CHAPTER 10. VECTORS AND COORDINATE GEOMETRY IN 3-SPACE

Section 10.1 Analytic Geometry in Three Dimensions (page 542)

1. The distance between $(0,0,0)$ and $(2,-1,-2)$ is

$$\sqrt{2^2+(-1)^2+(-2)^2}=3 \text{ units.}$$

2. The distance between $(-1,-1,-1)$ and $(1,1,1)$ is

$$\sqrt{(1+1)^2+(1+1)^2+(1+1)^2}=2\sqrt{3} \text{ units.}$$

3. The distance between $(1,1,0)$ and $(0,2,-2)$ is

$$\sqrt{(0-1)^2+(2-1)^2+(-2-0)^2}=\sqrt{6} \text{ units.}$$

4. The distance between $(3,8,-1)$ and $(-2,3,-6)$ is

$$\sqrt{(-2-3)^2+(3-8)^2+(-6+1)^2}=5\sqrt{3} \text{ units.}$$

5. a) The shortest distance from (x,y,z) to the xy-plane is $|z|$ units.

b) The shortest distance from (x,y,z) to the x-axis is $\sqrt{y^2+z^2}$ units.

6. If $A=(1,2,3)$, $B=(4,0,5)$, and $C=(3,6,4)$, then

$$\begin{aligned}|AB|&=\sqrt{3^2+(-2)^2+2^2}=\sqrt{17}\\ |AC|&=\sqrt{2^2+4^2+1^2}=\sqrt{21}\\ |BC|&=\sqrt{(-1)^2+6^2+(-1)^2}=\sqrt{38}.\end{aligned}$$

Since $|AB|^2+|AC|^2=17+21=38=|BC|^2$, the triangle ABC has a right angle at A.

7. If $A=(2,-1,-1)$, $B=(0,1,-2)$, and $C=(1,-3,1)$, then

$$\begin{aligned}c=|AB|&=\sqrt{(0-2)^2+(1+1)^2+(-2+1)^2}=3\\ b=|AC|&=\sqrt{(1-2)^2+(-3+1)^2+(1+1)^2}=3\\ a=|BC|&=\sqrt{(1-0)^2+(-3-1)^2+(1+2)^2}=\sqrt{26}.\end{aligned}$$

By the Cosine Law,

$$\begin{aligned}a^2&=b^2+c^2-2bc\cos\angle A\\ 26&=9+9-18\cos\angle A\\ \angle A&=\cos^{-1}\frac{26-18}{-18}\approx 116.4^\circ.\end{aligned}$$

8. If $A=(1,2,3)$, $B=(1,3,4)$, and $C=(0,3,3)$, then

$$\begin{aligned}|AB|&=\sqrt{(1-1)^2+(3-2)^2+(4-3)^2}=\sqrt{2}\\ |AC|&=\sqrt{(0-1)^2+(3-2)^2+(3-3)^2}=\sqrt{2}\\ |BC|&=\sqrt{(0-1)^2+(3-3)^2+(3-4)^2}=\sqrt{2}.\end{aligned}$$

All three sides being equal, the triangle is equilateral.

9. If$A=(1,1,0)$, $B=(1,0,1)$, and $C=(0,1,1)$, then

$$|AB|=|AC|=|BC|=\sqrt{2}.$$

Thus the triangle ABC is equilateral with sides $\sqrt{2}$. Its area is, therefore,

$$\frac{1}{2}\times\sqrt{2}\times\sqrt{2-\frac{1}{2}}=\frac{\sqrt{3}}{2} \text{ sq. units.}$$

10. The distance from the origin to $(1,1,1,\dots,1)$ in $\mathbb{R}^n$ is

$$\sqrt{1^2+1^2+1^2+\cdots+1}=\sqrt{n} \text{ units.}$$

11. The point on the x_1-axis closest to $(1,1,1,\dots,1)$ is $(1,0,0,\dots,0)$. The distance between these points is

$$\sqrt{0^2+1^2+1^2+\cdots+1^2}=\sqrt{n-1} \text{ units.}$$

12. $z=2$ is a plane, perpendicular to the z-axis at $(0,0,2)$.

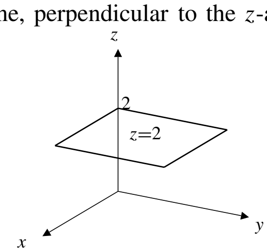

Fig. 10.1.12

13. $y\ge -1$ is the half-space consisting of all points on the plane $y=-1$ (which is perpendicular to the y-axis at $(0,-1,0)$) and all points on the same side of that plane as the origin.

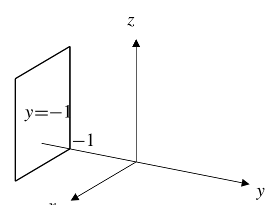

Fig. 10.1.13

14. $z=x$ is a plane containing the y-axis and making 45° angles with the positive directions of the x- and z-axes.

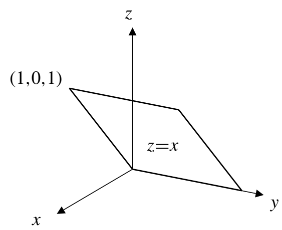

Fig. 10.1.14

15. $x + y = 1$ is a vertical plane (parallel to the z-axis) passing through the points $(1, 0, 0)$ and $(0, 1, 0)$.

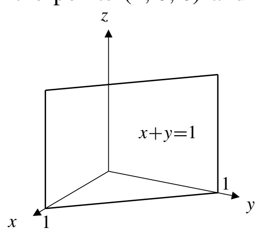

Fig. 10.1.15

16. $x^2 + y^2 + z^2 = 4$ is a sphere centred at the origin and having radius 2 (i.e., all points at distance 2 from the origin).

17. $(x-1)^2 + (y+2)^2 + (z-3)^2 = 4$ is a sphere of radius 2 with centre at the point $(1, -2, 3)$.

18. $x^2 + y^2 + z^2 = 2z$ can be rewritten

$$x^2 + y^2 + (z-1)^2 = 1,$$

and so it represents a sphere with radius 1 and centre at $(0, 0, 1)$. It is tangent to the xy-plane at the origin.

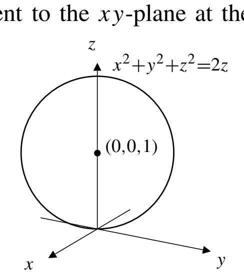

Fig. 10.1.18

19. $y^2+z^2 \le 4$ represents all points inside and on the circular cylinder of radius 2 with central axis along the x-axis (a solid cylinder).

20. $x^2 + z^2 = 4$ is a circular cylindrical surface of radius 2 with axis along the y-axis.

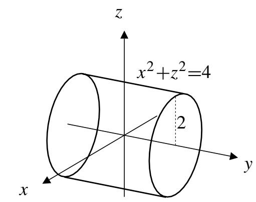

Fig. 10.1.20

21. $z = y^2$ is a "parabolic cylinder" — a surface all of whose cross-sections in planes perpendicular to the x-axis are parabolas.

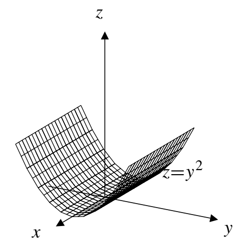

Fig. 10.1.21

22. $z \ge \sqrt{x^2+y^2}$ represents every point whose distance above the xy-plane is not less than its horizontal distance from the z-axis. It therefore consists of all points inside and on a circular cone with axis along the positive z-axis, vertex at the origin, and semi-vertical angle 45°.

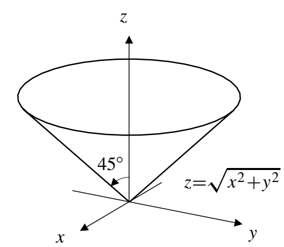

Fig. 10.1.22

23. $x + 2y + 3z = 6$ represents the plane that intersects the coordinate axes at the three points $(6, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 2)$. Only the part of the plane in the first octant is shown in the figure.

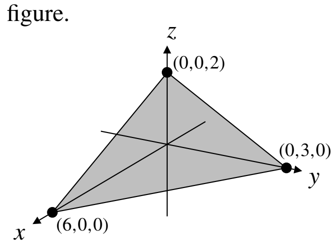

Fig. 10.1.23

24. $\begin{cases} x = 1 \\ y = 2 \end{cases}$ represents the vertical straight line in which the plane $x = 1$ intersects the plane $y = 2$.

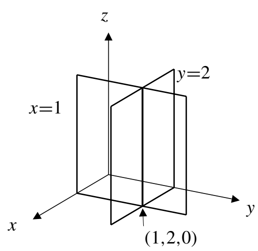

Fig. 10.1.24

25. $\begin{cases} x = 1 \\ y = z \end{cases}$ is the straight line in which the plane $z = 1$ intersects the plane $y = z$. It passes through the points $(1, 0, 0)$ and $(1, 1, 1)$.

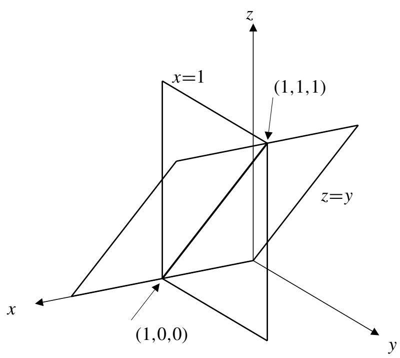

Fig. 10.1.25

26. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ z = 1 \end{cases}$ is the circle in which the horizontal plane $z = 1$ intersects the sphere of radius 2 centred at the origin. The circle has centre $(0, 0, 1)$ and radius $\sqrt{4-1} = \sqrt{3}$.

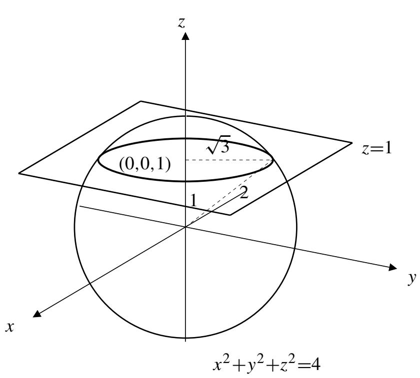

Fig. 10.1.26

27. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 + z^2 = 4z \end{cases}$ is the circle in which the sphere of radius 2 centred at the origin intersects the sphere of radius 2 centred at $(0, 0, 2)$. (The second equation can be rewritten $x^2 + y^2 + (z-2)^2 = 4$ for easier recognition.) Subtracting the equations of the two spheres we get $z = 1$, so the circle must lie in the plane $z = 1$ as well. Thus it is the same circle as in the previous exercise.

28. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + z^2 = 1 \end{cases}$ represents the two circles in which the cylinder $x^2 + z^2 - 1$ intersects the sphere $x^2 + y^2 + z^2 = 4$. Subtracting the two equations, we get $y^2 = 3$. Thus, one circle lies in the plane $y = \sqrt{3}$ and has centre $(0, \sqrt{3}, 0)$ and the other lies in the plane $y = -\sqrt{3}$ and has centre $(0, -\sqrt{3}, 0)$. Both circles have radius 1.

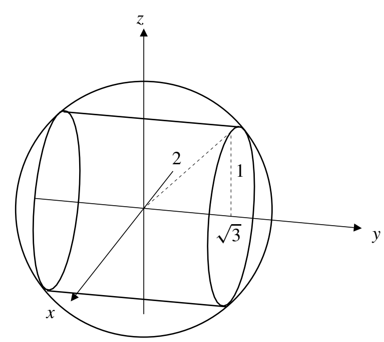

Fig. 10.1.28

29. $\begin{cases} x^2 + y^2 = 1 \\ z = x \end{cases}$ is the ellipse in which the slanted plane $z = x$ intersects the vertical cylinder $x^2 + y^2 = 1$.

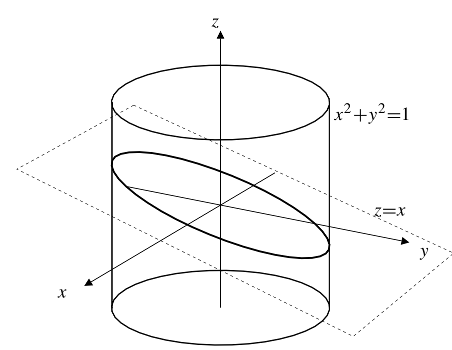

Fig. 10.1.29

30. $\begin{cases} y \ge x \\ z \le y \end{cases}$ is the quarter-space consisting of all points lying on or on the same side of the planes $y = x$ and $z = y$ as does the point $(0, 1, 0)$.

31. $\begin{cases} x^2+y^2 \le 1 \\ z \ge y \end{cases}$ represents all points which are inside or on the vertical cylinder $x^2+y^2=1$, and are also above or on the plane $z=y$.

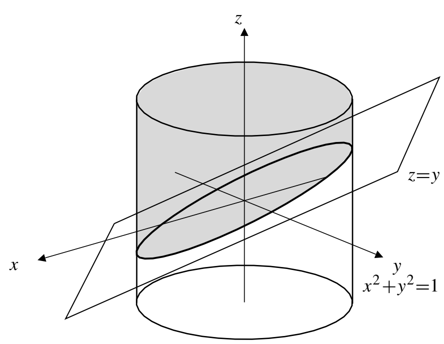

Fig. 10.1.31

32. $\begin{cases} x^2+y^2+z^2 \le 1 \\ \sqrt{x^2+y^2} \le z \end{cases}$ represents all points which are inside or on the sphere of radius 1 centred at the origin and which are also inside or on the upper half of the circular cone with axis along the z-axis, vertex at the origin, and semi-vertical angle 45°.

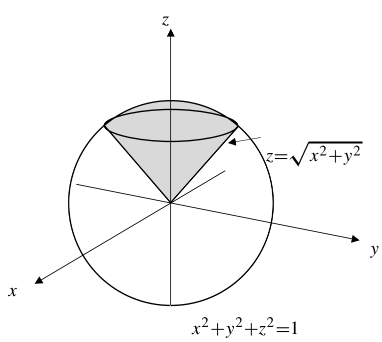

Fig. 10.1.32

33. $S = \{(x,y) : 0 < x^2+y^2 < 1\}$
The boundary of S consists of the origin and all points on the circle $x^2+y^2=1$. The interior of S is S, which is therefore open. S is bounded; all points in it are at distance less than 1 from the origin.

34. $S = \{(x,y) : x \ge 0,\ y < 0\}$
The boundary of S consists of points $(x,0)$ where $x \ge 0$, and points $(0,y)$ where $y \le 0$.
The interior of S consists of all points of S that are not on the y-axis, that is, all points (x,y) satisfying $x > 0$ and $y < 0$.
S is neither open nor closed; it contains some, but not all, of its boundary points.
S is not bounded; $(x,-1)$ belongs to S for $0 < x < \infty$.

35. $S = \{(x,y) : x+y=1\}$
The boundary of S is S. The interior of S is the empty set. S is closed, but not bounded. There are points on the line $x+y=1$ arbitrarily far away from the origin.

36. $S = \{(x,y) : |x|+|y| \le 1\}$
The boundary of S consists of all points on the edges of the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$.
The interior of S consists of all points inside that square.
S is closed since it contains all its boundary points. It is bounded since all points in it are at distance not greater than 1 from the origin.

37. $S = \{(x,y,z) : 1 \le x^2+y^2+z^2 \le 4\}$
Boundary: the spheres of radii 1 and 2 centred at the origin.
Interior: the region between these spheres. S is closed.

38. $S = \{(x,y,z) : x \ge 0,\ y > 1,\ z < 2\}$
Boundary: the quarter planes $x = 0$, $(y \ge 1,\ z \le 2)$, $y = 1$, $(x \ge 0,\ z \le 2)$, and $z = 2$, $(x \ge 0,\ y \ge 1)$.
Interior: the set of points (x,y,z) such that $x > 0$, $y > 1$, $z < 2$.
S is neither open nor closed.

39. $S = \{(x,y,z) : (x-z)^2 + (y-z)^2 = 0\}$
The boundary of S is S, that is, the line $x = y = z$. The interior of S is empty. S is closed.

40. $S = \{(x,y,z) : x^2+y^2 < 1,\ y+z > 2\}$
Boundary: the part of the cylinder $x^2+y^2=1$ that lies on or above the plane $y+z=2$ together with the part of that plane that lies inside the cylinder.
Interior: all points that are inside the cylinder $x^2+y^2=1$ and above the plane $y+z=2$. S is open.

Section 10.2 Vectors (page 551)

1. $A = (-1,2),\ B = (2,0),\ C = (1,-3),\ D = (0,4)$.

(a) $\overrightarrow{AB} = 3\mathbf{i} - 2\mathbf{j}$ (b) $\overrightarrow{BA} = -3\mathbf{i} + 2\mathbf{j}$

(c) $\overrightarrow{AC} = 2\mathbf{i} - 5\mathbf{j}$ (d) $\overrightarrow{BD} = -2\mathbf{i} + 4\mathbf{j}$

(e) $\overrightarrow{DA} = -\mathbf{i} - 2\mathbf{j}$ (f) $\overrightarrow{AB} - \overrightarrow{BC} = 4\mathbf{i} + \mathbf{j}$

(g) $\overrightarrow{AC} - 2\overrightarrow{AB} + 3\overrightarrow{CD} = -7\mathbf{i} + 20\mathbf{j}$

(h) $\frac{1}{3}\left(\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD}\right) = 2\mathbf{i} - \frac{5}{3}\mathbf{j}$

2. $\mathbf{u} = \mathbf{i} - \mathbf{j}$
$\mathbf{v} = \mathbf{j} + 2\mathbf{k}$

a)
$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{i} + 2\mathbf{k} \\ \mathbf{u} - \mathbf{v} &= \mathbf{i} - 2\mathbf{j} - 2\mathbf{k} \\ 2\mathbf{u} - 3\mathbf{v} &= 2\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} \end{aligned}$$

b) $|\mathbf{u}| = \sqrt{1+1} = \sqrt{2}$
$|\mathbf{v}| = \sqrt{1+4} = \sqrt{5}$

c) $\hat{\mathbf{u}} = \dfrac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j})$
$\hat{\mathbf{v}} = \dfrac{1}{\sqrt{5}}(\mathbf{j} + 2\mathbf{k})$

d) $\mathbf{u} \bullet \mathbf{v} = 0 - 1 + 0 = -1$

e) The angle between $\mathbf{u}$ and $\mathbf{v}$ is
$\cos^{-1} \dfrac{-1}{\sqrt{10}} \approx 108.4°$.

f) The scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ is
$\dfrac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|} = \dfrac{-1}{\sqrt{5}}$.

g) The vector projection of $\mathbf{v}$ along $\mathbf{u}$ is
$\dfrac{(\mathbf{v} \bullet \mathbf{u})\mathbf{u}}{|\mathbf{u}|^2} = -\dfrac{1}{2}(\mathbf{i} - \mathbf{j})$.

3. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$
$\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$

a)
$$\mathbf{u} + \mathbf{v} = 6\mathbf{i} - 10\mathbf{k}$$
$$\mathbf{u} - \mathbf{v} = 8\mathbf{j}$$
$$2\mathbf{u} - 3\mathbf{v} = -3\mathbf{i} + 20\mathbf{j} + 5\mathbf{k}$$

b) $|\mathbf{u}| = \sqrt{9+16+25} = 5\sqrt{2}$
$|\mathbf{v}| = \sqrt{9+16+25} = 5\sqrt{2}$

c) $\hat{\mathbf{u}} = \dfrac{1}{5\sqrt{2}}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$
$\hat{\mathbf{v}} = \dfrac{1}{5\sqrt{2}}(3\mathbf{i} - 4\mathbf{j} - 5\mathbf{k})$

d) $\mathbf{u} \bullet \mathbf{v} = 9 - 16 + 25 = 18$

e) The angle between $\mathbf{u}$ and $\mathbf{v}$ is
$\cos^{-1} \dfrac{18}{50} \approx 68.9°$.

f) The scalar projection of $\mathbf{u}$ in the direction of $\mathbf{v}$ is
$\dfrac{\mathbf{u} \bullet \mathbf{v}}{|\mathbf{v}|} = \dfrac{18}{5\sqrt{2}}$.

g) The vector projection of $\mathbf{v}$ along $\mathbf{u}$ is
$\dfrac{(\mathbf{v} \bullet \mathbf{u})\mathbf{u}}{|\mathbf{u}|^2} = \dfrac{9}{25}(3\mathbf{i} + 4\mathbf{j} - 5\mathbf{k})$.

4. If $a = (-1, 1)$, $B = (2, 5)$ and $C = (10, -1)$, then $\overrightarrow{AB} = 3\mathbf{i} + 4\mathbf{j}$ and $\overrightarrow{BC} = 8\mathbf{i} - 6\mathbf{j}$. Since $\overrightarrow{AB} \bullet \overrightarrow{BC} = 0$, therefore, $\overrightarrow{AB} \perp \overrightarrow{BC}$. Hence, $\triangle ABC$ has a right angle at B.

5. Let the triangle be ABC. If M and N are the midpoints of AB and AC respectively, then $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{AN} = \frac{1}{2}\overrightarrow{AC}$. Thus

$$\overrightarrow{MN} = \overrightarrow{AN} - \overrightarrow{AM} = \frac{\overrightarrow{AC} - \overrightarrow{AB}}{2} = \frac{\overrightarrow{BC}}{2}.$$

Thus MN is parallel to and half as long as BC.

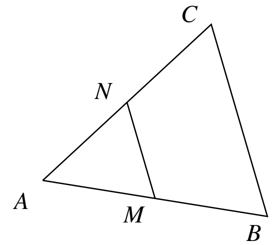

Fig. 10.2.5

6. We have

$$\overrightarrow{PQ} = \overrightarrow{PB} + \overrightarrow{BQ} = \tfrac{1}{2}\overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC} = \tfrac{1}{2}\overrightarrow{AC};$$
$$\overrightarrow{SR} = \overrightarrow{SD} + \overrightarrow{DR} = \tfrac{1}{2}\overrightarrow{AD} + \tfrac{1}{2}\overrightarrow{DC} = \tfrac{1}{2}\overrightarrow{AC}.$$

Therefore, $\overrightarrow{PQ} = \overrightarrow{SR}$. Similarly,

$$\overrightarrow{QR} = \overrightarrow{QC} + \overrightarrow{CR} = \tfrac{1}{2}\overrightarrow{BD};$$
$$\overrightarrow{PS} = \overrightarrow{PA} + \overrightarrow{AS} = \tfrac{1}{2}\overrightarrow{BD}.$$

Therefore, $\overrightarrow{QR} = \overrightarrow{PS}$. Hence, $PQRS$ is a parallelogram.

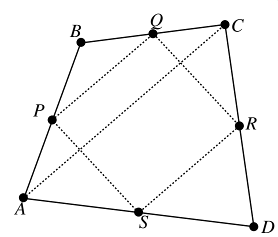

Fig. 10.2.6

7. Let the parallelogram be $ABCO$. Take the origin at O. The position vector of the midpoint of OB is

$$\frac{\overrightarrow{OB}}{2} = \frac{\overrightarrow{OB} + \overrightarrow{CB}}{2} = \frac{\overrightarrow{OC} + \overrightarrow{OA}}{2}.$$

The position vector of the midpoint of CA is

$$\overrightarrow{OC} + \frac{\overrightarrow{CA}}{2} = \overrightarrow{OC} + \frac{\overrightarrow{OA} - \overrightarrow{OC}}{2}$$
$$= \frac{\overrightarrow{OC} + \overrightarrow{OA}}{2}.$$

Thus the midpoints of the two diagonals coincide, and the diagonals bisect each other.

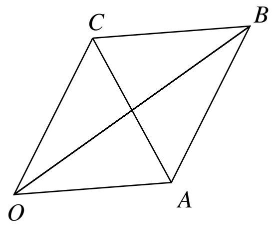

Fig. 10.2.7

8. Let X be the point of intersection of the medians AQ and BP as shown. We must show that CX meets AB in the midpoint of AB. Note that $\overrightarrow{PX} = \alpha\overrightarrow{PB}$ and $\overrightarrow{QX} = \beta\overrightarrow{QA}$ for certain real numbers α and β. Then

$$\begin{aligned}\overrightarrow{CX} &= \frac{1}{2}\overrightarrow{CB} + \beta\overrightarrow{QA} = \frac{1}{2}\overrightarrow{CB} + \beta\left(\frac{1}{2}\overrightarrow{CB} + \overrightarrow{BA}\right)\\ &= \frac{1+\beta}{2}\overrightarrow{CB} + \beta\overrightarrow{BA};\\ \overrightarrow{CX} &= \frac{1}{2}\overrightarrow{CA} + \alpha\overrightarrow{PB} = \frac{1}{2}\overrightarrow{CA} + \alpha\left(\frac{1}{2}\overrightarrow{CA} + \overrightarrow{AB}\right)\\ &= \frac{1+\alpha}{2}\overrightarrow{CA} + \alpha\overrightarrow{AB}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1+\beta}{2}\overrightarrow{CB} + \beta\overrightarrow{BA} &= \frac{1+\alpha}{2}\overrightarrow{CA} + \alpha\overrightarrow{AB}\\ (\beta+\alpha)\overrightarrow{BA} &= \frac{1+\alpha}{2}\overrightarrow{CA} - \frac{1+\beta}{2}\overrightarrow{CB}\\ (\beta+\alpha)(\overrightarrow{CA} - \overrightarrow{CB}) &= \frac{1+\alpha}{2}\overrightarrow{CA} - \frac{1+\beta}{2}\overrightarrow{CB}\\ \left(\beta+\alpha-\frac{1+\alpha}{2}\right)\overrightarrow{CA} &= \left(\beta+\alpha-\frac{1+\beta}{2}\right)\overrightarrow{CB}.\end{aligned}$$

Since $\overrightarrow{CA}$ is not parallel to $\overrightarrow{CB}$,

$$\begin{aligned}&\beta+\alpha-\frac{1+\alpha}{2} = \beta+\alpha-\frac{1+\beta}{2} = 0\\ &\Rightarrow \alpha = \beta = \frac{1}{3}.\end{aligned}$$

Since $\alpha = \beta$, x divides AQ and BP in the same ratio. By symmetry, the third median CM must also divide the other two in this ratio, and so must pass through X and $MX = \frac{1}{3}MC$.

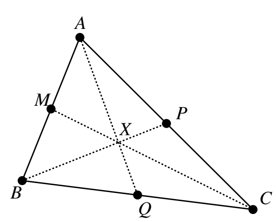

Fig. 10.2.8

9. Let $\mathbf{i}$ point east and $\mathbf{j}$ point north. Let the wind velocity be

$$\mathbf{v}_{\text{wind}} = a\mathbf{i} + b\mathbf{j}.$$

Now $\mathbf{v}_{\text{wind}} = \mathbf{v}_{\text{wind rel car}} + \mathbf{v}_{\text{car}}$.
When $\mathbf{v}_{\text{car}} = 50\mathbf{j}$, the wind appears to come from the west, so $\mathbf{v}_{\text{wind rel car}} = \lambda\mathbf{i}$. Thus

$$a\mathbf{i} + b\mathbf{j} = \lambda\mathbf{i} + 50\mathbf{j},$$

so $a = \lambda$ and $b = 50$.
When $\mathbf{v}_{\text{car}} = 100\mathbf{j}$, the wind appears to come from the northwest, so $\mathbf{v}_{\text{wind rel car}} = \mu(\mathbf{i}\text{-}\mathbf{j})$. Thus

$$a\mathbf{i} + b\mathbf{j} = \mu(\mathbf{i} - \mathbf{j}) + 100\mathbf{j},$$

so $a = \mu$ and $b = 100 - \mu$.
Hence $50 = 100 - \mu$, so $\mu = 50$. Thus $a = b = 50$. The wind is from the southwest at $50\sqrt{2}$ km/h.

10. Let the x-axis point east and the y-axis north. The velocity of the water is

$$\mathbf{v}_{\text{water}} = 3\mathbf{i}.$$

If you row through the water with speed 5 in the direction making angle θ west of north, then your velocity relative to the water will be

$$\mathbf{v}_{\text{boat rel water}} = -5\sin\theta\mathbf{i} + 5\cos\theta\mathbf{j}.$$

Therefore, your velocity relative to the land will be

$$\begin{aligned}\mathbf{v}_{\text{boat rel land}} &= \mathbf{v}_{\text{boat rel water}} + \mathbf{v}_{\text{water}}\\ &= (3 - 5\sin\theta)\mathbf{i} + 5\cos\theta\mathbf{j}.\end{aligned}$$

To make progress in the direction $\mathbf{j}$, choose θ so that $3 = 5\sin\theta$. Thus $\theta = \sin^{-1}(3/5) \approx 36.87°$. In this case, your actual speed relative to the land will be
$5\cos\theta = \frac{4}{5} \times 5 = 4$ km/h.
To row from A to B, head in the direction 36.87° west of north. The 1/2 km crossing will take $(1/2)/4 = 1/8$ of an hour, or about $7\frac{1}{2}$ minutes.

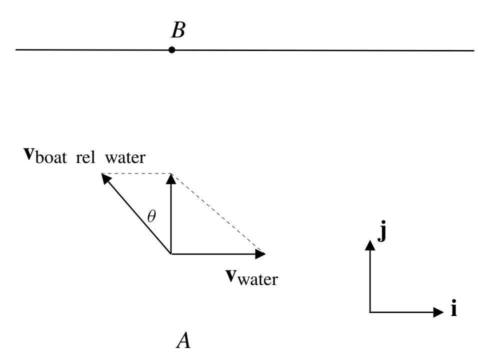

Fig. 10.2.10

11. We use the notations of the solution to Exercise 4. You now want to make progress in the direction $k\mathbf{i} + \frac{1}{2}\mathbf{j}$, that is, in the direction making angle

$$\phi = \tan^{-1}\frac{1}{2k}$$

with vector $\mathbf{i}$. Head at angle θ upstream of this direction. Since your rowing speed is 2, the triangle with angles θ and ϕ has sides 2 and 3 as shown in the figure. By the Sine Law, $\dfrac{3}{\sin\theta} = \dfrac{2}{\sin\phi}$, so

$$\sin\theta = \frac{3}{2}\sin\phi = \frac{3}{2}\,\frac{1}{2\sqrt{k^2+\frac{1}{4}}} = \frac{3}{2\sqrt{4k^2+1}}.$$

This is only possible if $\dfrac{3}{2\sqrt{4k^2+1}} \le 1$, that is, if $k \ge \dfrac{\sqrt{5}}{4}$.

Head in the direction $\theta = \sin^{-1}\dfrac{3}{2\sqrt{4k^2+1}}$ upstream of the direction of AC, as shown in the figure. The trip is not possible if $k < \sqrt{5}/4$.

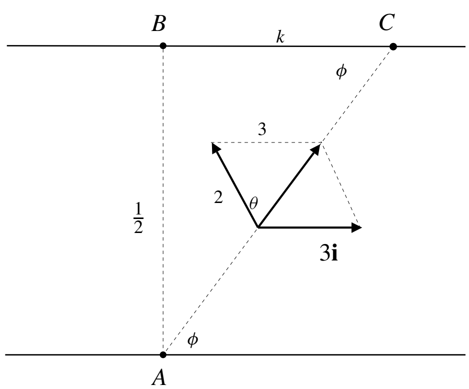

Fig. 10.2.11

12. Let $\mathbf{i}$ point east and $\mathbf{j}$ point north. If the aircraft heads in a direction θ north of east, then its velocity relative to the air is

$$750\cos\theta\mathbf{i} + 750\sin\theta\mathbf{j}.$$

The velocity of the air relative to the ground is

$$-\frac{100}{\sqrt{2}}\mathbf{i} + -\frac{100}{\sqrt{2}}\mathbf{j}.$$

Thus the velocity of the aircraft relative to the ground is

$$\left(750\cos\theta - \frac{100}{\sqrt{2}}\right)\mathbf{i} + \left(750\sin\theta - \frac{100}{\sqrt{2}}\right)\mathbf{j}.$$

If this velocity is true easterly, then

$$750\sin\theta = \frac{100}{\sqrt{2}},$$

so $\theta \approx 5.41^\circ$. The speed relative to the ground is

$$750\cos\theta - \frac{100}{\sqrt{2}} \approx 675.9 \text{ km/h}.$$

The time for the 1500 km trip is $\dfrac{1500}{675.9} \approx 2.22$ hours.

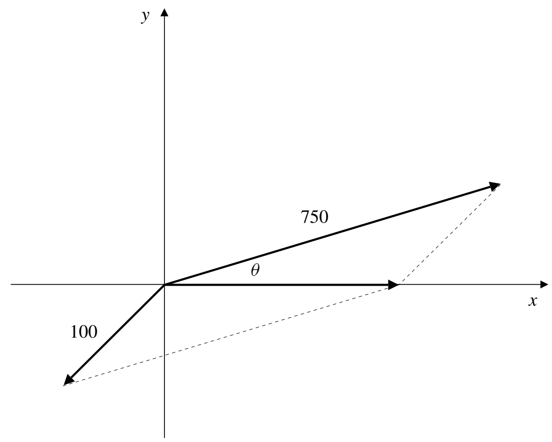

Fig. 10.2.12

13. The two vectors are perpendicular if their dot product is zero:

$$(2t\mathbf{i} + 4\mathbf{j} - (10+t)\mathbf{k}) \bullet (\mathbf{i} + t\mathbf{j} + \mathbf{k}) = 0$$
$$2t + 4t - 10 - t = 0 \quad \Rightarrow \quad t = 2.$$

The vectors are perpendicular if $t = 2$.

14. The cube with edges $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ has diagonal $\mathbf{i} + \mathbf{j} + \mathbf{k}$. The angle between $\mathbf{i}$ and the diagonal is

$$\cos^{-1}\frac{\mathbf{i}\bullet(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{3}} = \cos^{-1}\frac{1}{\sqrt{3}} \approx 54.7^\circ.$$

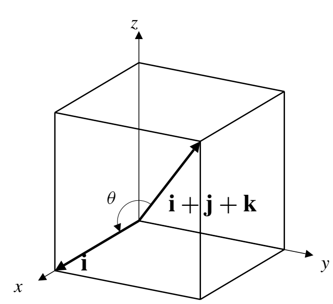

Fig. 10.2.14

15. The cube of Exercise 10 has six faces, each with 2 diagonals. The angle between $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and the face diagonal $\mathbf{i}+\mathbf{j}$ is

$$\cos^{-1}\frac{(\mathbf{i}+\mathbf{j})\bullet(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{2}\sqrt{3}} = \cos^{-1}\frac{2}{\sqrt{6}} \approx 35.26°.$$

Six of the face diagonals make this angle with $\mathbf{i}+\mathbf{j}+\mathbf{k}$. The face diagonal $\mathbf{i}-\mathbf{j}$ (and five others) make angle

$$\cos^{-1}\frac{(\mathbf{i}-\mathbf{j})\bullet(\mathbf{i}+\mathbf{j}+\mathbf{k})}{\sqrt{2}\sqrt{3}} = \cos^{-1}0 = 90°$$

with the cube diagonal $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

16. If $\mathbf{u} = u_1\mathbf{i}+u_2\mathbf{j}+u_3\mathbf{k}$, then $\cos\alpha\dfrac{\mathbf{u}\bullet\mathbf{i}}{|\mathbf{u}|} = \dfrac{u_1}{|\mathbf{u}|}$.
Similarly, $\cos\beta = \dfrac{u_2}{|\mathbf{u}|}$ and $\cos\gamma = \dfrac{u_3}{|\mathbf{u}|}$.
Thus, the unit vector in the direction of $\mathbf{u}$ is

$$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \cos\alpha\mathbf{i}+\cos\beta\mathbf{j}+\cos\gamma\mathbf{k},$$

and so $\cos^2\alpha+\cos^2\beta+\cos^2\gamma = |\hat{\mathbf{u}}|^2 = 1$.

17. If $\hat{\mathbf{u}}$ makes equal angles $\alpha = \beta = \gamma$ with the coordinate axes, then $3\cos^2\alpha = 1$, and $\cos\alpha = 1/\sqrt{3}$. Thus

$$\hat{\mathbf{u}} = \frac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}.$$

18. If $A = (1, 0, 0)$, $B = (0, 2, 0)$, and $C = (0, 0, 3)$, then

$$\angle ABC = \cos^{-1}\frac{\overrightarrow{BA}\bullet\overrightarrow{BC}}{|BA||BC|} = \cos^{-1}\frac{4}{\sqrt{5}\sqrt{13}} \approx 60.26°$$
$$\angle BCA = \cos^{-1}\frac{\overrightarrow{CB}\bullet\overrightarrow{CA}}{|CB||CA|} = \cos^{-1}\frac{9}{\sqrt{10}\sqrt{13}} \approx 37.87°$$
$$\angle CAB = \cos^{-1}\frac{\overrightarrow{AC}\bullet\overrightarrow{AB}}{|AC||AB|} = \cos^{-1}\frac{1}{\sqrt{10}\sqrt{5}} \approx 81.87°.$$

19. Since $\mathbf{r}-\mathbf{r}_1 = \lambda\mathbf{r}_1+(1-\lambda)\mathbf{r}_2-\mathbf{r}_1 = (1-\lambda)(\mathbf{r}_1-\mathbf{r}_2)$, therefore $\mathbf{r}-\mathbf{r}_1$ is parallel to $\mathbf{r}_1-\mathbf{r}_2$, that is, parallel to the line P_1P_2. Since P_1 is on that line, so must P be on it.
If $\lambda = \dfrac{1}{2}$, then $\mathbf{r} = \dfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)$, so P is midway between P_1 and P_2.
If $\lambda = \dfrac{2}{3}$, then $\mathbf{r} = \dfrac{2}{3}\mathbf{r}_1+\dfrac{1}{3}\mathbf{r}_2$, so P is two-thirds of the way from P_2 towards P_1 along the line.
If $\lambda = -1$, the $\mathbf{r} = -\mathbf{r}_1+2\mathbf{r}_2 = \mathbf{r}_2+(\mathbf{r}_2-\mathbf{r}_1)$, so P is such that P_2 bisects the segment P_1P.
If $\lambda = 2$, then $\mathbf{r} = 2\mathbf{r}_1-\mathbf{r}_2 = \mathbf{r}_1+(\mathbf{r}_1-\mathbf{r}_2)$, so P is such that P_1 bisects the segment P_2P.

20. If $\mathbf{a} \neq \mathbf{0}$, then $\mathbf{a}\bullet\mathbf{r} = 0$ implies that the position vector $\mathbf{r}$ is perpendicular to $\mathbf{a}$. Thus the equation is satisfied by all points on the plane through the origin that is normal (perpendicular) to $\mathbf{a}$.

21. If $\mathbf{r}\bullet\mathbf{a} = b$, then the vector projection of $\mathbf{r}$ along $\mathbf{a}$ is the constant vector

$$\frac{\mathbf{r}\bullet\mathbf{a}}{|\mathbf{a}|}\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{b}{|\mathbf{a}|^2}\mathbf{a} = \mathbf{r}_0, \quad \text{say.}$$

Thus $\mathbf{r}\bullet\mathbf{a} = b$ is satisfied by all points on the plane through $\mathbf{r}_0$ that is normal to $\mathbf{a}$.

In Exercises 22–24, $\mathbf{u} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$, $\mathbf{v} = \mathbf{i}+2\mathbf{j}-2\mathbf{k}$, and $\mathbf{w} = 2\mathbf{i}-2\mathbf{j}+\mathbf{k}$.

22. Vector $\mathbf{x} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ if

$$\begin{aligned}\mathbf{u}\bullet\mathbf{x} = 0 \quad &\Leftrightarrow \quad 2x+y-2z = 0\\ \mathbf{v}\bullet\mathbf{x} = 0 \quad &\Leftrightarrow \quad x+2y-2z = 0.\end{aligned}$$

Subtracting these equations, we get $x-y = 0$, so $x = y$. The first equation now gives $3x = 2z$. Now $\mathbf{x}$ is a unit vector if $x^2+y^2+z^2 = 1$, that is, if $x^2+x^2+\frac{9}{4}x^2 = 1$, or $x = \pm 2/\sqrt{17}$. The two unit vectors are

$$\mathbf{x} = \pm\left(\frac{2}{\sqrt{17}}\mathbf{i}+\frac{2}{\sqrt{17}}\mathbf{j}+\frac{3}{\sqrt{17}}\mathbf{k}\right).$$

23. Let $\mathbf{x} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. Then

$$\begin{aligned}\mathbf{x}\bullet\mathbf{u} = 9 \quad &\Leftrightarrow \quad 2x+y-2z = 9\\ \mathbf{x}\bullet\mathbf{v} = 4 \quad &\Leftrightarrow \quad x+2y-2z = 4\\ \mathbf{x}\bullet\mathbf{w} = 6 \quad &\Leftrightarrow \quad 2x-2y+z = 6.\end{aligned}$$

This system of linear equations has solution $x = 2$, $y = -3$, $z = -4$. Thus $\mathbf{x} = 2\mathbf{i}-3\mathbf{j}-4\mathbf{k}$.

24. Since $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ all have the same length (3), a vector $\mathbf{x} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ will make equal angles with all three if it has equal dot products with all three, that is, if

$$\begin{aligned}2x+y-2z = x+2y-2z \quad &\Leftrightarrow \quad x = y = 0\\ 2x+y-2z = 2x-2y+z \quad &\Leftrightarrow \quad 3y-3z = 0.\end{aligned}$$

Thus $x = y = z$. Two unit vectors satisfying this condition are

$$\mathbf{x} = \pm\left(\frac{1}{\sqrt{3}}\mathbf{i}+\frac{1}{\sqrt{3}}\mathbf{j}+\frac{1}{\sqrt{3}}\mathbf{k}\right).$$

25. Let $\hat{\mathbf{u}} = \mathbf{u}/|\mathbf{u}|$ and $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$.
Then $\hat{\mathbf{u}}+\hat{\mathbf{v}}$ bisects the angle between $\mathbf{u}$ and $\mathbf{v}$. A unit vector which bisects this angle is

$$\frac{\hat{\mathbf{u}}+\hat{\mathbf{v}}}{|\hat{\mathbf{u}}+\hat{\mathbf{v}}|} = \frac{\dfrac{\mathbf{u}}{|\mathbf{u}|}+\dfrac{\mathbf{v}}{|\mathbf{v}|}}{\left|\dfrac{\mathbf{u}}{|\mathbf{u}|}+\dfrac{\mathbf{v}}{|\mathbf{v}|}\right|} = \frac{|\mathbf{v}|\mathbf{u}+|\mathbf{u}|\mathbf{v}}{\big||\mathbf{v}|\mathbf{u}+|\mathbf{u}|\mathbf{v}\big|}.$$

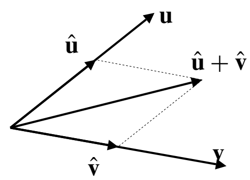

Fig. 10.2.25

26. If $\mathbf{u}$ and $\mathbf{v}$ are not parallel, then neither is the zero vector, and the origin and the two points with position vectors $\mathbf{u}$ and $\mathbf{v}$ lie on a unique plane. The equation $\mathbf{r} = \lambda\mathbf{u} + \mu\mathbf{v}$ (λ, μ real) gives the position vector of an arbitrary point on that plane.

27. a)
$$\begin{aligned}|\mathbf{u}+\mathbf{v}|^2 &= (\mathbf{u}+\mathbf{v})\bullet(\mathbf{u}+\mathbf{v})\\ &= \mathbf{u}\bullet\mathbf{u}+\mathbf{u}\bullet\mathbf{v}+\mathbf{v}\bullet\mathbf{u}+\mathbf{v}\bullet\mathbf{v}\\ &= |\mathbf{u}|^2+2\mathbf{u}\bullet\mathbf{v}+|\mathbf{v}|^2.\end{aligned}$$

b) If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta \le 1$, so

$$\mathbf{u}\bullet\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \le |\mathbf{u}||\mathbf{v}|.$$

c)
$$\begin{aligned}|\mathbf{u}+\mathbf{v}|^2 &= |\mathbf{u}|^2+2\mathbf{u}\bullet\mathbf{v}+|\mathbf{v}|^2\\ &\le |\mathbf{u}|^2+2|\mathbf{u}||\mathbf{v}|+|\mathbf{v}|^2\\ &= (|\mathbf{u}|+|\mathbf{v}|)^2.\end{aligned}$$
Thus $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}|+|\mathbf{v}|$.

28. a) $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ are the sides of a triangle. The triangle inequality says that the length of one side cannot exceed the sum of the lengths of the other two sides.

b) If $\mathbf{u}$ and $\mathbf{v}$ are parallel and point in the *same direction*, (or if at least one of them is the zero vector), then $|\mathbf{u}+\mathbf{v}| = |\mathbf{u}|+|\mathbf{v}|$.

29. $\mathbf{u} = \frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}$, $\mathbf{v} = \frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}$, $\mathbf{w} = \mathbf{k}$.

a) $|\mathbf{u}| = \sqrt{\frac{9}{25}+\frac{16}{25}} = 1$, $|\mathbf{v}| = \sqrt{\frac{16}{25}+\frac{9}{25}} = 1$, $|w| = 1$,

$\mathbf{u}\bullet\mathbf{v} = \frac{12}{25}-\frac{12}{25} = 0$, $\mathbf{u}\bullet\mathbf{w} = 0$, $\mathbf{v}\bullet\mathbf{w} = 0$.

b) If $\mathbf{r} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then

$$\begin{aligned}&(\mathbf{r}\bullet\mathbf{u})\mathbf{u}+(\mathbf{r}\bullet\mathbf{v})\mathbf{v}+(\mathbf{r}\bullet\mathbf{w})\mathbf{w}\\ &= \left(\frac{3}{5}x+\frac{4}{5}y\right)\left(\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}\right)\\ &\quad+\left(\frac{4}{5}x-\frac{3}{5}y\right)\left(\frac{4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}\right)+z\mathbf{k}\\ &= \frac{9x+16x}{25}\mathbf{i}+\frac{16y+9y}{25}\mathbf{j}+z\mathbf{k}\\ &= x\mathbf{i}+y\mathbf{j}+z\mathbf{k} = \mathbf{r}.\end{aligned}$$

30. Suppose $|\mathbf{u}| = |\mathbf{v}| = |\mathbf{w}| = 1$, and $\mathbf{u}\bullet\mathbf{v} = \mathbf{u}\bullet\mathbf{w} = \mathbf{v}\bullet\mathbf{w} = 0$, and let $\mathbf{r} = a\mathbf{u}+b\mathbf{v}+w\mathbf{w}$. Then

$$\mathbf{r}\bullet\mathbf{u} = a\mathbf{u}\bullet\mathbf{u}+b\mathbf{v}\bullet\mathbf{u}+c\mathbf{w}\bullet\mathbf{u} = a|\mathbf{u}|^2+0+0 = a.$$

Similarly, $\mathbf{r}\bullet\mathbf{v} = b$ and $\mathbf{r}\bullet\mathbf{w} = c$.

31. Let $\mathbf{u} = \dfrac{\mathbf{w}\bullet\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}$, (the vector projection of $\mathbf{w}$ along $\mathbf{a}$). Let $\mathbf{v} = \mathbf{w}-\mathbf{u}$. Then $\mathbf{w} = \mathbf{u}+\mathbf{v}$. Clearly $\mathbf{u}$ is parallel to $\mathbf{a}$, and

$$\mathbf{v}\bullet\mathbf{a} = \mathbf{w}\bullet\mathbf{a}-\frac{\mathbf{w}\bullet\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}\bullet\mathbf{a} = \mathbf{w}\bullet\mathbf{a}-\mathbf{w}\bullet\mathbf{a} = 0,$$

so $\mathbf{v}$ is perpendicular to $\mathbf{a}$.

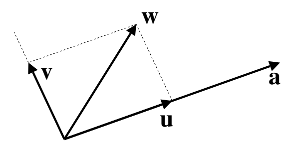

Fig. 10.2.31

32. Let $\hat{\mathbf{n}}$ be a unit vector that is perpendicular to $\mathbf{u}$ and lies in the plane containing the origin and the points U, V, and P. Then $\hat{\mathbf{u}} = \mathbf{u}/|\mathbf{u}|$ and $\hat{\mathbf{n}}$ constitute a standard basis in that plane, so each of the vectors $\mathbf{v}$ and $\mathbf{r}$ can be expressed in terms of them:

$$\begin{aligned}\mathbf{v} &= s\hat{\mathbf{u}}+t\hat{\mathbf{n}}\\ \mathbf{r} &= x\hat{\mathbf{u}}+y\hat{\mathbf{n}}.\end{aligned}$$

Since $\mathbf{v}$ is not parallel to $\mathbf{u}$, we have $t \ne 0$. Thus $\hat{\mathbf{n}} = (1/t)(\mathbf{v}-s\hat{\mathbf{u}})$ and

$$\mathbf{r} = x\hat{\mathbf{u}}+\frac{y}{t}(\mathbf{v}-s\hat{\mathbf{u}}) = \lambda\mathbf{u}+\mu\mathbf{v},$$

where $\lambda = (tx-ys)/(t|\mathbf{u}|)$ and $\mu = y/t$.

33. Let $|\mathbf{a}|^2-4rst = K^2$, where $K > 0$. Now

$$\begin{aligned}|\mathbf{a}|^2 &= \mathbf{a}\bullet\mathbf{a} = (r\mathbf{x}+s\mathbf{y})\bullet(r\mathbf{x}+s\mathbf{y})\\ &= r^2|\mathbf{x}|^2+s^2|\mathbf{y}|^2+2rs\mathbf{x}\bullet\mathbf{y}\\ K^2 &= |\mathbf{a}|^2-4rs\mathbf{x}\bullet\mathbf{y}\\ &= |r\mathbf{x}-s\mathbf{y}|^2\end{aligned}$$

(since $\mathbf{x}\bullet\mathbf{y} = t$).
Therefore $r\mathbf{x}-s\mathbf{y} = K\hat{\mathbf{u}}$, for some unit vector $\hat{\mathbf{u}}$.
Since $r\mathbf{x}+s\mathbf{y} = \mathbf{a}$, we have

$$\begin{aligned}2r\mathbf{x} &= \mathbf{a}+K\hat{\mathbf{u}}\\ 2s\mathbf{y} &= \mathbf{a}-K\hat{\mathbf{u}}.\end{aligned}$$

Thus

$$\mathbf{x} = \frac{\mathbf{a}+K\hat{\mathbf{u}}}{2r}, \qquad \mathbf{y} = \frac{\mathbf{a}-K\hat{\mathbf{u}}}{2s},$$

where $K = \sqrt{|\mathbf{a}|^2 - 4rst}$, and $\hat{\mathbf{u}}$ is any unit vector. (The solution is not unique.)

34. The derivation of the equation of the hanging cable given in the text needs to be modified by replacing $\mathbf{W} = -\delta g s \mathbf{j}$ with $\mathbf{W} = -\delta g x \mathbf{j}$. Thus $T_v = \delta g x$, and the slope of the cable satisfies

$$\frac{dy}{dx} = \frac{\delta g x}{H} = ax$$

where $a = \delta g/H$. Thus

$$y = \frac{1}{2}ax^2 + C;$$

the cable hangs in a parabola.

35. If $y = \dfrac{1}{a}\cosh(ax)$, then $y' = \sinh(ax)$, so

$$\begin{aligned} s &= \int_0^x \sqrt{1 + \sinh^2(au)}\, du = \int_0^x \cosh(au)\, du \\ &= \frac{\sinh(au)}{a}\bigg|_0^x = \frac{1}{a}\sinh(ax). \end{aligned}$$

As shown in the text, the tension $\mathbf{T}$ at P has horizontal and vertical components that satisfy $T_h = H = \dfrac{\delta g}{a}$ and $T_v = \delta g s = \dfrac{\delta g}{a}\sinh(ax)$. Hence

$$|\mathbf{T}| = \sqrt{T_h^2 + T_v^2} = \frac{\delta g}{a}\cosh(ax) = \delta g y.$$

36. The cable hangs along the curve $y = \dfrac{1}{a}\cosh(ax)$, and its length from the lowest point at $x = 0$ to the support tower at $x = 45$ m is 50 m. Thus

$$50 = \int_0^{45} \sqrt{1 + \sinh^2(ax)}\, dx = \frac{1}{a}\sinh(45a).$$

The equation $\sinh(45a) = 50a$ has approximate solution $a \approx 0.0178541$. The vertical distance between the lowest point on the cable and the support point is

$$\frac{1}{a}\Big(\cosh(45a) - 1\Big) \approx 19.07 \text{ m}.$$

37. The equation of the cable is of the form $y = \dfrac{1}{a}\cosh(ax)$. At the point P where $x = 10$ m, the slope of the cable is $\sinh(10a) = \tan(55°)$. Thus

$$a = \frac{1}{10}\sinh^{-1}(\tan(55°)) \approx 0.115423.$$

The length of the cable between $x = 0$ and $x = 10$ m is

$$\begin{aligned} L &= \int_0^{10} \sqrt{1 + \sinh^2(ax)}\, dx \\ &= \int_0^{10} \cosh(ax)\, dx = \frac{1}{a}\sinh(ax)\bigg|_0^{10} \\ &= \frac{1}{a}\sinh(10a) \approx 12.371 \text{ m}. \end{aligned}$$

Section 10.3 The Cross Product in 3-Space (page 559)

1. $(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (3\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = 5\mathbf{i} + 13\mathbf{j} + 7\mathbf{k}$

2. $(\mathbf{j} + 2\mathbf{k}) \times (-\mathbf{i} - \mathbf{j} + \mathbf{k}) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$

3. If $A = (1, 2, 0)$, $B = (1, 0, 2)$, and $C = (0, 3, 1)$, then $\overrightarrow{AB} = -2\mathbf{j} + 2\mathbf{k}$, $\overrightarrow{AC} = -\mathbf{i} + \mathbf{j} + \mathbf{k}$, and the area of triangle ABC is

$$\frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{|-4\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}|}{2} = \sqrt{6} \text{ sq. units}.$$

4. A vector perpendicular to the plane containing the three given points is

$$(-a\mathbf{i} + b\mathbf{j}) \times (-a\mathbf{i} + c\mathbf{k}) = bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}.$$

A unit vector in this direction is

$$\frac{bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}.$$

The triangle has area $\dfrac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$.

5. A vector perpendicular to $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + 2\mathbf{k}$ is

$$\pm(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + 2\mathbf{k}) = \pm(2\mathbf{i} - 2\mathbf{j} + \mathbf{k}),$$

which has length 3. A unit vector in that direction is

$$\pm\left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right).$$

6. A vector perpendicular to $\mathbf{u} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ and to $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ is the cross product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -7\mathbf{i} - 6\mathbf{j} - 4\mathbf{k},$$

which has length $\sqrt{101}$. A unit vector with positive $\mathbf{k}$ component that is perpenducular to $\mathbf{u}$ and $\mathbf{v}$ is

$$\frac{-1}{\sqrt{101}}\mathbf{u} \times \mathbf{v} = \frac{1}{\sqrt{101}}(7\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}).$$

7. Since $\mathbf{u}$ makes zero angle with itself, $|\mathbf{u}\times\mathbf{u}| = 0$ and $\mathbf{u}\times\mathbf{u} = \mathbf{0}$.

8. $$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = -\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = -\mathbf{v}\times\mathbf{u}.$$

9. $$\begin{aligned}(\mathbf{u}+\mathbf{v})\times\mathbf{w} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1+v_1 & u_2+v_2 & u_3+v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}.\end{aligned}$$

10. $$\begin{aligned}(t\mathbf{u})\times\mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ tu_1 & tu_2 & tu_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= t\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = t(\mathbf{u}\times\mathbf{v}),\\ \mathbf{u}\times(t\mathbf{v}) &= -(t\mathbf{v})\times\mathbf{u} \\ &= -t(\mathbf{v}\times\mathbf{u}) = t(\mathbf{u}\times\mathbf{v}).\end{aligned}$$

11. $$\begin{aligned}&\mathbf{u}\bullet(\mathbf{u}\times\mathbf{v}) \\ &= u_1\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - u_2\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + u_3\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \\ &= u_1u_2v_3 - u_1v_2u_3 - u_2u_1v_3 \\ &\quad + u_2v_1u_3 + u_3u_1v_2 - u_3v_1u_2 = 0,\\ &\mathbf{v}\bullet(\mathbf{u}\times\mathbf{v}) = -\mathbf{v}\bullet(\mathbf{v}\times\mathbf{u}) = 0.\end{aligned}$$

12. Both $\mathbf{u} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$ and $\mathbf{v} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$ are unit vectors. They make angles β and α, respectively, with the positive x-axis, so the angle between them is $|\alpha-\beta| = \alpha-\beta$, since we are told that $0 \le \alpha-\beta \le \pi$. They span a parallelogram (actually a rhombus) having area

$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin(\alpha-\beta) = \sin(\alpha-\beta).$$

But

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\beta & \sin\beta & 0 \\ \cos\alpha & \sin\alpha & 0 \end{vmatrix} = (\sin\alpha\cos\beta - \cos\alpha\sin\beta)\mathbf{k}.$$

Because $\mathbf{v}$ is displaced counterclockwise from $\mathbf{u}$, the cross product above must be in the positive k direction. Therefore its length is the k component. Therefore

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

13. Suppose that $\mathbf{u}+\mathbf{v}+\mathbf{w} = \mathbf{0}$. Then

$$\mathbf{u}\times\mathbf{v} + \mathbf{v}\times\mathbf{v} + \mathbf{w}\times\mathbf{v} = \mathbf{0}\times\mathbf{v} = \mathbf{0}.$$

Thus $\mathbf{u}\times\mathbf{v} + \mathbf{w}\times\mathbf{v} = \mathbf{0}$.
Thus $\mathbf{u}\times\mathbf{v} = -\mathbf{w}\times\mathbf{v} = \mathbf{v}\times\mathbf{w}$.
By symmetry, we also have $\mathbf{v}\times\mathbf{w} = \mathbf{w}\times\mathbf{u}$.

14. The base of the tetrahedron is a triangle spanned by $\mathbf{v}$ and $\mathbf{w}$, which has area

$$A = \frac{1}{2}|\mathbf{v}\times\mathbf{w}|.$$

The altitude h of the tetrahedron (measured perpendicular to the plane of the base) is equal to the length of the projection of $\mathbf{u}$ onto the vector $\mathbf{v}\times\mathbf{w}$ (which is perpendicular to the base). Thus

$$h = \frac{|\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})|}{|\mathbf{v}\times\mathbf{w}|}.$$

The volume of the tetrahedron is

$$\begin{aligned}V &= \frac{1}{3}Ah = \frac{1}{6}|\mathbf{u}\bullet(\mathbf{v}\times\mathbf{w})| \\ &= \frac{1}{6}\left|\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}\right|.\end{aligned}$$

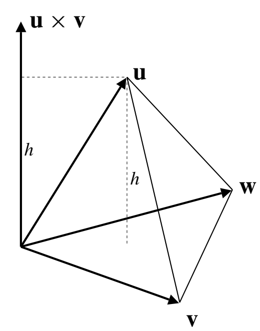

Fig. 10.3.14

15. The tetrahedron with vertices $(1,0,0)$, $(1,2,0)$, $(2,2,2)$, and $(0,3,2)$ is spanned by $\mathbf{u} = 2\mathbf{j}$, $\mathbf{v} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$, and $\mathbf{w} = -\mathbf{i}+3\mathbf{j}+2\mathbf{k}$. By Exercise 14, its volume is

$$V = \frac{1}{6}\left|\begin{vmatrix} 0 & 2 & 0 \\ 1 & 2 & 2 \\ -1 & 3 & 2 \end{vmatrix}\right| = \frac{4}{3} \text{ cu. units.}$$

16. Let the cube be as shown in the figure. The required parallelepiped is spanned by $a\mathbf{i}+a\mathbf{j}$, $a\mathbf{j}+a\mathbf{k}$, and $a\mathbf{i}+a\mathbf{k}$. Its volume is

$$V = \left|\begin{vmatrix} a & a & 0 \\ 0 & a & a \\ a & 0 & a \end{vmatrix}\right| = 2a^3 \text{ cu. units.}$$

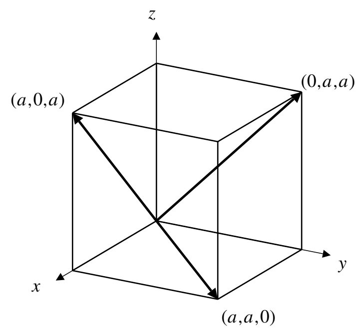

Fig. 10.3.16

17. The points $A = (1,1,-1)$, $B = (0,3,-2)$, $C = (-2,1,0)$, and $D = (k,0,2)$ are coplanar if $(\overrightarrow{AB} \times \overrightarrow{AC}) \bullet \overrightarrow{AD} = 0$. Now

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -1 \\ -3 & 0 & 1 \end{vmatrix} = 2\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}.$$

Thus the four points are coplanar if

$$2(k-1) + 4(0-1) + 6(2+1) = 0,$$

that is, if $k = -6$.

18.
$$\begin{aligned} \mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) &= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= -\begin{vmatrix} v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \\ &= \begin{vmatrix} v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \end{vmatrix} \\ &= \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) \\ &= \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}) \qquad \text{(by symmetry).} \end{aligned}$$

19. If $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) \neq 0$, and $\mathbf{x} = \lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w}$, then

$$\begin{aligned} &\mathbf{x} \bullet (\mathbf{v} \times \mathbf{w}) \\ &= \lambda\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) + \mu\mathbf{v} \bullet (\mathbf{v} \times \mathbf{w}) + \nu\mathbf{w} \bullet (\mathbf{v} \times \mathbf{w}) \\ &= \lambda\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}). \end{aligned}$$

Thus
$$\lambda = \frac{\mathbf{x} \bullet (\mathbf{v} \times \mathbf{w})}{\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})}.$$

Since $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})$, we have, by symmetry,

$$\mu = \frac{\mathbf{x} \bullet (\mathbf{w} \times \mathbf{u})}{\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})}, \qquad \nu = \frac{\mathbf{x} \bullet (\mathbf{u} \times \mathbf{v})}{\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})}.$$

20. If $\mathbf{v} \times \mathbf{w} \neq \mathbf{0}$, then $(\mathbf{v} \times \mathbf{w}) \bullet (\mathbf{v} \times \mathbf{w}) \neq 0$. By the previous exercise, there exist constants λ, μ and ν such that

$$\mathbf{u} = \lambda\mathbf{v} + \mu\mathbf{w} + \nu(\mathbf{v} \times \mathbf{w}).$$

But $\mathbf{v} \times \mathbf{w}$ is perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, so

$$\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = 0 + 0 + \nu(\mathbf{v} \times \mathbf{w}) \bullet (\mathbf{v} \times \mathbf{w}).$$

If $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = 0$, then $\nu = 0$, and

$$\mathbf{u} = \lambda\mathbf{v} + \mu\mathbf{w}.$$

21. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
$\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$
$\mathbf{w} = \mathbf{j} - \mathbf{k}$
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \times (3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = -2\mathbf{i} + 7\mathbf{j} - 4\mathbf{k}$
$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (9\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}) \times \mathbf{w} = \mathbf{i} + 9\mathbf{j} + 9\mathbf{k}$.
$\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ lies in the plane of $\mathbf{v}$ and $\mathbf{w}$;
$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ lies in the plane of $\mathbf{u}$ and $\mathbf{v}$.

22. $\mathbf{u} \bullet \mathbf{v} \times \mathbf{w}$ makes sense in that it must mean $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})$. ($(\mathbf{u} \bullet \mathbf{v}) \times \mathbf{w}$ makes no sense since it is the cross product of a scalar and a vector.)

$\mathbf{u} \times \mathbf{v} \times \mathbf{w}$ makes no sense. It is ambiguous, since $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ are not in general equal.

23. As suggested in the hint, let the x-axis lie in the direction of $\mathbf{v}$, and let the y-axis be such that $\mathbf{w}$ lies in the xy-plane. Thus

$$\mathbf{v} = v_1\mathbf{i}, \qquad \mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j}.$$

Thus $\mathbf{v} \times \mathbf{w} = v_1 w_2 \mathbf{i} \times \mathbf{j} = v_1 w_2 \mathbf{k}$, and

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1 w_2 \mathbf{k}) \\ &= u_1 v_1 w_2 \mathbf{i} \times \mathbf{k} + u_2 v_1 w_2 \mathbf{j} \times \mathbf{k} \\ &= -u_1 v_1 w_2 \mathbf{i} - u_1 v_1 w_2 \mathbf{j}. \end{aligned}$$

But
$$\begin{aligned} &(\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w} \\ &= (u_1 w_1 + u_2 w_2) v_1 \mathbf{i} - u_1 v_1 (w_1 \mathbf{i} + w_2 \mathbf{j}) \\ &= u_2 v_1 w_2 \mathbf{i} - u_1 v_1 w_2 \mathbf{j}. \end{aligned}$$

Thus $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w}$.

24. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are mutually perpendicular, then $\mathbf{v} \times \mathbf{w}$ is parallel to $\mathbf{u}$, so $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{0}$. In this case, $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \pm|\mathbf{u}||\mathbf{v}||\mathbf{w}|$; the sign depends on whether $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$ are in the same or opposite directions.

25. Applying the result of Exercise 23 three times, we obtain

$$\begin{aligned} &\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) \\ &= (\mathbf{u} \bullet \mathbf{w})\mathbf{v} - (\mathbf{u} \bullet \mathbf{v})\mathbf{w} + (\mathbf{v} \bullet \mathbf{u})\mathbf{w} - (\mathbf{v} \bullet \mathbf{w})\mathbf{u} \\ &\quad + (\mathbf{w} \bullet \mathbf{v})\mathbf{u} - (\mathbf{w} \bullet \mathbf{u})\mathbf{v} \\ &= \mathbf{0}. \end{aligned}$$

26. If $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, then

$$\begin{aligned}\mathbf{a} \times \mathbf{x} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 3 \\ x & y & z \end{vmatrix} \\ &= (2z - 3y)\mathbf{i} + (3x + z)\mathbf{y} - (y + 2x)\mathbf{k} \\ &= \mathbf{i} + 5\mathbf{j} - 3\mathbf{k},\end{aligned}$$

provided $2z - 3y = 1$, $3x + z = 5$, and $-y - 2x = -3$. This system is satisfied by $x = t$, $y = 3 - 2t$, $z = 5 - 3t$, for any real number t. Thus

$$x = t\mathbf{i} + (3 - 2t)\mathbf{j} + (5 - 3t)\mathbf{k}$$

gives a solution of $\mathbf{a} \times \mathbf{x} = \mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$ for any t. These solutions constitute a line parallel to $\mathbf{a}$.

27. Let $\mathbf{a} = -\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 5\mathbf{j}$. If $\mathbf{x}$ is a solution of $\mathbf{a} \times \mathbf{x} = \mathbf{b}$, then

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{a} \bullet (\mathbf{a} \times \mathbf{x}) = 0.$$

However, $\mathbf{a} \bullet \mathbf{b} \neq 0$, so there can be no such solution $\mathbf{x}$.

28. The equation $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ can be solved for $\mathbf{x}$ if and only if $\mathbf{a} \bullet \mathbf{b} = 0$. The "only if" part is demonstrated in the previous solution. For the "if" part, observe that if $\mathbf{a} \bullet \mathbf{b} = 0$ and $\mathbf{x}_0 = (\mathbf{b} \times \mathbf{a})/|\mathbf{a}|^2$, then by Exercise 23,

$$\mathbf{a} \times \mathbf{x}_0 = \frac{1}{|a|^2}\mathbf{a} \times (\mathbf{b} \times \mathbf{a}) = \frac{(\mathbf{a} \bullet \mathbf{a})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{a}}{|a|^2} = \mathbf{b}.$$

The solution $\mathbf{x}_0$ is not unique; as suggested by the example in Exercise 26, any multiple of $\mathbf{a}$ can be added to it and the result will still be a solution. If $\mathbf{x} = \mathbf{x}_0 + t\mathbf{a}$, then

$$\mathbf{a} \times \mathbf{x} = \mathbf{a} \times \mathbf{x}_0 + t\mathbf{a} \times \mathbf{a} = \mathbf{b} + 0 = \mathbf{b}.$$

Section 10.4 Planes and Lines (page 567)

1. a) $x^2 + y^2 + z^2 = z^2$ represents a line in 3-space, namely the z-axis.

b) $x + y + z = x + y + z$ is satisfied by every point in 3-space.

c) $x^2 + y^2 + z^2 = -1$ is satisfied by no points in (real) 3-space.

2. The plane through $(0, 2, -3)$ normal to $4\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ has equation

$$4(x - 0) - (y - 2) - 2(z + 3) = 0,$$

or $4x - y - 2z = 4$.

3. The plane through the origin having normal $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ has equation $x - y + 2z = 0$.

4. The plane passing through $(1, 2, 3)$, parallel to the plane $3x + y - 2z = 15$, has equation $3z + y - 2z = 3 + 2 - 6$, or $3x + y - 2z = -1$.

5. The plane through $(1, 1, 0)$, $(2, 0, 2)$, and $(0, 3, 3)$ has normal

$$(\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}) = 7\mathbf{i} + 5\mathbf{j} - \mathbf{k}.$$

It therefore has equation

$$7(x - 1) + 5(y - 1) - (z - 0) = 0,$$

or $7x + 5y - z = 12$.

6. The plane passing through $(-2, 0, 0)$, $(0, 3, 0)$, and $(0, 0, 4)$ has equation

$$\frac{x}{-2} + \frac{y}{3} + \frac{z}{4} = 1,$$

or $6x - 4y - 3z = -12$.

7. The normal $\mathbf{n}$ to a plane through $(1, 1, 1)$ and $(2, 0, 3)$ must be perpendicular to the vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ joining these points. If the plane is perpendicular to the plane $x + 2y - 3z = 0$, then $\mathbf{n}$ must also be perpendicular to $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, the normal to this latter plane. Hence we can use

$$\mathbf{n} = (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \times (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = -\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}.$$

The plane has equation

$$-(x - 1) + 5(y - 1) + 3(z - 1) = 0,$$

or $x - 5y - 3z = -7$.

8. Since $(-2, 0, -1)$ does not lie on $x - 4y + 2z = -5$, the required plane will have an equation of the form

$$2x + 3y - z + \lambda(x - 4y + 2z + 5) = 0$$

for some λ. Thus

$$-4 + 1 + \lambda(-2 - 2 + 5) = 0,$$

so $\lambda = 3$. The required plane is $5x - 9y + 5z = -15$.

9. A plane through the line $x + y = 2$, $y - z = 3$ has equation of the form

$$x + y - 2 + \lambda(y - z - 3) = 0.$$

This plane will be perpendicular to $2x + 3y + 4z = 5$ if

$$(2)(1) + (1 + \lambda)(3) - (\lambda)(4) = 0,$$

that is, if $\lambda = 5$. The equation of the required plane is

$$x + 6y - 5z = 17.$$

10. Three distinct points will not determine a unique plane through them if they all lie on a straight line. If the points have position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$, then they will all lie on a straight line if

$$(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1) = \mathbf{0}.$$

11. If the four points have position vectors $\mathbf{r}_i$, $(1 \le i \le 4)$, then they are coplanar if, for example,

$$(\mathbf{r}_2 - \mathbf{r}_1) \bullet \Big[(\mathbf{r}_3 - \mathbf{r}_1) \times (\mathbf{r}_4 - \mathbf{r}_1)\Big] = 0$$

(or if they satisfy any similar such condition that asserts that the tetrahedron whose vertices they are has zero volume).

12. $x + y + z = \lambda$ is the family of all (parallel) planes normal to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

13. $x + \lambda y + \lambda z = \lambda$ is the family of all planes containing the line of intersection of the planes $x = 0$ and $y + z = 1$, except the plane $y + z = 1$ itself. All these planes pass through the points $(0, 1, 0)$ and $(0, 0, 1)$.

14. The distance from the planes

$$\lambda x + \sqrt{1 - \lambda^2}\, y = 1$$

to the origin is $1/\sqrt{\lambda^2 + 1 - \lambda^2} = 1$. Hence the equation represents the family of all vertical planes at distance 1 from the origin. All such planes are tangent to the cylinder $x^2 + y^2 = 1$.

15. The line through $(1, 2, 3)$ parallel to $2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$ has equations given in vector parametric form by

$$\mathbf{r} = (1 + 2t)\mathbf{i} + (2 - 3t)\mathbf{j} + (3 - 4t)\mathbf{k},$$

or in scalar parametric form by

$$x = 1 + 2t, \quad y = 2 - 3t, \quad z = 3 - 4t,$$

or in standard form by

$$\frac{x-1}{2} = \frac{y-2}{-3} - \frac{z-3}{-4}.$$

16. The line through $(-1, 0, 1)$ perpendicular to the plane $2x - y + 7z = 12$ is parallel to the normal vector $2\mathbf{i} - \mathbf{j} + 7\mathbf{k}$ to that plane. The equations of the line are, in vector parametric form,

$$\mathbf{r} = (-1 + 2t)\mathbf{i} - t\mathbf{j} + (1 + 7t)\mathbf{k},$$

or in scalar parametric form,

$$x = -1 + 2t, \quad y = -t, \quad z = 1 + 7t,$$

or in standard form

$$\frac{x+1}{2} = \frac{y}{-1} = \frac{z-1}{7}.$$

17. A line parallel to the line with equations

$$x + 2y - z = 2, \qquad 2x - y + 4z = 5$$

is parallel to the vector

$$(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} + 4\mathbf{k}) = 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}.$$

Since the line passes through the origin, it has equations

$$\begin{aligned} &\mathbf{r} = 7t\mathbf{i} - 6t\mathbf{j} - 5t\mathbf{k} && \text{(vector parametric)}\\ &x = 7t, \quad y = -6t, \quad z = -5t && \text{(scalar parametric)}\\ &\frac{x}{7} = \frac{y}{-6} = \frac{z}{-5} && \text{(standard form).} \end{aligned}$$

18. A line parallel to $x + y = 0$ and to $x - y + 2z = 0$ is parallel to the cross product of the normal vectors to these two planes, that is, to the vector

$$(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = 2(\mathbf{i} - \mathbf{j} - \mathbf{k}).$$

Since the line passes through $(2, -1, -1)$, its equations are, in vector parametric form

$$\mathbf{r} = (2 + t)\mathbf{i} - (1 + t)\mathbf{j} - (1 + t)\mathbf{k},$$

or in scalar parametric form

$$x = 2 + t, \quad y = -(1 + t), \quad z = -(1 + t),$$

or in standard form

$$x - 2 = -(y + 1) = -(z + 1).$$

19. A line making equal angles with the positive directions of the coordinate axes is parallel to the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$. If the line passes through the point $(1, 2, -1)$, then it has equations

$$\begin{aligned} &\mathbf{r} = (1 + t)\mathbf{i} + (2 + t)\mathbf{j} + (-1 + t)\mathbf{k} && \text{(vector parametric)}\\ &x = 1 + t, \quad y = 2 + t, \quad z = -1 + t && \text{(scalar parametric)}\\ &x - 1 = y - 2 = z + 1 && \text{(standard form).} \end{aligned}$$

20. The line $\mathbf{r} = (1-2t)\mathbf{i} + (4+3t)\mathbf{j} + (9-4t)\mathbf{k}$ has standard form
$$\frac{x-1}{-2} = \frac{y-4}{3} = \frac{z-9}{-4}.$$

21. The line $\begin{cases} x = 4-5t \\ y = 3t \\ z = 7 \end{cases}$ has standard form
$$\frac{x-4}{-5} = \frac{y}{3}, \quad z = 7.$$

22. The line $\begin{cases} x - 2y + 3z = 0 \\ 2x + 3y - 4z = 4 \end{cases}$ is parallel to the vector
$$(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \times (2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) = -\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}.$$
We need a point on this line. Putting $z = 0$, we get
$$x - 2y = 0, \qquad 2x + 3y = 4.$$
The solution of this system is $y = 4/7$, $x = 8/7$. A possible standard form for the given line is
$$\frac{x - \frac{8}{7}}{-1} = \frac{y - \frac{4}{7}}{10} = \frac{z}{7},$$
though, of course, this answer is not unique as the coordinates of any point on the line could have been used.

23. The equations
$$\begin{cases} x = x_1 + t(x_2 - x_1) \\ y = y_1 + t(y_2 - y_1) \\ z = z_1 + t(z_2 - z_1) \end{cases}$$
certainly represent a straight line. Since $(x, y, z) = (x_1, y_1, z_1)$ if $t = 0$, and $(x, y, z) = (x_2, y_2, z_2)$ if $t = 1$, the line must pass through P_1 and P_2.

24. The point on the line corresponding to $t = -1$ is the point P_3 such that P_1 is midway between P_3 and P_2.
The point on the line corresponding to $t = 1/2$ is the midpoint between P_1 and P_2.
The point on the line corresponding to $t = 2$ is the point P_4 such that P_2 is the midpoint between P_1 and P_4.

25. Let $\mathbf{r}_i$ be the position vector of P_i $(1 \le i \le 4)$. The line P_1P_2 intersects the line P_3P_4 in a unique point if the four points are coplanar, and P_1P_2 is not parallel to P_3P_4. It is therefore sufficient that
$$(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_4 - \mathbf{r}_3) \ne \mathbf{0}, \text{ and}$$
$$(\mathbf{r}_3 - \mathbf{r}_1) \bullet \Big[(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_4 - \mathbf{r}_3)\Big] = 0.$$
(Other similar answers are possible.)

26. The distance from $(0, 0, 0)$ to $x + 2y + 3z = 4$ is
$$\frac{4}{\sqrt{1^2 + 2^2 + 3^2}} = \frac{4}{\sqrt{14}} \text{ units.}$$

27. The distance from $(1, 2, 0)$ to $3x - 4y - 5z = 2$ is
$$\frac{|3 - 8 - 0 - 2|}{\sqrt{3^2 + 4^2 + 5^2}} = \frac{7}{5\sqrt{2}} \text{ units.}$$

28. A vector parallel to the line $x + y + z = 0$, $2x - y - 5z = 1$ is
$$\mathbf{a} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \times (2\mathbf{i} - \mathbf{j} - 5\mathbf{k}) = -4\mathbf{i} + 7\mathbf{j} - 3\mathbf{k}.$$
We need a point on this line: if $z = 0$ then $x + y = 0$ and $2x - y = 1$, so $x = 1/3$ and $y = -1/3$. The position vector of this point is
$$\mathbf{r}_1 = \frac{1}{3}\mathbf{i} - \frac{1}{3}\mathbf{j}.$$
The distance from the origin to the line is
$$s = \frac{|\mathbf{r}_1 \times \mathbf{a}|}{|\mathbf{a}|} = \frac{|\mathbf{i} + \mathbf{j} + \mathbf{k}|}{\sqrt{74}} = \sqrt{\frac{3}{74}} \text{ units.}$$

29. The line $\begin{cases} x + 2y = 3 \\ y + 2z = 3 \end{cases}$ contains the points $(1, 1, 1)$ and $(3, 0, 3/2)$, so is parallel to the vector $2\mathbf{i} - \mathbf{j} + \frac{1}{2}\mathbf{k}$, or to $4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
The line $\begin{cases} x + y + z = 6 \\ x - 2z = -5 \end{cases}$ contains the points $(-5, 11, 0)$ and $(-1, 5, 2)$, and so is parallel to the vector $4\mathbf{i} - 6\mathbf{j} + 2\mathbf{k}$, or to $2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.
Using the values
$$\mathbf{r}_1 = \mathbf{i} + \mathbf{j} + \mathbf{k} \qquad \mathbf{a}_1 = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$
$$\mathbf{r}_2 = -\mathbf{i} + 5\mathbf{j} + 2\mathbf{k} \qquad \mathbf{a}_2 = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k},$$
we calculate the distance between the two lines by the formula in Section 10.4 as
$$\begin{aligned} s &= \frac{|(\mathbf{r}_1 - \mathbf{r}_2) \bullet (\mathbf{a}_1 \times \mathbf{a}_2)|}{|\mathbf{a}_1 \times \mathbf{a}_2|} \\ &= \frac{|(2\mathbf{i} - 4\mathbf{j} - \mathbf{k}) \bullet (\mathbf{i} - 2\mathbf{j} - 8\mathbf{k})|}{|\mathbf{i} - 2\mathbf{j} - 8\mathbf{k}|} \\ &= \frac{18}{\sqrt{69}} \text{ units.} \end{aligned}$$

30. The line $x-2=\dfrac{y+3}{2}=\dfrac{z-1}{4}$ passes through the point $(2,-3,1)$, and is parallel to $\mathbf{a}=\mathbf{i}+2\mathbf{j}+4\mathbf{k}$.
The plane $2y-z=1$ has normal $\mathbf{n}=2\mathbf{j}-\mathbf{k}$.
Since $\mathbf{a}\bullet\mathbf{n}=0$, the line is parallel to the plane.
The distance from the line to the plane is equal to the distance from $(2,-3,1)$ to the plane $2y-z=1$, so is

$$D=\frac{|-6-1-1|}{\sqrt{4+1}}=\frac{8}{\sqrt{5}}\text{ units.}$$

31. $(1-\lambda)(x-x_0)=\lambda(y-y_0)$ represents any line in the xy-plane passing through (x_0,y_0). Therefore, in 3-space the pair of equations

$$(1-\lambda)(x-x_0)=\lambda(y-y_0),\qquad z=z_0$$

represents all straight lines in the plane $z=z_0$ which pass through the point (x_0,y_0,z_0).

32. $\dfrac{x-x_0}{\sqrt{1-\lambda^2}}=\dfrac{y-y_0}{\lambda}=z-z_0$ represents all lines through (x_0,y_0,z_0) parallel to the vectors

$$\mathbf{a}=\sqrt{1-\lambda^2}\mathbf{i}+\lambda\mathbf{j}+\mathbf{k}.$$

All such lines are generators of the circular cone

$$(z-z_0)^2=(x-x_0)^2+(y-y_0)^2,$$

so the given equations specify all straight lines lying on that cone.

33. The equation

$$(A_1x+B_1y+C_1z+D_1)(A_2x+B_2y+C_2z+D_2)=0$$

is satisfied if *either* $A_1x+B_1y+C_1z+D_1=0$ *or* $A_2x+B_2y+C_2z+D_2=0$, that is, if (a,y,z) lies on either of these planes. It is not necessary that the point lie on both planes, so the given equation represents all the points on each of the planes, not just those on the line of intersection of the planes.

Section 10.5 Quadric Surfaces (page 570)

1. $x^2+4y^2+9z^2=36$
$\dfrac{x^2}{6^2}+\dfrac{y^2}{3^2}+\dfrac{z^2}{2^2}=1$
This is an ellipsoid with centre at the origin and semi-axes 6, 3, and 2.

2. $x^2+y^2+4z^2=4$ represents an oblate spheroid, that is, an ellipsoid with its two longer semi-axes equal. In this case the longer semi-axes have length 2, and the shorter one (in the z direction) has length 1. Cross-sections in planes perpendicular to the z-axis between $z=-1$ and $z=1$ are circles.

3. $2x^2+2y^2+2z^2-4x+8y-12z+27=0$
$2(x^2-2x+1)+2(y^2+4y+4)+2(z^2-6z+9)$
$=-27+2+8+18$
$(x-1)^2+(y+2)^2+(z-3)^2=\dfrac{1}{2}$
This is a sphere with radius $1/\sqrt{2}$ and centre $(1,-2,3)$.

4. $x^2+4y^2+9z^2+4x-8y=8$
$(x+2)^2+4(y-1)^2+9z^2=8+8=16$
$\dfrac{(x+2)^2}{4^2}+\dfrac{(y-1)^2}{2^2}+\dfrac{z^2}{(4/3)^2}=1$
This is an ellipsoid with centre $(-2,1,0)$ and semi-axes 4, 2, and 4/3.

5. $z=x^2+2y^2$ represents an elliptic paraboloid with vertex at the origin and axis along the positive z-axis. Cross-sections in planes $z=k>0$ are ellipses with semi-axes $\sqrt{k}$ and $\sqrt{k/2}$.

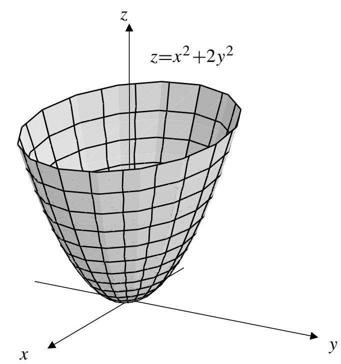

Fig. 10.5.5

6. $z=x^2-2y^2$ represents a hyperbolic paraboloid.

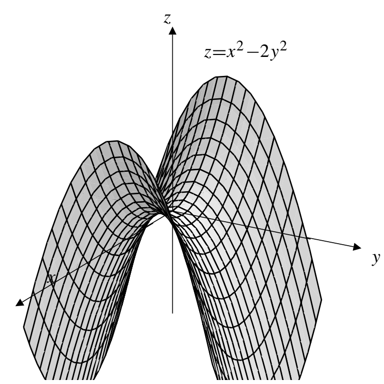

Fig. 10.5.6

7. $x^2-y^2-z^2=4$ represents a hyperboloid of two sheets with vertices at $(\pm2,0,0)$ and circular cross-sections in planes $x=k$, where $|k|>2$.

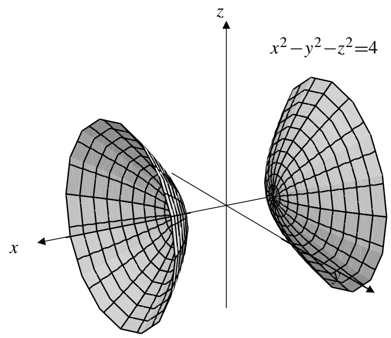

Fig. 10.5.7

8. $-x^2 + y^2 + z^2 = 4$ represents a hyperboloid of one sheet, with circular cross-sections in all planes perpendicular to the x-axis.

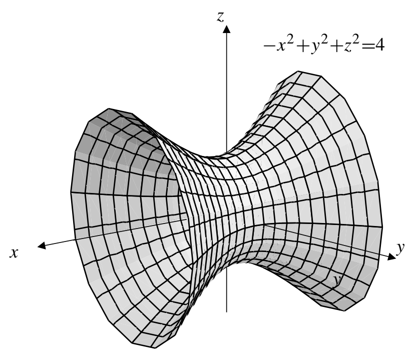

Fig. 10.5.8

9. $z = xy$ represents a hyperbolic paraboloid containing the x- and y-axes.

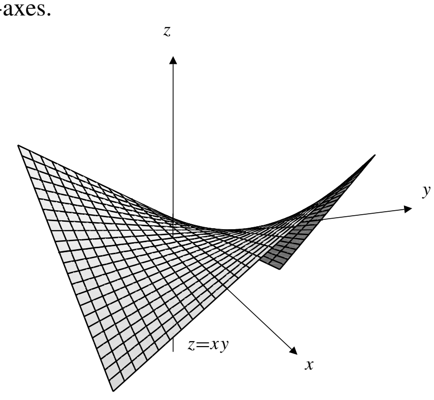

Fig. 10.5.9

10. $x^2 + 4z^2 = 4$ represents an elliptic cylinder with axis along the y-axis.

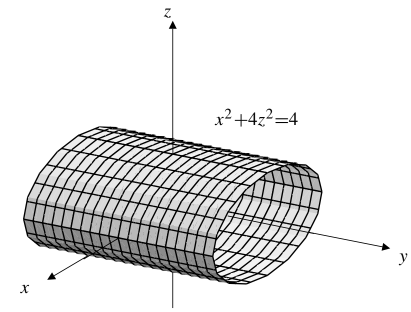

Fig. 10.5.10

11. $x^2 - 4z^2 = 4$ represents a hyperbolic cylinder with axis along the y-axis.

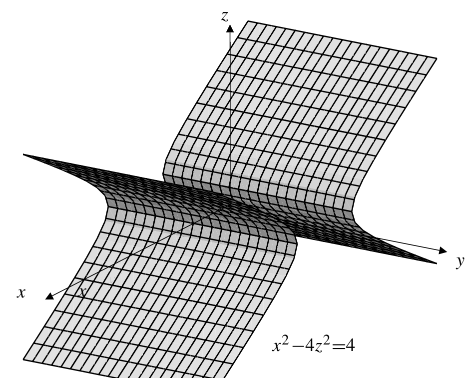

Fig. 10.5.11

12. $y = z^2$ represents a parabolic cylinder with vertex line along the x-axis.

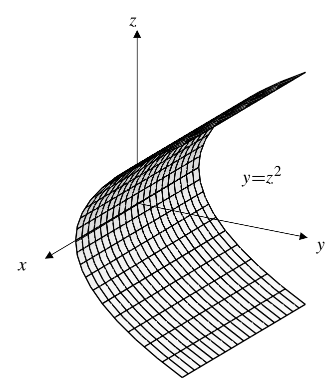

Fig. 10.5.12

13. $x = z^2+z = \left(z + \frac{1}{2}\right)^2 - \frac{1}{4}$ represents a parabolic cylinder with vertex line along the line $z = -1/2$, $x = -1/4$.

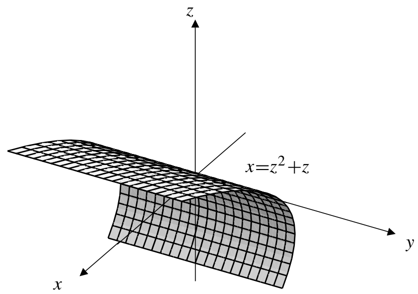

Fig. 10.5.13

14. $x^2 = y^2 + 2z^2$ represents an elliptic cone with vertex at the origin and axis along the x-axis.

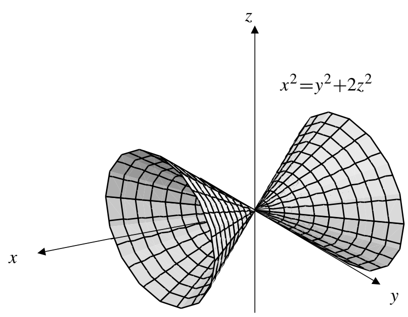

Fig. 10.5.14

15. $(z-1)^2 = (x-2)^2 + (y-3)^2$ represents a circular cone with axis along the line $x = 2$, $y = 3$, and vertex at $(2, 3, 1)$

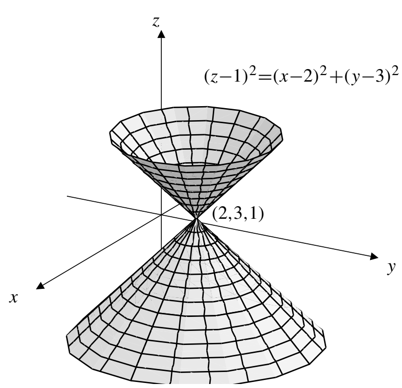

Fig. 10.5.15

16. $(z-1)^2 = (x-2)^2 + (y-3)^2 + 4$ represents a hyperboloid of two sheets with centre at $(2, 3, 1)$, axis along the line $x = 2$, $y = 3$, and vertices at $(2, 3, -1)$ and $(2, 3, 3)$.

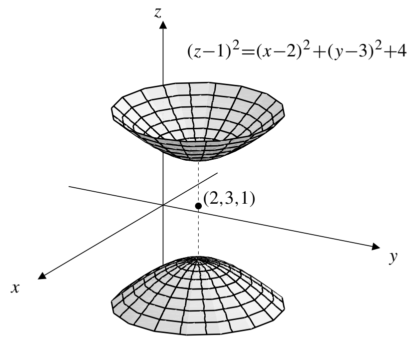

Fig. 10.5.16

17. $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x + y + z = 1 \end{cases}$ represents the circle of intersection of a sphere and a plane. The circle lies in the plane $x + y + z = 1$, and has centre $(1/3, 1/3, 1/3)$ and radius $\sqrt{4 - (3/9)} = \sqrt{11/3}$.

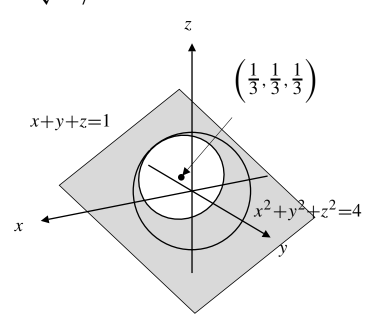

Fig. 10.5.17

18. $\begin{cases} x^2 + y^2 = 1 \\ z = x + y \end{cases}$ is the ellipse of intersection of the plane $z = x + y$ and the circular cylinder $x^2 + y^2 = 1$. The centre of the ellipse is at the origin, and the ends of the major axis are $\pm(1/\sqrt{2}, 1/\sqrt{2}, \sqrt{2})$.

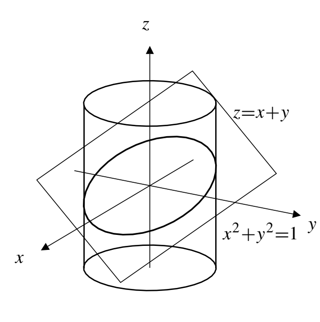

Fig. 10.5.18

19. $\begin{cases} z^2 = x^2 + y^2 \\ z = 1 + x \end{cases}$ is the parabola in which the plane $z = 1 + x$ intersects the circular cone $z^2 = x^2 + y^2$. (It is a parabola because the plane is parallel to a generator of the cone, namely the line $z = x$, $y = 0$.) The vertex of the parabola is $(-1/2, 0, 1/2)$, and its axis is along the line $y = 0$, $z = 1 + x$.

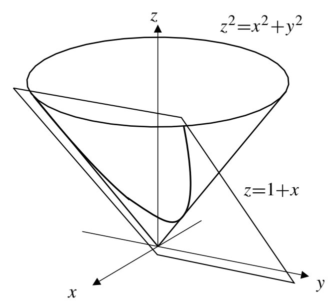

Fig. 10.5.19

20. $\begin{cases} x^2 + 2y^2 + 3z^2 = 6 \\ y = 1 \end{cases}$ is an ellipse in the plane $y = 1$. Its projection onto the xz-plane is the ellipse $x^2 + 3z^2 = 4$. One quarter of the ellipse is shown in the figure.

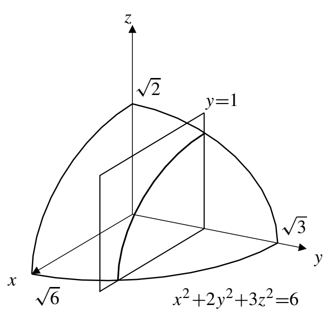

Fig. 10.5.20

21.
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y^2}{b^2}$$
$$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right)$$

$$\text{Family 1:} \quad \begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = \lambda\left(1 + \dfrac{y}{b}\right) \\ \lambda\left(\dfrac{x}{a} - \dfrac{z}{c}\right) = 1 - \dfrac{y}{b}. \end{cases}$$

$$\text{Family 2:} \quad \begin{cases} \dfrac{x}{a} + \dfrac{z}{c} = \mu\left(1 - \dfrac{y}{b}\right) \\ \mu\left(\dfrac{x}{a} - \dfrac{z}{c}\right) = 1 + \dfrac{y}{b}. \end{cases}.$$

22. $z = xy$

$$\text{Family 1:} \quad \begin{cases} z = \lambda x \\ \lambda = y. \end{cases}$$

$$\text{Family 2:} \quad \begin{cases} z = \mu y \\ \mu = x. \end{cases}$$

23. The cylinder $2x^2 + y^2 = 1$ intersects horizontal planes in ellipses with semi-axes 1 in the y direction and $1/\sqrt{2}$ in the x direction. Tilting the plane in the x direction will cause the shorter semi-axis to increase in length. The plane $z = cx$ intersects the cylinder in an ellipse with principal axes through the points $(0, \pm 1, 0)$ and $(\pm 1/\sqrt{2}, 0, \pm c/\sqrt{2})$. The semi-axes will be equal (and the ellipse will be a circle) if $(1/2) + (c^2/2) = 1$, that is, if $c = \pm 1$. Thus cross-sections of the cylinder perpendicular to the vectors $\mathbf{a} = \mathbf{i} \pm \mathbf{k}$ are circular.

24. The plane $z = cx + k$ intersects the elliptic cone $z^2 = 2x^2 + y^2$ on the cylinder

$$c^2x^2 + 2ckx + k^2 = 2x^2 + y^2$$
$$(2 - c^2)x^2 - 2ckx + y^2 = k^2$$
$$(2 - c^2)\left(x - \frac{ck}{2 - c^2}\right)^2 + y^2 = k^2 + \frac{c^2k^2}{2 - c^2} = \frac{2k^2}{2 - c^2}$$
$$\frac{(x - x_0)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $x_0 = \dfrac{ck}{2 - c^2}$, $a^2 = \dfrac{2k^2}{(2 - c^2)^2}$, and $b^2 = \dfrac{2k^2}{2 - c^2}$.

As in the previous exercise, $z = cx + k$ intersects the cylinder (and hence the cone) in an ellipse with principal axes joining the points

$$(x_0 - a, 0, c(x_0 - a) + k) \quad \text{to} \quad (x_0 + a, 0, c(x_0 + a) + k),$$
$$\text{and } (x_0, -b, cx_0 + k) \quad \text{to} \quad (x_0, b, cx_0 + k).$$

The centre of this ellipse is $(x_0, 0, cx_0 + k)$. The ellipse is a circle if its two semi-axes have equal lengths, that is, if

$$a^2 + c^2a^2 = b^2,$$

that is,

$$(1 + c^2)\frac{2k^2}{(2 - c^2)^2} = \frac{2k^2}{2 - c^2},$$

or $1 + c^2 = 2 - c^2$. Thus $c = \pm 1/\sqrt{2}$. A vector normal to the plane $z = \pm(x/\sqrt{2}) + k$ is $\mathbf{a} = \mathbf{i} \pm \sqrt{2}\mathbf{k}$.

Section 10.6 A Little Linear Algebra (page 579)

1. $$\begin{pmatrix} 3 & 0 & -2 \\ 1 & 1 & 2 \\ -1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 7 \\ 5 & -3 \\ 1 & 1 \end{pmatrix}$$

2. $$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

3. $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}$$

4. $\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aw+cx & bw+dx \\ ay+cz & by+dz \end{pmatrix}$

5.
$$\mathcal{A}\mathcal{A}^T = \begin{pmatrix} 1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0\\1&1&0&0\\1&1&1&0\\1&1&1&1 \end{pmatrix} = \begin{pmatrix} 4&3&2&1\\3&3&2&1\\2&2&2&1\\1&1&1&1 \end{pmatrix}$$
$$\mathcal{A}^2 = \begin{pmatrix} 1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1 \end{pmatrix}\begin{pmatrix} 1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1 \end{pmatrix} = \begin{pmatrix} 1&2&3&4\\0&1&2&3\\0&0&1&2\\0&0&0&1 \end{pmatrix}$$

6. $\mathbf{x} = \begin{pmatrix} x\\y\\z \end{pmatrix}$, $\quad \mathcal{A} = \begin{pmatrix} a&p&q\\p&b&r\\q&r&c \end{pmatrix}$
$$\mathbf{x}\mathbf{x}^T = \begin{pmatrix} x\\y\\z \end{pmatrix}(x,y,z) = \begin{pmatrix} x^2&xy&xz\\xy&y^2&yz\\xz&yz&z^2 \end{pmatrix}$$
$$\mathbf{x}^T\mathbf{x} = (x,y,z)\begin{pmatrix} x\\y\\z \end{pmatrix} = (x^2+y^2+z^2)$$
$$\mathbf{x}^T\mathcal{A}\mathbf{x} = (x,y,z)\begin{pmatrix} a&p&q\\p&b&r\\q&r&c \end{pmatrix}\begin{pmatrix} x\\y\\z \end{pmatrix}$$
$$= (x,y,z)\begin{pmatrix} ax+py+qz\\px+by+rz\\qx+ry+cz \end{pmatrix}$$
$$= ax^2+by^2+cz^2+2pxy+2qxz+2ryz$$

7.
$$\begin{vmatrix} 2&3&-1&0\\4&0&2&1\\1&0&-1&1\\-2&0&0&1 \end{vmatrix} = -3\begin{vmatrix} 4&2&1\\1&-1&1\\-2&0&1 \end{vmatrix}$$
$$= -3\left(-2\begin{vmatrix} 1&1\\-2&1 \end{vmatrix} - 1\begin{vmatrix} 4&1\\-2&1 \end{vmatrix}\right)$$
$$= 6(3)+3(6) = 36$$

8.
$$\begin{vmatrix} 1&1&1&1\\1&2&3&4\\-2&0&2&4\\3&-3&2&-2 \end{vmatrix}$$
$$= -2\begin{vmatrix} 1&1&1\\2&3&4\\-3&2&-2 \end{vmatrix} + 2\begin{vmatrix} 1&1&1\\1&2&4\\3&-3&-2 \end{vmatrix} - 4\begin{vmatrix} 1&1&1\\1&2&3\\3&-3&2 \end{vmatrix}$$
$$= -2\begin{vmatrix} 1&1&1\\0&1&2\\0&5&1 \end{vmatrix} + 2\begin{vmatrix} 1&1&1\\0&1&3\\0&-6&-5 \end{vmatrix} - 4\begin{vmatrix} 1&1&1\\0&1&2\\0&-6&-1 \end{vmatrix}$$
$$= -2\begin{vmatrix} 1&2\\5&1 \end{vmatrix} + 2\begin{vmatrix} 1&3\\-6&-5 \end{vmatrix} - 4\begin{vmatrix} 1&2\\-6&-1 \end{vmatrix}$$
$$= -2(-9)+2(13)-4(11) = 0$$

9.
$$\begin{vmatrix} a_{11}&a_{12}&a_{13}&\cdots&a_{1n}\\0&a_{22}&a_{23}&\cdots&a_{2n}\\0&0&a_{33}&\cdots&a_{3n}\\\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&\cdots&a_{nn} \end{vmatrix}$$
$$= a_{11}\begin{vmatrix} a_{22}&a_{23}&\cdots&a_{2n}\\0&a_{33}&\cdots&a_{3n}\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&a_{nn} \end{vmatrix}$$
$$= a_{11}a_{22}\begin{vmatrix} a_{33}&\cdots&a_{3n}\\\vdots&\ddots&\vdots\\0&\cdots&a_{nn} \end{vmatrix}$$
$$= a_{11}a_{22}a_{33}\cdots a_{nn}$$
(or use induction on n)

10. $\begin{vmatrix} 1&1\\x&y \end{vmatrix} = y-x$. If
$$f(x,y,z) = \begin{vmatrix} 1&1&1\\x&y&z\\x^2&y^2&z^2 \end{vmatrix},$$
then f is a polynomial of degree 2 in z.
Since $f(x,y,x)=0$ and $f(x,y,y)=0$, we must have $f(x,y,z) = A(z-x)(z-y)$ for some A independent of z. But
$$Axy = f(x,y,0) = \begin{vmatrix} 1&1&1\\x&y&0\\x^2&y^2&0 \end{vmatrix} = xy(y-x),$$
so $A = y-x$ and
$$f(x,y,z) = (y-x)(z-x)(z-y).$$

Generalization:

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{vmatrix} = \prod_{1\le i<j\le n} (x_j - x_i).$$

11. Let $\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} \ell & m \\ n & p \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Then

$$\begin{aligned} (\mathcal{AB})\mathcal{C} &= \begin{pmatrix} a\ell + bn & am + bp \\ c\ell + dn & cm + dp \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ &= \begin{pmatrix} a\ell w + bnw + amy + bpy & a\ell x + bnx + amz + bpz \\ c\ell w + dnw + cmy + dpy & c\ell x + dnx + cmz + dpz \end{pmatrix} \\ \mathcal{A}(\mathcal{BC}) &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \ell w + my & \ell x + mz \\ nw + py & nx + pz \end{pmatrix} \\ &= \begin{pmatrix} a\ell w + amy + bnw + bpy & a\ell x + amz + bnx + bpz \\ c\ell w + cmy + dnw + dpy & c\ell x + cmz + dnx + dpz \end{pmatrix} \end{aligned}$$

Thus $(\mathcal{AB})\mathcal{C} = \mathcal{A}(\mathcal{BC})$.

12. If $\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathcal{A}^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, and

$$\det(\mathcal{A}) = ad - bc = \det(\mathcal{A}^T).$$

We generalize this by induction.
Suppose $\det(\mathcal{B}^T)$=$\det(\mathcal{B})$ for any $(n-1)\times(n-1)$ matrix, where $n \ge 3$. Let

$$\mathcal{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

be an $n \times n$ matrix. If $\det(\mathcal{A})$ is expanded in minors about the first row, and $\det(\mathcal{A}^T)$ is expanded in minors about the first column, the corresponding terms in these expansions are equal by the induction hypothesis. (The $(n-1)\times(n-1)$ matrices whose determinants appear in one expansion are the transposes of those in the other expansion.) Therefore $\det(\mathcal{A}^T)$=$\det(\mathcal{A})$ for any square matrix $\mathcal{A}$.

13. Let $\mathcal{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Then

$$\mathcal{AB} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \det(\mathcal{A})\det(\mathcal{B}) &= (ad - bc)(wz - xy) \\ &= adwz - adxy - bcwz + bcxy \\ \det(\mathcal{AB}) &= (aw + by)(cx + dz) - (ax + bz)(cw + dy) \\ &= awcx + awdz + bycx + bydz \\ &\quad - axwc - axdy - bzcw - bzdy \\ &= adwz - adxy - bcwz + bcxy \\ &= \det(\mathcal{A})\det(\mathcal{B}). \end{aligned}$$

14. If $\mathcal{A}_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, then

$$\mathcal{A}_{-\theta} = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

and

$$\mathcal{A}_\theta \mathcal{A}_{-\theta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Thus $\mathcal{A}_{-\theta} = (\mathcal{A}_\theta)^{-1}$.

15. Let $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathcal{A}^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Since $\mathcal{AA}^{-1} = I$ we must have

$$\begin{aligned} a + d + g &= 1 & b + e + h &= 0 & c + f + i &= 0 \\ d + g &= 0 & e + h &= 1 & f + i &= 0 \\ g &= 0 & h &= 0 & i &= 1. \end{aligned}$$

Thus $a = 1$, $d = g = 0$, $h = 0$, $e = 1$, $b = -1$, $i = 1$, $f = -1$, $c = 0$, and so

$$\mathcal{A}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

16. Let $\mathcal{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 2 & 1 & 3 \end{pmatrix}$, $\mathcal{A}^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Since $\mathcal{AA}^{-1} = I$ we must have

$$\begin{aligned} a - g &= 1 & b - h &= 0 & c - i &= 0 \\ -a + d &= 0 & -b + e &= 1 & -c + f &= 0 \\ 2a + d + 3g &= 0 & 2b + e + 3h &= 0 & 2c + f + 3i &= 1. \end{aligned}$$

Solving these three systems of equations, we get

$$\mathcal{A}^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{5}{6} & \frac{1}{6} \\ -\frac{1}{2} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix}.$$

17. The given system of equations is

$$\mathcal{A}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 13 \end{pmatrix}.$$

Thus
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathcal{A}^{-1}\begin{pmatrix} -2 \\ 1 \\ 13 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$
so $x = 1$, $y = 2$, and $z = 3$.

18. If $\mathcal{A}$ is the matrix of Exercises 16 and 17 then $\det(\mathcal{A}) = 6$. By Cramer's Rule,

$$x = \frac{1}{6}\begin{vmatrix} -2 & 0 & -1 \\ 1 & 1 & 0 \\ 13 & 1 & 3 \end{vmatrix} = \frac{6}{6} = 1$$
$$y = \frac{1}{6}\begin{vmatrix} 1 & -2 & -1 \\ -1 & 1 & 0 \\ 2 & 13 & 3 \end{vmatrix} = \frac{12}{6} = 2$$
$$z = \frac{1}{6}\begin{vmatrix} 1 & 0 & -2 \\ -1 & 1 & 1 \\ 2 & 1 & 13 \end{vmatrix} = \frac{18}{6} = 3.$$

19. $$\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{pmatrix}$$
$$\det(\mathcal{A}) = \begin{vmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & -1 & -1 \end{vmatrix}$$
$$= -2\begin{vmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & -1 & -1 \end{vmatrix} = -4\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix} = 8$$

$$x_1 = \frac{1}{8}\begin{vmatrix} 0 & 1 & 1 & 1 \\ 4 & 1 & 1 & -1 \\ 6 & 1 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{vmatrix}$$
$$= \frac{1}{8}\begin{vmatrix} 0 & 1 & 1 & 1 \\ 4 & 0 & 0 & -2 \\ 6 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{vmatrix}$$
$$= -\frac{2}{8}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 2 & 0 & 0 \end{vmatrix} = -\frac{4}{8}\begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = 1$$

$$x_2 = \frac{1}{8}\begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 4 & 1 & -1 \\ 1 & 6 & -1 & -1 \\ 1 & 2 & -1 & -1 \end{vmatrix}$$
$$= \frac{1}{8}\begin{vmatrix} 2 & 0 & 0 & 1 \\ 0 & 4 & 2 & -1 \\ 0 & 6 & 0 & -1 \\ 0 & 2 & 0 & -1 \end{vmatrix}$$
$$= \frac{2}{8}\begin{vmatrix} 4 & 2 & -1 \\ 6 & 0 & -1 \\ 2 & 0 & -1 \end{vmatrix} = \frac{-4}{8}\begin{vmatrix} 6 & -1 \\ 2 & -1 \end{vmatrix} = 2$$

$$x_3 = \frac{1}{8}\begin{vmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 4 & -1 \\ 1 & 1 & 6 & -1 \\ 1 & -1 & 2 & -1 \end{vmatrix}$$
$$= \frac{1}{8}\begin{vmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 4 & -1 \\ 0 & 0 & 6 & -1 \\ 2 & -2 & 2 & -1 \end{vmatrix}$$
$$= -\frac{2}{8}\begin{vmatrix} 2 & 0 & 1 \\ 0 & 4 & -1 \\ 0 & 6 & -1 \end{vmatrix} = -\frac{4}{8}\begin{vmatrix} 4 & -1 \\ 6 & -1 \end{vmatrix} = -1$$
$$x_4 = -(x_1 + x_2 + x_3) = -2.$$

20. Let $F(x_1, x_2) = \mathcal{F}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, where $\mathcal{F} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Let $G(y_1, y_2) = \mathcal{G}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, where $\mathcal{G} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

If $y_1 = ax_1 + bx_2$ and $y_2 = cx_1 + dx_2$, then

$$G \circ F(x_1, x_2) = G(y_1, y_2)$$
$$= \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$$
$$= \begin{pmatrix} pax_1 + pbx_2 + qcx_1 + qdx_2 \\ rax_1 + rbx_2 + scx_1 + sdx_2 \end{pmatrix}$$
$$= \begin{pmatrix} pa + qc & pb + qd \\ ra + sc & rb + sd \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \mathcal{G}\mathcal{F}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus, $\mathcal{G} \circ \mathcal{F}$ is represented by the matrix $\mathcal{G}\mathcal{F}$.

21. $\mathcal{A} = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$. Use Theorem 8. $D_1 = -1 < 0$, $D_2 = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 1 > 0$. Thus $\mathcal{A}$ is negative definite.

22. $\mathcal{A} = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Use Theorem 8.

$$D_1 = 1 > 0, \quad D_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 < 0,$$
$$D_3 = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -3 < 0.$$

Thus $\mathcal{A}$ is indefinite.

23. $\mathcal{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$. Use Theorem 8.

$$D_1 = 2 > 0, \quad D_2 = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3 > 0,$$

$$D_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 4 > 0.$$

Thus $\mathcal{A}$ is positive definite.

24. $\mathcal{A} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$, we cannot use Theorem 8. The corresponding quadratic form is

$$Q(x, y, z) = x^2 + y^2 + 2xy + z^2 = (x + y)^2 + z^2,$$

which is positive semidefinite. ($Q(1, -1, 0) = 0$.). Thus $\mathcal{A}$ is positive semidefinite.

25. $\mathcal{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$. Use Theorem 8.

$$D_1 = 1 > 0, \quad D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0,$$

$$D_3 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix} = -1 < 0.$$

Thus $\mathcal{A}$ is indefinite.

26. $\mathcal{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 11 \\ 1 & -1 & 1 \end{pmatrix}$. Use Theorem 8.

$$D_1 = 2 > 0, \quad D_2 = \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} = 8 > 0,$$

$$D_3 = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 4 & 11 \\ 1 & -1 & 1 \end{vmatrix} = 2 > 0.$$

Thus $\mathcal{A}$ is positive definite.

Section 10.7 Using Maple for Vector and Matrix Calculations (page 588)

It is assumed that the Maple package **LinearAlgebra** has been loaded for all the calculations in this section.

1. We use the result of Example 9 of Section 10.4.

```
> r1 := <3|0|2>: v1 := <2,1,-2>:
> r2 := <1|2|4>: v2 := <1,3,4>:
> v1xv2 := v1 &x v2:
> dist :=
abs((r2-r1).v1xv2)/Norm(v1xv2,2);
```

$$dist := 2$$

The distance between the two lines is 2 units.

2. The plane P through the origin containing the vectors $\mathbf{v}_1 = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v}2 = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ has normal $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2$.

```
> n := <1|-2|-3> &x <2|3|4>;
```

$$n := [1, -10, 7]$$

The angle between $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{n}$ (in degrees) is

```
> angle
:= evalf((180/Pi)*VectorAngle(n,<1,-
1,2>));
```

$$angvn := 33.55730975$$

Since this angle is acute, the angle between $\mathbf{v}$ and the plane P is its complement.

```
> angle := 90 - angvn;
```

$$angle := 56.44269025$$

3. These calculations verify the identity:

```
> U := Vector[row](3,symbol=u): V :=
Vector[row](3,symbol=v):

> W := Vector[row](3,symbol=w):

> a := DotProduct(U,(V &x
W),conjugate=false):

> b := DotProduct(V,(W &x
U),conjugate=false):

> c := DotProduct(W,(U &x
V),conjugate=false):

> simplify(a-b); simplify(a-c);
```

$$0$$
$$0$$

4. These calculations verify the identity:

```
> U := Vector[row](3,symbol=u): V :=
Vector[row](3,symbol=v):

> W := Vector[row](3,symbol=w):

> LHS := (U &x V) &x (U &x W):
> RHS := (DotProduct(U,(V &x
W),conjugate=false))*U:

> simplify(LHS-RHS);
```

$$[0, 0, 0]$$

5.
```
sp := (U,V) -> DotProd-
uct(U,Normalize(V,2),conjugate=false)
```

6. `vp := (U,V) -> DotProduct(U,Normalize(V,2), conjugate=false)*Normalize(V,2)`

7. `ang := (u,v) -> evalf((180/Pi)*VectorAngle(U,V))`

8. `unitn := (U,V)->Normalize((U &x V),2)`

9. `VolT := (U,V,W)->(1/6)*abs(DotProduct(U,(V &x W), conjugate=false))`

10. `dist:=(A,B)->Norm(A-B,2)`

```
> dist(<1,1,1,1>,<3,-1,2,5>);
```

$$5$$

11. We use `LinearSolve`.

```
> A := Matrix([[1,2,3,4,5],
> [6,-1,6,2,-3],[2,8,-8,-2,1],
> [1,1,1,1,1],[10,-3,3,-2,2]]):
> X := LinearSolve(A,<20,0,6,5,5>,free=t);
```

$$X := \begin{bmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 2 \end{bmatrix}$$

The solution is $u = 1$, $v = 0$, $x = -1$, $y = 3$, $z = 2$.

12. We use `LinearSolve`.

```
> B := Matrix([[1,1,1,1,0],
> [1,0,0,1,1],[1,0,1,1,0],
> [1,1,1,0,1],[0,1,0,1,-1]]):
> X := LinearSolve(B,<10,10,8,11,1>,free=t);
```

$$X := \begin{bmatrix} 11 - 2t_5 \\ 2 \\ -2 + t_5 \\ -1 + t_5 \\ t_5 \end{bmatrix}$$

There is a one-parameter family of solutions: $u = 11-2t$, $v = 2$, $x = -2 + t$, $y = -1 + t$, $z = t$, for arbitrary t.

13.
```
> A := Matrix([[1,2,3,4,5],
> [6,-1,6,2,-3],[2,8,-8,-2,1],
> [1,1,1,1,1],[10,-3,3,-2,2]]):
> Determinant(A);
```

$$-935$$

14.
```
> B := Matrix([[1,1,1,1,0],
> [1,0,0,1,1],[1,0,1,1,0],
> [1,1,1,0,1],[0,1,0,1,-1]]):
> Digits := 5: evalf(Eigenvalues(B));
```

$$\begin{bmatrix} 0 \\ 3.3133 - 0.0000053418I \\ 0.8693 + 0.0000073520I \\ -1.2728 - 0.0000025143I \\ -1.9098 + 5.041\ 10^{-7} I \end{bmatrix}$$

The tiny imaginary parts are due to roundoff error in the calculations. They should all be 0. Since B is a real, symmetric matrix, its eigenvalues are all real. The eigenvalues, rounded to 5 decimal places are 0, 3.3133, 0.8693, -1.2728, and -1.9098.

15.
```
> A := Matrix([[1,1/2,1/3],
> [1/2,1/3,1/4],[1/3,1/4,1/5]]):
> Ainv := MatrixInverse(A);
```

$$Ainv := \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

16.
```
> A := Matrix([[1,1/2,1/3],
> [1/2,1/3,1/4],[1/3,1/4,1/5]]):
> Ainv := MatrixInverse(A):

> Digits := 10:
evalf(Eigenvalues(A));
```

$$\begin{bmatrix} 1.408318927 - 4\ 10^{-11} I \\ 0.00268734034 - 5.673502694\ 10^{-10} I \\ 0.1223270659 + 5.873502694\ 10^{-10} I \end{bmatrix}$$

```
> evalf(Eigenvalues(Ainv));
```

$$\begin{bmatrix} 372.1151279 - 2\ 10^{-9} I \\ 0.710066409 - 5.096152424\ 10^{-8} I \\ 8.174805711 + 5.296152424\ 10^{-8} I \end{bmatrix}$$

The small imaginary parts are due to round-off errors in the solution process. The eigenvalues are real since the matrix and its inverse are real and symmetric.

Although they appear in different orders, each eigenvalue of A^{-1} is the reciprocal of an eigenvalue of A. This is to be expected since

$$A^{-1}\mathbf{x} = \lambda\mathbf{x} \quad \equiv \quad (1/\lambda)\mathbf{x} = A\mathbf{x}.$$

Review Exercises 10 (page 589)

1. $x + 3z = 3$ represents a plane parallel to the y-axis and passing through the points $(3, 0, 0)$ and $(0, 0, 1)$.

2. $y - z \ge 1$ represents all points on or below the plane parallel to the x-axis that passes through the points $(0, 1, 0)$ and $(0, 0, -1)$.

3. $x + y + z \ge 0$ represents all points on or above the plane through the origin having normal vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

4. $x - 2y - 4z = 8$ represents all points on the plane passing through the three points $(8, 0, 0)$, $(0, -4, 0)$, and $(0, 0, -2)$.

5. $y = 1+x^2+z^2$ represents the circular paraboloid obtained by rotating about the y-axis the parabola in the xy-plane having equation $y = 1 + x^2$.

6. $y = z^2$ represents the parabolic cylinder parallel to the x-axis containing the curve $y = z^2$ in the yz-plane.

7. $x = y^2 - z^2$ represents the hyperbolic paraboloid whose intersections with the xy- and xz-planes are the parabolas $x = y^2$ and $x = -z^2$, respectively.

8. $z = xy$ is the hyperbolic paraboloid containing the x- and y-axes that results from rotating the hyperbolic paraboloid $z = (x^2 - y^2)/2$ through 45° about the z-axis.

9. $x^2 + y^2 + 4z^2 < 4$ represents the interior of the circular ellipsoid (oblate spheroid) centred at the origin with semi-axes 2, 2, and 1 in the x, y, and z directions, respectively.

10. $x^2 + y^2 - 4z^2 = 4$ represents a hyperboloid of one sheet with circular cross-sections in planes perpendicular to the z-axis, and asymptotic to the cone obtained by rotating the line $x = 2z$ about the z-axis.

11. $x^2 - y^2 - 4z^2 = 0$ represents an elliptic cone with axis along the x-axis whose cross-sections in planes $x = k$ are ellipses with semi-axes $|k|$ and $|k|/2$ in the y and z directions, respectively.

12. $x^2 - y^2 - 4z^2 = 4$ represents a hyperboloid of two sheets asymptotic to the cone of the previous exercise.

13. $(x - z)^2 + y^2 = 1$ represents an elliptic cylinder with oblique axis along the line $z = x$ in the xz-plane, having circular cross-sections of radius 1 in horizontal planes $z = k$.

14. $(x-z)^2+y^2 = z^2$ represents an elliptic cone with oblique axis along the line $z = x$ in the xz-plane, having circular cross-sections of radius $|k|$ in horizontal planes $z = k$. The z-axis lies on the cone.

15. $x + 2y = 0$, $z = 3$ together represent the horizontal straight line through the point $(0, 0, 3)$ parallel to the vector $2\mathbf{i} - \mathbf{j}$.

16. $x + y + 2z = 1$, $x + y + z = 0$ together represent the straight line through the points $(-1, 0, 1)$ and $(0, -1, 1)$.

17. $x^2 + y^2 + z^2 = 4$, $x + y + z = 3$ together represent the circle in which the sphere of radius 2 centred at the origin intersects the plane through $(1, 1, 1)$ with normal $\mathbf{i} + \mathbf{j} + \mathbf{k}$. Since this plane lies at distance $\sqrt{3}$ from the origin, the circle has radius $\sqrt{4 - 3} = 1$.

18. $x^2 + z^2 \le 1$, $x - y \ge 0$ together represent all points that lie inside or on the circular cylinder of radius 1 and axis along the y-axis and also either on the vertical plane $x - y = 0$ or on the side of that plane containing the positive x-axis.

19. The given line is parallel to the vector $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. The plane through the origin perpendicular to $\mathbf{a}$ has equation $2x - y + 3z = 0$.

20. A plane through $(2, -1, 1)$ and $(1, 0, -1)$ is parallel to $\mathbf{b} = (2 - 1)\mathbf{i} + (-1 - 0)\mathbf{j} + (1 - (-1))\mathbf{k} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. If it is also parallel to the vector $\mathbf{a}$ in the previous solution, then it is normal to

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k}.$$

The plane has equation $(x - 1) - (y - 0) - (z + 1) = 0$, or $x - y - z = 2$.

21. A plane perpendicular to $x-y+z = 0$ and $2x+y-3z = 2$ has normal given by the cross product of the normals of these two planes, that is, by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 2 & 1 & -3 \end{vmatrix} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}.$$

If the plane also passes through $(2, -1, 1)$, then its equation is

$$2(x - 2) + 5(y + 1) + 3(z - 1) = 0,$$

or $2x + 5y + 3z = 2$.

22. The plane through $A = (-1, 1, 0)$, $B = (0, 4, -1)$ and $C = (2, 0, 0)$ has normal

$$\overrightarrow{AC} \times \overrightarrow{AB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 0 \\ 1 & 3 & -1 \end{vmatrix} = \mathbf{i} + 3\mathbf{j} + 10\mathbf{k}.$$

Its equation is $(x-2)+3y+10z = 0$, or $x+3y+10z = 2$.

23. A plane containing the line of intersection of the planes $x + y + z = 0$ and $2x + y - 3z = 2$ has equation

$$2x + y - 3z - 2 + \lambda(x + y + z - 0) = 0.$$

This plane passes through $(2, 0, 1)$ if $-1 + 3\lambda = 0$. In this case, the equation is $7x + 4y - 8z = 6$.

24. A plane containing the line of intersection of the planes $x + y + z = 0$ and $2x + y - 3z = 2$ has equation

$$2x + y - 3z - 2 + \lambda(x + y + z - 0) = 0.$$

This plane is perpendicular to $x - 2y - 5z = 17$ if their normals are perpendicular, that is, if

$$1(2 + \lambda) - 2(1 + \lambda) - 5(-3 + \lambda) = 0,$$

or $9x + 7y - z = 4$.

25. The line through $(2, 1, -1)$ and $(-1, 0, 1)$ is parallel to the vector $3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, and has vector parametric equation

$$\mathbf{r} = (2 + 3t)\mathbf{i} + (1 + t)\mathbf{j} - (1 + 2t)\mathbf{k}.$$

26. A vector parallel to the planes $x - y = 3$ and $x + 2y + z = 1$ is $(\mathbf{i} - \mathbf{j}) \times (\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = -\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. A line through $(1, 0, -1)$ parallel to this vector is

$$\frac{x - 1}{-1} = \frac{y}{-1} = \frac{z + 1}{3}.$$

27. The line through the origin perpendicular to the plane $3x - 2y + 4z = 5$ has equations $x = 3t$, $y = -2t$, $z = 4t$.

28. The vector

$$\begin{aligned}\mathbf{a} &= (1 + t)\mathbf{i} - t\mathbf{j} - (2 + 2t)\mathbf{k} - \Big(2s\mathbf{i} + (s - 2)\mathbf{j} - (1 + 3s)\mathbf{k}\Big)\\ &= (1 + t - 2s)\mathbf{i} - (t + s - 2)\mathbf{j} - (1 + 2t - 3s)\mathbf{k}\end{aligned}$$

joins points on the two lines and is perpendicular to both lines if $\mathbf{a} \bullet (\mathbf{i} - \mathbf{j} - 2\mathbf{k}) = 0$ and $\mathbf{a} \bullet (2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = 0$, that is, if

$$\begin{aligned}1 + t - 2s + t + s - 2 + 2 + 4t - 6s &= 0\\ 2 + 2t - 4s - t - s + 2 + 3 + 6t - 9s &= 0,\end{aligned}$$

or, on simplification,

$$\begin{aligned}6t - 7s &= -1\\ 7t - 14s &= -7.\end{aligned}$$

This system has solution $t = 1$, $s = 1$. We would expect to use $\mathbf{a}$ as a vector perpendicular to both lines, but, as it happens, $\mathbf{a} = \mathbf{0}$ if $t = s = 1$, because the two given lines intersect at $(2, -1, -4)$. A nonzero vector perpendicular to both lines is

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & -2\\ 2 & 1 & -3\end{vmatrix} = 5\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

Thus the required line is parallel to this vector and passes through $(2, -1, -4)$, so its equation is

$$\mathbf{r} = (2 + 5t)\mathbf{i} - (1 + t)\mathbf{j} + (-4 + 3t)\mathbf{k}.$$

29. The points with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$ are collinear if the triangle having these points as vertices has zero area, that is, if

$$(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1) = \mathbf{0}.$$

(Any permutation of the subscripts 1, 2, and 3 in the above equation will do as well.)

30. The points with position vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, and $\mathbf{r}_4$ are coplanar if the tetrahedron having these points as vertices has zero volume, that is, if

$$\Big[(\mathbf{r}_2 - \mathbf{r}_1) \times (\mathbf{r}_3 - \mathbf{r}_1)\Big] \bullet (\mathbf{r}_4 - \mathbf{r}_1) = 0.$$

(Any permutation of the subscripts 1, 2, 3, and 4 in the above equation will do as well.)

31. The triangle with vertices $A = (1, 2, 1)$, $B = (4, -1, 1)$, and $C = (3, 4, -2)$ has area

$$\begin{aligned}\frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| &= \frac{1}{2}\left|\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -3 & 0\\ 2 & 2 & -3\end{vmatrix}\right|\\ &= \frac{1}{2}|9\mathbf{i} + 9\mathbf{j} + 12\mathbf{k}| = \frac{3\sqrt{34}}{2} \text{ sq. units.}\end{aligned}$$

32. The tetrahedron with vertices $A = (1, 2, 1)$, $B = (4, -1, 1)$, $C = (3, 4, -2)$, and $D = (2, 2, 2)$ has volume

$$\begin{aligned}\frac{1}{6}|(\overrightarrow{AB} \times \overrightarrow{AC}) \bullet \overrightarrow{AD}| &= \frac{1}{6}|(9\mathbf{i} + 9\mathbf{j} + 12\mathbf{k}) \bullet (\mathbf{i} + \mathbf{k})|\\ &= \frac{9 + 12}{6} = \frac{7}{2} \text{ cu. units.}\end{aligned}$$

33. The inverse of $\mathcal{A}$ satisfies

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 2 & 1 & 0 & 0\\ 3 & 2 & 1 & 0\\ 4 & 3 & 2 & 1\end{pmatrix}\begin{pmatrix}a & b & c & d\\ e & f & g & h\\ i & j & k & l\\ m & n & o & p\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Expanding the product on the left we get four systems of equations:

$$\begin{aligned}&a = 1, \quad 2a + e = 0, \quad 3a + 2e + i = 0, \quad 4a + 3e + 2i + m = 0.\\ &b = 0, \quad 2b + f = 1, \quad 3b + 2f + j = 0, \quad 4b + 3f + 2j + n = 0.\\ &c = 0, \quad 2c + g = 0, \quad 3c + 2g + k = 1, \quad 4c + 3g + 2k + o = 0.\\ &d = 0, \quad 2d + h = 0, \quad 3d + 2h + l = 0, \quad 4d + 3h + 2l + p = 1.\end{aligned}$$

These systems have solutions

$$\begin{aligned}&a = 1, \quad e = -2, \quad i = 1, \quad m = 0,\\ &b = 0, \quad f = 1, \quad j = -2, \quad n = 1,\\ &c = 0, \quad g = 0, \quad k = 1, \quad o = -2,\\ &d = 0, \quad h = 0, \quad l = 0, \quad p = 1.\end{aligned}$$

Thus

$$\mathcal{A}^{-1} = \begin{pmatrix}1 & 0 & 0 & 0\\ -2 & 1 & 0 & 0\\ 1 & -2 & 1 & 0\\ 0 & 1 & -2 & 1\end{pmatrix}.$$

34. Let $\mathcal{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.
Then

$$\mathcal{A}\mathbf{x} = \mathbf{b} \quad \Leftrightarrow \quad \begin{aligned} x_1 + x_2 + x_3 &= b_1 \\ 2x_1 + x_2 \quad\ \ &= b_2 \\ x_1 \quad\ \ - x_3 &= b_3. \end{aligned}$$

The sum of the first and third equations is $2x_1 + x_2 = b_1 + b_3$, which is incompatible with the second equation unless $b_2 = b_1 + b_3$, that is, unless

$$\mathbf{b} \bullet (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 0.$$

If $\mathbf{b}$ satisfies this condition then there will be a line of solutions; if $x_1 = t$, then $x_2 = b_2 - 2t$, and $x_3 = t - b_3$, so

$$\mathbf{x} = \begin{pmatrix} t \\ b_2 - 2t \\ t - b_3 \end{pmatrix}$$

is a solution for any t.

35. $\mathcal{A} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{pmatrix}$. We use Theorem 8.

$$D_1 = 3 > 0, \quad D_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 2 > 0,$$

$$D_3 = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2 > 0.$$

Thus $\mathcal{A}$ is positive definite.

Challenging Problems 10 (page 589)

1. If d is the distance from P to the line AB, then d is the altitude of the triangle APB measured perpendicular to the base AB. Thus the area of the triangle is

$$(1/2)d|\overrightarrow{BA}| = (1/2)d|\mathbf{r}_A - \mathbf{r}_B|.$$

On the other hand, the area is also given by

$$(1/2)|\overrightarrow{PA} \times \overrightarrow{PB}| = (1/2)|(\mathbf{r}_A - \mathbf{r}_P) \times (\mathbf{r}_B - \mathbf{r}_P)|.$$

Equating these two expressions for the area of the triangle and solving for d we get

$$d = \frac{|(\mathbf{r}_A - \mathbf{r}_P) \times (\mathbf{r}_B - \mathbf{r}_P)|}{|\mathbf{r}_A - \mathbf{r}_B|}.$$

2. By the formula for the vector triple product given in Exercise 23 of Section 1.3,

$$\begin{aligned} (\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{x}) &= [(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{x}]\mathbf{w} - [(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}]\mathbf{x} \\ (\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{x}) &= -(\mathbf{w} \times \mathbf{x}) \times (\mathbf{u} \times \mathbf{v}) \\ &= -[(\mathbf{w} \times \mathbf{x}) \bullet \mathbf{v}]\mathbf{u} + [(\mathbf{w} \times \mathbf{x}) \bullet \mathbf{u}]\mathbf{v}. \end{aligned}$$

In particular, if $\mathbf{w} = \mathbf{u}$, then, since $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{u} = 0$, we have

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{x}) = [(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{x}]\mathbf{u},$$

or, replacing x with w,

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{u} \times \mathbf{w}) = [(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w}]\mathbf{u}.$$

3. The triangle with vertices $(x_1, y_1, 0)$, $(x_2, y_2, 0)$, and $(x_3, y_3, 0)$, has two sides corresponding to the vectors $(x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j}$ and $(x_3 - x_1)\mathbf{i} + (y_3 - y_1)\mathbf{j}$. Thus the triangle has area given by

$$\begin{aligned} A &= \frac{1}{2}\left| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 - x_1 & y_2 - y_1 & 0 \\ x_3 - x_1 & y_3 - y_1 & 0 \end{vmatrix} \right| \\ &= \frac{1}{2}|[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]\mathbf{k}| \\ &= \frac{1}{2}|x_2y_3 - x_2y_1 - x_1y_3 - x_3y_2 + x_3y_1 + x_1y_2| \\ &= \frac{1}{2}\left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|. \end{aligned}$$

4. a) Let Q_1 and Q_2 be the points on lines L_1 and L_2, respectively, that are closest together. As observed in Example 9 of Section 1.4, $\overrightarrow{Q_1Q_2}$ is perpendicular to both lines.
Therefore, the plane P_1 through Q_1 having normal $\overrightarrow{Q_1Q_2}$ contains the line L_1. Similarly, the plane P_2 through Q_2 having normal $\overrightarrow{Q_1Q_2}$ contains the line L_2. These planes are parallel since they have the same normal. They are different planes because $Q_1 \neq Q_2$ (because the lines are skew).

b) Line L_1 through $(1, 1, 0)$ and $(2, 0, 1)$ is parallel to $\mathbf{i} - \mathbf{j} + \mathbf{k}$, and has parametric equation

$$\mathbf{r}_1 = (1+t)\mathbf{i} + (1-t)\mathbf{j} + t\mathbf{k}.$$

Line L_2 through $(0, 1, 1)$ and $(1, 2, 2)$ is parallel to $\mathbf{i} + \mathbf{j} + \mathbf{k}$, and has parametric equation

$$\mathbf{r}_2 = s\mathbf{i} + (1+s)\mathbf{j} + (1+s)\mathbf{k}.$$

Now $\mathbf{r}_2 - \mathbf{r}_1 = (s-t-1)\mathbf{i} + (s+t)\mathbf{j} + (1+s-t)\mathbf{k}$.

To find the points Q_1 on L_1 and Q_2 on L_2 for which $\overrightarrow{Q_1Q_2}$ is perpendicular to both lines, we solve

$$(s-t-1) - (s+t) + (1+s-t) = 0$$
$$(s-t-1) + (s+t) + (1+s-t) = 0.$$

Subtracting these equations gives $s + t = 0$, so $t = -s$. Then substituting into either equation gives $2s - 1 + 1 + 2s = 0$, so $s = -t = 0$. Thus $Q_1 = (1, 1, 0)$ and $Q_2 = (0, 1, 1)$, and $\overrightarrow{Q_1Q_2} = -\mathbf{i} + \mathbf{k}$. The required planes are $x - z = 1$ (containing L_1) and $x - z = -1$ (containing L_2).

5. This problem is similar to Exercise 28 of Section 1.3. The equation $\mathbf{a} \times \mathbf{x} = \mathbf{b}$ has no solution $\mathbf{x}$ unless $\mathbf{a} \bullet \mathbf{b} = 0$. If this condition is satisfied, then $\mathbf{x} = \mathbf{x}_0 + t\mathbf{a}$ is a solution for any scalar t, where $\mathbf{x}_0 = (\mathbf{b} \times \mathbf{a})/|\mathbf{a}|^2$.