquad y = 1 - \frac{\epsilon^2}{3e^2} + \cdots.$$

For $\epsilon = \frac{1}{100}$ we have

$$\begin{aligned} x &= 1 - \frac{1}{100e} + \frac{1}{30,000e^2} + \cdots \\ y &= 1 - \frac{1}{30,000e^2} + \cdots. \end{aligned}$$

## Section 13.6 Newton's Method (page 746)

For each of Exercises 1–6, and 9, we sketch the graphs of the two given equations, $f(x,y) = 0$ and $g(x,y) = 0$, and use their intersections to make initial guesses $x_0$ and $y_0$ for the solutions. These guesses are then refined using the formulas

$$x_{n+1} = x_n - \left.\frac{fg_2 - gf_2}{f_1 g_2 - g_1 f_2}\right|_{(x_n,y_n)}, \qquad y_{n+1} = y_n - \left.\frac{f_1 g - g_1 f}{f_1 g_2 - g_1 f_2}\right|_{(x_n,y_n)}.$$

NOTE: The numerical values in the tables below were obtained by programming a microcomputer to calculate the iterations of the above formulas. In most cases the computer was using more significant digits than appear in the tables, and did not truncate the values obtained at one step before using them to calculate the next step. If you use a calculator, and use the numbers as quoted on one line of a table to calculate the numbers on the next line, your results may differ slightly (in the last one or two decimal places).

**1.**

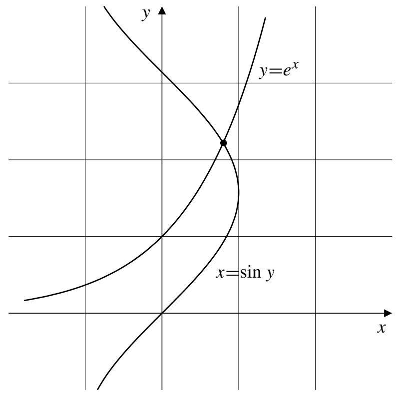


Fig. 13.6.1

$$\begin{array}{lll} f(x,y) = y - e^x & f_1(x,y) = -e^x & g_1(x,y) = 1 \\ g(x,y) = x - \sin y & f_2(x,y) = 1 & g_2(x,y) = -\cos y \end{array}.$$

We start with $x_0 = 0.9$, $y_0 = 2.0$.

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | 0.9000000 | 2.0000000 | −0.4596031 | −0.0092974 |
| 1 | 0.8100766 | 2.2384273 | −0.0096529 | 0.0247861 |
| 2 | 0.7972153 | 2.2191669 | −0.0001851 | 0.0001464 |
| 3 | 0.7971049 | 2.2191071 | 0.0000000 | 0.0000000 |
| 4 | 0.7971049 | 2.2191071 | 0.0000000 | 0.0000000 |

Thus $x = 0.7971049$, $y = 2.2191071$.

**2.**

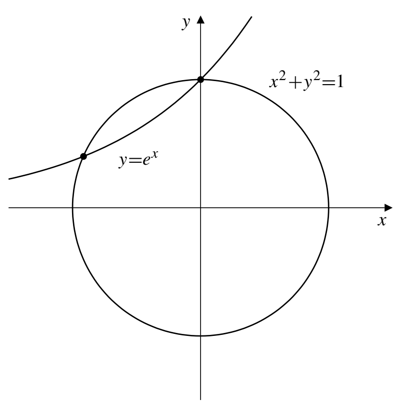


Fig. 13.6.2

$$\begin{array}{lll} f(x,y) = x^2 + y^2 - 1 & f_1(x,y) = 2x & g_1(x,y) = -e^x \\ g(x,y) = y - e^x & f_2(x,y) = 2y & g_2(x,y) = 1 \end{array}.$$

Evidently one solution is $x = 0$, $y = 1$. The second solution is near $(-1, 0)$. We try $x_0 = -0.9$, $y_0 = 0.2$.

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | −0.9000000 | 0.2000000 | −0.1500000 | −0.2065697 |
| 1 | −0.9411465 | 0.3898407 | 0.0377325 | −0.0003395 |
| 2 | −0.9170683 | 0.3995751 | 0.0006745 | −0.0001140 |

| | | | | |
|---|---|---|---|---|
| 3 | −0.9165628 | 0.3998911 | 0.0000004 | −0.0000001 |
| 4 | −0.9165626 | 0.3998913 | 0.0000000 | 0.0000000 |

The second solution is $x = -0.9165626$, $y = 0.3998913$.

**3.**

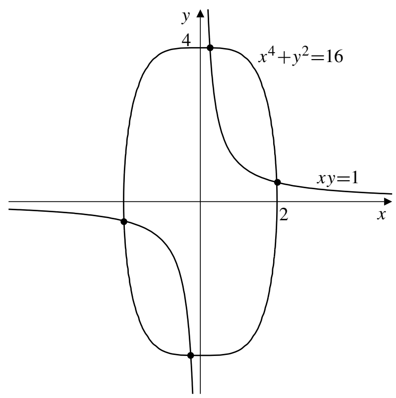


Fig. 13.6.3

$$f(x,y) = x^4 + y^2 - 16 \qquad f_1(x,y) = 4x^3 \qquad g_1(x,y) = y\,.$$
$$g(x,y) = xy - 1 \qquad f_2(x,y) = 2y \qquad g_2(x,y) = x$$

There are four solutions as shown in the figure. We will find the two in the first quadrant; the other two are the negatives of these by symmetry.
The first quadrant solutions appear to be near (1.9, 0.5) and (0.25, 3.9).

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | 1.9000000 | 0.5000000 | −2.7179000 | −0.0500000 |
| 1 | 1.9990542 | 0.5002489 | 0.2200049 | 0.0000247 |
| 2 | 1.9921153 | 0.5019730 | 0.0011548 | −0.0000120 |
| 3 | 1.9920783 | 0.5019883 | 0.0000000 | 0.0000000 |
| 4 | 1.9920783 | 0.5019883 | 0.0000000 | 0.0000000 |

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | 0.2500000 | 3.9000000 | −0.7860937 | −0.0250000 |
| 1 | 0.2499499 | 4.0007817 | 0.0101569 | −0.0000050 |
| 2 | 0.2500305 | 3.9995117 | 0.0000016 | −0.0000001 |
| 3 | 0.2500305 | 3.9995115 | 0.0000000 | 0.0000000 |

The four solutions are $x = \pm 1.9920783$, $\pm y = 0.5019883$, and $x = \pm 0.2500305$, $y = \pm 3.9995115$.

**4.**

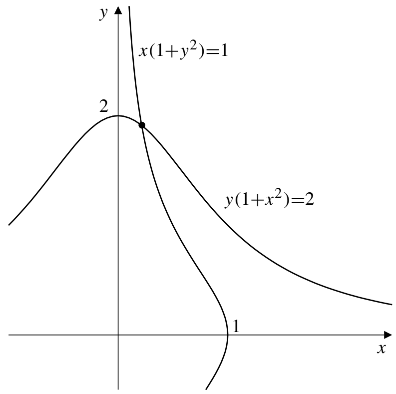


Fig. 13.6.4

$$\begin{array}{lll} f(x,y) = x(1+y^2) - 1 & f_1(x,y) = 1 + y^2 & g_1(x,y) = 2xy \\ g(x,y) = y(1+x^2) - 2 & f_2(x,y) = 2xy & g_2(x,y) = 1 + x^2 \end{array}.$$

The solution appears to be near $x = 0.2$, $y = 1.8$.

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | 0.2000000 | 1.8000000 | −0.1520000 | −0.1280000 |
| 1 | 0.2169408 | 1.9113487 | 0.0094806 | 0.0013031 |
| 2 | 0.2148268 | 1.9117785 | −0.0000034 | 0.0000081 |
| 3 | 0.2148292 | 1.9117688 | 0.0000000 | 0.0000000 |

The solution is $x = 0.2148292$, $y = 1.9117688$.

**5.**

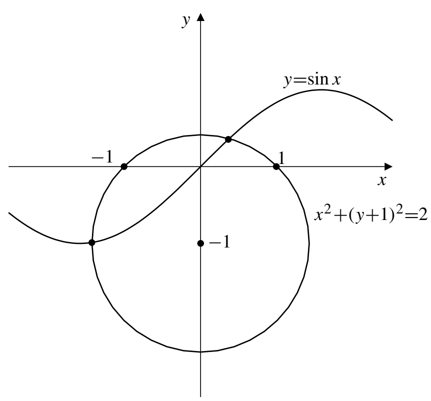


Fig. 13.6.5

$$\begin{array}{lll} f(x,y) = y - \sin x & f_1(x,y) = -\cos x & g_1(x,y) = 2x \\ g(x,y) = x^2 + (y+1)^2 - 2 & f_2(x,y) = 1 & g_2(x,y) = 2(y+1) \end{array}.$$

Solutions appear to be near $(0.5, 0.3)$ and $(-1.5, -1)$.

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | 0.5000000 | 0.3000000 | −0.1794255 | −0.0600000 |
| 1 | 0.3761299 | 0.3707193 | 0.0033956 | 0.0203450 |
| 2 | 0.3727877 | 0.3642151 | 0.0000020 | 0.0000535 |
| 3 | 0.3727731 | 0.3641995 | 0.0000000 | 0.0000000 |
| 4 | 0.3727731 | 0.3641995 | 0.0000000 | 0.0000000 |

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | $-1.5000000$ | $-1.0000000$ | $-0.0025050$ | 0.2500000 |
| 1 | $-1.4166667$ | $-0.9916002$ | $-0.0034547$ | 0.0070150 |
| 2 | $-1.4141680$ | $-0.9877619$ | $-0.0000031$ | 0.0000210 |
| 3 | $-1.4141606$ | $-0.9877577$ | 0.0000000 | 0.0000000 |
| 4 | $-1.4141606$ | $-0.9877577$ | 0.0000000 | 0.0000000 |

The solutions are $x = 0.3727731$, $y = 0.3641995$, and $x = -1.4141606$, $y = -0.9877577$.

**6.**

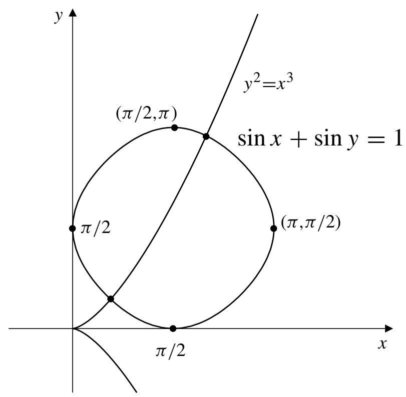


Fig. 13.6.6

$f(x, y) = \sin x + \sin y - 1 \qquad f_1(x, y) = \cos x \qquad g_1(x, y) = -3x^2.$

$g(x, y) = y^2 - x^3 \qquad f_2(x, y) = \cos y \qquad g_2(x, y) = 2y$

There are infinitely many solutions for the given pair of equations, since the level curve of $f(x, y) = 0$ is repeated periodically throughout the plane. We will find the two solutions closest to the origin in the first quadrant. From the figure, it appears that these solutions are near (0.6, 0.4) and (2, 3).

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | 0.6000000 | 0.4000000 | $-0.0459392$ | $-0.0560000$ |
| 1 | 0.5910405 | 0.4579047 | $-0.0007050$ | 0.0032092 |
| 2 | 0.5931130 | 0.4567721 | $-0.0000015$ | $-0.0000063$ |
| 3 | 0.5931105 | 0.4567761 | 0.0000000 | 0.0000000 |
| 4 | 0.5931105 | 0.4567761 | 0.0000000 | 0.0000000 |

| $n$ | $x_n$ | $y_n$ | $f(x_n, y_n)$ | $g(x_n, y_n)$ |
|---|---|---|---|---|
| 0 | 2.0000000 | 3.0000000 | 0.0504174 | 1.0000000 |
| 1 | 2.0899016 | 3.0131366 | $-0.0036336$ | $-0.0490479$ |
| 2 | 2.0854887 | 3.0116804 | $-0.0000086$ | $-0.0001199$ |
| 3 | 2.0854779 | 3.0116770 | 0.0000000 | 0.0000000 |
| 4 | 2.0854779 | 3.0116770 | 0.0000000 | 0.0000000 |

The solutions are $x = 0.5931105$, $y = 0.4567761$, and $x = 2.0854779$, $y = 3.0116770$.

**7.** By analogy with the two-dimensional case, the Newton's Method iteration formulas are

$$x_{n+1} = x_n - \frac{1}{\Delta}\left.\begin{vmatrix} f & f_2 & f_3 \\ g & g_2 & g_3 \\ h & h_2 & h_3 \end{vmatrix}\right|_{(x_n,y_n,z_n)} \qquad y_{n+1} = y_n - \frac{1}{\Delta}\left.\begin{vmatrix} f_1 & f & f_3 \\ g_1 & g & g_3 \\ h_1 & h & h_3 \end{vmatrix}\right|_{(x_n,y_n,z_n)}$$

$$z_{n+1} = z_n - \frac{1}{\Delta}\left.\begin{vmatrix} f_1 & f_2 & f \\ g_1 & g_2 & g \\ h_1 & h_2 & h \end{vmatrix}\right|_{(x_n,y_n,z_n)} \qquad \text{where } \Delta = \left.\begin{vmatrix} f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \\ h_1 & h_2 & h_3 \end{vmatrix}\right|_{(x_n,y_n,z_n)}$$

**8.** $f(x,y,z) = y^2 + z^2 - 3$ $\quad g(x,y,z) = x^2 + z^2 - 2$ $\quad h(x,y,z) = x^2 - z$

$f_1(x,y,z) = 0$ $\quad g_1(x,y,z) = 2x$ $\quad h_1(x,y,z) = 2x$

$f_2(x,y,z) = 2y$ $\quad g_2(x,y,z) = 0$ $\quad h_2(x,y,z) = 0$

$f_3(x,y,z) = 2z$ $\quad g_3(x,y,z) = 2z$ $\quad h_3(x,y,z) = -1$

It is easily seen that the system

$$f(x,y,z) = 0, \qquad g(x,y,z) = 0, \qquad h(x,y,z) = 0$$

has first-quadrant solution $x = z = 1$, $y = \sqrt{2}$. Let us start at the "guess" $x_0 = y_0 = z_0 = 2$.

| $n$ | $x_n$ | $y_n$ | $z_n$ | $f(x_n,y_n,z_n)$ | $g(x_n,y_n,z_n)$ | $h(x_n,y_n,z_n)$ |
|---|---|---|---|---|---|---|
| 0 | 2.0000000 | 2.0000000 | 2.0000000 | 5.0000000 | 6.0000000 | 2.0000000 |
| 1 | 1.3000000 | 1.5500000 | 1.2000000 | 0.8425000 | 1.1300000 | 0.4900000 |
| 2 | 1.0391403 | 1.4239564 | 1.0117647 | 0.0513195 | 0.1034803 | 0.0680478 |
| 3 | 1.0007592 | 1.4142630 | 1.0000458 | 0.0002313 | 0.0016104 | 0.0014731 |
| 4 | 1.0000003 | 1.4142136 | 1.0000000 | 0.0000000 | 0.0000006 | 0.0000006 |
| 5 | 1.0000000 | 1.4142136 | 1.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |

**9.** $f(x,y) = y - x^2$ $\quad f_1(x,y) = -2x$ $\quad g_1(x,y) = -3x^2$

$g(x,y) = y - x^3$ $\quad f_2(x,y) = 1$ $\quad g_2(x,y) = 1$

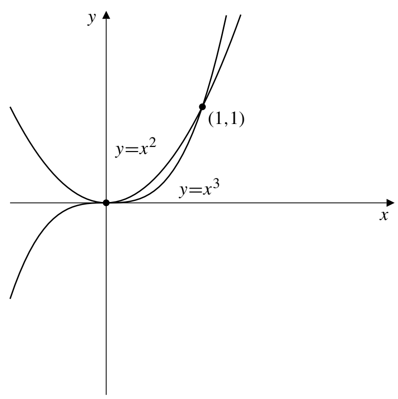


Fig. 13.6.9

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0.1000000 | 0.1000000 |
| 1 | 0.0470588 | −0.0005882 |
| 2 | 0.0229337 | −0.0000561 |
| 3 | 0.0113307 | −0.0000062 |
| 4 | 0.0056327 | −0.0000007 |
| 5 | 0.0028083 | −0.0000001 |
| ⋮ | | |
| 15 | 0.0000027 | 0.0000000 |
| 16 | 0.0000014 | 0.0000000 |
| 17 | 0.0000007 | 0.0000000 |
| 18 | 0.0000003 | 0.0000000 |

| $n$ | $x_n$ | $y_n$ |
|---|---|---|
| 0 | 0.9000000 | 0.9000000 |
| 1 | 1.0285714 | 1.0414286 |
| 2 | 1.0015038 | 1.0022771 |
| 3 | 1.0000045 | 1.0000068 |
| 4 | 1.0000000 | 1.0000000 |

Eighteen iterations were needed to obtain the solution $x = y = 0$ correct to six decimal places, starting from $x = y = 0.1$. This slow convergence is due to the fact that the curves $y = x^2$ and $y = x^3$ are tangent at $(0,0)$. Only four iterations were needed to obtain the solution $x = y = 1$ starting from $x = y = 0$, because, although the angle between the curves is small at $(1,1)$, it is not 0. The curves are not tangent there.

## Section 13.7 Calculations with Maple (page 751)

**1.** The equation $z = xy$ can be used to reduce the given system of three equations in three variables to a system of 2 equations in two variables:

$$x^2 + y^2 + x^2y^2 = 1$$
$$6x^2y = 1.$$

The first equation can only be satisfied by points $(x, y)$ satisfying $|x| \le 1$ and $|y| \le 1$.

```
> Digits := 6:

> eqns := {x^2+y^2+(x*y)^2=1,
6*x^2*y=1}:
```

We use `plots[implicitplot]` to locate suitable starting points for `fsolve`.

```
> plots[implicitplot](eqns,x=-1..1,
y=-1..1);
```

The resulting plot (omitted here) shows four roots; two in the first quadrant near (.9, .2) and (.5, .8), and two more that are reflections of these in the $y$-axis. We use `fsolve` to find the two first-quadrant roots and calculate the corresponding values for $z$ by substitution.

```
> vars := {x=0.9, y=0.2}:

> xy := fsolve(eqns,vars);

> z=evalf(subs(xy,x*y));
```

$$xy := \{x = 0.968971, y = 0.177512\}$$
$$z = 0.172004$$

```
> vars := {x=0.5, y=0.8}:

> xy := fsolve(eqns,vars);

> z=evalf(subs(xy,x*y));
```

$$xy := \{y = 0.812044, x = 0.453038\}$$
$$z = 0.367887$$

The four solutions are $(x, y, z) = (\pm 0.96897, 0.17751, \pm 0.17200)$ and $(x, y, z) = (\pm 0.45304, 81204, \pm 0.36789)$, rounded to five figures.

**2.** The equation $y = \sin z$ can be used to reduce the given system of three equations in three variables to a system of 2 equations in two variables:

$$x^4 + \sin^2 z + z^2 = 1$$
$$z + z^3 + z^4 = x + \sin z.$$

The first equation can only be satisfied by points $(x, z)$ satisfying $|x| \le 1$ and $|z| \le 1$.

```
> Digits := 6:

> eqns := {x^4+(sin(z))^2+z^2=1,

> z+z^3+z4̂=x+sin(z)}:
```

We use `plots[implicitplot]` to locate suitable starting points for `fsolve`.

```
> plots[implicitplot](eqns,x=-1..1,
z=-1..1);
```

The resulting plot shows two roots in the $xz$-plane, one near $(0.6, 0.7)$ and the other near $(-0.2, -0.7)$. We use `fsolve` to find them more precisely, and we then calculate the corresponding values for $y$ by substitution.

```
> vars := {x=0.6, z=0.7}:

> xz := fsolve(eqns,vars);

> y=evalf(subs(xz,sin(z)));
```

$$xy := \{z = 0.686259, x = 0.597601\}$$
$$y = 0.633648$$

```
> vars := {x=-0.2, z=-0.7}:

> xy := fsolve(eqns,vars);

> y=evalf(subs(xz,sin(z)));
```

$$xy := \{z = -0.738742, x = -0.170713\}$$
$$y = -0.673358$$

The two solutions are $(x, y, z) = (0.59760, 0.63365, 0.68626)$ and $(x, y, z) = (-0.17071, -0.67336, -0.73874)$, each rounded to five figures.

**3.** First define the expression $f$:

```
> f := (x*y-x-2*y)/(1+x^2+y^2)^2:
```

Because the numerator grows much more slowly than the denominator for large $x^2 + y^2$, global max and min values will be near the origin. We plot contours of $f$ on, say, the square $|x| \le 2$, $|y| \le 2$.

```
> contourplot(f(x,y), x=-2..2,
>      y=-2..2, contours=16);
```

The resulting plot (which we omit here) indicates the only likely critical points are near $(-0.3, -0.6)$ and $(0.2, 0.6)$. We determine them using `fsolve` and use substitution to evaluate $f$.

```
> Digits := 6:

> eqns := {diff(f,x), diff(f,y)}:

> vars := {x=-0.3, y=-0.6}:

> cp := fsolve(eqns,vars);

> val=evalf(subs(cp,f));
```

$$cp := \{x = -.338532, y = -.520621\}$$
$$val = 0.810414$$

```
> vars := {x=0.2, y=0.6}:

> cp := fsolve(eqns,vars);

> val=evalf(subs(cp,f));
```

$$cp := \{x = 0.133192, y = 0.536823\}$$
$$val = -.665721$$

There are only two critical points and the values of $f$ at them have opposite sign. Since $f \to 0$ as $x^2 + y^2 \to \infty$, $f$ has absolute maximum value 0.81041 at $(-0.33853, -0.52062)$ and absolute minimum value $-0.66572$ at $(0.13319, 0.53682)$, all numerical values rounded to five figures.

**4.** We begin with

```
> Digits := 6:

> f := 1 - 10*x^4 - 8*y^4 - 7*z^4:

> g := y*z - x*y*z - x - 2*y + z:

> h := f + g:
```

Since $h = 1$ at $(0, 0, 0)$ and $h \to -\infty$ as $x^2 + y^2 + z^2$ increases, the maximum value of $g$ will be near $(0, 0, 0)$.

We can try various choices of starting points including $(0, 0, 0)$ itself. It turns out they all lead to the same critical point:

```
> eqns :=
{diff(h,x),diff(h,y),diff(h,z)}:
> vars := x=0,y=0,z=0:
> cp := fsolve(eqns,vars); val =
evalf(subs(cp,h));
```

$$cp := \{x = -.28429, y = -.372953, z = 0.265109\}$$
$$val = 1.91367$$

The absolute maximum value of $h$ is 1.91367 (to five decimal places).

**5.** Because of the small coefficients on the $xy$ and $xz$ terms and the fact that without them $f$ would certainly have a minimum value near the origin, we can use `fsolve` starting with various points near the origin. It turns out they all lead to only one critical point.

```
> Digits := 6:

> f := x^2 + y^2 + z^2

> +0.2*x*y-0.3*x*z+4*x-y:

> eqns :=
{diff(f,x),diff(f,y),diff(f,z)}:
> vars := x=0,y=0,z=0:
> cp := fsolve(eqns,vars); val =
evalf(subs(cp,f));
```

$$cp := \{x = -2.11886, y = 0.711886, z = -.317829\}$$
$$val = -4.59368$$

To confirm that this CP does give a local minimum, you can calculate `VectorCalculus[Hessian](f,[x,y,z])` and then `evalf` the result of `LinearAlgebra[Eigenvalues](subs(cp,%))` and observe that all three eigenvalues are positive.

The minimum value of $f$ is $-4.59368$.

**6.** First define the function:

```
> f := (x+1.1y-0.9z+1)/(1+x^2+y^2);
```

Since $f(x,y,z) \to 0$ as $x^2+y^2+z^2 \to \infty$ we expect $f$ to have maximum and minimum values in some neighbourhood of the origin. If the numerator were instead $x+y-z$, we would expect the extreme values to occur along the line $x = y = -z$ by symmetry. Accordingly, we use starting points along this line.

```
> Digits := 6:

> f     :=
(x+1.1*y-0.9*z+1)/(1+x^2+y^2+z^2):


> eqns    :=
{diff(f,x),diff(f,y),diff(f,z)}:
> vars := x=1,y=1,z=-1:
> cp := fsolve(eqns,vars); val =
evalf(subs(cp,f));
```

This attempt fails; `fsolve` cannot locate a solution. We try a guess closer to the origin.

```
> vars := x=0.5,y=0.5,z=-0.5:
> cp := fsolve(eqns,vars); val =
evalf(subs(cp,f));
```

$$cp := \{y = 0.366057, z = -.299501, x = 0.332779\}$$
$$val = 1.50250$$

```
> vars := x=-0.5,y=-0.5,z=+0.5:
> cp := fsolve(eqns,vars); val =
evalf(subs(cp,f));
```

$$cp := \{x = -.995031, z = 0.895528, y = -1.09453\}$$
$$val = -.502494$$

The eigenvalues of the Hessian matrix of $f$ at each of these critical points confirms that the first is a local maximum and gives $f$ its absolute maximum value 1.50250 and the second is a local minimum so the absolute minimum value of $f$ is $-0.502494$.

## Review Exercises 13 (page 752)

**1.** $f(x,y) = xye^{-x+y}$

$f_1(x,y) = (y-xy)e^{-x+y} = y(1-x)e^{-x+y}$

$f_2(x,y) = (x+xy)e^{-x+y} = x(1+y)e^{-x+y}$

$A = f_{11} = (-2y+xy)e^{-x+y}$

$B = f_{12} = (1-x+y-xy)e^{-x+y}$

$C = f_{22} = (2x+xy)e^{-x+y}$.

For CP: either $y = 0$ or $x = 1$, and either $x = 0$ or $y = -1$. The CPs are $(0,0)$ and $(1,-1)$.

| CP | $A$ | $B$ | $C$ | $AC-B^2$ | class |
|---|---|---|---|---|---|
| $(0,0)$ | 0 | 1 | 0 | $-1$ | saddle |
| $(1,-1)$ | $e^{-2}$ | 0 | $e^{-2}$ | $e^{-4}$ | loc. min |

**2.** $f(x,y) = x^2y - 2xy^2 + 2xy$

$f_1(x,y) = 2xy - 2y^2 + 2y = 2y(x-y+1)$

$f_2(x,y) = x^2 - 4xy + 2x = x(x-4y+2)$

$A = f_{11} = 2y$

$B = f_{12} = 2x - 4y + 2$

$C = f_{22} = -4x$.

For CP: either $y = 0$ or $x - y + 1 = 0$, and either $x = 0$ or $x - 4y + 2 = 0$. The CPs are $(0,0)$, $(0,1)$, $(-2,0)$, and $(-2/3, 1/3)$.

| CP | $A$ | $B$ | $C$ | $AC-B^2$ | class |
|---|---|---|---|---|---|
| $(0,0)$ | 0 | 2 | 0 | $-4$ | saddle |
| $(0,1)$ | 2 | $-2$ | 0 | $-4$ | saddle |
| $(-2,0)$ | 0 | $-2$ | 8 | $-4$ | saddle |
| $(-\frac{2}{3},\frac{1}{3})$ | $\frac{2}{3}$ | $-\frac{2}{3}$ | $\frac{8}{3}$ | $\frac{4}{3}$ | loc. min |

**3.** $f(x,y) = \dfrac{1}{x} + \dfrac{4}{y} + \dfrac{9}{4-x-y}$

$f_1(x,y) = -\dfrac{1}{x^2} + \dfrac{9}{(4-x-y)^2}$

$f_2(x,y) = -\dfrac{4}{y^2} + \dfrac{9}{(4-x-y)^2}$

$A = f_{11} = \dfrac{2}{x^3} + \dfrac{18}{(4-x-y)^3}$

$B = f_{12} = \dfrac{18}{(4-x-y)^3}$

$C = f_{22} = \dfrac{8}{y^3} + \dfrac{18}{(4-x-y)^3}$.

For CP: $y^2 = 4x^2$ so that $y = \pm 2x$. If $y = 2x$, then $9x^2 = (4-3x)^2$, from which $x = 2/3$, $y = 4/3$. If $y = -2x$, then $9x^2 = (4+x)^2$, from which $x = -1$ or $x = 2$. The CPs are $(2/3, 4/3)$, $(-1,2)$, and $(2,-4)$.

| CP | $A$ | $B$ | $C$ | $AC-B^2$ | class |
|---|---|---|---|---|---|
| $(-1,2)$ | $-\frac{4}{3}$ | $\frac{2}{3}$ | $\frac{5}{3}$ | $-\frac{8}{3}$ | saddle |
| $(2,-4)$ | $\frac{1}{3}$ | $\frac{1}{12}$ | $-\frac{1}{24}$ | $-\frac{1}{48}$ | saddle |
| $(\frac{2}{3},\frac{4}{3})$ | 9 | $\frac{9}{4}$ | $\frac{45}{8}$ | $\frac{729}{16}$ | loc. min |

**4.** $f(x,y) = x^2y(2-x-y) = 2x^2y - x^3y - x^2y^2$
$f_1(x,y) = 4xy - 3x^2y - 2xy^2 = xy(4-3x-2y)$
$f_2(x,y) = 2x^2 - x^3 - 2x^2y = x^2(2-x-2y)$
$A = f_{11} = 4y - 6xy - 2y^2$
$B = f_{12} = 4x - 3x^2 - 4xy$
$C = f_{22} = -2x^2.$
$(0,y)$ is a CP for any $y$. If $x \neq 0$ but $y = 0$, then $x = 2$ from the second equation. Thus $(2,0)$ is a CP.

If neither $x$ nor $y$ is 0, then $x + 2y = 2$ and $3x + 2y = 4$, so that $x = 1$ and $y = 1/2$. The third CP is $(1, 1/2)$.

| CP | $A$ | $B$ | $C$ | $AC - B^2$ | class |
|---|---|---|---|---|---|
| $(0,y)$ | $4y-2y^2$ | 0 | 0 | 0 | ? |
| $(2,0)$ | 0 | $-4$ | $-8$ | $-16$ | saddle |
| $(1,\frac{1}{2})$ | $-\frac{3}{2}$ | $-1$ | $-2$ | 2 | loc. max |

The second derivative test is unable to classify the line of critical points along the $y$-axis. However, direct inspection of $f(x,y)$ shows that these are local minima if $y(2-y) > 0$ (that is, if $0 < y < 2$) and local maxima if $y(2-y) < 0$ (that is, if $y < 0$ or $y > 2$). The points $(0,0)$ and $(0,2)$ are neither maxima nor minima, so they are saddle points.

**5.** $f(x,y,z) = g(s) = s + (1/s)$, where $s = x^2+y^2+z^2$. Since $g(s) \to \infty$ as $s \to \infty$ or $s \to 0+$, $g$ must have a minimum value at a critical point in $(0,\infty)$. For CP: $0 = g'(s) = 1 - (1/s^2)$, that is, $s = 1$. $g(1) = 2$. The minimum value of $f$ is 2, and is assumed at every point of the sphere $x^2+y^2+z^2 = 1$.

**6.**
$$\begin{aligned} x^2 + y^2 + z^2 - xy - xz - yz &\\ = \frac{1}{2}\big[(x^2-2xy+y^2) + (x^2-2xz+z^2) &\\ + (y^2-2yz+z^2)\big] &\\ = \frac{1}{2}\big[(x-y)^2 + (x-z)^2 + (y-z)^2\big] \geq 0.& \end{aligned}$$
The minimum value, 0, is assumed at the origin and at all points of the line $x = y = z$.

**7.** $f(x,y) = xye^{-x^2-4y^2}$ satisfies $\displaystyle\lim_{x^2+y^2\to\infty} f(x,y) = 0$. Since $f(1,1) > 0$ and $f(-1,1) < 0$, $f$ must have maximum and minimum values and these must occur at critical points. For CP:

$$0 = f_1 = e^{-x^2-4y^2}(y - 2x^2y) = e^{-x^2-4y^2}y(1-2x^2)$$
$$0 = f_2 = e^{-x^2-4y^2}(x - 8xy^2) = e^{-x^2-4y^2}x(1-8y^2).$$

The CPs are $(0,0)$ (where $f = 0$), $\pm\big(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\big)$ (where $f = 1/4e$), and $\pm\big(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}\big)$ (where $f = -1/4e$). Thus $f$ has maximum value $1/4e$ and minimum value $-1/4e$.

**8.** $f(x,y) = (4x^2 - y^2)e^{-x^2+y^2}$
$f_1(x,y) = e^{-x^2+y^2}2x(4 - 4x^2 + y^2)$
$f_2(x,y) = e^{-x^2+y^2}(-2y)(1 - 4x^2 + y^2).$
$f$ has CPs $(0,0)$, $(\pm1, 0)$. $f(0,0) = 0$.
$f(\pm1, 0) = 4/e$.

a) Since $f(0,y) = -y^2e^{y^2} \to -\infty$ as $y \to \pm\infty$, and since $f(x,x) = 3x^2e^0 = 3x^2 \to \infty$ as $x \to \pm\infty$, $f$ does not have a minimum or a maximum value on the $xy$-plane.

b) On $y = 3x$, $f(x,3x) = -5x^2e^{8x^2} \to -\infty$ as $x \to \infty$. Thus $f$ can have no minimum value on the wedge $0 \leq y \leq 3x$. However, as noted in (a), $f(x,x) \to \infty$ as $x \to \infty$. Since $(x,x)$ is in the wedge for $x > 0$, $f$ cannot have a maximum value on the wedge either.

**9.** Let the three pieces of wire have lengths $x$, $y$, and $L - x - y$ cm, respectively. The sum of areas of the squares is

$$S = \frac{1}{16}\big(x^2 + y^2 + (L-x-y)^2\big),$$

for which we must find extreme values over the triangle $x \geq 0$, $y \geq 0$, $x + y \leq L$. For critical points:

$$0 = \frac{\partial S}{\partial x} = \frac{1}{8}\big(x - (L-x-y)\big)$$
$$0 = \frac{\partial S}{\partial y} = \frac{1}{8}\big(y - (L-x-y)\big),$$

from which we obtain $x = y = L/3$. This CP is inside the triangle, and $S = L^2/48$ at it.

On the boundary segment $x = 0$, we have

$$S = \frac{1}{16}\big(y^2 + (L-y)^2\big), \quad (0 \leq y \leq L).$$

At $y = 0$ or $y = L$, we have $S = L^2/16$. For critical points

$$0 = \frac{dS}{dy} = \frac{1}{8}\big(y - (L-y)\big),$$

so $y = L/2$ and $S = L^2/32$. By symmetry the extreme values of $S$ on the other two boundary segments are the same.
Thus the minimum value of $S$ is $L^2/48$, and corresponds to three equal squares. The maximum value of $S$ is $L^2/16$, and corresponds to using the whole wire for one square.

**10.** Let the length, width, and height of the box be $x$, $y$, and $z$ in, respectively. Then the girth is $g = 2x + 2y$. We require $g + z \le 120$ in. The volume $V = xyz$ of the box will be maximized under the constraint $2x + 2y + z = 120$, so we look for CPs of

$$L = xyz + \lambda(2x + 2y + z - 120).$$

For CPs:

$$0 = \frac{\partial L}{\partial x} = yz + 2\lambda \qquad \text{(A)}$$
$$0 = \frac{\partial L}{\partial y} = xz + 2\lambda \qquad \text{(B)}$$
$$0 = \frac{\partial L}{\partial z} = xy + \lambda \qquad \text{(C)}$$
$$0 = \frac{\partial L}{\partial \lambda} = 2x + 2y + z - 120. \qquad \text{(D)}$$

Comparing (A), (B), and (C), we see that $x = y = z/2$. Then (D) implies that $3z = 120$, so $z = 40$ and $x = y = 20$ in. The largest box has volume

$$V = (20)(20)(40) = 16,000 \text{ in}^3,$$

or, about 9.26 cubic feet.

**11.** The ellipse $(x/a)^2+(y/b)^2 = 1$ contains the rectangle $-1 \le x \le 1$, $-2 \le y \le 2$, if $(1/a^2) + (4/b^2) = 1$. The area of the ellipse is $A = \pi ab$. We minimize $A$ by looking for critical points of

$$L = \pi ab + \lambda\left(\frac{1}{a^2} + \frac{4}{b^2} - 1\right).$$

For CPs:

$$0 = \frac{\partial L}{\partial a} = \pi b - \frac{2\lambda}{a^3} \qquad \text{(A)}$$
$$0 = \frac{\partial L}{\partial b} = \pi a - \frac{8\lambda}{b^3} \qquad \text{(B)}$$
$$0 = \frac{\partial L}{\partial \lambda} = \frac{1}{a^2} + \frac{4}{b^2} - 1. \qquad \text{(C)}$$

Multiplying (A) by $a$ and (B) by $b$, we obtain $2\lambda/a^2 = 8\lambda/b^2$, so that either $\lambda = 0$ or $b = 2a$. Now $\lambda = 0$ implies $b = 0$, which is inconsistent with (C). If $b = 2a$, then (C) implies that $2/a^2 = 1$, so $a = \sqrt{2}$. The smallest area of the ellipse is $V = 4\pi$ square units.

**12.** The ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$ contains the rectangle $-1 \le x \le 1$, $-2 \le y \le 2$, $-3 \le z \le 3$, provided $(1/a^2) + (4/b^2) + (9/c)^2 = 1$. The volume of the ellipsoid is $V = 4\pi abc/3$. We minimize $V$ by looking for critical points of

$$L = \frac{4\pi}{3}abc + \lambda\left(\frac{1}{a^2} + \frac{4}{b^2} + \frac{9}{c^2} - 1\right).$$

For CPs:

$$0 = \frac{\partial L}{\partial a} = \frac{4\pi}{3}bc - \frac{2\lambda}{a^3} \qquad \text{(A)}$$
$$0 = \frac{\partial L}{\partial b} = \frac{4\pi}{3}ac - \frac{8\lambda}{b^3} \qquad \text{(B)}$$
$$0 = \frac{\partial L}{\partial c} = \frac{4\pi}{3}ab - \frac{18\lambda}{c^3} \qquad \text{(C)}$$
$$0 = \frac{\partial L}{\partial \lambda} = \frac{1}{a^2} + \frac{4}{b^2} + \frac{9}{c^2} - 1. \qquad \text{(D)}$$

Multiplying (A) by $a$, (B) by $b$, and (C) by $c$, we obtain $2\lambda/a^2 = 8\lambda/b^2 = 18\lambda/c^2$, so that either $\lambda = 0$ or $b = 2a$, $c = 3a$. Now $\lambda = 0$ implies $bc = 0$, which is inconsistent with (D). If $b = 2a$ and $c = 3a$, then (D) implies that $3/a^2 = 1$, so $a = \sqrt{3}$. The smallest volume of the ellipsoid is

$$V = \frac{4\pi}{3}(\sqrt{3})(2\sqrt{3})(3\sqrt{3}) = 24\sqrt{3}\pi \text{ cubic units.}$$

**13.** The box $-1 \le x \le 1$, $-2 \le y \le 2$, $0 \le z \le 2$ is contained in the region

$$0 \le z \le a\left(1 - \frac{x^2}{b^2} - \frac{y^2}{c^2}\right)$$

provided that $(2/a) + (1/b^2) + (4/c^2) = 1$. The volume of the region would normally be calculated via a "double integral" which we have not yet encountered. (See Chapter 5.) It can also be done directly by slicing. A horizontal plane at height $z$ (where $0 \le z \le a$) intersects the region in an elliptic disk bounded by the ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1 - \frac{z}{a}.$$

The area of this disk is

$$A(z) = \pi \left(b\sqrt{1-\frac{z}{a}}\right)\left(c\sqrt{1-\frac{z}{a}}\right) = \pi bc\left(1-\frac{z}{a}\right).$$

Thus the region has volume

$$V = \pi bc \int_0^a \left(1-\frac{z}{a}\right) dz = \frac{\pi abc}{2}.$$

Thus we look for critical points of

$$L = \frac{\pi abc}{2} + \lambda\left(\frac{2}{a}+\frac{1}{b^2}+\frac{4}{c^2}-1\right).$$

For critical points:

$$0 = \frac{\partial L}{\partial a} = \frac{\pi}{2}bc - \frac{2\lambda}{a^2} \qquad \text{(A)}$$
$$0 = \frac{\partial L}{\partial b} = \frac{\pi}{2}ac - \frac{2\lambda}{b^3} \qquad \text{(B)}$$
$$0 = \frac{\partial L}{\partial c} = \frac{\pi}{2}ab - \frac{8\lambda}{c^3} \qquad \text{(C)}$$
$$0 = \frac{\partial L}{\partial \lambda} = \frac{2}{a}+\frac{1}{b^2}+\frac{4}{c^2}-1. \qquad \text{(D)}$$

Multiplying (A) by $a$, (B) by $b$, and (C) by $c$, we obtain $2\lambda/a = 2\lambda/b^2 = 8\lambda/c^2$, so that either $\lambda = 0$ or $b^2 = a$, $c^2 = 4a$. Now $\lambda = 0$ implies $bc = 0$, which is inconsistent with (D). If $b^2 = a$ and $c^2 = 4a$, then (D) implies that $4/a = 1$, so $a = 4$. The smallest volume of the region is $V = \pi(4)(2)(4)/2 = 16\pi$ cubic units.

**14.**

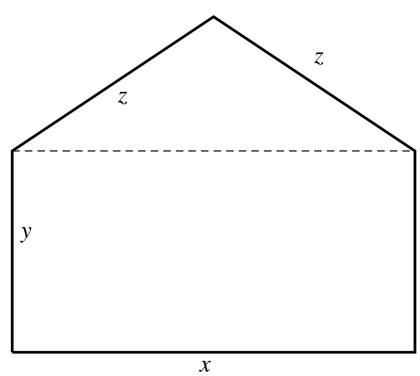


Fig. R-13.14

The area of the window is

$$A = xy + \frac{x}{2}\sqrt{z^2 - \frac{x^2}{4}},$$

or, since $x + 2y + 2z = L$,

$$A = \frac{x}{2}\left(L - x - 2z + \sqrt{z^2 - \frac{x^2}{4}}\right).$$

For maximum $A$, we look for critical points:

$$\begin{aligned} 0 = \frac{\partial A}{\partial x} &= \frac{1}{2}\left(L - x - 2z + \sqrt{z^2 - \frac{x^2}{4}}\right) \\ &\quad + \frac{x}{2}\left(-1 - \frac{x}{4\sqrt{z^2 - \frac{x^2}{4}}}\right) \\ &= \frac{L}{2} - x - z + \frac{2z^2 - x^2}{4\sqrt{z^2 - \frac{x^2}{4}}} \qquad \text{(A)} \\ 0 = \frac{\partial A}{\partial z} &= -x + \frac{xz}{2\sqrt{z^2 - \frac{x^2}{4}}}. \qquad \text{(B)} \end{aligned}$$

Now (B) implies that either $x = 0$ or $z = 2\sqrt{z^2 - (x^2/4)}$. But $x = 0$ gives zero area rather than maximum area, so the second alternative must hold, and it implies that $z = x/\sqrt{3}$. Then (A) gives

$$\frac{L}{2} = \left(1 + \frac{1}{\sqrt{3}}\right)x + \frac{x}{2\sqrt{3}},$$

from which we obtain $x = L/(2+\sqrt{3})$. The maximum area of the window is, therefore,

$$A\bigg|_{x=\frac{L}{2+\sqrt{3}},\ z=\frac{L/\sqrt{3}}{2+\sqrt{3}}} = \frac{1}{4}\,\frac{L^2}{2+\sqrt{3}} \approx 0.0670L^2 \text{ sq. units.}$$

**15.** If $1,000x widgets per month are manufactured and sold for $y per widget, then the monthly profit is $1,000P, where

$$P = xy - \frac{x^2y^3}{27} - x.$$

We are required to maximize $P$ over the rectangular region $R$ satisfying $0 \le x \le 3$ and $0 \le y \le 2$. First look for critical points:

$$0 = \frac{\partial P}{\partial x} = y - \frac{2xy^3}{27} - 1 \qquad \text{(A)}$$
$$0 = \frac{\partial P}{\partial y} = x - \frac{x^2y^2}{9}. \qquad \text{(B)}$$

(B) implies that $x = 0$, which yields zero profit, or $xy^2 = 9$, which, when substituted into (A), gives $y = 3$ and $x = 1$. Unfortunately, the critical point $(1, 3)$ lies outside of $R$. Therefore the maximum $P$ must occur on the boundary of $R$.
We consider all four boundary segments of $R$.
On segment $x = 0$, we have $P = 0$.
On segment $y = 0$, we have $P = -x \le 0$.
On segment $x = 3$, $0 \le y \le 2$, we have $P = 3y - (y^3/3) - 3$, which has values $P = -3$ at $y = 0$ and $P = 1/3$ at $y = 2$. It also has a critical point given by

$$0 = \frac{dP}{dy} = 3 - y^2,$$

so $y = \sqrt{3}$ and $P = 2\sqrt{3} - 3 \approx 0.4641$.
On segment $y = 2$, $0 \le x \le 3$, we have $P = x - (8x^2/27)$, which has values $P = 0$ at $x = 0$ and $P = 1/3$ at $x = 3$. It also has a critical point given by

$$0 = \frac{dP}{dx} = 1 - \frac{16x}{27},$$

so $x = 27/16$ and $P = 27/32 \approx 0.84375$.
It appears that the greatest monthly profit corresponds to manufacturing $27,000/16 \approx 1,688$ widgets/month and selling them for \$2 each.

**16.** The envelope of $y = (x - c)^3 + 3c$ is found by eliminating $c$ from that equation and

$$0 = \frac{\partial}{\partial c}[(x - c)^3 + 3c] = -3(x - c)^2 + 3.$$

This later equation implies that $(x - c)^2 = 1$, so $x - c = \pm 1$.
The envelope is $y = (\pm 1)^3 + 3(x \mp 1)$, or $y = 3x \pm 2$.

**17.** Look for a solution of $y + \epsilon x e^y = -2x$ in the form of a Maclaurin series

$$y = y(x, \epsilon) = y(x, 0) + \epsilon y_\epsilon(x, 0) + \frac{\epsilon^2}{2!} y_{\epsilon\epsilon}(x, 0) + \cdots.$$

Putting $\epsilon = 0$ in the given equation, we get $y(x, 0) = -2x$. Now differentiate the given equation with respect to $\epsilon$ twice:

$$y_\epsilon + xe^y + \epsilon xe^y y_\epsilon = 0$$
$$y_{\epsilon\epsilon} + 2xe^y y_\epsilon + \epsilon xe^y y_\epsilon^2 + \epsilon xe^y y_{\epsilon\epsilon} = 0.$$

The first of these equations gives

$$y_\epsilon(x, 0) = -xe^{y(x,0)} = -xe^{-2x}.$$

The second gives

$$y_{\epsilon\epsilon}(x, 0) = -2xe^{y(x,0)} y_\epsilon(x, 0) = 2x^2 e^{-4x}.$$

Thus $y = -2x - 2\epsilon x e^{-2x} + \epsilon^2 x^2 e^{-4x} + \cdots$.

**18.** a) $$G(y) = \int_0^\infty \frac{\tan^{-1}(xy)}{x}\, dx$$

$$G'(y) = \int_0^\infty \frac{1}{x} \frac{x}{1 + x^2 y^2}\, dx \qquad \text{Let } u = xy,\ du = y\, dx$$

$$= \frac{1}{y} \int_0^\infty \frac{du}{1 + u^2} = \frac{\pi}{2y} \quad \text{for } y > 0.$$

b) $$\int_0^\infty \frac{\tan^{-1}(\pi x) - \tan^{-1} x}{x}\, dx$$

$$= G(\pi) - G(1) = \int_1^\pi G'(y)\, dy = \frac{\pi}{2} \int_1^\pi \frac{dy}{y} = \frac{\pi \ln \pi}{2}.$$

## Challenging Problems 13 (page 753)

**1.** To minimize

$$I_n = \int_{-\pi}^{\pi} \left[ f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right]^2 dx$$

we choose $a_k$ and $b_k$ to satisfy

$$0 = \frac{\partial I_n}{\partial a_0}$$
$$= -\int_{-\pi}^{\pi} \left[ f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right] dx$$
$$= \left[ \pi a_0 - \int_{-\pi}^{\pi} f(x)\, dx \right]$$

$$0 = \frac{\partial I_n}{\partial a_m}$$
$$= -2 \int_{-\pi}^{\pi} \left[ f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right] \cos mx\, dx$$
$$= 2a_m \int_{-\pi}^{\pi} \cos^2 mx\, dx - \int_{-\pi}^{\pi} f(x) \cos mx\, dx$$

$$0 = \frac{\partial I_n}{\partial b_m}$$
$$= -2 \int_{-\pi}^{\pi} \left[ f(x) - \frac{a_0}{2} - \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \right] \sin mx\, dx$$
$$= 2b_m \int_{-\pi}^{\pi} \sin^2 mx\, dx - \int_{-\pi}^{\pi} f(x) \sin mx\, dx.$$

The simplifications in the integrals above resulted from the facts that for any integers $k$ and $m$,

$$\int_{-\pi}^{\pi} \cos kx \cos mx\, dx = 0 \text{ unless } k = m$$
$$\int_{-\pi}^{\pi} \sin kx \sin mx\, dx = 0 \text{ unless } k = m, \text{ and}$$
$$\int_{-\pi}^{\pi} \cos kx \sin mx\, dx = 0.$$

Since

$$\int_{-\pi}^{\pi} \cos^2 mx\, dx = \int_{-\pi}^{\pi} \sin^2 mx\, dx = \pi,$$

$I_n$ is minimized when

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\, dx \text{ for } 0 \le m \le n, \text{ and}$$
$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx \text{ for } 1 \le m \le n.$$

Because of the properties of trigonometric integrals listed in the solution to Problem 1,

$$\int_{-\pi}^{\pi}\left(\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)\right)^2 dx$$
$$= \frac{\pi a_0^2}{2} + \pi\sum_{k=0}^{n}(a_k^2+b_k^2)$$
$$\int_{-\pi}^{\pi} f(x)\left(\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)\right) dx$$
$$= \frac{\pi a_0^2}{2} + \pi\sum_{k=0}^{n}(a_k^2+b_k^2).$$

Therefore

$$I_n = \int_{-\pi}^{\pi}\left[f(x)-\left(\frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx + b_k\sin kx)\right)\right]^2 dx$$
$$= \int_{-\pi}^{\pi}\big(f(x)\big)^2 dx - 2\left(\frac{\pi a_0^2}{2}+\pi\sum_{k=0}^{n}(a_k^2+b_k^2)\right)$$
$$+ \frac{\pi a_0^2}{2}+\pi\sum_{k=0}^{n}(a_k^2+b_k^2)$$
$$= \int_{-\pi}^{\pi}\big(f(x)\big)^2 dx - \left(\frac{\pi a_0^2}{2}+\pi\sum_{k=0}^{n}(a_k^2+b_k^2)\right).$$

In fact, it can be shown that $I_n \to 0$ as $n \to \infty$.

**2.** If $f(x) = \begin{cases} 0 & \text{for } -\pi \le x < 0 \\ x & \text{for } 0 \le x \le \pi \end{cases}$, then

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\, dx = \frac{\pi}{2}$$
$$a_k = \frac{1}{\pi}\int_0^{\pi} x\cos kx\, dx$$
$$\begin{matrix} U = x & dV = \cos kx\, dx \\ dU = dx & V = \frac{1}{k}\sin kx \end{matrix}$$
$$= \frac{1}{\pi k}\left(x\sin kx\Big|_0^{\pi} - \int_0^{\pi}\sin kx\, dx\right)$$
$$= \frac{\cos k\pi - 1}{\pi k^2} = \begin{cases} 0 & \text{if } k \text{ is even} \\ -\dfrac{2}{\pi k^2} & \text{if } k \text{ is odd} \end{cases}$$
$$b_k = \frac{1}{\pi}\int_0^{\pi} x\sin kx\, dx$$
$$\begin{matrix} U = x & dV = \sin kx\, dx \\ dU = dx & V = -\frac{1}{k}\cos kx \end{matrix}$$
$$= -\frac{1}{\pi k}\left(x\cos kx\Big|_0^{\pi} - \int_0^{\pi}\cos kx\, dx\right)$$
$$= \frac{(-1)^{k+1}}{k}.$$

**3.** Let $I(x) = \displaystyle\int_0^x \frac{\ln(1+tx)}{1+t^2}\, dt$. Then

$$I'(x) = \frac{\ln(1+x^2)}{1+x^2} + \int_0^x \frac{t}{(1+t^2)(1+tx)}\, dt.$$

If we expand the latter integrand in partial fractions with respect to $t$, we obtain

$$\frac{t}{(1+t^2)(1+tx)} = \frac{x+t}{(1+x^2)(1+t^2)} - \frac{x}{(1+x^2)(1+tx)}.$$

Now we have

$$\int_0^x \frac{(x+t)\,dt}{(1+x^2)(1+t^2)} = \left.\frac{2x\tan^{-1}t+\ln(1+t^2)}{2(1+x^2)}\right|_0^x$$
$$= \frac{2x\tan^{-1}x+\ln(1+x^2)}{2(1+x^2)}$$
$$= \frac{1}{2}\frac{d}{dx}\tan^{-1}x\ln(1+x^2)$$

$$\int_0^x \frac{x\,dt}{(1+x^2)(1+tx)} = \frac{x}{1+x^2}\int_0^x \frac{dt}{1+tx}$$

Let $u = 1+tx$

$du = x\,dt$

$$= \frac{1}{1+x^2}\int_1^{1+x^2}\frac{du}{u} = \frac{\ln(1+x^2)}{1+x^2}.$$

Thus

$$I'(x) = \frac{\ln(1+x^2)}{1+x^2} + \frac{1}{2}\frac{d}{dx}\tan^{-1}x\ln(1+x^2) - \frac{\ln(1+x^2)}{1+x^2}$$
$$= \frac{1}{2}\frac{d}{dx}\tan^{-1}x\ln(1+x^2).$$

Therefore, $I(x) = \frac{1}{2}\tan^{-1}x\ln(1+x^2)+C$. Since $I(0)=0$, we have $C=0$, and

$$\int_0^x \frac{\ln(1+tx)}{1+t^2}\,dx = \frac{1}{2}\tan^{-1}x\ln(1+x^2).$$

**4.**

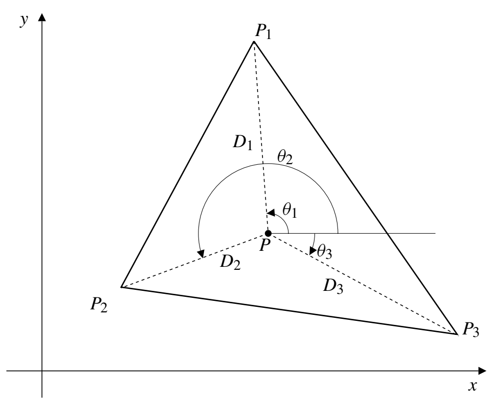


Fig. C-13.4

If $D_i = |PP_i|$ for $i = 1, 2, 3$, then

$$D_i^2 = (x-x_i)^2 + (y-y_i)^2$$
$$2D_i\frac{\partial D_i}{\partial x} = 2(x-x_i)$$
$$\frac{\partial D_i}{\partial x} = \frac{x-x_i}{D_i} = \cos\theta_i$$

where $\theta_i$ is the angle between $\overrightarrow{PP_i}$ and $\mathbf{i}$. Similarly $\partial D_i/\partial y = \sin\theta_i$. To minimize $S = D_1+D_2+D_3$ we look for critical points:

$$0 = \frac{\partial S}{\partial x} = \cos\theta_1+\cos\theta_2+\cos\theta_3$$
$$0 = \frac{\partial S}{\partial y} = \sin\theta_1+\sin\theta_2+\sin\theta_3.$$

Thus $\cos\theta_1+\cos\theta_2 = -\cos\theta_3$ and $\sin\theta_1+\sin\theta_2 = -\sin\theta_3$. Squaring and adding these two equations we get

$$2+2(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2) = 1,$$

or $\cos(\theta_1-\theta_2) = -1/2$. Thus $\theta_1-\theta_2 = \pm 2\pi/3$. Similarly $\theta_1-\theta_3 = \theta_2-\theta_3 = \pm 2\pi/3$. Thus $P$ should be chosen so that $\overrightarrow{PP_1}$, $\overrightarrow{PP_2}$,and $\overrightarrow{PP_3}$ make $120°$ angles with each other. This is possible only if all three angles of the triangle are less than $120°$. If the triangle has an angle of $120°$ or more (say at $P_1$), then $P$ should be that point on the side $P_2P_3$ such that $PP_1 \perp P_2P_3$.