CHAPTER 1. LIMITS AND CONTINUITY

Section 1.1 Examples of Velocity, Growth Rate, and Area (page 61)

1. Average velocity $= \dfrac{\Delta x}{\Delta t} = \dfrac{(t+h)^2 - t^2}{h}$ m/s.

2.

h	Avg. vel. over $[2, 2+h]$
1	5.0000
0.1	4.1000
0.01	4.0100
0.001	4.0010
0.0001	4.0001

3. Guess velocity is $v = 4$ m/s at $t = 2$ s.

4. Average volocity on $[2, 2+h]$ is

$$\frac{(2+h)^2 - 4}{(2+h) - 2} = \frac{4 + 4h + h^2 - 4}{h} = \frac{4h + h^2}{h} = 4 + h.$$

As h approaches 0 this average velocity approaches 4 m/s

5. $x = 3t^2 - 12t + 1$ m at time t s.
Average velocity over interval $[1, 2]$ is
$\dfrac{(3 \times 2^2 - 12 \times 2 + 1) - (3 \times 1^2 - 12 \times 1 + 1)}{2 - 1} = -3$ m/s.
Average velocity over interval $[2, 3]$ is
$\dfrac{(3 \times 3^2 - 12 \times 3 + 1) - (3 \times 2^2 - 12 \times 2 + 1)}{3 - 2} = 3$ m/s.
Average velocity over interval $[1, 3]$ is
$\dfrac{(3 \times 3^2 - 12 \times 3 + 1) - (3 \times 1^2 - 12 \times 1 + 1)}{3 - 1} = 0$ m/s.

6. Average velocity over $[t, t+h]$ is

$$\frac{3(t+h)^2 - 12(t+h) + 1 - (3t^2 - 12t + 1)}{(t+h) - t}$$
$$= \frac{6th + 3h^2 - 12h}{h} = 6t + 3h - 12 \text{ m/s}.$$

This average velocity approaches $6t - 12$ m/s as h approaches 0.
At $t = 1$ the velocity is $6 \times 1 - 12 = -6$ m/s.
At $t = 2$ the velocity is $6 \times 2 - 12 = 0$ m/s.
At $t = 3$ the velocity is $6 \times 3 - 12 = 6$ m/s.

7. At $t = 1$ the velocity is $v = -6 < 0$ so the particle is moving to the left.
At $t = 2$ the velocity is $v = 0$ so the particle is stationary.
At $t = 3$ the velocity is $v = 6 > 0$ so the particle is moving to the right.

8. Average velocity over $[t-k, t+k]$ is

$$\frac{3(t+k)^2 - 12(t+k) + 1 - [3(t-k)^2 - 12(t-k) + 1]}{(t+k) - (t-k)}$$
$$= \frac{1}{2k}\Big(3t^2 + 6tk + 3k^2 - 12t - 12k + 1 - 3t^2 + 6tk - 3k^2$$
$$+ 12t - 12k + 1\Big)$$
$$= \frac{12tk - 24k}{2k} = 6t - 12 \text{ m/s},$$

which is the velocity at time t from Exercise 7.

9.

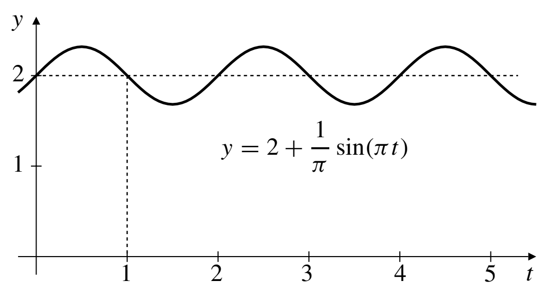

Fig. 1.1.9

At $t = 1$ the height is $y = 2$ ft and the weight is moving downward.

10. Average velocity over $[1, 1+h]$ is

$$\frac{2 + \frac{1}{\pi}\sin\pi(1+h) - \left(2 + \frac{1}{\pi}\sin\pi\right)}{h}$$
$$= \frac{\sin(\pi + \pi h)}{\pi h} = \frac{\sin\pi\cos(\pi h) + \cos\pi\sin(\pi h)}{\pi h}$$
$$= -\frac{\sin(\pi h)}{\pi h}.$$

h	Avg. vel. on $[1, 1+h]$
1.0000	0
0.1000	-0.983631643
0.0100	-0.999835515
0.0010	-0.999998355

11. The velocity at $t = 1$ is about $v = -1$ ft/s. The "$-$" indicates that the weight is moving downward.

12. We sketched a tangent line to the graph on page 55 in the text at $t = 20$. The line appeared to pass through the points $(10, 0)$ and $(50, 1)$. On day 20 the biomass is growing at about $(1 - 0)/(50 - 10) = 0.025$ mm^2/d.

13. The curve is steepest, and therefore the biomass is growing most rapidly, at about day 45.

14. a)

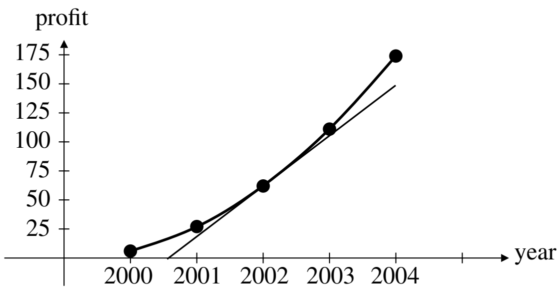

Fig. 1.1.14

b) Average rate of increase in profits between 2002 and 2004 is

$$\frac{174 - 62}{2004 - 2002} = \frac{112}{2} = 56 \text{ (thousand\$/yr).}$$

c) Drawing a tangent line to the graph in (a) at $t = 2002$ and measuring its slope, we find that the rate of increase of profits in 1992 is about 43 thousand\$/year.

Section 1.2 Limits of Functions (page 68)

1. From inspecting the graph

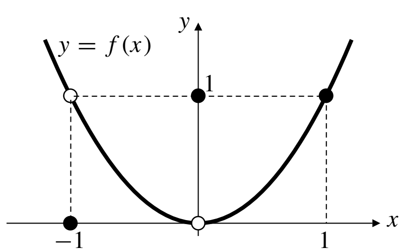

Fig. 1.2.1

we see that

$$\lim_{x\to -1} f(x) = 1, \quad \lim_{x\to 0} f(x) = 0, \quad \lim_{x\to 1} f(x) = 1.$$

2. From inspecting the graph

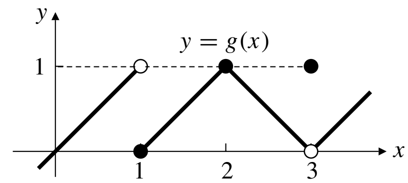

Fig. 1.2.2

we see that

$$\lim_{x\to 1} g(x) \text{ does not exist}$$

(left limit is 1, right limit is 0)

$$\lim_{x\to 2} g(x) = 1, \qquad \lim_{x\to 3} g(x) = 0.$$

3. $\displaystyle\lim_{x\to 1-} g(x) = 1$

4. $\displaystyle\lim_{x\to 1+} g(x) = 0$

5. $\displaystyle\lim_{x\to 3+} g(x) = 0$

6. $\displaystyle\lim_{x\to 3-} g(x) = 0$

7. $\displaystyle\lim_{x\to 4} (x^2 - 4x + 1) = 4^2 - 4(4) + 1 = 1$

8. $\displaystyle\lim_{x\to 2} 3(1 - x)(2 - x) = 3(-1)(2 - 2) = 0$

9. $\displaystyle\lim_{x\to 3} \frac{x + 3}{x + 6} = \frac{3 + 3}{3 + 6} = \frac{2}{3}$

10. $\displaystyle\lim_{t\to -4} \frac{t^2}{4 - t} = \frac{(-4)^2}{4 + 4} = 2$

11. $\displaystyle\lim_{x\to 1} \frac{x^2 - 1}{x + 1} = \frac{1^2 - 1}{1 + 1} = \frac{0}{2} = 0$

12. $\displaystyle\lim_{x\to -1} \frac{x^2 - 1}{x + 1} = \lim_{x\to -1} (x - 1) = -2$

13. $$\lim_{x\to 3} \frac{x^2 - 6x + 9}{x^2 - 9} = \lim_{x\to 3} \frac{(x - 3)^2}{(x - 3)(x + 3)} = \lim_{x\to 3} \frac{x - 3}{x + 3} = \frac{0}{6} = 0$$

14. $\displaystyle\lim_{x\to -2} \frac{x^2 + 2x}{x^2 - 4} = \lim_{x\to -2} \frac{x}{x - 2} = \frac{-2}{-4} = \frac{1}{2}$

15. $\displaystyle\lim_{h\to 2} \frac{1}{4 - h^2}$ does not exist; denominator approaches 0 but numerator does not approach 0.

16. $\displaystyle\lim_{h\to 0} \frac{3h + 4h^2}{h^2 - h^3} = \lim_{h\to 0} \frac{3 + 4h}{h - h^2}$ does not exist; denominator approaches 0 but numerator does not approach 0.

17. $$\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}=\lim_{x\to 9}\frac{(\sqrt{x}-3)(\sqrt{x}+3)}{(x-9)(\sqrt{x}+3)} =\lim_{x\to 9}\frac{x-9}{(x-9)(\sqrt{x}+3)}=\lim_{x\to 9}\frac{1}{\sqrt{x}+3}=\frac{1}{6}$$

18. $$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} =\lim_{h\to 0}\frac{4+h-4}{h(\sqrt{4+h}+2)} =\lim_{h\to 0}\frac{1}{\sqrt{4+h}+2}=\frac{1}{4}$$

19. $$\lim_{x\to\pi}\frac{(x-\pi)^2}{\pi x}=\frac{0^2}{\pi^2}=0$$

20. $$\lim_{x\to -2}|x-2|=|-4|=4$$

21. $$\lim_{x\to 0}\frac{|x-2|}{x-2}=\frac{|-2|}{-2}=-1$$

22. $$\lim_{x\to 2}\frac{|x-2|}{x-2}=\lim_{x\to 2}\begin{cases}1, & \text{if } x>2\\ -1, & \text{if } x<2.\end{cases}$$
Hence, $\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

23. $$\lim_{t\to 1}\frac{t^2-1}{t^2-2t+1}$$
$\lim_{t\to 1}\frac{(t-1)(t+1)}{(t-1)^2}=\lim_{t\to 1}\frac{t+1}{t-1}$ does not exist (denominator $\to 0$, numerator $\to 2$.)

24. $$\lim_{x\to 2}\frac{\sqrt{4-4x+x^2}}{x-2}$$
$=\lim_{x\to 2}\frac{|x-2|}{x-2}$ does not exist.

25. $$\lim_{t\to 0}\frac{t}{\sqrt{4+t}-\sqrt{4-t}}=\lim_{t\to 0}\frac{t(\sqrt{4+t}+\sqrt{4-t})}{(4+t)-(4-t)} =\lim_{t\to 0}\frac{\sqrt{4+t}+\sqrt{4-t}}{2}=2$$

26. $$\lim_{x\to 1}\frac{x^2-1}{\sqrt{x+3}-2}=\lim_{x\to 1}\frac{(x-1)(x+1)(\sqrt{x+3}+2)}{(x+3)-4} =\lim_{x\to 1}(x+1)(\sqrt{x+3}+2)=(2)(\sqrt{4}+2)=8$$

27. $$\lim_{t\to 0}\frac{t^2+3t}{(t+2)^2-(t-2)^2} =\lim_{t\to 0}\frac{t(t+3)}{t^2+4t+4-(t^2-4t+4)} =\lim_{t\to 0}\frac{t+3}{8}=\frac{3}{8}$$

28. $$\lim_{s\to 0}\frac{(s+1)^2-(s-1)^2}{s}=\lim_{s\to 0}\frac{4s}{s}=4$$

29. $$\lim_{y\to 1}\frac{y-4\sqrt{y}+3}{y^2-1} =\lim_{y\to 1}\frac{(\sqrt{y}-1)(\sqrt{y}-3)}{(\sqrt{y}-1)(\sqrt{y}+1)(y+1)}=\frac{-2}{4}=\frac{-1}{2}$$

30. $$\lim_{x\to -1}\frac{x^3+1}{x+1} =\lim_{x\to -1}\frac{(x+1)(x^2-x+1)}{x+1}=3$$

31. $$\lim_{x\to 2}\frac{x^4-16}{x^3-8} =\lim_{x\to 2}\frac{(x-2)(x+2)(x^2+4)}{(x-2)(x^2+2x+4)} =\frac{(4)(8)}{4+4+4}=\frac{8}{3}$$

32. $$\lim_{x\to 8}\frac{x^{2/3}-4}{x^{1/3}-2} =\lim_{x\to 8}\frac{(x^{1/3}-2)(x^{1/3}+2)}{(x^{1/3}-2)} =\lim_{x\to 8}(x^{1/3}+2)=4$$

33. $$\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right) =\lim_{x\to 2}\frac{x+2-4}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}$$

34. $$\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{1}{x^2-4}\right) =\lim_{x\to 2}\frac{x+2-1}{(x-2)(x+2)}$$
$=\lim_{x\to 2}\frac{x+1}{(x-2)(x+2)}$ does not exist.

35. $$\lim_{x\to 0}\frac{\sqrt{2+x^2}-\sqrt{2-x^2}}{x^2} =\lim_{x\to 0}\frac{(2+x^2)-(2-x^2)}{x^2(\sqrt{2+x^2}+\sqrt{2-x^2})} =\lim_{x\to 0}\frac{2x^2}{x^2\left(\sqrt{2+x^2})+\sqrt{2-x^2}\right)} =\frac{2}{\sqrt{2}+\sqrt{2}}=\frac{1}{\sqrt{2}}$$

36. $$\lim_{x\to 0}\frac{|3x-1|-|3x+1|}{x} =\lim_{x\to 0}\frac{(3x-1)^2-(3x+1)^2}{x\,(|3x-1|+|3x+1|)} =\lim_{x\to 0}\frac{-12x}{x\,(|3x-1|+|3x+1|)}=\frac{-12}{1+1}=-6$$

37. $$f(x)=x^2$$
$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} =\lim_{h\to 0}\frac{2hx+h^2}{h}=\lim_{h\to 0}2x+h=2x$$

38.
$$f(x) = x^3$$
$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}\\ &= \lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}\\ &= \lim_{h\to 0} 3x^2+3xh+h^2 = 3x^2\end{aligned}$$

39.
$$f(x) = 1/x$$
$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\ &= \lim_{h\to 0}\frac{x-(x+h)}{h(x+h)x}\\ &= \lim_{h\to 0} -\frac{1}{(x+h)x} = -\frac{1}{x^2}\end{aligned}$$

40.
$$f(x) = 1/x^2$$
$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}\\ &= \lim_{h\to 0}\frac{x^2-(x^2+2xh+h^2)}{h(x+h)^2x^2}\\ &= \lim_{h\to 0} -\frac{2x+h}{(x+h)^2x^2} = -\frac{2x}{x^4} = -\frac{2}{x^3}\end{aligned}$$

41.
$$f(x) = \sqrt{x}$$
$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \lim_{h\to 0}\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}\end{aligned}$$

42.
$$f(x) = 1/\sqrt{x}$$
$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}\\ &= \lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}\\ &= \lim_{h\to 0}\frac{x-(x+h)}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\\ &= \lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}\\ &= \frac{-1}{2x^{3/2}}\end{aligned}$$

43. $\displaystyle\lim_{x\to\pi/2}\sin x = \sin\pi/2 = 1$

44. $\displaystyle\lim_{x\to\pi/4}\cos x = \cos\pi/4 = 1/\sqrt{2}$

45. $\displaystyle\lim_{x\to\pi/3}\cos x = \cos\pi/3 = 1/2$

46. $\displaystyle\lim_{x\to 2\pi/3}\sin x = \sin 2\pi/3 = \sqrt{3}/2$

47.

x	$(\sin x)/x$
± 1.0	0.84147098
± 0.1	0.99833417
± 0.01	0.99998333
± 0.001	0.99999983
0.0001	1.00000000

It appears that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$.

48.

x	$(1-\cos x)/x^2$
± 1.0	0.45969769
± 0.1	0.49958347
± 0.01	0.49999583
± 0.001	0.49999996
0.0001	0.50000000

It appears that $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2} = \frac{1}{2}$.

49. $\displaystyle\lim_{x\to 2-}\sqrt{2-x} = 0$

50. $\displaystyle\lim_{x\to 2+}\sqrt{2-x}$ does not exist.

51. $\displaystyle\lim_{x\to -2-}\sqrt{2-x} = 2$

52. $\displaystyle\lim_{x\to -2+}\sqrt{2-x} = 2$

53. $\displaystyle\lim_{x\to 0}\sqrt{x^3-x}$ does not exist.
$(x^3 - x < 0$ if $0 < x < 1)$

54. $\displaystyle\lim_{x\to 0-}\sqrt{x^3-x} = 0$

55. $\displaystyle\lim_{x\to 0+}\sqrt{x^3-x}$ does not exist. (See # 9.)

56. $\displaystyle\lim_{x\to 0+}\sqrt{x^2-x^4} = 0$

57.
$$\lim_{x\to a-}\frac{|x-a|}{x^2-a^2}$$
$$= \lim_{x\to a-}\frac{|x-a|}{(x-a)(x+a)} = -\frac{1}{2a} \qquad (a\neq 0)$$

58. $\displaystyle\lim_{x\to a+}\frac{|x-a|}{x^2-a^2} = \lim_{x\to a+}\frac{x-a}{x^2-a^2} = \frac{1}{2a}$

59. $\displaystyle\lim_{x\to 2-}\frac{x^2-4}{|x+2|} = \frac{0}{4} = 0$

60. $\displaystyle\lim_{x\to 2+}\frac{x^2-4}{|x+2|} = \frac{0}{4} = 0$

61. $f(x) = \begin{cases} x-1 & \text{if } x \le -1 \\ x^2+1 & \text{if } -1 < x \le 0 \\ (x+\pi)^2 & \text{if } x > 0 \end{cases}$

$\lim\limits_{x\to-1-} f(x) = \lim\limits_{x\to-1-} x - 1 = -1 - 1 = -2$

62. $\lim\limits_{x\to-1+} f(x) = \lim\limits_{x\to-1+} x^2 + 1 = 1 + 1 = 2$

63. $\lim\limits_{x\to0+} f(x) = \lim\limits_{x\to0+} (x+\pi)^2 = \pi^2$

64. $\lim\limits_{x\to0-} f(x) = \lim\limits_{x\to0-} x^2 + 1 = 1$

65. If $\lim\limits_{x\to4} f(x) = 2$ and $\lim\limits_{x\to4} g(x) = -3$, then

a) $\lim\limits_{x\to4}\Big(g(x)+3\Big) = -3+3 = 0$

b) $\lim\limits_{x\to4} xf(x) = 4 \times 2 = 8$

c) $\lim\limits_{x\to4}\Big(g(x)\Big)^2 = (-3)^2 = 9$

d) $\lim\limits_{x\to4} \dfrac{g(x)}{f(x)-1} = \dfrac{-3}{2-1} = -3$

66. If $\lim x \to a\, f(x) = 4$ and $\lim\limits_{x\to a} g(x) = -2$, then

a) $\lim\limits_{x\to a}\Big(f(x)+g(x)\Big) = 4 + (-2) = 2$

b) $\lim\limits_{x\to a} f(x) \cdot g(x) = 4 \times (-2) = -8$

c) $\lim\limits_{x\to a} 4g(x) = 4(-2) = -8$

d) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{4}{-2} = -2$

67. If $\lim\limits_{x\to2} \dfrac{f(x)-5}{x-2} = 3$, then

$$\lim_{x\to2}\Big(f(x)-5\Big) = \lim_{x\to2}\frac{f(x)-5}{x-2}(x-2) = 3(2-2) = 0.$$

Thus $\lim_{x\to2} f(x) = 5$.

68. If $\lim\limits_{x\to0} \dfrac{f(x)}{x^2} = -2$ then

$\lim_{x\to0} f(x) = \lim_{x\to0} x^2 \dfrac{f(x)}{x^2} = 0 \times (-2) = 0$,

and similarly,

$\lim_{x\to0} \dfrac{f(x)}{x} = \lim\limits_{x\to0} x \dfrac{f(x)}{x^2} = 0 \times (-2) = 0.$

69.

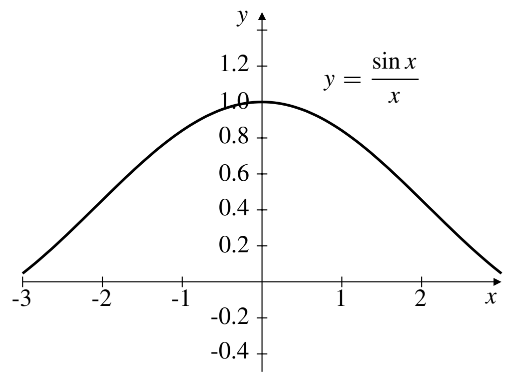

Fig. 1.2.69

$\lim\limits_{x\to0} \dfrac{\sin x}{x} = 1$

70.

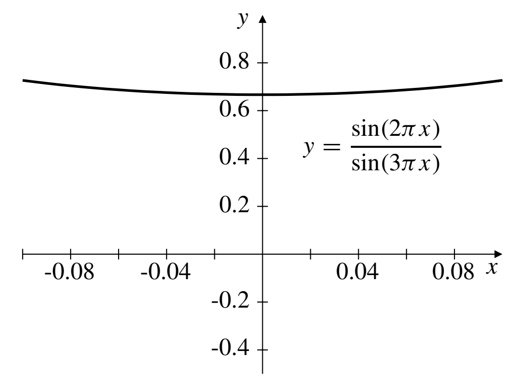

Fig. 1.2.70

$\lim_{x\to0} \sin(2\pi x)/\sin(3\pi x) = 2/3$

71.

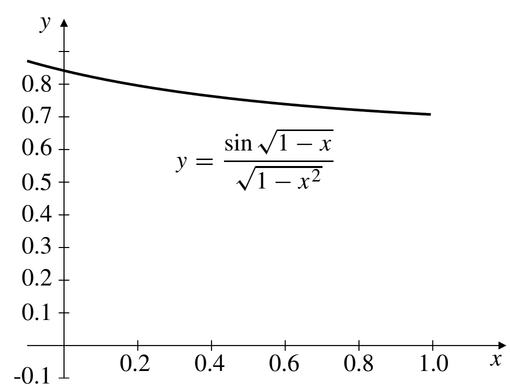

Fig. 1.2.71

$\lim\limits_{x\to1-} \dfrac{\sin\sqrt{1-x}}{\sqrt{1-x^2}} \approx 0.7071$

72.

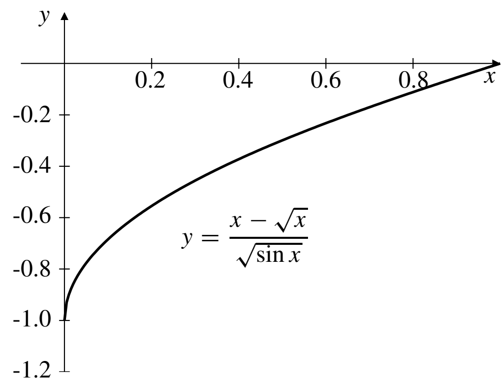

Fig. 1.2.72

$$\lim_{x\to 0+} \frac{x - \sqrt{x}}{\sqrt{\sin x}} = -1$$

73.

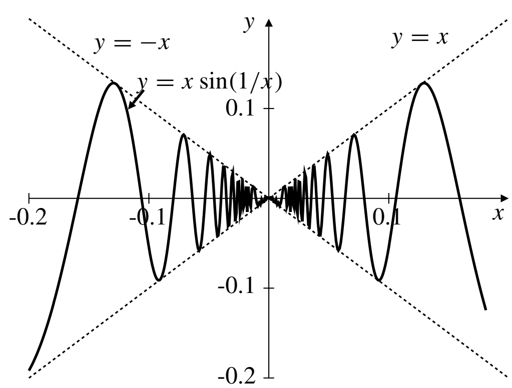

Fig. 1.2.73

$f(x) = x\sin(1/x)$ oscillates infinitely often as x approaches 0, but the amplitude of the oscillations decreases and, in fact, $\lim_{x\to 0} f(x) = 0$. This is predictable because $|x\sin(1/x)| \le |x|$. (See Exercise 95 below.)

74. Since $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $-1 \le x \le 1$, and $\lim_{x\to 0} \sqrt{5 - 2x^2} = \lim_{x\to 0} \sqrt{5 - x^2} = \sqrt{5}$, we have $\lim_{x\to 0} f(x) = \sqrt{5}$ by the squeeze theorem.

75. Since $2 - x^2 \le g(x) \le 2\cos x$ for all x, and since $\lim_{x\to 0}(2 - x^2) = \lim_{x\to 0} 2\cos x = 2$, we have $\lim_{x\to 0} g(x) = 2$ by the squeeze theorem.

76. a)

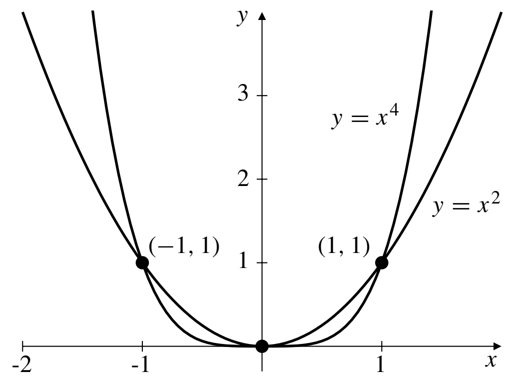

Fig. 1.2.76

b) Since the graph of f lies between those of x^2 and x^4, and since these latter graphs come together at $(\pm 1, 1)$ and at $(0, 0)$, we have $\lim_{x\to\pm 1} f(x) = 1$ and $\lim_{x\to 0} f(x) = 0$ by the squeeze theorem.

77. $x^{1/3} < x^3$ on $(-1, 0)$ and $(1, \infty)$. $x^{1/3} > x^3$ on $(-\infty, -1)$ and $(0, 1)$. The graphs of $x^{1/3}$ and x^3 intersect at $(-1, -1)$, $(0, 0)$, and $(1, 1)$. If the graph of $h(x)$ lies between those of $x^{1/3}$ and x^3, then we can determine $\lim_{x\to a} h(x)$ for $a = -1$, $a = 0$, and $a = 1$ by the squeeze theorem. In fact

$$\lim_{x\to -1} h(x) = -1, \qquad \lim_{x\to 0} h(x) = 0, \qquad \lim_{x\to 1} h(x) = 1.$$

78. $f(x) = s\sin\dfrac{1}{x}$ is defined for all $x \ne 0$; its domain is $(-\infty, 0) \cup (0, \infty)$. Since $|\sin t| \le 1$ for all t, we have $|f(x)| \le |x|$ and $-|x| \le f(x) \le |x|$ for all $x \ne 0$. Since $\lim_{x\to 0} = (-|x|) = 0 = \lim_{x\to 0} |x|$, we have $\lim_{x\to 0} f(x) = 0$ by the squeeze theorem.

79. $|f(x)| \le g(x) \Rightarrow -g(x) \le f(x) \le g(x)$
Since $\lim\limits_{x\to a} g(x) = 0$, therefore $0 \le \lim\limits_{x\to a} f(x) \le 0$.
Hence, $\lim\limits_{x\to a} f(x) = 0$.
If $\lim\limits_{x\to a} g(x) = 3$, then either $-3 \le \lim\limits_{x\to a} f(x) \le 3$ or $\lim_{x\to a} f(x)$ does not exist.

Section 1.3 Limits at Infinity and Infinite Limits (page 75)

1. $\displaystyle \lim_{x\to\infty} \frac{x}{2x - 3} = \lim_{x\to\infty} \frac{1}{2 - (3/x)} = \frac{1}{2}$

2. $\displaystyle \lim_{x\to\infty} \frac{x}{x^2 - 4} = \lim_{x\to\infty} \frac{1/x}{1 - (4/x^2)} = \frac{0}{1} = 0$

3. $\displaystyle \lim_{x\to\infty} \frac{3x^3 - 5x^2 + 7}{8 + 2x - 5x^3}$

$$= \lim_{x\to\infty} \frac{3 - \dfrac{5}{x} + \dfrac{7}{x^3}}{\dfrac{8}{x^3} + \dfrac{2}{x^2} - 5} = -\frac{3}{5}$$

4. $\displaystyle \lim_{x\to-\infty}\frac{x^2-2}{x-x^2}$

$\displaystyle =\lim_{x\to-\infty}\frac{1-\frac{2}{x^2}}{\frac{1}{x}-1}=\frac{1}{-1}=-1$

5. $\displaystyle \lim_{x\to-\infty}\frac{x^2+3}{x^3+2}=\lim_{x\to-\infty}\frac{\frac{1}{x}+\frac{3}{x^3}}{1+\frac{2}{x^3}}=0$

6. $\displaystyle \lim_{x\to\infty}\frac{x^2+\sin x}{x^2+\cos x}=\lim_{x\to\infty}\frac{1+\frac{\sin x}{x^2}}{1+\frac{\cos x}{x^2}}=\frac{1}{1}=1$

We have used the fact that $\lim_{x\to\infty}\frac{\sin x}{x^2}=0$ (and similarly for cosine) because the numerator is bounded while the denominator grows large.

7. $\displaystyle \lim_{x\to\infty}\frac{3x+2\sqrt{x}}{1-x}$

$\displaystyle =\lim_{x\to\infty}\frac{3+\frac{2}{\sqrt{x}}}{\frac{1}{x}-1}=-3$

8. $\displaystyle \lim_{x\to\infty}\frac{2x-1}{\sqrt{3x^2+x+1}}$

$\displaystyle =\lim_{x\to\infty}\frac{x\left(2-\frac{1}{x}\right)}{|x|\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}$ (but $|x|=x$ as $x\to\infty$)

$\displaystyle =\lim_{x\to\infty}\frac{2-\frac{1}{x}}{\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}=\frac{2}{\sqrt{3}}$

9. $\displaystyle \lim_{x\to-\infty}\frac{2x-1}{\sqrt{3x^2+x+1}}$

$\displaystyle =\lim_{x\to-\infty}\frac{2-\frac{1}{x}}{-\sqrt{3+\frac{1}{x}+\frac{1}{x^2}}}=-\frac{2}{\sqrt{3}},$

because $x\to-\infty$ implies that $x<0$ and so $\sqrt{x^2}=-x$.

10. $\displaystyle \lim_{x\to-\infty}\frac{2x-5}{|3x+2|}=\lim_{x\to-\infty}\frac{2x-5}{-(3x+2)}=-\frac{2}{3}$

11. $\displaystyle \lim_{x\to3}\frac{1}{3-x}$ does not exist.

12. $\displaystyle \lim_{x\to3}\frac{1}{(3-x)^2}=\infty$

13. $\displaystyle \lim_{x\to3-}\frac{1}{3-x}=\infty$

14. $\displaystyle \lim_{x\to3+}\frac{1}{3-x}=-\infty$

15. $\displaystyle \lim_{x\to-5/2}\frac{2x+5}{5x+2}=\frac{0}{\frac{-25}{2}+2}=0$

16. $\displaystyle \lim_{x\to-2/5}\frac{2x+5}{5x+2}$ does not exist.

17. $\displaystyle \lim_{x\to-(2/5)-}\frac{2x+5}{5x+2}=-\infty$

18. $\displaystyle \lim_{x\to-2/5+}\frac{2x+5}{5x+2}=\infty$

19. $\displaystyle \lim_{x\to2+}\frac{x}{(2-x)^3}=-\infty$

20. $\displaystyle \lim_{x\to1-}\frac{x}{\sqrt{1-x^2}}=\infty$

21. $\displaystyle \lim_{x\to1+}\frac{1}{|x-1|}=\infty$

22. $\displaystyle \lim_{x\to1-}\frac{1}{|x-1|}=\infty$

23. $\displaystyle \lim_{x\to2}\frac{x-3}{x^2-4x+4}=\lim_{x\to2}\frac{x-3}{(x-2)^2}=-\infty$

24. $\displaystyle \lim_{x\to1+}\frac{\sqrt{x^2-x}}{x-x^2}=\lim_{x\to1+}\frac{-1}{\sqrt{x^2-x}}=-\infty$

25. $\displaystyle \lim_{x\to\infty}\frac{x+x^3+x^5}{1+x^2+x^3}$

$\displaystyle =\lim_{x\to\infty}\frac{\frac{1}{x^2}+1+x^2}{\frac{1}{x^3}+\frac{1}{x}+1}=\infty$

26. $\displaystyle \lim_{x\to\infty}\frac{x^3+3}{x^2+2}=\lim_{x\to\infty}\frac{x+\frac{3}{x^2}}{1+\frac{2}{x^2}}=\infty$

27. $\displaystyle \lim_{x\to\infty}\frac{x\sqrt{x+1}\left(1-\sqrt{2x+3}\right)}{7-6x+4x^2}$

$\displaystyle =\lim_{x\to\infty}\frac{x^2\left(\sqrt{1+\frac{1}{x}}\right)\left(\frac{1}{\sqrt{x}}-\sqrt{2+\frac{3}{x}}\right)}{x^2\left(\frac{7}{x^2}-\frac{6}{x}+4\right)}$

$\displaystyle =\frac{1(-\sqrt{2})}{4}=-\frac{1}{4}\sqrt{2}$

28. $\displaystyle \lim_{x\to\infty}\left(\frac{x^2}{x+1}-\frac{x^2}{x-1}\right)=\lim_{x\to\infty}\frac{-2x^2}{x^2-1}=-2$

29. $\displaystyle \lim_{x\to-\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$
$\displaystyle =\lim_{x\to-\infty}\frac{(x^2+2x)-(x^2-2x)}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$
$\displaystyle =\lim_{x\to-\infty}\frac{4x}{(-x)\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}\right)}$
$\displaystyle =-\frac{4}{1+1}=-2$

30. $\displaystyle \lim_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-2x}\right)$
$\displaystyle =\lim_{x\to\infty}\frac{x^2+2x-x^2+2x}{\sqrt{x^2+2x}+\sqrt{x^2-2x}}$
$\displaystyle =\lim_{x\to\infty}\frac{4x}{x\sqrt{1+\frac{2}{x}}+x\sqrt{1-\frac{2}{x}}}$
$\displaystyle =\lim_{x\to\infty}\frac{4}{\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{2}{x}}}=\frac{4}{2}=2$

31. $\displaystyle \lim_{x\to\infty}\frac{1}{\sqrt{x^2-2x}-x}$
$\displaystyle =\lim_{x\to\infty}\frac{\sqrt{x^2-2x}+x}{(\sqrt{x^2-2x}+x)(\sqrt{x^2-2x}-x)}$
$\displaystyle =\lim_{x\to\infty}\frac{\sqrt{x^2-2x}+x}{x^2-2x-x^2}$
$\displaystyle =\lim_{x\to\infty}\frac{x(\sqrt{1-(2/x)}+1)}{-2x}=\frac{2}{-2}=-1$

32. $\displaystyle \lim_{x\to-\infty}\frac{1}{\sqrt{x^2+2x}-x}=\lim_{x\to-\infty}\frac{1}{|x|(\sqrt{1+(2/x)}+1}=0$

33. By Exercise 35, $y=-1$ is a horizontal asymptote (at the right) of $y=\dfrac{1}{\sqrt{x^2-2x}-x}$. Since

$$\lim_{x\to-\infty}\frac{1}{\sqrt{x^2-2x}-x}=\lim_{x\to-\infty}\frac{1}{|x|(\sqrt{1-(2/x)}+1}=0,$$

$y=0$ is also a horizontal asymptote (at the left).
Now $\sqrt{x^2-2x}-x=0$ if and only if $x^2-2x=x^2$, that is, if and only if $x=0$. The given function is undefined at $x=0$, and where $x^2-2x<0$, that is, on the interval $[0,2]$. Its only vertical asymptote is at $x=0$, where $\displaystyle \lim_{x\to0-}\frac{1}{\sqrt{x^2-2x}-x}=\infty$.

34. Since $\displaystyle \lim_{x\to\infty}\frac{2x-5}{|3x+2|}=\frac{2}{3}$ and $\displaystyle \lim_{x\to-\infty}\frac{2x-5}{|3x+2|}=-\frac{2}{3}$, $y=\pm(2/3)$ are horizontal asymptotes of $y=(2x-5)/|3x+2|$. The only vertical asymptote is $x=-2/3$, which makes the denominator zero.

35. $\displaystyle \lim_{x\to0+}f(x)=1$

36. $\displaystyle \lim_{x\to1}f(x)=\infty$

37.

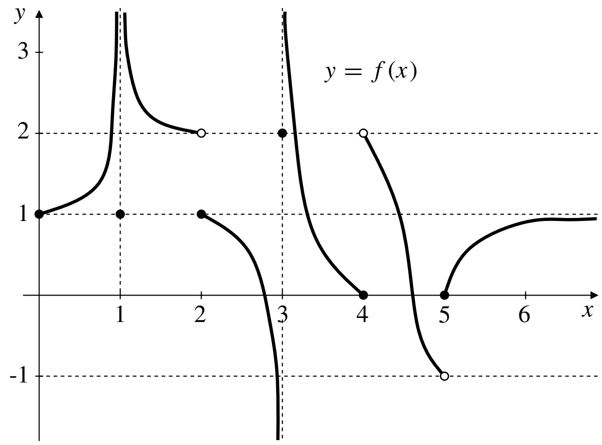

Fig. 1.3.37

$\lim_{x\to2+}f(x)=1$

38. $\displaystyle \lim_{x\to2-}f(x)=2$

39. $\displaystyle \lim_{x\to3-}f(x)=-\infty$

40. $\displaystyle \lim_{x\to3+}f(x)=\infty$

41. $\displaystyle \lim_{x\to4+}f(x)=2$

42. $\displaystyle \lim_{x\to4-}f(x)=0$

43. $\displaystyle \lim_{x\to5-}f(x)=-1$

44. $\displaystyle \lim_{x\to5+}f(x)=0$

45. $\displaystyle \lim_{x\to\infty}f(x)=1$

46. horizontal: $y=1$; vertical: $x=1$, $x=3$.

47. $\displaystyle \lim_{x\to3+}\lfloor x\rfloor=3$

48. $\displaystyle \lim_{x\to3-}\lfloor x\rfloor=2$

49. $\displaystyle \lim_{x\to3}\lfloor x\rfloor$ does not exist

50. $\displaystyle \lim_{x\to2.5}\lfloor x\rfloor=2$

51. $\displaystyle \lim_{x\to0+}\lfloor 2-x\rfloor=\lim_{x\to2-}\lfloor x\rfloor=1$

52. $\displaystyle \lim_{x\to-3-}\lfloor x\rfloor=-4$

53. $\displaystyle \lim_{t\to t_0}C(t)=C(t_0)$ except at integers t_0
$\displaystyle \lim_{t\to t_0-}C(t)=C(t_0)$ everywhere
$\displaystyle \lim_{t\to t_0+}C(t)=C(t_0)$ if $t_0\neq$ an integer
$\displaystyle \lim_{t\to t_0+}C(t)=C(t_0)+1.5$ if t_0 is an integer

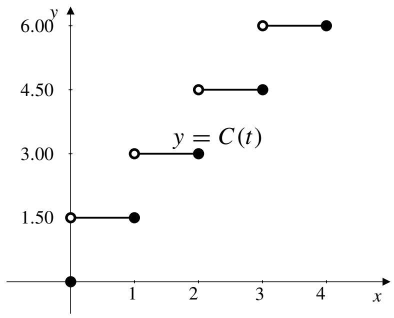

Fig. 1.3.53

54. $\lim_{x \to 0+} f(x) = L$
(a) If f is even, then $f(-x) = f(x)$.
Hence, $\lim_{x \to 0-} f(x) = L$.
(b) If f is odd, then $f(-x) = -f(x)$.
Therefore, $\lim_{x \to 0-} f(x) = -L$.

55. $\lim_{x \to 0+} f(x) = A, \quad \lim_{x \to 0-} f(x) = B$

a) $\lim_{x \to 0+} f(x^3 - x) = B$ (since $x^3 - x < 0$ if $0 < x < 1$)

b) $\lim_{x \to 0-} f(x^3 - x) = A$ (because $x^3 - x > 0$ if $-1 < x < 0$)

c) $\lim_{x \to 0-} f(x^2 - x^4) = A$

d) $\lim_{x \to 0+} f(x^2 - x^4) = A$ (since $x^2 - x^4 > 0$ for $0 < |x| < 1$)

Section 1.4 Continuity (page 85)

1. g is continuous at $x = -2$, discontinuous at $x = -1$, 0, 1, and 2. It is left continuous at $x = 0$ and right continuous at $x = 1$.

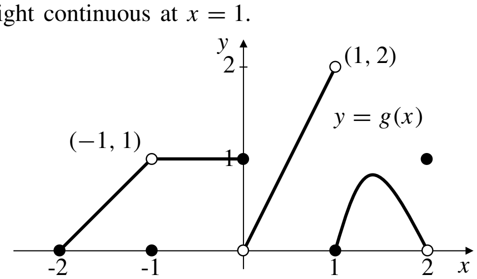

Fig. 1.4.1

2. g has removable discontinuities at $x = -1$ and $x = 2$. Redefine $g(-1) = 1$ and $g(2) = 0$ to make g continuous at those points.

3. g has no absolute maximum value on $[-2, 2]$. It takes on every positive real value less than 2, but does not take the value 2. It has absolute minimum value 0 on that interval, assuming this value at the three points $x = -2$, $x = -1$, and $x = 1$.

4. Function f is discontinuous at $x = 1$, 2, 3, 4, and 5. f is left continuous at $x = 4$ and right continuous at $x = 2$ and $x = 5$.

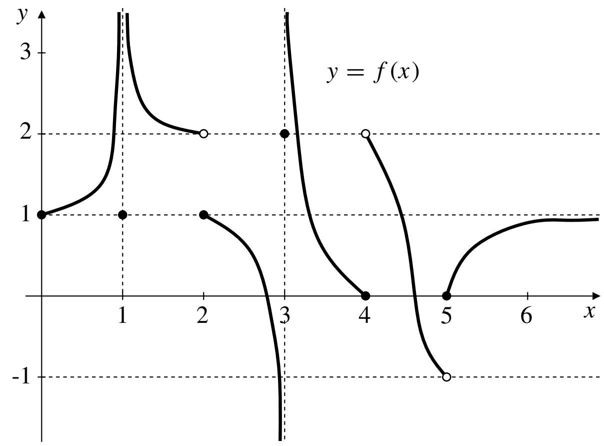

Fig. 1.4.4

5. f cannot be redefined at $x = 1$ to become continuous there because $\lim_{x \to 1} f(x)$ $(= \infty)$ does not exist. (∞ is not a real number.)

6. $\operatorname{sgn} x$ is not defined at $x = 0$, so cannot be either continuous or discontinuous there. (Functions can be continuous or discontinuous only at points in their domains!)

7. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } x \geq 0 \end{cases}$ is continuous everywhere on the real line, even at $x = 0$ where its left and right limits are both 0, which is $f(0)$.

8. $f(x) = \begin{cases} x & \text{if } x < -1 \\ x^2 & \text{if } x \geq -1 \end{cases}$ is continuous everywhere on the real line except at $x = -1$ where it is right continuous, but not left continuous.

$$\begin{aligned} \lim_{x \to -1-} f(x) &= \lim_{x \to -1-} x = -1 \neq 1 \\ &= f(-1) = \lim_{x \to -1+} x^2 = \lim_{x \to -1+} f(x). \end{aligned}$$

9. $f(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous everywhere except at $x = 0$, where it is neither left nor right continuous since it does not have a real limit there.

10. $f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 0.987 & \text{if } x > 1 \end{cases}$ is continuous everywhere except at $x = 1$, where it is left continuous but not right continuous because $0.987 \neq 1$. Close, as they say, but no cigar.

11. The least integer function $\lceil x \rceil$ is continuous everywhere on $\mathbb{R}$ except at the integers, where it is left continuous but not right continuous.

12. $C(t)$ is discontinuous only at the integers. It is continuous on the left at the integers, but not on the right.

13. Since $\dfrac{x^2-4}{x-2} = x+2$ for $x \neq 2$, we can define the function to be $2+2=4$ at $x=2$ to make it continuous there. The continuous extension is $x+2$.

14. Since $\dfrac{1+t^3}{1-t^2} = \dfrac{(1+t)(1-t+t^2)}{(1+t)(1-t)} = \dfrac{1-t+t^2}{1-t}$ for $t \neq -1$, we can define the function to be $3/2$ at $t=-1$ to make it continuous there. The continuous extension is $\dfrac{1-t+t^2}{1-t}$.

15. Since $\dfrac{t^2-5t+6}{t^2-t-6} = \dfrac{(t-2)(t-3)}{(t+2)(t-3)} = \dfrac{t-2}{t+2}$ for $t \neq 3$, we can define the function to be $1/5$ at $t=3$ to make it continuous there. The continuous extension is $\dfrac{t-2}{t+2}$.

16. Since $\dfrac{x^2-2}{x^4-4} = \dfrac{(x-\sqrt{2})(x+\sqrt{2})}{(x-\sqrt{2})(x+\sqrt{2})(x^2+2)} = \dfrac{x+\sqrt{2}}{(x+\sqrt{2})(x^2+2)}$ for $x \neq \sqrt{2}$, we can define the function to be $1/4$ at $x=\sqrt{2}$ to make it continuous there. The continuous extension is $\dfrac{x+\sqrt{2}}{(x+\sqrt{2})(x^2+2)}$. (Note: cancelling the $x+\sqrt{2}$ factors provides a further continuous extension to $x=-\sqrt{2}$.

17. $\lim_{x\to 2+} f(x) = k-4$ and $\lim_{x\to 2-} f(x) = 4 = f(2)$. Thus f will be continuous at $x=2$ if $k-4=4$, that is, if $k=8$.

18. $\lim_{x\to 3-} g(x) = 3-m$ and $\lim_{x\to 3+} g(x) = 1-3m = g(3)$. Thus g will be continuous at $x=3$ if $3-m=1-3m$, that is, if $m=-1$.

19. x^2 has no maximum value on $-1<x<1$; it takes all positive real values less than 1, but it does not take the value 1. It does have a minimum value, namely 0 taken on at $x=0$.

20. The Max-Min Theorem says that a continuous function defined on a closed, finite interval must have maximum and minimum values. It does not say that other functions cannot have such values. The Heaviside function is not continuous on $[-1,1]$ (because it is discontinuous at $x=0$), but it still has maximum and minimum values. Do not confuse a theorem with its converse.

21. Let the numbers be x and y, where $x \geq 0$, $y \geq 0$, and $x+y=8$. If P is the product of the numbers, then

$$P = xy = x(8-x) = 8x - x^2 = 16-(x-4)^2.$$

Therefore $P \leq 16$, so P is bounded. Clearly $P=16$ if $x=y=4$, so the largest value of P is 16.

22. Let the numbers be x and y, where $x \geq 0$, $y \geq 0$, and $x+y=8$. If S is the sum of their squares then

$$\begin{aligned} S &= x^2+y^2 = x^2+(8-x)^2 \\ &= 2x^2-16x+64 = 2(x-4)^2+32. \end{aligned}$$

Since $0 \leq x \leq 8$, the maximum value of S occurs at $x=0$ or $x=8$, and is 64. The minimum value occurs at $x=4$ and is 32.

23. Since $T = 100-30x+3x^2 = 3(x-5)^2+25$, T will be minimum when $x=5$. Five programmers should be assigned, and the project will be completed in 25 days.

24. If x desks are shipped, the shipping cost per desk is

$$\begin{aligned} C = \frac{245x-30x^2+x^3}{x} &= x^2-30x+245 \\ &= (x-15)^2+20. \end{aligned}$$

This cost is minimized if $x=15$. The manufacturer should send 15 desks in each shipment, and the shipping cost will then be \$20 per desk.

25. $f(x) = \dfrac{x^2-1}{x} = \dfrac{(x-1)(x+1)}{x}$
$f=0$ at $x=\pm 1$. f is not defined at 0.
$f(x)>0$ on $(-1,0)$ and $(1,\infty)$.
$f(x)<0$ on $(-\infty,-1)$ and $(0,1)$.

26. $f(x) = x^2+4x+3 = (x+1)(x+3)$
$f(x)>0$ on $(-\infty,-3)$ and $(-1,\infty)$
$f(x)<0$ on $(-3,-1)$.

27. $f(x) = \dfrac{x^2-1}{x^2-4} = \dfrac{(x-1)(x+1)}{(x-2)(x+2)}$
$f=0$ at $x=\pm 1$.
f is not defined at $x=\pm 2$.
$f(x)>0$ on $(-\infty,-2)$, $(-1,1)$, and $(2,\infty)$.
$f(x)<0$ on $(-2,-1)$ and $(1,2)$.

28. $f(x) = \dfrac{x^2+x-2}{x^3} = \dfrac{(x+2)(x-1)}{x^3}$
$f(x)>0$ on $(-2,0)$ and $(1,\infty)$
$f(x)<0$ on $(-\infty,-2)$ and $(0,1)$.

29. $f(x) = x^3+x-1$, $f(0)=-1$, $f(1)=1$.
Since f is continuous and changes sign between 0 and 1, it must be zero at some point between 0 and 1 by IVT.

30. $f(x) = x^3-15x+1$ is continuous everywhere.
$f(-4)=-3$, $f(-3)=19$, $f(1)=-13$, $f(4)=5$.
Because of the sign changes f has a zero between -4 and -3, another zero between -3 and 1, and another between 1 and 4.

31. $F(x) = (x-a)^2(x-b)^2 + x$. Without loss of generality, we can assume that $a < b$. Being a polynomial, F is continuous on $[a, b]$. Also $F(a) = a$ and $F(b) = b$. Since $a < \frac{1}{2}(a+b) < b$, the Intermediate-Value Theorem guarantees that there is an x in (a, b) such that $F(x) = (a+b)/2$.

32. Let $g(x) = f(x) - x$. Since $0 \le f(x) \le 1$ if $0 \le x \le 1$, therefore, $g(0) \ge 0$ and $g(1) \le 0$. If $g(0) = 0$ let $c = 0$, or if $g(1) = 0$ let $c = 1$. (In either case $f(c) = c$.) Otherwise, $g(0) > 0$ and $g(1) < 0$, and, by IVT, there exists c in $(0, 1)$ such that $g(c) = 0$, i.e., $f(c) = c$.

33. The domain of an even function is symmetric about the y-axis. Since f is continuous on the right at $x = 0$, therefore it must be defined on an interval $[0, h]$ for some $h > 0$. Being even, f must therefore be defined on $[-h, h]$. If $x = -y$, then

$$\lim_{x\to 0-} f(x) = \lim_{y\to 0+} f(-y) = \lim_{y\to 0+} f(y) = f(0).$$

Thus, f is continuous on the left at $x = 0$. Being continuous on both sides, it is therefore continuous.

34. f odd $\Leftrightarrow f(-x) = -f(x)$
f continuous on the right $\Leftrightarrow \lim_{x\to 0+} f(x) = f(0)$
Therefore, letting $t = -x$, we obtain

$$\begin{aligned}\lim_{x\to 0-} f(x) &= \lim_{t\to 0+} f(-t) = \lim_{t\to 0+} -f(t) \\ &= -f(0) = f(-0) = f(0).\end{aligned}$$

Therefore f is continuous at 0 and $f(0) = 0$.

35. max 1.593 at -0.831, min -0.756 at 0.629

36. max 0.133 at $x = 1.437$; min -0.232 at $x = -1.805$

37. max 10.333 at $x = 3$; min 4.762 at $x = 1.260$

38. max 1.510 at $x = 0.465$; min 0 at $x = 0$ and $x = 1$

39. root $x = 0.682$

40. root $x = 0.739$

41. roots $x = -0.637$ and $x = 1.410$

42. roots $x = -0.7244919590$ and $x = 1.220744085$

43. fsolve gives an approximation to the single real root to 10 significant figures; solve gives the three roots (including a complex conjugate pair) in exact form involving the quantity $\left(108 + 12\sqrt{69}\right)^{1/3}$; evalf(solve) gives approximations to the three roots using 10 significant figures for the real and imaginary parts.

Section 1.5 The Formal Definition of Limit (page 90)

1. We require $39.9 \le L \le 40.1$. Thus

$$\begin{aligned}39.9 &\le 39.6 + 0.025T \le 40.1 \\ 0.3 &\le 0.025T \le 0.5 \\ 12 &\le T \le 20.\end{aligned}$$

The temperature should be kept between 12 °C and 20 °C.

2. Since 1.2% of 8,000 is 96, we require the edge length x of the cube to satisfy $7904 \le x^3 \le 8096$. It is sufficient that $19.920 \le x \le 20.079$. The edge of the cube must be within 0.079 cm of 20 cm.

3. $$\begin{aligned}3 - 0.02 &\le 2x - 1 \le 3 + 0.02 \\ 3.98 &\le 2x \le 4.02 \\ 1.99 &\le x \le 2.01\end{aligned}$$

4. $$\begin{aligned}4 - 0.1 &\le x^2 \le 4 + 0.1 \\ 1.9749 &\le x \le 2.0024\end{aligned}$$

5. $$\begin{aligned}1 - 0.1 &\le \sqrt{x} \le 1.1 \\ 0.81 &\le x \le 1.21\end{aligned}$$

6. $$\begin{aligned}-2 - 0.01 &\le \frac{1}{x} \le -2 + 0.01 \\ -\frac{1}{2.01} &\ge x \ge -\frac{1}{1.99} \\ -0.5025 &\le x \le -0.4975\end{aligned}$$

7. We need $-0.03 \le (3x+1)-7 \le 0.03$, which is equivalent to $-0.01 \le x - 2 \le 0.01$ Thus $\delta = 0.01$ will do.

8. We need $-0.01 \le \sqrt{2x+3} - 3 \le 0.01$. Thus

$$\begin{aligned}2.99 &\le \sqrt{2x+3} \le 3.01 \\ 8.9401 &\le 2x + 3 \le 9.0601 \\ 2.97005 &\le x \le 3.03005 \\ 3 - 0.02995 &\le x - 3 \le 0.03005.\end{aligned}$$

Here $\delta = 0.02995$ will do.

9. We need $8 - 0.2 \le x^3 \le 8.2$, or $1.9832 \le x \le 2.0165$. Thus, we need $-0.0168 \le x - 2 \le 0.0165$. Here $\delta = 0.0165$ will do.

10. We need $1 - 0.05 \le 1/(x+1) \le 1 + 0.05$, or $1.0526 \ge x + 1 \ge 0.9524$. This will occur if $-0.0476 \le x \le 0.0526$. In this case we can take $\delta = 0.0476$.

11. To be proved: $\lim_{x\to 1}(3x+1) = 4$.
Proof: Let $\epsilon > 0$ be given. Then $|(3x+1)-4| < \epsilon$ holds if $3|x-1| < \epsilon$, and so if $|x-1| < \delta = \epsilon/3$. This confirms the limit.

12. To be proved: $\lim_{x\to 2}(5-2x)=1$.
Proof: Let $\epsilon>0$ be given. Then $|(5-2x)-1|<\epsilon$ holds if $|2x-4|<\epsilon$, and so if $|x-2|<\delta=\epsilon/2$. This confirms the limit.

13. To be proved: $\lim_{x\to 0}x^2=0$.
Let $\epsilon>0$ be given. Then $|x^2-0|<\epsilon$ holds if $|x-0|=|x|<\delta=\sqrt{\epsilon}$.

14. To be proved: $\lim_{x\to 2}\dfrac{x-2}{1+x^2}=0$.
Proof: Let $\epsilon>0$ be given. Then

$$\left|\frac{x-2}{1+x^2}-0\right|=\frac{|x-2|}{1+x^2}\le|x-2|<\epsilon$$

provided $|x-2|<\delta=\epsilon$.

15. To be proved: $\lim_{x\to 1/2}\dfrac{1-4x^2}{1-2x}=2$.
Proof: Let $\epsilon>0$ be given. Then if $x\ne 1/2$ we have

$$\left|\frac{1-4x^2}{1-2x}-2\right|=|(1+2x)-2|=|2x-1|=2\left|x-\frac{1}{2}\right|<\epsilon$$

provided $|x-\frac{1}{2}|<\delta=\epsilon/2$.

16. To be proved: $\lim_{x\to -2}\dfrac{x^2+2x}{x+2}=-2$.
Proof: Let $\epsilon>0$ be given. For $x\ne -2$ we have

$$\left|\frac{x^2+2x}{x+2}-(-2)\right|=|x+2|<\epsilon$$

provided $|x+2|<\delta=\epsilon$. This completes the proof.

17. To be proved: $\lim_{x\to 1}\dfrac{1}{x+1}=\dfrac{1}{2}$.
Proof: Let $\epsilon>0$ be given. We have

$$\left|\frac{1}{x+1}-\frac{1}{2}\right|=\left|\frac{1-x}{2(x+1)}\right|=\frac{|x-1|}{2|x+1|}.$$

If $|x-1|<1$, then $0<x<2$ and $1<x+1<3$, so that $|x+1|>1$. Let $\delta=\min(1,2\epsilon)$. If $|x-1|<\delta$, then

$$\left|\frac{1}{x+1}-\frac{1}{2}\right|=\frac{|x-1|}{2|x+1|}<\frac{2\epsilon}{2}=\epsilon.$$

This establishes the required limit.

18. To be proved: $\lim_{x\to -1}\dfrac{x+1}{x^2-1}=-\dfrac{1}{2}$.
Proof: Let $\epsilon>0$ be given. If $x\ne -1$, we have

$$\left|\frac{x+1}{x^2-1}-\frac{1}{2}\right|=\left|\frac{1}{x-1}-\left(-\frac{1}{2}\right)\right|=\frac{|x+1|}{2|x-1|}.$$

If $|x+1|<1$, then $-2<x<0$, so $-3<x-1<-1$ and $|x-1|>1$. Ler $\delta=\min(1,2\epsilon)$. If $0<|x-(-1)|<\delta$ then $|x-1|>1$ and $|x+1|<2\epsilon$. Thus

$$\left|\frac{x+1}{x^2-1}-\frac{1}{2}\right|=\frac{|x+1|}{2|x-1|}<\frac{2\epsilon}{2}=\epsilon.$$

This completes the required proof.

19. To be proved: $\lim_{x\to 1}\sqrt{x}=1$.
Proof: Let $\epsilon>0$ be given. We have

$$|\sqrt{x}-1|=\left|\frac{x-1}{\sqrt{x}+1}\right|\le|x-1|<\epsilon$$

provided $|x-1|<\delta=\epsilon$. This completes the proof.

20. To be proved: $\lim_{x\to 2}x^3=8$.
Proof: Let $\epsilon>0$ be given. We have $|x^3-8|=|x-2||x^2+2x+4|$. If $|x-2|<1$, then $1<x<3$ and $x^2<9$. Therefore $|x^2+2x+4|\le 9+2\times 3+4=19$. If $|x-2|<\delta=\min(1,\epsilon/19)$, then

$$|x^3-8|=|x-2||x^2+2x+4|<\frac{\epsilon}{19}\times 19=\epsilon.$$

This completes the proof.

21. We say that $\lim_{x\to a-}f(x)=L$ if the following condition holds: for every number $\epsilon>0$ there exists a number $\delta>0$, depending on ϵ, such that

$$a-\delta<x<a \quad\text{implies}\quad |f(x)-L|<\epsilon.$$

22. We say that $\lim_{x\to -\infty}f(x)=L$ if the following condition holds: for every number $\epsilon>0$ there exists a number $R>0$, depending on ϵ, such that

$$x<-R \quad\text{implies}\quad |f(x)-L|<\epsilon.$$

23. We say that $\lim_{x\to a}f(x)=-\infty$ if the following condition holds: for every number $B>0$ there exists a number $\delta>0$, depending on B, such that

$$0<|x-a|<\delta \quad\text{implies}\quad f(x)<-B.$$

24. We say that $\lim_{x\to\infty}f(x)=\infty$ if the following condition holds: for every number $B>0$ there exists a number $R>0$, depending on B, such that

$$x>R \quad\text{implies}\quad f(x)>B.$$

25. We say that $\lim_{x\to a+}f(x)=-\infty$ if the following condition holds: for every number $B>0$ there exists a number $\delta>0$, depending on R, such that

$$a<x<a+\delta \quad\text{implies}\quad f(x)<-B.$$

26. We say that $\lim_{x\to a-} f(x) = \infty$ if the following condition holds: for every number $B > 0$ there exists a number $\delta > 0$, depending on B, such that

$$a - \delta < x < a \quad \text{implies} \quad f(x) > B.$$

27. To be proved: $\lim_{x\to 1+} \dfrac{1}{x-1} = \infty$. Proof: Let $B > 0$ be given. We have $\dfrac{1}{x-1} > B$ if $0 < x - 1 < 1/B$, that is, if $1 < x < 1 + \delta$, where $\delta = 1/B$. This completes the proof.

28. To be proved: $\lim_{x\to 1-} \dfrac{1}{x-1} = -\infty$. Proof: Let $B > 0$ be given. We have $\dfrac{1}{x-1} < -B$ if $0 > x - 1 > -1/B$, that is, if $1 - \delta < x < 1$, where $\delta = 1/B$.. This completes the proof.

29. To be proved: $\lim_{x\to\infty} \dfrac{1}{\sqrt{x^2+1}} = 0$. Proof: Let $\epsilon > 0$ be given. We have

$$\left|\frac{1}{\sqrt{x^2+1}}\right| = \frac{1}{\sqrt{x^2+1}} < \frac{1}{x} < \epsilon$$

provided $x > R$, where $R = 1/\epsilon$. This completes the proof.

30. To be proved: $\lim_{x\to\infty} \sqrt{x} = \infty$. Proof: Let $B > 0$ be given. We have $\sqrt{x} > B$ if $x > R$ where $R = B^2$. This completes the proof.

31. To be proved: if $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} f(x) = M$, then $L = M$.
Proof: Suppose $L \ne M$. Let $\epsilon = |L - M|/3$. Then $\epsilon > 0$. Since $\lim\limits_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x) - L| < \epsilon$ if $|x - a| < \delta_1$. Since $\lim\limits_{x\to a} f(x) = M$, there exists $\delta_2 > 0$ such that $|f(x) - M| < \epsilon$ if $|x - a| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. If $|x - a| < \delta$, then

$$\begin{aligned} 3\epsilon = |L - M| &= |(f(x) - M) + (L - f(x)| \\ &\le |f(x) - M| + |f(x) - L| < \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

This implies that $3 < 2$, a contradiction. Thus the original assumption that $L \ne M$ must be incorrect. Therefore $L = M$.

32. To be proved: if $\lim\limits_{x\to a} g(x) = M$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x)| < 1 + |M|$.
Proof: Taking $\epsilon = 1$ in the definition of limit, we obtain a number $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x) - M| < 1$. It follows from this latter inequality that

$$|g(x)| = |(g(x) - M) + M| \le |G(x) - M| + |M| < 1 + |M|.$$

33. To be proved: if $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$, then $\lim\limits_{x\to a} f(x)g(x) = LM$.
Proof: Let $\epsilon > 0$ be given. Since $\lim\limits_{x\to a} f(x) = L$, there exists $\delta_1 > 0$ such that $|f(x) - L| < \epsilon/(2(1 + |M|))$ if $0 < |x - a| < \delta_1$. Since $\lim\limits_{x\to a} g(x) = M$, there exists $\delta_2 > 0$ such that $|g(x) - M| < \epsilon/(2(1 + |L|))$ if $0 < |x - a| < \delta_2$. By Exercise 32, there exists $\delta_3 > 0$ such that $|g(x)| < 1 + |M|$ if $0 < |x - a| < \delta_3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $|x - a| < \delta$, then

$$\begin{aligned} |f(x)g(x) - LM &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\ &= |(f(x) - L)g(x) + L(g(x) - M)| \\ &\le |(f(x) - L)g(x)| + |L(g(x) - M)| \\ &= |f(x) - L||g(x)| + |L||g(x) - M| \\ &< \frac{\epsilon}{2(1 + |M|)}(1 + |M|) + |L|\frac{\epsilon}{2(1 + |L|)} \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\lim\limits_{x\to a} f(x)g(x) = LM$.

34. To be proved: if $\lim\limits_{x\to a} g(x) = M$ where $M \ne 0$, then there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x)| > |M|/2$.
Proof: By the definition of limit, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x) - M| < |M|/2$ (since $|M|/2$ is a positive number). This latter inequality implies that

$$|M| = |g(x) + (M - g(x))| \le |g(x)| + |g(x) - M| < |g(x)| + \frac{|M|}{2}.$$

It follows that $|g(x)| > |M| - (|M|/2) = |M|/2$, as required.

35. To be proved: if $\lim\limits_{x\to a} g(x) = M$ where $M \ne 0$, then $\lim\limits_{x\to a} \dfrac{1}{g(x)} = \dfrac{1}{M}$.
Proof: Let $\epsilon > 0$ be given. Since $\lim\limits_{x\to a} g(x) = M \ne 0$, there exists $\delta_1 > 0$ such that $|g(x) - M| < \epsilon|M|^2/2$ if $0 < |x - a| < \delta_1$. By Exercise 34, there exists $\delta_2 > 0$ such that $|g(x)| > |M|/2$ if $0 < |x - a| < \delta_3$. Let $\delta = \min(\delta_1, \delta_2)$. If $0 < |x - a| < \delta$, then

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| = \frac{|M - g(x)|}{|M||g(x)|} < \frac{\epsilon|M|^2}{2}\,\frac{2}{|M|^2} = \epsilon.$$

This completes the proof.

36. To be proved: if $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} f(x) = M \ne 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$.
Proof: By Exercises 33 and 35 we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x) \times \frac{1}{g(x)} = L \times \frac{1}{M} = \frac{L}{M}.$$

37. To be proved: if f is continuous at L and $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f(g(x)) = f(L)$.
Proof: Let $\epsilon > 0$ be given. Since f is continuous at L, there exists a number $\gamma > 0$ such that if $|y-L| < \gamma$, then $|f(y) - f(L)| < \epsilon$. Since $\lim_{x\to c} g(x) = L$, there exists $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|g(x) - L| < \gamma$. Taking $y = g(x)$, it follows that if $0 < |x-c| < \delta$, then $|f(g(x)) - f(L)| < \epsilon$, so that $\lim_{x\to c} f(g(x)) = f(L)$.

38. To be proved: if $f(x) \le g(x) \le h(x)$ in an open interval containing $x = a$ (say, for $a - \delta_1 < x < a + \delta_1$, where $\delta_1 > 0$), and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then also $\lim_{x\to a} g(x) = L$.
Proof: Let $\epsilon > 0$ be given. Since $\lim_{x\to a} f(x) = L$, there exists $\delta_2 > 0$ such that if $0 < |x-a| < \delta_2$, then $|f(x) - L| < \epsilon/3$. Since $\lim_{x\to a} h(x) = L$, there exists $\delta_3 > 0$ such that if $0 < |x-a| < \delta_3$, then $|h(x) - L| < \epsilon/3$. Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x-a| < \delta$, then

$$\begin{aligned}
|g(x) - L| &= |g(x) - f(x) + f(x) - L| \\
&\le |g(x) - f(x)| + |f(x) - L| \\
&\le |h(x) - f(x)| + |f(x) - L| \\
&= |h(x) - L + L - f(x)| + |f(x) - L| \\
&\le |h(x) - L| + |f(x) - L| + |f(x) - L| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Thus $\lim_{x\to a} g(x) = L$.

Review Exercises 1 (page 91)

1. The average rate of change of x^3 over $[1, 3]$ is

$$\frac{3^3 - 1^3}{3 - 1} = \frac{26}{2} = 13.$$

2. The average rate of change of $1/x$ over $[-2, -1]$ is

$$\frac{(1/(-1)) - (1/(-2))}{-1 - (-2)} = \frac{-1/2}{1} = -\frac{1}{2}.$$

3. The rate of change of x^3 at $x = 2$ is

$$\begin{aligned}
\lim_{h\to 0} \frac{(2+h)^3 - 2^3}{h} &= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\
&= \lim_{h\to 0} (12 + 6h + h^2) = 12.
\end{aligned}$$

4. The rate of change of $1/x$ at $x = -3/2$ is

$$\begin{aligned}
\lim_{h\to 0} \frac{\dfrac{1}{-(3/2)+h} - \left(\dfrac{1}{-3/2}\right)}{h} &= \lim_{h\to 0} \frac{\dfrac{2}{2h-3} + \dfrac{2}{3}}{h} \\
&= \lim_{h\to 0} \frac{2(3 + 2h - 3)}{3(2h-3)h} \\
&= \lim_{h\to 0} \frac{4}{3(2h-3)} = -\frac{4}{9}.
\end{aligned}$$

5. $\lim_{x\to 1} (x^2 - 4x + 7) = 1 - 4 + 7 = 4$

6. $\lim_{x\to 2} \dfrac{x^2}{1-x^2} = \dfrac{2^2}{1-2^2} = -\dfrac{4}{3}$

7. $\lim_{x\to 1} \dfrac{x^2}{1-x^2}$ does not exist. The denominator approaches 0 (from both sides) while the numerator does not.

8. $\lim_{x\to 2} \dfrac{x^2-4}{x^2-5x+6} = \lim_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)(x-3)} = \lim_{x\to 2} \dfrac{x+2}{x-3} = -4$

9. $\lim_{x\to 2} \dfrac{x^2-4}{x^2-4x+4} = \lim_{x\to 2} \dfrac{(x-2)(x+2)}{(x-2)^2} = \lim_{x\to 2} \dfrac{x+2}{x-2}$ does not exist. The denominator approaches 0 (from both sides) while the numerator does not.

10. $\lim_{x\to 2-} \dfrac{x^2-4}{x^2-4x+4} = \lim_{x\to 2-} \dfrac{x+2}{x-2} = -\infty$

11. $\lim_{x\to -2+} \dfrac{x^2-4}{x^2+4x+4} = \lim_{x\to -2+} \dfrac{x-2}{x+2} = -\infty$

12. $\lim_{x\to 4} \dfrac{2-\sqrt{x}}{x-4} = \lim_{x\to 4} \dfrac{4-x}{(2+\sqrt{x})(x-4)} = -\dfrac{1}{4}$

13.
$$\begin{aligned}
\lim_{x\to 3} \frac{x^2-9}{\sqrt{x}-\sqrt{3}} &= \lim_{x\to 3} \frac{(x-3)(x+3)(\sqrt{x}+\sqrt{3})}{x-3} \\
&= \lim_{x\to 3} (x+3)(\sqrt{x}+\sqrt{3}) = 12\sqrt{3}
\end{aligned}$$

14.
$$\begin{aligned}
\lim_{h\to 0} \frac{h}{\sqrt{x+3h}-\sqrt{x}} &= \lim_{h\to 0} \frac{h(\sqrt{x+3h}+\sqrt{x})}{(x+3h)-x} \\
&= \lim_{h\to 0} \frac{\sqrt{x+3h}+\sqrt{x}}{3} = \frac{2\sqrt{x}}{3}
\end{aligned}$$

15. $\lim_{x\to 0+} \sqrt{x-x^2} = 0$

16. $\lim_{x\to 0} \sqrt{x-x^2}$ does not exist because $\sqrt{x-x^2}$ is not defined for $x < 0$.

17. $\lim_{x\to 1} \sqrt{x-x^2}$ does not exist because $\sqrt{x-x^2}$ is not defined for $x > 1$.

18. $\lim_{x\to 1-} \sqrt{x-x^2} = 0$

19. $\lim_{x\to\infty} \dfrac{1-x^2}{3x^2-x-1} = \lim_{x\to\infty} \dfrac{(1/x^2)-1}{3-(1/x)-(1/x^2)} = -\dfrac{1}{3}$

20. $\displaystyle\lim_{x\to-\infty}\frac{2x+100}{x^2+3}=\lim_{x\to-\infty}\frac{(2/x)+(100/x^2)}{1+(3/x^2)}=0$

21. $\displaystyle\lim_{x\to-\infty}\frac{x^3-1}{x^2+4}=\lim_{x\to-\infty}\frac{x-(1/x^2)}{1+(4/x^2)}=-\infty$

22. $\displaystyle\lim_{x\to\infty}\frac{x^4}{x^2-4}=\lim_{x\to\infty}\frac{x^2}{1-(4/x^2)}=\infty$

23. $\displaystyle\lim_{x\to0+}\frac{1}{\sqrt{x-x^2}}=\infty$

24. $\displaystyle\lim_{x\to1/2}\frac{1}{\sqrt{x-x^2}}=\frac{1}{\sqrt{1/4}}=2$

25. $\displaystyle\lim_{x\to\infty}\sin x$ does not exist; $\sin x$ takes the values -1 and 1 in any interval (R,∞), and limits, if they exist, must be unique.

26. $\displaystyle\lim_{x\to\infty}\frac{\cos x}{x}=0$ by the squeeze theorem, since

$$-\frac{1}{x}\le\frac{\cos x}{x}\le\frac{1}{x}\quad\text{for all } x>0$$

and $\lim_{x\to\infty}(-1/x)=\lim_{x\to\infty}(1/x)=0$.

27. $\displaystyle\lim_{x\to0}x\sin\frac{1}{x}=0$ by the squeeze theorem, since

$$-|x|\le x\sin\frac{1}{x}\le|x|\quad\text{for all } x\ne0$$

and $\lim_{x\to0}(-|x|)=\lim_{x\to0}|x|=0$.

28. $\displaystyle\lim_{x\to0}\sin\frac{1}{x^2}$ does not exist; $\sin(1/x^2)$ takes the values -1 and 1 in any interval $(-\delta,\delta)$, where $\delta>0$, and limits, if they exist, must be unique.

29. $\displaystyle\lim_{x\to-\infty}[x+\sqrt{x^2-4x+1}]$

$$\begin{aligned}
&=\lim_{x\to-\infty}\frac{x^2-(x^2-4x+1)}{x-\sqrt{x^2-4x+1}}\\
&=\lim_{x\to-\infty}\frac{4x-1}{x-|x|\sqrt{1-(4/x)+(1/x^2)}}\\
&=\lim_{x\to-\infty}\frac{x[4-(1/x)]}{x+x\sqrt{1-(4/x)+(1/x^2)}}\\
&=\lim_{x\to-\infty}\frac{4-(1/x)}{1+\sqrt{1-(4/x)+(1/x^2)}}=2.
\end{aligned}$$

Note how we have used $|x|=-x$ (in the second last line), because $x\to-\infty$.

30. $\displaystyle\lim_{x\to\infty}[x+\sqrt{x^2-4x+1}]=\infty+\infty=\infty$

31. $f(x)=x^3-4x^2+1$ is continuous on the whole real line and so is discontinuous nowhere.

32. $f(x)=\dfrac{x}{x+1}$ is continuous everywhere on its domain, which consists of all real numbers except $x=-1$. It is discontinuous nowhere.

33. $f(x)=\begin{cases}x^2 & \text{if } x>2\\ x & \text{if } x\le2\end{cases}$ is defined everywhere and discontinuous at $x=2$, where it is, however, left continuous since $\lim_{x\to2-}f(x)=2=f(2)$.

34. $f(x)=\begin{cases}x^2 & \text{if } x>1\\ x & \text{if } x\le1\end{cases}$ is defined and continuous everywhere, and so discontinuous nowhere. Observe that $\lim_{x\to1-}f(x)=1=\lim_{x\to1+}f(x)$.

35. $f(x)=H(x-1)=\begin{cases}1 & \text{if } x\ge1\\ 0 & \text{if } x<1\end{cases}$ is defined everywhere and discontinuous at $x=1$ where it is, however, right continuous.

36. $f(x)=H(9-x^2)=\begin{cases}1 & \text{if } -3\le x\le3\\ 0 & \text{if } x<-3 \text{ or } x>3\end{cases}$ is defined everywhere and discontinuous at $x=\pm3$. It is right continuous at -3 and left continuous at 3.

37. $f(x)=|x|+|x+1|$ is defined and continuous everywhere. It is discontinuous nowhere.

38. $f(x)=\begin{cases}|x|/|x+1| & \text{if } x\ne-1\\ 1 & \text{if } x=-1\end{cases}$ is defined everywhere and discontinuous at $x=-1$ where it is neither left nor right continuous since $\lim_{x\to-1}f(x)=\infty$, while $f(-1)=1$.

Challenging Problems 1 (page 92)

1. Let $0<a<b$. The average rate of change of x^3 over $[a,b]$ is

$$\frac{b^3-a^3}{b-a}=b^2+ab+a^2.$$

The instantaneous rate of change of x^3 at $x=c$ is

$$\lim_{h\to0}\frac{(c+h)^3-c^3}{h}=\lim_{h\to0}\frac{3c^2h+3ch^2+h^3}{h}=3c^2.$$

If $c=\sqrt{(a^2+ab+b^2)/3}$, then $3c^2=a^2+ab+b^2$, so the average rate of change over $[a,b]$ is the instantaneous rate of change at $\sqrt{(a^2+ab+b^2)/3}$.
Claim: $\sqrt{(a^2+ab+b^2)/3}>(a+b)/2$.
Proof: Since $a^2-2ab+b^2=(a-b)^2>0$, we have

$$\begin{aligned}
4a^2+4ab+4b^2&>3a^2+6ab+3b^2\\
\frac{a^2+ab+b^2}{3}&>\frac{a^2+2ab+b^2}{4}=\left(\frac{a+b}{2}\right)^2\\
\sqrt{\frac{a^2+ab+b^2}{3}}&>\frac{a+b}{2}.
\end{aligned}$$

2. For x near 0 we have $|x-1|=1-x$ and $|x+1|=x+1$. Thus

$$\lim_{x\to0}\frac{x}{|x-1|-|x+1|}=\lim_{x\to0}\frac{x}{(1-x)-(x+1)}=-\frac{1}{2}.$$

3. For x near 3 we have $|5-2x| = 2x-5$, $|x-2| = x-2$, $|x-5| = 5-x$, and $|3x-7| = 3x-7$. Thus

$$\begin{aligned}\lim_{x\to 3}\frac{|5-2x|-|x-2|}{|x-5|-|3x-7|} &= \lim_{x\to 3}\frac{2x-5-(x-2)}{5-x-(3x-7)}\\ &= \lim_{x\to 3}\frac{x-3}{4(3-x)} = -\frac{1}{4}.\end{aligned}$$

4. Let $y = x^{1/6}$. Then we have

$$\begin{aligned}\lim_{x\to 64}\frac{x^{1/3}-4}{x^{1/2}-8} &= \lim_{y\to 2}\frac{y^2-4}{y^3-8}\\ &= \lim_{y\to 2}\frac{(y-2)(y+2)}{(y-2)(y^2+2y+4)}\end{aligned}$$

$$= \lim_{y\to 2}\frac{y+2}{y^2+2y+4} = \frac{4}{12} = \frac{1}{3}.$$

5. Use $a-b = \dfrac{a^3-b^3}{a^2+ab+b^2}$ to handle the denominator. We have

$$\begin{aligned}&\lim_{x\to 1}\frac{\sqrt{3+x}-2}{\sqrt[3]{7+x}-2}\\ &= \lim_{x\to 1}\frac{3+x-4}{\sqrt{3+x}+2}\times\frac{(7+x)^{2/3}+2(7+x)^{1/3}+4}{(7+x)-8}\\ &= \lim_{x\to 1}\frac{(7+x)^{2/3}+2(7+x)^{1/3}+4}{\sqrt{3+x}+2} = \frac{4+4+4}{2+2} = 3.\end{aligned}$$

6. $r_+(a) = \dfrac{-1+\sqrt{1+a}}{a}$, $r_-(a) = \dfrac{-1-\sqrt{1+a}}{a}$.

a) $\lim_{a\to 0} r_-(a)$ does not exist. Observe that the right limit is $-\infty$ and the left limit is ∞.

b) From the following table it appears that $\lim_{a\to 0} r_+(a) = 1/2$, the solution of the linear equation $2x-1=0$ which results from setting $a=0$ in the quadratic equation $ax^2+2x-1=0$.

a	$r_+(a)$
1	0.41421
0.1	0.48810
−0.1	0.51317
0.01	0.49876
−0.01	0.50126
0.001	0.49988
−0.001	0.50013

c) $$\begin{aligned}\lim_{a\to 0} r_+(a) &= \lim_{a\to 0}\frac{\sqrt{1+a}-1}{a}\\ &= \lim_{a\to 0}\frac{(1+a)-1}{a(\sqrt{1+a}+1)}\\ &= \lim_{a\to 0}\frac{1}{\sqrt{1+a}+1} = \frac{1}{2}.\end{aligned}$$

7. TRUE or FALSE

a) If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.
TRUE, because if $\lim_{x\to a}\big(f(x)+g(x)\big)$ were to exist then

$$\begin{aligned}\lim_{x\to a} g(x) &= \lim_{x\to a}\Big(f(x)+g(x)-f(x)\Big)\\ &= \lim_{x\to a}\Big(f(x)+g(x)\Big) - \lim_{x\to a} f(x)\end{aligned}$$

would also exist.

b) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}\big(f(x)+g(x)\big)$ does not exist.
FALSE. Neither $\lim_{x\to 0} 1/x$ nor $\lim_{x\to 0}(-1/x)$ exist, but $\lim_{x\to 0}\big((1/x)+(-1/x)\big) = \lim_{x\to 0} 0 = 0$ exists.

c) If f is continuous at a, then so is $|f|$.
TRUE. For any two real numbers u and v we have

$$\Big||u|-|v|\Big| \le |u-v|.$$

This follows from

$$\begin{aligned}&|u| = |u-v+v| \le |u-v|+|v|, \quad \text{and}\\ &|v| = |v-u+u| \le |v-u|+|u| = |u-v|+|u|.\end{aligned}$$

Now we have

$$\Big||f(x)|-|f(a)|\Big| \le |f(x)-f(a)|$$

so the left side approaches zero whenever the right side does. This happens when $x\to a$ by the continuity of f at a.

d) If $|f|$ is continuous at a, then so is f.
FALSE. The function $f(x) = \begin{cases}-1 & \text{if } x<0\\ 1 & \text{if } x\ge 0\end{cases}$ is discontinuous at $x=0$, but $|f(x)| = 1$ everywhere, and so is continuous at $x=0$.

e) If $f(x) < g(x)$ in an interval around a and if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ both exist, then $L < M$.
FALSE. Let $g(x) = \begin{cases}x^2 & \text{if } x\ne 0\\ 1 & \text{if } x=0\end{cases}$ and let $f(x) = -g(x)$. Then $f(x) < g(x)$ for all x, but $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} g(x)$. (Note: under the given conditions, it is TRUE that $L \le M$, but not necessarily true that $L < M$.)

8. a) To be proved: if f is a continuous function defined on a closed interval $[a, b]$, then the range of f is a closed interval.
Proof: By the Max-Min Theorem there exist numbers u and v in $[a, b]$ such that $f(u) \le f(x) \le f(v)$ for all x in $[a, b]$. By the Intermediate-Value Theorem, $f(x)$ takes on all values between $f(u)$ and $f(v)$ at values of x between u and v, and hence at points of $[a, b]$. Thus the range of f is $[f(u), f(v)]$, a closed interval.

b) If the domain of the continuous function f is an open interval, the range of f can be any interval (open, closed, half open, finite, or infinite).

9. $f(x) = \dfrac{x^2 - 1}{|x^2 - 1|} = \begin{cases} -1 & \text{if } -1 < x < 1 \\ 1 & \text{if } x < -1 \text{ or } x > 1 \end{cases}.$
f is continuous wherever it is defined, that is at all points except $x = \pm 1$. f has left and right limits -1 and 1, respectively, at $x = 1$, and has left and right limits 1 and -1, respectively, at $x = -1$. It is not, however, discontinuous at any point, since -1 and 1 are not in its domain.

10. $f(x) = \dfrac{1}{x - x^2} = \dfrac{1}{\frac{1}{4} - \left(\frac{1}{4} - x + x^2\right)} = \dfrac{1}{\frac{1}{4} - \left(x - \frac{1}{2}\right)^2}.$
Observe that $f(x) \ge f(1/2) = 4$ for all x in $(0, 1)$.

11. Suppose f is continuous on $[0, 1]$ and $f(0) = f(1)$.

a) To be proved: $f(a) = f(a + \frac{1}{2})$ for some a in $[0, \frac{1}{2}]$.
Proof: If $f(1/2) = f(0)$ we can take $a = 0$ and be done. If not, let

$$g(x) = f(x + \tfrac{1}{2}) - f(x).$$

Then $g(0) \ne 0$ and

$$g(1/2) = f(1) - f(1/2) = f(0) - f(1/2) = -g(0).$$

Since g is continuous and has opposite signs at $x = 0$ and $x = 1/2$, the Intermediate-Value Theorem assures us that there exists a between 0 and 1/2 such that $g(a) = 0$, that is, $f(a) = f(a + \frac{1}{2})$.

b) To be proved: if $n > 2$ is an integer, then $f(a) = f(a + \frac{1}{n})$ for some a in $[0, 1 - \frac{1}{n}]$.
Proof: Let $g(x) = f(x + \frac{1}{n}) - f(x)$. Consider the numbers $x = 0$, $x = 1/n$, $x = 2/n$, ..., $x = (n - 1)/n$. If $g(x) = 0$ for any of these numbers, then we can let a be that number. Otherwise, $g(x) \ne 0$ at any of these numbers. Suppose that the values of g at all these numbers has the same sign (say positive). Then we have

$$f(1) > f(\tfrac{n-1}{n}) > \cdots > f(\tfrac{2}{n}) > \tfrac{1}{n} > f(0),$$

which is a contradiction, since $f(0) = f(1)$. Therefore there exists j in the set $\{0, 1, 2, \ldots, n - 1\}$ such that $g(j/n)$ and $g((j + 1)/n)$ have opposite sign. By the Intermediate-Value Theorem, $g(a) = 0$ for some a between j/n and $(j + 1)/n$, which is what we had to prove.