CHAPTER P. PRELIMINARIES

Section P.1 Real Numbers and the Real Line (page 10)

1. $\frac{2}{9} = 0.22222222\cdots = 0.\overline{2}$

2. $\frac{1}{11} = 0.09090909\cdots = 0.\overline{09}$

3. If $x = 0.121212\cdots$, then $100x = 12.121212\cdots = 12 + x$. Thus $99x = 12$ and $x = 12/99 = 4/33$.

4. If $x = 3.277777\cdots$, then $10x - 32 = 0.77777\cdots$ and $100x - 320 = 7 + (10x - 32)$, or $90x = 295$. Thus $x = 295/90 = 59/18$.

5. $1/7 = 0.142857142857\cdots = 0.\overline{142857}$
$2/7 = 0.285714285714\cdots = 0.\overline{285714}$
$3/7 = 0.428571428571\cdots = 0.\overline{428571}$
$4/7 = 0.571428571428\cdots = 0.\overline{571428}$
note the same cyclic order of the repeating digits
$5/7 = 0.714285714285\cdots = 0.\overline{714285}$
$6/7 = 0.857142857142\cdots = 0.\overline{857142}$

6. Two different decimal expansions can represent the same number. For instance, both $0.999999\cdots = 0.\overline{9}$ and $1.000000\cdots = 1.\overline{0}$ represent the number 1.

7. $x \ge 0$ and $x \le 5$ define the interval $[0, 5]$.

8. $x < 2$ and $x \ge -3$ define the interval $[-3, 2)$.

9. $x > -5$ or $x < -6$ defines the union $(-\infty, -6) \cup (-5, \infty)$.

10. $x \le -1$ defines the interval $(-\infty, -1]$.

11. $x > -2$ defines the interval $(-2, \infty)$.

12. $x < 4$ or $x \ge 2$ defines the interval $(-\infty, \infty)$, that is, the whole real line.

13. If $-2x > 4$, then $x < -2$. Solution: $(-\infty, -2)$

14. If $3x + 5 \le 8$, then $3x \le 8 - 5 - 3$ and $x \le 1$. Solution: $(-\infty, 1]$

15. If $5x - 3 \le 7 - 3x$, then $8x \le 10$ and $x \le 5/4$. Solution: $(-\infty, 5/4]$

16. If $\frac{6-x}{4} \ge \frac{3x-4}{2}$, then $6 - x \ge 6x - 8$. Thus $14 \ge 7x$ and $x \le 2$. Solution: $(-\infty, 2]$

17. If $3(2 - x) < 2(3 + x)$, then $0 < 5x$ and $x > 0$. Solution: $(0, \infty)$

18. If $x^2 < 9$, then $|x| < 3$ and $-3 < x < 3$. Solution: $(-3, 3)$

19. Given: $1/(2 - x) < 3$.
CASE I. If $x < 2$, then $1 < 3(2 - x) = 6 - 3x$, so $3x < 5$ and $x < 5/3$. This case has solutions $x < 5/3$.
CASE II. If $x > 2$, then $1 > 3(2-x) = 6-3x$, so $3x > 5$ and $x > 5/3$. This case has solutions $x > 2$.
Solution: $(-\infty, 5/3) \cup (2, \infty)$.

20. Given: $(x + 1)/x \ge 2$.
CASE I. If $x > 0$, then $x + 1 \ge 2x$, so $x \le 1$.
CASE II. If $x < 0$, then $x + 1 \le 2x$, so $x \ge 1$. (not possible)
Solution: $(0, 1]$.

21. Given: $x^2 - 2x \le 0$. Then $x(x - 2) \le 0$. This is only possible if $x \ge 0$ and $x \le 2$. Solution: $[0, 2]$.

22. Given $6x^2 - 5x \le -1$, then $(2x - 1)(3x - 1) \le 0$, so either $x \le 1/2$ and $x \ge 1/3$, or $x \le 1/3$ and $x \ge 1/2$. The latter combination is not possible. The solution set is $[1/3, 1/2]$.

23. Given $x^3 > 4x$, we have $x(x^2 - 4) > 0$. This is possible if $x < 0$ and $x^2 < 4$, or if $x > 0$ and $x^2 > 4$. The possibilities are, therefore, $-2 < x < 0$ or $2 < x < \infty$. Solution: $(-2, 0) \cup (2, \infty)$.

24. Given $x^2 - x \le 2$, then $x^2 - x - 2 \le 0$ so $(x-2)(x+1) \le 0$. This is possible if $x \le 2$ and $x \ge -1$ or if $x \ge 2$ and $x \le -1$. The latter situation is not possible. The solution set is $[-1, 2]$.

25. Given: $\frac{x}{2} \ge 1 + \frac{4}{x}$.
CASE I. If $x > 0$, then $x^2 \ge 2x + 8$, so that $x^2 - 2x - 8 \ge 0$, or $(x - 4)(x + 2) \ge 0$. This is possible for $x > 0$ only if $x \ge 4$.
CASE II. If $x < 0$, then we must have $(x-4)(x+2) \le 0$, which is possible for $x < 0$ only if $x \ge -2$.
Solution: $[-2, 0) \cup [4, \infty)$.

26. Given: $\frac{3}{x-1} < \frac{2}{x+1}$.
CASE I. If $x > 1$ then $(x - 1)(x + 1) > 0$, so that $3(x+1) < 2(x-1)$. Thus $x < -5$. There are no solutions in this case.
CASE II. If $-1 < x < 1$, then $(x - 1)(x + 1) < 0$, so $3(x + 1) > 2(x - 1)$. Thus $x > -5$. In this case all numbers in $(-1, 1)$ are solutions.
CASE III. If $x < -1$, then $(x - 1)(x + 1) > 0$, so that $3(x + 1) < 2(x - 1)$. Thus $x < -5$. All numbers $x < -5$ are solutions.
Solutions: $(-\infty, -5) \cup (-1, 1)$.

27. If $|x| = 3$ then $x = \pm 3$.

28. If $|x - 3| = 7$, then $x - 3 = \pm 7$, so $x = -4$ or $x = 10$.

29. If $|2t + 5| = 4$, then $2t + 5 = \pm 4$, so $t = -9/2$ or $t = -1/2$.

30. If$|1 - t| = 1$, then $1 - t = \pm 1$, so $t = 0$ or $t = 2$.

31. If $|8-3s|=9$, then $8-3s=\pm 9$, so $3s=-1$ or 17, and $s=-1/3$ or $s=17/3$.

32. If $\left|\frac{s}{2}-1\right|=1$, then $\frac{s}{2}-1=\pm 1$, so $s=0$ or $s=4$.

33. If $|x|<2$, then x is in $(-2,2)$.

34. If $|x|\le 2$, then x is in $[-2,2]$.

35. If $|s-1|\le 2$, then $1-2\le s\le 1+2$, so s is in $[-1,3]$.

36. If $|t+2|<1$, then $-2-1<t<-2+1$, so t is in $(-3,-1)$.

37. If $|3x-7|<2$, then $7-2<3x<7+2$, so x is in $(5/3,3)$.

38. If $|2x+5|<1$, then $-5-1<2x<-5+1$, so x is in $(-3,-2)$.

39. If $\left|\frac{x}{2}-1\right|\le 1$, then $1-1\le \frac{x}{2}\le 1+1$, so x is in $[0,4]$.

40. If $\left|2-\frac{x}{2}\right|<\frac{1}{2}$, then $x/2$ lies between $2-(1/2)$ and $2+(1/2)$. Thus x is in $(3,5)$.

41. The inequality $|x+1|>|x-3|$ says that the distance from x to -1 is greater than the distance from x to 3, so x must be to the right of the point half-way between -1 and 3. Thus $x>1$.

42. $|x-3|<2|x| \Leftrightarrow x^2-6x+9=(x-3)^2<4x^2$ $\Leftrightarrow 3x^2+6x-9>0 \Leftrightarrow 3(x+3)(x-1)>0$. This inequality holds if $x<-3$ or $x>1$.

43. $|a|=a$ if and only if $a\ge 0$. It is false if $a<0$.

44. The equation $|x-1|=1-x$ holds if $|x-1|=-(x-1)$, that is, if $x-1<0$, or, equivalently, if $x<1$.

45. The triangle inequality $|x+y|\le |x|+|y|$ implies that

$$|x|\ge |x+y|-|y|.$$

Apply this inequality with $x=a-b$ and $y=b$ to get

$$|a-b|\ge |a|-|b|.$$

Similarly, $|a-b|=|b-a|\ge |b|-|a|$. Since $\big||a|-|b|\big|$ is equal to either $|a|-|b|$ or $|b|-|a|$, depending on the sizes of a and b, we have

$$|a-b|\ge \big||a|-|b|\big|.$$

Section P.2 Cartesian Coordinates in the Plane (page 16)

1. From $A(0,3)$ to $B(4,0)$, $\Delta x=4-0=4$ and $\Delta y=0-3=-3$. $|AB|=\sqrt{4^2+(-3)^2}=5$.

2. From $A(-1,2)$ to $B(4,-10)$, $\Delta x=4-(-1)=5$ and $\Delta y=-10-2=-12$. $|AB|=\sqrt{5^2+(-12)^2}=13$.

3. From $A(3,2)$ to $B(-1,-2)$, $\Delta x=-1-3=-4$ and $\Delta y=-2-2=-4$. $|AB|=\sqrt{(-4)^2+(-4)^2}=4\sqrt{2}$.

4. From $A(0.5,3)$ to $B(2,3)$, $\Delta x=2-0.5=1.5$ and $\Delta y=3-3=0$. $|AB|=1.5$.

5. Starting point: $(-2,3)$. Increments $\Delta x=4$, $\Delta y=-7$. New position is $(-2+4,3+(-7))$, that is, $(2,-4)$.

6. Arrival point: $(-2,-2)$. Increments $\Delta x=-5$, $\Delta y=1$. Starting point was $(-2-(-5),-2-1)$, that is, $(3,-3)$.

7. $x^2+y^2=1$ represents a circle of radius 1 centred at the origin.

8. $x^2+y^2=2$ represents a circle of radius $\sqrt{2}$ centred at the origin.

9. $x^2+y^2\le 1$ represents points inside and on the circle of radius 1 centred at the origin.

10. $x^2+y^2=0$ represents the origin.

11. $y\ge x^2$ represents all points lying on or above the parabola $y=x^2$.

12. $y<x^2$ represents all points lying below the parabola $y=x^2$.

13. The vertical line through $(-2,5/3)$ is $x=-2$; the horizontal line through that point is $y=5/3$.

14. The vertical line through $(\sqrt{2},-1.3)$ is $x=\sqrt{2}$; the horizontal line through that point is $y=-1.3$.

15. Line through $(-1,1)$ with slope $m=1$ is $y=1+1(x+1)$, or $y=x+2$.

16. Line through $(-2,2)$ with slope $m=1/2$ is $y=2+(1/2)(x+2)$, or $x-2y=-6$.

17. Line through $(0,b)$ with slope $m=2$ is $y=b+2x$.

18. Line through $(a,0)$ with slope $m=-2$ is $y=0-2(x-a)$, or $y=2a-2x$.

19. At $x=2$, the height of the line $2x+3y=6$ is $y=(6-4)/3=2/3$. Thus $(2,1)$ lies above the line.

20. At $x=3$, the height of the line $x-4y=7$ is $y=(3-7)/4=-1$. Thus $(3,-1)$ lies on the line.

21. The line through $(0,0)$ and $(2,3)$ has slope $m=(3-0)/(2-0)=3/2$ and equation $y=(3/2)x$ or $3x-2y=0$.

22. The line through $(-2,1)$ and $(2,-2)$ has slope $m=(-2-1)/(2+2)=-3/4$ and equation $y=1-(3/4)(x+2)$ or $3x+4y=-2$.

23. The line through $(4,1)$ and $(-2,3)$ has slope $m=(3-1)/(-2-4)=-1/3$ and equation $y=1-\frac{1}{3}(x-4)$ or $x+3y=7$.

24. The line through $(-2, 0)$ and $(0, 2)$ has slope $m = (2 - 0)/(0 + 2) = 1$ and equation $y = 2 + x$.

25. If $m = -2$ and $b = \sqrt{2}$, then the line has equation $y = -2x + \sqrt{2}$.

26. If $m = -1/2$ and $b = -3$, then the line has equation $y = -(1/2)x - 3$, or $x + 2y = -6$.

27. $3x + 4y = 12$ has x-intercept $a = 12/3 = 4$ and y-intercept $b = 12/4 = 3$. Its slope is $-b/a = -3/4$.

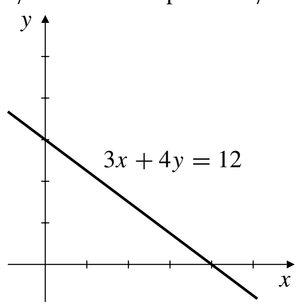


Fig. P.2.27

28. $x + 2y = -4$ has x-intercept $a = -4$ and y-intercept $b = -4/2 = -2$. Its slope is $-b/a = 2/(-4) = -1/2$.

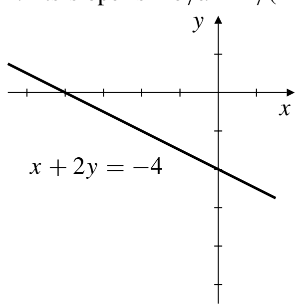


Fig. P.2.28

29. $\sqrt{2}x - \sqrt{3}y = 2$ has x-intercept $a = 2/\sqrt{2} = \sqrt{2}$ and y-intercept $b = -2/\sqrt{3}$. Its slope is $-b/a = 2/\sqrt{6} = \sqrt{2/3}$.

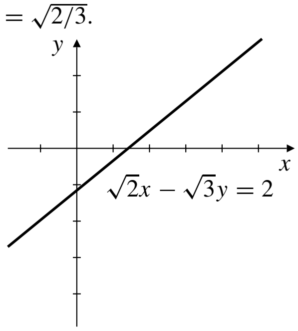


Fig. P.2.29

30. $1.5x - 2y = -3$ has x-intercept $a = -3/1.5 = -2$ and y-intercept $b = -3/(-2) = 3/2$. Its slope is $-b/a = 3/4$.

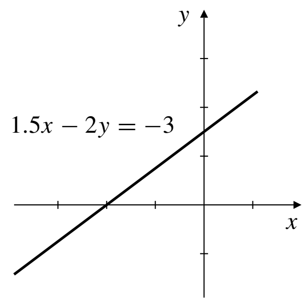


Fig. P.2.30

31. line through $(2, 1)$ parallel to $y = x + 2$ is $y = x - 1$; line perpendicular to $y = x + 2$ is $y = -x + 3$.

32. line through $(-2, 2)$ parallel to $2x + y = 4$ is $2x + y = -2$; line perpendicular to $2x + y = 4$ is $x - 2y = -6$.

33. We have

$$\begin{aligned} 3x + 4y &= -6 & \implies \quad 6x + 8y &= -12 \\ 2x - 3y &= 13 & 6x - 9y &= 39. \end{aligned}$$

Subtracting these equations gives $17y = -51$, so $y = -3$ and $x = (13-9)/2 = 2$. The intersection point is $(2, -3)$.

34. We have

$$\begin{aligned} 2x + y &= 8 & \implies \quad 14x + 7y &= 56 \\ 5x - 7y &= 1 & 5x - 7y &= 1. \end{aligned}$$

Adding these equations gives $19x = 57$, so $x = 3$ and $y = 8 - 2x = 2$. The intersection point is $(3, 2)$.

35. If $a \neq 0$ and $b \neq 0$, then $(x/a) + (y/b) = 1$ represents a straight line that is neither horizontal nor vertical, and does not pass through the origin. Putting $y = 0$ we get $x/a = 1$, so the x-intercept of this line is $x = a$; putting $x = 0$ gives $y/b = 1$, so the y-intercept is $y = b$.

36. The line $(x/2) - (y/3) = 1$ has x-intercept $a = 2$, and y-intercept $b = -3$.

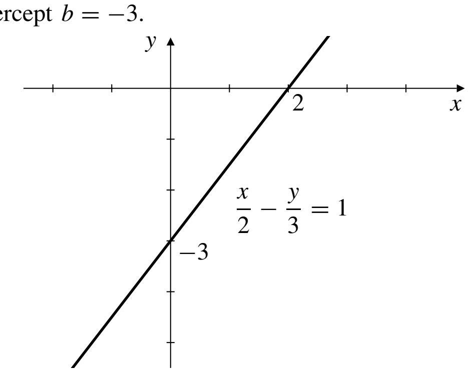


Fig. P.2.36

37. The line through $(2, 1)$ and $(3, -1)$ has slope $m = (-1 - 1)/(3 - 2) = -2$ and equation $y = 1 - 2(x - 2) = 5 - 2x$. Its y-intercept is 5.

38. The line through $(-2,5)$ and $(k,1)$ has x-intercept 3, so also passes through $(3,0)$. Its slope m satisfies

$$\frac{1-0}{k-3} = m = \frac{0-5}{3+2} = -1.$$

Thus $k-3=-1$, and so $k=2$.

39. $C = Ax+B$. If $C = 5,000$ when $x = 10,000$ and $C = 6,000$ when $x = 15,000$, then

$$10,000A + B = 5,000$$
$$15,000A + B = 6,000$$

Subtracting these equations gives $5,000A = 1,000$, so $A = 1/5$. From the first equation, $2,000 + B = 5,000$, so $B = 3,000$. The cost of printing 100,000 pamphlets is $\$100,000/5 + 3,000 = \$23,000$.

40. $-40°$ and $-40°$ is the same temperature on both the Fahrenheit and Celsius scales.

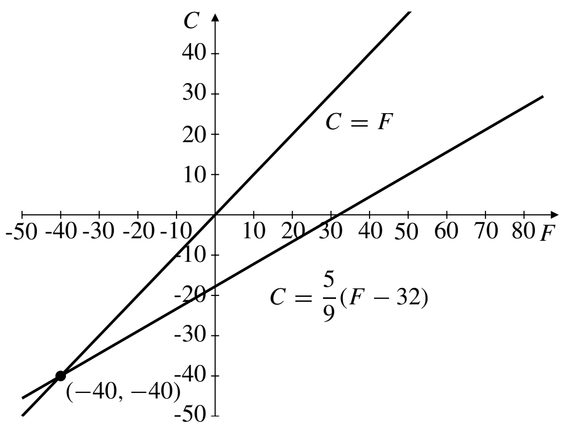


Fig. P.2.40

41. $A=(2,1),\quad B=(6,4),\quad C=(5,-3)$

$$|AB| = \sqrt{(6-2)^2+(4-1)^2} = \sqrt{25} = 5$$
$$|AC| = \sqrt{(5-2)^2+(-3-1)^2} = \sqrt{25} = 5$$
$$|BC| = \sqrt{(6-5)^2+(4+3)^2} = \sqrt{50} = 5\sqrt{2}.$$

Since $|AB| = |AC|$, triangle ABC is isosceles.

42. $A=(0,0),\quad B=(1,\sqrt{3}),\quad C=(2,0)$

$$|AB| = \sqrt{(1-0)^2+(\sqrt{3}-0)^2} = \sqrt{4} = 2$$
$$|AC| = \sqrt{(2-0)^2+(0-0)^2} = \sqrt{4} = 2$$
$$|BC| = \sqrt{(2-1)^2+(0-\sqrt{3})^2} = \sqrt{4} = 2.$$

Since $|AB| = |AC| = |BC|$, triangle ABC is equilateral.

43. $A=(2,-1),\quad B=(1,3),\quad C=(-3,2)$

$$|AB| = \sqrt{(1-2)^2+(3+1)^2} = \sqrt{17}$$
$$|AC| = \sqrt{(-3-2)^2+(2+1)^2} = \sqrt{34} = \sqrt{2}\sqrt{17}$$
$$|BC| = \sqrt{(-3-1)^2+(2-3)^2} = \sqrt{17}.$$

Since $|AB| = |BC|$ and $|AC| = \sqrt{2}|AB|$, triangle ABC is an isosceles right-angled triangle with right angle at B. Thus $ABCD$ is a square if D is displaced from C by the same amount A is from B, that is, by increments $\Delta x = 2-1 = 1$ and $\Delta y = -1-3 = -4$. Thus $D = (-3+1, 2+(-4)) = (-2,-2)$.

44. If $M = (x_m, y_m)$ is the midpoint of P_1P_2, then the displacement of M from P_1 equals the displacement of P_2 from M:

$$x_m - x_1 = x_2 - x_m, \quad y_m - y_1 = y_2 - y_m.$$

Thus $x_m = (x_1+x_2)/2$ and $y_m = (y_1+y_2)/2$.

45. If $Q = (x_q, y_q)$ is the point on P_1P_2 that is two thirds of the way from P_1 to P_2, then the displacement of Q from P_1 equals twice the displacement of P_2 from Q:

$$x_q - x_1 = 2(x_2 - x_q), \quad y_q - y_1 = 2(y_2 - y_q).$$

Thus $x_q = (x_1+2x_2)/3$ and $y_q = (y_1+2y_2)/3$.

46. Let the coordinates of P be $(x,0)$ and those of Q be $(X,-2X)$. If the midpoint of PQ is $(2,1)$, then

$$(x+X)/2 = 2, \quad (0-2X)/2 = 1.$$

The second equation implies that $X=-1$, and the second then implies that $x=5$. Thus P is $(5,0)$.

47. $\sqrt{(x-2)^2+y^2} = 4$ says that the distance of (x,y) from $(2,0)$ is 4, so the equation represents a circle of radius 4 centred at $(2,0)$.

48. $\sqrt{(x-2)^2+y^2} = \sqrt{x^2+(y-2)^2}$ says that (x,y) is equidistant from $(2,0)$ and $(0,2)$. Thus (x,y) must lie on the line that is the right bisector of the line from $(2,0)$ to $(0,2)$. A simpler equation for this line is $x=y$.

49. The line $2x+ky=3$ has slope $m = -2/k$. This line is perpendicular to $4x+y=1$, which has slope -4, provided $m = 1/4$, that is, provided $k=-8$. The line is parallel to $4x+y=1$ if $m=-4$, that is, if $k=1/2$.

50. For any value of k, the coordinates of the point of intersection of $x+2y=3$ and $2x-3y=-1$ will also satisfy the equation

$$(x+2y-3) + k(2x-3y+1) = 0$$

because they cause both expressions in parentheses to be 0. The equation above is linear in x and y, and so represents a straight line for any choice of k. This line will pass through $(1, 2)$ provided $1+4-3+k(2-6+1)=0$, that is, if $k = 2/3$. Therefore, the line through the point of intersection of the two given lines and through the point $(1, 2)$ has equation

$$x + 2y - 3 + \frac{2}{3}(2x - 3y + 1) = 0,$$

or, on simplification, $x = 1$.

Section P.3 Graphs of Quadratic Equations (page 22)

1. $x^2 + y^2 = 16$

2. $x^2 + (y-2)^2 = 4$, or $x^2 + y^2 - 4y = 0$

3. $(x+2)^2 + y^2 = 9$, or $x^2 + y^2 + 4y = 5$

4. $(x-3)^2 + (y+4)^2 = 25$, or $x^2 + y^2 - 6x + 8y = 0$.

5. $x^2 + y^2 - 2x = 3$
$x^2 - 2x + 1 + y^2 = 4$
$(x-1)^2 + y^2 = 4$
centre: $(1, 0)$; radius 2.

6. $x^2 + y^2 + 4y = 0$
$x^2 + y^2 + 4y + 4 = 4$
$x^2 + (y+2)^2 = 4$
centre: $(0, -2)$; radius 2.

7. $x^2 + y^2 - 2x + 4y = 4$
$x^2 - 2x + 1 + y^2 + 4y + 4 = 9$
$(x-1)^2 + (y+2)^2 = 9$
centre: $(1, -2)$; radius 3.

8. $x^2 + y^2 - 2x - y + 1 = 0$
$x^2 - 2x + 1 + y^2 - y + \frac{1}{4} = \frac{1}{4}$
$(x-1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{1}{4}$
centre: $(1, 1/2)$; radius $1/2$.

9. $x^2 + y^2 > 1$ represents all points lying outside the circle of radius 1 centred at the origin.

10. $x^2 + y^2 < 4$ represents the open disk consisting of all points lying inside the circle of radius 2 centred at the origin.

11. $(x+1)^2 + y^2 \le 4$ represents the closed disk consisting of all points lying inside or on the circle of radius 2 centred at the point $(-1, 0)$.

12. $x^2 + (y-2)^2 \le 4$ represents the closed disk consisting of all points lying inside or on the circle of radius 2 centred at the point $(0, 2)$.

13. Together, $x^2 + y^2 > 1$ and $x^2 + y^2 < 4$ represent annulus (washer-shaped region) consisting of all points that are outside the circle of radius 1 centred at the origin and inside the circle of radius 2 centred at the origin.

14. Together, $x^2 + y^2 \le 4$ and $(x+2)^2 + y^2 \le 4$ represent the region consisting of all points that are inside or on both the circle of radius 2 centred at the origin and the circle of radius 2 centred at $(-2, 0)$.

15. Together, $x^2 + y^2 < 2x$ and $x^2 + y^2 < 2y$ (or, equivalently, $(x-1)^2 + y^2 < 1$ and $x^2 + (y-1)^2 < 1$) represent the region consisting of all points that are inside both the circle of radius 1 centred at $(1, 0)$ and the circle of radius 1 centred at $(0, 1)$.

16. $x^2 + y^2 - 4x + 2y > 4$ can be rewritten $(x-2)^2 + (y+1)^2 > 9$. This equation, taken together with $x + y > 1$, represents all points that lie both outside the circle of radius 3 centred at $(2, -1)$ and above the line $x + y = 1$.

17. The interior of the circle with centre $(-1, 2)$ and radius $\sqrt{6}$ is given by $(x+1)^2 + (y-2)^2 < 6$, or $x^2 + y^2 + 2x - 4y < 1$.

18. The exterior of the circle with centre $(2, -3)$ and radius 4 is given by $(x-2)^2 + (y+3)^2 > 16$, or $x^2 + y^2 - 4x + 6y > 3$.

19. $x^2 + y^2 < 2, \quad x \ge 1$

20. $x^2 + y^2 > 4, \quad (x-1)^2 + (y-3)^2 < 10$

21. The parabola with focus $(0, 4)$ and directrix $y = -4$ has equation $x^2 = 16y$.

22. The parabola with focus $(0, -1/2)$ and directrix $y = 1/2$ has equation $x^2 = -2y$.

23. The parabola with focus $(2, 0)$ and directrix $x = -2$ has equation $y^2 = 8x$.

24. The parabola with focus $(-1, 0)$ and directrix $x = 1$ has equation $y^2 = -4x$.

25. $y = x^2/2$ has focus $(0, 1/2)$ and directrix $y = -1/2$.

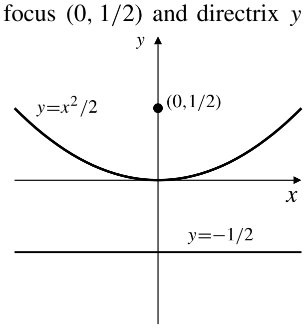


Fig. P.3.25

26. $y = -x^2$ has focus $(0, -1/4)$ and directrix $y = 1/4$.

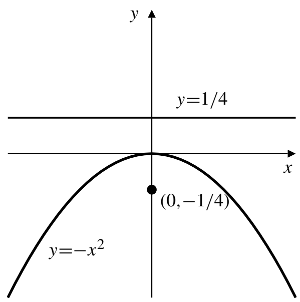


Fig. P.3.26

27. $x = -y^2/4$ has focus $(-1, 0)$ and directrix $x = 1$.

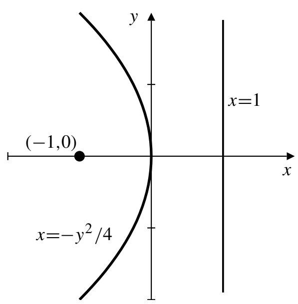


Fig. P.3.27

28. $x = y^2/16$ has focus $(4, 0)$ and directrix $x = -4$.

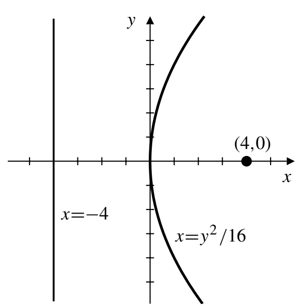


Fig. P.3.28

29.

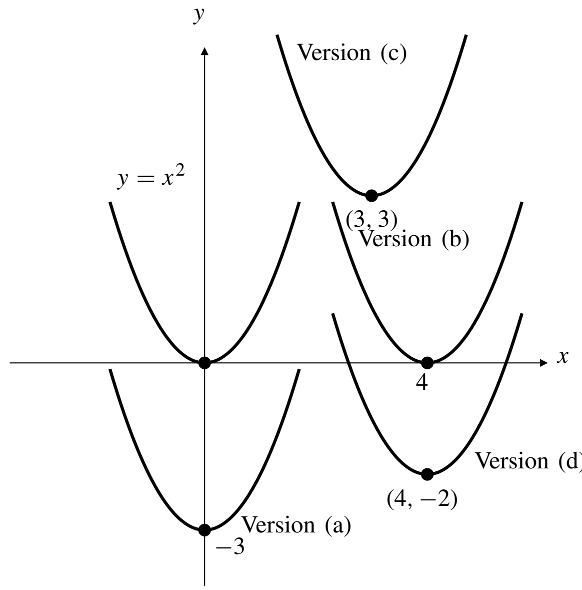


Fig. P.3.29

a) has equation $y = x^2 - 3$.

b) has equation $y = (x - 4)^2$ or $y = x^2 - 8x + 16$.

c) has equation $y = (x - 3)^2 + 3$ or $y = x^2 - 6x + 12$.

d) has equation $y = (x - 4)^2 - 2$, or $y = x^2 - 8x + 14$.

30. a) If $y = mx$ is shifted to the right by amount x_1, the equation $y = m(x - x_1)$ results. If (a, b) satisfies this equation, then $b = m(a - x_1)$, and so $x_1 = a - (b/m)$. Thus the shifted equation is
$y = m(x - a + (b/m)) = m(x - a) + b$.

b) If $y = mx$ is shifted vertically by amount y_1, the equation $y = mx + y_1$ results. If (a, b) satisfies this equation, then $b = ma + y_1$, and so $y_1 = b - ma$. Thus the shifted equation is $y = mx + b - ma = m(x - a) + b$, the same equation obtained in part (a).

31. $y = \sqrt{(x/3) + 1}$

32. $4y = \sqrt{x + 1}$

33. $y = \sqrt{(3x/2) + 1}$

34. $(y/2) = \sqrt{4x + 1}$

35. $y = 1 - x^2$ shifted down 1, left 1 gives $y = -(x + 1)^2$.

36. $x^2 + y^2 = 5$ shifted up 2, left 4 gives
$(x + 4)^2 + (y - 2)^2 = 5$.

37. $y = (x - 1)^2 - 1$ shifted down 1, right 1 gives
$y = (x - 2)^2 - 2$.

38. $y = \sqrt{x}$ shifted down 2, left 4 gives $y = \sqrt{x + 4} - 2$.

39. $y = x^2 + 3$, $y = 3x + 1$. Subtracting these equations gives
$x^2 - 3x + 2 = 0$, or $(x - 1)(x - 2) = 0$. Thus $x = 1$ or $x = 2$. The corresponding values of y are 4 and 7. The intersection points are $(1, 4)$ and $(2, 7)$.

40. $y = x^2 - 6$, $y = 4x - x^2$. Subtracting these equations gives
$2x^2 - 4x - 6 = 0$, or $2(x - 3)(x + 1) = 0$. Thus $x = 3$ or $x = -1$. The corresponding values of y are 3 and -5. The intersection points are $(3, 3)$ and $(-1, -5)$.

41. $x^2 + y^2 = 25$, $3x + 4y = 0$. The second equation says that $y = -3x/4$. Substituting this into the first equation gives $25x^2/16 = 25$, so $x = \pm 4$. If $x = 4$, then the second equation gives $y = -3$; if $x = -4$, then $y = 3$. The intersection points are $(4, -3)$ and $(-4, 3)$. Note that having found values for x, we substituted them into the linear equation rather than the quadratic equation to find the corresponding values of y. Had we substituted into the quadratic equation we would have got more solutions (four points in all), but two of them would have failed to satisfy $3x + 4y = 12$. When solving systems of nonlinear equations you should always verify that the solutions you find do satisfy the given equations.

42. $2x^2 + 2y^2 = 5$, $xy = 1$. The second equation says that $y = 1/x$. Substituting this into the first equation gives $2x^2 + (2/x^2) = 5$, or $2x^4 - 5x^2 + 2 = 0$. This equation factors to $(2x^2 - 1)(x^2 - 2) = 0$, so its solutions are $x = \pm 1/\sqrt{2}$ and $x = \pm\sqrt{2}$. The corresponding values of y are given by $y = 1/x$. Therefore, the intersection points are $(1/\sqrt{2}, \sqrt{2})$, $(-1/\sqrt{2}, -\sqrt{2})$, $(\sqrt{2}, 1/\sqrt{2})$, and $(-\sqrt{2}, -1/\sqrt{2})$.

43. $(x^2/4) + y^2 = 1$ is an ellipse with major axis between $(-2, 0)$ and $(2, 0)$ and minor axis between $(0, -1)$ and $(0, 1)$.

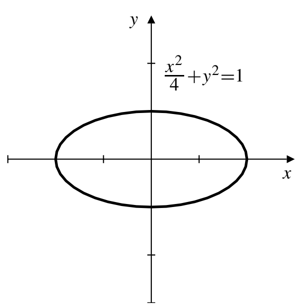


Fig. P.3.43

44. $9x^2 + 16y^2 = 144$ is an ellipse with major axis between $(-4, 0)$ and $(4, 0)$ and minor axis between $(0, -3)$ and $(0, 3)$.

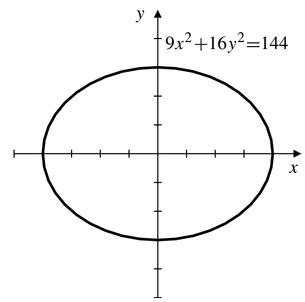


Fig. P.3.44

45. $\dfrac{(x-3)^2}{9} + \dfrac{(y+2)^2}{4} = 1$ is an ellipse with centre at $(3, -2)$, major axis between $(0, -2)$ and $(6, -2)$ and minor axis between $(3, -4)$ and $(3, 0)$.

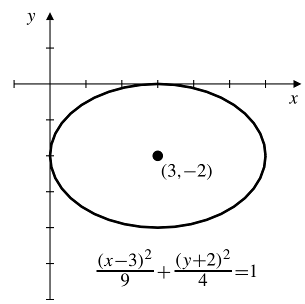


Fig. P.3.45

46. $(x - 1)^2 + \dfrac{(y+1)^2}{4} = 4$ is an ellipse with centre at $(1, -1)$, major axis between $(1, -5)$ and $(1, 3)$ and minor axis between $(-1, -1)$ and $(3, -1)$.

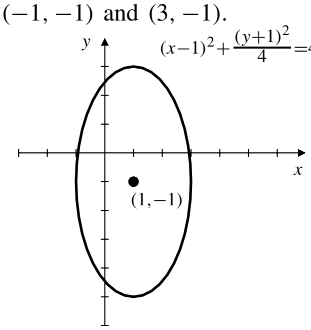


Fig. P.3.46

47. $(x^2/4) - y^2 = 1$ is a hyperbola with centre at the origin and passing through $(\pm 2, 0)$. Its asymptotes are $y = \pm x/2$.

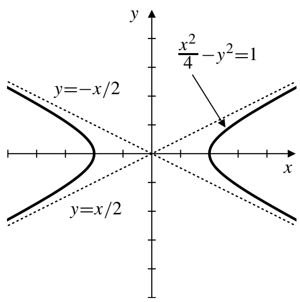


Fig. P.3.47

48. $x^2 - y^2 = -1$ is a rectangular hyperbola with centre at the origin and passing through $(0, \pm 1)$. Its asymptotes are $y = \pm x$.

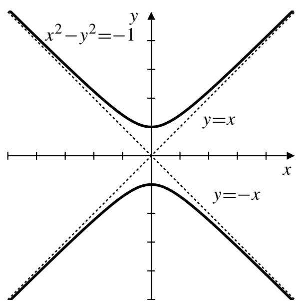


Fig. P.3.48

49. $xy = -4$ is a rectangular hyperbola with centre at the origin and passing through $(2, -2)$ and $(-2, 2)$. Its asymptotes are the coordinate axes.

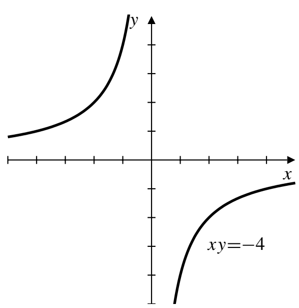


Fig. P.3.49

50. $(x - 1)(y + 2) = 1$ is a rectangular hyperbola with centre at $(1, -2)$ and passing through $(2, -1)$ and $(0, -3)$. Its asymptotes are $x = 1$ and $y = -2$.

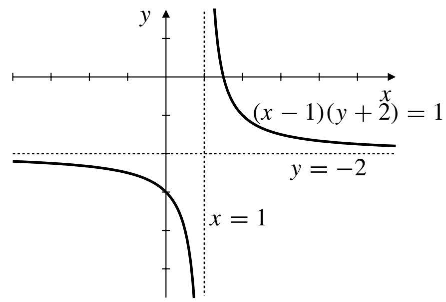


Fig. P.3.50

51. a) Replacing x with $-x$ replaces a graph with its reflection across the y-axis.

b) Replacing y with $-y$ replaces a graph with its reflection across the x-axis.

52. Replacing x with $-x$ and y with $-y$ reflects the graph in both axes. This is equivalent to rotating the graph 180° about the origin.

53. $|x| + |y| = 1$.
In the first quadrant the equation is $x + y = 1$.
In the second quadrant the equation is $-x + y = 1$.
In the third quadrant the equation is $-x - y = 1$.
In the fourth quadrant the equation is $x - y = 1$.

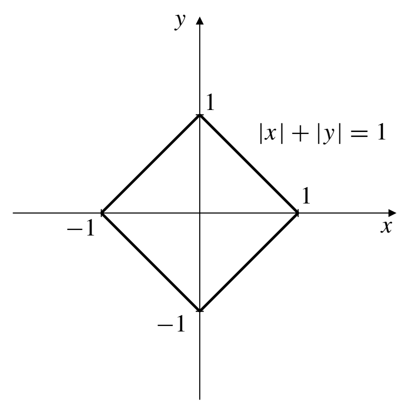


Fig. P.3.53

Section P.4 Functions and Their Graphs (page 31)

1. $f(x) = 1 + x^2$; domain $\mathbb{R}$, range $[1, \infty)$

2. $f(x) = 1 - \sqrt{x}$; domain $[0, \infty)$, range $(-\infty, 1]$

3. $G(x) = \sqrt{8 - 2x}$; domain $(-\infty, 4]$, range $[0, \infty)$

4. $F(x) = 1/(x - 1)$; domain $(-\infty, 1) \cup (1, \infty)$, range $(-\infty, 0) \cup (0, \infty)$

5. $h(t) = \dfrac{t}{\sqrt{2-t}}$; domain $(-\infty, 2)$, range $\mathbb{R}$. (The equation $y = h(t)$ can be squared and rewritten as $t^2 + y^2t - 2y^2 = 0$, a quadratic equation in t having real solutions for every real value of y. Thus the range of h contains all real numbers.)

6. $g(x) = \dfrac{1}{1-\sqrt{x-2}}$; domain $(2, 3) \cup (3, \infty)$, range $(-\infty, 0) \cup (0, \infty)$. The equation $y = g(x)$ can be solved for $x = 2 - (1 - (1/y))^2$ so has a real solution provided $y \neq 0$.

7.

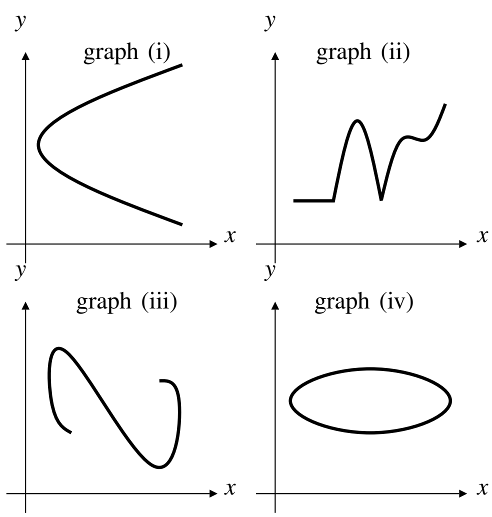


Fig. P.4.7

Graph (ii) is the graph of a function because vertical lines can meet the graph only once. Graphs (i), (iii), and (iv) do not have this property, so are not graphs of functions.

8.

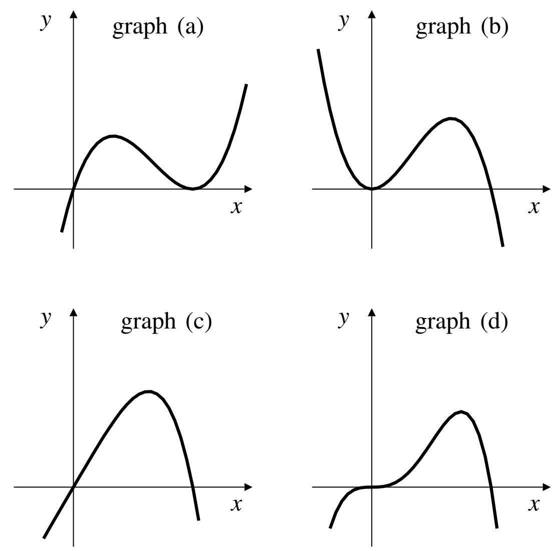


Fig. P.4.8

a) is the graph of $x(1-x)^2$, which is positive for $x > 0$.

b) is the graph of $x^2 - x^3 = x^2(1-x)$, which is positive if $x < 1$.

c) is the graph of $x - x^4$, which is positive if $0 < x < 1$ and behaves like x near 0.

d) is the graph of $x^3 - x^4$, which is positive if $0 < x < 1$ and behaves like x^3 near 0.

9.

x	$f(x) = x^4$
0	0
± 0.5	0.0625
± 1	1
± 1.5	5.0625
± 2	16

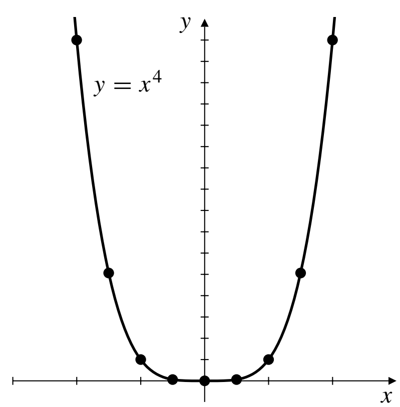


Fig. P.4.9

10.

x	$f(x) = x^{2/3}$
0	0
± 0.5	0.62996
± 1	1
± 1.5	1.3104
± 2	1.5874

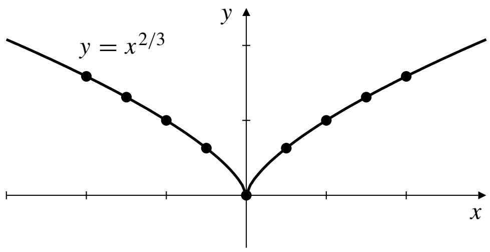


Fig. P.4.10

11. $f(x) = x^2 + 1$ is even: $f(-x) = f(x)$

12. $f(x) = x^3 + x$ is odd: $f(-x) = -f(x)$

13. $f(x) = \dfrac{x}{x^2 - 1}$ is odd: $f(-x) = -f(x)$

14. $f(x) = \dfrac{1}{x^2 - 1}$ is even: $f(-x) = f(x)$

15. $f(x) = \dfrac{1}{x - 2}$ is odd about $(2, 0)$: $f(2-x) = -f(2+x)$

16. $f(x) = \dfrac{1}{x + 4}$ is odd about $(-4, 0)$:
$f(-4-x) = -f(-4+x)$

17. $f(x) = x^2 - 6x$ is even about $x = 3$: $f(3-x) = f(3+x)$

18. $f(x) = x^3 - 2$ is odd about $(0, -2)$:
$f(-x) + 2 = -(f(x) + 2)$

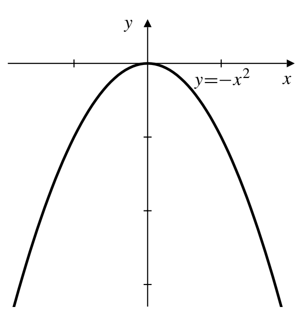
19. $f(x) = |x^3| = |x|^3$ is even: $f(-x) = f(x)$

20. $f(x) = |x + 1|$ is even about $x = -1$:
$f(-1-x) = f(-1+x)$

21. $f(x) = \sqrt{2x}$ has no symmetry.

22. $f(x) = \sqrt{(x-1)^2}$ is even about $x = 1$:
$f(1-x) = f(1+x)$

23.



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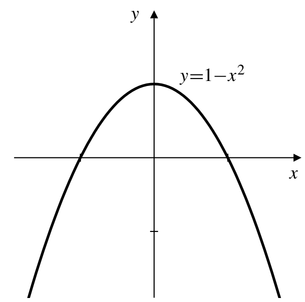

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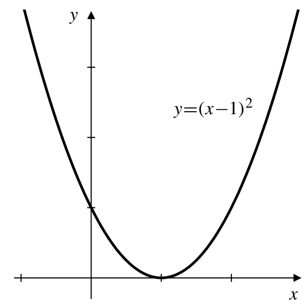

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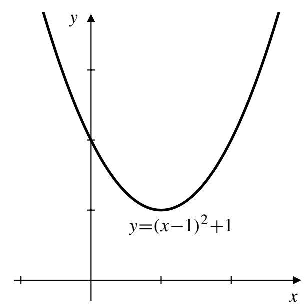

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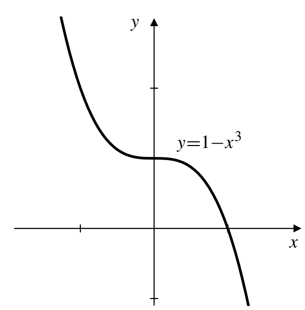

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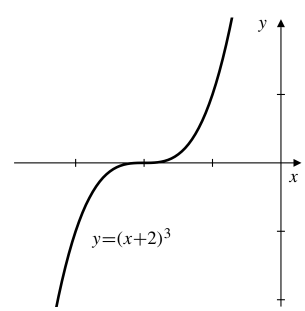

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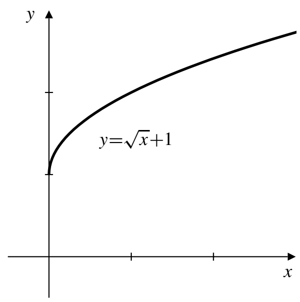

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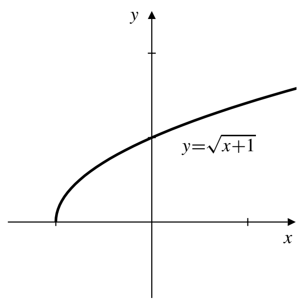

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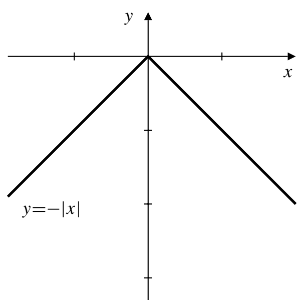

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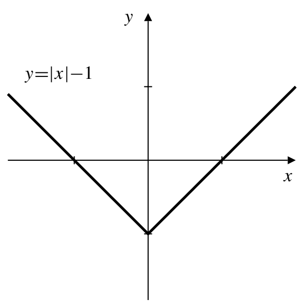

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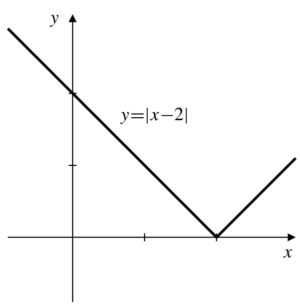

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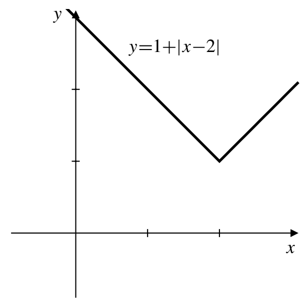

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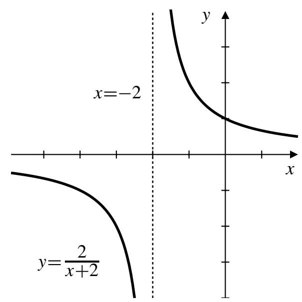

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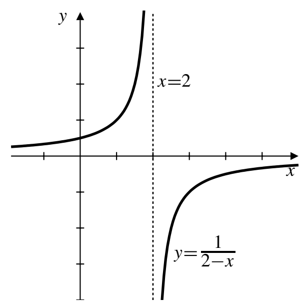

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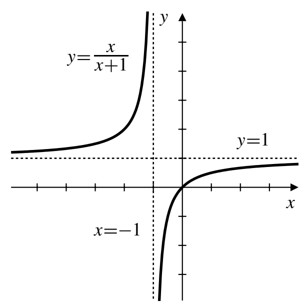

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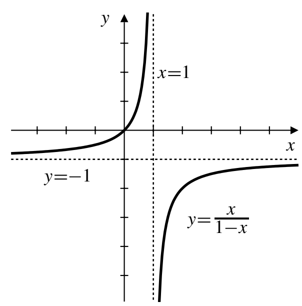

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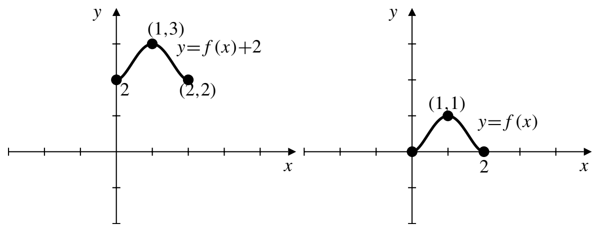


Fig. P.4.39(a) Fig. P.4.39(b)

40.

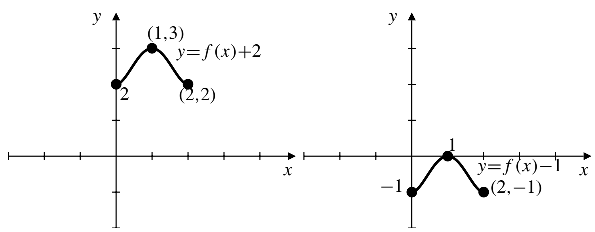
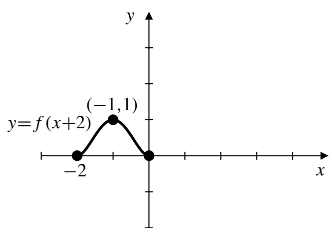


Fig. P.4.40(a) Fig. P.4.40(b)

41.



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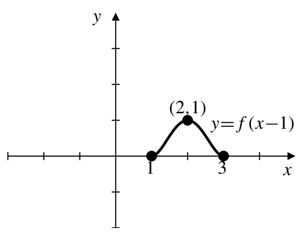

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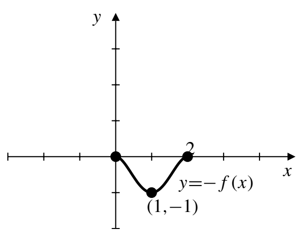

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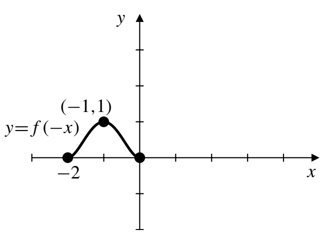

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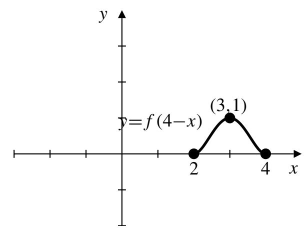

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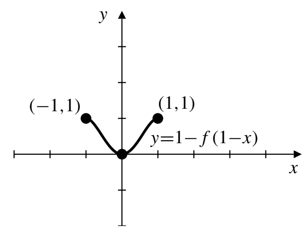


47. Range is approximately $[-0.18, 0.68]$.

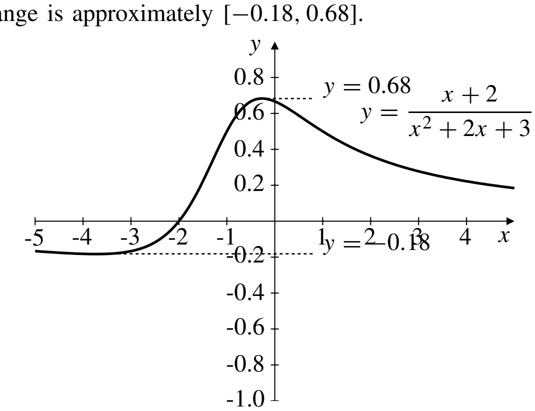


Fig. P.4.47

48. Range is approximately $(-\infty, 0.17]$.

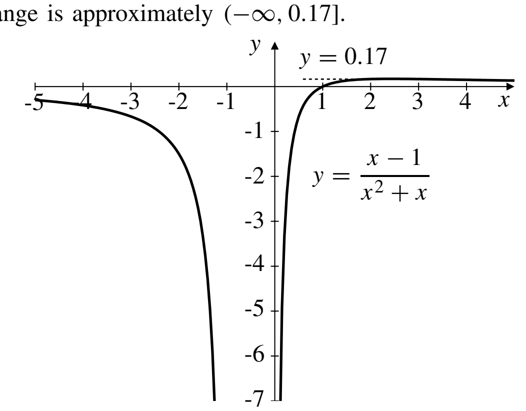


Fig. P.4.48

49.

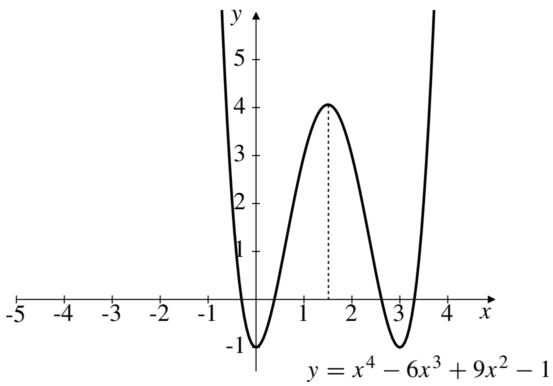


Fig. P.4.49

Apparent symmetry about $x = 1.5$.
This can be confirmed by calculating $f(3 - x)$, which turns out to be equal to $f(x)$.

50.

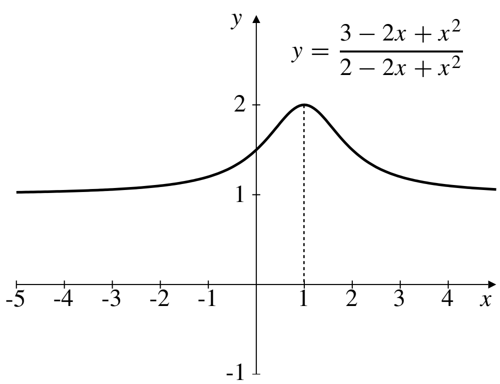


Fig. P.4.50

Apparent symmetry about $x = 1$.
This can be confirmed by calculating $f(2 - x)$, which turns out to be equal to $f(x)$.

51.

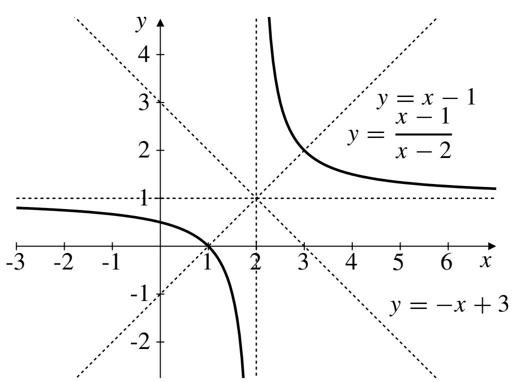


Fig. P.4.51

Apparent symmetry about $(2, 1)$, and about the lines $y = x - 1$ and $y = 3 - x$.

These can be confirmed by noting that $f(x) = 1 + \dfrac{1}{x - 2}$, so the graph is that of $1/x$ shifted right 2 units and up one.

52.

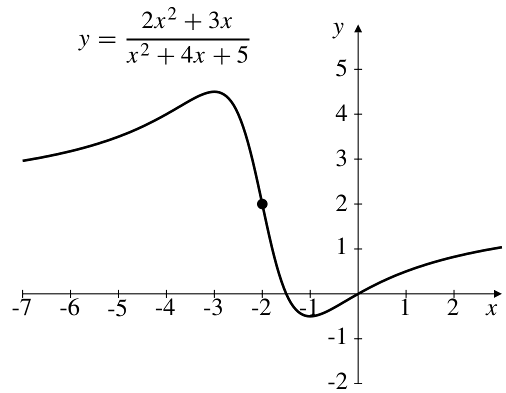


Fig. P.4.52

Apparent symmetry about $(-2, 2)$.
This can be confirmed by calculating shifting the graph right by 2 (replace x with $x - 2$) and then down 2 (subtract 2). The result is $-5x/(1 + x^2)$, which is odd.

53. If f is both even and odd the $f(x) = f(-x) = -f(x)$, so $f(x) = 0$ identically.

Section P.5 Combining Functions to Make New Functions (page 37)

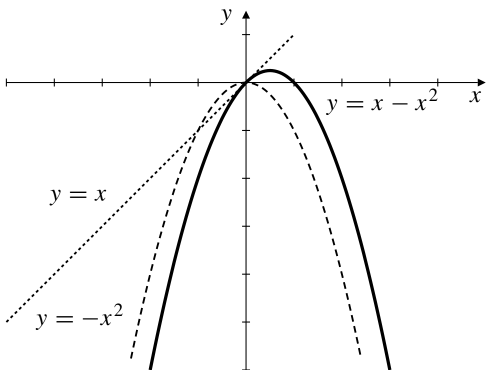
1. $f(x) = x$, $g(x) = \sqrt{x - 1}$.
$\mathcal{D}(f) = \mathbb{R}$, $\mathcal{D}(g) = [1, \infty)$.
$\mathcal{D}(f + g) = \mathcal{D}(f - g) = \mathcal{D}(fg) = \mathcal{D}(g/f) = [1, \infty)$,
$\mathcal{D}(f/g) = (1, \infty)$.

$$(f + g)(x) = x + \sqrt{x - 1}$$
$$(f - g)(x) = x - \sqrt{x - 1}$$
$$(fg)(x) = x\sqrt{x - 1}$$
$$(f/g)(x) = x/\sqrt{x - 1}$$
$$(g/f)(x) = (\sqrt{1 - x})/x$$

2. $f(x) = \sqrt{1 - x}$, $g(x) = \sqrt{1 + x}$.
$\mathcal{D}(f) = (-\infty, 1]$, $\mathcal{D}(g) = [-1, \infty)$.
$\mathcal{D}(f + g) = \mathcal{D}(f - g) = \mathcal{D}(fg) = [-1, 1]$,
$\mathcal{D}(f/g) = (-1, 1]$, $\mathcal{D}(g/f) = [-1, 1)$.

$$(f + g)(x) = \sqrt{1 - x} + \sqrt{1 + x}$$
$$(f - g)(x) = \sqrt{1 - x} - \sqrt{1 + x}$$
$$(fg)(x) = \sqrt{1 - x^2}$$
$$(f/g)(x) = \sqrt{(1 - x)/(1 + x)}$$
$$(g/f)(x) = \sqrt{(1 + x)/(1 - x)}$$

3.



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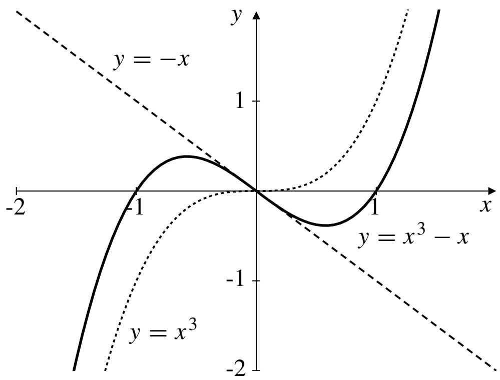

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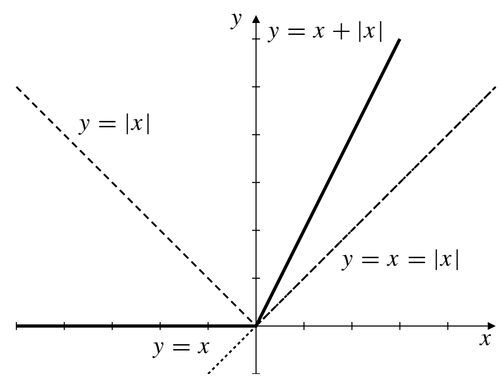

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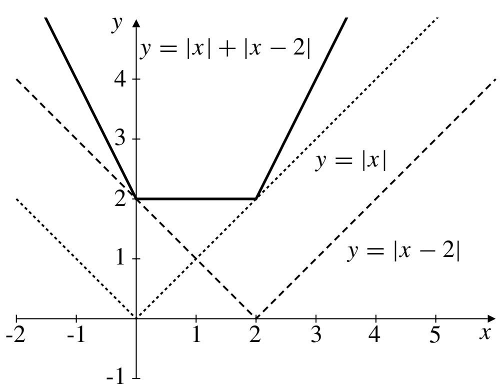


7. $f(x) = x + 5,\ g(x) = x^2 - 3.$

$f \circ g(0) = f(-3) = 2, \quad g(f(0)) = g(5) = 22$

$f(g(x)) = f(x^2 - 3) = x^2 + 2$

$g \circ f(x) = g(f(x)) = g(x + 5) = (x + 5)^2 - 3$

$f \circ f(-5) = f(0) = 5, \quad g(g(2)) = g(1) = -2$

$f(f(x)) = f(x + 5) = x + 10$

$g \circ g(x) = g(g(x)) = (x^2 - 3)^2 - 3$

8. $f(x) = 2/x,\ g(x) = x/(1 - x).$

$f \circ f(x) = 2/(2/x) = x; \quad \mathcal{D}(f \circ f) = \{x\ :\ x \neq 0\}$

$f \circ g(x) = 2/(x/(1 - x)) = 2(1 - x)/x;$

$\mathcal{D}(f \circ g) = \{x\ :\ x \neq 0, 1\}$

$g \circ f(x) = (2/x)/(1 - (2/x)) = 2/(x - 2);$

$\mathcal{D}(g \circ f) = \{x\ :\ x \neq 0, 2\}$

$g \circ g(x) = (x/(1 - x))/(1 - (x/(1 - x))) = x/(1 - 2x);$

$\mathcal{D}(g \circ g) = \{x\ :\ x \neq 1/2, 1\}$

9. $f(x) = 1/(1 - x),\ g(x) = \sqrt{x - 1}.$

$f \circ f(x) = 1/(1 - (1/(1 - x))) = (x - 1)/x;$

$\mathcal{D}(f \circ f) = \{x\ :\ x \neq 0, 1\}$

$f \circ g(x) = 1/(1 - \sqrt{x - 1});$

$\mathcal{D}(f \circ g) = \{x\ :\ x \geq 1,\ x \neq 2\}$

$g \circ f(x) = \sqrt{(1/(1 - x)) - 1} = \sqrt{x/(1 - x)};$

$\mathcal{D}(g \circ f) = [0, 1)$

$g \circ g(x) = \sqrt{\sqrt{x - 1} - 1}; \quad \mathcal{D}(g \circ g) = [2, \infty)$

10. $f(x) = (x + 1)/(x - 1) = 1 + 2/(x - 1),\ g(x) = \operatorname{sgn}(x).$

$f \circ f(x) = 1 + 2/(1 + (2/(x - 1)) - 1)) = x;$

$\mathcal{D}(f \circ f) = \{x\ :\ x \neq 1\}$

$$f \circ g(x) = \frac{\operatorname{sgn} x + 1}{\operatorname{sgn} x - 1} = 0; \quad \mathcal{D}(f \circ g) = (-\infty, 0)$$

$$g \circ f(x) = \operatorname{sgn}\left(\frac{x + 1}{x - 1}\right) = \begin{cases} 1 & \text{if } x < -1 \text{ or } x > 1 \\ -1 & \text{if } -1 < x < 1 \end{cases};$$

$\mathcal{D}(g \circ f) = \{x\ :\ x \neq -1,\ 1\}$

$g \circ g(x) = \operatorname{sgn}(\operatorname{sgn}(x)) = \operatorname{sgn}(x); \quad \mathcal{D}(g \circ g) = \{x\ :\ x \neq 0\}$

	$f(x)$	$g(x)$	$f \circ g(x)$
11.	x^2	$x + 1$	$(x + 1)^2$
12.	$x - 4$	$x + 4$	x
13.	$\sqrt{x}$	x^2	$\|x\|$
14.	$2x^3 + 3$	$x^{1/3}$	$2x + 3$
15.	$(x + 1)/x$	$1/(x - 1)$	x
16.	$1/(x + 1)^2$	$x - 1$	$1/x^2$

17. $y = \sqrt{x}.$

$y = 2 + \sqrt{x}$: previous graph is raised 2 units.

$y = 2 + \sqrt{3 + x}$: previous graph is shiftend left 3 units.

$y = 1/(2 + \sqrt{3 + x})$: previous graph turned upside down and shrunk vertically.

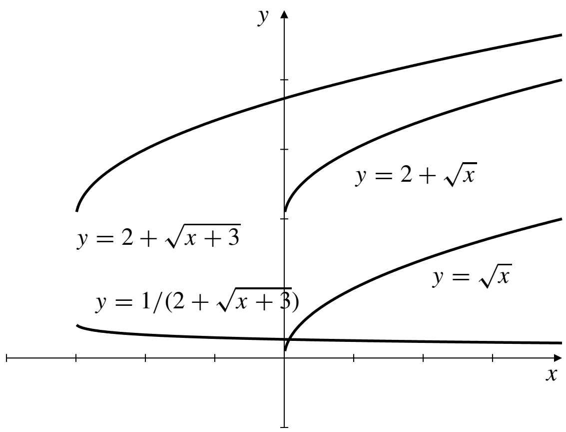


Fig. P.5.17

18.

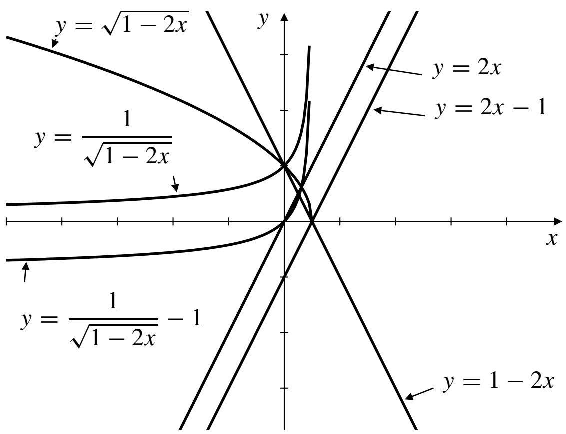
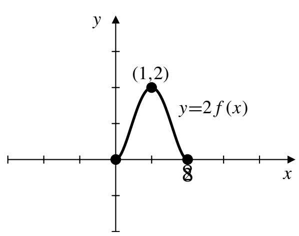


Fig. P.5.18

19.



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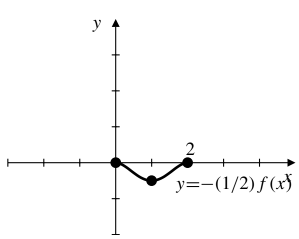

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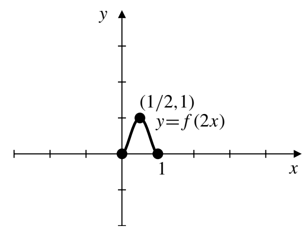

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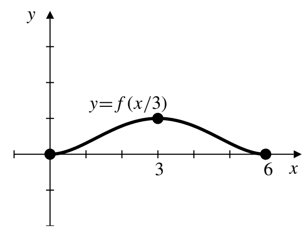

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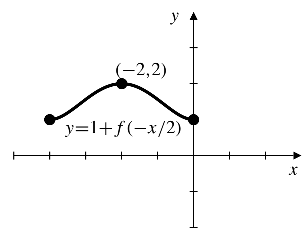

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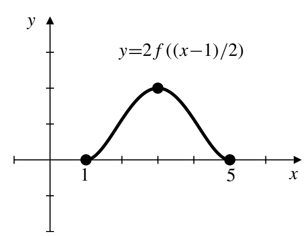

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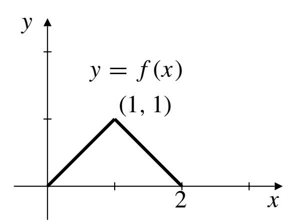

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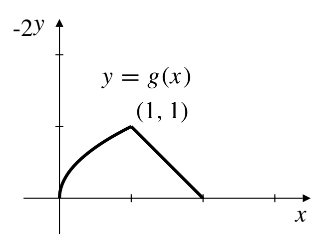


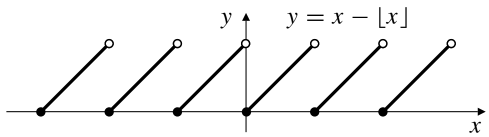
27. $F(x) = Ax + B$
(a) $F \circ F(x) = F(x)$
$\Rightarrow \ A(Ax + B) + B = Ax + B$
$\Rightarrow \ A[(A - 1)x + B] = 0$
Thus, either $A = 0$ or $A = 1$ and $B = 0$.
(b) $F \circ F(x) = x$
$\Rightarrow \ A(Ax + B) + B = x$
$\Rightarrow \ (A^2 - 1)x + (A + 1)B = 0$
Thus, either $A = -1$ or $A = 1$ and $B = 0$

28. $\lfloor x \rfloor = 0$ for $0 \le x < 1$; $\lceil x \rceil = 0$ for $-1 \le x < 0$.

29. $\lfloor x \rfloor = \lceil x \rceil$ for all integers x.

30. $\lceil -x \rceil = -\lfloor x \rfloor$ is true for all real x; if $x = n + y$ where n is an integer and $0 \le y < 1$, then $-x = -n - y$, so that $\lceil -x \rceil = -n$ and $\lfloor x \rfloor = n$.

31.



32. $f(x)$ is called the integer part of x because $|f(x)|$ is the largest integer that does not exceed x; i.e. $|x| = |f(x)| + y$, where $0 \le y < 1$.

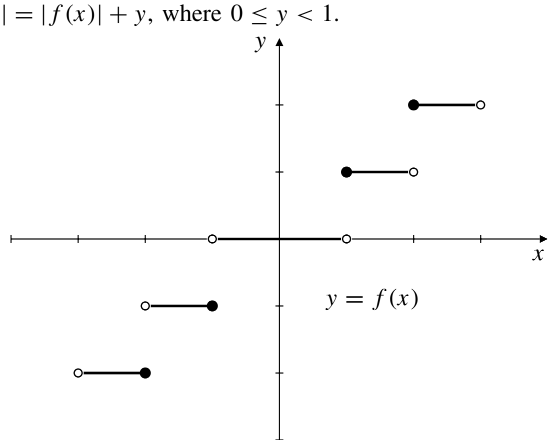


Fig. P.5.32

33. If f is even and g is odd, then: f^2, g^2, $f \circ g$, $g \circ f$, and $f \circ f$ are all even. fg, f/g, g/f, and $g \circ g$ are odd, and $f + g$ is neither even nor odd. Here are two typical verifications:

$$f \circ g(-x) = f(g(-x)) = f(-g(x)) = f(g(x)) = f \circ g(x)$$
$$(fg)(-x) = f(-x)g(-x) = f(x)[-g(x)]$$
$$= -f(x)g(x) = -(fg)(x).$$

The others are similar.

34. f even $\Leftrightarrow f(-x) = f(x)$
f odd $\Leftrightarrow f(-x) = -f(x)$
f even and odd $\Rightarrow f(x) = -f(x) \Rightarrow 2f(x) = 0$
$\Rightarrow f(x) = 0$

35. a) Let $E(x) = \frac{1}{2}[f(x) + f(-x)]$.
Then $E(-x) = \frac{1}{2}[f(-x) + f(x)] = E(x)$. Hence, $E(x)$ is even.
Let $O(x) = \frac{1}{2}[f(x) - f(-x)]$.
Then $O(-x) = \frac{1}{2}[f(-x) - f(x)] = -O(x)$ and $O(x)$ is odd.

$$\begin{aligned} &E(x) + O(x) \\ &\quad = \tfrac{1}{2}[f(x) + f(-x)] + \tfrac{1}{2}[f(x) - f(-x)] \\ &\quad = f(x). \end{aligned}$$

Hence, $f(x)$ is the sum of an even function and an odd function.

b) If $f(x) = E_1(x) + O_1(x)$ where E_1 is even and O_1 is odd, then

$$E_1(x) + O_1(x) = f(x) = E(x) + O(x).$$

Thus $E_1(x) - E(x) = O(x) - O_1(x)$. The left side of this equation is an even function and the right side is an odd function. Hence both sides are both even and odd, and are therefore identically 0 by Exercise 36. Hence $E_1 = E$ and $O_1 = O$. This shows that f can be written in only one way as the sum of an even function and an odd function.

Section P.6 Polynomials and Rational Functions (page 43)

1. $x^2 - 7x + 10 = (x + 5)(x + 2)$
The roots are -5 and -2.

2. $x^2 - 3x - 10 = (x - 5)(x + 2)$
The roots are 5 and -2.

3. If $x^2 + 2x + 2 = 0$, then $x = \dfrac{-2 \pm \sqrt{4 - 8}}{2} = -1 \pm i$.
The roots are $-1 + i$ and $-1 - i$.
$x^2 + 2x + 2 = (x + 1 - i)(x + 1 + i)$.

4. Rather than use the quadratic formula this time, let us complete the square.

$$\begin{aligned} x^2 - 6x + 13 &= x^2 - 6x + 9 + 4 \\ &= (x - 3)^2 + 2^2 \\ &= (x - 3 - 2i)(x - 3 + 2i). \end{aligned}$$

The roots are $3 + 2i$ and $3 - 2i$.

5. $16x^4 - 8x^2 + 1 = (4x^2 - 1)^2 = (2x - 1)^2(2x + 1)^2$. There are two double roots: $1/2$ and $-1/2$.

6. $x^4 + 6x^3 + 9x^2 = x^2(x^2 + 6x + 9) = x^2(x + 3)^2$. There are two double roots, 0 and -3.

7. $x^3+1=(x+1)(x^2-x+1)$. One root is -1. The other two are the solutions of $x^2-x+1=0$, namely

$$x=\frac{1\pm\sqrt{1-4}}{2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i.$$

We have

$$x^3+1=(x+1)\left(x-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(x-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right).$$

8. $x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x-i)(x+i)$. The roots are 1, -1, i, and $-i$.

9. $x^6-3x^4+3x^2-1=(x^2-1)^3=(x-1)^3(x+1)^3$. The roots are 1 and -1, each with multiplicity 3.

10.
$$\begin{aligned}x^5-x^4-16x+16&=(x-1)(x^4-16)\\&=(x-1)(x^2-4)(x^4+4)\\&=(x-1)(x-2)(x+2)(x-2i)(x+2i).\end{aligned}$$

The roots are 1, 2, -2, $2i$, and $-2i$.

11.
$$\begin{aligned}x^5+x^3+8x^2+8&=(x^2+1)(x^3+8)\\&=(x+2)(x-i)(x+i)(x^2-2x+4)\end{aligned}$$
Three of the five roots are -2, i and $-i$. The remaining two are solutions of $x^2-2x+4=0$, namely $x=\dfrac{2\pm\sqrt{4-16}}{2}=1\pm\sqrt{3}i$. We have

$$x^5+x^3+8x^2+8=(x+2)(x-i)(x+i)(x-a+\sqrt{3}i)(x-a-\sqrt{3}i).$$

12.
$$\begin{aligned}x^9-4x^7-x^6+4x^4&=x^4(x^5-x^2-4x^3+4)\\&=x^4(x^3-1)(x^2-4)\\&=x^4(x-1)(x-2)(x+2)(x^2+x+1).\end{aligned}$$

Seven of the nine roots are: 0 (with multiplicity 4), 1, 2, and -2. The other two roots are solutions of $x^2+x+1=0$, namely

$$x=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i.$$

The required factorization of $x^9-4x^7-x^6+4x^4$ is

$$x^4(x-1)(x-2)(x+2)\left(x-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(x-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right).$$

13.
$$\begin{aligned}\frac{x^3-1}{x^2-2}&=\frac{x^3-2x+2x-1}{x^2-2}\\&=\frac{x(x^2-2)+2x-1}{x^2-2}\\&=x+\frac{2x-1}{x^2-2}.\end{aligned}$$

14.
$$\begin{aligned}\frac{x^2}{x^2+5x+3}&=\frac{x^2+5x+3-5x-3}{x^2+5x+3}\\&=1+\frac{-5x-3}{x^2+5x+3}.\end{aligned}$$

15.
$$\begin{aligned}\frac{x^3}{x^2+2x+3}&=\frac{x^3+2x^2+3x-2x^2-3x}{x^2+2x+3}\\&=\frac{x(x^2+2x+3)-2x^2-3x}{x^2+2x+3}\\&=x-\frac{2(x^2+2x+3)-4x-6-3x}{x^2+2x+3}\\&=x-2+\frac{7x+6}{x^2+2x+3}.\end{aligned}$$

16.
$$\begin{aligned}\frac{x^4+x^2}{x^3+x^2+1}&=\frac{x(x^3+x^2+1)-x^3-x+x^2}{x^3+x^2+1}\\&=x+\frac{-(x^3+x^2+1)+x^2+1-x+x^2}{x^3+x^2+1}\\&=x-1+\frac{2x^2-x+1}{x^3+x^2+1}.\end{aligned}$$

17. Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $n\ge 1$. By the Factor Theorem, $x-1$ is a factor of $P(x)$ if and only if $P(1)=0$, that is, if and only if $a_n+a_{n-1}+\cdots+a_1+a_0=0$.

18. Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where $n\ge 1$. By the Factor Theorem, $x+1$ is a factor of $P(x)$ if and only if $P(-1)=0$, that is, if and only if $a_0-a_1+a_2-a_3+\cdots+(-1)^na_n=0$. This condition says that the sum of the coefficients of even powers is equal to the sum of coefficients of odd powers.

19. Let $P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, where the coefficients a_k, $0\le k\le n$ are all real numbers, so that $\bar{a_k}=a_k$. Using the facts about conjugates of sums and products mentioned in the statement of the problem, we see that if $z=x+iy$, where x and y are real, then

$$\begin{aligned}\overline{P(z)}&=\overline{a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0}\\&=a_n\bar{z}^n+a_{n-1}\bar{z}^{n-1}+\cdots+a_1\bar{z}+a_0\\&=P(\bar{z}).\end{aligned}$$

If z is a root of P, then $P(\bar{z})=\overline{P(z)}=\bar{0}=0$, and $\bar{z}$ is also a root of P.

20. By the previous exercise, $\bar{z}=u-iv$ is also a root of P. Therefore $P(x)$ has two linear factors $x-u-iv$ and $x-u+iv$. The product of these factors is the real quadratic factor $(x-u)^2-i^2v^2=x^2-2ux+u^2+v^2$, which must also be a factor of $P(x)$.

21. By the previous exercise

$$\frac{P(x)}{x^2-2ux+u^2+v^2}=\frac{P(x)}{(x-u-iv)(x-u+iv)}=Q_1(x),$$

where Q_1, being a quotient of two polynomials with real coefficients, must also have real coefficients. If $z = u+iv$ is a root of P having multiplicity $m > 1$, then it must also be a root of Q_1 (of multiplicity $m-1$), and so, therefore, $\bar{z}$ must be a root of Q_1, as must be the real quadratic $x^2 - 2ux + u^2 + v^2$. Thus

$$\frac{P(x)}{(x^2 - 2ux + u^2 + v^2)^2} = \frac{Q_1(x)}{x^2 - 2ux + u^2 + v^2} = Q_2(x),$$

where Q_2 is a polynomial with real coefficients. We can continue in this way until we get

$$\frac{P(x)}{(x^2 - 2ux + u^2 + v^2)^m} = Q_m(x),$$

where Q_m no longer has z (or $\bar{z}$) as a root. Thus z and $\bar{z}$ must have the same multiplicity as roots of P.

Section P.7 The Trigonometric Functions (page 55)

1. $\cos\left(\frac{3\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

2. $\tan\frac{-3\pi}{4} = -\tan\frac{3\pi}{4} = -1$

3. $\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

4.
$$\begin{aligned}\sin\left(\frac{7\pi}{12}\right) &= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\\ &= \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3}\\ &= \frac{1}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} = \frac{1+\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

5.
$$\begin{aligned}\cos\frac{5\pi}{12} &= \cos\left(\frac{2\pi}{3} - \frac{\pi}{4}\right)\\ &= \cos\frac{2\pi}{3}\cos\frac{\pi}{4} + \sin\frac{2\pi}{3}\sin\frac{\pi}{4}\\ &= -\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

6.
$$\begin{aligned}\sin\frac{11\pi}{12} &= \sin\frac{\pi}{12}\\ &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right)\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

7.
$$\begin{aligned}\cos(\pi + x) &= \cos\Big(2\pi - (\pi - x)\Big)\\ &= \cos\Big(-(\pi - x)\Big)\\ &= \cos(\pi - x) = -\cos x\end{aligned}$$

8. $\sin(2\pi - x) = -\sin x$

9.
$$\begin{aligned}\sin\left(\frac{3\pi}{2} - x\right) &= \sin\left(\pi - \left(x - \frac{\pi}{2}\right)\right)\\ &= \sin\left(x - \frac{\pi}{2}\right)\\ &= -\sin\left(\frac{\pi}{2} - x\right)\\ &= -\cos x\end{aligned}$$

10.
$$\begin{aligned}\cos\left(\frac{3\pi}{2} + x\right) &= \cos\frac{3\pi}{2}\cos x - \sin\frac{3\pi}{2}\sin x\\ &= (-1)(-\sin x) = \sin x\end{aligned}$$

11.
$$\begin{aligned}\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}\\ &= \frac{1}{\cos x \sin x}\end{aligned}$$

12.
$$\begin{aligned}\frac{\tan x - \cot x}{\tan x + \cot x} &= \frac{\left(\frac{\sin x}{\cos x} - \frac{\cos x}{\sin x}\right)}{\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)}\\ &= \frac{\left(\frac{\sin^2 x - \cos^2 x}{\cos x \sin x}\right)}{\left(\frac{\sin^2 x + \cos^2 x}{\cos x \sin x}\right)}\\ &= \sin^2 x - \cos^2 x\end{aligned}$$

13.
$$\begin{aligned}\cos^4 x - \sin^4 x &= (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)\\ &= \cos^2 x - \sin^2 x = \cos(2x)\end{aligned}$$

14. $(1 - \cos x)(1 + \cos x) = 1 - \cos^2 x = \sin^2 x$ implies $\frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$. Now

$$\begin{aligned}\frac{1 - \cos x}{\sin x} &= \frac{1 - \cos 2\left(\frac{x}{2}\right)}{\sin 2\left(\frac{x}{2}\right)}\\ &= \frac{1 - \left(1 - 2\sin^2\left(\frac{x}{2}\right)\right)}{2\sin\frac{x}{2}\cos\frac{x}{2}}\\ &= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} = \tan\frac{x}{2}\end{aligned}$$

15. $\frac{1 - \cos x}{1 + \cos x} = \frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)} = \tan^2\left(\frac{x}{2}\right)$

16.
$$\begin{aligned}\frac{\cos x - \sin x}{\cos x + \sin x} &= \frac{(\cos x - \sin x)^2}{(\cos x + \sin x)(\cos x - \sin x)}\\ &= \frac{\cos^2 x - 2\sin x\cos x + \sin^2 x}{\cos^2 x - \sin^2 x}\\ &= \frac{1-\sin(2x)}{\cos(2x)}\\ &= \sec(2x) - \tan(2x)\end{aligned}$$

17.
$$\begin{aligned}\sin 3x &= \sin(2x + x)\\ &= \sin 2x\cos x + \cos 2x \sin x\\ &= 2\sin x\cos^2 x + \sin x(1 - 2\sin^2 x)\\ &= 2\sin x(1-\sin^2 x) + \sin x - 2\sin^3 x\\ &= 3\sin x - 4\sin^3 x\end{aligned}$$

18.
$$\begin{aligned}\cos 3x &= \cos(2x + x)\\ &= \cos 2x\cos x - \sin 2x\sin x\\ &= (2\cos^2 x - 1)\cos x - 2\sin^2 x\cos x\\ &= 2\cos^3 x - \cos x - 2(1-\cos^2 x)\cos x\\ &= 4\cos^3 x - 3\cos x\end{aligned}$$

19. $\cos 2x$ has period π.

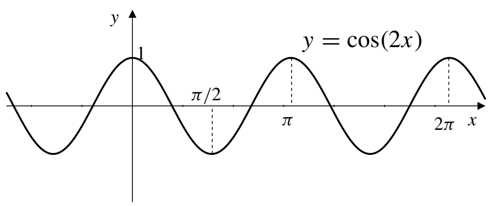


Fig. P.7.19

20. $\sin\frac{x}{2}$ has period 4π.

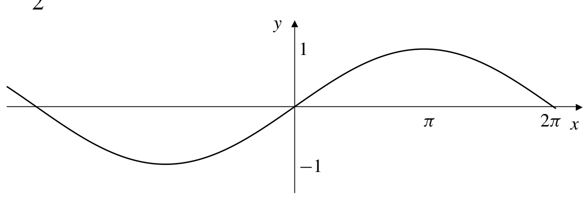


Fig. P.7.20

21. $\sin \pi x$ has period 2.

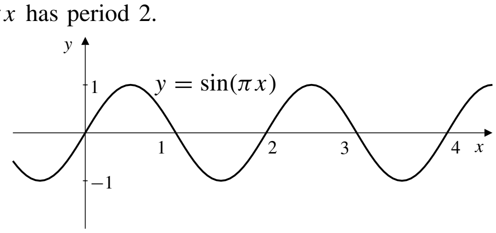


Fig. P.7.21

22. $\cos\frac{\pi x}{2}$ has period 4.

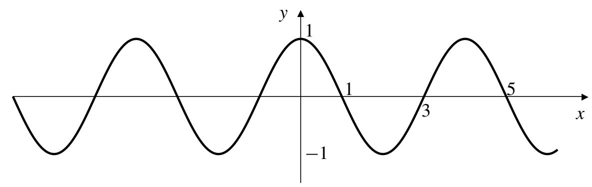
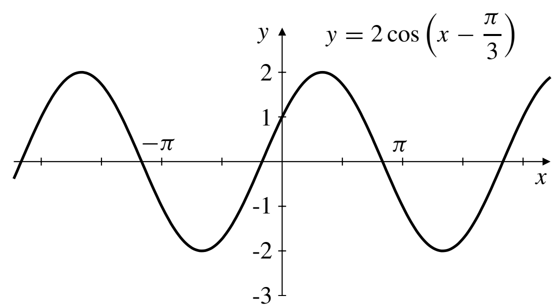
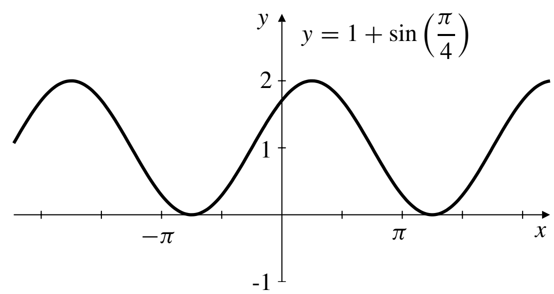


Fig. P.7.22

23.



24.



25. $\sin x = \frac{3}{5}, \quad \frac{\pi}{2} < x < \pi$

$\cos x = -\frac{4}{5}, \quad \tan x = -\frac{3}{4}$

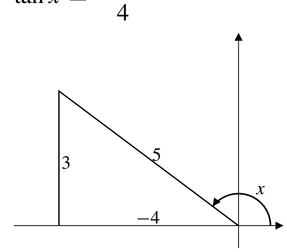


Fig. P.7.25

26. $\tan x = 2$ where x is in $[0, \frac{\pi}{2}]$. Then $\sec^2 x = 1 + \tan^2 x = 1 + 4 = 5$. Hence, $\sec x = \sqrt{5}$ and $\cos x = \frac{1}{\sec x} = \frac{1}{\sqrt{5}}$,

$\sin x = \tan x \cos x = \frac{2}{\sqrt{5}}$.

27. $\cos x = \frac{1}{3}, \quad -\frac{\pi}{2} < x < 0$

$\sin x = -\frac{\sqrt{8}}{3} = -\frac{2}{3}\sqrt{2}$

$\tan x = -\frac{\sqrt{8}}{1} = -2\sqrt{2}$

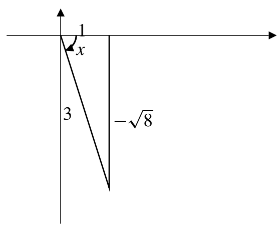


Fig. P.7.27

28. $\cos x = -\frac{5}{13}$ where x is in $\left[\frac{\pi}{2}, \pi\right]$. Hence,

$\sin x = \sqrt{1-\cos^2 x} = \sqrt{1-\frac{25}{169}} = \frac{12}{13},$

$\tan x = -\frac{12}{5}.$

29. $\sin x = -\frac{1}{2}, \quad \pi < x < \frac{3\pi}{2}$

$\cos x = -\frac{\sqrt{3}}{2}$

$\tan x = \frac{1}{\sqrt{3}}$

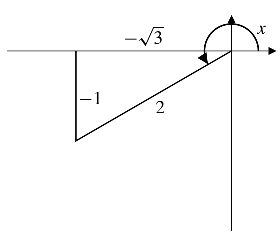


Fig. P.7.29

30. $\tan x = \frac{1}{2}$ where x is in $[\pi, \frac{3\pi}{2}]$. Then,

$\sec^2 x = 1 + \frac{1}{4} = \frac{5}{4}$. Hence,

$\sec x = -\frac{\sqrt{5}}{2}, \quad \cos x = -\frac{2}{\sqrt{5}},$

$\sin x = \tan x \cos x = -\frac{1}{\sqrt{5}}.$

31. $c = 2, \quad B = \frac{\pi}{3}$

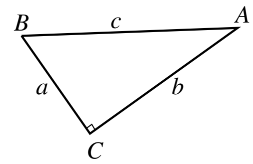
$a = c\cos B = 2 \times \frac{1}{2} = 1$

$b = c\sin B = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

32. $b = 2, \quad B = \frac{\pi}{3}$

$\frac{2}{a} = \tan B = \sqrt{3} \Rightarrow a = \frac{2}{\sqrt{3}}$

$\frac{2}{c} = \sin B = \frac{\sqrt{3}}{2} \Rightarrow c = \frac{4}{\sqrt{3}}$



33. $a = 5, \quad B = \frac{\pi}{6}$

$b = a\tan B = 5 \times \frac{1}{\sqrt{3}} = \frac{5}{\sqrt{3}}$

$c = \sqrt{a^2+b^2} = \sqrt{25+\frac{25}{3}} = \frac{10}{\sqrt{3}}$

34. $\sin A = \frac{a}{c} \Rightarrow a = c\sin A$

35. $\frac{a}{b} = \tan A \Rightarrow a = b\tan A$

36. $\cos B = \frac{a}{c} \Rightarrow a = c\cos B$

37. $\frac{b}{a} = \tan B \Rightarrow a = b\cot B$

38. $\sin A = \frac{a}{c} \Rightarrow c = \frac{a}{\sin A}$

39. $\frac{b}{c} = \cos A \Rightarrow c = b\sec A$

40. $\sin A = \frac{a}{c}$

41. $\sin A = \frac{a}{c} = \frac{\sqrt{c^2-b^2}}{c}$

42. $\sin A = \frac{a}{c} = \frac{a}{\sqrt{a^2+b^2}}$

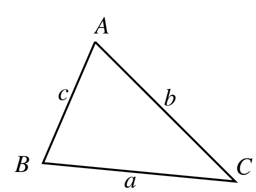
43. $a = 4,\ b = 3,\ A = \frac{\pi}{4}$

$\sin B = b\frac{\sin A}{a} = \frac{3}{4}\frac{1}{\sqrt{2}} = \frac{3}{4\sqrt{2}}$

44. Given that $a = 2, b = 2, c = 3$.
Since $a^2 = b^2 + c^2 - 2bc\cos A$,

$\cos A = \frac{a^2-b^2-c^2}{-2bc}$

$= \frac{4-4-9}{-2(2)(3)} = \frac{3}{4}.$



45. $a = 2,\ b = 3,\ c = 4$
$b^2 = a^2 + c^2 - 2ac\cos B$
Thus $\cos B = \frac{4+16-9}{2\times 2\times 4} = \frac{11}{16}$

$\sin B = \sqrt{1-\frac{11^2}{16^2}} = \frac{\sqrt{256-121}}{16} = \frac{\sqrt{135}}{16}$

46. Given that $a = 2,\ b = 3,\ C = \dfrac{\pi}{4}$.
$c^2 = a^2+b^2-2ab\cos C = 4+9-2(2)(3)\cos\dfrac{\pi}{4} = 13-\dfrac{12}{\sqrt{2}}$.
Hence, $c = \sqrt{13-\dfrac{12}{\sqrt{2}}} \approx 2.12479$.

47. $c = 3,\ A = \dfrac{\pi}{4},\ B = \dfrac{\pi}{3}$ implies $C = \dfrac{5\pi}{12}$
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow a = \frac{1}{\sqrt{2}}\frac{3}{\sin\left(\frac{5\pi}{12}\right)}$$
$$\begin{aligned} a &= \frac{3}{\sqrt{2}}\frac{1}{\sin\left(\frac{7\pi}{12}\right)} \\ &= \frac{3}{\sqrt{2}}\frac{2\sqrt{2}}{1+\sqrt{3}} \text{ (by \#5)} \\ &= \frac{6}{1+\sqrt{3}} \end{aligned}$$

48. Given that $a = 2,\ b = 3,\ C = 35°$. Then
$c^2 = 4+9-2(2)(3)\cos 35°$, hence $c \approx 1.78050$.

49. $a = 4,\ B = 40°,\ C = 70°$
Thus $A = 70°$.
$\dfrac{b}{\sin 40°} = \dfrac{4}{\sin 70°}$ so $b = 4\dfrac{\sin 40°}{\sin 70°} = 2.736$

50. If $a = 1, b = \sqrt{2}, A = 30°$, then $\dfrac{\sin B}{b} = \dfrac{\sin A}{a} = \dfrac{1}{2}$.
Thus $\sin B = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$, $B = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$, and
$C = \pi - \left(\dfrac{\pi}{4}+\dfrac{\pi}{6}\right) = \dfrac{7\pi}{12}$ or $C = \pi - \left(\dfrac{3\pi}{4}+\dfrac{\pi}{6}\right) = \dfrac{\pi}{12}$.
Thus, $\cos C = \cos\dfrac{7\pi}{12} = \cos\left(\dfrac{\pi}{4}+\dfrac{\pi}{3}\right) = \dfrac{1-\sqrt{3}}{2\sqrt{2}}$ or
$\cos C = \cos\dfrac{\pi}{12} = \cos\left(\dfrac{\pi}{3}-\dfrac{\pi}{4}\right) = \dfrac{1+\sqrt{3}}{2\sqrt{2}}$.
Hence,
$$\begin{aligned} c^2 &= a^2+b^2-2ab\cos C \\ &= 1+2-2\sqrt{2}\cos C \\ &= 3-(1-\sqrt{3}) \text{ or } 3-(1+\sqrt{3}) \\ &= 2+\sqrt{3} \text{ or } 2-\sqrt{3}. \end{aligned}$$
Hence, $c = \sqrt{2+\sqrt{3}}$ or $\sqrt{2-\sqrt{3}}$.

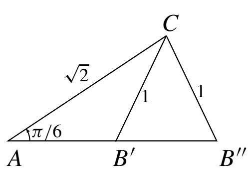


Fig. P.7.50

51. Let h be the height of the pole and x be the distance from C to the base of the pole.
Then $h = x\tan 50°$ and $h = (x+10)\tan 35°$
Thus $x\tan 50° = x\tan 35° + 10\tan 35°$ so
$$\begin{aligned} x &= \frac{10\tan 35°}{\tan 50° - \tan 35°} \\ h &= \frac{10\tan 50°\tan 35°}{\tan 50° - \tan 35°} \approx 16.98 \end{aligned}$$
The pole is about 16.98 metres high.

52. See the following diagram. Since $\tan 40° = h/a$, therefore $a = h/\tan 40°$. Similarly, $b = h/\tan 70°$.
Since $a + b = 2$ km, therefore,
$$\begin{aligned} \frac{h}{\tan 40°} + \frac{h}{\tan 70°} &= 2 \\ h = \frac{2(\tan 40°\tan 70°)}{\tan 70° + \tan 40°} &\approx 1.286 \text{ km.} \end{aligned}$$

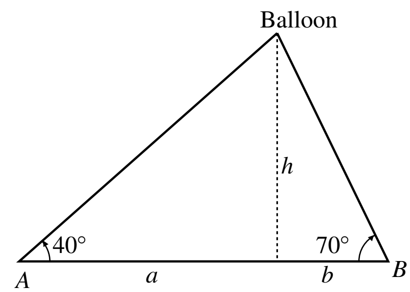


Fig. P.7.52

53. Area $\triangle ABC = \dfrac{1}{2}|BC|h = \dfrac{ah}{2} = \dfrac{ac\sin B}{2} = \dfrac{ab\sin C}{2}$
By symmetry, area $\triangle ABC$ also $= \dfrac{1}{2}bc\sin A$

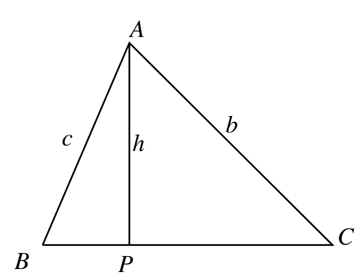


Fig. P.7.53

54. From Exercise 53, area $= \frac{1}{2}ac\sin B$. By Cosine Law, $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$. Thus,

$$\begin{aligned}\sin B &= \sqrt{1-\left(\frac{a^2+c^2-b^2}{2ac}\right)^2}\\ &= \frac{\sqrt{-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2a^2c^2}}{2ac}.\end{aligned}$$

Hence, Area $= \dfrac{\sqrt{-a^4-b^4-c^4+2a^2b^2+2b^2c^2+2a^2c^2}}{4}$ square units. Since,

$$\begin{aligned}&s(s-a)(s-b)(s-c)\\ &\quad= \frac{b+c+a}{2}\,\frac{b+c-a}{2}\,\frac{a-b+c}{2}\,\frac{a+b-c}{2}\\ &\quad= \frac{1}{16}\Big((b+c)^2-a^2\Big)\Big(a^2-(b-c)^2\Big)\\ &\quad= \frac{1}{16}\Big(a^2\Big((b+c)^2+(b-c)^2\Big)-a^4-(b^2-c^2)^2\Big)\\ &\quad= \frac{1}{16}\Big(2a^2b^2+2a^2c^2-a^4-b^4-c^4+2b^2c^2\Big)\end{aligned}$$

Thus $\sqrt{s(s-a)(s-b)(s-c)}$ = Area of triangle.